An explicit state-space solution to the one-block super-optimal distance problem

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1 Abstract

An explicit state-space approach is presented for solving the super-optimal Nehari-extension problem. The approach is based on the all-pass dilation technique developed in [JL93] which offers considerable advantages compared to traditional methods relying on a diagonalisation procedure via a Schmidt pair of the Hankel operator associated with the problem. As a result, all derivations presented in this work rely only on simple linear-algebraic arguments. Further, when the simple structure of the one-block problem is taken into account, this approach leads to a detailed and complete state-space analysis which clearly illustrates the structure of the optimal solution and allows for the removal of all technical assumptions (minimality, multiplicity of largest Hankel singular value, positive-definiteness of the solutions of certain Riccati equations) made in previous work [LHG89],[HLG93]. The advantages of the approach are illustrated with a numerical example. Finally, the paper presents a short survey of super-optimization, the various techniques developed for its solution and some of its applications in the area of modern robust control.

Keywords: super-optimal Nehari-extension problems, Hankel operator, all-pass dilations, $\mathcal{H}_\infty$ - optimal control, maximally robust stabilization.

2 Notation

Here we define the main notation used in the paper. Additional notation is introduced in subsequent sections as needed. All systems considered in this paper are assumed linear, time invariant and finite dimensional. Let $\mathcal{R}^{p\times m}(s)$ denote the space of proper $p \times m$ rational matrix functions in s with real coefficients. Associated with $\boldsymbol{P} \in \mathcal{R}^{p\times m}(s)$ of McMillan degree n is a state-space realization:

$$\boldsymbol{P} = C(sI - A)^{-1}B + D$$

where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$ and $D \in \mathcal{R}^{p\times m}$. For $\boldsymbol{P} \in \mathcal{R}(s)^{p\times m}$ let $\boldsymbol{P}(s)^{\sim} := \boldsymbol{P}'(-\bar{s})$ denotes the *para-hermitian conjugate* of $\boldsymbol{P}$. Throughout the paper we distinguish transfer matrices by making use of bold lettering which shall imply the s dependence. Let $\boldsymbol{P}$ be partitioned in 2×2 sub-blocks $\boldsymbol{P}_{ij}$, $i = \{1, 2\}$, $j = \{1, 2\}$. Then a state space realization of $\boldsymbol{P}$ can be written as:

$$\boldsymbol{P} := \begin{pmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

and

$$\boldsymbol{P}_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$$

is a state-space realization of $\boldsymbol{P}_{ij}$. A lower linear fractional transformation of $\boldsymbol{P}$ and $\boldsymbol{K}$ is defined as

$$\mathcal{F}_l(\boldsymbol{P}, \boldsymbol{K}) = \boldsymbol{P}_{11} + \boldsymbol{P}_{12}\boldsymbol{K}(I - \boldsymbol{P}_{22}\boldsymbol{K})^{-1}\boldsymbol{P}_{21}$$

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where $\boldsymbol{K}$ is of dimension $m \times p$ if $\boldsymbol{P}_{22}$ has dimension $p \times m$ and the indicated inverse exists. Similarly we define the upper linear fractional transformation of $\boldsymbol{P}$ and $\boldsymbol{K}$ as:

$$\mathcal{F}_u(\boldsymbol{P}, \boldsymbol{K}) = \boldsymbol{P}_{22} + \boldsymbol{P}_{21}\boldsymbol{K}(I - \boldsymbol{P}_{11}\boldsymbol{K})^{-1}\boldsymbol{P}_{12}$$

for a compatible partitioning of $\boldsymbol{P}$ with $\boldsymbol{K}$ and provided that the indicated inverse exists.

The spaces $\mathcal{RL}_2$ consist of all real-rational matrix functions $G(s)$ which are square-integrable on the imaginary axis, i.e. whose $\mathcal{L}_2$ norm:

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}(G^*(j\omega)G(j\omega))d\omega}$$

is finite. This coincides with the space of all strictly proper real-rational matrix functions which are analytic on the imaginary axis. Similarly, $\mathcal{RH}_2$ ($\mathcal{RH}_2^{\perp}$) denotes the spaces of all strictly proper real-rational transfer matrix functions which are analytic in closed right-half complex plane (closed left-half complex plane), respectively. We let $\|\cdot\|_2$ stand simultaneously for the $\mathcal{L}_2$-norm, the $\mathcal{H}_2$-norm or the $\mathcal{H}_2^{\perp}$-norm (for G belonging to the appropriate space). $\mathcal{RH}_2$ and $\mathcal{RH}_2^{\perp}$ are subspaces of $\mathcal{RL}_2$ and we define P_+ and P_- to be the orthogonal projections from $\mathcal{RL}_2$ to $\mathcal{RH}_2$ and $\mathcal{RH}_2^{\perp}$, respectively.

The space $\mathcal{RL}_\infty$ consists of all proper real-rational transfer matrix functions which are analytic on the imaginary axis. $\mathcal{RH}_\infty^+$ and $\mathcal{RH}_\infty^-$ are the subspaces of $\mathcal{RL}_\infty$ consisting of all real-rational proper matrix functions which are analytic in the closed right-half plane and closed left-half plane, respectively. Thus $\mathcal{RL}_\infty = \mathcal{RH}_\infty^+ \oplus \mathcal{RH}_\infty^-$ where $\oplus$ denotes direct sum of subspaces. The norm $\|\cdot\|_\infty$ denotes either the $\mathcal{L}_\infty$-norm of a function in $\mathcal{L}_\infty$ or the $\mathcal{H}_\infty$-norm of a function in $\mathcal{H}_\infty^+$, depending on context. $\mathcal{RH}_\infty(k)$ is the subset of $\mathcal{RL}_\infty$ consisting of all functions with no more than k poles in the right-half plane. If Γ is an operator, then Γ^* denotes its adjoint and $\|\Gamma\|$ denotes its induced norm. Here we make use of the induced norm of the Hankel operator with symbol $\boldsymbol{G}$ defined in section 3.1, which will also be denoted as $\sigma_1(\Gamma_{\boldsymbol{G}})$ or as $\|\boldsymbol{G}\|_H$, where σ_1 denotes the largest singular value of $\Gamma_{\boldsymbol{G}}$. A square matrix function $\boldsymbol{G} \in \mathcal{RL}_\infty$ is called γ-allpass if $\boldsymbol{G}\boldsymbol{G}^\sim = \boldsymbol{G}^\sim\boldsymbol{G} = \gamma^2 I$. A square all-pass function with $\gamma = 1$ is called inner if it lies in $\mathcal{RH}_\infty^+$ and anti-inner if it lies in $\mathcal{RH}_\infty^-$.

Let $\mathcal{F}^{m\times n}$ be the set of matrices with elements in field $\mathcal{F}$. In this context the field will be either the set of real numbers $\mathcal{R}$ or the set of complex numbers $\mathcal{C}$. Here by $\mathcal{C}_+$ ($\mathcal{C}_-$) we shall denote the set of complex numbers which are analytic in the open right (left) half plane. For a matrix $A \in \mathcal{F}^{m\times n}$ its transpose is denoted by A'. Further, we define $\mathcal{R}(A)$ to be the range of A and $\mathcal{N}(A)$ the null-space (kernel) of A, respectively. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are subspaces of $\mathcal{F}^m$ and $\mathcal{F}^n$, respectively, whose corresponding dimensions are denoted as $\text{rank}(A)$ and $\text{null}(A)$. For a square matrix A, $\lambda(A)$ denotes the spectrum of A, i.e. the set of its eigenvalues $\lambda_i(A)$, and $\rho(A)$ is the spectral radius of A.

The acronyms ARE, CIF, LFT, LTI and SODP stand for *algebraic Riccati equation*, *complementary inner factorization*, *linear fractional transformation*, *linear time-invariant* and *super-optimal distance problem*, respectively.

3 Introduction

In Nehari approximation problems we seek

$$\inf_{\boldsymbol{Q}\in\mathcal{H}_\infty^{+,\,p\times m}} \|\boldsymbol{R}+\boldsymbol{Q}\|_\infty \tag{1}$$

where $\boldsymbol{R} \in \mathcal{RL}_\infty^{p\times m}$ (or $\boldsymbol{R} \in \mathcal{RH}_\infty^{-,\,p\times m}$ without loss of generality). Throughout this paper we study the matrix case $\min(p, m) > 1$. Further, depending on the kind of application $\boldsymbol{Q}$ may be further constrained to have a zero block row and/or column. Then the problem is said to be a *two-block* or a *four-block* distance problem. In this work only *one-block* problems are considered, where no further constraints on Q are imposed.

By introducing the new notation $s_1^\infty(\boldsymbol{R}) = \|\boldsymbol{R}\|_\infty$ the approximation problem posed in (1) can be rewritten as:

$$s_1(\boldsymbol{R}) := \inf_{\boldsymbol{Q}\in\mathcal{H}_\infty^{+,\,p\times m}} s_1^\infty(\boldsymbol{R}+\boldsymbol{Q}) \tag{2}$$

where $s_1(\boldsymbol{R})$ will be referred to as the optimal level of $\boldsymbol{R}$. The set of all optimal approximations of $\boldsymbol{R}$ is defined by

$$\mathcal{S}_1(\boldsymbol{R}) := \{\boldsymbol{Q} \in \mathcal{H}_\infty^{+,\,p\times m} : s_1^\infty(\boldsymbol{R}+\boldsymbol{Q}) = s_1\} \tag{3}$$

Note that $s_1(\boldsymbol{R}) := \sigma_1(\Gamma_{\boldsymbol{R}^\sim})$, the Hankel norm of $\boldsymbol{R}^\sim$. Since, in general, the solution of this problem is not unique, we can define a stronger version of optimality, by requiring that the sequence of the suprema (taken over $\omega \in \mathcal{R}\cup\{\infty\}$) of all singular values of the "error" system $(R+Q)(j\omega)$ is minimized lexicographically. This stronger version of the problem was first proposed by Young and was defined as *super-optimization.* The main motivation, arising from esthetic considerations, was to restore uniqueness to the solution of the matrix Nehari problem, by showing in [You86] the existence of a unique super-optimal approximation $\boldsymbol{Q_{sup}}$. Nevertheless, in the present work and also others (e.g. [PF85], [KHJ07]) it is argued that super-optimization fits naturally within the modern robust control-theoretic framework, and can be used to define hierarchical optimization problems in which additional performance and stability objectives can be addressed [PF85], [GHJ00].

Given $\boldsymbol{G} \in \mathcal{RL}_\infty^{p\times m}$, the Hankel operator with symbol $\boldsymbol{G}$ is defined as:

$$\Gamma_G : \mathcal{H}_2^{\perp,m} \to \mathcal{H}_2^p, \quad \Gamma_G\hat{f} := (P_+M_G)\hat{f} = P_+(\boldsymbol{G}\hat{f}) \ \text{ for } \ \hat{f} \in \mathcal{H}_2^{\perp,m}$$

where M_G denotes the multiplication operator. Since $\boldsymbol{G} \in \mathcal{RL}_\infty^{p\times m}$ is analytic on a vertical strip containing the imaginary axis, we can define its two-sided Laplace transform, $g(t) \in l_2^{p\times m}(-\infty,\infty)$, containing both causal and anti-causal parts. Here $l_2(-\infty,\infty)$ denotes the space of all square-integrable functions with support (∞,∞). The equivalent definition of the Hankel operator in the time-domain is:

$$\Gamma_g : l_2^m(-\infty,0] \to l_2^p[0,\infty), \ (\Gamma_g f)(t) = P_+(g*f), \ \text{ for } \ f(t) \in l_2^m(-\infty,0]$$

where $*$ denotes convolution. Define $\sigma_i^2(\Gamma_G) = \lambda_i(\Gamma_g\Gamma_g^*) = \lambda_i(PQ)$. Here the $\sigma_i(\Gamma_G)$'s (denoted simply as σ_i) are the singular values of Γ_G (Hankel singular values of $\boldsymbol{G}$) and P and Q are the controllability and observability gramians of the system (A,B,C) which satisfy the Lyapunov equations $AP+PA'+BB' = 0$ and $A'Q+QA+C'C = 0$ respectively. Let these be ordered as $\sigma_1 = \ldots = \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_n > 0$ where n is the McMillan degree of $\boldsymbol{G}$. Then, $\sigma_1 = \|\Gamma_G\|$ is the Hankel norm of $\boldsymbol{G}$. Next, let $u_i(t) \in l_2^m(-\infty,0]$, $u_i(t) \neq 0$, be an eigenvector of $\Gamma_g^*\Gamma_g$ corresponding to the eigenvalue σ_i^2. Then $\Gamma_g^*\Gamma_g u_i = \sigma_i^2 u_i$. Define $v_i = (1/\sigma_i)\Gamma_g u_i \in l_2^p[0,\infty)$. Then the pair (u_i,v_i) satisfies $\Gamma_g u_i = \sigma_i v_i$ and $\Gamma_g^* v_i = \sigma_i u_i$ and is called a Schmidt pair of Γ_G. Thus $u_i(t) = \sigma_i^{-1}B'e^{-A't}Qx_i \in l_2^m(-\infty,0]$ and $v_i(t) = Ce^{At}x_i \in l_2^p[0,\infty)$. Let $\{u_1,u_2,\ldots,u_r\}$ and $\{v_1,v_2,\ldots,v_r\}$ be a collection of r $(\leq n)$ linearly independent eigenvectors of $\Gamma_g^*\Gamma_g$ and $\Gamma_g\Gamma_g^*$, respectively, corresponding to the eigenvalue σ_1^2. Then [GL95],[ZDG96]:

$$U(t) = \begin{bmatrix} u_1 & \ldots & u_r \end{bmatrix}(t) = \sigma_1^{-1}B'e^{-A't}Q\begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix} \in l_2^{m\times r}(-\infty,0]$$

and

$$V(t) = \begin{bmatrix} v_1 & \ldots & v_r \end{bmatrix}(t) = Ce^{At}\begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix} \in l_2^{p\times r}[0,\infty)$$

Taking the (bilateral) Laplace transform shows that

$$\boldsymbol{U} = -B'(sI+A')^{-1}\Xi \in \mathcal{RH}_2^{\perp,m\times r}, \ \ \Xi = \sigma_1^{-1}Q\begin{bmatrix} x_1 & x_2 & \ldots & x_r \end{bmatrix}$$

and

$$\boldsymbol{V} = C(sI-A)^{-1}\Theta \in \mathcal{H}_2^{p\times r}, \ \ \Theta = \begin{bmatrix} x_1 & x_2 & \ldots & x_r \end{bmatrix}$$

Next, we invoke Nehari's theorem:

Theorem 3.1.

$$\inf_{\boldsymbol{Q}\in\mathcal{H}_\infty^-} \|\boldsymbol{G}+\boldsymbol{Q}\|_\infty = \|\Gamma_{\boldsymbol{G}}\| = \sigma_1 \tag{4}$$

Proof. The theorem was first proved in [Neh57] for the case of SISO discrete-time systems. See [Fra87], [Glo84], [Pel03] for the complete proof of the multivariable case. □

Remark 3.1. *In Theorem 3.1, $\boldsymbol{G}$ need not be minimal. See for example [Glo89] and [JL93], where minimality is not assumed.*

It can be shown that the infimum in (4) is attained; further [Glo89]:

$$\text{rank}[\boldsymbol{U}] = \text{rank}[\boldsymbol{V}] := l \leq \min(p, m, r) \tag{5}$$

and

$$(\boldsymbol{G}+\boldsymbol{Q})\boldsymbol{U} = \sigma_1 \boldsymbol{V} \tag{6}$$

for *every* (optimal) $\boldsymbol{Q}$ which achieves the infimum in (4). Equation (6) may be used to show that in the scalar case the optimal Nehari extension is unique and is given by $\boldsymbol{Q} = \boldsymbol{G} + \sigma_1 \boldsymbol{V}/\boldsymbol{U}$. In the matrix case the equation has been used to derive the parametrization of all optimal solutions of the Nehari extension problem [Glo89], and has also inspired most methods used to solve the super-optimal distance problem, typically based on the construction of all-pass diagonalising transformations of $\boldsymbol{G}+\boldsymbol{Q}$ using $\boldsymbol{U}$ and $\boldsymbol{V}$.

3.1 Statement of the problem

A formal definition of the problem follows. Let $\boldsymbol{R} \in \mathcal{RH}_\infty^{-,p\times m}$. Then, define

$$s_i^\infty(\boldsymbol{R}) := \sup_{\omega\in\mathbb{R}} \sigma_i[R(j\omega)], \qquad i = 1, 2, \ldots, \min(p, m).$$

If p and m are both greater than 1, then we define recursively the first and subsequent super-optimal levels of $\boldsymbol{R}$ as

$$s_i(\boldsymbol{R}) := \inf_{\boldsymbol{Q}\in\mathcal{S}_{i-1}(\boldsymbol{R})} s_i^\infty(\boldsymbol{R}+\boldsymbol{Q}) \quad i = 1, 2, \ldots, \min(p, m) \tag{7}$$

and the set of all i-th level super-optimal approximations of $\boldsymbol{R}$ as

$$\mathcal{S}_i(\boldsymbol{R}) := \{\boldsymbol{Q} \in \mathcal{S}_{i-1}(\boldsymbol{R}) : s_i^\infty(\boldsymbol{R}+\boldsymbol{Q}) = s_i(\boldsymbol{R})\} \quad i = 1, 2, \ldots, \min(p, m).$$

In other words, we seek among all super-optimal approximations at the $(i-1)$-th level $\mathcal{S}_{i-1}(\boldsymbol{R})$ a set for which $s_i(\boldsymbol{R})$ is minimized (it turns out that the infimum in (7) is always attained). This set is not a singleton in general (apart from the case of $i = \min(p, m)$), but forms a subset of all $(i-1)$-th level super-optimal approximations of $\boldsymbol{R}$, $\mathcal{S}_{i-1}(\boldsymbol{R})$. Note that for $i = 1$, (7) is taken to be a Nehari extension problem and hence we define $\mathcal{S}_0(\boldsymbol{R}) := \mathcal{H}_\infty^{+,\,p\times m}$. Due to the lexicographic nature of the problem, it is clear that every element of $\mathcal{S}_i(\boldsymbol{R})$ is also an element of $\mathcal{S}_{i-1}(\boldsymbol{R})$, i.e. that the super-optimal approximation sets nest as:

$$\mathcal{S}_0(\boldsymbol{R}) \supseteq \mathcal{S}_1(\boldsymbol{R}) \supseteq \ldots \supseteq \mathcal{S}_i(\boldsymbol{R}) \supseteq \ldots \supseteq \mathcal{S}_{\min(p,m)}(\boldsymbol{R})$$

The super-optimal approximation problem ([SODP]) considered in this paper can be formally defined as follows:

Problem 3.1. [SODP]. *Given an $\boldsymbol{R} \in \mathcal{RH}_\infty^{-,\,p\times m}$, find the (unique) matrix-function $\boldsymbol{Q_{sup}} \in \mathcal{H}_\infty^{+,\,p\times m}$ which minimizes the sequence*

$$s^\infty(\boldsymbol{R}+\boldsymbol{Q}) = (s_1^\infty(\boldsymbol{R}+\boldsymbol{Q}), s_2^\infty(\boldsymbol{R}+\boldsymbol{Q}), \ldots, s_k^\infty(\boldsymbol{R}+\boldsymbol{Q}))$$

with respect to the lexicographic ordering, where $k = \min(p, m)$.

The approach followed here involves the reduction of the lexicographic minimization into a hierarchy of ordinary $\mathcal{H}_\infty$-optimization (Nehari-extension) problems of progressively reduced input-output dimensions, whose solution is well known in the literature [Glo84], [Glo89], [ZDG96], [GL95]. In particular, for the case of $i = 2$ in (7), two all-pass system matrices $\boldsymbol{V}^\sim$ and $\boldsymbol{W}$ are constructed (depending on $\boldsymbol{R}$) which diagonalise every optimal "error system" $\boldsymbol{R} + \boldsymbol{Q}$, $\boldsymbol{Q} \in \mathcal{S}_1(\boldsymbol{R})$, i.e.

$$\boldsymbol{V}^\sim(\boldsymbol{R} + \boldsymbol{Q})\boldsymbol{W} = \begin{pmatrix} s_1(\boldsymbol{R})\boldsymbol{\alpha} & 0 \\ 0 & \widehat{\boldsymbol{R}} + \overline{\boldsymbol{Q}} \end{pmatrix}$$

Denoting the multiplicities of distinct super-optimal levels by $l_1, l_2, \ldots, l_k$ (i.e. $l_1 + l_2 + \cdots + l_k = min(p, m)$) we have that $\widehat{\boldsymbol{R}} \in \mathcal{RH}_\infty^{-,(p-l_1)\times(m-l_1)}$, $\overline{\boldsymbol{Q}} \in \mathcal{RH}_\infty^{+,(p-l_1)\times(m-l_1)}$, $l_1 \geq 1$. Note that $\boldsymbol{\alpha}$ is anti-inner of dimension $l_1 \times l_1$; also $\boldsymbol{\alpha}$ and $\widehat{\boldsymbol{R}}$ are fixed (i.e. they do not depend on $\boldsymbol{Q} \in \mathcal{S}_1(\boldsymbol{R})$). It is further shown that $\|\widehat{\boldsymbol{R}}^\sim\|_H < s_1(\boldsymbol{R})$ and that as $\boldsymbol{Q}$ varies over $\mathcal{S}_1(\boldsymbol{R})$, $\overline{\boldsymbol{Q}}$ varies over the set of all $s_1(\boldsymbol{R})$ sub-optimal Nehari approximations of $\widehat{\boldsymbol{R}}$, i.e. over the set

$$\mathcal{S}(\widehat{\boldsymbol{R}}, s_1(\boldsymbol{R})) := \{\boldsymbol{\Psi} \in \mathcal{H}_\infty^{+\,(p-l_1)\times(m-l_1)} : \|\widehat{\boldsymbol{R}} + \boldsymbol{\Psi}\|_\infty \leq s_1(\boldsymbol{R})\}$$

Thus (in the generic case $l_1 = 1$),

$$s_{l_1+1}(\boldsymbol{R}) = \inf_{\boldsymbol{Q} \in \mathcal{S}_1(\boldsymbol{R})} s_{l_1+1}^\infty(\boldsymbol{V}^\sim(\boldsymbol{R} + \boldsymbol{Q})\boldsymbol{W}) = \inf_{\overline{\boldsymbol{Q}} \in \mathcal{S}(\widehat{\boldsymbol{R}}, s_1(\boldsymbol{R}))} s_1^\infty(\widehat{\boldsymbol{R}} + \overline{\boldsymbol{Q}})$$

and so in this case (as all optimal Nehari approximations of $\widehat{\boldsymbol{R}}$ are also $s_1(\boldsymbol{R})$-suboptimal)

$$s_{l_1+1}(\boldsymbol{R}) = s_1(\widehat{\boldsymbol{R}})$$

A recursive application of this procedure generates all super-optimal levels.

The super-optimal distance problem has been proposed in the context of $\mathcal{H}_\infty$-optimal control as a means of restoring uniqueness to the optimal controller in the multivariable case. Although the key theoretical and computational aspects of the linear $\mathcal{H}_\infty$ theory have been resolved (while the theory has even been extended to more general settings), the choice of the "best" optimal controller is still an open problem. Note that, in this respect, most solution techniques, including those based on Linear Matrix Inequalities [CSC97], [IS94], are essentially suboptimal in nature and do not differentiate between different near-optimal solutions). In cases where strong directionality information is available in the model of the disturbance signal (which must be rejected) or the uncertainty model of the plant (which must be robustly stabilized), the super-optimal solution may offer important advantages, apart from mathematical elegance in restoring uniqueness.

3.2 Overview

The paper considers the super-optimal Nehari-extension problem for real-rational continuous-time systems. All results are established via simple linear algebraic methods. The main steps of the algorithm are first developed purely at a transfer-function level, although this construction is subsequently supported via a detailed state-space analysis in order to develop efficient numerical algorithms for the solution of the problem. The main features of our approach and the contribution of the work are briefly described below:

- We remove all main assumptions made in previous state-space based solutions to the problem. Specifically: (i) The realization of the system which is approximated ($\boldsymbol{R}$) is not assumed to be minimal or balanced; (ii) The largest Hankel singular value of $\boldsymbol{R}$ is here assumed to have arbitrary multiplicity; and (iii) no assumption is made about the invertibility of the controllability and observability gramians of certain realizations arising at intermediate steps of the algorithm; in previous work, these conditions were assumed to facilitate the state-space analysis of the algorithm and (unnecessarily) qualified the derived degree bound of the super-optimal approximation [LHG89].

- We have investigated pathological non-generic cases related to Hankel singular value multiplicities and the degree of the optimal solution. This allows for the development of algorithms with improved numerical properties.

- The all-pass dilation approach [JL93] adopted here provides conceptual and computational advantages over existing methods, e.g. [TGP88], [Kwa86], [LHG89]. The starting point of these methods is invariably the diagonalisation of the Nehari optimal solution set with the help of the Schmidt-pair of the Hankel operator associated with the problem, which is in fact conceptually and computationally redundant. The present construction is entirely based on the properties of the dilated system. This simplifies the exposition and allows us to keep the argument entirely at the transfer function level, although a state-space construction is also developed in parallel for computational purposes.

- The structure of the Nehari approximation ("one-block") problem is exploited to develop a concrete state-space implementation of the algorithm which relies on the duality between two spectral factorization-type Riccati equations and their their corresponding Hamiltonians. The analysis is used to derive degree bounds of the super-optimal approximation and establish certain interlacing inequalities between super-optimal levels and Hankel singular values [LHG88], [LHG89] without imposing unnecessary assumptions.

- The paper briefly discusses applications of super-optimization in control theory. Early references report applications in the areas of disturbance rejection [Kwa86], robust stabilization [KN89], [Nym95] and hierarchical $\mathcal{H}_\infty$ design [HJ98a], [HJW97]. Applications of super-optimization in the areas of robust stabilization and structured-singular value approximations can be found in [GHJ00] and [JHMG06].

3.3 Brief survey of literature

The first published results in super-optimization can be found in [You86] and are based on operator theoretic methods. In subsequent years, linear-algebraic algorithms for the real-rational problem appeared in a series of papers [PF85], [PTG89], [TGP88], [LHG88], [LHG89], [GTP90], [TGPA90]. These all relied on state-space methods and addressed the problem both in continuous and discrete-time settings. A parallel approach using a polynomial framework was developed in references [Kwa86], [KN89]. Investigations on cancellation analysis, degree-bounds and "interlacing inequalities" between Hankel singular values and super-optimal levels can be found in [LHG88], [LHG89] and [Pel03]. Generalizations of super-optimization to the two-block and four-block problems were first reported in [GTP89], [Nym94] and [JL93]. Reference [GTP89] follows the early state-space approach for solving the two-block $\mathcal{H}_\infty$ problem, by reducing it to an equivalent one-block problem via a spectral and an inner-outer factorization. In contrast, the approach of [Nym94] is based on the "equalization-principle", widely used in early $\mathcal{H}_\infty$ polynomial methods [Kwa86], while [JL93] relies on a state-space all-pass dilation technique, proposed in [GLD$^+$91] for solving the general-distance $\mathcal{H}_\infty$ problem. An interesting state-feedback approach based on Riccati inequalities, in the spirit of recent LMI developments, can be found in [Foo04]. Extensions of super-optimization to the Hankel-norm approximation (AAK) problem, originating with the work of [PY96], [Tre95] were further developed in an algorithmic state-space setting in [HLG93] and [HJ98b]. Despite its similarity to its Nehari counterpart, the super-optimal Hankel-norm problem is considerably more intricate; it is known that in pathological cases, even uniqueness of the super-optimal approximation can be lost [Tre95],[HJ98b], which was the original motivating factor for introducing super-optimization.

Applications of super-optimization in control theory were first reported in the areas of disturbance rejection [Kwa86] and robust stabilization [Nym95]. The stronger version of optimality resulting from super-optimal approximations has been used in [Hal93], [HJW97], [HJ98a], [DH98] to address hierarchical optimization problems in an $\mathcal{H}_\infty$ or mixed-norm setting. In [Nym99] a multidirectional gap-metric is defined for multivariable systems under gap and coprime-factor perturbations using super-optimization ideas. In [Gom95] an inverse-robust stabilization problem is addressed: Given a super-optimal controller, determine the set of plants which it stabilizes. Reference [GHJ00] applies super-optimization techniques in the area of maximal robust-stabilization of LTI systems under additive perturbations: Explicit expressions for the improved robust stability radius are derived by imposing structure on the perturbation set via a uniform frequency constraint in the most-critical direction which is identified. The method is also used in [GHJ00], [JHMG06] to derive an upper bound on the structured singular value for multivariable systems in the case of complex structured block-diagonal perturbations, which is tighter than the convex upper bound provided by the "D-iteration". In this context, the multiplicity of the largest Hankel singular value becomes a crucial consideration, which motivates the detailed analysis of the general problem presented in this paper. An overview of these results and extensions to the case of normalized coprime-factor uncertainty models will be reported in a future publication.

4 The 1-block Super-Optimal Distance Problem

The approach for solving the SODP adopted in this paper is based on all-pass dilation techniques. First the system to be approximated, $\boldsymbol{R}$, is embedded in an all-pass system $\boldsymbol{H}$ of higher dimensions (note that $\boldsymbol{R}$ is taken to lie in $\mathcal{H}_\infty^-$ for compatibility with the existing $\mathcal{H}_\infty$ optimal-control literature). This acts as a "generator" of the optimal solution set of the Nehari extension problem, as all solutions can be obtained via a LFT of $\boldsymbol{H}$ with the ball of $\mathcal{H}_\infty$ of radius s_1^{-1} (i.e. the set of all stable s_1^{-1}-contractions) [Glo89]. Next, a sub-block of the optimal generator $\boldsymbol{H}$ is dilated to define a new square all-pass system $\overline{\boldsymbol{H}}$, of lower dimensions compared to those of $\boldsymbol{H}$. Exploiting the all-pass nature of $\boldsymbol{H}$ and $\overline{\boldsymbol{H}}$ and the fact that they share a common block, two diagonalizing transformations of $\boldsymbol{H}$ can be defined from certain sub-blocks of $\boldsymbol{H}$ and $\overline{\boldsymbol{H}}$. The diagonalization is analogous to the partial singular-value decomposition of constant matrices and makes the minimization of the second super-optimal level transparent. First, the general solution of the optimal Nehari-extension problem is given under minimal assumptions:

Theorem 4.1 (Optimal Nehari approximation). *Consider $\boldsymbol{R} \in \mathcal{RH}_\infty^{-,\,p\times m}$ with realization $\boldsymbol{R} \stackrel{s}{=} (A,B,C,0)$ where $\lambda(A) \subset \mathcal{C}_+$. Then there exists $\boldsymbol{Q_a} \in \mathcal{RH}_\infty^{+,(p+m-l)\times(p+m-l)}$ such that all $\boldsymbol{Q} \in \mathcal{H}_\infty^{+,\,p\times m}$ which satisfy $\|\boldsymbol{R}+\boldsymbol{Q}\|_\infty = \|\boldsymbol{R}^\sim\|_H = s_1$ (Nehari optimal approximations of $\boldsymbol{R}$) are given by*

$$\boldsymbol{Q} = \mathcal{F}_l(\boldsymbol{Q_a}, s_1^{-1}\, \mathcal{BH}_\infty^{(p-l)\times(m-l)})$$

where r denotes the multiplicity of the largest Hankel singular value of $\boldsymbol{R}^\sim$, l is defined in (5), and

$$\boldsymbol{Q_a} := \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A_q & B_{q1} & B_{q2} \\ \hline C_{q1} & D_{11} & D_{12} \\ C_{q2} & D_{21} & 0 \end{array}\right] \tag{8}$$

The corresponding "error" system is given by

$$\begin{aligned} \boldsymbol{H} \;&:=\; \begin{pmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{R}+\boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \hline \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{array}\right) \\ &\stackrel{s}{=} \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A_q & B_{q1} & B_{q2} \\ \hline C & C_{q1} & D_{11} & D_{12} \\ 0 & C_{q2} & D_{21} & 0 \end{array}\right] \stackrel{s}{=}: \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] \end{aligned} \tag{9}$$

where $\|\boldsymbol{H}_{22}\|_\infty < s_1$ and $\boldsymbol{Q}_{ij} \in \mathcal{H}_\infty^+$, for $i,j \in \{1,2\}$. Further, $\boldsymbol{H}\boldsymbol{H}^\sim = \boldsymbol{H}^\sim\boldsymbol{H} = s_1^2 I$ and the following set of equations is satisfied

$$
\begin{aligned}
&P_H Q_H = Q_H P_H = s_1^2 I \\
&D_H D_H' = D_H' D_H = s_1^2 I \\
&A_H' Q_H + Q_H A_H + C_H' C_H = 0 \\
&A_H P_H + P_H A_H' + B_H B_H' = 0 \\
&D_H' C_H + B_H' Q_H = 0 \\
&D_H B_H' + C_H P_H = 0
\end{aligned}
\tag{10}
$$

Here P_H and Q_H are the gramians of the realization of $\boldsymbol{H}$ given in (10).

Proof. The proof is constructive. See [Glo84] in which explicit state-space realisation of $\boldsymbol{Q_a}$ is given. See also [JL93] and [GLD+91] for a more general setting. □

Remark 4.1. *The realization of $\boldsymbol{R}$ need not be assumed minimal. However, we require that $\lambda(A) \subset \mathcal{C}_+$. If $\boldsymbol{R}$ has McMillan degree n, it can be shown [Glo89] that $\boldsymbol{Q_a}$ given in (8) has degree $n-r$; in addition, $\sigma_i(\boldsymbol{Q_a}) = \sigma_{i+r}(\boldsymbol{R}^\sim)$, $i = 1, 2, \ldots, n-r$ [Glo89], [GL95].*

Remark 4.2. *The integer parameter l which is used to define the input and output dimension of $\boldsymbol{Q}_{22}$ is the normal rank of the Laplace transform of the matrix formed by the r Schmidt vectors of $\Gamma_{\boldsymbol{R}^\sim}$ corresponding to σ_1, defined in equation (5). In the notation of Theorem 4.1 $\boldsymbol{R}^\sim = (-A', C', -B')$ and hence $\boldsymbol{U}$ and $\boldsymbol{V}$ are given as*

$$\boldsymbol{U} = -C(sI-A)^{-1}\Xi \in \mathcal{RH}_2^{\perp, m\times r}, \;\; \Xi = \sigma_1^{-1} P \begin{bmatrix} x_1 & x_2 & \ldots & x_r \end{bmatrix}$$

and

$$\boldsymbol{V} = -B'(sI+A')^{-1}\Theta \in \mathcal{H}_2^{p\times r}, \;\; \Theta = \begin{bmatrix} x_1 & x_2 & \ldots & x_r \end{bmatrix}$$

where P and Q are the controllability and observability matrices of $\boldsymbol{R} \stackrel{s}{=} (A, B, C)$ and the x_i's are r linearly independent eigenvectors of QP corresponding to the eigenvalue σ_1^2. In particular, if (A, B, C) is balanced, $P = Q = -\mathrm{diag}(\sigma_1 I_r, \Sigma_2)$, and thus $\Xi = -E_r$ and $\Theta = \sigma_1^2 E_r$ (where E_r denotes the first r-columns of the $n \times n$ unit matrix), so that $\boldsymbol{U} = C(sI-A)^{-1}E_r \in \mathcal{H}_2^\perp$ and $\boldsymbol{V} = -s_1^2 B'(sI+A')^{-1}E_r \in \mathcal{H}_2$. Thus,

$$rank_{\mathcal{R}(s)}\boldsymbol{U}^\sim \geq \lim_{s\to\infty}[s\boldsymbol{U}^\sim] = rank\,(CE_r)$$

and

$$rank_{\mathcal{R}(s)}\boldsymbol{V} \geq \lim_{s\to\infty}[s\boldsymbol{V}] = rank\,(E_r'B)$$

It is shown in [Glo89] that these two inequalities are actually equalities; further, the normal ranks of $\boldsymbol{U}$ and $\boldsymbol{V}$ are equal, since Rank (CE_r) = Rank $(E_r'B)$, as can be verified by the equality $E_r'C'CE_r = E_r'BB'E_r$, which follows easily from the all-pass equations (10). Thus $l \leq \min(p, m, r)$ and l can be easily determined from the balanced realization of $\boldsymbol{R}$.

Remark 4.3. *In the present work, the gramians of $\boldsymbol{H}$ are not considered to be balanced. The above set of equations is known as the set of "all-pass" equations. Partitioning conformally with (8), these can be written in* *full (for easy future reference) as:*

$$
\begin{aligned}
&\textbf{\textit{(i)}} \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} = \begin{bmatrix} s_1^2 I & 0 \\ 0 & s_1^2 I \end{bmatrix} \\
&\textbf{\textit{(ii)}} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \begin{bmatrix} D_{11}' & D_{21}' \\ D_{12}' & 0 \end{bmatrix} = \begin{bmatrix} s_1^2 I & 0 \\ 0 & s_1^2 I \end{bmatrix} = \begin{bmatrix} D_{11}' & D_{21}' \\ D_{12}' & 0 \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \\
&\textbf{\textit{(iii)}} \begin{bmatrix} A' & 0 \\ 0 & A_q' \end{bmatrix} \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} + \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A_q \end{bmatrix} + \begin{bmatrix} C' & 0 \\ C_{q1}' & C_{q2}' \end{bmatrix} \begin{bmatrix} C & C_{q1} \\ 0 & C_{q2} \end{bmatrix} = 0 \\
&\textbf{\textit{(iv)}} \begin{bmatrix} A & 0 \\ 0 & A_q \end{bmatrix} \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} + \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & A_q' \end{bmatrix} + \begin{bmatrix} B & 0 \\ B_{q1} & B_{q2} \end{bmatrix} \begin{bmatrix} B' & B_{q1}' \\ 0 & B_{q2}' \end{bmatrix} = 0 \\
&\textbf{\textit{(v)}} \begin{bmatrix} D_{11}' & D_{21}' \\ D_{12}' & 0 \end{bmatrix} \begin{bmatrix} C & C_{q1} \\ 0 & C_{q2} \end{bmatrix} + \begin{bmatrix} B' & B_{q1}' \\ 0 & B_{q2}' \end{bmatrix} \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} = 0 \\
&\textbf{\textit{(vi)}} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \begin{bmatrix} B' & B_{q1}' \\ 0 & B_{q2}' \end{bmatrix} + \begin{bmatrix} C & C_{q1} \\ 0 & C_{q2} \end{bmatrix} \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} = 0
\end{aligned}
\tag{11}
$$

In the following, keeping $\boldsymbol{H}_{22} = \boldsymbol{Q}_{22} \in \mathcal{H}_\infty^{+,(m-l)\times(p-l)}$ (with $\|\boldsymbol{Q}_{22}\| < s_1$ from Theorem 4.1), we construct an s_1-allpass matrix function $\overline{\boldsymbol{H}}$, corresponding to a new system $\widehat{\boldsymbol{R}} \in \mathcal{H}_\infty^{-,(p-l)\times(m-l)}$ defined from its $(1,1)$ block. It is shown that $\overline{\boldsymbol{H}}$ acts as a s_1-suboptimal Nehari generator of $\widehat{\boldsymbol{R}}$, i.e. that the LFT of $\overline{\boldsymbol{H}}$ with the s_1^{-1}-ball of $\mathcal{H}_\infty$ generates the set $\{\boldsymbol{\Psi} \in \mathcal{H}_\infty : \|\widehat{\boldsymbol{R}} + \boldsymbol{\Psi}\| \leq s_1\}$. Using this structure, it is possible to construct all level-two super-optimal approximations of $\boldsymbol{R}$, which lie inside the set of all optimal approximations, $\boldsymbol{Q}$, of $\boldsymbol{R}$. By choosing all $\boldsymbol{Q}$ inside the subset, the corresponding "error" systems $\boldsymbol{R} + \boldsymbol{Q}$ will now minimize the first as well as the second singular values of $\boldsymbol{R}$ (for $l = 1$), i.e. this subset defines the super-optimal approximations of $\boldsymbol{R}$ with respect to the first two levels. The method can be repeated using a recursive procedure until all degrees of freedom have been exhausted.

The construction of $\overline{\boldsymbol{H}}$ is based on the following proposition, first stated at a transfer function level. A state-space construction of $\overline{\boldsymbol{H}}$ follows, proving that it acts as an s_1-suboptimal Nehari generator of the anti-stable projection of its $(1,1)$ block.

Proposition 4.1. *Let $\boldsymbol{H}_{22}$ be defined in 4.1. Recall $\|\boldsymbol{H}_{22}\|_\infty < s_1$; then,*

1. *There exists a square transfer matrix $\overline{\boldsymbol{H}}_{21} \in \mathcal{RH}_\infty$ such that $\overline{\boldsymbol{H}}_{21}\overline{\boldsymbol{H}}_{21}^\sim = s_1^2 I - \boldsymbol{H}_{22}\boldsymbol{H}_{22}^\sim$ and $\overline{\boldsymbol{H}}_{21}^{-1} \in \mathcal{RH}_\infty$.*
2. *There exists a square transfer matrix $\overline{\boldsymbol{H}}_{12} \in \mathcal{RH}_\infty$ such that $\overline{\boldsymbol{H}}_{12}^\sim\overline{\boldsymbol{H}}_{12} = s_1^2 I - \boldsymbol{H}_{22}^\sim\boldsymbol{H}_{22}$ and $\overline{\boldsymbol{H}}_{12}^{-1} \in \mathcal{RH}_\infty$.*
3. *The system*

$$\overline{\boldsymbol{H}} = \begin{pmatrix} \overline{\boldsymbol{H}}_{11} & \overline{\boldsymbol{H}}_{12} \\ \overline{\boldsymbol{H}}_{21} & \boldsymbol{H}_{22} \end{pmatrix} := \begin{pmatrix} -\overline{\boldsymbol{H}}_{12}\boldsymbol{H}_{22}^\sim\overline{\boldsymbol{H}}_{21}^\sim & \overline{\boldsymbol{H}}_{12} \\ \overline{\boldsymbol{H}}_{21} & \boldsymbol{H}_{22} \end{pmatrix}$$

is in $\mathcal{RL}_\infty$ and is s_1-allpass. Further, let $-\overline{\boldsymbol{H}}_{12}\boldsymbol{H}_{22}^\sim\overline{\boldsymbol{H}}_{21}^\sim = \widehat{\boldsymbol{R}} + \overline{\boldsymbol{Q}}_{11}$ where $\hat{R} \in \mathcal{RH}_\infty^-$ and $\overline{\boldsymbol{Q}}_{11} \in \mathcal{RH}_\infty^+$. Then $\|\widehat{\boldsymbol{R}}^\sim\|_H < s_1$.

Proof. For parts (1) and (2) see [ZDG96], Corollary 13.22. The proof follows from a detailed construction involving elements from the theory of algebraic Riccati equations and spectral factorization, which is briefly discussed in the following section. The proof that $\overline{\boldsymbol{H}}$ is in $\mathcal{L}_\infty$ and is s_1-allpass follows from [Glo84] and can be verified directly by showing that $\overline{\boldsymbol{H}}\overline{\boldsymbol{H}}^\sim = s_1^2 I$. Finally, to show that $\|\widehat{\boldsymbol{R}}^\sim\|_H < s_1$, note that since $\overline{\boldsymbol{H}}_{12}$ (or $\overline{\boldsymbol{H}}_{21}$) is a unit of $\mathcal{H}_\infty$ and $\overline{\boldsymbol{H}}$ is s_1-allpass, then $\|\overline{\boldsymbol{H}}_{11}\|_\infty < s_1$. Write $\overline{\boldsymbol{H}}_{11} = \widehat{\boldsymbol{R}} + \overline{\boldsymbol{Q}}_{11}$ where $\widehat{\boldsymbol{R}} \in \mathcal{H}_\infty^-$ and $\overline{\boldsymbol{Q}}_{11} \in \mathcal{H}_\infty^+$. Then, using Nehari's theorem

$$\|\widehat{\boldsymbol{R}}^\sim\|_H = \inf_{\boldsymbol{X} \in \mathcal{H}_\infty^-} \|\widehat{\boldsymbol{R}}^\sim + \boldsymbol{X}\|_\infty \leq \|\widehat{\boldsymbol{R}}^\sim + \overline{\boldsymbol{Q}}_{11}^\sim\|_\infty = \|\overline{\boldsymbol{H}}_{11}^\sim\|_\infty < s_1$$

which completes the proof. □

Remark 4.4. *Since* $s_1 = \sigma_1(\boldsymbol{R}^{\sim})$ *the inequality of part (3.) implies that* $\sigma_1(\widehat{\boldsymbol{R}}^{\sim}) < \sigma_1(\boldsymbol{R}^{\sim})$. *As shown later in this section this can be strengthened to* $\sigma_1(\widehat{\boldsymbol{R}}^{\sim}) < \sigma_{r+1}(\boldsymbol{R}^{\sim})$, *where* r *is the multiplicity of the largest Hankel singular value of* $\boldsymbol{R}^{\sim}$.

A detailed state-space construction of $\overline{\boldsymbol{H}}$ and its properties are given in Theorem 4.2 below.

Theorem 4.2. *Consider*

$$\boldsymbol{H}_{22} = \boldsymbol{Q}_{22} \stackrel{s}{=} \left[\begin{array}{c|c} A_q & B_{q2} \\ \hline C_{q2} & 0 \end{array}\right] \in \mathcal{H}_\infty^{+,(m-l)\times(p-l)}, \quad \|\boldsymbol{Q}_{22}\|_\infty < s_1$$

defined in Theorem 4.1. Then there exist unique stabilizing solutions $\overline{P}_2$ *and* $\overline{Q}_2$ *to the following algebraic Riccati equations:*

$$\begin{aligned} A_q\overline{P}_2 + \overline{P}_2A_q' + B_{q2}B_{q2}' + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 = 0 \\ A_q'\overline{Q}_2 + \overline{Q}_2A_q + C_{q2}'C_{q2} + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2 = 0 \end{aligned} \tag{12}$$

respectively. Define:

$$\overline{R} := \overline{Q}_2\overline{P}_2 - s_1^2 I \tag{13}$$

Then $\overline{R}$ *is non-singular. Further, there exists a* $\overline{\boldsymbol{Q}}_{\boldsymbol{a}} \in \mathcal{H}_\infty^{+,(p+m-2l)\times(p+m-2l)}$ *with realization*

$$\overline{\boldsymbol{Q}}_{\boldsymbol{a}} := \begin{pmatrix} \overline{\boldsymbol{Q}}_{11} & \overline{\boldsymbol{Q}}_{12} \\ \overline{\boldsymbol{Q}}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A_q & \overline{B}_{q1} & B_{q2} \\ \hline \overline{C}_{q1} & 0 & s_1 I \\ C_{q2} & s_1 I & 0 \end{array}\right] \tag{14}$$

where

$$\overline{C}_{q1} = -s_1^{-1}B_{q2}'\overline{Q}_2, \qquad \overline{B}_{q1} = -s_1^{-1}\overline{P}_2C_{q2}' \tag{15}$$

so that $\overline{\boldsymbol{Q}} = \mathcal{F}_l(\overline{\boldsymbol{Q}}_{\boldsymbol{a}}, s_1^{-1}\ \mathcal{BH}_\infty^{(p-l)\times(m-l)})$ *is the set of all* s_1*−suboptimal Nehari extensions of a system* $\widehat{\boldsymbol{R}} \in \mathcal{H}_\infty^{-,(p-l)\times(m-l)}$ *defined as:*

$$\widehat{\boldsymbol{R}} \stackrel{s}{=} \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & 0 \end{array}\right] \tag{16}$$

in which

$$\widehat{A} = -(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2})', \qquad \widehat{B} = -s_1^{-1}C_{q2}', \qquad \widehat{C} = s_1^{-1}B_{q2}'\overline{R} \tag{17}$$

The corresponding "error system"

$$\overline{\boldsymbol{H}} = \widehat{\boldsymbol{R}}_a + \overline{\boldsymbol{Q}}_{\boldsymbol{a}} = \begin{pmatrix} \widehat{\boldsymbol{R}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \overline{\boldsymbol{Q}}_{11} & \overline{\boldsymbol{Q}}_{12} \\ \overline{\boldsymbol{Q}}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \tag{18}$$

is s_1*-allpass and has a realization*

$$\begin{aligned} \overline{\boldsymbol{H}} \ &:= \begin{pmatrix} \overline{\boldsymbol{H}}_{11} & \overline{\boldsymbol{H}}_{12} \\ \overline{\boldsymbol{H}}_{21} & \boldsymbol{H}_{22} \end{pmatrix} = \begin{pmatrix} \widehat{\boldsymbol{R}} + \overline{\boldsymbol{Q}}_{11} & \overline{\boldsymbol{Q}}_{12} \\ \overline{\boldsymbol{Q}}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} \\ &\stackrel{s}{=} \left[\begin{array}{c|c} A_{\overline{H}} & B_{\overline{H}} \\ \hline C_{\overline{H}} & D_{\overline{H}} \end{array}\right] \stackrel{s}{=} \left[\begin{array}{cc|cc} \widehat{A} & 0 & \widehat{B} & 0 \\ 0 & A_q & \overline{B}_{q1} & B_{q2} \\ \hline \widehat{C} & \overline{C}_{q1} & 0 & s_1 I \\ 0 & C_{q2} & s_1 I & 0 \end{array}\right] \end{aligned} \tag{19}$$

which satisfies the following set of all-pass equations:

$$\begin{aligned} &A_{\overline{H}}'Q_{\overline{H}} + Q_{\overline{H}}A_{\overline{H}} + C_{\overline{H}}'C_{\overline{H}} = 0 \\ &A_{\overline{H}}P_{\overline{H}} + P_{\overline{H}}A_{\overline{H}}' + B_{\overline{H}}B_{\overline{H}}' = 0 \\ &D_{\overline{H}}'C_{\overline{H}} + B_{\overline{H}}'Q_{\overline{H}} = 0 \\ &D_{\overline{H}}B_{\overline{H}}' + C_{\overline{H}}P_{\overline{H}}' = 0 \\ &D_{\overline{H}}D_{\overline{H}}' = D_{\overline{H}}'D_{\overline{H}} = s_1^2 I \\ &P_{\overline{H}}Q_{\overline{H}} = Q_{\overline{H}}P_{\overline{H}} = s_1^2 I \end{aligned} \tag{20}$$

in which $Q_{\overline{H}}$ and $P_{\overline{H}}$ are the gramians of the realization of $\overline{\boldsymbol{H}}$ given in (19).

Proof. The proof is based on [Glo84]; see also [JL93] and [GLD+91] for a more general setting. Here we outline the sequence of logical arguments. The existence of solutions of the two Riccati equations (12) follows from standard theory of spectral factorization and the bounded real-lemma (see Lemma 4.1 in the next section) and relies on the fact that $\|\boldsymbol{Q}_{22}\|_\infty < s_1$. Details and additional properties of the two solutions are included in the following section. Since the two stabilising solutions are chosen, $\hat{A}$ defined in equation (17) is anti-stable and thus $\widehat{\boldsymbol{R}} \in \mathcal{H}_\infty^-$. Systems $\overline{\boldsymbol{Q}}_{\boldsymbol{a}}$ and $\widehat{\boldsymbol{R}}$ correspond to the stable and anti-stable projections of $\overline{\boldsymbol{H}}$ given in Proposition 4.1 which also shows that $\overline{\boldsymbol{H}}$ is s_1-all pass. For a state-space based proof one needs to verify the all-pass equations given in (20) and expanded in (21) below; this is straightforward using the realizations given in Theorem 4.1 and the two Riccati equations (12). To show that $\overline{R}$ is non-singular, first note that $\overline{P}_2$ and $\overline{Q}_2$ are the controllability and observability gramians, respectively, of the realization of $\overline{\boldsymbol{Q}}_{\boldsymbol{a}}$ given in equation (14), so that $\sigma_1^2(\overline{\boldsymbol{Q}}_{\boldsymbol{a}}) = \lambda_{\max}(\overline{P}_2\overline{Q}_2)$. A standard argument (e.g. see the early part of the proof of Theorem 4.4 which does not rely on any state-space arguments) shows that $\sigma_1(\overline{\boldsymbol{Q}}_{\boldsymbol{a}}) \le \sigma_{r+1}(\widehat{\boldsymbol{R}}^{\sim}) < \sigma_1(\boldsymbol{R}^{\sim}) = s_1$. Thus $\rho(\overline{P}_2\overline{Q}_2) < s_1^2$ and thus $\overline{R}$ is nonsingular. Finally, the fact that $\overline{\boldsymbol{Q}}_{\boldsymbol{a}}$ generates all s_1-suboptimal Nehari extensions of $\widehat{\boldsymbol{R}}$ follows from the inertia properties of A and $\hat{A}$ and the all-pass nature of $\overline{\boldsymbol{H}}$ [Glo84]; the proof reduces to showing that the invariant zeros of the realizations of $\overline{\boldsymbol{Q}}_{12}$ (or $\overline{\boldsymbol{Q}}_{21}$) given in (19) lie in the open right-half plane, which follows readily by a simple calculation using the fact that $\lambda(\hat{A}) \subset \mathcal{C}_+$. □

Remark 4.5. *Expanding the compact form of the all-pass equations given in Theorem 4.2 we get*

$$\begin{aligned}
&\textit{(i)} \begin{bmatrix} \widehat{A}' & 0 \\ 0 & A_q' \end{bmatrix}\begin{bmatrix} \overline{Q}_1 & -\overline{R}' \\ -\overline{R} & \overline{Q}_2 \end{bmatrix} + \begin{bmatrix} \overline{Q}_1 & -\overline{R}' \\ -\overline{R} & \overline{Q}_2 \end{bmatrix}\begin{bmatrix} \widehat{A} & 0 \\ 0 & A_q \end{bmatrix} + \begin{bmatrix} \widehat{C}' & 0 \\ \overline{C}_{q1}' & C_{q2}' \end{bmatrix}\begin{bmatrix} \widehat{C} & \overline{C}_{q1} \\ 0 & C_{q2} \end{bmatrix} = 0 \\
&\textit{(ii)} \begin{bmatrix} \widehat{A} & 0 \\ 0 & A_q \end{bmatrix}\begin{bmatrix} \widehat{P}_1 & I \\ I & \overline{P}_2 \end{bmatrix} + \begin{bmatrix} \widehat{P}_1 & I \\ I & \overline{P}_2 \end{bmatrix}\begin{bmatrix} \widehat{A}' & 0 \\ 0 & A_q' \end{bmatrix} + \begin{bmatrix} \widehat{B} & 0 \\ \overline{B}_{q1} & B_{q2} \end{bmatrix}\begin{bmatrix} \widehat{B}' & \overline{B}_{q1}' \\ 0 & B_{q2}' \end{bmatrix} = 0 \\
&\textit{(iii)} \begin{bmatrix} 0 & s_1 I \\ s_1 I & 0 \end{bmatrix}\begin{bmatrix} \widehat{C} & \overline{C}_{q1} \\ 0 & C_{q2} \end{bmatrix} + \begin{bmatrix} \widehat{B}' & \overline{B}_{q1}' \\ 0 & B_{q2}' \end{bmatrix}\begin{bmatrix} \overline{Q}_1 & -\overline{R}' \\ -\overline{R} & \overline{Q}_2 \end{bmatrix} = 0 \\
&\textit{(iv)} \begin{bmatrix} 0 & s_1 I \\ s_1 I & 0 \end{bmatrix}\begin{bmatrix} \widehat{B}' & \overline{B}_{q1}' \\ 0 & B_{q2}' \end{bmatrix} + \begin{bmatrix} \widehat{C} & \overline{C}_{q1} \\ 0 & C_{q2} \end{bmatrix}\begin{bmatrix} \widehat{P}_1 & I \\ I & \overline{P}_2 \end{bmatrix} = 0 \\
&\textit{(v)} \begin{bmatrix} \overline{Q}_2\overline{R}^{-\prime} & I \\ I & \overline{P}_2 \end{bmatrix}\begin{bmatrix} \overline{P}_2\overline{R} & -\overline{R}' \\ -\overline{R} & \overline{Q}_2 \end{bmatrix} = \begin{bmatrix} s_1^2 I & 0 \\ 0 & s_1^2 I \end{bmatrix}
\end{aligned} \tag{21}$$

where $\widehat{P}_1 = \overline{Q}_2\overline{R}^{-\prime}$ and $\overline{Q}_1 = \overline{P}_2\overline{R}$.

The following theorem constructs a diagonalising transformation of $\overline{\boldsymbol{H}}$ and solves the level-two SODP.

Theorem 4.3. *Let $\boldsymbol{H}$ and $\overline{\boldsymbol{H}}$ be as defined in Theorems 4.1 and 4.2, respectively. Then*

$$\|\boldsymbol{R}^{\sim}\|_H = s_1(\boldsymbol{R}) = s_2(\boldsymbol{R}) = \ldots = s_l(\boldsymbol{R}) > s_{l+1}(\boldsymbol{R}) = \|\widehat{\boldsymbol{R}}^{\sim}\|_H$$

Further,

$$\mathcal{S}_1(\boldsymbol{R}) = \mathcal{S}_2(\boldsymbol{R}) = \ldots = \mathcal{S}_l(\boldsymbol{R}) = \mathcal{F}_l(\boldsymbol{Q}_{\boldsymbol{a}}, s_1^2\, \mathcal{BH}_\infty^{(p-l)\times(m-l)})$$

and

$$\mathcal{S}_{l+1}(\boldsymbol{R}) = \mathcal{F}_l[\boldsymbol{Q}_{\boldsymbol{a}}, \mathcal{F}_u(\overline{\boldsymbol{Q}}_{\boldsymbol{a}}^{-1}, \mathcal{S}_1(\widehat{\boldsymbol{R}}))] \subseteq \mathcal{S}_1(\boldsymbol{R})$$

where $\boldsymbol{Q}_{\boldsymbol{a}}$ and $\overline{\boldsymbol{Q}}_{\boldsymbol{a}}$ are defined in Theorems 4.1 and 4.2.

Proof. We adapt the proof of [JL93] Theorem 3 to our setting. First note that since $\boldsymbol{H}\boldsymbol{H}^{\sim} = \boldsymbol{H}^{\sim}\boldsymbol{H} = s_1^2 I$ and $\overline{\boldsymbol{H}}\overline{\boldsymbol{H}}^{\sim} = \overline{\boldsymbol{H}}^{\sim}\overline{\boldsymbol{H}} = s_1^2 I$, it follows that

$$\boldsymbol{H}_{11}\boldsymbol{H}_{21}^{\sim} = -\boldsymbol{H}_{21}\boldsymbol{H}_{22}^{\sim}, \qquad \overline{\boldsymbol{H}}_{11} = -\overline{\boldsymbol{H}}_{12}\boldsymbol{H}_{22}^{\sim}\overline{\boldsymbol{H}}_{21}^{-\sim}, \tag{22}$$

$$\overline{\boldsymbol{H}}_{21}\overline{\boldsymbol{H}}_{21}^{\sim} = s_1^2 I - \boldsymbol{H}_{22}\boldsymbol{H}_{22}^{\sim} = \boldsymbol{H}_{21}\boldsymbol{H}_{21}^{\sim} \tag{23}$$

and

$$\overline{\boldsymbol{H}}_{12}^{\sim}\overline{\boldsymbol{H}}_{12} = s_1^2 I - \boldsymbol{H}_{22}^{\sim}\boldsymbol{H}_{22} = \boldsymbol{H}_{12}^{\sim}\boldsymbol{H}_{12} \tag{24}$$

Define

$$\boldsymbol{V}_{\perp} := \boldsymbol{H}_{12}\overline{\boldsymbol{H}}_{12}^{-1} \qquad \text{and} \qquad \boldsymbol{W}_{\perp} := \boldsymbol{H}_{21}^{\sim}\overline{\boldsymbol{H}}_{21}^{-\sim} \tag{25}$$

Then (23) implies that

$$\boldsymbol{V}_{\perp}^{\sim}\boldsymbol{V}_{\perp} = I_{p-l} \qquad \text{and} \qquad \boldsymbol{W}_{\perp}^{\sim}\boldsymbol{W}_{\perp} = I_{m-l} \tag{26}$$

It can be readily verified from a state-space calculation (see next section) that $\boldsymbol{V}_{\perp} \in \mathcal{H}_{\infty}^{+,(p-l)\times p}$ and $\boldsymbol{W}_{\perp} \in \mathcal{H}_{\infty}^{-,(m-l)\times m}$. Thus there exist complementary inner and co-inner factors, respectively, such that

$$\boldsymbol{V} := \begin{pmatrix} \boldsymbol{v} & \boldsymbol{V}_{\perp} \end{pmatrix} \in \mathcal{H}_{\infty}^{+,\,p\times p} \qquad \text{and} \qquad \boldsymbol{W} := \begin{pmatrix} \boldsymbol{w} & \boldsymbol{W}_{\perp} \end{pmatrix} \in \mathcal{H}_{\infty}^{-,\,m\times m}$$

are square-inner and square anti-inner, respectively [ZDG96], [GL95]. Thus, using (22) and the definitions (25), we obtain

$$\begin{aligned} \boldsymbol{V}_{\perp}^{\sim}\boldsymbol{H}_{12} &= \overline{\boldsymbol{H}}_{12}^{-\sim}\boldsymbol{H}_{12}^{\sim}\boldsymbol{H}_{12} = \overline{\boldsymbol{H}}_{12}^{-\sim}\overline{\boldsymbol{H}}_{12}^{\sim}\overline{\boldsymbol{H}}_{12} = \overline{\boldsymbol{H}}_{12} \\ \boldsymbol{H}_{21}\boldsymbol{W}_{\perp} &= \boldsymbol{H}_{21}\boldsymbol{H}_{21}^{\sim}\overline{\boldsymbol{H}}_{21}^{-\sim} = \overline{\boldsymbol{H}}_{21}\overline{\boldsymbol{H}}_{21}^{\sim}\overline{\boldsymbol{H}}_{21}^{-\sim} = \overline{\boldsymbol{H}}_{21} \end{aligned} \tag{27}$$

and

$$\boldsymbol{V}_{\perp}^{\sim}\boldsymbol{H}_{11}\boldsymbol{W}_{\perp} = \boldsymbol{V}_{\perp}^{\sim}\boldsymbol{H}_{11}\boldsymbol{H}_{21}^{\sim}\overline{\boldsymbol{H}}_{21}^{-\sim} = -\boldsymbol{V}_{\perp}^{\sim}\boldsymbol{H}_{12}\boldsymbol{H}_{22}^{\sim}\overline{\boldsymbol{H}}_{21}^{-\sim} = -\overline{\boldsymbol{H}}_{12}H_{22}^{\sim}\overline{\boldsymbol{H}}_{21}^{-\sim} = \overline{\boldsymbol{H}}_{11} \tag{28}$$

It follows that

$$\begin{pmatrix} \boldsymbol{V}^{\sim} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{W} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}^{\sim}\boldsymbol{H}_{11}\boldsymbol{w} & \boldsymbol{v}^{\sim}\boldsymbol{H}_{11}\boldsymbol{W}_{\perp} & \boldsymbol{v}^{\sim}\boldsymbol{H}_{12} \\ \boldsymbol{V}_{\perp}^{\sim}\boldsymbol{H}_{11}\boldsymbol{w} & \overline{\boldsymbol{H}}_{11} & \overline{\boldsymbol{H}}_{12} \\ \boldsymbol{H}_{21}\boldsymbol{w} & \overline{\boldsymbol{H}}_{21} & \overline{\boldsymbol{H}}_{22} \end{pmatrix} \tag{29}$$

Now, since $\boldsymbol{V}$ and $\boldsymbol{W}$ are all-pass and $\boldsymbol{H}$ is s_1-allpass, the system on the RHS of equation (29) is s_1-allpass. But since $\overline{\boldsymbol{H}}$ is also s_1-allpass (Theorem 4.2), we have that $\boldsymbol{v}^{\sim}\boldsymbol{H}_{11}\boldsymbol{W}_{\perp} = 0$, $\boldsymbol{v}^{\sim}\boldsymbol{H}_{12} = 0$, $\boldsymbol{V}_{\perp}^{\sim}\boldsymbol{H}_{11}\boldsymbol{w} = 0$, $\boldsymbol{H}_{21}\boldsymbol{w} = 0$, and $\boldsymbol{v}^{\sim}\boldsymbol{H}_{11}\boldsymbol{w}$ is s_1-allpass and can be written as $\boldsymbol{v}^{\sim}\boldsymbol{H}_{11}\boldsymbol{w} = s_1\boldsymbol{\alpha}$, for some $l \times l$ all-pass matrix-function $\boldsymbol{\alpha}$ (generically $l = 1$ and hence $\boldsymbol{\alpha}$ is scalar). Taking linear fractional transformations with the set $s_1^{-1}\,\mathcal{BH}_{\infty}^{(p-l)\times(m-l)}$ and using the results of Theorem 4.2 and Theorem 4.1 shows that:

$$\boldsymbol{V}^{\sim}[\mathcal{F}_l(\boldsymbol{H}, s_1^{-1}\,\mathcal{BH}_{\infty}^{(p-l)\times(m-l)})]\boldsymbol{W} = \begin{pmatrix} s_1\boldsymbol{\alpha} & 0 \\ 0 & \mathcal{F}_l(\overline{\boldsymbol{H}}, s_1^{-1}\,\mathcal{BH}_{\infty}^{(p-l)\times(m-l)}) \end{pmatrix} \tag{30}$$

or equivalently,

$$\boldsymbol{V}^{\sim}[\boldsymbol{R} + \mathcal{S}_1(\boldsymbol{R})]\boldsymbol{W} = \begin{pmatrix} s_1\boldsymbol{\alpha} & 0 \\ 0 & \widehat{\boldsymbol{R}} + \mathcal{S}(\widehat{\boldsymbol{R}}, s_1) \end{pmatrix} \tag{31}$$

Since $\boldsymbol{\alpha} \in \mathcal{RL}_{\infty}^{l\times l}$ and is all-pass (in fact anti-inner as shown in the next section), it follows that:

$$\|\boldsymbol{R}^{\sim}\|_H = s_1(\boldsymbol{R}) = s_2(\boldsymbol{R}) = \ldots = s_l(\boldsymbol{R}) > s_{l+1}(\boldsymbol{R}) = \|\widehat{\boldsymbol{R}}^{\sim}\|_H$$

and

$$\mathcal{S}_1(\boldsymbol{R}) = \mathcal{S}_2(\boldsymbol{R}) = \ldots = \mathcal{S}_l(\boldsymbol{R}) = \mathcal{F}_l(\boldsymbol{Q}_{\boldsymbol{a}}, s_1^{-1}\,\mathcal{BH}_{\infty}^{(p-l)\times(m-l)})$$

which is the set of all optimal Nehari extensions of $\boldsymbol{R}$. Further, since all optimal Nehari extensions of $\widehat{\boldsymbol{R}}$ are also s_1-suboptimal extensions of $\widehat{\boldsymbol{R}}$, i.e. $\mathcal{S}_1(\widehat{\boldsymbol{R}}) \subseteq \mathcal{S}(\widehat{\boldsymbol{R}}, s_1)$, it follows that

$$s_{l+1}(\boldsymbol{R}) = s_1(\widehat{\boldsymbol{R}}) = \|\boldsymbol{R}^{\sim}\|_H$$

and

$$\begin{aligned}
\boldsymbol{R}+\mathcal{S}_2(\boldsymbol{R}) &= \begin{pmatrix}\boldsymbol{v} & \boldsymbol{V}_\perp\end{pmatrix}\begin{pmatrix}s_1\boldsymbol{\alpha} & 0\\ 0 & \widehat{\boldsymbol{R}}+\mathcal{S}_1(\widehat{\boldsymbol{R}})\end{pmatrix}\begin{pmatrix}\boldsymbol{w}^\sim\\ \boldsymbol{W}_\perp^\sim\end{pmatrix}\\
&= \begin{pmatrix}\boldsymbol{v} & \boldsymbol{V}_\perp\end{pmatrix}\begin{pmatrix}s_1\boldsymbol{\alpha} & 0\\ 0 & \widehat{\boldsymbol{R}}+\overline{\boldsymbol{Q}}\end{pmatrix}\begin{pmatrix}\boldsymbol{w}^\sim\\ \boldsymbol{W}_\perp^\sim\end{pmatrix}+\begin{pmatrix}\boldsymbol{v} & \boldsymbol{V}_\perp\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \mathcal{S}_1(\widehat{\boldsymbol{R}})-\overline{\boldsymbol{Q}}\end{pmatrix}\begin{pmatrix}\boldsymbol{w}^\sim\\ \boldsymbol{W}_\perp^\sim\end{pmatrix} \qquad (32)\\
&= \boldsymbol{R}+\boldsymbol{Q}_{11}+\boldsymbol{V}_\perp(\mathcal{S}_1(\widehat{\boldsymbol{R}})-\overline{\boldsymbol{Q}})\boldsymbol{W}_\perp^\sim
\end{aligned}$$

by observing that

$$\boldsymbol{V}^\sim\boldsymbol{H}_{11}\boldsymbol{W} = \begin{pmatrix}s_1\boldsymbol{\alpha} & 0\\ 0 & \overline{\boldsymbol{H}}_{11}\end{pmatrix} \Rightarrow \boldsymbol{R}+\boldsymbol{Q}_{11} = \boldsymbol{V}\begin{pmatrix}s_1\boldsymbol{\alpha} & 0\\ 0 & \widehat{\boldsymbol{R}}+\overline{\boldsymbol{Q}}_{11}\end{pmatrix}\boldsymbol{W}^\sim$$

Using the definitions of of $\boldsymbol{V}_\perp$ and $\boldsymbol{W}_\perp^\sim$ in (25) and cancelling $\boldsymbol{R}$ from both sides of equation (32), we can write:

$$\mathcal{S}_2(\boldsymbol{R}) = \boldsymbol{Q}_{11}+\boldsymbol{Q}_{12}\overline{\boldsymbol{Q}}_{12}^{-1}(\mathcal{S}_1(\widehat{\boldsymbol{R}})-\overline{\boldsymbol{Q}})\overline{\boldsymbol{Q}}_{21}^{-1}\boldsymbol{Q}_{21} =: \mathcal{F}_l(\boldsymbol{K},\mathcal{S}_1(\widehat{\boldsymbol{R}}))$$

where

$$\boldsymbol{K} := \begin{pmatrix}\boldsymbol{Q}_{11}-\boldsymbol{Q}_{12}\overline{\boldsymbol{Q}}_{12}^{-1}\overline{\boldsymbol{Q}}_{11}\overline{\boldsymbol{Q}}_{21}^{-1}\boldsymbol{Q}_{21} & \boldsymbol{Q}_{12}\overline{\boldsymbol{Q}}_{12}^{-1}\\ \overline{\boldsymbol{Q}}_{21}^{-1}\boldsymbol{Q}_{21} & 0\end{pmatrix} = \mathcal{F}_l(\boldsymbol{Q_a},\overline{\boldsymbol{Q}}_{\boldsymbol{a}}{}^{-1})$$

This completes the proof. □

The following Theorem establishes bounds on the super-optimal levels. The proof is similar to a parallel result in [LHG89], but the assumption involving the multiplicity of the largest Hankel singular value of $\boldsymbol{R}^\sim$ is removed.

Theorem 4.4 (Super-optimal level bounds)**.** *The* $(l+1)$*-th super-optimal level is bounded above by the* $(r+1)$*-th Hankel singular value of* $\boldsymbol{R}^\sim$*, i.e.*

$$\sigma_1(\widehat{\boldsymbol{R}}^\sim) = s_{l+1}(\boldsymbol{R}) \leq \sigma_{r+1}(\boldsymbol{R}^\sim) < s_1(\boldsymbol{R}) = s_2(\boldsymbol{R}) = \ldots = s_l(\boldsymbol{R}) = \sigma_1(\boldsymbol{R}^\sim)$$

Proof. The proof follows from the following sequence of inequalities:

$$\begin{aligned}
\sigma_{i+r}(\boldsymbol{R}^\sim) &= \sigma_i(\boldsymbol{Q_a}) \qquad\qquad i=1,2,\ldots,n-r\\
&= \inf_{\boldsymbol{\Psi}\in\mathcal{H}_\infty^-(i-1)}\|\boldsymbol{Q_a}+\boldsymbol{\Psi}\|_\infty\\
&= \inf_{\boldsymbol{\Psi}\in\mathcal{H}_\infty^-(i-1)}\|\boldsymbol{R}+\boldsymbol{Q_a}+\boldsymbol{\Psi}\|_\infty\\
&\geq \inf_{\boldsymbol{\Psi}\in\mathcal{H}_\infty^-(i-1)}\left\|\begin{pmatrix}\boldsymbol{V}_\perp^\sim & 0\\ 0 & I\end{pmatrix}(\boldsymbol{R}+\boldsymbol{Q_a}+\boldsymbol{\Psi})\begin{pmatrix}\boldsymbol{W}_\perp & 0\\ 0 & I\end{pmatrix}\right\|_\infty\\
&\geq \inf_{\widehat{\boldsymbol{\Psi}}\in\mathcal{H}_\infty^-(i-1)}\|\widehat{\boldsymbol{R}}_a+\overline{\boldsymbol{Q}}_{\boldsymbol{a}}+\widehat{\boldsymbol{\Psi}}\|_\infty\\
&\geq \inf_{\widehat{\boldsymbol{\Psi}}\in\mathcal{H}_\infty^-(i-1)}\|\overline{\boldsymbol{Q}}_{\boldsymbol{a}}+\widehat{\boldsymbol{\Psi}}\|_\infty\\
&= \sigma_i(\overline{\boldsymbol{Q}}_{\boldsymbol{a}})
\end{aligned}$$

The first equality follows from Theorem 4.1. The second equality is a statement of the AAK Theorem [Glo89], while the third equality holds since $\boldsymbol{R}\in\mathcal{H}_\infty^-$ and can be absorbed in $\boldsymbol{\Psi}$. The first inequality follows from the fact that $\boldsymbol{V}_\perp$ and $\boldsymbol{W}_\perp$ are contractive, while the second inequality follows from Theorem 4.3 and the fact that $\boldsymbol{V}_\perp^\sim$ and $\boldsymbol{W}_\perp$ are both in $\mathcal{RH}_\infty^-$. Finally, the third inequality follows from the fact that $\widehat{\boldsymbol{R}}\in\mathcal{RH}_\infty^-$, while the last equality is a restatement of the AAK Theorem.

Setting $i=1$ in the above inequality shows that $\sigma_{r+1}(\boldsymbol{R}^\sim)\geq\sigma_1(\overline{\boldsymbol{Q}}_{\boldsymbol{a}})$. Now, using (21), it follows that

$$\sigma_i^2(\widehat{\boldsymbol{R}}^\sim) = \lambda_i(\widehat{P}_1\overline{Q}_1) = \lambda_i(\overline{Q}_2\overline{R}^{-\prime}\overline{P}_2\overline{R}) = \lambda_i(\overline{Q}_2\overline{P}_2) = \sigma_i^2(\overline{\boldsymbol{Q}}_{\boldsymbol{a}})$$

and so $\widehat{\boldsymbol{R}}^\sim$ and $\overline{\boldsymbol{Q}}_{\boldsymbol{a}}$ have identical Hankel singular values. In particular, $s_{l+1}(\boldsymbol{R}) = \sigma_1(\widehat{\boldsymbol{R}}^\sim)\leq\sigma_{r+1}(\boldsymbol{R}^\sim)$ using the result of Theorem 4.3. □

Remark 4.6. *The result of Theorem 4.4 may be propagated to establish upper bounds for the subsequent super-optimal levels $s_i(\boldsymbol{R})$, $i > l+1$.*

Remark 4.7. *The early part of the proof (which does not rely on any state-space based arguments) may be used to show that $\sigma_1(\overline{\boldsymbol{Q}}_{\boldsymbol{a}}) \leq \sigma_{r+1}(\boldsymbol{R}^{\sim}) < \sigma_1(\boldsymbol{R}^{\sim}) = s_1$, from which it follows immediately that $\overline{R}$ defined in Theorem 4.2 is non-singular.*

4.1 State-space analysis

In this section we develop a state-space analysis of the solution to the super-optimal distance problem. This can be used to define an algorithm for constructing the super-optimal approximation based on standard linear-algebraic routines and analysing its complexity. We start by summarizing the results of the section and explain briefly how they are related to the solution of the super-optimal distance problem outlined in the previous section: First, some background material is briefly presented related to algebraic Riccati equations, Hamiltonean matrices and the solution of the spectral factorization problem. This, together with the "Bounded Real Lemma" (Lemma 4.1) is used to establish the existence (and various properties) of the solutions of two Lyapunov equations (P_2 and Q_2) and two Algebraic Riccati Equations ($\bar{P}_2$ and $\bar{Q}_2$) needed in the construction of the optimal and suboptimal generators in Theorem 4.1 and 4.2 (Propositions 4.2 and 4.3). In particular, Proposition 4.3 proves that the two inner matrices $V_\perp$ and $W_\perp^{\sim}$ used to diagonalize the set of all optimal approximations have identical poles which leads to significant simplifications in the subsequent state-space construction. Proposition 4.4 and Corollary 4.1 give concrete realisations of these two transformations and their inner complements (see Theorem 4.3). Propositions 4.5 and 4.6 establish some technical results used in the construction of the super-optimal approximation in Theorem 4.3 (Proposition 4.7). Parts of the state-space construction in this section are long and tedious and for this reason certain details in the proofs have been omitted.

Let A, Q and R be real n-by-n matrices with Q and R symmetric. The *Algebraic Riccati equation* (ARE) is the matrix equation:

$$A'X + XA + XRX + Q = 0$$

Associated with this equation, the *Hamiltonian matrix* is defined as:

$$H := \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix} \in \mathcal{R}^{2n \times 2n}$$

Introduce the matrix:

$$J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

Then $J' = J^{-1}$ or $J^2 = -I_n$. It follows easily that $J^{-1}HJ = -JHJ = -H'$ and hence the spectrum of H is symmetric with respect to the imaginary axis. A solution of the ARE is called stabilizing if the matrix $A + RX$ is stable (i.e. $\lambda(A + RX) \subset \mathcal{C}_-$) and in this case we write $H \in \text{dom(Ric)}$. Note that if a stabilising solution exists then it is unique and in this case H has no eigenvalues on the imaginary axis. For necessary and sufficient conditions for the existence of a stabilizing solution see [ZDG96], [Kim97] and [Fra87].

We start our state-space analysis by quoting the following well-known result ("Bounded-real lemma"):

Lemma 4.1. *Let $\boldsymbol{G} \in \mathcal{RH}_\infty$ with $\boldsymbol{G} = C(sI - A)^{-1}B$ and assume that (A, B) and (C, A) are stabilisable and detectable, respectively. Then, the following conditions are equivalent:*

1. $\|\boldsymbol{G}\|_\infty < \gamma$
2. *The Hamiltonian* $H = \begin{bmatrix} A & \gamma^{-2}BB' \\ -C'C & A' \end{bmatrix}$ *has no pure imaginary eigenvalues*
3. $H \in$ *dom(Ric)*

Proof. $1 \Leftrightarrow 2$. See [ZDG96], Lemma 4.7. $2 \Leftrightarrow 3$. See [ZDG96], Theorem 13.6. □

As an immediate consequence of the above Lemma we get the following result:

Proposition 4.2. *The algebraic Riccati equations (12) (Theorem 4.2) have (unique) positive-semidefinite stabilising solutions $\overline{P}_2$ and $\overline{Q}_2$ respectively.*

Proof. Since A_q is asymptotically stable, the conditions of stabilizability and detectability of Lemma 4.1 are trivially satisfied. Further, the fact that $\|\boldsymbol{Q}_{22}\|_\infty < s_1$ (see Theorem 4.1) shows that the two Hamiltonians associated with equations (12) are free of imaginary axis eigenvalues and that (unique) stabilizing solutions $\overline{P}_2$ and $\overline{Q}_2$ to these two equations exist. The fact that $\overline{P}_2 \geq 0$ and $\overline{Q}_2 \geq 0$ follows from [ZDG96]. □

Our next result shows that the two Riccati equations (12) are intimately related.

Proposition 4.3. *Let $\overline{P}_2$ be the stabilizing solution of* ***Ric1****:*

$$A_q\overline{P}_2 + \overline{P}_2 A_q' + s_1^{-2}\overline{P}_2 C_{q2}' C_{q2}\overline{P}_2 + B_{q2}B_{q2}' = 0$$

so that $\lambda(A_q + s_1^{-2}\overline{P}_2 C_{q2}' C_{q2}) \subset \mathcal{C}_-$ and its associated Hamiltonian

$$H_1 = \begin{bmatrix} A_q' & s_1^{-2}C_{q2}'C_{q2} \\ -B_{q2}B_{q2}' & -A_q \end{bmatrix} \tag{33}$$

Let also $\overline{Q}_2$ be the stabilizing solution of ***Ric2****:*

$$A_q'\overline{Q}_2 + \overline{Q}_2 A_q + s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}'\overline{Q}_2 + C_{q2}'C_{q2} = 0$$

so that $\lambda(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2) \subset \mathcal{C}_-$ and its associated Hamiltonian

$$H_2 = \begin{bmatrix} A_q & s_1^{-2}B_{q2}B_{q2}' \\ -C_{q2}'C_{q2} & -A_q' \end{bmatrix} \tag{34}$$

Then H_1 and H_2 have identical spectra. In particular there exist a similarity transformation $\overline{R}'$ so that

$$(A_q + s_1^{-2}\overline{P}_2 C_{q2}'C_{q2}) = \overline{R}'(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2)(\overline{R}')^{-1} \tag{35}$$

where $\overline{R}$ is defined (13).

Proof. Take

$$T = \begin{bmatrix} 0 & s_1^{-1}I \\ s_1 I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} s_1 I & 0 \\ 0 & s_1^{-1}I \end{bmatrix}$$

Note that $T = T^{-1}$. Then by inspection the first claim is true. Define

$$T_P := \begin{bmatrix} I & 0 \\ -\overline{P}_2 & I \end{bmatrix} \quad \Rightarrow \quad T_P^{-1} = \begin{bmatrix} I & 0 \\ \overline{P}_2 & I \end{bmatrix}$$

and observe that

$$\begin{bmatrix} I & 0 \\ -\overline{P}_2 & I \end{bmatrix}\begin{bmatrix} A_q' & s_1^{-2}C_{q2}'C_{q2} \\ -B_{q2}B_{q2}' & -A_q \end{bmatrix}\begin{bmatrix} I & 0 \\ \overline{P}_2 & I \end{bmatrix} = \begin{bmatrix} A_q' + s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 & s_1^{-2}C_{q2}'C_{q2} \\ 0 & -(A_q + s_1^{-2}\overline{P}_2 C_{q2}'C_{q2}) \end{bmatrix} =: \widehat{H}_1$$

Similarly, define

$$T_Q := \begin{bmatrix} I & 0 \\ -\overline{Q}_2 & I \end{bmatrix} \quad \Rightarrow \quad T_Q^{-1} = \begin{bmatrix} I & 0 \\ \overline{Q}_2 & I \end{bmatrix}$$

so that

$$\begin{bmatrix} I & 0 \\ -\overline{Q}_2 & I \end{bmatrix} \begin{bmatrix} A_q & s_1^{-2}B_{q2}B_{q2}' \\ -C_{q2}'C_{q2} & -A_q' \end{bmatrix} \begin{bmatrix} I & 0 \\ \overline{Q}_2 & I \end{bmatrix} = \begin{bmatrix} A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2 & s_1^{-2}B_{q2}B_{q2}' \\ 0 & -(A_q' + s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}') \end{bmatrix} =: \widehat{H}_2$$

Summarizing,

$$H_1 = -TH_2T^{-1}, \;\; \widehat{H}_1 = T_P H_1 T_P^{-1} \;\text{ and }\; \widehat{H}_2 = T_Q H_2 T_Q^{-1}$$

Using these three equations:

$$\widehat{H}_1(T_P T T_Q^{-1}) = -(T_P T T_Q^{-1})\widehat{H}_2 \tag{36}$$

with:

$$T_P T T_Q^{-1} = \begin{bmatrix} I & 0 \\ -\overline{P}_2 & I \end{bmatrix} \begin{bmatrix} 0 & s_1^{-1}I \\ s_1 I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \overline{Q}_2 & I \end{bmatrix} = s_1^{-1} \begin{bmatrix} \overline{Q}_2 & I \\ -\overline{R}' & -\overline{P}_2 \end{bmatrix}$$

and

$$T_Q T^{-1} T_P^{-1} = \begin{bmatrix} I & 0 \\ -\overline{Q}_2 & I \end{bmatrix} \begin{bmatrix} 0 & s_1^{-1}I \\ s_1 I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \overline{P}_2 & I \end{bmatrix} = s_1^{-1} \begin{bmatrix} \overline{P}_2 & I \\ -\overline{R} & -\overline{Q}_2 \end{bmatrix}$$

Writing equation (36) in full:

$$\begin{bmatrix} A_q' + s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 & s_1^{-2}C_{q2}'C_{q2} \\ 0 & -(A_q + s_1^{-2}\overline{P}_2 C_{q2}'C_{q2}) \end{bmatrix} \begin{bmatrix} \overline{Q}_2 & I \\ -\overline{R}' & -\overline{P}_2 \end{bmatrix}$$
$$= \begin{bmatrix} -\overline{Q}_2 & -I \\ \overline{R}' & \overline{P}_2 \end{bmatrix} \begin{bmatrix} A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2 & s_1^{-2}B_{q2}B_{q2}' \\ 0 & -(A_q + s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}') \end{bmatrix}$$

From the $(2,1)$ partition of the above equation, we have $(A_q + s_1^{-2}\overline{P}_2 C_{q2}'C_{q2})\overline{R}' = \overline{R}'(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2)$. So,

$$(A_q + s_1^{-2}\overline{P}_2 C_{q2}'C_{q2}) = \overline{R}'(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2)(\overline{R}')^{-1}$$

which proves the second claim. □

Remark 4.8. *Note that this proposition implies that the "A" matrices of the state space realizations of* $\boldsymbol{V}_\perp$ *and* $\boldsymbol{W}_\perp$ *have the same spectrum.*

Proposition 4.4. *Define*

$$\boldsymbol{V}_\perp := \boldsymbol{H}_{12}\overline{\boldsymbol{H}}_{12}^{-1} \quad\text{and}\quad \boldsymbol{W}_\perp := \boldsymbol{H}_{21}^\sim \overline{\boldsymbol{H}}_{21}^{-\sim}$$

Then, $\boldsymbol{V}_\perp$ *and* $\boldsymbol{W}_\perp^\sim$ *have, the following realizations:*

$$\boldsymbol{V}_\perp \stackrel{s}{=} \left[\begin{array}{c|c} A_q - s_1^{-1}B_{q2}\overline{C}_{q1} & s_1^{-1}B_{q2} \\ \hline C_{q1} - s_1^{-1}D_{12}\overline{C}_{q1} & s_1^{-1}D_{12} \end{array}\right]$$

and

$$\boldsymbol{W}_\perp^\sim \stackrel{s}{=} \left[\begin{array}{c|c} A_q - s_1^{-1}\overline{B}_{q1}C_{q2} & B_{q1} - s_1^{-1}\overline{B}_{q1}D_{21} \\ \hline s_1^{-1}C_{q2} & s_1^{-1}D_{21} \end{array}\right]$$

with corresponding controllability and observability gramians:

$$Y_v = -(\overline{R}')^{-1}\overline{P}_2, \qquad X_v = Q_2 - \overline{Q}_2$$
$$Y_w = P_2 - \overline{P}_2, \qquad X_w = -\widehat{P}_1.$$

In particular, the following matrix inequalities hold: $P_2 \geq \overline{P}_2$ *and* $Q_2 \geq \overline{Q}_2$*.*

Proof. This follows through a long and tedious sequence of straightforward state-space manipulations which are omitted. □

$\boldsymbol{V}_\perp$ and $\boldsymbol{W}_\perp^\sim$ constructed in proposition 4.4 are parts of inner matrix functions. Theorem 4.3 relies on the construction of two inner complements $\boldsymbol{v}$ and $\boldsymbol{w}^\sim$ so that $\begin{pmatrix} \boldsymbol{v} & \boldsymbol{V}_\perp \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{w}^\sim \\ \boldsymbol{W}_\perp^\sim \end{pmatrix}$ are square inner. To find realizations for $\boldsymbol{v}$ and $\boldsymbol{w}$, we can apply Lemma 13.31 from [ZDG96] which uses the gramians of the realizations of $\boldsymbol{V}_\perp$ and $\boldsymbol{W}_\perp^\sim$. This is outlined next, together with concrete realizations of $\boldsymbol{v}$ and $\boldsymbol{w}^\sim$.

Corollary 4.1. *Let $\boldsymbol{V}_\perp, \boldsymbol{W}_\perp^\sim$ be as defined in proposition 4.4. Then there exists a complementary inner factor of $\boldsymbol{v}$ and a complementary co-inner factor of $\boldsymbol{w}$, respectively, such that*

$$\boldsymbol{V} := \begin{pmatrix} \boldsymbol{v} & \boldsymbol{V}_\perp \end{pmatrix}, \quad \boldsymbol{W} := \begin{pmatrix} \boldsymbol{w}^\sim \\ \boldsymbol{W}_\perp^\sim \end{pmatrix}$$

are square inner. Further, $\boldsymbol{V} \in \mathcal{RH}_\infty^{-,\, p\times p}$ and $\boldsymbol{W} \in \mathcal{RH}_\infty^{+,\, m\times m}$. Concrete realizations of $\boldsymbol{v}^\sim$ and $\boldsymbol{w}$ are given as:

$$\boldsymbol{v}^\sim \stackrel{s}{=} \left[\begin{array}{c|c} -A_q' - s_1^{-2}\overline{Q}_2 B_{q2} B_{q2}' & C_{q1}' + s_1^{-2}\overline{Q}_2 B_{q2} D_{12}' \\ \hline (D_{12}^\perp)' C_{q1} (Q_2 - \overline{Q}_2)^\dagger & (D_{12}^\perp)' \end{array}\right]$$

and

$$\boldsymbol{w} \stackrel{s}{=} \left[\begin{array}{c|c} -A_q' - s_1^{-2} C_{q2}' C_{q2} \overline{P}_2 & (\overline{P}_2 - P_2)^\dagger B_{q1} D_{21}^\perp \\ \hline -B_{q1}' - s_1^{-2} D_{21}' C_{q2} \overline{P}_2 & D_{21}^\perp \end{array}\right]$$

respectively.

Proof. This follows immediately from Lemma 13.31 in [ZDG96]. □

Remark 4.9. *The pair ($\boldsymbol{v}$, $\boldsymbol{w}$) as constructed in corollary 4.1 forms a scaled Schmidt pair corresponding to the largest Hankel singular value of $\boldsymbol{R}^\sim$.*

In the final part of this section we develop a state space realisation of the allpass system $\boldsymbol{\alpha}$ defined in the proof of Theorem 4.3 and show that it is anti-inner. The proof is based on a lengthy state space calculation and numerous pole-zero cancellations. We first need the following two results.

Proposition 4.5. *Let Q, P be the observability and the controllability gramians, respectively, of a system having state space realization $\boldsymbol{G} \stackrel{s}{=} (A, B, C)$. Then, (i) $\mathcal{N}(Q) \subseteq \mathcal{N}(C)$ and (ii) $\mathcal{N}(P) \subseteq \mathcal{N}(B')$.*

Proof. *(i)* Let $\underline{\xi}_o \in Ker(Q), \underline{\xi}_o \neq 0$. Then, $Q\xi_o = 0$. Consider the Lyapunov equation:

$$A'Q + QA + C'C = 0 \Rightarrow \xi_o'(A'Q + QA + C'C)\xi_o = 0 \Rightarrow C\xi_o = 0$$

and hence $\mathcal{N}(Q) \subseteq \mathcal{N}(C)$. A similar argument proves part *(ii)*. □

Proposition 4.6. *In previously defined notation:*

(i) $\left[I - (Q_2 - \overline{Q}_2)^\dagger (Q_2 - \overline{Q}_2)\right] C_{q1}' D_{12}^\perp = 0$, *and*

(ii) $\left[I - (\overline{P}_2 - P_2)^\dagger (\overline{P}_2 - P_2)\right] B_{q1} D_{21}^\perp = 0$.

Proof. *(i)* First note that from Proposition 4.4 $(Q_2 - \overline{Q}_2)$ is the observability gramian of $(A_q + s_1^{-2} B_{q2} B_{q2} \overline{Q}_2, C_{q1} + s_1^{-2} D_{12} B_{q2}' \overline{Q}_2)$. It follows, using Proposition 4.5 that $\mathcal{N}[Q_2 - \overline{Q}_2] \subseteq \mathcal{N}[C_{q1} + s_1^{-2} D_{12} B_{q2}' \overline{Q}_2]$, or equivalently, $\mathcal{R}[C_{q1}' + s_1^{-2}\overline{Q}_2 B_{q2} D_{12}'] \subseteq \mathcal{R}[Q_2 - \overline{Q}_2]$. Thus,

$$\mathcal{R}[(C_{q1}' + s_1^{-2}\overline{Q}_2 B_{q2} D_{12}') D_{12}^\perp] = \mathcal{R}[C_{q1}' D_{12}^\perp] \subseteq \mathcal{R}[C_{q1}' + s_1^{-2}\overline{Q}_2 B_{q2} D_{12}']$$

and hence $\mathcal{R}[C_{q1}' D_{12}^\perp] \subseteq \mathcal{R}[Q_2 - \overline{Q}_2]$. The result now follows on noting that $\left[I - (Q_2 - \overline{Q}_2)^\dagger (Q_2 - \overline{Q}_2)\right]$ projects orthogonally onto $\mathcal{N}[Q_2 - \overline{Q}_2]$. Part (ii) follows dually on noting that $P_2 - \overline{P}_2$ is the controllability gramian of the realization of $\boldsymbol{W}_\perp^\sim$ given in Proposition 4.4. □

Proposition 4.7. *The s_1-allpass system $s_1\boldsymbol{\alpha} \in \mathcal{RL}_\infty^{l\times l}$ defined in the proof of Theorem 4.3 can be written as a parallel system interconnection $s_1\boldsymbol{\alpha} = \boldsymbol{\alpha_1} + \boldsymbol{\alpha_2}$,*

$$s_1\boldsymbol{\alpha} \stackrel{s}{=} \left[\begin{array}{cc|c} A & 0 & B_{\alpha_1} \\ 0 & -A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 & B_{\alpha_2} \\ \hline C_{\alpha_1} & C_{\alpha_2} & (D_{12}^{\perp})'D_{11}D_{21}^{\perp} \end{array}\right]$$

in which

$$\begin{aligned} B_{\alpha_1} &:= BD_{21}^{\perp} + P_3(\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp} \\ B_{\alpha_2} &:= (\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp} \\ C_{\alpha_1} &:= -(D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger}Q_3' + (D_{12}^{\perp})'C \\ C_{\alpha_2} &:= -(D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger}\overline{R} \end{aligned}$$

In particular, $\boldsymbol{\alpha} \in \mathcal{H}_\infty^{-,\, l\times l}$ and $deg(\boldsymbol{\alpha}) \leq 2n - r$.

Proof. The proof follows a sequence of detailed state-space calculations and is omitted. □

5 Numerical Example

Consider $\boldsymbol{R} \in \mathcal{RH}_\infty^{-,2\times 2}$ with state-space realization:

$$\boldsymbol{R} \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right] := \left[\begin{array}{ccc|cc} 1 & 1 & \frac{3\sqrt{10}}{5(1+\varrho)} & \frac{\sqrt{10}}{5} & \frac{2\sqrt{10}}{5} \\ 3 & 4 & \frac{6\sqrt{10}}{5(1+\varrho)} & \frac{2\sqrt{10}}{5} & \frac{4\sqrt{10}}{5} \\ \frac{3\sqrt{10}}{5(1+\varrho)} & \frac{6\sqrt{10}}{5(1+\varrho)} & \frac{1}{\varrho} & 1 & 1 \\ \hline \frac{\sqrt{10}}{5} & \frac{2\sqrt{10}}{5} & 1 & 0 & 0 \\ \frac{2\sqrt{10}}{5} & \frac{4\sqrt{10}}{5} & 1 & 0 & 0 \end{array}\right]$$

in which $0 < \varrho < 1$. It can be easily verified that this realization is minimal and balanced with gramians $\Sigma = \mathrm{diag}(1, 1, \varrho)$. Here, the multiplicity on the largest Hankel singular value is $r = 2$ and $l = rank(B_1) = rank(C_1) = 1 < r$. This is a pathological case, as discussed in Remark 4.2. The generator of all optimal Nehari extensions of $\boldsymbol{R}$ is computed as [Glo89]:

$$\boldsymbol{Q_a} = \left(\begin{array}{c|c} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \hline \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{array}\right) \stackrel{s}{=} \left[\begin{array}{c|cc} A_q & B_{q1} & B_{q2} \\ \hline C_{q1} & D_{11} & D_{12} \\ C_{q2} & D_{21} & 0 \end{array}\right]$$

where,

$$A_q = \frac{\varrho^2 - (9/5)\varrho + 1}{\varrho(\varrho^2 - 1)}, \quad B_{q1} = \frac{1}{5(1-\varrho^2)}\begin{bmatrix} 5\varrho - 3 & 5\varrho - 6 \end{bmatrix}, \quad B_{q2} = \frac{1}{\sqrt{5}(1-\varrho^2)}$$

and

$$C_{q1} = \begin{bmatrix} \frac{5\varrho-3}{5} \\ \frac{5\varrho-6}{5} \end{bmatrix}, \quad C_{q2} = \frac{1}{\sqrt{5}}, \quad D = \left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0.2 & 0.4 & -\frac{2}{\sqrt{5}} \\ 0.4 & 0.8 & \frac{1}{\sqrt{5}} \\ \hline -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{array}\right]$$

The generator of all optimal approximations, $\boldsymbol{Q_a}$, (see Theorem 4.1) is:

$$\boldsymbol{Q_a} == \frac{1}{s + \frac{5\varrho^2-9\varrho+5}{5\varrho(1-\varrho^2)}}\left(\begin{array}{cc|c} 0.2s - \frac{25\varrho^3-25\varrho^2+5}{25\varrho(\varrho^2-1)} & 0.4s - \frac{25\varrho^3-35\varrho^2+10}{25\varrho(\varrho^2-1)} & -\frac{2}{\sqrt{5}}s - \frac{5\varrho^2-15\varrho+10}{5\sqrt{5}\varrho(\varrho^2-1)} \\ 0.4s - \frac{25\varrho^3-35\varrho^2+10}{25\varrho(\varrho^2-1)} & 0.8s - \frac{25\varrho^3-40\varrho^2+20}{25\varrho(\varrho^2-1)} & \frac{1}{\sqrt{5}}s - \frac{10\varrho^2-15\varrho+5}{5\sqrt{5}\varrho(\varrho^2-1)} \\ \hline -\frac{2}{\sqrt{5}}s - \frac{5\varrho^2-15\varrho+10}{5\sqrt{5}\varrho(\varrho^2-1)} & \frac{1}{\sqrt{5}}s - \frac{10\varrho^2-15\varrho+5}{5\sqrt{5}\varrho(\varrho^2-1)} & \frac{\varrho}{5\varrho(1-\varrho^2)} \end{array}\right)$$

and hence:

$$\boldsymbol{Q}_{22} = C_{q2}(sI - A_q)^{-1}B_{q2} = \frac{\frac{\varrho}{5\varrho(1-\varrho^2)}}{s + \frac{\varrho^2-(9/5)\varrho+1}{\varrho(1-\varrho^2)}}$$

Using the "all-pass" equations given in (11), we obtain the gramians as:

$$P = \left[\begin{array}{c|c} P_1 & P_3 \\ \hline P_3' & P_2 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\varrho & 1 \\ \hline 0 & 0 & 1 & \frac{\varrho}{1-\varrho^2} \end{array}\right]$$

and

$$Q = \left[\begin{array}{c|c} Q_1 & Q_3 \\ \hline Q_3' & Q_2 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\varrho & 1-\varrho^2 \\ \hline 0 & 0 & 1-\varrho^2 & \varrho(1-\varrho^2) \end{array}\right]$$

respectively. The solution of the ARE's in (12) reduces to the solution of the two quadratics:

$$\bar{P}_2^2 + \frac{2(5\varrho^2-9\varrho+5)}{\varrho(\varrho^2-1)}\bar{P}_2 + \frac{1}{(\varrho^2-1)^2} = 0$$

and

$$\bar{Q}_2^2 + \frac{2(5\varrho^2-9\varrho+5)(\varrho^2-1)}{\varrho}\bar{Q}_2 + (\varrho^2-1)^2 = 0.$$

and hence:

$$\bar{P}_2 = \frac{1}{\varrho(1-\varrho^2)}\left\{5\varrho^2-9\varrho+5 \pm \sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}\right\}$$

and

$$\bar{Q}_2 = \frac{1-\varrho^2}{\varrho}\left\{5\varrho^2-9\varrho+5 \pm \sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}\right\}$$

respectively. Using Proposition 4.1 and equation (14) (Theorem 4.2) we obtain:

$$\overline{\boldsymbol{Q}}_{\boldsymbol{a}} = \begin{pmatrix} \overline{\boldsymbol{Q}}_{11} & \overline{\boldsymbol{Q}}_{12} \\ \overline{\boldsymbol{Q}}_{21} & \boldsymbol{Q}_{22} \end{pmatrix} = \frac{1}{s+\frac{5\varrho^2-9\varrho+5}{5\varrho(1-\varrho^2)}} \left(\begin{array}{c|c} \frac{\left(5\varrho^2-9\varrho+5-\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}\right)^2}{5\varrho^2(1-\varrho^2)} & s+\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)} \\ \hline s+\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)} & \frac{\varrho}{5\varrho(1-\varrho^2)} \end{array}\right)$$

The realisation of $\widehat{\boldsymbol{R}}$ (equation (17), Theorem 4.2) is:

$$\widehat{\boldsymbol{R}} = \frac{\frac{\varrho^2-\left(5\varrho^2-9\varrho+5-\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}\right)^2}{5\varrho^2(1-\varrho^2)}}{s-\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)}}$$

The second super-optimal level of $\boldsymbol{R}$ is the Hankel norm of $\widehat{\boldsymbol{R}}$, i.e.

$$s_2(\boldsymbol{R}) = s_1(\widehat{\boldsymbol{R}}) = \frac{\frac{\varrho^2-\left(5\varrho^2-9\varrho+5-\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}\right)^2}{5\varrho^2(1-\varrho^2)}}{2\cdot\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)}}$$

Note also that the (unique) Nehari extension of $\widehat{\boldsymbol{R}}$ is constant and hence

$$\widehat{\boldsymbol{R}} + s_2(\boldsymbol{R}) = s_2\cdot\frac{s+\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)}}{s-\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)}} =: s_2\boldsymbol{\beta}$$

Next, construct:

$$\boldsymbol{V}_\perp = \begin{pmatrix} \frac{-\frac{2}{\sqrt{5}}s-\frac{5\varrho^2-15\varrho+10}{5\sqrt{5}\varrho(\varrho^2-1)}}{s+\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)}} \\ \\ \frac{\frac{1}{\sqrt{5}}s-\frac{10\varrho^2-15\varrho+5}{5\sqrt{5}\varrho(\varrho^2-1)}}{s+\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)}} \end{pmatrix} = (\boldsymbol{W}_\perp^\sim)' \quad \text{and} \quad \boldsymbol{v}^\sim = \boldsymbol{w} = \begin{pmatrix} \frac{-\frac{1}{\sqrt{5}}s+\frac{10\varrho^2-15\varrho+5}{5\sqrt{5}\varrho(\varrho^2-1)}}{s-\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)}} & \frac{-\frac{2}{\sqrt{5}}s-\frac{5\varrho^2-15\varrho+10}{5\sqrt{5}\varrho(\varrho^2-1)}}{s-\frac{\sqrt{(5\varrho^2-9\varrho+5)^2-\varrho^2}}{5\varrho(1-\varrho^2)}} \end{pmatrix}$$

Using equation (32) and specializing to the case $\varrho = 0.7$, we obtain the super-optimal Nehari extension of $\boldsymbol{R}$ as:

$$\boldsymbol{Q_{sopt}} = \boldsymbol{Q}_{11} - \boldsymbol{V}_{\perp}(\overline{\boldsymbol{Q}}_{11} - s_2(\boldsymbol{R}))\boldsymbol{W}_{\perp}^{\sim} = \frac{1}{s+0.5112}\begin{pmatrix} 0.47153(s+0.5165) & 0.26424(s+0.6258) \\ 0.26424(s+0.6258) & 0.86788(s+0.933) \end{pmatrix}$$

Note finally that $\boldsymbol{V}^{\sim}(\boldsymbol{R}+\boldsymbol{Q_{sopt}})\boldsymbol{W} = \mathrm{diag}(s_1\boldsymbol{\alpha}, s_2\boldsymbol{\beta})$ with α and β all-pass so that $s_1^{\infty}(R+Q_{sopt}) = 1$ and $s_2^{\infty}(R+Q_{sopt}) = 0.3394$

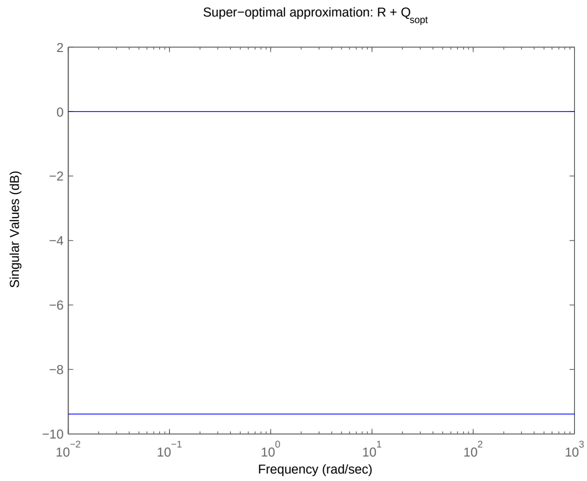

Figure 1: Plot of singular values of $\boldsymbol{R}+\boldsymbol{Q_{sopt}}$.

6 Conclusion

By means of conclusions we summarize the main contributions of this work:

- We have presented an explicit solution to the 1-block (Nehari) SODP which is easily implementable using state-space techniques. All assumptions made in previous work (minimal realization of the system which is approximated, non-repeated largest singular value of the associated Hankel operator, invertibility of certain gramians arising at intermediate steps of the algorithm) have been removed.
- The solution methodology is based on all-pass dilation techniques [JL93] and provides considerable conceptual and numerical simplifications compared to existing methods. In particular, the diagonalisation of the optimal solution set, normally carried out via the Schmidt pair of the Hankel operator associated with the problem now relies exclusively on the the generators of all optimal and suboptimal solutions, constructed directly from the data of the problem. As a result, all preliminary steps requiring a sequence of Schmidt vector scalings are completely avoided and related technical issues do not arise.
- By exploiting the simple structure of the problem and the intimate relation between the stabilising solutions of two algebraic Riccati equations, a detailed state-space analysis of the algorithm is developed and bounds on the complexity of the super-optimal solution are obtained. This approach can also be used to illuminate various pathological and non-generic problems, and also the structure and complexity of the super-optimal solution
- We have briefly discussed applications of super-optimization in the areas of robust control. Additional applications will be reported in planned future publications.

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