$\mathcal{PT}$-symmetric noncommutative spaces with minimal volume uncertainty relations

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ABSTRACT: We provide a systematic procedure to relate a three dimensional q-deformed oscillator algebra to the corresponding algebra satisfied by canonical variables describing noncommutative spaces. The large number of possible free parameters in these calculations is reduced to a manageable amount by imposing various different versions of $\mathcal{PT}$-symmetry on the underlying spaces, which are dictated by the specific physical problem under consideration. The representations for the corresponding operators are in general non-Hermitian with regard to standard inner products and obey algebras whose uncertainty relations lead to minimal length, areas or volumes in phase space. We analyze in particular one three dimensional solution which may be decomposed to a two dimensional noncommutative space plus one commuting space component and also into a one dimensional noncommutative space plus two commuting space components. We study some explicit models on these type of noncommutative spaces.

1. Introduction

The simplest and most commonly studied version of noncommutative spaces replaces the standard set of commuting coordinates by new ones obeying $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, with $\theta^{\mu\nu}$ being a constant antisymmetric tensor. However, even in the very first proposals on noncommutative spaces [1] the tensor $\theta^{\mu\nu}$ was taken to be a function of the position coordinates, i.e. $\theta^{\mu\nu}(x)$. Further possibilities arise when one breaks the Lorentz invariance of the tensor and allows for a general dependence of position and momenta [2, 3, 4, 5]. It is known for some time that such a scenario leads to the interesting versions of a generalized version of Heisenberg's uncertainty relations [2, 3]. In particular when the commutation relations are modified in such a way that their structure constants involve higher powers of the momenta or coordinates one encounters minimal lengths or momenta, respectively. As a consequence of these type of relations lead to more radical changes in the interpretation

of possible measurements than the conclusions usually drawn from the standard relations. Whereas the conventional relations, which simply have Planck's constant $\hbar$ as structure constant, only prevent that two quantities commuting in this manner, e.g. x and p, can be known simultaneously, the modified versions prohibit that the observables can be known at all below a certain scale, the minimal length or minimal momentum. This scale is usually identified to be of the order of the Planck scale. Combining some of these minimal lengths in two dimension leads to minimal areas and in three dimensions to minimal volumes. The need for such type of noncommutative space structures has arisen in many contexts, such as for instance in certain string theories [6] and models investigating gravitational stability [7]. For a general review on noncommutative quantum mechanics see for instance [8] and for a review on noncommutative quantum field theories see [9, 10].

By now many studies on the structure of such type of generalized canonical relations have been carried out [2, 3, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25], albeit mostly in dimensions less than three. Besides leading to different physical results, a furher crucial difference between the Snyder type noncommutative spaces and those with broken Lorentz invariance is the way they are constructed. Whereas the former spaces can be thought of as arising naturally from deformations based on general twists [26, 27], the construction of the latter is less systematic and is usually based on the deformation of oscillator algebras [28, 11, 3]. In [16, 18, 19] it was shown in one and two dimensions, respectively, how to map q-deformed oscillator algebras onto canonical variables. The main purpose of this manuscript is to extend these considerations to the full three dimensional space. This approach has the advantage that it allows for the explicit construction of the entire Fock space [12, 13, 14].

Our manuscript is organized as follows: As an introduction we explain in section 2 how one can systematically construct canonical variables on a flat noncommutative space starting from standard generators in Fock space by exploiting $\mathcal{PT}$-symmetry to reduce the number of free parameters. In section 3 we extend these considerations to q-deformed oscillator algebras and present in particular one solution in more detail for which we construct in a nontrivial limit a non-Hermitian representation. In section 4 we demonstrate that, depending on the dimension, this solution leads to minimal length, areas or volumes. In section 5 we study the non-Hermitian $\mathcal{PT}$-symmetric versions of the harmonic oscillator on these spaces and in section 6 we state our conclusions.

2. Deformed oscillator algebras and noncommutative spaces

Our starting point is a q-deformed oscillator algebra for the creation and annihilation operators $A_i^\dagger$, A_i as studied for instance in [12, 13, 14, 16, 19]

$$A_i A_j^\dagger - q^{2\delta_{ij}} A_j^\dagger A_i = \delta_{ij}, \quad [A_i^\dagger, A_j^\dagger] = 0, \quad [A_i, A_j] = 0, \qquad \text{for } i,j = 1,2,3; q \in \mathbb{R}. \quad (2.1)$$

In the limit $q \to 1$ we denote $A_i \to a_i$ and recover the standard Fock space commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = 0, \qquad [a_i^\dagger, a_j^\dagger] = 0, \qquad \text{for } i,j = 1,2,3. \quad (2.2)$$

We assume further that the relations (2.2) are linearly related to the standard three dimensional flat noncommutative space characterized by the relations

$$[x_0, y_0] = i\theta_1, \qquad [x_0, z_0] = i\theta_2, \qquad [y_0, z_0] = i\theta_3, \qquad \text{for } \theta_1, \theta_2, \theta_3 \in \mathbb{R}, \quad (2.3)$$
$$[x_0, p_{x_0}] = i\hbar, \qquad [y_0, p_{y_0}] = i\hbar, \qquad [z_0, p_{z_0}] = i\hbar,$$

with all remaining commutators to be zero and the $\theta_1, \theta_2, \theta_3$ denoting the noncommutative constants. The most general linear Ansatz to relate the generators of relations (2.3) and (2.2) reads

$$\varphi_i = \sum_{j=1}^{3} \kappa_{ij} \, a_j + \lambda_{ij} a_j^\dagger, \qquad \text{for } \vec{\varphi} = \{x_0, y_0, z_0, p_{x_0}, p_{y_0}, p_{z_0}\}, \qquad (2.4)$$

where the κ_{ij}, λ_{ij} having dimensions of length or momentum for $i = 1, 2, 3$ or $i = 4, 5, 6$ respectively. The commutation relations obeyed by the canonical variables X, Y, Z, P_x, P_y, P_z associated to the deformed algebra (2.1) are yet unknown and are subject to construction. The algebra they satisfy may be related to (2.1) by similar relations as (2.4), but since the constants κ_{ij} and λ_{ij} are in general complex, this amounts to finding 72 real parameters. To reduce this number to a manageable quantity one can utilize $\mathcal{PT}$-symmetry.

2.1 The role of $\mathcal{PT}$-symmetry

Whereas the momenta and coordinates in (2.3) are Hermitian operators acting on a Hilbert space with standard inner product, this is no longer true for the variables associated to the deformed algebra (2.1) as they become in general non-Hermitian with regard to these inner products. Thus a quantum mechanical or quantum field theoretical models on these spaces will, in general, not be Hermitian in that space. However, it is by now well accepted that one may consider complex $\mathcal{PT}$ symmetric non-Hermitian systems as self-consistent descriptions of physical systems [29, 30]. Guided by these results one may try to identify this symmetry for the noncommutative space relations (2.3). In [31] the authors argue that this would not be possible and one is therefore forced to take the noncommutative constants to be complex. We reason here that this is incorrect and even the standard noncommutative space relations are in fact symmetric under many different versions of $\mathcal{PT}$-symmetry. All one requires to formulate a consistent quantum description is an antilinear involutory map [32] that leaves the relations (2.3) invariant. We identify here the several possibilities:

Taking for instance $\theta_2 = 0$, the algebra (2.3) remains invariant under the following antilinear transformations

$$\mathcal{PT}_\pm : \quad x_0 \to \pm x_0, \quad y_0 \to \mp y_0, \quad z_0 \to \pm z_0, \quad i \to -i,$$
$$p_{x_0} \to \mp p_{x_0}, \; p_{y_0} \to \pm p_{y_0}, \; p_{z_0} \to \mp p_{z_0}. \qquad (2.5)$$

We may also attempt to keep θ_2 different from zero, in which case we have to transform the θ_2 as well in order to achieve the invariance of (2.3)

$$\mathcal{PT}_{\theta_\pm} : \quad x_0 \to \pm x_0, \quad y_0 \to \mp y_0, \quad z_0 \to \pm z_0, \quad i \to -i,$$
$$p_{x_0} \to \mp p_{x_0}, \; p_{y_0} \to \pm p_{y_0}, \; p_{z_0} \to \mp p_{z_0}, \; \theta_2 \to -\theta_2. \qquad (2.6)$$

A further option, which also allows to keep θ_2 different from zero, would be to introduce permutations amongst the different directions

$$\mathcal{PT}_{xz}: \quad \begin{aligned} &x_0 \to z_0, \quad y_0 \to y_0, \quad z_0 \to x_0, \quad i \to -i, \\ &p_{x_0} \to -p_{z_0}, \; p_{y_0} \to -p_{y_0}, \; p_{z_0} \to -p_{x_0}. \end{aligned} \tag{2.7}$$

Clearly all of these maps are involutions $\mathcal{PT}^2 = \mathbb{I}$. In fact, there might in fact be more options. The occurrence of various possibilities to implement the antilinear symmetry is a known feature previously observed for many examples [33, 34, 35] in dimensions larger than one. More restrictions and the explicit choice of symmetry result from the specific physical situation one wishes to describe. For instance $\mathcal{PT}_\pm$ might be appropriate when one deals with a problem in which one direction is singled out, $\mathcal{PT}_{\theta_\pm}$ requires the noncommutative constant θ_2 to appear as a parameter in the model and $\mathcal{PT}_{xz}$ suggest a symmetry along the line $x_0 = z_0$. For the creation and annihilation operators this symmetry could manifest itself in different ways, for instance as $a_i \to \pm a_i$, $a_i^\dagger \to \pm a_i^\dagger$ or by the permutation of indices $a_i \to a_j^\dagger$, $a_i^\dagger \to a_j$ when they label for instance particles in different potentials, see e.g. [36]. Once again the underlying physics will dictate which version one should select. The general reason for the occurrence of these different possibilities are just manifestations of the ambiguities in defining a metric to which the $\mathcal{PT}$-operator is directly related. What needs to be kept in mind is that we only require the symmetry of some antilinear involution [32] in order to obtain a meaningful quantum mechanical description.

2.2 Oscillator algebras of flat noncommutative spaces

Let us first see how to represent a three dimensional oscillator algebra in terms of the canonical variables in three dimensional flat noncommutative space. For definiteness we seek at first a description which is invariant under $\mathcal{PT}_\pm$. The most generic linear Ansatz for the creation and annihilation operators to achieve this is

$$a_1 = \alpha_1 x_0 + i\alpha_2 y_0 + \alpha_3 z_0 + i\alpha_4 p_{x_0} + \alpha_5 p_{y_0} + i\alpha_6 p_{z_0}, \tag{2.8}$$

$$a_2 = \alpha_7 x_0 + i\alpha_8 y_0 + \alpha_9 z_0 + i\alpha_{10} p_{x_0} + \alpha_{11} p_{y_0} + i\alpha_{12} p_{z_0}, \tag{2.9}$$

$$a_3 = \alpha_{13} x_0 + i\alpha_{14} y_0 + \alpha_{15} z_0 + i\alpha_{16} p_{x_0} + \alpha_{17} p_{y_0} + i\alpha_{18} p_{z_0}, \tag{2.10}$$

with dimensional real constants α_i. We note that we have $\mathcal{PT}_\pm : a_i \to \pm a_i, a_i^\dagger \to \pm a_i^\dagger$ for $i = 1, 2, 3$. The nonsequential ordering of the constants in (2.8)-(2.10) is chosen to perform the limit to the two dimensional case in a convenient way. For $\alpha_9, \ldots, \alpha_{18} \to 0$ we recover equation (2.4) in [19]. It is useful to invoke this limit at various stages of the calculation as a consistency check. We then compute that the operators (2.8)-(2.10), expressed on the three dimensional flat noncommutative space (2.3), satisfy the standard Fock space commutation relations (2.2) provided that the following nine constraints hold

$$1 = 2\sum_{j=1}^{3} \left[(2-j)\alpha_{2+k}\alpha_{j+k}\theta_j - (-1)^j \hbar \alpha_{j+k}\alpha_{j+k+3} \right] \qquad \text{for } k = 0, 6, 12, \tag{2.11}$$

$$0 = i(\alpha_p\alpha_q + \alpha_{p+2}\alpha_{q-2})\theta_2 + \sum_{j=1}^{2}(\alpha_{j+p}\alpha_{j+q-p+2} - \alpha_{j+p-1}\alpha_{j+q-2})\theta_{2j-1} \tag{2.12}$$

$$+ \sum_{j=1}^{3}(\alpha_{j+p+2}\alpha_{j+q-p-2} - \alpha_{j+p-1}\alpha_{j+q})\hbar \qquad \text{for } \{p,q\} = \{1,9\}, \{1,15\}, \{7,15\},$$

$$0 = i(\alpha_p\alpha_q + \alpha_{p+2}\alpha_{q-2})\theta_2 - \sum_{j=1}^{2}(-1)^j(\alpha_{j+p}\alpha_{j+q-p+2} + \alpha_{j+p-1}\alpha_{j+q-2})\theta_{2j-1} \tag{2.13}$$

$$- \sum_{j=1}^{3}(-1)^j(\alpha_{j+p+2}\alpha_{j+q-p-2} + \alpha_{j+p-1}\alpha_{j+q})\hbar \quad \text{for } \{p,q\} = \{1,9\}, \{1,15\}, \{7,15\}.$$

It turns out that when keeping $\theta_2 \neq 0$ these equations do not admit a nontrivial solution. However, setting θ_2 to zero we can solve (2.11)-(2.13) for instance by

$$\alpha_2 = -\frac{\alpha_{14}\left(\alpha_7\left(2h\alpha_{14}\alpha_{17} - 2\alpha_{13}\Delta' + 1\right) - 2\alpha_9\alpha_{13}\Delta''\right)}{2\Delta\Delta''}, \tag{2.14}$$

$$\alpha_4 = \frac{h\alpha_7\alpha_{16}\left(-2h\alpha_{14}\alpha_{17} + 2\alpha_{13}\Delta' - 1\right) + \alpha_9\left(2h\alpha_{13}\alpha_{16} - 1\right)\Delta''}{2h\Delta\Delta''}, \tag{2.15}$$

$$\alpha_5 = \frac{\alpha_1\Delta' + \alpha_3\Delta''}{h\alpha_{14}}, \tag{2.16}$$

$$\alpha_6 = \frac{2h\alpha_9\alpha_{13}\alpha_{18}\Delta'' + \alpha_7\left(2h\alpha_{13}\alpha_{18}\Delta' - \alpha_{14}\left(2\alpha_{17}\alpha_{18}h^2 + \theta_3\right)\right)}{2h\Delta\Delta''}, \tag{2.17}$$

$$\alpha_8 = \frac{\alpha_{14}\left(\alpha_1\left(2h\alpha_{14}\alpha_{17} - 2\alpha_{13}\Delta' + 1\right) + 2\alpha_3\alpha_{13}\left(\alpha_{14}\theta_3 - h\alpha_{18}\right)\right)}{2\Delta\Delta''}, \tag{2.18}$$

$$\alpha_{10} = \frac{h\alpha_1\alpha_{16}\left(2h\alpha_{14}\alpha_{17} - 2\alpha_{13}\Delta' + 1\right) - \alpha_3\left(2h\alpha_{13}\alpha_{16} - 1\right)\Delta''}{2h\Delta\Delta''}, \tag{2.19}$$

$$\alpha_{11} = \frac{\alpha_7\Delta' + \alpha_9\Delta''}{h\alpha_{14}}, \tag{2.20}$$

$$\alpha_{12} = \frac{-2h\alpha_3\alpha_{13}\alpha_{18}\Delta'' + \alpha_1\left(\alpha_{14}\left(2\alpha_{17}\alpha_{18}h^2 + \theta_3\right) - 2h\alpha_{13}\alpha_{18}\Delta'\right)}{2h\Delta\Delta''}, \tag{2.21}$$

$$\alpha_{15} = \frac{2h\alpha_{14}\alpha_{17} - 2\alpha\Delta' + 1}{2\Delta''}, \tag{2.22}$$

where we abbreviated $\Delta := \alpha_3\alpha_7 - \alpha_1\alpha_9$, $\Delta' := h\alpha_{16} + \alpha_{14}\theta_1$, $\Delta'' := h\alpha_{18} - \alpha_{14}\theta_3$. Thus we still have nine parameters left at our disposal. In other words the Ansatz (2.8)-(2.10) together with (2.2) enforces the $\mathcal{PT}$-symmetry of the type (2.5).

Inverting the relations (2.8)-(2.10) we may express the dynamical variables in terms of the creation and annihilation operators

$$x_0 = \frac{\alpha_9\alpha_{17} - \alpha_{11}\alpha_{15}}{2\det M_1}(a_1 + a_1^\dagger) + \frac{\alpha_5\alpha_{15} - \alpha_3\alpha_{17}}{2\det M_1}(a_2 + a_2^\dagger) + \frac{\alpha_3\alpha_{11} - \alpha_5\alpha_9}{2\det M_1}(a_3 + a_3^\dagger), \tag{2.23}$$

$$y_0 = \frac{\alpha_{10}\alpha_{18} - \alpha_{12}\alpha_{16}}{2i\det M_2}(a_1 - a_1^\dagger) + \frac{\alpha_6\alpha_{16} - \alpha_4\alpha_{18}}{2i\det M_2}(a_2 - a_2^\dagger) + \frac{\alpha_4\alpha_{12} - \alpha_6\alpha_{10}}{2i\det M_2}(a_3 - a_3^\dagger), \tag{2.24}$$

$$z_0 = \frac{\alpha_{11}\alpha_{13} - \alpha_7\alpha_{17}}{2\det M_1}(a_1 + a_1^\dagger) + \frac{\alpha_1\alpha_{17} - \alpha_5\alpha_{13}}{2\det M_1}(a_2 + a_2^\dagger) + \frac{\alpha_5\alpha_7 - \alpha_1\alpha_{11}}{2\det M_1}(a_3 + a_3^\dagger), \tag{2.25}$$

$$p_{x_0} = \frac{\alpha_{12}\alpha_{14} - \alpha_8\alpha_{18}}{2i \det M_2}(a_1 - a_1^\dagger) + \frac{\alpha_2\alpha_{18} - \alpha_6\alpha_{14}}{2i \det M_2}(a_2 - a_2^\dagger) + \frac{\alpha_6\alpha_8 - \alpha_2\alpha_{12}}{2i \det M_2}(a_3 - a_3^\dagger), \quad (2.26)$$

$$p_{y_0} = \frac{\alpha_7\alpha_{15} - \alpha_9\alpha_{13}}{2 \det M_1}(a_1 + a_1^\dagger) + \frac{\alpha_3\alpha_{13} - \alpha_1\alpha_{15}}{2 \det M_1}(a_2 + a_2^\dagger) + \frac{\alpha_1\alpha_9 - \alpha_3\alpha_7}{2 \det M_1}(a_3 + a_3^\dagger), \quad (2.27)$$

$$p_{z_0} = \frac{\alpha_8\alpha_{16} - \alpha_{10}\alpha_{14}}{2i \det M_2}(a_1 - a_1^\dagger) + \frac{\alpha_4\alpha_{14} - \alpha_2\alpha_{16}}{2i \det M_2}(a_2 - a_2^\dagger) + \frac{\alpha_2\alpha_{10} - \alpha_4\alpha_8}{2i \det M_2}(a_3 - a_3^\dagger), \quad (2.28)$$

where the matrices $M_{1/2}$ have entries

$$(M_l)_{jk} = 6j + 2k + l - 8 \qquad \text{for} \qquad l = 1, 2. \qquad (2.29)$$

These expressions satisfy the commutation relations (2.3) when we invoke the constraints (2.11)-(2.13) and the standard Fock space relation (2.2). In that case we also have the simple relation $\det M_1 \det M_2 = -1/8\hbar^3$. By changing the Ansatz (2.8)-(2.10) appropriately one may also obtain $\mathcal{PT}_{\theta_\pm}$ or $\mathcal{PT}_{xz}$-invariant solutions.

3. Noncommutative space-time from q-deformed creation and annihilation operators

Next we construct the commutation relations for the deformed noncommutative space satisfied by the canonical variables X, Y, Z, P_x, P_y, P_z, which we express linearly in terms of the creation and annihilation operators obeying the deformed algebra (2.1). Guided by the fact that in the limit $q \to 1$ we should recover the relations (2.23)-(2.28) of the previous subsection. We therefore make the Ansatz

$$X = \hat{\kappa}_1(A_1^\dagger + A_1) + \hat{\kappa}_2(A_2^\dagger + A_2) + \hat{\kappa}_3(A_3^\dagger + A_3), \qquad (3.1)$$

$$Y = i\hat{\kappa}_4(A_1^\dagger - A_1) + i\hat{\kappa}_5(A_2^\dagger - A_2) + i\hat{\kappa}_6(A_3^\dagger - A_3), \qquad (3.2)$$

$$Z = \hat{\kappa}_7(A_1^\dagger + A_1) + \hat{\kappa}_8(A_2^\dagger + A_2) + \hat{\kappa}_9(A_3^\dagger + A_3), \qquad (3.3)$$

$$P_x = i\check{\kappa}_{10}(A_1^\dagger - A_1) + i\check{\kappa}_{11}(A_2^\dagger - A_2) + i\check{\kappa}_{12}(A_3^\dagger - A_3), \qquad (3.4)$$

$$P_y = \check{\kappa}_{13}(A_1^\dagger + A_1) + \check{\kappa}_{14}(A_2^\dagger + A_2) + \check{\kappa}_{15}(A_3^\dagger + A_3), \qquad (3.5)$$

$$P_z = i\check{\kappa}_{16}(A_1^\dagger - A_1) + i\check{\kappa}_{17}(A_2^\dagger - A_2) + i\check{\kappa}_{18}(A_3^\dagger - A_3), \qquad (3.6)$$

with $\hat{\kappa}_i = \kappa_i\sqrt{\hbar/(m\omega)}$ for $i = 1, \ldots, 9$ having the dimension of a length and $\check{\kappa}_i = \kappa_i\sqrt{m\omega\hbar}$ for $i = 10, \ldots, 18$ possessing the dimension of a momentum. The constants κ_i for $i = 1, \ldots, 18$ are therefore dimensionless. We deliberately keep here all dimensional constants different from 1. With the help of the q-deformed oscillator algebra (2.1) we compute

$$[X, Y] = 2i\sum_{j=1}^3 \hat{\kappa}_j\hat{\kappa}_{3+j}\left[1 + \left(q^2 - 1\right)\right]A_j^\dagger A_j, \qquad (3.7)$$

$$[Y, Z] = -2i\sum_{j=1}^3 \hat{\kappa}_{3+j}\hat{\kappa}_{6+j}\left[1 + \left(q^2 - 1\right)\right]A_j^\dagger A_j, \qquad (3.8)$$

$$[X, P_x] = 2i\sum_{j=1}^3 \hat{\kappa}_j\check{\kappa}_{9+j}\left[1 + \left(q^2 - 1\right)\right]A_j^\dagger A_j, \qquad (3.9)$$

$$[Y, P_y] = -2i\sum_{j=1}^3 \hat{\kappa}_{3+j}\check{\kappa}_{12+j}\left[1 + \left(q^2 - 1\right)\right]A_j^\dagger A_j, \qquad (3.10)$$

$$[Z, P_z] = 2i\sum_{j=1}^3 \hat{\kappa}_{6+j}\check{\kappa}_{15+j}\left[1 + \left(q^2 - 1\right)\right]A_j^\dagger A_j, \qquad (3.11)$$

$$[P_x, P_y] = -2i \sum_{j=1}^{3} \check{\kappa}_{9+j} \check{\kappa}_{12+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \tag{3.12}$$

$$[P_y, P_z] = 2i \sum_{j=1}^{3} \check{\kappa}_{12+j} \check{\kappa}_{15+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \tag{3.13}$$

$$[X, P_z] = 2i \sum_{j=1}^{3} \hat{\kappa}_j \check{\kappa}_{15+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \tag{3.14}$$

$$[Z, P_x] = 2i \sum_{j=1}^{3} \hat{\kappa}_{6+j} \check{\kappa}_{9+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \tag{3.15}$$

$$[X, Z] = [P_x, P_z] = [X, P_y] = [Y, P_x] = [Y, P_z] = [Z, P_y] = 0. \tag{3.16}$$

Inverting now the relations (3.1)-(3.6) we find that it is indeed possible to eliminate entirely the creations and annihilation operators from these relations. However, this leads to very lengthy expressions, which we will not present here. Instead we report some special, albeit still quite general, solutions obtained by setting some of the constants to zero and imposing further constraints.

3.1 A particular $\mathcal{PT}_\pm$-symmetric solution

We now make the assumption that $\kappa_1 = \kappa_4 = \kappa_5 = \kappa_8 = \kappa_{10} = \kappa_{12} = \kappa_{13} = \kappa_{14} = \kappa_{17} = \kappa_{18} = 0$. This choice still guarantees that none of the canonical variables become mutually identical. The consistency with the direct limit $q \to 1$ in which we want to recover (2.3) enforces the constraints

$$\hat{\kappa}_2 = \frac{\hbar}{2\check{\kappa}_{11}}, \quad \hat{\kappa}_3 = \frac{\theta_1}{2\hat{\kappa}_6}, \quad \hat{\kappa}_9 = -\frac{\theta_3}{2\hat{\kappa}_6}, \quad \check{\kappa}_{15} = -\frac{\hbar}{2\hat{\kappa}_6}, \quad \check{\kappa}_{16} = \frac{\hbar}{2\hat{\kappa}_7}. \tag{3.17}$$

The only non-vanishing commutators we obtain in this case are

$$[X, Y] = i\theta_1 + i\frac{q^2 - 1}{q^2 + 1}\frac{\theta_1}{\hbar}\left(\frac{m\omega}{2\kappa_6^2}Y^2 + \frac{2\kappa_6^2}{m\omega}P_y^2\right), \tag{3.18}$$

$$[Y, Z] = i\theta_3 + i\frac{q^2 - 1}{q^2 + 1}\frac{\theta_3}{\hbar}\left(\frac{m\omega}{2\kappa_6^2}Y^2 + \frac{2\kappa_6^2}{m\omega}P_y^2\right), \tag{3.19}$$

$$[X, P_x] = i\hbar + i\frac{q^2 - 1}{q^2 + 1}2m\omega\left(\kappa_{11}^2 X^2 + \frac{1}{4m^2\omega^2\kappa_{11}^2}P_x^2 + \frac{\theta_1^2\kappa_{11}^2}{\hbar^2}P_y^2 + 2\frac{\theta_1\kappa_{11}^2}{\hbar}XP_y\right), \tag{3.20}$$

$$[Y, P_y] = i\hbar + i\frac{q^2 - 1}{q^2 + 1}2m\omega\left(\frac{1}{4\kappa_6^2}Y^2 + \frac{\kappa_6^2}{m^2\omega^2}P_y^2\right), \tag{3.21}$$

$$[Z, P_z] = i\hbar + i\frac{q^2 - 1}{q^2 + 1}2m\omega\left(\frac{1}{4\kappa_7^2}Z^2 + \frac{\kappa_7^2}{m^2\omega^2}P_z^2 + \frac{\theta_3^2}{4\hbar^2\kappa_7^2}P_y^2 - \frac{\theta_3}{2\hbar^2\kappa_7^2}ZP_y\right). \tag{3.22}$$

Notice that we still have the three free parameters κ_6, κ_7 and κ_{11} at our disposal. It is easily verified that the relations (3.18)-(3.22) are left invariant under a $\mathcal{PT}_\pm$-symmetry (2.3) in the variables X, Y, Z, P_x, P_y, P_z.

3.1.1 Reduced three dimensional solution for $q \to 1$

The solution (3.18)-(3.22) possesses a non-trivial limit leading to an even simpler set of commutation relations. For this purpose we impose some additional constraints by setting

first $\check{\kappa}_{11} = m\omega\hat{\kappa}_6$, $\kappa_7 = 1/2\kappa_6$, $q = \exp(2\tau\kappa_6^2)$ and subsequently we take the limit $\kappa_6 \to 0$. The relations (3.18)-(3.22) then reduce to

$$[X, Y] = i\theta_1 \left(1 + \hat{\tau}Y^2\right), \quad [Y, Z] = i\theta_3 \left(1 + \hat{\tau}Y^2\right), \tag{3.23}$$

$$[X, P_x] = i\hbar \left(1 + \check{\tau}P_x^2\right), \quad [Y, P_y] = i\hbar \left(1 + \hat{\tau}Y^2\right), \quad [Z, P_z] = i\hbar \left(1 + \check{\tau}P_z^2\right), \tag{3.24}$$

where $\hat{\tau} = \tau m\omega/\hbar$ has the dimension of an inverse squared length, $\check{\tau} = \tau/(m\omega\hbar)$ has the dimension of an inverse squared momentum and τ is dimensionless. We find a concrete representation for this algebra in terms of the generators of the standard three dimensional flat noncommutative space (2.3)

$$\begin{aligned}
X &= (1 + \check{\tau}p_{x_0}^2)x_0 + \tfrac{\theta_1}{\hbar}\left(\check{\tau}p_{x_0}^2 - \hat{\tau}y_0^2\right)p_{y_0}, & P_x &= p_{x_0}, \\
Z &= (1 + \check{\tau}p_{z_0}^2)z_0 + \tfrac{\theta_3}{\hbar}\left(\hat{\tau}y_0^2 - \check{\tau}p_{z_0}^2\right)p_{y_0}, & P_z &= p_{z_0}, \\
P_y &= (1 + \check{\tau}p_{y_0}^2)p_{y_0}, & Y &= y_0.
\end{aligned} \tag{3.25}$$

Evidently the quantities X, Z and P_y are non-Hermitian in the space in which the x_0, y_0, z_0, p_{x_0}, p_{y_0}, p_{z_0} are Hermitian. In order to study concrete models it is very convenient to carry out a subsequent Bopp-shift of the form $x_0 \to x_s - \tfrac{\theta_1}{\hbar}p_{y_s}$, $y_0 \to y_s$, $z_0 \to z_s + \tfrac{\theta_3}{\hbar}p_{y_s}$, $p_{x_0} \to p_{x_s}$, $p_{y_0} \to p_{y_s}$, $p_{z_0} \to p_{z_s}$ and express the generators in (3.25) in terms of the standard canonical variables. Since there is no explicit occurrence of θ_2, the representation (3.25) is trivially invariant under $\mathcal{PT}_{\pm}$ as well as $\mathcal{PT}_{\theta_{\pm}}$. Taking, however, the representation (3.25) and in addition $\theta_2 \neq 0$ this evidently changes, as by direct computation one of the commutation relations is altered to $[X, Z] = i\theta_2 \left(1 + \check{\tau}P_x^2\right)\left(1 + \check{\tau}P_z^2\right)$. Setting furthermore $\theta_1 = \theta_3$ the representation in (3.25) is also invariant under the $\mathcal{PT}_{xz}$-symmetry stated in (2.7).

3.1.2 Reduction into a decoupled two dimensional plus a one dimensional space

The algebra (3.18)-(3.22) provides a larger three dimensional setting for a noncommutative two dimensional space decoupled from a standard one dimensional space. This is achieved by parameterizing $q = \exp(2\tau\kappa_6^2)$, setting $\check{\kappa}_{11} = m\omega\hat{\kappa}_6$, $\theta := \theta_1$ and subsequently taking the limit $(\theta_3, \kappa_6) \to 0$ reduces the algebra to a two noncommutative dimensional space in the X,Y-direction

$$[X, Y] = i\theta \left(1 + \hat{\tau}Y^2\right), \qquad [Y, P_y] = i\hbar \left(1 + \hat{\tau}Y^2\right), \qquad [X, P_x] = i\hbar \left(1 + \check{\tau}P_x^2\right), \tag{3.26}$$

decoupled from a standard one dimensional space in the Z-direction

$$[Z, P_z] = i\hbar, \qquad [Y, Z] = 0. \tag{3.27}$$

As a representation for the algebra (3.26) in flat noncommutative space we may simply use (3.25) with the appropriate limit $\theta_3 \to 0$. Carrying out the corresponding Bopp-shift $x_0 \to x_s - \tfrac{\theta}{\hbar}p_{y_s}$, $y_0 \to y_s$, $p_{x_0} \to p_{x_s}$ and $p_{y_0} \to p_{y_s}$ yields the operators

$$X = x_s - \frac{\theta}{\hbar}p_{y_s} + \check{\tau}p_{x_s}^2 x_s - \hat{\tau}\frac{\theta}{\hbar}y_s^2 p_{y_s}, \quad Y = y_s, \quad P_x = p_{x_s}, \quad \text{and} \quad P_y = p_{y_s} + \hat{\tau}y_s^2 p_{y_s}, \tag{3.28}$$

which are of course still non-Hermitian with regard to the standard inner product.

3.1.3 Reduction into three decoupled one dimensional spaces

We conclude this section by noting that all three directions in the algebra (3.18)-(3.22) can be decoupled, of which one becomes a one dimensional noncommutative space previously investigated by many authors, e.g. [2, 16]. It is easy to verify that this scenario is obtained from (3.18)-(3.22) when parameterizing $q = \exp(2\tau\kappa_{11}^2)$ and subsequently taking the limit $(\theta_1, \theta_3, \kappa_{11}) \to 0$. The remaining non-vanishing commutators are then

$$[X, P_x] = i\hbar\left(1 + \check{\tau}P_x^2\right), \qquad [Y, P_y] = i\hbar, \qquad \text{and} \qquad [Z, P_z] = i\hbar. \qquad (3.29)$$

Thus all three space directions are decoupled from each other. It is known that the choices $X = (1 + \check{\tau}p_s^2)x_s$, $P_x = p_s$ or $X' = X^\dagger = x_s(1 + \check{\tau}p_s^2)$, $P'_x = p_s$ constitute representations for the commutation relations (3.29) in the X-direction.

3.2 A particular $\mathcal{PT}_{\theta_\pm}$-symmetric solution

Instead of solving the complicated relations (3.7)-(3.16) one may also start by making directly an Ansatz of a similar form as in (3.25) without elaborating on the relation to the q-deformed oscillator algebra. Proceeding in this manner with an Ansatz respecting the $\mathcal{PT}_{\theta_\pm}$-symmetry we find for instance the representation

$$\begin{aligned}
X &= x_0 - \hat{\tau}\frac{\theta_1}{\hbar}y_0^2 p_{y_0} - \hat{\tau}\frac{\theta_2}{\hbar}y_0^2 p_{z_0}, & P_x &= p_{x_0}, \\
Z &= z_0 + \hat{\tau}\frac{\theta_3}{\hbar}y_0^2 p_{y_0} + \hat{\tau}\frac{\theta_2}{\hbar}\frac{\theta_3}{\theta_1}y_0^2 p_{z_0}, & P_z &= p_{z_0}, \\
P_y &= p_{y_0} + \hat{\tau}y_0^2 p_{y_0}, & Y &= y_0,
\end{aligned} \qquad (3.30)$$

yielding the closed algebra

$$\begin{aligned}
[X, Y] &= i\theta_1\left(1 + \hat{\tau}Y^2\right), & [X, Z] &= i\theta_2\left(1 + \hat{\tau}Y^2\right), & [Y, Z] &= i\theta_3\left(1 + \hat{\tau}Y^2\right), \\
[X, P_x] &= i\hbar, & [Y, P_y] &= i\hbar\left(1 + \hat{\tau}Y^2\right), & [Z, P_z] &= i\hbar,
\end{aligned} \qquad (3.31)$$

with all remaining commutators vanishing. Notice that if we set $\theta_1 = -\theta_3$ the generators in (3.30) are also invariant under the $\mathcal{PT}_{xz}$-symmetry.

4. Minimal length, minimal areas and minimal volumes

Let us now investigate the generalized uncertainty relations associated to the algebras constructed above. In general, the uncertainties ΔA and ΔB resulting from a simultaneous measurement of two observables A and B have to obey the inequality

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle[A, B]\rangle_\rho\right|. \qquad (4.1)$$

Here $\langle.\rangle_\rho$ denotes the inner product on a Hilbert space with metric ρ in which the operators A and B are Hermitian, as discussed in more detail in [2, 16, 18, 19]. The minimal length $\Delta A_{\min}$, that is the precision up to which the observable A can be known by giving up all the information on B is then computed by minimising $\Delta A\Delta B - \frac{1}{2}\left|\langle[A, B]\rangle_\rho\right|$ as a function of ΔB. In the standard scenario, i.e. when A and B commute up to a constant, the result is therefore usually zero. This outcome changes when the commutator $[A, B]$ involves higher

powers of ΔB, in which case we encounter the interesting scenario of non-vanishing $\Delta A_{\min}$. We now investigate some of the solutions presented above. Depending now on the question we ask, i.e. which quantities we attempt to measure, the minimal uncertainties for some specific operators turn out to be different.

4.1 A three dimensional noncommutative space giving rise to minimal areas

We start with our simplest three dimensional solution, that is the algebra (3.23)-(3.24). If we just want to measure the position of the particle on such a space independently of its momentum we only have to investigate the relations (3.23). Taking $\tau > 0$ and following the logic of [16, 18, 19], we obtain from (3.23) for a simultaneous measurement of all space coordinates non-vanishing minimal length in two directions

$$\Delta X_{\min} = |\theta_1|\sqrt{\hat{\tau}}\sqrt{1 + \hat{\tau}\langle Y\rangle_\rho^2}, \quad \Delta Y_{\min} = 0, \quad \text{and} \quad \Delta Z_{\min} = |\theta_3|\sqrt{\hat{\tau}}\sqrt{1 + \hat{\tau}\langle Y\rangle_\rho^2}. \quad (4.2)$$

Thus any measurement of space will involve an unavoidable uncertainty of an area A of size $\Delta A_0 = 4\hat{\tau}|\theta_1\theta_3|$ in the XZ-plane and no uncertainty in the Y-direction. Changing our question and attempt to measure instead all coordinates and all components of the momenta, we need to analyze the entire set of relations (3.23)-(3.24). The analysis of the equations (3.24) alone yields

$$\Delta X_{\min} = \hbar\sqrt{\tilde{\tau}}\sqrt{1 + \hat{\tau}\langle Y\rangle_\rho^2}, \quad \Delta Y_{\min} = 0, \quad \text{and} \quad \Delta Z_{\min} = \hbar\sqrt{\tilde{\tau}}\sqrt{1 + \hat{\tau}\langle Y\rangle_\rho^2}, \quad (4.3)$$

$$\Delta(P_x)_{\min} = 0, \qquad \Delta(P_y)_{\min} = \hbar\sqrt{\hat{\tau}}\sqrt{1 + \hat{\tau}\langle Y\rangle_\rho^2}, \quad \text{and} \quad \Delta(P_z)_{\min} = 0. \quad (4.4)$$

Thus, depending now on whether $|\theta_1|, |\theta_3| < 1$ or $|\theta_1|, |\theta_3| > 1$ the uncertainties in (4.2) or (4.3) will be smaller, respectively. For any type of measurement the region of uncertainty will be an area.

4.2 A three dimensional noncommutative space giving rise to minimal volumes

Let us now analyze our solution (3.23)-(3.24) before taking the limit $q \to 0$. We compute the uncertainties with regard to a measurement of all components of the coordinates and all components of the momenta. Since now the quantities are all coupled, in the sense that we do not have any nontrivial subalgebra, we will encounter uncertainties for all of them and observe a different type of behaviour as indicated in the previous subsection. Starting with a simultaneous Y, P_y-measurement we compute from (4.1) with (3.21) the uncertainties

$$\Delta Y_{\min} = |\hat{\kappa}_6|\sqrt{\frac{1}{2}(q^2 - q^{-2}) + (q - q^{-1})^2\left(\frac{1}{4\hat{\kappa}_6^2}\langle Y\rangle_\rho^2 + \frac{\hat{\kappa}_6^2}{\hbar^2}\langle P_y^2\rangle_\rho\right)}, \qquad (4.5)$$

$$\Delta(P_y)_{\min} = \frac{\hbar}{2|\hat{\kappa}_6|}\sqrt{\frac{1}{2}(q^2 - q^{-2}) + (q - q^{-1})^2\left(\frac{1}{4\hat{\kappa}_6^2}\langle Y\rangle_\rho^2 + \frac{\hat{\kappa}_6^2}{\hbar^2}\langle P_y^2\rangle_\rho\right)}, \qquad (4.6)$$

under the assumption that $q > 1$. The absolute minimal uncertainties resulting from these expressions are therefore

$$\Delta Y_0 = \frac{|\hat{\kappa}_6|}{\sqrt{2}}\sqrt{q^2 - q^{-2}}, \qquad \text{and} \qquad \Delta(P_y)_0 = \frac{\hbar}{2\sqrt{2}|\hat{\kappa}_6|}\sqrt{q^2 - q^{-2}}. \qquad (4.7)$$

Next we carry out a simultaneous X,Y-measurement and a Y,Z-measurement by employing (3.18) and (3.19), respectively. We find the minimal lengths

$$\Delta X_{\min} = \left| \frac{\theta_1}{\hat{\kappa}_6} \right| \sqrt{\frac{1}{2} \frac{q - q^{-1}}{q + q^{-1}} + \left[\frac{q - q^{-1}}{q + q^{-1}} \right]^2 \left[\frac{1}{4\hat{\kappa}_6^2} \langle Y \rangle_\rho^2 + \frac{\hat{\kappa}_6^2}{\hbar^2} \left(\langle P_y^2 \rangle_\rho + \Delta \left(P_y \right)_0^2 \right) \right]}, \quad (4.8)$$

$$\Delta Z_{\min} = \left| \frac{\theta_3}{\hat{\kappa}_6} \right| \sqrt{\frac{1}{2} \frac{q - q^{-1}}{q + q^{-1}} + \left[\frac{q - q^{-1}}{q + q^{-1}} \right]^2 \left[\frac{1}{4\hat{\kappa}_6^2} \langle Y \rangle_\rho^2 + \frac{\hat{\kappa}_6^2}{\hbar^2} \left(\langle P_y^2 \rangle_\rho + \Delta \left(P_y \right)_0^2 \right) \right]}. \quad (4.9)$$

There is no minimal length in the Y-direction resulting from these relations. Using the expression for $\Delta \left(P_y \right)_0$ from (4.7), the absolute minimal values for these uncertainties are

$$\Delta X_0 = \frac{1}{2\sqrt{2}} \left| \frac{\theta_1}{\hat{\kappa}_6} \right| \sqrt{q^2 - q^{-2}}, \qquad \text{and} \qquad \Delta Z_0 = \frac{1}{2\sqrt{2}} \left| \frac{\theta_3}{\hat{\kappa}_6} \right| \sqrt{q^2 - q^{-2}}. \qquad (4.10)$$

Thus a measurement of the position in space will be accompanied by an uncertainty volume V of the size

$$\Delta V_0 = \frac{1}{\sqrt{2}} \left| \frac{\theta_1 \theta_3}{\hat{\kappa}_6} \right| \left(q^2 - q^{-2} \right)^{3/2}. \qquad (4.11)$$

The evaluation for the simultaneous X,P_x and Z,P_z-measurements are slightly more complicated due to the occurrence of the XP_y and ZP_y terms in (3.20) and (3.22), respectively. We proceed similarly as before and make also use of the well known inequalities $|A + B| \geq |A| - |B|$ and $|\langle AB \rangle| \leq \Delta A \Delta B + |\langle A \rangle \langle B \rangle|$. We report here only the final result of the absolute minimal values

$$\Delta \left(P_i \right)_0 = \frac{\gamma_i \Delta \left(P_y \right)_0 - \sqrt{\beta_i \left[\alpha_i \gamma_i^2 \Delta \left(P_y \right)_0^2 + \lambda_i (1 - 4\alpha_i \beta_i) \right]}}{4\alpha_i \beta_i - 1} \qquad \text{for } i = x, z, \qquad (4.12)$$

with

$$\begin{array}{llll} \alpha_x = \alpha_2, & \beta_x = \alpha_{11}, & \gamma_x = \frac{2|\theta_1|}{\hbar} \alpha_{11}, & \lambda_x = \frac{\hbar}{2} + \alpha_{11} \frac{\theta_1^2}{\hbar^2} \Delta \left(P_y \right)_0^2, \\ \alpha_z = \alpha_7, & \beta_z = \alpha_{16}, & \gamma_z = \frac{|\theta_3| \hbar}{2} \alpha_{16}, & \lambda_z = \frac{\hbar}{2} + \alpha_{16} \theta_3^2 \Delta \left(P_y \right)_0^2, \end{array} \qquad (4.13)$$

where $\alpha_i = \hat{\kappa}_i^2 (q - q^{-1})/(q + q^{-1})\hbar$ for $i = 2, 7$ and $\alpha_i = \check{\kappa}_i^2 (q - q^{-1})/(q + q^{-1})\hbar$ for $i = 11, 16$. Further restrictions do not emerge.

By similar reasoning one finds non-vanishing $\Delta X_{\min}$, $\Delta Z_{\min}$ and $\Delta P_{y\,\min}$ for the PT_{xz}-invariant algebra (3.31).

5. Models on PT-symmetric noncommutative spaces

5.1 The one dimensional harmonic oscillator on a noncommutative space

We commence with the one-dimensional harmonic oscillator on the $PT_\pm$-symmetric noncommutative space described by (3.29). The corresponding Hamiltonian

$$H_{ncho}^{1D} = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 = H_{ho}^{1D} + \frac{m\omega^2}{2} \left(\check{\tau} p_s^2 x_s^2 + \check{\tau} x_s p_s^2 x_s + \check{\tau}^2 p_s^2 x_s p_s^2 x_s \right) = H_{ho}^{1D} + H_{nc}^{1D}, \qquad (5.1)$$

is evidently non-Hermitian with regard to the standard inner product. However, it is $\mathcal{PT}_{\pm}$-symmetric, such that it might constitutes a well-defined self-consistent description of a physical system. The associated Schrödinger equation $H_{ncho}^{1D}\psi = E\psi$ is most conveniently solved in p-space, i.e. with $x_s = i\hbar\partial_{p_s}$ it reads

$$\frac{m\omega^2\hbar^2}{2}(1 + \check{\tau}p_s^2)^2\psi'' + \tau\omega\hbar p_s(1 + \check{\tau}p_s^2)\psi' + \left(E - \frac{p_s^2}{2m}\right)\psi = 0. \tag{5.2}$$

Using the transformation

$$\mu = \frac{\sqrt{1 + 2Em\check{\tau}}}{\tau}, \qquad \nu = \frac{\sqrt{4 + \tau^2}}{2\tau} - \frac{1}{2} \quad \text{and} \quad z = ip_s\sqrt{\check{\tau}}, \tag{5.3}$$

we convert (5.2) into

$$(1 - z^2)\psi'' - 2z\psi' + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - z^2}\right]\psi = 0, \tag{5.4}$$

which is the standard differential equation for the associated Legendre polynomials $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ admitting the general solution

$$\psi(z) = c_1 P_\nu^\mu(z) + c_2 Q_\nu^\mu(z). \tag{5.5}$$

Seeking asymptotically vanishing solutions gives rise to the quantization condition $\mu + \nu = -n - 1$ with $n \in \mathbb{N}$. With (5.3) it follows therefore that the eigenenergies becomes

$$E_n = \omega\hbar\left(\frac{1}{2} + n\right)\sqrt{1 + \frac{\tau^2}{4}} + \tau\frac{\omega\hbar}{4}(1 + 2n + 2n^2) \qquad \text{for } n \in \mathbb{N}_0. \tag{5.6}$$

The expression agrees with the one found in [2]. The polynomial $Q_\nu^\mu(z)$ is not defined for these values, such that $c_2 = 0$ and $P_\nu^\mu(z)$ reduces to

$$\psi_{2n-i}(z) = c_1 \sum_{k=i}^{2n-i} \frac{1}{k!}\left[\prod_{l=i}^{\lceil\frac{k+i-2}{2}\rceil} 2(n-l)(2n + 2\nu - 2l + 1)\right] z^k \frac{(z^2 - 1)^{\frac{\nu-2n-1+i}{2}}}{(-1)^{1+i}(1 - z^2)^\nu}, \tag{5.7}$$

with $i = 0, 1$. Clearly the $\psi_{2n-i}(z)$ vanish for $|z| \to \infty$ if $\nu > -1$, which is always guaranteed for $\tau m\omega > 0$. The Dyson map η which adjointly maps H_{ncho}^{1D} to a Hermitian operator was easily found [2, 16] to be $\eta = \left(1 + \tau P_x^2\right)^{-1/2}$. In addition, we note that the solutions are square integrable $\psi_{2n-i}(z) \in L^2(i\mathbb{R})$ on $\langle\cdot|\,\eta^2\cdot\rangle$ and form an orthonormal basis.

An exact treatment for models in the higher dimensions is more difficult, but we may resort to perturbation theory to obtain some useful insight on the solutions. As a quality gauge we compare here the exact solution against perturbation theory around the standard Fock space harmonic oscillator solution with normalized eigenstates

$$|n\rangle = \frac{(a_{x_s}^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad a_{x_s}|0\rangle = 0, \quad a_{x_s}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a_{x_s}|n\rangle = \sqrt{n}|n-1\rangle. \tag{5.8}$$

A straightforward, albeit lengthy, computation yields the following corrections to the harmonic oscillator energy $E_n^{(0)} = \omega\hbar\left(n + \frac{1}{2}\right)$ for the eigenenergies of H_{ncho}^{1D}

$$E_n^{(p)} = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \mathcal{O}(\tau^3) \tag{5.9}$$

with

$$E_n^{(1)} = \langle n|\, H_{nc}^{1D}\, |n\rangle = \frac{\tau\omega\hbar}{4}\left(1 + 2n + 2n^2\right) + \frac{\tau^2\omega\hbar}{16}\left(3 + 8n + 6n^2 + 4n^3\right), \tag{5.10}$$

$$E_n^{(2)} = \sum_{p\neq n} \frac{\langle n|\, H_{nc}^{1D}\, |p\rangle \langle p|\, H_{nc}^{1D}\, |n\rangle}{E_n^{(0)} - E_p^{(0)}} = -\frac{1}{8}\tau^2\omega\hbar\left(1 + 3n + 3n^2 + 2n^3\right) + \mathcal{O}(\tau^3). \tag{5.11}$$

As it should be, the expression for E_n in (5.6) when expanded up to order τ^3 coincides precisely with $E_n^{(p)}$. We further note that also in a perturbative treatment the eigenenergies are strictly positive.

The validity of these expansions is governed by the well-known sufficient conditions for the applicability of the Rayleigh-Schrödinger perturbation theory to a Hamiltonian of the form $H = H_0 + H_1$ around the solutions of $H_0\,|n\rangle = E_n^{(0)}\,|n\rangle$

$$\left|\frac{\langle p|\, H_1\, |n\rangle}{E_n^{(0)} - E_p^{(0)}}\right| \ll 1 \qquad \text{for all } p \neq n. \tag{5.12}$$

This is guaranteed for (5.1) when $\tau^2 \ll 32/(2n + 13)\sqrt{(4 + n)(3 + n)(2 + n)(1 + n)}$, such that perturbation theory will break down for large values of n.

5.2 The two dimensional harmonic oscillator on a noncommutative space

Next we consider the two-dimensional harmonic oscillator on the $\mathcal{PT}_\pm$-symmetric noncommutative space described by the algebra (3.26). Using the representation (3.28) for this algebra, the corresponding Hamiltonian reads

$$\begin{aligned}
H_{ncho}^{2D} &= \frac{1}{2m}(P_x^2 + P_y^2) + \frac{m\omega^2}{2}(X^2 + Y^2) \\
&= H_{fncho}^{2D} + \frac{\tau\omega}{2\hbar}\left[\{p_{x_0}^2 x_0, x_0\} + \{y_0^2 p_{y_0}, p_{y_0}\} + \frac{\theta}{\hbar}\{p_{x_0}^2 p_{y_0}, x_0\} - \frac{m^2\omega^2\theta}{\hbar}\{y_0^2 p_{y_0}, x_0\}\right] \\
&\quad + \frac{\tau^2\omega^2 m}{2\hbar^2}\left[\left\{y_0^2 p_{y_0}, (1 + \Omega)y_0^2 p_{y_0} - \frac{\theta p_{x_0}^2 x_0}{\hbar} - \frac{\theta^2 p_{x_0}^2 p_{y_0}}{\hbar^2}\right\} + \left(\frac{p_{x_0}^2 x_0}{m\omega} + \frac{\theta p_{x_0}^2 p_{y_0}}{m\omega\hbar}\right)^2\right]
\end{aligned} \tag{5.13}$$

where we used the standard notation for the anti-commutator $\{A, B\} = AB + BA$. Once again this Hamiltonian is non-Hermitian with regard to the inner product on the flat noncommutative space, but it respects a $\mathcal{PT}_\pm$-symmetry. In order to be able to perturb around the standard harmonic oscillator solution we still need to convert flat noncommutative space into the canonical variable x_s, y_s, p_{x_s} and p_{y_s}. Thus when using the representation (3.28)

this Hamiltonian is converted into

$$
\begin{aligned}
H_{ncho}^{2D} = {} & H_{ho}^{2D} + \frac{m\theta^2\omega^2}{2\hbar^2}p_{y_s}^2 - \frac{m\theta\omega^2}{2\hbar}\{x_s, p_{y_s}\} + \frac{\tau}{2}\left[m\omega^2\{p_{x_s}^2 x_s, x_s\}\right. \\
& + \left(\frac{1}{m} + \frac{m\theta^2\omega^2}{\hbar^2}\right)\{y_s^2 p_{y_s}, p_{y_s}\} - \frac{m\theta\omega^2}{\hbar}\left(\{y_s^2 p_{y_s}, x_s\} + \{p_{x_s}^2 x_s, p_{y_s}\}\right)\bigg] \\
& + \frac{\tau^2}{2}\left[\left(\frac{1}{m} + \frac{m\theta^2\omega^2}{\hbar^2}\right)(y_s^2 p_{y_s})^2 + m\omega^2\left(p_{x_s}^2 x_s\right)^2 - \frac{m\theta\omega^2}{\hbar}\{y_s^2 p_{y_s}, p_{x_s}^2 x_s\}\right], \\
= {} & H_{ho}^{2D}(x_s, y_s, p_{x_s}, p_{y_s}) + H_{nc}^{2D}(x_s, y_s, p_{x_s}, p_{y_s}).
\end{aligned}
$$

In this formulation we may now proceed as in the previous subsection and expand perturbatively around the standard two dimensional Fock space harmonic oscillator solution with normalized eigenstates

$$
|n_1 n_2\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}}|00\rangle, \quad a_i^\dagger|n_1 n_2\rangle = \sqrt{n_i + 1}\,|(n_1 + \delta_{i1})(n_2 + \delta_{i2})\rangle, \qquad (5.14)
$$

$$
a_i|00\rangle = 0, \qquad\qquad a_i|n_1 n_2\rangle = \sqrt{n_i}\,|(n_1 - \delta_{i1})(n_2 - \delta_{i2})\rangle, \qquad (5.15)
$$

for $i = 1, 2$. The energy eigenvalues for the Hamiltonian H_{ncho}^{2D} then result to

$$
\begin{aligned}
E_{nl}^p = {} & E_{nl}^{(0)} + E_{nl}^{(1)} + E_{nl}^{(2)} + \mathcal{O}(\tau^2) \qquad\qquad\qquad\qquad\qquad (5.16) \\
= {} & \hbar\omega(n + l + 1) + \langle nl|\, H_{nc}^{2D}\,|nl\rangle + \sum_{p,q \neq n+l=p+q} \frac{\langle nl|\, H_{nc}^{2D}\,|pq\rangle\,\langle pq|\, H_{nc}^{2D}\,|nl\rangle}{E_{nl}^{(0)} - E_{pq}^{(0)}} + \mathcal{O}(\tau^2) \\
= {} & E_{nl}^{(0)} + \frac{\Omega\omega\hbar}{8}\left[(3 + n + 5l) - \Omega(l + \frac{1}{2})\right] + \frac{\tau}{2}\omega\hbar\left[1 + n + n^2 + l + l^2\right. \\
& \frac{\Omega}{4}\left(4 + 3n + n^2 + 7l + 4nl + 5l^2\right)\bigg] + \mathcal{O}(\tau^2),
\end{aligned}
$$

where we introduced the dimensionless quantity $\Omega_i = m^2\theta_i^2\omega^2/\hbar^2$. Notice that unlike as in the one dimensional case the perturbation beyond H_{ho}^{2D} also involves terms of order $\mathcal{O}(\tau^0)$, such that we need to compute also $E_{nl}^{(2)}$ to achieve a precision of first order in τ. We also notice that the energy E_{nl}^p is only bounded from below for $\Omega < 5$. The minus sign is an indication that we will encounter exceptional points [37] and broken $\mathcal{PT}$-symmetry in some parameter range.

5.3 The three dimensional harmonic oscillator on a noncommutative space

Let us finally consider the three-dimensional harmonic oscillator on the noncommutative space described by the algebra (3.18) and (3.19). Using the representation (3.25) together with a subsequent Bopp-shift, the corresponding Hamiltonian can be expressed in terms

of the standard canonical coordinates

$$
\begin{aligned}
H_{ncho}^{3D} &= \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + \frac{m\omega^2}{2}(X^2 + Y^2 + Z^2) = H_{ho}^{3D} + H_{nc}^{3D} \qquad (5.17)\\
&= H_{ho}^{3D} + \frac{m\omega^2}{2\hbar}\left[\theta_3\{p_{y_s},z_s\} - \theta_1\{x_s,p_{y_s}\} + \frac{\theta_1^2 + \theta_3^2}{\hbar}p_{y_s}^2\right]\\
&\quad + \tau\frac{\omega}{2\hbar}\left[\{p_{x_s}^2 x_s, x_s\} + \{p_{z_s}^2 z_s, z_s\} + (1 + \Omega_1 + \Omega_3)\{y_s^2 p_{y_s}, p_{y_s}\}\right.\\
&\quad \left. - \frac{\theta_1}{\hbar}\left(m^2\omega^2\{y_s^2 p_{y_s}, x_s\} + \{p_{x_s}^2 x_s, p_{y_s}\}\right) + \frac{\theta_3}{\hbar}\left(m^2\omega^2\{y_s^2 p_{y_s}, z_s\} + \{p_{z_s}^2 z_s, p_{y_s}\}\right)\right]\\
&\quad + \tau^2\frac{1}{2m\hbar^2}\left[p_{x_s}^2 x_s p_{x_s}^2 x_s + p_{z_s}^2 z_s p_{z_s}^2 z_s + m^2\omega^2(1 + \Omega_1 + \Omega_3)y_s^2 p_{y_s}y_s^2 p_{y_s}\right.\\
&\quad \left. + \frac{\theta_3}{\hbar}m^2\omega^2\{p_{z_s}^2 z_s, y_s^2 p_{y_s}\} - \frac{\theta_1}{\hbar}m^2\omega^2\{p_{x_s}^2 x_s, y_s^2 p_{y_s}\}\right]
\end{aligned}
$$

We expand now around the standard three dimensional Fock space harmonic oscillator solution with normalized eigenstates

$$
|n_1 n_2 n_3\rangle = \prod_{i=1}^3 \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!}}|000\rangle, \quad a_i^\dagger|n_1 n_2 n_3\rangle = \sqrt{n_i + 1}\left|\prod_{j=1}^3 (n_j + \delta_{ij})\right\rangle, \quad (5.18)
$$

$$
a_i|000\rangle = 0, \qquad\qquad a_i|n_1 n_2 n_3\rangle = \sqrt{n_i}\left|\prod_{j=1}^3 (n_j - \delta_{ij})\right\rangle. \qquad (5.19)
$$

for $i = 1, 2, 3$ and compute the energy eigenvalues for H_{ncho}^{3D} to

$$
\begin{aligned}
E_{nlr}^{(p)} &= E_{nlr}^{(0)} + E_{nlr}^{(1)} + E_{nlr}^{(2)} + \mathcal{O}(\tau^2) \qquad (5.20)\\
&= E_{nl}^{(0)} + \langle nlr|H_{nc}^{3D}|nlr\rangle + \sum_{s,p,q\neq n+l+r=p+q+s}\frac{\langle nlr|H_{nc}^{3D}|pqs\rangle\langle pqs|H_{nc}^{3D}|nlr\rangle}{E_{nlr}^{(0)} - E_{pqr}^{(0)}} + \mathcal{O}(\tau^2)\\
&= \omega\hbar\left[\frac{3}{2} + n + l + r + \frac{1}{8}(\Omega_1 + \Omega_3)(3 + 5l) - \frac{1}{16}(2l + 1)(\Omega_1 + \Omega_3)^2 + \frac{1}{8}(n\Omega_1 + r\Omega_3)\right.\\
&\quad + \frac{\tau}{2}\left(n^2 + n + l^2 + r^2 + l + r + \frac{1}{4}(n^2 + 4ln + 3n + 5l^2 + 7l + 4)\Omega_1\right.\\
&\quad \left.\left. + \frac{1}{4}(5l^2 + 4rl + 7l + r^2 + 3r + 4)\Omega_3 + \frac{3}{2}\right)\right].
\end{aligned}
$$

As in the two dimensional case we encounter negative terms in this expression, thus indicating that broken $\mathcal{PT}$-symmetry will be broken in some parameter range.

6. Conclusions

Contrary to some claims in the literature [31], we have demonstrated that it is indeed possible to implement $\mathcal{PT}$-symmetry on noncommutative spaces while keeping the noncommutative constants real. Starting from a generic Ansatz for the canonical variables obeying a q-deformed oscillator algebra, we employed $\mathcal{PT}$-symmetry to limit the amount of free parameters. The relations (3.7)-(3.16) resulting from this Ansatz turned out to be solvable. A specific $\mathcal{PT}_\pm$-symmetric solution was presented in (3.18)-(3.22). Clearly there

exist more solutions with different kinds of properties. We constructed an explicit representation for the algebra obtained in the nontrivial limit $q \to 1$ in terms of the generators of a flat noncommutative space. With regard to the standard inner product for this space, the operators are non-Hermitian. We computed the minimal length and momenta resulting from the generalized uncertainty relations, which overall give rise to minimal areas or minimal volumes in phase space.

Despite being non-Hermitian, due to the built-in $\mathcal{PT}$-symmetry any model formulated in terms of these variables is a candidate for a self-consistent theory with real eigenvalue spectrum. We have studied the harmonic oscillator on these spaces in one, two and three dimensions. The perturbative computation of the energy eigenvalues indicates that there exists a parameter regime for which the $\mathcal{PT}$-symmetry is broken. It would be interesting to investigate this further and determine when this transition precisely occurs. The eigenvalues will also be useful in further investigations [38] allowing for the construction of coherent states related to the algebras presented in section 4.

Obviously there are many more solutions to (3.7)-(3.16), which might be studied in their own right together with models formulated on them. Minor modifications would also allow to investigate the occurrence of upper bounds, i.e. maximal length and momenta [24], giving rise to a second scale in special relativity, so-called double special relativity [39] constituting a possibility to explain the cosmic-ray paradox [40].

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