Measurement and Pricing of Risk in Insurance Markets*

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Abstract

The theory and practice of risk measurement provides a point of intersection between risk management, economic theories of choice under risk, financial economics and actuarial pricing theory. This paper provides a review of these interrelationships, from the perspective of an insurance company seeking to price the risks that it underwrites. We examine three distinct approaches to insurance risk pricing, all being contingent on the concept of risk measures. Risk measures can be interpreted as representations of risk orderings, as well as absolute (monetary) quantifiers of risk. The first approach can be called an 'axiomatic' one, whereby the price for risks is calculated according to a functional determined by a set of desirable properties. The price of a risk is directly interpreted as a risk measure and may be induced by an economic theory of price under risk. The second approach consists in contextualising the considerations of the risk bearer by embedding them in the market where risks are traded. Prices are calculated by equilibrium arguments, where each economic agent's optimisation problem follows from the minimisation of a risk measure. Finally, in the third approach, weaknesses of the equilibrium approach are addressed by invoking alternative valuation techniques, the leading paradigm among which is arbitrage pricing. Such models move the focus from individual decision takers to abstract market price systems and are thus more parsimonious in the amount of information that they require. In this context, risk

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measures, instead of characterising individual agents, are used for determining the set of price systems that would be viable in a market.

Keywords: Risk measures, Insurance Pricing, Choice under Risk, Risk Exchange, Good Deals.

1 Introduction

The theory of risk measures provides a point of intersection between economic theories of choice under risk, actuarial pricing theory and financial economics. In this paper a review of these interrelationships is provided, through the lens of insurance pricing theory. The present exposition starts with the concept of actuarial premium calculation principles, continues with risk exchange equilibrium models, and concludes with some elements of financial economics. In each of these different approaches risk measures play an important role, which we seek to highlight.

The present investigation aspires to give an account of an intellectual journey undertaken by economists and actuaries over the last fifty years and discuss the evolution and interrelationships of insurance pricing models. Historically, insurance pricing models were proposed as responses and correctives to the ones that preceded them. Premium calculation principles were initially defined via an axiomatic view that sought to mimic market prices. This ignored though the interactions between market participants, namely primary insurers, reinsurers and policyholders. To overcome this shortcoming, equilibrium asset pricing models were developed, where premium calculation principles enter insurers' decision making process through their objective function or budget constraint. Such models have their own drawbacks, as they make excessive assumptions on the availability of information on the preferences and risk endowments of market participants, while also assuming perfect competitive markets and linearity of market prices. Financial economics provide the tools to overcome such weaknesses. Arbitrage pricing theory disposes of the need for agent-specific information, market imperfections are accounted for via the theory of incomplete markets, and the resulting price functionals are no longer necessarily linear and thus reflect more consistently actual prices in insurance and financial markets. In the context of this dialogue, risk measures form a recurring theme, emerging within each class of models in a different guise. A discussion from the perspective of risk measures useful, as it provides a common framework through which the different models can be examined and compared. Hence, the current study focuses on models that explicitly utilise the concept of a risk measure, while avoiding a great level of mathematical detail in an attempt to present the concept in an accessible way.

The structure of the rest of the paper is as follows. In the next section, an overview of the

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economic arguments underlying the evolution of insurance pricing theories is given. Premium calculation principles and risk measures are presented in section 3, along with their fundamental properties, representations and links to economic theories of choice. In section 4 the role of premium principles in equilibrium pricing models is discussed. Section 5 examines the link between arbitrage pricing, good deals and risk measures. Finally, a brief summary can be found in Section 6.

2 Perspectives of insurance pricing

Calculating the price of insurance has been one of the central concerns of actuarial science. Traditionally the fair premium in insurance pricing is equated with the expected loss resulting from the underwritten risk. However, as the expected loss (or *net premium*) does not account for the variability of risks nor for the risk aversion of economic agents, it is apparent that more sophisticated mechanisms for the calculation of insurance premiums are called for. Systematic approach to premium calculation were first proposed by Markowitz [47],[48] and Bühlmann [8], who introduced the concept of premium calculation principles. A premium calculation principle is a function that takes as an argument (the probability distribution of) a risk and returns the premium that should be charged for it.

The specification of appropriate functional forms for premium calculation principles has been the subject of much discussion in the actuarial community [37]. There are two distinct, but interrelated, perspectives from which premium calculation could be viewed. One is to require that the properties of a premium principle should reflect the properties of the actual prices charged in insurance markets, as proposed in [56]. This does of course produce a circular argument: premium should be calculated according to a premium principle, which reflects the way that premium is actually calculated. The apparent discrepancy is partially resolved if we accept that the premium calculation principle is not used to calculate actual insurance prices, but to produce an actuarial benchmark. That this benchmark should, at least to some extent, be consistent with market prices is a natural requirement.

An alternative view is to determine insurance premiums via indifference arguments. This means that premium rates should be set in a way such that the insurer is 'not worse off' after selling a contract; the premium, safely invested, should thus offset the potential losses from the contract. What 'not worse off' exactly means will depend on the way in which the insurer's preferences are modelled. Economics provides a variety of such models, termed *theories of choice under risk*, of which von Neumann and Morgenstern's [55] expected utility theory is the most commonly used. Different theories of choice induce different premium principles with alternative sets of properties; this topic is extensively discussed in [54]. [1]

The interpretation of premium as capital used to offset potential insurance losses invites an interpretation of premium calculation principles as risk measures. Risk measures are defined as functions giving the amount of capital that the holder of a risky position should prudently invest so that he is allowed (e.g. by a regulator) to proceed with his investment plans [5]. Insofar, the parallel between premium principles and risk measures reflects the relationship between pricing and capital allocation by an insurance company.

A notable absentee in the preceding discussion is the market in which insurance risks are traded. Premium principles in general depend on the probability distribution of the risk that is to be priced. Moreover, it is sometimes required that the premium depends *only* on the probability distribution of the underlying risk [56]. However, the concept of determining insurance prices in isolation of the market in which the insurance is traded is unrealistic; insurance prices, like much else, are determined by supply and demand. The economics of insurance and financial markets provide us with the tools for calculating insurance premiums, as well as a framework in which to associate premium principles, risk measures and market conditions. Equilibrium asset pricing models are based on the premise that each market participant decides on an investment (which in the case of an insurer will correspond to both underwriting policy and asset allocation) that is optimal with respect to its preferences. Market prices are then determined by the condition that the market clears, that is, every risk is finally either ceded or retained. An individual insurer's investment decision, which is treated as a preference maximization problem, can equivalently be treated as a risk minimization problem. Here risk is defined via the risk measure derived from the theory of choice used to model the insurer's preferences.

Thus equilibrium asset pricing models provide a powerful framework in which we can examine the relationship between premium rates and risk measures in a market setting. It is then fair to ask whether the properties of equilibrium prices actually conform with those of observed prices in insurance markets. The answer given to that question has in general to be a negative one. An important discrepancy is that in equilibrium models asset and liability prices are defined as being linear functions, while it is widely accepted that insurance prices are non-linear [56]. The

[1] It is worth noting that premium calculation principles are not only monetary quantifiers of risk, but through their association with theories of choice also provide an ordering of risks. A key reference on ordering of risk is Kaas et al. [45], while Dhaene and Goovaerts [28] associate stochastic orderings of risk to dependence structures.

inconsistency between actual and theoretical insurance prices is generally attributed to market imperfections. One such imperfection can be due to the fact that not every potentially desirable underwriting/investment portfolio is attainable by trading in the market. Another is the presence of transaction costs, which de facto render prices non-linear [16].

An alternative approach to premium calculation relies on trying to harness the potential of financial economics, specifically arbitrage pricing theory. Such an approach moves the focus from the action of individual decision makers (and the effect of those actions on market prices) to sets of price systems that could possibly exist in a market. Such market prices are characterized by the no-arbitrage condition, which postulates that prices should not create opportunities for making sure trading gains at zero price. In perfect (complete) markets there is only one price system that satisfies the no-arbitrage property; in imperfect markets there will be many (e.g. [29]). Although each of these price systems will be linear, the presence of several of them induces bounds for insurance prices and prices evaluated at these bounds will not be linear. Making these bounds sharper and thus more realistic goes through the concept of 'good deals', which are trading positions generally accepted as worth taking without necessarily being arbitrage opportunities [18]. In defining a 'good deal' one imposes more structure on market prices than the one implied by no arbitrage. This additional structure can be provided by considering risk preferences, potentially modelled by an actuarial premium principle.

3 Premium Calculation Principles

3.1 Premium principles and risk measures

Let $\mathcal{X}$ be a set of random variables, with elements of $\mathcal{X}$ standing for insurance risks. Whenever $X \in \mathcal{X}$ is positive it will be considered to represent a loss. We assume that insurers calculate premia based solely on the distribution of future claims and that premia are subsequently invested with zero interest. The mechanism for calculating insurance prices is given by a *premium calculation principle*:

Definition 1. *A* premium calculation principle *is defined as a function $\pi : \mathcal{X} \mapsto \mathbb{R}$, $\pi(X)$ representing the price (premium) that an insurer would charge for insuring risk X.*

Given that premiums are invested in order to pay for future claims, the choice of π by an insurer represents its risk preferences and is also likely to reflect some stability criterion such as the probability of ruin [8].

Risk measures are mathematical constructs in many ways similar to premium principles, even though they have a different interpretation (e.g. [5]).

Definition 2. *A risk measure is defined as a function $\rho : \mathcal{X} \mapsto \mathbb{R}$, $\rho(X)$ representing the amount of capital that the holder of risk X should add to its position and safely invest in order to satisfy a regulator.*

A risk measure, as defined above, reflects the preferences of a regulator rather than those of an individual agent. The requirement that an insurer holds capital equal to its risk measure reflects the constraints imposed on economic agents by regulation. The acceptability of risky portfolios can then be formulated, similarly to [5], by a requirement that measure of risk plus regulatory capital is non-positive, e.g. $\rho(X - \rho(X)) \leq 0$. We note here that the definition of the acceptability of risks is not contingent on the presence of a regulator; the constraint $\rho(X - \rho(X)) \leq 0$ can be effectively imposed by entities such as rating agencies or the risk management team of the holder (e.g. insurance company) itself.

If X is an insurance liability, then in the simplified insurance pricing framework discussed here the amount of capital $\rho(X)$ is raised from the insurer's premium income, that is, $\pi(X) = \rho(X)$. This produces a conceptual link between risk measures and premium calculation principles; in this section we will actually consider them as being essentially identical constructs and will use the terms risk measure and premium principle interchangeably. Note, though, that in Sections 4 and 5, a risk measure and a price functional take very different meanings.

Note that the premium, as calculated by a premium calculation principle, is not being directly associated with market conditions and actual prices of traded liabilities. One way of dealing with this apparent discrepancy is to differentiate between the *actuarial premium*, which is given by a premium calculation principle, and the *underwriting premium*, which is the actual price of insurance. The actuarial premium can then be considered as a benchmark that the underwriter should consult. It is reasonable to require that the actuarial and underwriting price should agree on aggregate, i.e. across portfolios and accounting years, with the insurer's investment income also being taken into account.

Even though a premium calculation principle essentially gives a benchmark price for an insured risk, it is a matter of consistency to require that it satisfies a set of properties similar to those of actual prices prevailing in insurance markets. Such constructions of premium calculation principles (and associated risk measures) are discussed in the following section (3.2) and functional representations of premium principles are discussed in section 3.3. Additionally, the premium principle should be consistent with the insurer's risk preferences, i.e. with the way that it quantifies and compares risks. Premium principles derived on such a premise are presented in section 3.4.

3.2 Properties of premium principles

In this section we present two alternative axiomatizations of premium calculation principles and, consequently, risk measures. The properties that one requires premium principles to satisfy should thus make sense both from a pricing and a capital allocation perspective.

An important issue is how to deal with portfolios of dependent risks. Thus, before we proceed with discussing properties of premium principles and risk measures, we briefly present some notions of dependence between risks. [44] is a textbook on dependencies between random variables, while [60] and [31] provide interesting discussions on the relationship between risk measures and dependence structures.

A useful characterization of positive dependence is *positive quadrant dependence* (PQD). Two risks $X,\ Y$ are PQD whenever:

$$\mathbb{P}(X \leq x \textit{ and } Y \leq y) \geq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y),\ \forall x, y \in \mathbb{R}.$$

Essentially two risks being PQD means that their probability of assuming low (or high) values simultaneously is higher than it would be, were they independent. The negative analogue of PQD is *negative quadrant dependence* (NQD):

$$\mathbb{P}(X \leq x \textit{ and } Y \leq y) \leq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y),\ \forall x, y \in \mathbb{R}.$$

Finally, *comonotonicity* is defined as the strongest possible form of positive dependence. Two random variables $X,\ Y$ are called comonotonic if there is a random variable U and non-decreasing functions g, h such that:

$$X = g(U), Y = h(U).$$

A popular characterization of premium principles and risk measures stems from the requirement that they satisfy the following set of five properties [23], [56]:

Monotonicity: If $X \leq Y$, then $\pi(X) \leq \pi(Y)$.

Translation Invariance: If $a \in \mathbb{R}$ then $\pi(X + a) = \pi(X) + a$.

Positive Homogeneity: If $a \in \mathbb{R}_+$ then $\pi(aX) = a\pi(X)$.

Subadditivity: $\pi(X + Y) \leq \pi(X) + \pi(Y)$.

Comonotonic Additivity: If X, Y are comonotonic then $\pi(X+Y) = \pi(X) + \pi(Y)$.

Monotonicity implies that liabilities that always yield higher losses should always be insured at a higher price. Translation invariance formalizes the requirement that an increase of the insured risk by a constant amount should induce an equal increase in premium. Positive homogeneity suggests that the premium principle is sensitive only to the composition of portfolios of liabilities and not to their size. Subadditivity has the consequence that the pooling of liabilities always produces diversification benefits and that there should thus be no incentive for the splitting of portfolios. From a pricing perspective, subadditivity can also be seen as representing frictional costs, when risks are traded in imperfect markets. It is in fact widely accepted that prices in imperfect markets are subadditive (e.g. [16], [56]). Finally, comonotonic additivity implies that no diversification takes place when pooled risks are totally positively dependent. Note that the first four properties define what Artzner et al. [5] call a *coherent measure of risk*.

The above axiomatization of premium principles and risk measures has not been without its detractors [37], with the positive homogeneity and subadditivity properties being generally issues of contention. The insensitivity of the risk measure to the scale of potential losses, manifested by positive homogeneity, implies insensitivity to liquidity risk, that is, to the risk that potential losses are so high that the liability holder cannot raise the capital to pay for them. From an insurance pricing perspective this concern is reflected in capacity constraints; in practice a (re)insurer will be weary of underwriting a treaty providing infinite coverage above a deductible.

On the other hand, subadditivity disregards the risk of aggregating many dependent positions, for example insurance contracts contingent on the same event. Also the requirement that portfolios should never be split seems unrealistic. Buying excess-of-loss reinsurance, provides exactly such a splitting, providing limits up to which losses from each portfolio will draw upon the aggregate resources of an insurer.

Reflecting these concerns an alternative axiomatization of premium principles is proposed by [38]:

Monotonicity: If $X \leq Y$, then $\pi(X) \leq \pi(Y)$.

Translation Invariance: If $a \in \mathbb{R}$ then $\pi(X+a) = \pi(X) + a$.

Subadditivity for NQD risks: If X, Y are negative quadrant dependent, $\pi(X+Y) \leq \pi(X) + \pi(Y)$.

Additivity for independent risks: If X, Y are independent, $\pi(X+Y) = \pi(X) + \pi(Y)$.

Superadditivity for PQD risks: If X, Y are positive quadrant dependent, $\pi(X+Y) \geq \pi(X) + \pi(Y)$.

The above axiomatization acknowledges diversification only in the case of negatively dependent risks. It is also a direct consequence that a premium principle satisfying these properties will not be positively homogenous and thus be sensitive to portfolio size.

It can be argued that this second set of properties is too strict in its penalization of liquidity and aggregation risk. Especially if exposures are relatively small, liquidity and aggregation risk become less of an issue. It could thus be said that a satisfactory set of properties should lie somewhere in between the two extremes presented above. A weaker set of axioms has been proposed by Föllmer and Schied [32], [33], in their definition of convex measures of risk. From an insurance perspective, a class of (convex) risk measures were defined in [54], which are subadditive and positively homogenous for small portfolios, while becoming superlinear for larger ones.

It is furthermore noted that more general objections have been formulated regarding axiomatic approaches to premium calculation and risk measures. It has been argued by Goovaerts et al. [38] that the properties of risk measures should not be postulated in an abstract way, but rather depend on the specific problem studied. Characteristically, it is possible that a set of properties is assumed with regard to the risk measure used for determining the aggregate capital held by an insurance company, while a totally different approach is used in order to allocate this amount of capital to different portfolios. The latter could, for example, consist of minimisation of residual risk exposure (see [46] and [39] for such an approach), which yields portfolios' Risk Based Capital requirements.

An additional point where axiomatic approaches can be misleading is their treatment of the dependence between risks. Although in the last axiomatization presented (i.e. with additivity for independent risks) the dependence between risks is to some extent taken into account, the interrelationship between dependence and diversification is a much more complicate issue, which is not necessarily captured by a set of axioms. In [28] it is shown that portfolios of risks which are characterised by a higher dependence between their constituents (as reflected by the correlation order on sets of random vectors with fixed marginals) are also riskier in the stop-loss order sense. Therefore one should require that the properties of risk measures also include preservation of stop-loss order (the distortion and exponential principles defined in subsequent sections do satisfy this property).

3.3 Representations of premium principles

For practical purposes, it is necessary to derive specific functional forms for premium principles that satisfy a set of axioms such as the ones defined in the previous section.

We start with the first axiomatization of premium principles that we presented. Let $\mathbb{P}$ be the actuarial probability measure. Then it is shown ([23], [56]) that (subject to a technical condition) any premium principle satisfying monotonicity, translation invariance, positive homogeneity, subadditivity and comonotonic additivity can be represented by:

$$\pi(X) = \int_{-\infty}^{0} (g(\mathbb{P}(X > x)) - 1)dx + \int_{0}^{\infty} g(\mathbb{P}(X > x))dx, \tag{1}$$

where g is an increasing and concave function such that $g(0) = 0$ and $g(1) = 1$. The function g is called a *distortion function*, as it distorts the risk's probability distribution assigning a higher probability to unfavorable events. Correspondingly, the above premium principle is called a *distortion premium principle*. Thus, the distortion premium principle is essentially the expected loss of the liability, calculated under a transformed probability distribution. Note that in expression (1) we take into account the case of $X < 0$, representing the possibility of gains. If we are concerned with pure liabilities, i.e. $X \geq 0$ a.s., then the first term in (1) vanishes.

An alternative representation goes through the definition of sets of probability measures. In [5] it is shown that any coherent measure of risk can be written as:

$$\pi(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}}[X]. \tag{2}$$

Thus, premium is calculated as the maximal expected loss over a set of alternative probability measures $\mathbb{Q} \in \mathcal{Q}$. This is essentially a way of stress testing, where event scenarios are replaced by scenarios of probability distributions, termed *generalized scenarios* [5]. When comonotonic additivity is also satisfied, that is, we are dealing with a distortion principle, the set of generalized scenarios is characterized by [24]:

$$\mathcal{Q} = \{\mathbb{Q} : \mathbb{Q}(A) \leq g(\mathbb{P}(A)), \forall A \in 2^{\Omega}\} \tag{3}$$

Turning now our attention to the second set of axioms presented in the previous section, we consider premium principles satisfying monotonicity, translation invariance, subadditivity for NQD risks, additivity for independent risks and superadditivity for PQD ones. Such premium principles can be represented by [38]:

$$\pi(X) = \frac{1}{a} \ln E[e^{aX}], \tag{4}$$

where a is a non-negative number and it is understood that for $a = 0$ the premium principle reduces to $\pi(X) = E[X]$. Equation (4) defines the *exponential premium principle*, which is well-established in the actuarial literature and has an additional interpretation as a mechanism for bounding the insurer's probability of ruin ([34], [11]).

3.4 Premium principles derived from preferences

In this section we present an economic justification for the previously defined premium principles. Consider an insurer's preferences being characterized by a *preference operator*, V, such that $V(X) \geq V(Y)$ means that risk X is preferable to Y. Thus, preference operators provide an ordering of risks. As premium principles essentially perform the same service, it is desirable to link the two.

Premium principles can be derived from indifference arguments, as first demonstrated by Bühlmann [8]. Consider an insurer with initial surplus (cash) w, who insures a liability X. The premium $\pi(X)$ that the insurer will charge, should be the amount for which it is indifferent to the insurer whether to go ahead with the contract or not. Such indifference can be formalized by requiring that evaluations of the preference functional before and after the contract yield the same result:

$$V(w) = V(w - X + \pi(X)) \tag{5}$$

The premium $\pi(X)$ can then be obtained as a solution to the above equation.

The best-known preference operator is expected utility, as defined in [55]:

$$V(X) = E[u(X)], \tag{6}$$

where u is an increasing and (for a risk averse agent) concave function. The equation (5) can then be re-written as

$$u(w) = E[u(w - X + \pi(X))] \tag{7}$$

and its solution, $\pi(X)$ is called the *principle of equivalent utility.* Note that when the utility function is of exponential type,

$$u(x) = \begin{cases} \frac{1}{a}(1 - e^{-ax}), & a > 0 \\ x, & a = 0 \end{cases} \tag{8}$$

the exponential premium principle (4) is recovered.

An alternative characterization of preferences is Yaari's [61] dual theory of choice under risk, which proposes the following preference operator:

$$V(X) = \int_{-\infty}^{0} (h(\mathbb{P}(X > x)) - 1)dx + \int_{0}^{\infty} h(\mathbb{P}(X > x))dx, \tag{9}$$

where h is an increasing and (for a risk averse agent) convex function with $h(0) = 0$ and $h(1) = 1$. It is easy to show that, using this preference functional, the solution of (5) is the distortion premium principle (1) with $g(s) = 1 - h(1 - s)$.

The relationship between premium principles and economic models of preferences has been dealt with in detail in [54], where the authors also proposed a premium principle based on rank-dependent utility theory, which combines both expected utility and probability distortion functions.

3.5 Discussion

Premium principles and risk measures provide a sophisticated theory of premium calculation. However, two important criticisms of the use of the axiomatic approach to premium calculation presented here can be formulated.

One criticism is that pricing by a (re)insurer of each contract with a premium calculation principle does not acknowledge the diversification that holding different portfolios of liabilities implies. Indeed, regardless of whether a sub- or super-additive risk measure is used, the aggregate risk to the insurer will be different than the sum of its parts. It is not unreasonable to require that contracts which, by some measure, contribute to the diversification of the insurer's portfolio should be insured at lower (and thus more competitive) prices than contracts which do not produce diversification benefits.

A way of countering this deficiency is to advocate a 'top-down' approach to premium calculation [11]. This would mean that a premium calculation principle is used to calculate the aggregate premium that the insurer requires in order to be able to meet its liabilities. This aggregate premium is then broken down by a linear rule in order to calculate prices for individual policies.

The 'top-down' approach to pricing bears a formal as well as economic relationship to the capital allocation exercise performed by insurance companies. First the aggregate capital that the company should hold is calculated, by applying a risk measure on the aggregate risk, and is thereafter allocated to the different portfolios of risks. The use of concepts from cooperative game theory for determining a capital allocation mechanism was proposed in [22], while in [52] explicit capital allocation formulae in the case of distortion principles are obtained and the links between capital allocation and the pricing of pooled liabilities are discussed. An alternative approach to the allocation economic capital within a financial conglomerate is given by [46] and [39], who determine capital allocations by minimising the residual risk exposure of individual portfolios. In this capital allocation method, the aggregate amount of capital that the conglomerate has to hold is determined by a different risk measure than the one that is used for evaluating residual risk. The approach furthermore shows that imposing axiomatic requirements on the allocation mechanism (such as subadditivity) can lead to serious pitfalls.

A second, and possibly more important, criticism arises from the axiomatic view of premium calculation presented in this section, which seeks to emulate the properties of market prices, while not taking into account the conditions prevailing in the market where (insurance) risks are traded. This inconsistency creates an obvious paradox: premium calculation principles produce 'market' prices, while ignoring the presence of the market. A way of addressing this problem is to embed the premium principles in the decision making process that forms part of an economic market model. That approach is presented in the next section.

4 Risk Measures in Equilibrium Models

4.1 Equilibrium asset pricing models

One of the fundamental weaknesses of the pricing models described in the previous section is that they disregard the presence of a market where risks are traded. Market prices are determined through the interactions of market participants, that is, primary insurers, reinsures and policyholders. Such interactions can be modelled using the theory of competitive equilibrium, whereon a range of asset and insurance pricing models are based (for reviews see [19], [35] and [4]). These models are often highly stylized versions of reality, but provide useful insight. For that reason they have proved popular both with academics and practitioners, with the Capital Asset Pricing Model (CAPM) being a leading example (for an excellent review of the application of the CAPM to insurance pricing see [19]).

Equilibrium asset pricing models are derived on a number of premises. It is assumed that each market participant, or 'agent', decides on its exposure to sources of risk by maximising its preference functional under a budget constraint. Market prices are given by a linear functional and equilibrium is achieved when all agents maximize their preferences and the market clears. In a (re)insurance context such models have been studied extensively in [7], [9], [10], [3].

Preference maximization has often been associated with risk minimization, as in the CAPM's ‘risk vs return’ arguments. Recall that in section 3.4 it was shown how the modelling of risk preferences can be associated with the definition of risk measures such as the well-known exponential and distortion premium principles. In this section we revisit the equilibrium insurance pricing models [9] and [53] and reformulate them from such a perspective.

4.2 Equilibrium models with risk measures

Let an insurance market consist of a finite number N of agents, each exposed to a liability $X_i \in \mathcal{X}$ and holding cash (surplus) $w_i > 0$, for $i = 1, \dots, N$. Market prices are given by a function π, which is linear, i.e. $\pi(a + bX) = a + b\pi(X),\ \forall X \in \mathcal{X},\ a, b \in \mathbb{R}$. Furthermore, the ith agent’s preferences are characterized by a preference functional V_i such as the ones discussed in section 3.4. Let ρ_i be the risk measure defined by the indifference argument $V_i(w) = V_i(w - X + \rho_i(X)),\ X \in \mathcal{X}$. We note that the risk measure is in this context not anymore equivalent to a premium principle. The definition of a risk measure via indifference arguments reflects the preferences of either a regulator or the agent himself, depending on the market setting (if the risk measure is defined by a regulator, it is not specific to each agent and the subscript i can be dropped from ρ).

The agents trade their risks in the insurance market. After trading, each will hold a liability Y_i, which includes cash. Y_i can be any, possibly non-linear, function of $X_1, \dots, X_N$. We define the aggregate risk in the market as $Z = \sum_i X_i$. It is required that the market clears, that is, the aggregate liabilities after trading equal the aggregate liabilities before the exchange minus the aggregate insurers’ surplus w_i:

$$\sum_i Y_i = \sum_i X_i - \sum_i w_i = Z - \sum_i w_i \tag{10}$$

Each agent decides on the level of its liabilities after trading, or *risk allocation*, Y_i, by solving the optimization problem:

$$\max_{Y_i} V_i(-Y_i), \quad \text{such that} \quad \pi(X_i) \leq w_i + \pi(Y_i) \tag{11}$$

The objective function $V_i(-Y_i)$ quantifies the ith agent’s preferences after trading (the minus sign is due to the fact that Y_i is considered to be a liability). $\pi(Y_i)$ is interpreted as the premium that the agent receives for (re)insuring liability Y_i, while the $\pi(X_i)$ is the premium that it pays for (re)insuring its initial liability X_i. Thus the constraint in (11) means that the agent cannot pay for reinsurance more than its initial capital plus its premium income.

A related equilibrium problem is the following:

$$\min_{Y_i} \rho_i(Y_i), \quad \text{such that} \quad \pi(X_i) \leq w_i + \pi(Y_i) \tag{12}$$

If the risk measures are defined via indifference arguments of the type $V_i(w) = V_i(w - X + \rho_i(X))$, and in the case that the preference functional corresponds to either an exponential utility (8) or to Yaari's dual theory of choice under risk (9), the optimisation programs (11) and (12) are in fact equivalent. [2] Thus each agent's preference maximization problem can actually be rephrased as a risk minimization problem.

In the context of regulated insurance markets, where a common risk measure ρ is imposed on all agents, an alternative formulation of agents' optimisation problems is:

$$\max_{Y_i} \pi(Y_i), \quad \text{such that} \quad \rho(Y_i) \leq w_i - \pi(X_i) \tag{13}$$

The agent makes investment decisions with the aim of maximising premium income, while the risk measure now enters as a regulatory constraint on the risk that the insurer is allowed to retain after trading. Specifically, in order to be acceptable, the agent's risk after trading should not exceed the capital that it holds, that is, the initial surplus minus reinsurance expenditure. Even though optimisation problem (13) is not equivalent to the two previous ones, it has been shown that, for the distortion risk measures discussed here, it yields the same equilibrium prices [53].

4.3 Equilibrium with the exponential premium principle

Let the preference functional be given by an exponential expected utility, that is, for each agent:

$$V_i(X) = E\left[\frac{1}{a_i}(1 - e^{-a_i X})\right], \quad a_i > 0, \ X \in \mathcal{X} \tag{14}$$

The associated risk measure then is the exponential premium principle:

$$\rho_i(X) = \frac{1}{a_i} \ln E[e^{a_i X}] \tag{15}$$

Under these assumptions, Bühlmann [9], solved problem (11) (in extension, also solving (12)) and, using the clearing condition (10), determined the form of the price functional π as:

$$\pi(X) = \frac{E[Xe^{aZ}]}{E[e^{aZ}]}, \tag{16}$$

where $Z = \sum_i X_i$ and $a = (\sum_i \frac{1}{a_i})^{-1}$. Note that equation (16) can again be considered as a premium calculation principle. However premium does not anymore depend only on the distribution of the underwritten risk, but also on the conditions prevailing in the insurance market as exemplified by the aggregate risk Z and the 'market risk aversion' coefficient a.

[2] In the case of exponential utility, this follows from the observation that $\rho_i(Y_i)$ is an increasing function of the quantity $E[\exp^{a_i Y_i}]$ while V_i is an increasing function of $E[-\exp^{a_i Y_i}]$. In the case of the dual theory of choice, the equivalence follows directly from $\rho(Y_i) = -V_i(-Y_i)$.

Equation (16) defines the Esscher transform, which is a transformation of the probability distribution frequently used in actuarial mathematics. A dynamic version of the Esscher transform has been used in [36] for the pricing of financial derivatives.

The term $\zeta = \frac{e^{aZ}}{E[e^{aZ}]}$ is called a *price density*. We can rewrite the 'economic' premium principle (16) as:

$$\pi(X) = E[\zeta X] = E[X] + Cov(X, \zeta) \tag{17}$$

Thus, premium is calculated as expected loss (net premium) plus a risk loading, which increases whenever X and Z are highly correlated (observe that ζ is increasing in Z). Thus the risk loading is high whenever the underwritten risk provides a bad hedge against market losses.

4.4 Equilibrium with the distortion premium principle

We now let the risk measure be a distortion premium principle, as in (1). Furthermore we assume that the risk measure is imposed on market agents by a regulator and thus is the same for every one. The resulting equilibrium prices are given in [53]:

$$\pi(X) = E[Xg'(1 - F_Z(Z))], \tag{18}$$

where $F_Z(z) = \mathbb{P}(Z \leq z)$ is the cumulative distribution function of the aggregate liabilities Z.

For the distortion function:

$$g(s) = \frac{1 - e^{-as}}{1 - e^{-a}}, \quad a > 0, \tag{19}$$

the price functional becomes:

$$\pi(X) = \frac{E[Xe^{aF_Z(Z)}]}{E[e^{aF_Z(Z)}]}. \tag{20}$$

One can observe the formal similarity between the prices as given by (16) and (20). The difference is that, while in the case of equilibrium with an exponential premium principle market prices depend on the scale of potential market losses, in the case of the distortion principle they depend on the random variable $F_Z(Z)$ representing the rank of market losses in the set of possible outcomes.

4.5 Discussion

This section was motivated by the need to address the presence of market conditions in insurance premium calculation, a presence that is not accounted for in the theory of premium calculation principles. It is seen however from optimisation programs (12) and (13) that premium calculation principles are utilised in equilibrium pricing models, where they are respectively viewed as either objective functions for insurers' decision making problems or constraints on their exposure.

An important difference in comparison to section 3 is that, in the equilibrium models presented, the premium principle is not only applied to individual insurance contracts, but to the aggregate risk carried by an insurer. This does bear some similarity with the 'top-down' premium calculation and capital allocation problems discussed in section 3.5. On the one hand, equilibrium models incorporating premium principles provide a way of determining a 'top-down' premium calculation approach. On the other hand, one might ask whether the capital allocation performed by an insurer is consistent with the way that its retained risks are priced in the market. [53] concluded that the capital allocation and equilibrium pricing approaches can be consistent, when insurers are well enough diversified, so that their portfolios are very similar to (i.e. increasing in) the market portfolio.

In sections 4.3 and 4.4 equilibrium models based on utility and distortion functions respectively were discussed. A question arises as to whether these two approaches are reconcilable. One approach, followed in [53], is to study an equilibrium model where agent' preferences are characterized by both utilities and distortions and derive prices depending both on the utilities and probability distortions of market participants. [3] An alternative approach has been developed by a series of papers by Wang [57], [58], [59]. In [57] a probability distortion function is introduced, such that the resulting distortion premium principles replicates under certain assumptions the CAPM and Black-Scholes pricing formulas. The application of this method to insurance pricing and capital allocation was presented in [58]. Furthermore, it is shown in [59] that the above probability distortion function can be derived under a number of justifiable assumptions from the Esscher transform (16), which is based on equilibrium with exponential utilities.

There are three major drawbacks of the equilibrium pricing models discussed in this section. A significant weakness of equilibrium models is that they rely on a fair amount of information with respect to the risk traded in the market and the preferences of its participants. However, this weakness can be moderated. It can be seen that equilibrium prices, such as the ones given by (20) and (16), are functions only of the aggregate market risk Z, and of some characterization of aggregated preferences. This creates the possibility of analysing the market as if it consisted only of one agent and calibration problems become more tractable. This corresponds to the well-known representative agent paradigm of the asset pricing literature, e.g. [29]. Furthermore note that in the case that the agent's optimization programs are induced by the same risk measure imposed by a regulator, market participants' decision problems can be realistically assumed to be quite fairly similar. As for the aggregate market risk Z, it will not be exactly known but can be reasonably

[3] Note that such preferences give rise to a class of *convex* measures of risk [54].

approximated by using an index.

Another drawback is that the equilibrium problems presented here are idealized versions of reality, via the implicit assumption of perfect competitive markets. No constraints have been imposed on the possible contracts that an insurer can buy or sell and the presence of transaction costs has been ignored. Even though the emergence of innovative instruments such as CAT bonds, double-trigger options and other insurance derivatives has significantly expanded insurers' possibilities of asset and liability management, insurance markets are still far from perfect.

Finally, one of the main assumptions behind equilibrium asset pricing models is that price functionals are linear functions of risks on which they are defined. This contradicts the observation that insurance prices tend to be non-linear, an observation reflected in the definition of premium calculation principles. Non-linearity of prices is widely considered to be a product of market imperfections, such as the ones discussed above. Thus, in our effort to include the market in premium calculation considerations, we made a significant compromise: the equilibrium prices obtained actually look less like actual market prices.

In a recent paper [25], it is shown that market equilibrium can be achieved if trading takes place under a non-linear pricing rule based on a signed Choquet integral, which is closely related to the distortion premium principle. Another possible method of producing non-linear prices is, instead of using perfect market equilibrium models, to move to other classes of models that deal with imperfect markets and utilise tools from game theory. Such approaches are presented in [51] and [4]. They provide useful insight and are particularly appealing because they are derived from first principles and do not impose ad hoc non-linearity of prices. However these models are possibly too complicated for practical use.

In summary, the main problems that equilibrium insurance pricing models present are the need for advanced knowledge of market risks and market participants' preferences, the assumption of perfect competitive markets, and the linearity of derived price systems. In the next section we will attempt to address these three problems, using concepts from the field of financial economics.

5 Risk Measures and Price Systems

5.1 No-arbitrage pricing

The pricing formulae that were derived in the previous section are examples of valuation methods that rely on complete knowledge of the constituent parts of the market examined: risk exposures and individual preferences. An alternative approach is to price risks via valuation techniques, which do not rely on agent-specific knowledge but on information revealed by observed prices of risks actually traded in the market. In order to price a financial product, such as an option or a reinsurance contract, one requires that the price of the product is, in some sense, consistent with the observed market prices. This is the pricing framework which has proved most popular in the financial literature and practice over the last twenty years, with the Black-Scholes option pricing model being a leading paradigm.

A classic way of defining such consistency is via the no-arbitrage condition. Broadly speaking, it is said that no arbitrage opportunities exist in a market when it is not possible to make, by cleverly trading, a certain profit at zero cost. If an arbitrage opportunity existed, it is reasonable to assume that it would be spotted instantly and exploited by market participants and thus rapidly disappear. Market prices would then adjust such that the arbitrage opportunity disappears. Thus, as opposed to equilibrium, no-arbitrage refers to the properties of price systems used in the market rather than on agent's individual decision making. No-arbitrage is a necessary, but not sufficient condition for equilibrium; it is evident that if an arbitrage opportunity emerges any rational agent would move to exploit it and equilibrium would be disturbed. Insofar, no-arbitrage is a weaker condition than equilibrium.

The absence of arbitrage opportunities in a financial market has profound implications for the existence and admissibility of prices in the market. Suppose that a number of risks $X_1, X_2, \ldots, X_n$ are traded in the market. The nature of these risks depends on the economic and mathematical context; in a dynamic case they might stand for stochastic processes such as stock-prices, while in a one period setting they may be random variables representing a pay-off at a fixed future point in time. [4] Even though the bulk of financial pricing theory focuses on the dynamic setting, it has been shown that the theory is general enough to accommodate both the one-period and multi-period models [43]. In the subsequent discussion the potentially dynamic character of risks is not explicitly treated, with the implicit understanding that the concepts discussed (if not the mathematical technology) apply to both the static and dynamic cases.

A classic result from financial economics states that no-arbitrage implies the existence of a linear pricing functional π [29]. Consider now an insurer who participates in the market where $X_1, X_2, \ldots, X_n$ are traded, being exposed to a risk Y. If it is assumed that all traded risks are discounted at the risk free rate of interest, then the price $\pi(Y)$ of risk Y can be written as its expected value with respect to an alternative probability measure $\mathbb{Q}$. [5] We denote as $E_{\mathbb{Q}}[\cdot]$ the

[4] The annual renewal of reinsurance programs arguably falls somewhere between those two cases.

[5] This representation is subject to a technical condition on the integrability of Y. The probability measure

expectation operator with respect to $\mathbb{Q}$. The price of Y can be written as:

$$\pi(X) = E[\zeta Y] = E_{\mathbb{Q}}[Y], \tag{21}$$

where ζ is a price density such as the ones discussed in section 4.3 (in probabilistic terms ζ is called a Radon-Nikodym derivative; if we consider a dynamic model, ζ is itself a stochastic process). $\mathbb{Q}$ is usually referred to as a *risk neutral probability measure*, as prices under $\mathbb{Q}$ look like the net premium that a risk-neutral agent would ask for. The device of changing the probability measure is very useful in harnessing the theoretical potential of probability theory. This is especially significant in the dynamic case, where no-arbitrage has the consequence of all traded positions being martingales under the risk neutral measure.

So far we have not commented on whether the no-arbitrage price functional π (and by extension the risk neutral measure $\mathbb{Q}$) is unique. No-arbitrage implies the existence of a unique price functional only if the market where risks are traded is *complete*. Market completeness can be defined in two, essentially equivalent, ways. On the one hand, a market is complete if and only if the number of traded instruments is not exceeded by the number of sources of uncertainty. On the other hand, market completeness is characterized by the ability to hedge (or replicate) any position by trading other instruments available in the market. Hedging (replicating) a position is understood as trading (that is, buying and selling in the market risks $X_1, X_2, \ldots, X_n$) in such a way that the final portfolio produces the same payoff as that position in all states of the world, so that the risk from holding it is eliminated. The link between pricing and hedging comes from the argument that the price of a risk should equal the cost of replicating it; if that is not the case an arbitrage opportunity emerges.

The equivalence between the two definitions of market completeness can be understood by the example of a reinsurer exposed to financial and insurance, e.g. catastrophic, risk. Typically, the reinsurer will be able to hedge its financial risk by trading in the equities and derivatives market. However, the catastrophic risk will be much more difficult to hedge by trading (assuming that the reinsurer will not be able or willing to enter a retrocession contract). The possibility of suffering a loss from a catastrophic event induces a source of uncertainty external to the financial markets. Consider now the idealized case where an insurance derivative (such as a CAT bond or option) is traded in the market and is perfectly correlated with the catastrophic risk held by the reinsurer. Then it is possible that the reinsurer, using the insurance derivative, can hedge its exposure. It is

$\mathbb{Q}$ is equivalent to the real-world (actuarial) probability measure $\mathbb{P}$, meaning that the two measures assign zero probability to the same events.

said that the additional instrument ‘completes’ the market, in the sense that it provides a means for replicating the risk arising from the additional uncertainty induced by the insurance risk.

The application of financial pricing techniques to insurance risks has attracted a lot of interest and generated an important volume of research, an excellent review being [30]. In the following two sections, we present a perspective of the intersection between financial economics, risk measurement and insurance pricing corresponding to the present paper’s scope. A number of alternative approaches are briefly discussed in section 5.4.

5.2 Super-replication and good deals

Typically risks are traded in markets that are not complete, with insurance markets being a characteristic example. In the case of *market incompleteness* the absence of arbitrage does not guarantee the existence of a unique price functional. There might be many, possibly infinite, prices that are consistent with the no-arbitrage requirement and thus the calculation of an appropriate price is not unequivocal. Other considerations will now contribute to the determination of the price of a risk. In the context of incomplete markets, preference modelling, premium calculation principles and risk measures reclaim their significance by providing these ‘other considerations’ which are necessary for the pricing of risks.

It is interesting to note how the concept of replication adapts to the incomplete setting. In an incomplete market it is not possible to compose, by trading in the marketed risks $X_1, X_2, \dots, X_n$, a portfolio that exactly replicates every additional exposure Y. It might however be possible to produce a portfolio whose payoff is *at least as much* as the financial obligation of its holder. Obtaining such a portfolio by trading is called *super-replication*. Super-replication, as a generalization of replication, is also closely related to market prices. Let Π be the set containing all price functionals that are consistent with the no-arbitrage requirement and $\mathcal{Q}_{NA}$ the corresponding set of risk-neutral probability measures. Define the *super-replication price* $\overline{\pi}(Y)$ of a risk Y as the highest price consistent with no-arbitrage:

$$\overline{\pi}(Y) = \sup_{\pi \in \Pi} \pi(Y) = \sup_{\mathbb{Q} \in \mathcal{Q}_{NA}} E_{\mathbb{Q}}[Y]. \tag{22}$$

$\overline{\pi}(Y)$ is the highest price at which a reinsurer would possibly be able to sell an insurance contract covering liability Y. Correspondingly, the price $\underline{\pi}(Y) = \inf_{\mathbb{Q} \in \mathcal{Q}_{NA}} E_{\mathbb{Q}}[Y]$ is the lowest price consistent with no-arbitrage and thus the lowest price that a reinsurer would accept for insuring a risk Y. It can then be proved that the price $\overline{\pi}(Y)$ equals the minimum cost of super-replicating the risk Y [43].

Super-replication, even though it provides useful insight to incomplete market problems, is a strategy rarely employed in practice, mainly because it is very expensive. Recall that super-replication guarantees that the final payoff of the portfolio acquired by trading is at least as much as the claim one is trying to hedge. Thus, super-replication will typically yield a portfolio with a higher payoff than the hedged risk and this — not necessarily desirable — excessive payoff incurs the higher cost.

From the above argument it also follows that the bound on market prices induced by the super-replication price is unrealistically high (one pays for far more than one really wants). Even though it might not be realistic in the incomplete market case to determine a unique pricing functional, it would still be desirable to produce a sharper bound on market prices than the one implied by the no-arbitrage condition. This can be achieved by reducing the number of price systems that are considered viable in the market. It is clear from formula (22) that if the number of probability measures in $\mathcal{Q}_{NA}$ is reduced, the upper pricing bound will also be lower.

Such reduction can be achieved through the definition of *good deals*, proposed in [18]. A good deal is a traded position which might not be an arbitrage opportunity but is attractive enough so that one can safely assume that any conceivable investor would want to acquire it. Good deal pricing bounds are obtained by excluding price systems that, besides arbitrage opportunities, allow good deals. Consider for example a position obtainable at zero price, which pays \$1000 with probability 0.9 and $-\$1$ with probability 0.1. This is not an arbitrage opportunity, since it is possible to make a loss from investing in it. However, it is clear that such a position would not be viable in the market, as every investor would try to buy it and thus its price would increase.

In defining what actually constitutes a good deal, one needs to employ the orderings of risks employed by economic theories of choice and premium calculation principles (risk measures). In [18] good deals are defined as investment opportunities characterized by high Sharpe ratios. As discussed in [15], such a definition of good deals is consistent with assuming preference modelling consistent with a quadratic utility functions. Hence, we can consider good-deal pricing based on high Sharpe ratios as being an extension of the Capital Asset Pricing Model. Insofar, good deal pricing using Sharpe ratios is subject to criticisms similar to the ones faced by the CAPM, namely that it only works in a universe of elliptically distributed risks [31]. In a non-elliptical framework, there might be arbitrage opportunities that are not recognized as good deals [15]. To remedy this drawback, it was proposed to use Generalized Sharpe Ratios based on exponential utility functions [42], while this approach was extended to broader classes of utility functions in [14]. It is furthermore significant that when the valuation bounds implied by the utility function are made sharper, the good-deal price converges to the equilibrium price. In the papers referred to above, it is shown that the bounds on the (generalized) Sharpe ratios induce bounds on the price density ζm, which this makes explicit calculation of prices possible.

In this way, the economic modelling of preferences and the associated risk measures re-enter the calculation of prices for risks. In general it can be said that a good deal is a position, which, according to some risk measure, bears a low risk. Using an exponential utility function as a means of characterising good deals, as in [42], is of course equivalent to using the exponential premium principle for the same purpose.

5.3 Valuation bounds as measures of risk

In the preceding section it was discussed how a realistic bound on prices can be obtained in an incomplete market via the definition of good deals. Good deals are derived via economic arguments implied by preferences and associated measures of risk. Here we turn the argument on its head and ask the following question: can valuation bounds themselves be interpreted as risk measures and what are the implications for risk management and pricing?

Let the upper bound $\overline{\pi}_{RM}(Y)$ on the price of risk Y as be of the form:

$$\overline{\pi}_{RM}(Y) = \sup_{\mathbb{Q} \in \mathcal{Q}_{RM}} E_{\mathbb{Q}}[Y] \tag{23}$$

The set of measures $\mathcal{Q}_{RM}$ is no longer the set of no-arbitrage measures $\mathcal{Q}_{NA}$, but is derived by other arguments, such as the ones employed in the previous section. Observe now the formal resemblance between equations (23) and (2). It is apparent that the valuation bound $\overline{\pi}$ is a coherent measure of risk.

Conversely, let $\mathcal{Q}_{RM}$ be a set of generalized scenarios defining a coherent measure of risk. We can then examine the implications of this choice of risk measure for pricing risks. Recall the definition of a risk measure as the amount of capital that the holder of a risky position is obliged to safely invest. Thus, if Y is an insurance loss, we can interpret $\overline{\pi}(Y)$ as the maximal price for reinsuring Y, consistent with no arbitrage, and $\overline{\pi}_{RM}(Y)$ as the amount of capital that the holder of loss Y has to hold. It is reasonable to assume that an insurer would like to recover the amount of capital that he needs from his premium income. This implies that, in a competitive market, the price of a risk X should be less that or equal to its risk measure. It could be less since the subadditivity of coherent risk measures implies that the holder of a diversified insurance portfolio can offer discounts for individual policies, under the condition that the aggregate premium equals the aggregate risk. On the other hand, the premium cannot be higher than the risk measure, since in that case the buyer of the insurance contract would be better off by retaining the risk and investing safely an amount equal to the risk measure. Thus, the coherent measure of risk implicitly defines a valuation bound. A detailed discussion of the relationship between coherent measures of risk, valuation bounds and no-arbitrage is by [43], while a brief technical discussion in [21] is highly interesting.

Let $\mathcal{Q}_{RM} \subset \mathcal{Q}_{NA}$, that is, the set of generalized scenarios defining the coherent measure of risk is a subset of the set of no-arbitrage measures. This is equivalent to saying that $\overline{\pi}_{RM}(Y) \leq \overline{\pi}(Y)$ for all Y [21], i.e. the risk measure of a liability Y (and the associated valuation bound) is always lower than its super-replication price. This is reasonable since with capital $\overline{\pi}(Y)$ the holder of Y could hedge its risk completely. The risk measure being less than the super-replication price implies that agents will in general retain some risk after trading, a premise on which the operation of risk markets is largely based. Thus the requirement $\mathcal{Q}_{RM} \subset \mathcal{Q}_{NA}$ gives a clue for how a (coherent) measure of risk should be chosen.

5.4 Discussion

Financial economics provide invaluable tools for addressing the weaknesses of equilibrium asset pricing models that were identified in section 4.5. First of all, the restrictive assumption of a perfect competitive market is dropped and market imperfections can be treated via the theory of incomplete markets. Secondly, by moving from an equilibrium setting to arbitrage-pricing techniques, the unrealistically strong assumption that one knows the specific preferences and exposures of all market participants is no longer required. Thirdly, the resulting price functionals, such as (22) and (23) are no longer linear, reflecting to some extent actual prices in insurance and financial markets. Moreover, the non-linearity of prices is not imposed ad hoc but emerges naturally from economic concepts, such as no-arbitrage and good deals.

In an incomplete market setting, a unique price for a risk cannot be defined. This makes the determination of prices (or bounds on them) reliant on other considerations, relating to individuals' perception of risk. Such considerations have of course been the domain of actuaries for several decades and were formalized through the concept of premium calculation principles. We attempted to highlight two distinct ways in which premium calculation principles enter the pricing exercise, via the concept of good deals. On the one hand, the definition of a good deal relies on evaluating the riskyness of traded positions; an evaluation which goes through preference models and risk measures such as the ones studied in section 3. On the other hand, the good-deal bounds imposed on prices can themselves be interpreted as measures of risk.

The literature on the financial pricing of insurance is quite wide and there are a number of approaches that we did not discuss here. For a review of the subject we refer the reader to [30]. We note that the papers on the subject generally deal with the dynamic setting, where financial pricing theory attains its full force. Thus, [20] and [50] characterise no-arbitrage prices in dynamic insurance markets, making use of the stochastic processes particular to insurance losses. In the classic paper [20], the set of no-arbitrage pricing measures is characterised. It is then shown how different choices of the pricing measure relate to different actuarial premium calculation principles. Alternative discussions of dynamic insurance markets are provided in [1] and [2], where dynamic equilibrium models are studied.

There are other approaches to pricing in incomplete markets, where premium calculation principles and preference modelling are utilized. These include the extension of indifference arguments to a dynamic setting via stochastic control methods and hedging in incomplete markets via risk minimization criteria. We did not discuss these approaches here mainly because excellent reviews already exist in the actuarial literature. We refer the interested reader to the reviews [49] and [62], as well as to the dynamic generalization of the exponential principle [6]. A parallel strand in the literature, related to the role of risk measures in capital allocation and pricing, utilizes the concept of 'frictional costs', see e.g. [17]. Other papers related to some degree to the present exposition are [12], [13] and [41].

6 Conclusion

Insurance pricing models have evolved greatly over the last fifty years, moving from simple pricing rules called premium calculation principles to economic competitive equilibrium models and, finally, to models inspired by the advances in financial economics. The evolution of these models has been driven by the need to incorporate market information into the pricing of risks performed by insurance companies. It was argued that risk measures, that is, functionals which characterise the inherent dangerousness of random losses, appear and fulfil a different function in each of those classes of models. Thus, while risk measures can be viewed as essentially equivalent to premium calculation principles, in the context of equilibrium models they become part of the decision problem facing insurers. Finally, the in the no-arbitrage pricing of insurance liabilities, risk measures re-appear due to the incompleteness of insurance markets and offer methods for narrowing down the range of reasonable prices.

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