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# Abelian Hall Fluids and Edge States: a Conformal Field Theory Approach

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**Abstract**

We show that a Coulomb gas Vertex Operator representation of 2D Conformal Field Theory gives a complete description of abelian Hall fluids: as an euclidean theory in two space dimensions leads to the construction of the ground state wave function for planar and toroidal geometry and characterizes the spectrum of low energy excitations; as a $1+1$ Minkowski theory gives the corresponding dynamics of the edge states. The difference between a generic Hall fluid and states of the Jain's sequences is emphasized and the presence, in the latter case, of of an $\hat{U}(1) \otimes \hat{SU}(n)$ extended algebra and the consequent propagation on the edges of a single charged mode and $n-1$ neutral modes is discussed.

## 1. Introduction

Since the fractional quantized Hall effect (FQHE) was first observed by Tsui, Stormer and Gossard[1] in 1982, considerable experimental and theoretical progress[2] has been made toward a physical understanding and a formal characterization of this intrinsically collective phenomenon. An important element, common to many recent developments, is a better understanding of the interplay between the physics of the incompressible Hall fluid of the bulk and the dynamics of the gapless excitations on the edges of the sample. Indeed, for the simple case of filling $\nu = 1$, the quantization of the Hall conductance may be related to the perfect transmission of free electrons edge states[3] and, more generally, even in the case of multiple edge channels, stability of the edge currents against impurity perturbation is expected to be crucial for explaining the experimentally observed Hall plateaux.

A natural and unified description of the properties of the bulk and of the edges of a Hall system is provided by a 2D Conformal Field Theory (2DCFT) approach[4]. In this framework the main tool is a Coulomb gas Vertex Operator representation, that translates in a precise mathematical form the physical idea that the quasiparticles of the Hall fluid arise from the binding of electrons and magnetic vortices. In the case of the Laughlin states[5], corresponding to a filling factor $\nu = 1/m$, the electron Vertex Operator (VO) is a $U(1)$ field of conformal weight $m/2$ and the corresponding ground state wave function (gswf) is given by an appropriate correlator of such fields. Therefore the Laughlin wave function satisfies a Knizhnik-Zamolodchikov equation for an abelian Wess-Zumino model[6], expressing the existence of a non trivial connection due to the presence of infinitely thin magnetic flux tubes located at the positions of the particles. Furthermore, one can show that in order to build the Laughlin gswf's on a compact Riemann surface one needs a set of VO's, characterized by the integer lattice $\mathbf{Z}/m\mathbf{Z}$, in one-to-one correspondence with the $m$-fold degeneracy of the gswf's on a torus, leading to a field theoretical characterization of topological order and providing the spectrum of low lying excitations.

It has been stressed[7,8] that FQHE on closed Riemann surfaces, although experimentally inaccessible, dictates the structure of the edge dynamics. The same $U(1)$ 2DCFT, that as an euclidean theory in two space dimensions leads to the construction of the gswf's, gives, as a theory in $1+1$ space-time dimensions, the dynamics of the chiral edge states. Indeed, by looking at the edge states on a cylinder, one recognizes the existence of $m$ sectors of fractionally charged excitations, that are described by means of the same set of $m$ VO's required to build the gswf's on a torus, expressed in this case in terms of a chiral field propagating on the edge.

The purpose of this paper is to extend this picture in the general case of an abelian quantum Hall fluid, that can be characterized[9] by a symmetric integer valued $n \times n$ matrix, $K$. We shall see that in order to have well defined gswf's on the torus the $K$ matrix must be positive definite. We may recall that this requirement implies that the multiple channel edge currents move all in the same direction[10], and, as a consequence, the total current is not altered by scattering events and the conductance is quantized. Furthermore it has been shown[11,12], for Jain's states[13] corresponding to $K$ matrix not positive definite, that it is necessary to take into account the effect of disorder to explain the observed Hall conductance quantization. Therefore in the following we shall assume that the positivity condition for the $K$ matrix is fulfilled and we shall show that a complete 2DCFT description of a multi component abelian Hall fluid can be achieved by introducing $n$ properly compactified, holomorphic scalar fields. The complete set of inequivalent VO's will be characterized by the $n$ dimensional integer lattice $\mathbf{Z}^n/K\mathbf{Z}^n$, whose points are in one-to-one correspondence with the $\det K$ degenerate gswf's on the torus. This set of VO's will also lead to a full characterization of the low energy fractionally charged excitations and the corresponding edge states.

However there are important differences in the structure of the 2DCFT between the case of a *generic*- in a sense to be later specified- $K$ matrix and the case of the matrix corresponding to a Jain's state of filling $\nu = n/(2np+1)$. We shall see that in the latter case the lattice characterizing the inequivalent VO's may be taken as $\mathbf{Z}/(2np+1)\mathbf{Z}$, in complete analogy with the Laughlin states. Furthermore, an extended algebra $\hat{U}(1) \otimes \hat{SU}(n)$ will naturally appear and, by introducing a formal description of the transport mechanism, we shall see that only the $\hat{U}(1)$ mode contributes to the conductance, while the ones corresponding to $\hat{SU}(n)$ are neutral. The important consequence for the dynamics of the edge states is that, while in the generic case there will be $n$ charged modes propagating on the edges, in the Jain's case there will be a single charged mode and $n-1$ neutral ones. This picture, that confirms the results obtained in refs. 12,14, may be a starting point for understanding the experimental evidence showing that the most prominent Hall plateaux belong to the Jain's sequences.

This paper is organized as follows. In sec. **2** we briefly review the results relative to the Laughlin states, stressing the correspondence between the 2DCFT description on the bulk and on the edge. Sec. **3** is devoted to a general abelian Hall fluid. Once a complete set of VO's is identified, the corresponding gswf's on the torus are constructed, their properties under modular transformations and magnetic translations are analyzed and the response to an applied electric field is evaluated. In sec. **4** the corresponding description for the edge states is given. The Jain's states are discussed in sec. **5**, where it is shown that in a basis that diagonalizes $K$ an $\hat{U}(1) \otimes \hat{SU}(n)$ structure naturally arises; only the $\hat{U}(1)$ mode contributes to the conductance and the center of charge theta function can be decomposed as a sum of products of factors relative to the charged and the neutral modes. Taking the $\hat{SU}(2)$ and $\hat{SU}(3)$ cases as specific examples, the corresponding character decomposition is exhibited. Finally, sec. **6** is devoted to concluding remarks and perspectives.

## 2. Laughlin sequence

In order to present the basic ideas relative to the description of the quantum Hall effect by means of 2DCFT and fix the basic notations, we recall briefly the well known results for the Laughlin sequence[4]. The starting point is the introduction of a holomorphic scalar field $\phi(z)$, with two point correlator:

$$\langle \phi(z_1)\, \phi(z_2) \rangle = -\ln(z_1 - z_2)\ . \tag{1}$$

The mode expansion for the field $\phi(z)$ is given by

$$\phi(z) = \hat{q} - i\hat{p}\ln z + i\sum_{n\neq 0}\frac{a_n}{n}z^{-n}, \tag{2}$$

with coefficients satisfying the usual commutation relations. By taking the field $\phi(z)$ compactified on a circle of radius $R = \sqrt{m}$, i.e. $\phi \equiv \phi + 2\pi\sqrt{m}$, one identifies a set of $m$ inequivalent VO's:

$$V_l(z) =: \exp\left[i\frac{l}{\sqrt{m}}\phi(z)\right] :=: \exp\left[il\sqrt{\nu}\phi(z)\right] :, \qquad l \in \frac{\mathbf{Z}}{m\mathbf{Z}}\,. \tag{3}$$

The operators $V_l$ are primary fields of conformal weight $\Delta = l^2/2m$. Their physical meaning is easily recognized by looking at the braiding relation:

$$V_l(z)\, V_{l'}(z') = (z - z')^{\frac{ll'}{m}} : V_l(z)\, V_{l'}(z') := e^{i\pi\frac{ll'}{m}} V_{l'}(z')\, V_l(z)\ , \tag{4}$$

showing that the VO $V_l$ describes a particle with statistical factor $l^2/2m$, carrying $l$ units of magnetic flux and electric charge $l/m$. The electron field corresponds to the choice $l = m$ and the analytic part of the Laughlin gswf is given by

$$\left\langle V_{\sqrt{m}}(z_1)\ldots V_{\sqrt{m}}(z_N)\right\rangle = \prod_{i<j}^{N}(z_i - z_j)^m\,, \tag{5}$$

where by definition

$$\left\langle \prod_{i=1}^{N} V_{\sqrt{m}}(z_i)\right\rangle \equiv \left\langle N\sqrt{m}\left|\prod_{i=1}^{N} V_{\sqrt{m}}(z_i)\right| 0\right\rangle \tag{6}$$

and

$$\langle p|\, \hat{p} = p\, \langle p|\,, \quad \langle p|\, a_{-n} = 0\,, \quad n > 0\,. \tag{7}$$

Due to the finite energy gap existing for an incompressible Hall fluid, the full spectrum of low energy excitations, corresponding to the full set of operators given by eq. (3), will not play a crucial role at very low temperature. However, their theoretical relevance is due to the one-to-one correspondence between the set of

VO's eq. (3) and the set of vacuum states $|g=1,\, l\rangle$, that enter in the construction of the Laughlin gswf's on a compact genus one Riemann surface:

$$\left\langle \prod_{i=1}^{N} V_{\sqrt{m}}\left(z_i\right) \right\rangle_l^{g=1} \equiv \left\langle N\sqrt{m} \left| \prod_{i=1}^{N} V_{\sqrt{m}}\left(z_i\right) \right| g=1, l \right\rangle \tag{8}$$

More explicitly, on a torus described by $z/L = \xi + \tau\eta$, $\mathrm{Im}\tau > 0$, with $\xi \equiv \xi + 1$ and $\eta \equiv \eta + 1$, pearced by an integer number $N_\Phi$ of magnetic fluxes, in the Landau gauge $\vec{A} = 2\pi N_\Phi(\eta, 0)$, one has

$$\left\langle \prod_{i=1}^{N} V_{\sqrt{m}}\left(z_i\right) \right\rangle_l^{g=1} = \prod_{i<j}^{N} \left[\theta_1\left(z_{ij}|\tau\right)\right]^m \Theta \left[ \begin{array}{c} l/m \\ 0 \end{array} \right] (mZ|m\tau) \equiv F_l\left(\{z_i\}\,|\tau\right) \,, \tag{9}$$

where $Z = \sum_{i=1}^{N} z_i/L$ is the center of charge variable and $z_{ij} = (z_i - z_j)/L$. Here we have introduced the theta functions with rational characteristics[15] defined by

$$\begin{aligned} \Theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z|\tau) &= \sum_{h \in \mathbf{Z}} \exp\left[\pi i \left(h+a\right)^2 \tau + 2\pi i \left(h+a\right)\left(z+b\right)\right] = \\ &= \mathcal{S}_b \mathcal{T}_a \Theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (z|\tau) \,. \end{aligned} \tag{10}$$

where $a, b \in \mathbf{Q}$.

The above equation defines implicitly the magnetic translation operators $\mathcal{S}$ and $\mathcal{T}$. The presence of the product of theta functions of the relative particle positions in eq. (9) is expected because $[-\ln\, \theta_1(z_i - z_j|\tau)]$ is the singular part of the Coulomb propagator on the torus as $[-\ln(z_i - z_j)]$ is on the plane. On the other hand, the center of charge theta function is required in order to satisfy the correct boundary conditions in each particle variable and to give a uniform charge distribution in the thermodynamical limit, as can be easily shown provided the condition $N_\Phi = mN$, expressing the cancellation (in the average) between the external magnetic field and the statistical one, is fulfilled.

The gswf's eq. (9) realize a natural splitting between local and global properties: while the local part is unaffected by total magnetic translations, the center of charge theta functions, and henceforth the complete gswf's, provide an irreducible representation of the subgroup of the total magnetic translation group generated by $\mathcal{S}_{L/N_\Phi}$ and $\mathcal{T}_{L/N_\Phi}$:

$$\begin{aligned} \mathcal{S}_{\frac{L}{N_\Phi}} F_l &= \exp\left[2\pi i \tfrac{l}{m}\right] F_l \,, \\ \mathcal{T}_{\frac{L}{N_\Phi}} F_l &= F_{l+1} \,. \end{aligned} \tag{11}$$

The Hall conductance quantization can be seen as a global property: by introducing, along the $B$ cycle of the torus, a magnetic flux tube of strength $2\pi\lambda$ the Laughin gswf's are transformed as follows:

$$F_l \to F_{l+\lambda} = \mathcal{T}_{\frac{\lambda L}{N_\Phi}} F_l \,, \tag{12}$$

implying[4] a pure Hall conductance $\sigma_H = 1/m$ in natural units.

Although the Hall effect on closed Riemann surfaces is not experimentally accessible, the above argument has immediate consequence on the structure of the edge state excitations. Indeed, consider the cylinder resulting from cutting the torus along the $A$ cycle; then, from eq. (12), one sees that the variation of one unit of flux implies the transfer of a charge $1/m$ from one edge to the other. The charge spectrum of edge states excitations is then in one-to-one correspondence with the set of VO's eq. (3). Furthermore, the same VO's give *bosonized* field operators representing the charged edge excitations, where now $\phi = \phi(x - vt)$ is a chiral field in $1+1$ space-time, and $x$ is a coordinate along the edge.

The 2DCFT provides a complete description of the dynamics of the edge states as the Hamiltonian density of the neutral excitations and the electric current are completely expressed in terms of the chiral field $\phi = \phi(x - vt)$:

$$\mathcal{H} = \frac{v}{8\pi}\left[\frac{1}{v^2}\left(\partial_0\phi\right)^2 + \left(\partial_x\phi\right)^2\right] \,, \tag{13}$$

$$j = \frac{\sqrt{\nu}}{2\pi}\partial_x\phi \,, \tag{14}$$

The current satisfies a $\hat{U}(1)$ Kac-Moody algebra that in terms of Fourier components reads

$$[j_n, j_m] = \nu n \delta_{n+m,0} \,, \tag{15}$$

and the following commutation relation with the Hamiltonian

$$[H, j_n] = -vnj_n \,. \tag{16}$$

In order to stress the difference with an ordinary Fermi liquid, Wen[7] has introduced the term *chiral Luttinger liquid* to describe such a system.

### 3. Abelian Hall Fluid

A general $n$ component Hall fluid may be characterized[9] by a symmetric integer valued matrix $K$, where the element $K_{IJ}$ is the braiding factor between an electron of the $I^{th}$ component and an electron of $J^{th}$ component. To describe such a system in the framework of 2DCFT we introduce a set of $n$ independent holomorphic scalar fields $\phi_i(z)$ whit correlators

$$\langle \phi_i\left(z_1\right) \phi_j\left(z_2\right)\rangle = -\delta_{ij} \ln\left(z_1 - z_2\right) \,. \tag{17}$$

The correct braiding properties between any two electrons are obtained by defining the VO's

$$V_{\vec{\beta}_I}\left(z\right) =: \exp\left[i\vec{\beta}_I \cdot \vec{\phi}\left(z\right)\right] :, \tag{18}$$

where the vectors $\vec{\beta}_I$ satisfies $\vec{\beta}_I \cdot \vec{\beta}_J = K_{IJ}$. A more explicit form for $\vec{\beta}_I$ will be given below. The conformal weight, and therefore the spin, of the VO's eq. (18) is given by $\vec{\beta}_I^2/2$, leading to the requirement $K_{II}$ odd. The explicit form of the generalized Laughlin wave function on the plane is given by

$$\left\langle \prod_{I=1}^{n} \prod_{i=1}^{N_I} V_{\vec{\beta}_I}\left(z_i^I\right)\right\rangle = \prod_{I=1}^{n} \prod_{i<j}^{N_I} \left(z_i^I - z_j^I\right)^{K_{II}} \prod_{I<J}^{n} \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} \left(z_i^I - z_j^J\right)^{K_{IJ}} \,. \tag{19}$$

In order to obtain a complete characterization of the low energy excitations of the system, we proceed as in the case of the Laughlin states and introduce a full set of VO's. To this purpose, we compactify the field $\vec{\phi}$ as follows: $\vec{\phi} \equiv \vec{\phi} + 2\pi R\vec{h}$, where $\vec{h} \in \mathbf{Z}^n$, and $R_T R = K$; the explicit form of $R$ can be easily obtained by the diagonalization of $K$. As a consequence, a complete set of inequivalent VO's is given by

$$V_{\vec{l}}(z) =: \exp\left[i\vec{l}_T R^{-1}\vec{\phi}\left(z\right)\right] :, \qquad \vec{l} \in \frac{\mathbf{Z}^n}{K\mathbf{Z}^n} \equiv \mathbf{Z}_K^n \,. \tag{20}$$

Notice that the electronic VO's $V_{\vec{\beta}_I}$ correspond to the choice $\vec{l} = K\vec{\delta}_I$, where $(\vec{\delta}_I)_J = \delta_{IJ}$. The VO's eq. (20) represent excitations currying a vector of magnetic flux $\vec{l}$ and "charges"

$$Q_I = \left(K^{-1}\right)_{IJ} l_J \,, \tag{21}$$

as it can be seen from the braiding relation

$$V_{\vec{l}}(z)\, V_{\vec{l}'}\,(z') = \exp\left[i\pi \vec{l}_T K^{-1} \vec{l}'\right] V_{\vec{l}'}\,(z')\, V_{\vec{l}}(z) \ . \tag{22}$$

By evaluating the Hall conductance, we shall see shortly that the electrical charge is given by $Q = \sum_I Q_I$. Notice that the electron charge is one as it should.

The set of VO's eq. (20) realizes a consistent description of the system, as it can be seen analyzing the Laughlin gswf's on the torus. We work in the Landau gauge $\vec{A} = 2\pi N_\Phi(\eta, 0)$ and we assume that the condition

$$K_{IJ} N_J = N_\Phi \,, \tag{23}$$

expressing the cancellation (in the average) between statistical and external magnetic field, is fulfilled, implying that the system is at filling factor $\nu = \sum_I N_I/N_\Phi = \sum_{I,J} \left(K^{-1}\right)_{IJ}$.

Then the Laughlin gswf's on the torus are given by

$$F_{\vec{l}}\left(\left\{z_i^I\right\} | \tau\right) = \left\langle \prod_{I=1}^{n} \prod_{i=1}^{N_I} V_{\vec{\beta}_I}\left(z_i^I\right) \right\rangle_{\vec{l}}^{g=1} \,, \tag{24}$$

where we have used again the one-to-one correspondence existing in 2DCFT between the set of VO's (20) and the $g = 1$ vacuum states on the torus. The explicit form of the gswf's is given by

$$\begin{aligned} F_{\vec{l}}\left(\left\{z_i^I\right\} | \tau\right) &= \prod_{I=1}^{n} \prod_{i<j}^{N_I} \left[\theta_1\left(z_{ij}^{II} | \tau\right)\right]^{K_{II}} \prod_{I<J}^{n} \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} \left[\theta_1\left(z_{ij}^{IJ} | \tau\right)\right]^{K_{IJ}} \times \\ &\times \ \Theta\left[\begin{array}{c} K^{-1}\vec{l} \\ 0 \end{array}\right]\left(K\vec{Z} | K\tau\right), \qquad \vec{l} \in \mathbf{Z}_K^n \,, \end{aligned} \tag{25}$$

where $z_{ij}^{IJ} = (z_i^I - z_j^J)/L$, and $Z^I = \sum_{i=1}^{N_I} z_i^I/L$ is the center of charge coordinate of the $I^{th}$ component. For the sake of simplicity, in the following we will use the shorthand notation $\prod' [\theta_1]$ for the products of $\theta_1$-functions appearing in the right hand side of the eq. (25). Here we also have introduced the theta functions of several variables with rational characteristics[15]:

$$\Theta\left[\begin{array}{c}\vec{a}\\ \vec{b}\end{array}\right](\vec{z}|\Omega)=\sum_{\vec{h}\in\mathbf{Z}^n}\exp\left[\pi i\left(\vec{h}+\vec{a}\right)_T\Omega\left(\vec{h}+\vec{a}\right)+2\pi i\left(\vec{h}+\vec{a}\right)\cdot\left(\vec{z}+\vec{b}\right)\right], \quad (26)$$

where $\vec{a},\vec{b}\in\mathbf{Q}^n$ and $\Omega$ is a $n\times n$ symmetric complex valued matrix with positive definite imaginary part. As $Im\tau>0$ , it is now clear why we have taken $K$ to be positive definite. Provided that the condition (23) is verified, one can show that the gswf's eq. (25) give a complete set of states with the correct boundary conditions in each particle variable[16].

There is a strict correspondence between modular invariance of 2DCFT and gauge invariance of the physical system. Indeed, by using the usual modular transformations for the $\theta_1$-functions[15] and appropriate transformation properties for the center of charge theta functions under the transformation $\tau\rightarrow\tilde{\tau}=-\frac{1}{\tau}$, and $z\rightarrow\tilde{z}=\frac{z}{\tau}$, namely

$$\Theta\left[\begin{array}{c}K^{-1}\vec{l}\\ \vec{0}\end{array}\right]\left(K\tau^{-1}\vec{Z}|-K\tau^{-1}\right)=\det\left(\frac{K^{-1}\tau}{i}\right)^{1/2}\exp\left[\pi i\vec{Z}_TK\tau^{-1}\vec{Z}\right]\times$$

$$\times\sum_{\vec{l}'\in\,\mathbf{Z}^n_K}\exp\left[-\pi i\vec{l}_TK^{-1}\vec{l}'\right]\Theta\left[\begin{array}{c}K^{-1}\vec{l}\\ \vec{0}\end{array}\right]\left(K\vec{Z}|K\tau\right), \quad (27)$$

one sees that, provided the condition (23) is satisfied, the following equation holds

$$F_{\vec{l}}\left(\left\{\tilde{z}^I_i\right\}|\tilde{\tau}\right)=\frac{c}{\sqrt{\det K}}\exp\left[\frac{\pi i}{\tau}N_\Phi\sum_{I=1}^{n}\sum_{i=1}^{N_I}\left(z^I_i\right)^2\right]\sum_{\vec{l}'\in\,\mathbf{Z}^n_K}\exp\left[-2\pi i\vec{l}_TK^{-1}\vec{l}'\right]F_{\vec{l}'}\left(\left\{z^I_i\right\}|\tau\right), \quad (28)$$

where $c$ is an inessential constant. The physical meaning of the above equation is more easily understood when we complete the gswf $\Psi_{\vec{l}}$ by multiplying its analytic part eq. (25) by the appropriate gaussian factor, obtaining

$$\Psi_{\vec{l}}\rightarrow\tilde{\Psi}_{\vec{l}}=\frac{c'}{\sqrt{\det K}}\exp\left[2\pi iN_\Phi\sum_{I=1}^{n}\sum_{i=1}^{N_I}\left(\xi^I_i\eta^I_i\right)\right]\sum_{\vec{l}'\in\,\mathbf{Z}^n_K}\exp\left[-2\pi i\vec{l}_TK^{-1}\vec{l}'\right]\Psi_{\vec{l}'} \quad (29)$$

As the exponential factor correspond to the gauge transformation taking from the gauge $\vec{A}_1 = 2\pi N_\Phi (\eta, 0)$ to the gauge $\vec{A}_2 = 2\pi N_\Phi (0, -\xi)$, we see that the eq. (29) express the gauge invariance of the theory.

The global properties of the gswf's are encoded in the center of charge theta function. Let us introduce the total magnetic translation group

$$
\begin{aligned}
\mathcal{S}_b F_{\vec{l}}\left(\left\{z_i^I\right\}\right) &= F_{\vec{l}}\left(\left\{z_i^I + b\right\}\right), \\
\mathcal{T}_a F_{\vec{l}}\left(\left\{z_i^I\right\}\right) &= \exp\left[2\pi i N_\Phi \left(\tfrac{a}{L}\sum_{I=1}^n Z^I + \tau \tfrac{a^2}{2L^2}\sum_{I=1}^n N_I\right)\right] F_{\vec{l}}\left(\left\{z_i^I + a\tau\right\}\right) .
\end{aligned} \tag{30}
$$

$\mathcal{S}_b$ e $\mathcal{T}_a$ are non commuting operators, in particular by taking $a = b = L/N_\Phi$, one has

$$
\mathcal{S}_{\frac{L}{N_\Phi}} \mathcal{T}_{\frac{L}{N_\Phi}} = \exp\left[2\pi i \sum_{I,J=1}^{n} \left(K^{-1}\right)^{IJ}\right] \mathcal{T}_{\frac{L}{N_\Phi}} \mathcal{S}_{\frac{L}{N_\Phi}} = e^{2\pi i \nu} \mathcal{T}_{\frac{L}{N_\Phi}} \mathcal{S}_{\frac{L}{N_\Phi}} . \tag{31}
$$

The gswf's $\left\{F_{\vec{l}},\, \vec{l} \in \mathbf{Z}_K^n\right\}$ provide a basis for an irreducible representation of the subgroup of the magnetic translation group generated by $\mathcal{S}_{L/N_\Phi}$ e $\mathcal{T}_{L/N_\Phi}$. In the language of 2DCFT such operators are explicitly realized in terms of holomorphic field $\vec{\phi}\,(z)$ as follows

$$
\begin{aligned}
\mathcal{S}_{\frac{L}{N_\Phi}} &= \exp\left[\vec{1}_T R^{-1} \oint_A dz \partial \vec{\phi}\,(z)\right] , \\
\mathcal{T}_{\frac{L}{N_\Phi}} &= \exp\left[\vec{1}_T R^{-1} \oint_B dz \partial \vec{\phi}\,(z)\right] ,
\end{aligned} \tag{32}
$$

where $\vec{1}_T = (1, ..., 1)$, and the integrals are taken along the homology cycles of the torus. The translation operator along the $B$ cycle enters directly the evaluation of the conductance of the system. As for the Laughlin states, we introduce a magnetic flux line of strength $\Delta\Phi = 2\pi\lambda$ along the $B$ cycle of the torus. The corresponding change in the periodicity of the gswf's along the $A$ cycle implies the transformation

$$
F_{\vec{l}} \longrightarrow F_{\vec{l}+\lambda\vec{1}} , \tag{33}
$$

that can be explicitly realized by means of the action of the operator $\mathcal{T}_{\lambda L/N_\Phi}$ on $F_{\vec{l}}$ :

$$
F_{\vec{l}+\lambda\vec{1}}\left(\left\{z_i^I\right\}|\tau\right) = \mathcal{T}_{\frac{\lambda L}{N_\Phi}} F_{\vec{l}}\left(\left\{z_i^I\right\}|\tau\right) = \left[\prod{}' \theta_1\right] \Theta \begin{bmatrix} K^{-1}\left(\vec{l} + \lambda\vec{1}\right) \\ 0 \end{bmatrix} \left(K\vec{Z}|K\tau\right) . \tag{34}
$$

As the electric field corresponding to the change of flux is along the A cycle and the translation is along the $B$ cycle, the conductance is purely transverse and is given by Faraday law:

$$\sigma_H = \frac{Q}{\Delta\Phi} = \sum_{I,J=1}^{n} \left(K^{-1}\right)_{IJ} = \frac{\nu}{2\pi} \,. \tag{35}$$

**4. Edge states for the generic abelian case**

As in the simple case of the Laughlin states $\nu = 1/m$, the dynamics of the edge states is, for the generic abelian case, completely dictated by the same 2DCFT leading to the bulk gswf's. As a consequence the hamiltonian for the edge excitations is simply given as a superposition of the free hamiltonian relative to each component of the chiral field $\vec{\phi}$ propagating on the edge of the sample, $\phi_I = \phi_I(x - v_I t)$:

$$\mathcal{H} = \sum_{I=1}^{n} \mathcal{H}^I = \sum_{I=1}^{n} \frac{v_I}{8\pi} \left[ \frac{1}{v_I^2} \left(\partial_0 \phi_I\right)^2 + \left(\partial_x \phi_I\right)^2 \right] \tag{36}$$

where $v_I$ is the propagation velocity of the $I^{th}$ mode. The electromagnetic field does not couples with the same strength to each component of the field $\vec{\phi}$. In order to find the correct structure of the electromagnetic current we require that it should lead to the physical value of the Hall conductance. We can therefore read its structure out of the form of the magnetic translation operator eq. (32). Then

$$J = \sum_{I=1}^{n} j^I \,, \tag{37}$$

where

$$j^I = \sum_{J=1}^{n} \frac{1}{2\pi} \left(R^{-1}\right)_{IJ} \partial_x \phi_J \,. \tag{38}$$

Notice that currents satisfy the $\hat{U}(1)^n$ Kac-Moody algebra, that in terms of Fourier components reads

$$\left[ j_k^I, j_{k'}^J \right] = \left(K^{-1}\right)_{IJ} k \delta_{k+k',0} \,, \tag{39}$$

in agreement with the results of ref. 10.

The hamiltonian eq. (36) can be rewritten in terms of the physical edge currents instead of the chiral field $\vec{\phi}$:

$$H = 2\pi \sum_{I,J=1}^{n} \sum_{k>0} V^{IJ} j_k^I j_{-k}^J, \tag{40}$$

where

$$\left(R_T^{-1} V R^{-1}\right)_{IJ} = v_I \delta_{IJ}. \tag{41}$$

Notice that a non diagonal compactification matrix implies an interaction between edge currents.

The spectrum of all possible charged edge excitations is again given by the set of $\det K$ VO's, eq. (20), written in terms of the chiral field $\vec{\phi}$ defined on the edge, and the corresponding charge is obtained trough the equation

$$\left[J\left(x\right), V_{\vec{l}}(x')\right] = \delta\left(x - x'\right) \sum_I \left(K^{-1}\right)_{IJ} l_J V_{\vec{l}}(x) \ . \tag{42}$$

We see that the normalization of the currents chosen in eqs. 37, (38) in order to be consistent with the Hall conductance determines the correct value of the charge as is given by eq. (21). The introduction of the $n$ component vector $\vec{\phi}$ leads then for a generic abelian Hall fluid to the existence of $n$ branches of charged excitations, each corresponding to a $\hat{U}(1)$ current. We shall see in next section that for a certain set of $K$ matrices a different structure for 2DCFT arises. As a consequence, the structure of the edge states will be correspondingly modified.

## 5. Jain's sequences

The description of a generic abelian Hall fluid, as a $n$ component 2DCFT characterized by an integer valued, non singular symmetric matrix $K$, is essentially kinematical in nature, and does not lead to specific dynamical prediction on the stability of the different states. In contrast, there is a striking experimental evidence that the most prominent Hall plateaux are seen at the fillings of the principal sequence $\nu = n/(2n \pm 1)$ and of the next stable sequence at $\nu = n/(4n \pm 1)$. It was first suggested by Jain[13] the idea of looking at the FQHE for electrons at filling $n/(2pn \pm 1)$ as a manifestation of the integer effect for *composite fermions*, obtained by attaching to each electron an even number of flux units opposite to the external magnetic field. This approach has found further evidence in observation that the energy gaps, measured for the principal sequence[17], correspond to the cyclotron energies relative to the reduced magnetic field $B - B_{1/2}$ and from the analysis of the accumulation point of the principal sequence at $\nu = 1/2$, showing the existence of many features typical of a Fermi surface[18]. Furthermore, in this framework there is a natural explanation for measurements of non-local four terminal magneto-resistance[19], showing that there is no indication of edge state dissipationless conduction near filling $\nu = 1/2$. However it has been shown that Jain's approach can be extended to an arbitrary abelian Hall fluid by introducing a non trivial connection that takes into account the braiding factor between any two particles[20]. All efforts to clarify the specific nature of Jain's states is, therefore, extremely relevant.

Let us then recall that for the Jain's sequences corresponding to filling factors $\nu = n/(2pn+1)$ the structure of the $K$ matrix is the following:

$$K = \mathbf{1} + 2pC\,, \tag{43}$$

where $\mathbf{1}$ is the unit $n \times n$ matrix and $C$ is an $n \times n$ matrix with all entries equal to 1. The corresponding compactification matrix is given by

$$R = \mathbf{1} + \tfrac{1}{n}\left(\sqrt{\tfrac{n}{\nu}} - 1\right)C\,. \tag{44}$$

Notice that $R$ is transformed in its inverse by sending $\nu$ into the integer value $\nu' = n^2/\nu$. By recalling that $\vec{\beta}_I = R\vec{\delta}_I$ we see that

$$\vec{\beta}_I - \vec{\beta}_J = \vec{\delta}_I - \vec{\delta}_J\,. \tag{45}$$

We shall call a matrix $K$, such that the difference between two of the corresponding vectors $\vec{\beta}_I$ has integer entries, *degenerate*. The rational behind this denomination is that, as it has been shown by Cappelli, Trugenberger and Zemba[14], the corresponding Hall fluid is characterized by a degenerate representation of the $W_{1+\infty}$ algebra.

In order to unveil the consequence of the peculiar structure of the $K$ matrix given by the eq. (43), we recall that it can be diagonalized by means of an orthogonal transformation, $K = O_T K_{diag} O$, where

$$O = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & ... & ... & ... & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 & ... & ... & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -4/\sqrt{12} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & 0 \\ 1/\sqrt{n} & 1/\sqrt{n} & ... & ... & ... & ... & 1/\sqrt{n} \end{pmatrix} \tag{46}$$

We introduce the new set of fields

$$\Phi_i = O_{ij}\phi_j, \quad i = 1, 2, \dots n-1\,, \tag{47}$$

$$\Phi_+ = O_{nj}\phi_j = \frac{1}{\sqrt{n}}(\phi_1 + \phi_2 \dots \phi_n)\,. \tag{48}$$

As the matrix $O$ is orthogonal the new fields are still independent:

$$\langle \Phi_i(z)\Phi_j(z')\rangle = -\delta_{ij}\ln(z - z')\,. \tag{49}$$

We can then write the scalar product $\vec{\beta}_I \cdot \vec{\phi}(z)$ in the form

$$\vec{\beta}_I \cdot \vec{\phi}(z) = \frac{1}{\sqrt{\nu}}\Phi_+(z) + \vec{u}_J \cdot \vec{\Phi}(z)\,, \tag{50}$$

where the $n$, $(n-1)$-dimensional vectors $\vec{u}_I$ are given by the columns of the matrix $O$ with the last row omitted:

$$O = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & ... & \vec{u}_n \\ 1/\sqrt{n} & 1/\sqrt{n} & ... & 1/\sqrt{n} \end{pmatrix}. \tag{51}$$

They satisfy the following relations

$$\sum_{I=1}^{n} \vec{u}_I = 0, \quad \sum_{I=1}^{n} (\vec{u}_I)_a(\vec{u}_I)_b = \delta_{ab}, \quad \vec{u}_I \cdot \vec{u}_J = \delta_{IJ} - \frac{1}{n}, \tag{52}$$

and are strictly related to the lattice of roots and weights of $SU(n)$: the simple roots $\vec{\alpha}_I$ are explicitly given by

$$\vec{\alpha}_I = \vec{u}_I - \vec{u}_{I+1}, \tag{53}$$

and the fundamental weight $\vec{\Lambda}_I$ by

$$\vec{\Lambda}_I = \sum_{J=1}^{I} \vec{u}_J, \quad I = 1, ..., n-1\,. \tag{54}$$

Out of the VO's $V_{\vec{u}_I} =: \exp(i\,\vec{u}_I \cdot \vec{\Phi}) :$ one can build the currents corresponding to the off-diagonal generators of the affine $\hat{SU}(n)$ algebra[21]

$$J_{\vec{\alpha}} =: e^{i\vec{u}_I\cdot\vec{\Phi}} e^{-i\vec{u}_J\cdot\vec{\Phi}} :, \tag{55}$$

where $\vec{\alpha} = \vec{u}_I - \vec{u}_J$ is a root of $SU(n)$. In order to close the $\hat{SU}(n)$ affine algebra, one has to introduce the diagonal currents

$$J_I =: e^{i\vec{u}_I\cdot\vec{\Phi}} e^{-i\vec{u}_I\cdot\vec{\Phi}} := i\partial\Phi_I, \quad I = 1, \ldots, n-1. \tag{56}$$

The group structure can be easily seen by looking at the operator product expansion (OPE) for the currents:

$$J_I(z)J_J(w) \sim 0\,, \tag{57}$$

$$J_{\vec{\alpha}}(z)J_{-\vec{\alpha}}(w) \sim \frac{1}{(z-w)^2} + \frac{\sum_{I=1}^{n-1}(\vec{\alpha})_I J_I(w)}{(z-w)}\,, \tag{58}$$

$$J_I(z)J_{\vec{\alpha}}(w) \sim -\frac{(\vec{\alpha})_I J_{\vec{\alpha}}(w)}{(z-w)}\,, \tag{59}$$

$$J_{\vec{\alpha}}(z)J_{\vec{\alpha}'}(w) \sim \frac{J_{\vec{\alpha}+\vec{\alpha}'}(w)}{(z-w)}\,, \tag{60}$$

where the last equation holds if $\vec{\alpha} + \vec{\alpha}'$ belongs to the root lattice, otherwise the right hand side is zero.

It is interesting to notice that only the field $\Phi_+$ contributes to the Hall conductance, implying that it is the only charged mode. This can be seen by recalling that the conductance is obtained through the action of the magnetic translation operator along the $B$ cycle of the torus, eq. (34), and noticing that

$$\mathcal{T}_{\frac{L_1}{N_\phi}} = \exp\left[\vec{1}_t R^{-1} \oint_B dz \partial\vec{\phi}\,(z)\right] = \exp\left[\sqrt{\nu}\oint_B dz\partial\Phi_+\,(z)\right] \tag{61}$$

This suggests of looking in some detail the structure of the center of charge theta functions responsible for the global properties of the system. We first show that they will decompose into a sum of terms, that are products of the contribution of the charge and neutral modes. To this purpose, let us introduce the matrix $\Lambda$

given by $\Lambda = DO$ where $O$ is given by eq. (46) and $D$ is a diagonal matrix with the following entries

$$D = diag\left(\sqrt{2}, \sqrt{6}, ..., \sqrt{i\,(i+1)}, ..., \sqrt{(n-1)\,n}, \sqrt{n}\right). \tag{62}$$

Its usefulness can be seen by noticing that it diagonalizes both $K$ and $K^{-1}$, $\Lambda K = K_{diag}\Lambda$, $\Lambda K^{-1} = K^{-1}_{diag}\Lambda$, that the new variables $\vec{W} = \Lambda\vec{Z}$ are such that only $W_n = Z_1 + \ldots + Z_n$ is affected by total translations, while the remaining are left invariant, and finally that the center of charge theta function can be written as a theta function on the lattice $\Lambda\mathbf{Z}^n$, namely:

$$\Theta\left[\begin{array}{c} K^{-1}\vec{l} \\ \vec{0} \end{array}\right]\left(K\vec{Z}|K\tau\right) = \Theta_\Lambda\left[\begin{array}{c} K^{-1}_{diag}\vec{L} \\ \vec{0} \end{array}\right]\left(K_{diag}D^{-2}\vec{W}|K_{diag}D^{-2}\tau\right) \tag{63}$$

where $\vec{L} = \Lambda\vec{l}$, and the function $\Theta_\Lambda$ is defined as in eq. (26), except that the sum over $\vec{h}$ runs on $\vec{h} \in \Lambda\mathbf{Z}^n$ instead of $\vec{h} \in \mathbf{Z}^n$. The vectors $\vec{L}$, that belong to $\Lambda\mathbf{Z}^n$, are defined modulo $\Lambda K\mathbf{Z}^n = K_{diag}\Lambda\mathbf{Z}^n$ and one can show that a set of inequivalent values is given by $\left\{\vec{L} = ln\vec{\delta}_n; l = 1, \ldots, 2pn+1\right\}$, corresponding to $\left\{\vec{l} = l\vec{1}; l = 1, \ldots, 2pn+1\right\}$. Since the lattice $\Lambda\mathbf{Z}^n$ can be expressed as follows:

$$\Lambda\mathbf{Z}^n = \bigcup_{a=1}^{\det\Lambda}\left\{\vec{r}^{(a)} + D^2\mathbf{Z}^n\right\} \tag{64}$$

where $\det\Lambda = n!$, and the explicit expression for the integer vectors $\vec{r}^{(a)}$ is easily evaluated in any specific case, we obtain the following decomposition of the center of charge theta function:

$$\begin{aligned}\Theta\left[\begin{array}{c} \frac{l}{2pn+1}\vec{1} \\ \vec{0} \end{array}\right]\left(K\vec{Z}|K\tau\right) &= \sum_{a=1}^{\det\Lambda}\left\{\prod_{i=1}^{n-1}\Theta\left[\begin{array}{c} \frac{r_i^{(a)}}{i(i+1)} \\ 0 \end{array}\right](W_i|i\,(i+1)\,\tau) \times\right. \\ &\times\left.\Theta\left[\begin{array}{c} \frac{r_n^{(a)}}{n} + \frac{\nu l}{n} \\ 0 \end{array}\right]\left(\frac{n}{\nu}W_n|\frac{n^2}{\nu}\tau\right)\right\}.\end{aligned} \tag{65}$$

It is clear that in each term of the sum only the factor depending on $W_n$ contributes to the Hall conductance and corresponds then to the charged mode, while the remaining $n-1$ correspond to the neutral ones.

In order to stress the connection of this decomposition with the structure $\hat{U}(1)\otimes \hat{SU}(n)$ that we have discussed with reference to VO's, we take $\vec{Z}=0$, recovering the characters of our 2DCFT. For the sake of simplicity, we will analyze the case $n=2$ and $n=3$. In the first case one has:

$$\begin{aligned}\Theta\left[\begin{array}{c}\frac{l}{4p+1}\vec{1}\\ \vec{0}\end{array}\right](\vec{0}|K\tau) &= \Theta\left[\begin{array}{c}0\\ 0\end{array}\right](0|2\tau)\,\Theta\left[\begin{array}{c}\frac{1}{2}+\frac{l}{4p+1}\\ 0\end{array}\right](0|4\tau/\nu)+\\ &+\ \Theta\left[\begin{array}{c}1/2\\ 0\end{array}\right](0|2\tau)\,\Theta\left[\begin{array}{c}\frac{l}{4p+1}\\ 0\end{array}\right](0|4\tau/\nu)\ ,\end{aligned} \tag{66}$$

that correspond to the character decomposition

$$\chi^{l}=\chi_{SU(2)_1,\Lambda=0}\chi^{l,1}_{U(1)}+\chi_{SU(2)_1,\Lambda=1/2}\chi^{l,0}_{U(1)} \tag{67}$$

where $\chi_{SU(2)_1,s}$ is the character of the affine $\hat{SU}(2)$ level 1 representation of highest weight $s$ and $\chi^{l,i}_{U(1)}$ is the the character of a $\hat{U}(1)$ theory of conformal weight $1/2\nu$, summed over the lattice $2\mathbf{Z}+i+l\nu$, where $l$ (resp. $i$) is defined modulo $4p+1$ (resp. 2). In a similar fashion for $n=3$ we have:

$$\chi^{l}=\chi_{SU(3)_1,\vec{\Lambda}=(0,0)}\chi^{l,0}_{U(1)}+\chi_{SU(3)_1,\vec{\Lambda}=(1/\sqrt{2},1/\sqrt{6})}\chi^{l,1}_{U(1)}+\chi_{SU(3)_1,\vec{\Lambda}=(0,1/\sqrt{6})}\chi^{l,2}_{U(1)} \tag{68}$$

where the notation relative to $\hat{SU}(3)$ characters is self explanatory and $\chi^{l,i}_{U(1)}$ is the character of a $\hat{U}(1)$ theory of conformal weight $1/2\nu$ summed over the lattice $3\mathbf{Z}+i+l\nu$, with $l$ modulo $6p+1$ and $i$ modulo 3.

Finally, by eq. 36, it immediately seen that the extended algebra $\hat{U}(1)\otimes\hat{SU}(n)$ introduced in terms of the VO's correspond to a symmetry of the edge dynamics as long as the difference in velocity among the components is disregarded. Furthermore the structure of the edge excitations is again determined by the general forms of chiral the VO's eq. (20), on the edge. In terms of the new basis the generic VO is given by

$$V_{\vec{l}}(z)=:e^{i\left(\frac{\sqrt{\nu}}{n}\sum_{I=1}^{n-1}l_I\right)\Phi_+}::e^{i\sum_{I=1}^{n-1}l_I\vec{u}_I\cdot\vec{\Phi}}:,\qquad \vec{l}\in\frac{\mathbf{Z}^n}{K\mathbf{Z}^n}\equiv\mathbf{Z}^n_K \tag{69}$$

Notice that the electric current, eq. 37, is written only in terms of the $\hat{U}(1)$ mode:

$$J = \frac{1}{2\pi}\sqrt{\nu}\partial_x \Phi_+ \tag{70}$$

and has the same structure as in the Laughlin case. Therefore, the more general excitation correspond to the propagation of a single charged mode and $n-1$ neutral ones. It is also suggestive to notice that for the choice of inequivalent vectors $\vec{l} = l\vec{1}$ with $l = 1, 2, \ldots, \det K = 2np+1$ the set of independent VO's takes the form

$$V_l(z) =: e^{il\sqrt{\nu}\Phi_+} :, \qquad l \in \frac{\mathbf{Z}}{(2np+1)\mathbf{Z}}, \tag{71}$$

in complete analogy to the case of filling $\nu = 1/m$, eq. 3.

## 6. Concluding remarks

In this paper we have shown how construct a consistent 2DCFT description of an arbitrary abelian Hall fluid, under the requirement that the corresponding $K$ matrix is positive definite. Under this condition, that express the basic requirements of having a consistent description on higher genus Riemann surfaces, one obtains a complete characterization of all universal properties, such as the Hall conductance and the spectrum of low energy excitations, that are experimentally accessible as edge state excitations of the system. This theory is kinematical in nature and therefore is not expected to give any dynamical information on the relative stability of the different Hall fluids. However it already leads to a natural distinction between the case of a generic abelian fluid and the case of a fluid belonging to a Jain's sequence, where an extended $\hat{U}(1) \otimes \hat{SU}(n)$ algebra appears. In view of the experimental evidence showing that the most prominent Hall plateaux correspond to Jain's sequences, it is then natural to ask whether this difference may play a dynamical role in selecting the most stable abelian quantum Hall fluids. At an intuitive level the analysis of sec. **5** seems to indicate that this should be the case. In fact any excitation of a Hall fluid belonging to a Jain's sequence may be described, as shown by eq. (71), in complete analogy to a single component Laughlin's state, regardless of the number $n$ of components.

Looking at this problem more closely, it is interesting to notice that the conformal properties of the VO's corresponding to tunneling transitions between different channels determine the relevance of the corresponding impurity driven transition amplitudes, at least at a perturbative level. Therefore, the $K$ matrix contains at a perturbative level full information on the role of disorder for an abelian Hall fluid. What is then typical of the Jain's states[12], under the positivity condition of the $K$ matrix, is that the renormalization group (RG) behaviour for the disorder driven tunneling amplitudes is *universal*, that is independent on the values of $p$ and $n$ and of the edge velocities. This leads to a new fixed point quadratic action, that takes completely into accounts disorder and is insensitive to interactions between different channels, leaving the charged mode uncoupled from the neutral ones.

The situation in completely different for a generic abelian Hall fluid, for which all modes are charged and the RG behaviour of the transition amplitudes is determined by the detailed structure of the $K$ matrix. This may leads, for a given system, to transition amplitudes that are relevant or irrelevant according to the choice of different couples of channels. This suggest that the study of the role of the disorder for a generic abelian Hall fluid may be useful for understanding the experimental observed stability of different fillings.

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