Efficient Groundness Analysis in Prolog

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Abstract

Boolean functions can be used to express the groundness of, and trace grounding dependencies between, program variables in (constraint) logic programs. In this paper, a variety of issues pertaining to the efficient Prolog implementation of groundness analysis are investigated, focusing on the domain of definite Boolean functions, *Def*. The systematic design of the representation of an abstract domain is discussed in relation to its impact on the algorithmic complexity of the domain operations; the most frequently called operations should be the most lightweight. This methodology is applied to *Def*, resulting in a new representation, together with new algorithms for its domain operations utilising previously unexploited properties of *Def* – for instance, quadratic-time entailment checking. The iteration strategy driving the analysis is also discussed and a simple, but very effective, optimisation of induced magic is described. The analyser can be implemented straightforwardly in Prolog and the use of a non-ground representation results in an efficient, scalable tool which does not require widening to be invoked, even on the largest benchmarks. An extensive experimental evaluation is given.

Keywords: Abstract interpretation, groundness analysis, definite Boolean functions, fixpoint algorithms.

1 Introduction

Groundness analysis is an important theme of logic programming and abstract interpretation. Groundness analyses identify those program variables bound to terms that contain no variables (ground terms). Groundness information is typically inferred by tracking dependencies among program variables. These dependencies are commonly expressed as Boolean functions. For example, the function $x \wedge (y \leftarrow z)$ describes a state in which x is definitely ground, and there exists a grounding dependency such that whenever z becomes ground then so does y.

Groundness analyses usually track dependencies using either *Pos*, the class of positive Boolean functions (Bagnara & Schachte, 1999; Baker & Søndergaard, 1993; Codish & Demoen, 1995; Fecht & Seidl, 1999; Marriott & Søndergaard, 1993; Van Hentenryck *et al.*, 1995), or *Def*, the class of definite positive functions (Armstrong *et al.*, 1998; García de la Banda *et al.*, 1996; Genaim & Codish, 2001; Howe &

King, 2000). *Pos* is more expressive than *Def*, but studies have shown that *Def* analysers can be faster than comparable *Pos* analysers (Armstrong *et al.*, 1998) and, in practice, the loss of precision for goal-dependent groundness analysis is usually small (Heaton *et al.*, 2000). This paper is a development of (Howe & King, 2000) and is an exploration of the representation of Boolean functions for groundness analysis and the use of Prolog as a medium for implementing all the components of a groundness analyser.

The rationale for this work was to develop an analyser with conceptually simple domain operations, with a small and simple (thus easily maintained) Prolog implementation based on a meta-interpreter and with performance comparable to that of BDD-based analysers. Moreover, since Prolog is well suited to symbolic manipulation, it should provide an appropriate medium for implementing a symbolic analysis, such as groundness analysis. Any analysis that can be quickly prototyped in Prolog is particularly attractive. The main drawback of this approach has traditionally been performance. The efficiency of an analyser can be guaranteed by including a widening (the controlled exchange of precision for scalability). However, a successful analyser should fire widening infrequently to maximise precision.

The efficiency of groundness analysis depends critically on the way dependencies are represented. Representation has two aspects: the theoretical representation (BDDs, Blake Canonical Form, etc.) of the Boolean functions and the data-structures of the implementation language that are used to support this representation. The theoretical representation determines the complexity of the domain operations, but the implementation requires the specific data-structures used to be amenable to efficient implementation in the chosen language. That is, the implementation can push the complexity into a higher class, or introduce a prohibitive constant factor in the complexity function. This paper considers how a representation should be chosen for the intended application (groundness analysis) by balancing the size of the representation (and its impact) with the complexity of the abstract operations and the frequency with which these operations are applied. The paper also explains how Prolog can be used to implement a particularly efficient *Def*-based groundness analysis. The orthogonal issue of the iteration strategy used to drive the analysis is also considered. Specifically, this paper makes the following contributions:

- A representation of *Def* formulae as non-canonical conjunctions of clauses is chosen by following a methodology that advocates: 1) ensuring that the most commonly called domain operations are the most lightweight; 2) that the abstractions that arise in practice should be dense; 3) that, where possible, expensive domain operations should be filtered by lightweight special cases.
- A fast Prolog implementation of *Def*-based groundness analysis is given founded on the methodology above, using a compact, factorised representation.
- Representing Boolean functions as non-ground formulae allows succinct implementation of domain operations. In particular a constant-time meet is

achieved using difference lists and a quadratic-time entailment check is built using delay declarations.
- A new join algorithm is presented which does not require formulae to be preprocessed into a canonical form.
- The use of entailment checking as a filter for join is described, as is the use of a filtered projection.
- Various iteration strategies are systematically compared and it is suggested (at least for groundness analysis) that good performance can be obtained by a surprisingly simple analysis framework.
- An extensive experimental evaluation of groundness analysis using a variety of combinations of domains, representations and iteration strategies is given.
- As a whole, the work presented in this paper strongly suggests that the implementor can produce a robust, fast, precise, scalable analyser for goal-dependent groundness analysis written in Prolog. The analyser presented does not require widening to be applied for any programs in the benchmarks suite.

The rest of the paper is structured as follows: Section 2 details the necessary preliminaries. Section 3 reviews the methods used for choosing the representation of *Def*. It goes on to describe various representations of *Def* in relation to a frequency analysis of the operations; a non-canonical representation as conjunctions of clauses is detailed. Section 4 describes a new join algorithm, along with filtering techniques for join and for projection. Section 5 discusses a variety of iteration strategies for driving an analysis. Section 6 gives an experimental evaluation of the various combinations of domain representations and iteration strategy for *Def* (and also for the domains *EPos* and *Pos*). Section 7 surveys related work and Section 8 concludes.

2 Preliminaries

A Boolean function is a function $f : \mathit{Bool}^n \to \mathit{Bool}$ where $n \geq 0$. Let V denote a denumerable universe of variables. A Boolean function can be represented by a propositional formula over $X \subseteq V$ where $|X| = n$. The set of propositional formulae over X is denoted by Bool_X. Throughout this paper, Boolean functions and propositional formulae are used interchangeably without worrying about the distinction. The convention of identifying a truth assignment with the set of variables M that it maps to true is also followed. Specifically, a map $\psi_X(M) : \wp(X) \to \mathit{Bool}_X$ is introduced defined by: $\psi_X(M) = (\wedge M) \wedge \neg(\vee(X \setminus M))$. In addition, the formula $\wedge Y$ is often abbreviated as Y.

Definition 1
The (bijective) map $\mathit{model}_X : \mathit{Bool}_X \to \wp(\wp(X))$ is defined by: $\mathit{model}_X(f) = \{M \subseteq X \mid \psi_X(M) \models f\}$.

Example 1
If $X = \{x, y\}$, then the function $\{\langle \mathit{true}, \mathit{true}\rangle \mapsto \mathit{true}$, $\langle \mathit{true}, \mathit{false}\rangle \mapsto \mathit{false}$, $\langle \mathit{false}, \mathit{true}\rangle \mapsto \mathit{false}$, $\langle \mathit{false}, \mathit{false}\rangle \mapsto \mathit{false}\}$ can be represented by the formula $x \wedge y$. Also, $\mathit{model}_X(x \wedge y) = \{\{x, y\}\}$ and $\mathit{model}_X(x \vee y) = \{\{x\}, \{y\}, \{x, y\}\}$.

The focus of this paper is on the use of sub-classes of $Bool_X$ in tracing groundness dependencies. These sub-classes are defined below:

Definition 2
A function f is positive iff $X \in model_X(f)$. Pos_X is the set of positive Boolean functions over X. A function f is definite iff $M \cap M' \in model_X(f)$ for all $M, M' \in model_X(f)$. Def_X is the set of positive functions over X that are definite. A function f is GE iff f is definite positive and for all $M, M' \in model_{var(f)}(f)$, $|M \setminus M'| \neq 1$. $EPos_X$ is the set of GE functions over X.

Note that $EPos_X \subseteq Def_X \subseteq Pos_X$. One useful representational property of Def_X is that each $f \in Def_X$ can be described as a conjunction of definite (propositional) clauses, that is, $f = \wedge_{i=1}^{n}(y_i \leftarrow \wedge Y_i)$ (Dart, 1991). Note that the y_is are not necessarily distinct. Finally, Def abbreviates Def_V. Also notice that $EPos_X = \{\wedge F \mid F \subseteq X \cup E_X\}$, where $E_X = \{x \leftrightarrow y \mid x, y \in X\}$.

Example 2
Suppose $X = \{x, y, z\}$ and consider the following table, which states, for some Boolean functions, whether they are in $EPos_X$, Def_X or Pos_X and also gives $model_X$.

f	$EPos_X$	Def_X	Pos_X	$model_X(f)$
$false$				$\emptyset$
$x \wedge y$	•	•	•	$\{\{x,y\}, \{x,y,z\}\}$
$x \vee y$			•	$\{\{x\}, \{y\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}\}$
$x \leftarrow y$		•	•	$\{\emptyset, \{x\}, \{z\}, \{x,y\}, \{x,z\}, \{x,y,z\}\}$
$x \vee (y \leftarrow z)$			•	$\{\emptyset, \{x\}, \{y\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}\}$
$true$	•	•	•	$\{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}\}$

Note, in particular, that $x \vee y$ is not in Def_X (since its set of models is not closed under intersection) and that $false$ is neither in $EPos_X$, nor Pos_X nor Def_X.

Defining $f_1 \dot{\vee} f_2 = \wedge\{f \in Def_X \mid f_1 \models f \wedge f_2 \models f\}$, the 4-tuple $\langle Def_X, \models, \wedge, \dot{\vee}\rangle$ is a finite lattice (Armstrong *et al.*, 1998), where $true$ is the top element and $\wedge X$ is the bottom element. Existential quantification is defined by Schröder's Elimination Principle, that is, $\exists x.f = f[x \mapsto true] \dot{\vee} f[x \mapsto false]$. Note that if $f \in Def_X$ then $\exists x.f \in Def_X$ (Armstrong *et al.*, 1998).

Example 3
If $X = \{x, y\}$ then $x \dot{\vee} (x \leftrightarrow y) = \wedge\{(x \leftarrow y), true\} = (x \leftarrow y)$, as can be seen in the Hasse diagram for dyadic Def_X (Fig. 1). Note also that $x \dot{\vee} y = \wedge\{true\} = true \neq (x \vee y)$.

The set of (free) variables in a syntactic object o is denoted by $var(o)$. Also, $\exists\{y_1, \ldots, y_n\}.f$ (project out) abbreviates $\exists y_1. \ldots .\exists y_n.f$ and $\overline{\exists} Y.f$ (project onto) denotes $\exists var(f) \setminus Y.f$. Let ρ_1, ρ_2 be fixed renamings such that $X \cap \rho_1(X) = X \cap \rho_2(X) = \rho_1(X) \cap \rho_2(X) = \emptyset$. Renamings are bijective and therefore invertible.

Downward closure, $\downarrow$, relates Pos and Def and is useful when tracking sharing with Boolean functions (Codish *et al.*, 1999). It is defined by $\downarrow f = model_X^{-1}(\{\cap S \mid$

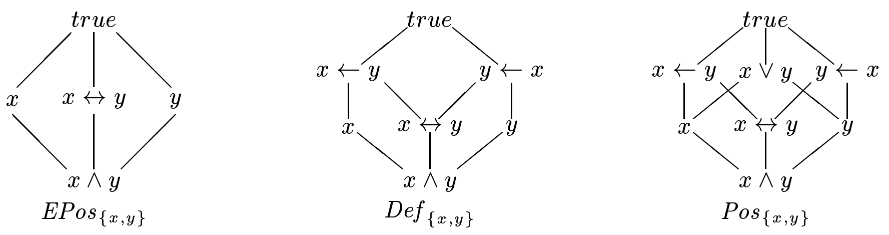

Fig. 1. Hasse Diagrams

$\emptyset \subset S \subseteq model_X(f)\})$. Note that $\downarrow f$ has the useful computational property that $\downarrow f = \wedge\{f' \in Def_X \mid f \models f'\}$ if $f \in Pos_X$. That is, $\downarrow$ takes a *Pos* formula to its best *Def* approximation. Finally, for any $f \in Bool_X$, $coneg(f) = model_X^{-1}(\{X \setminus M \mid M \in model_X(f)\})$ (Codish *et al.*, 1999).

The following pieces of logic programming terminology will also be needed. Let T denote the set of terms constructed from V and a set of function symbols F. An equation e is a pair $(s = t)$ where $s, t \in T$. A substitution is a (total) map $\theta : V \to T$ such that $\{v \in V \mid \theta(v) \neq v\}$ is finite. Let *Sub* denote the set of substitutions and let E denote a finite set of equations. Let $\theta(t)$ denote the term obtained by simultaneously replacing each occurrence of v in t with $\theta(v)$, and let $\theta(E) = \{\theta(s) = \theta(t) \mid (s = t) \in E\}$.

Composition of substitutions induces the (more general than) relation $\leq$ defined by: $\theta \leq \psi$ if there exists $\delta \in Sub$ such that $\psi = \delta \circ \theta$. More general than lifts to terms by $s \leq t$ iff there exists $\theta \in Sub$ such that $\theta(s) = t$. The set of unifiers of E, $unify(E)$, is defined by: $unify(E) = \{\theta \in Sub \mid \forall (s = t) \in E.\theta(s) = \theta(t)\}$ and the set of most general unifiers, $mgu(E)$, is defined by: $mgu(E) = \{\theta \in unify(E) \mid \forall \psi \in unify(E).\theta \leq \psi\}$. Finally, the set of generalisations of two terms is defined by: $gen(t_1, t_2) = \{t \in T | t \leq t_1 \wedge t \leq t_2\}$ and the set of most specific generalisations is defined by: $msg(t_1, t_2) = \{t \in gen(t_1, t_2) | \forall s \in gen(t_1, t_2).s \leq t\}$.

3 Choosing a Representation for *Def*

3.1 Review of Design Methods

The efficiency of an analyser depends critically on the algorithmic complexities of its abstract domain operations. These in turn are determined by the representation of the abstract domain. The representation also determines the size of the inputs to the domain operations, as well as impacting on memory usage. Because of this, the choice of representation is fundamental to the efficiency of an analyser and is therefore of great importance. The remainder of this subsection reviews three factors which should help the implementor arrive at a suitable representation and suggest where domain operations might be refined.

3.1.1 Frequency Analysis of the Domain Operations

There are typically many degrees of freedom in designing an analyser, even for a given domain. Furthermore, work can often be shifted from one abstract operation into another. For example, Boolean formulae can be represented in either conjunctive normal form (CNF) or disjunctive normal form (DNF). In CNF, conjunction is constant-time and disjunction is quadratic-time, whereas in DNF, conjunction is quadratic-time and disjunction is constant-time. Ideally, an analysis should be designed so that the most frequently used operations have low complexity and are therefore fast. This motivates the following approach:

1. Prototype an analyser for the given domain.
2. Instrument the analyser to count the number of times each domain operation is invoked.
3. Generate these counts for a number of programs (the bigger the better).
4. Choose a representation which gives a good match between the frequency and the complexity of the domain operations.

Because the frequency analysis is solely concerned with generated instruction counts, the efficiency of the prototype analyser is not a significant issue. The objective is to choose a representation for which the most frequently occurring operations are also the fastest. However, this criterion needs to be balanced with others, such as the density of the representation.

3.1.2 Density of the Domain Representation

The complexity of the domain operations is a function of the size of their inputs. Large inputs nullify the value of good complexities, hence a balance between size of representation and complexity of domain operations is needed. The following factors impact on this relationship:

1. The abstractions which typically arise should be represented compactly.
2. A factorised representation with an expressive, high density, low maintenance component is attractive.
3. Maintaining the representation (for example, as a canonical form) should not come with a high overhead.
4. The representation should fit with machinery available in the implementation language.

A domain is said to be factorised if its information is represented as a product of subdomains. It may not always be possible to fulfill all these requirements. Moreover, these factors needs to balanced with others, such as their impact on the complexities of frequently called domain operations.

3.1.3 Filtering the Domain Operations

In many analyses it is inevitable that some domain operations will have high complexity. However, it is sometimes possible to reduce the impact of this by filtering the operation, as follows:

1. For a high complexity domain operation identify special cases where the operation can be calculated using a lower complexity algorithm.

2. Instrument the analyser to quantify how often the lower complexity algorithm can be applied.
3. If it appears that the special case occurs frequently, then implement the special case and measure the impact on performance.

The bottom line is that the cost of detecting the special case should not outweigh the benefit of applying the specialised domain algorithm.

3.2 ***Frequency Analysis for*** *Def*

In order to balance the frequency of abstract operations against their cost, an existing *Def* analyser was instrumented to count the number of calls to the various abstract operations. The analyser used for this is based on Armstrong and Schachte's BDD-based domain operations for *Pos* and *Sharing* (Armstrong *et al.*, 1998; Schachte, 1999). Using the domain operations provided for these domains, a *Def* analyser can easily be derived. This analyser is coded in Prolog as a simple meta-interpreter that uses induced magic-sets (Codish, 1999a) and eager evaluation (Wunderwald, 1995) to perform goal-dependent bottom-up evaluation and call the C implemented domain operations.

Induced magic is a refinement of the magic set transformation, avoiding much of the re-computation that arises because of the repetition of literals in the bodies of magicked clauses (Codish, 1999a). Eager evaluation (Wunderwald, 1995) is a fixpoint iteration strategy which proceeds as follows: whenever an atom is updated with a new (weaker) abstraction, a recursive procedure is invoked to ensure that every clause that has that atom in its body is re-evaluated. An advantage of induced magic is that it can be coded straightforwardly in Prolog.

Table 1 gives a breakdown of the relative frequency (in percentages) of the calls to each abstract operation in the BDD-based *Def* analysis of eight large programs. Meet, join, equiv, project and rename are the obvious Boolean operations. Join (diff) is those calls to a join $f_1 \dot{\vee} f_2$ where $f_1 \dot{\vee} f_2 \neq f_1$ and $f_1 \dot{\vee} f_2 \neq f_2$ (this will be useful in section 4). Total details the total number of calls to these domain operations.

file	rubik	chat_parser	sim_v5-2	peval	aircraft	essln	chat_80	aqua_c
meet	30.9	31.6	35.9	32.5	28.5	42.7	34.0	34.2
join	10.4	10.4	8.8	9.7	11.1	8.4	10.2	10.5
join (diff)	1.1	1.7	0.0	2.9	0.1	0.9	1.5	1.6
equiv	10.4	10.4	8.8	9.7	11.1	8.4	10.2	10.5
project	12.6	12.5	13.0	12.5	13.0	10.5	12.1	11.7
rename	34.7	33.4	33.6	32.8	36.2	29.2	32.0	31.6
total	14336	14124	5943	6275	24758	19051	45444	280485

Table 1. *Frequency Analysis: BDD-based Def Analyser (Figures in %)*

Observe that meet and rename are called most frequently. Join, equiv and project are called with a similar frequency, but less frequently than meet and rename. Note that it is rare for a join to differ from both its arguments. Join is always followed by an equivalence and this explains why the join and equiv rows coincide. Since meet and rename are the most frequently called operations, ideally they should be the most lightweight. As join, equiv and project are called less frequently, a higher algorithmic complexity is more acceptable for these operations.

3.3 Representations of Def

This section reviews a number of representations of *Def* in terms of the algorithmic complexity of the domain operations. The representations considered are reduced ordered binary decision diagrams, dual Blake canonical form (specialised for *Def* (Armstrong *et al.*, 1998)) and a non-canonical definite propositional clause representation.

ROBDD A reduced ordered binary decision diagram (ROBDD) is a rooted, directed acyclic graph. Terminal nodes are labelled 0 or 1 and non-terminal nodes are labelled by a variable and have edges directed towards two child nodes. ROBDDs have the additional properties that: 1) each path from the root to a node respects a given ordering on the variables, 2) a variable cannot occur multiply in a path, 3) no subBDD occurs multiply. ROBDDs give a unique representation for every Boolean function (up to variable ordering).

DBCF Dual Blake Canonical Form (DBCF) represents *Def* functions as conjunctions of definite (propositional) clauses (Armstrong *et al.*, 1998; Dart, 1991; García de la Banda *et al.*, 1996) maintained in a canonical (orthogonal) form that makes explicit transitive variable dependencies and uses a reduced monotonic body form. For example, the function $(x \leftarrow y) \wedge (y \leftarrow z)$ is represented as $(x \leftarrow (y \vee z)) \wedge (y \leftarrow z)$. Again, DBCF gives a unique representation for every *Def* function (up to variable ordering).

Non-canonical The non-canonical clausal representation expresses *Def* functions as conjunctions of propositional clauses, but does not maintain a canonical form. This does not give a unique representation.

Table 2 details the complexities of the domain operations for *Def* for the three representations. Notice that the complexities are in terms of the size of the representations and that these are all potentially exponential in the number of variables. Also, observe that since DBCF maintains transitive dependencies, whereas the non-canonical representation does not, the DBCF of a *Def* function has the potential to be considerably larger than the non-canonical representation. As ROBDDs are represented in a fundamentally different way, their size cannot be directly compared with clausal representations.

Both ROBDDs and DBCF are maintained in a canonical form. Canonical forms reduce the cost of operations such as equivalence checking and projection by factoring out search. However, canonical forms need to be maintained and this maintenance has an associated cost in meet and join. That is, ROBDDs and DBCF buy low complexity equivalence checking and projection at the cost of higher complexity meet and join.

Representation	meet	join	equiv	rename	project
ROBDD	$O(N^2)$	$O(N^2)$	$O(1)$	$O(N^2)$	$O(N^2)$
DBCF	$O(N^4)$	$O(2^{2N})$	$O(N)$	$O(N)$	$O(N)$
Non-canonical	$O(1)$	$O(2^{2N})$	$O(N^2)$	$O(N)$	$O(2^N)$

Table 2. *Complexity of Def Operations for Various Representations (where N is the size of the representation – number of nodes/variable occurrences).*

As discussed in the previously, the lowest cost operations should be those that are most frequently called. Table 1 shows that for *Def* based groundness analysis, meet and renaming are called significantly more often than the other operations. Hence these should be the most lightweight. This suggests that the non-canonical representation is better suited to *Def*-based goal-dependent groundness analysis than ROBDDs and DBCF. The following sections will detail the non-canonical representation.

3.4 GEP Representation

This section outlines how the non-canonical representation is used in an analysis for call and answer patterns. Implementing call and answer patterns with a non-ground representation enables the non-canonical representation to be factorised at little overhead.

A call (or answer) pattern is a pair $\langle a, f\rangle$ where a is an atom and $f \in \mathit{Def}$. Normally the arguments of a are distinct variables. The formula f is a conjunction (list) of propositional clauses. In a non-ground representation the arguments of a can be instantiated and aliased to express simple dependency information (Heaton *et al.*, 2000). For example, if $a = p(x_1, ..., x_5)$, then the atom $p(x_1, \mathit{true}, x_1, x_4, \mathit{true})$ represents a coupled with the formula $(x_1 \leftrightarrow x_3) \wedge x_2 \wedge x_5$. This enables the abstraction $\langle p(x_1, ..., x_5), (x_1 \leftrightarrow x_3) \wedge x_2 \wedge x_5 \wedge (x_4 \rightarrow x_1)\rangle$ to be collapsed to $\langle p(x_1, \mathit{true}, x_1, x_4, \mathit{true}), x_4 \rightarrow x_1\rangle$. This encoding leads to a more compact representation and is similar to the GER factorisation of ROBDDs proposed by Bagnara and Schachte (Bagnara & Schachte, 1999). The representation of call and answer patterns described above is called GEP (groundness, equivalences and propositional clauses) where the atom captures the first two properties and the formula the latter.

Formally, let $GEP = \{\langle p(t_1, ..., t_n), f\rangle \mid p \in \Pi, t_i \in V \cup \{\mathit{true}\}, f \in \mathit{Def} \setminus GE\}$. Define $\models$ by $\langle p(\vec{a}_1), f_1\rangle \models \langle p(\vec{a}_2), f_2\rangle$ iff $\overline{\exists}\vec{x}.((\vec{a}_1 \leftrightarrow \vec{x}) \wedge f_1) \models \overline{\exists}\vec{x}.((\vec{a}_2 \leftrightarrow \vec{x}) \wedge f_2)$ and $\mathit{var}(\vec{x}) \cap (\mathit{var}(\vec{a}_1) \cup \mathit{var}(\vec{a}_2) \cup \mathit{var}(f_1) \cup \mathit{var}(f_2)) = \emptyset$. Then $\langle GEP, \models\rangle$ is a preorder. The preorder induces the equivalence relation $\equiv$ defined by $\langle p(\vec{a}_1), f_1\rangle \equiv \langle p(\vec{a}_2), f_2\rangle$ iff $\langle p(\vec{a}_1), f_1\rangle \models \langle p(\vec{a}_2), f_2\rangle$ and $\langle p(\vec{a}_2), f_2\rangle \models \langle p(\vec{a}_1), f_1\rangle$. Let $GEP_\equiv$ denotes GEP quotiented by the equivalence. Define $\wedge : GEP_\equiv \times GEP_\equiv \rightarrow GEP_\equiv$ by $[\langle a_1, f_1\rangle]_\equiv \wedge [\langle a_2, f_2\rangle]_\equiv = [\langle \theta(a_1), \theta(f_1) \wedge \theta(f_2)\rangle]_\equiv$, where $\theta \in \mathit{mgu}(a_1, a_2)$. Then $\langle GEP_\equiv, \models, \wedge\rangle$ is a finite lattice.

The meet of the pairs $\langle p(\vec{a}_1), f_1\rangle$ and $\langle p(\vec{a}_2), f_2\rangle$ can be computed by unifying a_1 and a_2 and concatenating f_1 and f_2. The unification is nearly linear in the arity of p (using rational tree unification (Jaffar, 1984)) and concatenation is constant-time (using difference lists). Since the arguments $\vec{a}_1$ and $\vec{a}_2$ are necessarily distinct, the analyser would unify $\vec{a}_1$ and $\vec{a}_2$ even in a non-factorised representation, hence no extra overhead is incurred. The objects that require renaming are formulae and call (answer) pattern GEP pairs. If a dynamic database is used to store the pairs (Hermenegildo *et al.*, 1992), then renaming is automatically applied each time a pair is looked-up in the database. Formulae can be renamed with a single call to the Prolog builtin copy_term. Renaming is therefore linear.

The GEP factorisation defined above is true, that is, all the GE dependencies are factored into the atom. An alternative definition would be $GEP = \{\langle p(t_1, ..., t_n), f\rangle \mid p \in \Pi, t_i \in V \cup \{true\}, f \in Def\}$. Here the factorisation is not necessarily true, in the sense that GE dependencies may exists in the P component, e.g. $\langle p(x, x, true), true\rangle$ may also be correctly expressed as $\langle p(u, v, w), (u \leftrightarrow v) \wedge w\rangle$. A non-true factorisation may be adventageous when it comes to implementing the domain and from henceforth GEP will refer to the non-true factorisation version unless stated otherwise. The P component may contain redundant (indeed, repeated) clauses and these may impact adversely on performance. In order to avert unconstrained growth of P, a redundancy removal step may be applied to P at a convenient point (via entailment checking). Since the non-canonical formulae do not need to be maintained in a canonical form and since the factorisation is not necessarily true, the representation is flexible in that it can be maintained on demand, that is, the implementor can choose to move dependencies from P into GE at exactly those points in the analysis where true factorisation gives a performance benefit.

4 Filtering and Algorithms

The non-canonical representation has high cost join and projection algorithms. Therefore it is sensible to focus on improving the efficiency of these operations. This is accomplished through filtering following the strategy described in section 3.1. This section presents a new approach to calculating join and describes the use of entailment checking as a filter in the join algorithm. It also describes a filtering method for projection.

4.1 Join

This section describes a new approach to calculating join, inspired by a convex hull algorithm for polyhedra used in disjunctive constraint solving (De Backer & Beringer, 1993). The new join algorithm is first described for formulae and is then lifted to the GEP representation.

4.1.1 Join for Formulae

Calculating join in *Def* is not straightforward. It is not enough to take the join each possible pair of clauses and conjoin them – transitive dependencies also need to be taken into account. This is illustrated by the following example (adapted from (Armstrong *et al.*, 1998)).

Example 4

Put $f_1 = (x \leftarrow u) \wedge (u \leftarrow y)$ and $f_2 = (x \leftarrow v) \wedge (v \leftarrow y)$. Then $f_1 \dot{\vee} f_2 = (x \leftarrow (u \wedge v)) \wedge (x \leftarrow y)$. The clause $(x \leftarrow (u \wedge v))$ comes from $(x \leftarrow u) \dot{\vee} (x \leftarrow v)$, but the clause $x \leftarrow y$ is not the result of the join of any pair of clauses in f_1, f_2. It arises since $f_1 \models x \leftarrow y$ and $f_2 \models x \leftarrow y$, that is, from clauses which appear in transitive closure.

One way in which to address the problem of ensuring that the transitive dependencies are captured is to make the explicit in the representation (this idea is captured in the orthogonal form requirement of (Armstrong *et al.*, 1998)). However, this leads to redundancy in the formula which ideally should be avoided.

It is insightful to consider $\dot{\vee}$ as an operation on the models of f_1 and f_2. Since both $model_X(f_i)$ are closed under intersection, $\dot{\vee}$ essentially needs to extend $model_X(f_1) \cup model_X(f_2)$ with new models $M_1 \cap M_2$ where $M_i \in model_X(f_i)$ to compute $f_1 \dot{\vee} f_2$. The following definition expresses this observation and leads to a new way of computing $\dot{\vee}$ in terms of meet, renaming and projection, that does not require formulae to be first put into orthogonal form.

Definition 3
The map $\dot{\curlyvee} : {Bool_X}^2 \rightarrow Bool_X$ is defined by: $f_1 \dot{\curlyvee} f_2 = \overline{\exists} Y. f_1 \curlyvee f_2$ where $Y = var(f_1) \cup var(f_2)$ and $f_1 \curlyvee f_2 = \rho_1(f_1) \wedge \rho_2(f_2) \wedge \wedge_{y \in Y} y \leftrightarrow (\rho_1(y) \wedge \rho_2(y))$.

The following example illustrates the $\dot{\curlyvee}$ operator.

Example 5
Let $f_1 = (x \leftarrow u) \wedge (u \leftarrow y)$, $f_2 = (x \leftarrow v) \wedge (v \leftarrow y)$. Then $Y = \{u, v, x, y\}$. The following substitutions rename the functions apart, $\rho_1 = \{u \mapsto u', v \mapsto v', x \mapsto x', y \mapsto y'\}$, $\rho_2 = \{u \mapsto u'', v \mapsto v'', x \mapsto x'', y \mapsto y''\}$. Using Definition 3, $f_1 \curlyvee f_2 = (x' \leftarrow u') \wedge (u' \leftarrow y') \wedge (x'' \leftarrow v)'' \wedge (v'' \leftarrow y'') \wedge u \leftrightarrow (u' \wedge u'') \wedge v \leftrightarrow (v' \wedge v'') \wedge x \leftrightarrow (x' \wedge x'') \wedge y \leftrightarrow (y' \wedge y'')$. Projection onto Y gives $f_1 \dot{\curlyvee} f_2 = \overline{\exists}\{u, v, x, y\}. f_1 \curlyvee f_2 = (x \leftarrow (u \wedge v)) \wedge (x \leftarrow y)$.

Note that $\dot{\curlyvee}$ operates on $Bool_X$ rather than Def_X. This is required for the downward closure operator in section 5.3. Lemma 1 expresses a key relationship between $\dot{\curlyvee}$ and the models of f_1 and f_2.

Lemma 1
Let $f_1, f_2 \in Bool_X$. $M \in model_X(f_1 \dot{\curlyvee} f_2)$ if and only if there exists $M_1 \in model_X(f_1)$ and $M_2 \in model_X(f_2)$ such that $M = M_1 \cap M_2$.

Proof
Put $X' = X \cup \rho_1(X) \cup \rho_2(X)$. Let $M \in model_X(f_1 \dot{\curlyvee} f_2)$. There exists $M \subseteq M' \subseteq X'$ such that $M' \in model_{X'}(f_1 \curlyvee f_2)$. Let $M_i = \rho_i^{-1}(M' \cap \rho_i(Y))$, for $i \in \{1, 2\}$. Thus $M_i \in model_X(F_i)$ for $i \in \{1, 2\}$. Observe that $M \subseteq M_1 \cap M_2$ since $f_1 \curlyvee f_2 \models y \rightarrow (\rho_1(y) \wedge \rho_2(y))$. Also observe that $M_1 \cap M_2 \subseteq M$ since $f_1 \curlyvee f_2 \models (\rho_1(y) \wedge \rho_2(y)) \rightarrow y$. Thus $M = M_1 \cap M_2$, as required.

Let $M_i \in model_X(f_i)$ for $i \in \{1, 2\}$ and put $M = M_1 \cap M_2$ and $M' = M \cup \rho_1(M_1) \cup \rho_1(M_2)$. Observe $M' \in model_{X'}(f_1 \curlyvee f_2)$ so that $M \in model_X(f_1 \dot{\curlyvee} f_2)$. $\square$

From lemma 1 flows the following corollary and also the useful result that $\dot{\curlyvee}$ is monotonic.

Corollary 1
Let $f \in Pos_X$. Then $f = f \dot{\curlyvee} f$ if and only if $f \in Def_X$.

Lemma 2
$\dot{\curlyvee}$ is monotonic, that is, $f_1 \dot{\curlyvee} f_2 \models f'_1 \dot{\curlyvee} f'_2$ whenever $f_1 \models f'_1$ and $f_2 \models f'_2$.

Proof
Let $M \in model_X(f_1 \dot{\curlyvee} f_2)$. By lemma 1, there exist $M_i \in model_X(f_i)$ such that $M = M_1 \cap M_2$. Since $f_i \models f'_i$, $M_i \in model_X(f'_i)$ and hence, by lemma 1, $M \in model_X(f'_1 \dot{\curlyvee} f'_2)$. $\square$

The following proposition states that $\dot{\curlyvee}$ coincides with $\dot{\vee}$ on Def_X. This gives a simple algorithm for calculating $\dot{\vee}$ that does not depend on the representation of a formula.

Proposition 1
Let $f_1, f_2 \in Def_X$. Then $f_1 \dot{\curlyvee} f_2 = f_1 \dot{\vee} f_2$.

Proof
Since $X \models f_2$ it follows by monotonicity that $f_1 = f_1 \dot{\curlyvee} X \models f_1 \dot{\curlyvee} f_2$ and similarly $f_2 \models f_1 \dot{\curlyvee} f_2$. Hence $f_1 \dot{\vee} f_2 \models f_1 \dot{\curlyvee} f_2$ by the definition of $\dot{\vee}$.

Now let $M \in model_X(f_1 \dot{\curlyvee} f_2)$. By lemma 1, there exists $M_i \in model_X(f_i)$ such that $M = M_1 \cap M_2 \in model_X(f_1 \dot{\vee} f_2)$. Hence $f_1 \dot{\curlyvee} f_2 \models f_1 \dot{\vee} f_2$. $\square$

4.1.2 Join for GEP

Join, $\vee : GEP_\equiv \times GEP_\equiv \to GEP_\equiv$, in the GEP representation can be defined in terms of $\wedge$ and $\models$ in the usual way, i. e.

$$[\langle a_1, f_1\rangle]_\equiv \vee [\langle a_2, f_2\rangle]_\equiv = \wedge \left\{ [\langle a, f\rangle]_\equiv \in GEP_\equiv \;\middle|\; \begin{array}{l} [\langle a_1, f_1\rangle]_\equiv \models [\langle a, f\rangle]_\equiv, \\ [\langle a_2, f_2\rangle]_\equiv \models [\langle a, f\rangle]_\equiv \end{array} \right\}$$

In practice quotienting manifests itself through the dynamic database. Each time a pattern is read from the database it is renamed. Join is lifted to quotients by reformulated GEP pairs as follows: $\langle p(\vec{a}_1), f_1\rangle$ becomes $\langle p(\vec{a}), (\vec{a} \leftrightarrow \vec{a}_1) \wedge f_1\rangle$ where $p(\vec{a}) = msg(p(\vec{a}_1), p(\vec{a}_2))$. $p(\vec{a})$ is computed using Plotkin's anti-unification algorithm in $O(N \log(N))$ time, where N is the arity of p (Plotkin, 1970). The following lemma formalises this lifting of the join algorithm to the GEP representation.

Lemma 3
$[\langle p(\vec{t_1}), f_1\rangle]_\equiv \vee [\langle p(\vec{t_2}), f_2\rangle]_\equiv = [\langle p(\vec{t}), (f_1 \wedge (\vec{t_1} \leftrightarrow \vec{t})) \dot{\curlyvee} (f_2 \wedge (\vec{t_2} \leftrightarrow \vec{t}))\rangle]_\equiv$, where $\vec{t} \in msg(\vec{t_1}, \vec{t_2})$.

Proof

The first equality holds by the definition of $\equiv$ in $GEP_{\equiv}$, the second by the definition of join in $GEP_{\equiv}$, the third by the definition of $\models$ in $GEP_{\equiv}$, the fourth by the definition of $\wedge$ in $GEP_{\equiv}$, and the last by Proposition 1.

$$\begin{array}{ll}
\multicolumn{2}{l}{[\langle p(\vec{t_1}), f_1\rangle]_{\equiv} \vee [\langle p(\vec{t_2}), f_2\rangle]_{\equiv}} \\
= & [\langle p(\vec{t}), (\vec{t_1} \leftrightarrow \vec{t}) \wedge f_1\rangle]_{\equiv} \vee [\langle p(\vec{t}), (\vec{t_2} \leftrightarrow \vec{t}) \wedge f_2\rangle]_{\equiv} \\
= & \wedge \left\{ [\langle p(\vec{t'}), f'\rangle]_{\equiv} \;\middle|\; \begin{array}{l} [\langle p(\vec{t}), (\vec{t_1} \leftrightarrow \vec{t}) \wedge f_1\rangle]_{\equiv} \models [\langle p(\vec{t'}), f'\rangle]_{\equiv}, \\ [\langle p(\vec{t}), (\vec{t_2} \leftrightarrow \vec{t}) \wedge f_2\rangle]_{\equiv} \models [\langle p(\vec{t'}), f'\rangle]_{\equiv} \end{array} \right\} \\
= & \wedge \left\{ [\langle p(\vec{t}), f'\rangle]_{\equiv} \;\middle|\; \begin{array}{l} [\langle p(\vec{t}), (\vec{t_1} \leftrightarrow \vec{t}) \wedge f_1\rangle]_{\equiv} \models [\langle p(\vec{t}), f'\rangle]_{\equiv}, \\ [\langle p(\vec{t}), (\vec{t_2} \leftrightarrow \vec{t}) \wedge f_2\rangle]_{\equiv} \models [\langle p(\vec{t}), f'\rangle]_{\equiv} \end{array} \right\} \\
= & [\langle p(\vec{t}), \wedge\{f' \in Def \mid (\vec{t_1} \leftrightarrow \vec{t}) \wedge f_1 \models f', (\vec{t_2} \leftrightarrow \vec{t}) \wedge f_2 \models f'\}\rangle]_{\equiv} \\
= & [\langle p(\vec{t}), (f_1 \wedge (\vec{t_1} \leftrightarrow \vec{t})) \dot{\curlyvee} (f_2 \wedge (\vec{t_2} \leftrightarrow \vec{t}))\rangle]_{\equiv}
\end{array}$$

$\square$

4.2 Filtering Join using Entailment Checking

In section 3.3 it was observed that some high complexity domain operations have special cases where the operation can be calculated using a lower complexity algorithm. Join for *Def* in the non-canonical GEP representation is one such operation. Specifically, $\dot{\vee}$ is exponential (see Table 2), however, if $f_1 \models f_2$, then $f_1 \dot{\vee} f_2 = f_2$. Entailment checking is quadratic in the number of variable occurrences (using a forward chaining algorithm), hence by using this test, join can be refined. Table 1 shows that the majority of calls to join will be caught by the cheaper entailment checking case. The following proposition explains how this filtering is lifted to the GEP representation. Observe that this proposition has three cases. The third case is when the entailment check fails. The first case is when entailment checking reduces to a lightweight match on the GE component followed by an entailment check on the P component. The second case is more expensive, requiring a most specific generalisation to be computed as well as an entailment check on more complicated formulae. In the context of the analyser, the pair $[\langle p(\vec{t_2}), f_2\rangle]_{\equiv}$ corresponds to an abstraction in the database and these abstractions have the property that the variables in the P component are contained in those of the GE component. This is not necessarily the case for $[\langle p(\vec{t_1}), f_1\rangle]_{\equiv}$, since in the induced magic framework f_1 represents the state of the variables of the clause to the left of the call to $p(\vec{t_1})$. Variable disjointness follows since renaming automatically occurs every time a fact is read from the dynamic database.

Proposition 2

Suppose $var(f_2) \subseteq var(p(\vec{t_2}))$ and $var(\langle p(\vec{t_1}), f_1\rangle) \cap var(\langle p(\vec{t_2}), f_2\rangle) = \emptyset$. Then,

$$[\langle p(\vec{t_1}), f_1\rangle]_\equiv \vee [\langle p(\vec{t_2}), f_2\rangle]_\equiv$$
$$= \begin{cases} [\langle p(\vec{t_2}), f_2\rangle]_\equiv & \text{if} & \theta \in mgu(p(\vec{t_1}), p(\vec{t_2})), \\ & & p(\vec{t_1}) = \theta(p(\vec{t_2})), \\ & & \theta(f_1) \models \theta(f_2) \\ [\langle p(\vec{t_2}), f_2\rangle]_\equiv & \text{if} & p(\vec{t}) \in msg(p(\vec{t_1}), p(\vec{t_2})), \\ & & f_1 \wedge (\vec{t_1} \leftrightarrow \vec{t}) \models f_2 \wedge (\vec{t_2} \leftrightarrow \vec{t}) \\ [\langle p(\vec{t}), f\rangle]_\equiv & \text{otherwise} & \text{where } p(\vec{t}) = msg(p(\vec{t_1}), p(\vec{t_2})), \\ & & f = (f_1 \wedge (\vec{t_1} \leftrightarrow \vec{t}))\dot{\Upsilon}(f_2 \wedge (\vec{t_2} \leftrightarrow \vec{t})) \end{cases}$$

Proof
Case 1

$$\begin{array}{rrcll} & \theta(f_1) & \models & \theta(f_2) & \\ \Rightarrow & (\theta(\vec{t_1}) \leftrightarrow \vec{x}) \wedge \theta(f_1) & \models & (\theta(\vec{t_2}) \leftrightarrow \vec{x}) \wedge \theta(f_2) & \text{by assumption} \\ \Rightarrow & (\vec{t_1} \leftrightarrow \vec{x}) \wedge \theta(f_1) & \models & (\theta(\vec{t_2}) \leftrightarrow \vec{x}) \wedge \theta(f_2) & \vec{t_1} = \theta(\vec{t_2}) = \theta(\vec{t_1}) \\ \Rightarrow & (\vec{t_1} \leftrightarrow \vec{x}) \wedge f_1 & \models & (\theta(\vec{t_2}) \leftrightarrow \vec{x}) \wedge \theta(f_2) & var(f_1) \cap var(\vec{t_2}) = \emptyset \\ \Rightarrow & (\vec{t_1} \leftrightarrow \vec{x}) \wedge f_1 & \models & (\vec{t_2} \leftrightarrow \vec{x}) \wedge f_2 & \models \text{ is transitive} \\ \Rightarrow & \overline{\exists}\vec{x}.((\vec{t_1} \leftrightarrow \vec{x}) \wedge f_1) & \models & \overline{\exists}\vec{x}.((\vec{t_2} \leftrightarrow \vec{x}) \wedge f_2) & \overline{\exists} \text{ is monotonic} \\ \Rightarrow & [\langle p(\vec{t_1}), f_1\rangle]_\equiv & \models & [\langle p(\vec{t_2}), f_2\rangle]_\equiv & \text{by definition} \end{array}$$

Case 2

$$\begin{array}{rrcll} & (\vec{t_1} \leftrightarrow \vec{t}) \wedge f_1 & \models & (\vec{t_2} \leftrightarrow \vec{t}) \wedge f_2 & \\ \Rightarrow & (\vec{t} \leftrightarrow \vec{x}) \wedge (\vec{t_1} \leftrightarrow \vec{t}) \wedge f_1 & \models & (\vec{t} \leftrightarrow \vec{x}) \wedge (\vec{t_2} \leftrightarrow \vec{t}) \wedge f_2 & \\ \Rightarrow & \overline{\exists}\vec{x}.((\vec{t} \leftrightarrow \vec{x}) \wedge (\vec{t_1} \leftrightarrow \vec{t}) \wedge f_1) & \models & \overline{\exists}\vec{x}.((\vec{t} \leftrightarrow \vec{x}) \wedge (\vec{t_2} \leftrightarrow \vec{t}) \wedge f_2) & \overline{\exists} \text{ is monotonic} \\ \Rightarrow & \overline{\exists}\vec{x}.((\vec{t_1} \leftrightarrow \vec{x}) \wedge f_1) & \models & \overline{\exists}\vec{x}.((\vec{t_2} \leftrightarrow \vec{x}) \wedge f_2) & \text{since } \vec{x} \text{ are fresh} \\ \Rightarrow & [\langle p(\vec{t_1}), f_1\rangle]_\equiv & \models & [\langle p(\vec{t_2}), f_2\rangle]_\equiv & \text{by definition} \end{array}$$

Case 3 Immediate from lemma 3. □

A non-ground representation allows chaining to be implemented efficiently using block declarations. To check that $\wedge_{i=1}^{n} y_i \leftarrow Y_i$ entails $z \leftarrow Z$ the variables of Z are first grounded. Next, a process is created for each clause $y_i \leftarrow Y_i$ that suspends until Y_i is ground. When Y_i is ground, the process resumes and grounds y_i. If z is ground after a single pass over the clauses, then $(\wedge_{i=1}^{n} y_i \leftarrow Y_i) \models z \leftarrow Z$. Suspending and resuming a process declared by a block is constant-time (in SICStus). By calling the check under negation, no problematic bindings or suspended processes are created.

4.3 Downward Closure

A useful spin-off of the join algorithm in section 5.1 is a result that shows how to calculate succinctly the downward closure operator that arises in BDD-based set sharing analysis (Codish *et al.*, 1999). Downward closure is closely related to $\dot{\Upsilon}$ and, in fact, $\dot{\Upsilon}$ can be used repeatedly to compute a finite iterative sequence that converges to $\downarrow$. This is stated in proposition 3. Finiteness follows from bounded chain length of Pos_X.

Proposition 3

Let $f \in Pos_X$. Then $\downarrow f = \vee_{i \geq 1} f_i$ where $f_i \in Pos_X$ is the increasing chain given by: $f_1 = f$ and $f_{i+1} = f_i \dot{\curlyvee} f_i$.

Proof
Let $M \in model_X(\downarrow f)$. Thus there exists $M_j \in model_X(f)$ such that $M = \cup_{j=1}^{m} M_j$. Observe $M_1 \cap M_2, M_3 \cap M_4, \ldots \in model_X(f_2)$ and therefore $M \in model_X(f_{\lceil \log_2(m) \rceil})$. Since $m \leq 2^{2^n}$ where $n = |X|$ it follows that $\downarrow f \models f_{2^n}$.

Proof by induction is used for the opposite direction. Observe that $f_1 \models \downarrow f$. Suppose $f_i \models \downarrow f$. Let $M \in model_X(f_{i+1})$. By lemma 1 there exists $M_1, M_2 \in model_X(f_i)$ such that $M = M_1 \cap M_2$. By the inductive hypothesis $M_1, M_2 \in model_X(\downarrow f)$ thus $M \in model_X(\downarrow f)$. Hence $f_{i+1} \models \downarrow f$.

Finally, $\vee_{i=1} f_i \in Def_X$ since $f_1 \in Pos_X$ and $\dot{\curlyvee}$ is monotonic and thus $X \in model_X(\vee_{i=1} f_i)$. $\square$

The significance of this is that it enables $\downarrow$ to be implemented straightforwardly with standard BDD operations. This saves the implementor the task of coding another BDD operation.

4.4 Projection

Projection is only applied to the P component of the GEP representation (since projection is onto the variables of the GE component). Projection is another exponential operation. Again, this operation can be filtered by recognising special cases where the projection can be calculated with lower complexity. The projection algorithm implemented is based on a Fourier-Motzkin style algorithm (as opposed to a Schröder variable elimination algorithm). The algorithm is syntactic and each of the variables to be projected out is eliminated in turn. The first two steps collect clauses with the variable to be projected out occurring in them, the third performs the projection by syllogising and the fourth increases efficiency by removing redundant clauses. Suppose that $f = \wedge F$, where F is a set of clauses, and suppose x is to be projected out of f.

1. All those clauses with x as their head are found, giving $H = \{x \leftarrow X_i \mid i \in I\}$, where I is a (possibly empty) index set.
2. All those clauses with x in the body are found, giving $B = \{y \leftarrow Y_j \mid j \in J\}$, where J is a (possibly empty) index set and $x \in Y_j$ for each $j \in J$.
3. Let $Z_{i,j} = X_i \cup (Y_j \setminus \{x\})$. Then $N = \{y \leftarrow Z_{i,j} \mid i \in I \wedge j \in J \wedge y \notin Z_{i,j}\}$ (syllogising). Put $F' = ((F \setminus H) \setminus B) \cup N$. (Then $\exists x.f = \wedge F'$.)
4. A compact representation is maintained by eliminating redundant clauses from F' (compaction).

All four steps can be performed in a single pass over f. A final pass over f retracts clauses such as $x \leftarrow true$ by binding x to true and also removes clause pairs such as $y \leftarrow z$ and $z \leftarrow y$ by unifying y and z.

At each pass the cost of step 4, the compaction process, is quadratic in the size of the formula to be compacted (since the compaction can be reduced to a linear number of entailment checks, each of which is linear). The point of compaction is to keep the representation small. Therefore, if the result of projecting out a variable (prior to compaction) is smaller than the original formula, then compaction appears to be unnecessary. Thus, step 4 is only applied when the number of clauses in the result of the projection is strictly greater than the number of clauses in the original formula. Notice also that in the filtered case the number of syllogisms is linear in the number of occurrences of the variable being projected out. Table 3 details the relative frequency with which the filtered and compaction cases are encountered. Observe that the vast majority of cases do not require compaction. Finally notice that join is defined in terms of projection, hence the filter for projection is inherited by join.

file	strips	chat_parser	sim_v5-2	peval	aircraft	essln	chat_80	aqua_c
filt	100.0	99.8	100.0	97.4	100.0	99.4	99.7	96.1
elim	0.0	0.2	0.0	2.6	0.0	0.6	0.3	3.9

Table 3. *Frequency Analysis of Compaction in Projection (induced magic)*

Notice that filtered algorithms break up an operation into several components of increasing complexity. The filtered algorithm then suggests natural places at which to widen, i. e. the high complexity component is widened from above using a cheap approximation. This approximation might be acceptable since the high complexity case will be called infrequently. For example, widening might be used to improve the worst case complexity of projection (and hence join) for non-canonical *Def*.

5 Implementation of the Iteration Strategy

Sections 3 and 4 are concerned with the representation of the abstract domain and the design and implementation of domain operations. The overall efficiency of an analyser depends not only on these operations, but also on the iteration strategy employed within the fixpoint engine. A fixpoint engine has to trade off the complexity of its data-structures against the degree of recomputation that these data-structures factor out. For example, semi-naïve iteration (Bancilhon & Ramakrishnan, 1986) has very simple data-structures, but entails a degree of recomputation, whereas PLAI (Hermenegildo *et al.*, 2000) tracks dependencies with dynamically generated graphs to dramatically reduce the amount of recomputation.

Fixpoint engines with dependency tracking which have been applied to logic programming analyses include: PLAI (Muthukumar & Hermenegildo, 1992; Hermenegildo *et al.*, 2000), GAIA (Le Charlier & Van Hentenryck, 1994), the CLP($\mathcal{R}$) engine (Kelly *et al.*, 1998) and GENA (Fecht & Seidl, 1996; Fecht, 1997; Fecht & Seidl, 1999). An alternative to on-the-fly dependency tracking is to use semi-naïve iteration driven by a redo worklist detailing which call and answer patterns need to be re-evaluated and (possibly) in which order. One instance of this is induced magic (Codish, 1999a) under eager evaluation (Wunderwald, 1995), which factors out much of the recomputation that arises through magic transformation. Other instances use knowledge of the dependencies to help order the redo list and thereby reduce unnecessary computation – this is typically done by statically calculating

SCCs (Gallagher & de Waal, 1994), possibly recursively (Bourdoncle, 1993), on the call graph or on the call graph of the magic program.

The benefit of reduced recomputation is dependent upon the cost of the abstract domain operations. Therefore the sophistication of the iteration strategies of engines such as PLAI and GENA is of most value when the domain operations are complex. The present paper has designed its analysis so that heavyweight domain operations are infrequently called, hence an iteration strategy employing simpler data-structures, but possibly introducing extra computation, is worthy of consideration. The analyser described in (Howe & King, 2000) used induced magic under eager evaluation. The current analyser builds on this work by adopting tactics inspired by PLAI, GAIA and GENA into the induced magic framework. Importantly these tactics require no extra data-structures and little computational effort. Experimental results suggest that this choice of iteration strategy is well suited to Def-based groundness analysis.

5.1 Ordered Induced Magic

Induced magic was introduced in (Codish, 1999a), where a meta-interpreter for semi-naïve, goal-dependent, bottom-up evaluation is presented. The analyser described in (Howe & King, 2000) implements a variant of this scheme using eager evaluation. In that paper, eager evaluation was implemented without an explicit redo list as follows: each time a new call or answer pattern is generated, the meta-interpreter invokes a predicate, solve, which re-evaluates the appropriate clauses. The re-evaluation of a clause may in turn generate new calls to solve so that one call may start before another finishes. The status of these calls is maintained on the stack, which simulates a redo list. Henceforth, this strategy is referred to as *eager induced magic.*

As noted by other authors, simple optimisations can significantly impact on performance. In particular, as noted in (Hermenegildo *et al.*, 2000), evaluations resulting from new calls should be performed before those resulting from new answers, and a call to solve for one rule should finish before another call to solve for another rule starts. These optimisations cannot be integrated with stack based eager evaluation because they rely on reordering the calls to solve. Hence a redo list is reintroduced in order to make these optimisations.

The meta-interpreter listed in Fig. 2 illustrates how a redo list can be integrated with induced magic. Four of the predicates are represented as atoms in the dynamic database: `redo/2`, the redo list; `fact/4`, the call and answer patterns, where propositional formulae are represented as difference lists – specifically, the fourth argument is an open list with the third argument being its tail; `head_to_clause/2`, representing the head and body for each clause; `atom_to_clause/4`, representing the clauses with a given atom in the body. Before invoking `oim_solve/0`, a call to `cond_assert/3` is required. This has the effect of adding the top-level call to the `fact/4` database and adding the call pattern to the `redo/2` database, thereby initialising the fixpoint calculation. Evaluation is driven by the redo list. If the redo list contains call patterns, then the first (most recently introduced) is removed

```
oim_solve :-
    retract(redo(call, Atom)), !, (call_solve(Atom); oim_solve).
oim_solve :-
    retract(redo(answ, Atom)), !, (answ_solve(Atom); oim_solve).

call_solve(Head) :-
    head_to_clause(Head, Body), fact(call, Head, [], Form1),
    solve_right(Body, Form1, Form2), cond_assert(answ, Head, Form2), fail.

answ_solve(Atom) :-
    atom_to_clause(Atom, Head, Left, Right),
    fact(call, Head, [], Form1), fact(answ, Atom, Form1, Form2),
    solve_left(Left, Form2, Form3), solve_right(Right, Form3, Form4),
    cond_assert(answ, Head, Form4), fail.

solve_left([], Form, Form).
solve_left([Atom | Atoms], Form1, Form3) :-
    fact(answ, Atom, Form1, Form2), solve_left(Atoms, Form2, Form3).

solve_right([], Form, Form).
solve_right([Atom | Atoms], Form1, Form2) :-
    solve_right(Atom, Atoms, Form1, Form2).

solve_right(Atom, _, Form, _) :-
    cond_assert(call, Atom, Form), !, redo_assert(answ, Atom), fail.
solve_right(Atom, Atoms, Form1, Form3) :-
    fact(answ, Atom, Form1, Form2), solve_right(Atoms, Form2, Form3).
```

Fig. 2. A Meta-interpreter for Ordered Induced Magic

and `call_solve/1` is invoked. If the redo list contains only answer patterns, then the first is removed and control is passed to `answ_solve/1`. The meta-interpreter terminates (with failure) when the redo list is empty.

The predicate `call_solve/1` re-evaluates those clauses whose heads match a new call pattern. It first looks up a body for a clause with a given head followed by the current call pattern for head, then solves the body in induced magic fashion with `solve_right/3`. If `cond_assert/3` is called with a call (answer) pattern that does not entail the call (answer) pattern in `fact/4`, then it succeeds, updating `fact/4` with the join of the call (answer) patterns. In this event, the new call (answer) pattern is added to the beginning of the `redo/2` database. The predicate `answ_solve/1` re-evaluates those clauses containing a body atom which matches a new answer pattern. It looks up a clause with a body that contains a given atom, solves the body to the left of the atom and then to the right of the atom. If a new call pattern is encountered in `solve_right/4`, then the evaluation of the clause is aborted, as the new call may give a new answer for this body atom. In this situation, calculating an answer for the head with the old body answer will result in an answer that needs to be re-calculated. To ensure that the clause is re-evaluated, an answer for the body atom is put in the redo list by `redo_assert/2`. This iteration strategy is referred to as *o*rdered induced magic.

5.2 SCC-based Strategies

In order to assess the suitability of ordered induced magic as a fixpoint strategy for *Def*-based groundness analysis, it has been compared with a variety of popular SCC-based methods. The fixpoint engine can be driven either by considering the top-level SCCs (Gallagher & de Waal, 1994) or by considering the recursive nesting of SCCs, for example (Bourdoncle, 1993). The SCCs can be statically calculated either on the call graph of the magicked program or on the call graph of the original program.

SCCs for the call graph of the magicked program (in topological order) are calculated using Tarjan's algorithm (Tarjan, 1972). The fixpoint calculation then proceeds bottom-up, stabilising on the (call and answer) predicates in each SCC in topological order. If an SCC contains a single, non-recursive, (call or answer) predicate, then the predicate must stabilise immediately, hence no fixpoint check is needed. This strategy is henceforth referred to as *SCC magic*.

A more sophisticated SCC-based tactic is to calculate SCCs within an SCC, as suggested by Bourdoncle (Bourdoncle, 1993). The 'recursive strategy' described by Bourdoncle recursively applies Tarjan's algorithm to each non-trivial SCC having removed an appropriate node (the head node) and corresponding edges. The fixpoint calculation proceeds bottom-up, stabilising on the (call and answer) predicates in each component recursively. The fixpoint check need only be made at the head node. This is strategy has potential for reaching a fixpoint in a particularly small number of updates. This strategy is henceforth referred to as *Bourdoncle magic*.

Since both SCC magic and Bourdoncle magic work on the call graph of the magic program, they cannot be combined with induced magic; the ordering of the re-evaluations conflicts. Calculating SCCs on the call graph of the original program may be combined with (ordered) induced magic. The order in which the calls are encountered is determined by the top-down left-to-right execution of the program and the evaluation of a call may add new answers to the redo list. SCCs can be used to order new answers as they are added to the redo list. This strategy is henceforth referred to as *SCC induced magic*. However, since calls are re-evaluated in preference to answers, the order of answers in the redo list is largely determined by the order of the calls. Consequently, SCCs should have a negligible effect on performance.

5.3 Dynamic Dependency Tracking

One test of the efficacy of an iteration strategy is the number of iterations required to reach the fixpoint. In order to assess how well ordered induced magic behaves, a more sophisticated iteration strategy based on dynamic dependency tracking was implemented. The strategy chosen was that of WRT solver of GENA (Fecht, 1997; Fecht & Seidl, 1999) since this recent work is particularly well described, has extensive experimental results and conveniently fits with the worklist model.

The WRT strategy utilises a worklist, which is effectively reordered on-the-fly. To quote Fecht and Seidl (Fecht & Seidl, 1996), "The worklist now is organized as

file	rubik	chat_parser	sim_v5-2	peval	aircraft	essln	chat_80	aqua_c
meet	39.3	40.5	41.5	44.6	35.4	48.3	41.0	43.5
join	8.7	8.7	10.0	6.4	10.5	8.0	9.1	8.7
join (diff)	1.0	2.0	0.1	2.6	0.2	0.7	1.8	1.3
equiv	8.7	8.7	10.0	6.4	10.5	8.0	9.1	8.7
proj	5.8	4.7	4.5	7.1	4.1	4.0	4.4	4.2
rename	36.5	35.4	34.1	33.0	39.3	31.0	34.5	33.6
total	6646	11324	5748	3992	12550	11754	32906	109612

Table 4. *Frequency Analysis: Non-canonical Def Analyser with Ordered Induced Magic*

file	strips	chat_parser	sim_v5-2	peval	aircraft	essln	chat_80	aqua_c
filt	100.0	99.7	100.0	98.4	100.0	99.7	99.7	98.0
elim	0.0	0.3	0.0	1.6	0.0	0.3	0.3	2.0

Table 5. *Frequency Analysis of Compaction in Projection (Ordered Induced Magic)*

a (max) priority queue where the priority of an element [call pattern] is given by its time stamp," where the time stamp records the last time the solver was called for that call pattern. If, whilst solving for a call pattern, new call patterns are encountered, then the bottom answer pattern is not simply returned. Instead the solver tries to recursively compute a better approximation to this answer pattern. This tactic is also applied in PLAI and GAIA, though realised differently.

The WRT strategy of GENA gives a small number of updates, hence is an attractive iteration strategy. However, its implementation in a backtrack driven meta-interpreter requires extensive use of the dynamic database for the auxiliary data-structures. In Prolog this is potentially expensive (Hermenegildo *et al.*, 1992).

5.4 Frequency Analysis for Def: Reprise

In section 4 a frequency analysis of the abstract domain operations in *Def*-based groundness analysis was given. It was then argued that in light of these results certain choices about the abstract domain operations should be made. These results are dependent on the iteration strategy of the analyser. In this section several different iteration strategies have been proposed and it needs to be checked that these give similar proportions of calls to the abstract domain operations – that is, that the choices for the abstract domain operations remain justified. Table 4 gives the frequency analysis for ordered induced magic driving non-canonical *Def* and indicates that the choices of domain operation remain valid. Note that for the BDD analyser, each rename is accompanied by a projection – this is not the case for non-canonical *Def*, explaining the lesser frequency of projection. This makes the non-canonical *Def* representation appear even more suitable. Table 5 demonstrates that projection still almost always avoids compaction. Similar distributions are found with the other iteration strategies and for brevity these tables are omitted.

6 Experimental Evaluation

This section gives experimental results for a number of analysers with the objective of comparing the analysis proposed in the previous sections with existing techniques and evaluating the impact of the various tactics utilised. These analysers are built by selecting appropriate combinations of: abstract domain, domain representation, iteration strategy and optimisations. The analysers are evaluated in terms of both execution time and the underlying behaviour (i. e. the number of updates). All implementations are coded in SICStus Prolog 3.8.3 with the exception of the domain operations for *Pos*, which were written in C by Schachte (Schachte, 1999). The analysers were run on a 296MHz Sun UltraSPARC-II with 1GByte of RAM running Solaris 2.6. Programs are abstracted following the elegant (two program) scheme of (Bueno *et al.*, 1996) to guarantee correctness. Programs are normalised to definite clauses. Timings are the arithmetic mean over 10 runs. Timeouts were set at five minutes.

6.1 Domains: Timings and Precision

Tables 6 and 7 give timing and precision results for the domains *EPos*, *Def* represented in DBCF, non-canonical *Def* (denoted GEP after the representation) and *Pos*. In these tables, file is the name of the program analysed; size is the number of abstract clauses in the normalised program; abs is the time taken to read, parse and normalise the input file, producing the abstract program; fixpoint details the analysis time for the various domains; precision gives the total number of ground arguments in the call and answer patterns found by each analysis (excluding those introduced by normalising the program); % prec. loss gives the loss of precision of *EPos* and *Def* as compared to *Pos* – to emphasise where precision is lost, entries are only made when there is a precison loss. All the analyses were driven by the ordered induced magic iteration strategy.

First consider precision. As is well known, in practice, for goal-dependent groundness analysis, the precision of *Def* is very close to that of *Pos*. In the benchmark suite used here, *Def* loses ground arguments in only two programs: rotate.pl, which loses three arguments, and sim_v5-2.pl, where two arguments are lost. *EPos* loses precision in several programs, but still performs reasonably well. (Goal-independent analysis precision comparisons for *EPos* and *Def* are given in (Heaton *et al.*, 2000) and (Genaim & Codish, 2001). These show that *EPos* loses significant precision, whereas *Def* gives precision close to that of *Pos*.)

The non-canonical *Def* analyser appears to be fast and scalable – taking more than a second to analyse only the largest benchmark program. This analyser does not employ widening (however, incorporating a widening would guarantee robustness of the analyser, even for pathological programs (Genaim *et al.*, 2001)). Notice that the analysis times for all the programs is close to the abstraction time – this suggests that a large speed up in the analysis time needs to be coupled with a commensurate speedup in the abstracter.

The non-canonical *Def* analysis times are comparable to those for *EPos* for

			fixpoint				precision				% prec. loss	
file	size	abs	*EPos*	DBCF	GEP	*Pos*	*EPos*	DBCF	GEP	*Pos*	*EPos*	*Def*
append.pl	2	0.00	0.00	0.01	0.01	0.01	3	4	4	4	25.0	
rotate.pl	3	0.00	0.00	0.01	0.01	0.01	2	3	3	6	66.6	50.0
mortgage.clpr	4	0.00	0.00	3.31	0.00	0.04	6	6	6	6		
qsort.pl	6	0.01	0.00	0.00	0.00	0.01	11	11	11	11		
rev.pl	6	0.01	0.00	0.01	0.01	0.01	0	0	0	0		
queens.pl	9	0.00	0.00	0.04	0.00	0.02	3	3	3	3		
zebra.pl	9	0.01	0.00	0.06	0.01	0.10	19	19	19	19		
laplace.clpr	10	0.01	0.00	0.08	0.01	0.01	0	0	0	0		
shape.pl	11	0.00	0.00	0.04	0.00	0.03	6	6	6	6		
parity.pl	12	0.01	0.00	3.24	0.52	–	0	0	0	–	–	–
treeorder.pl	12	0.00	0.00	0.20	0.01	0.03	0	0	0	0		
fastcolor.pl	13	0.04	0.00	0.00	0.01	0.01	14	14	14	14		
music.pl	13	0.01	0.01	–	0.02	0.07	2	–	2	2		
serialize.pl	13	0.01	0.00	0.12	0.00	0.06	3	3	3	3		
crypt_wamcc.pl	19	0.02	0.01	0.03	0.01	0.04	31	31	31	31		
option.clpr	19	0.02	0.00	1.27	0.02	0.07	42	42	42	42		
circuit.clpr	20	0.02	0.00	52.69	0.02	0.12	3	3	3	3		
air.clpr	20	0.01	0.00	44.63	0.02	0.09	9	9	9	9		
dnf.clpr	22	0.02	0.01	0.01	0.00	0.03	8	8	8	8		
dcg.pl	23	0.02	0.00	0.01	0.00	0.02	59	59	59	59		
hamiltonian.pl	23	0.02	0.00	0.01	0.00	0.02	37	37	37	37		
nandc.pl	31	0.03	0.02	0.03	0.01	0.05	34	37	37	37	8.1	
semi.pl	31	0.03	0.02	0.75	0.04	0.23	28	28	28	28		
life.pl	32	0.02	0.00	0.03	0.01	0.05	58	58	58	58		
poly10.pl	32	0.03	0.00	0.02	0.00	0.04	45	45	45	45		
meta.pl	33	0.02	0.01	0.02	0.02	0.03	1	1	1	1		
rings-on-pegs.clpr	34	0.02	0.02	1.20	0.02	0.11	11	11	11	11		
browse.pl	35	0.02	0.01	0.04	0.02	0.04	41	41	41	41		
gabriel.pl	38	0.03	0.02	0.06	0.02	0.07	37	37	37	37		
tsp.pl	38	0.02	0.02	0.07	0.02	0.11	122	122	122	122		
map.pl	41	0.02	0.02	0.03	0.01	0.05	17	17	17	17		
csg.clpr	42	0.04	0.01	0.00	0.00	0.02	8	8	8	8		
disj_r.pl	48	0.03	0.01	0.02	0.02	0.08	97	97	97	97		
ga.pl	48	0.08	0.00	0.03	0.02	0.09	141	141	141	141		
critical.clpr	49	0.03	0.01	–	0.04	0.21	14	–	14	14		
robot.pl	51	0.04	0.00	0.01	0.00	0.03	41	41	41	41		
scc1.pl	51	0.03	0.01	0.08	0.01	0.14	89	89	89	89		
ime_v2-2-1.pl	53	0.04	0.02	0.30	0.03	0.20	100	101	101	101	0.9	
cs_r.pl	54	0.06	0.01	0.06	0.01	0.09	149	149	149	149		
tictactoe.pl	55	0.05	0.01	0.08	0.02	0.09	60	60	60	60		
flatten.pl	56	0.04	0.02	0.22	0.04	0.13	27	27	27	27		
mastermind.pl	56	0.03	0.02	0.04	0.02	0.09	43	43	43	43		
dialog.pl	61	0.03	0.01	0.03	0.02	0.05	45	45	45	45		
neural.pl	67	0.06	0.03	0.13	0.02	0.08	121	123	123	123	1.6	
bridge.clpr	68	0.10	0.00	0.07	0.01	0.09	24	24	24	24		
conman.pl	76	0.05	0.00	0.00	0.00	0.02	6	6	6	6		
unify.pl	77	0.05	0.02	0.19	0.05	0.38	70	70	70	70		
kalah.pl	78	0.04	0.02	0.05	0.02	0.10	199	199	199	199		
nbody.pl	85	0.07	0.03	0.08	0.04	0.19	113	113	113	113		
peep.pl	85	0.11	0.02	0.13	0.03	0.14	10	10	10	10		
sdda.pl	89	0.05	0.02	0.13	0.04	0.12	17	17	17	17		
bryant.pl	94	0.07	0.06	0.23	0.14	0.76	99	99	99	99		
boyer.pl	95	0.07	0.02	0.08	0.05	0.08	3	3	3	3		
read.pl	101	0.09	0.03	0.15	0.05	0.20	99	99	99	99		
qplan.pl	108	0.09	0.02	0.07	0.02	0.16	216	216	216	216		
trs.pl	108	0.14	0.06	–	0.09	2.46	13	–	13	13		
press.pl	109	0.08	0.07	0.40	0.10	0.36	52	53	53	53	1.8	
reducer.pl	113	0.07	0.05	3.47	0.04	0.30	41	41	41	41		
parser_dcg.pl	122	0.09	0.04	2.27	0.08	0.24	28	43	43	43	34.8	
simple_analyzer.pl	140	0.11	0.05	0.28	0.10	0.58	89	89	89	89		

Table 6. *Groundness Results: Smaller Programs*

			fixpoint				precision				% prec. loss	
file	size	abs	*EPos*	DBCF	GEP	*Pos*	*EPos*	DBCF	GEP	*Pos*	*EPos*	*Def*
dbqas.pl	143	0.09	0.02	0.54	0.03	0.09	18	18	18	18		
ann.pl	146	0.10	0.05	0.77	0.09	0.32	71	71	71	71		
asm.pl	160	0.17	0.04	0.08	0.04	0.17	90	90	90	90		
nand.pl	179	0.14	0.04	0.19	0.05	0.37	402	402	402	402		
lnprolog.pl	220	0.10	0.07	0.16	0.07	0.21	110	143	143	143	23.0	
ili.pl	221	0.15	0.07	1.29	0.17	0.36	4	4	4	4		
strips.pl	240	0.22	0.02	0.04	0.03	0.14	142	142	142	142		
sim.pl	244	0.20	0.08	1.69	0.18	1.38	100	100	100	100		
rubik.pl	255	0.20	0.12	–	0.16	0.46	158	–	158	158		
chat_parser.pl	281	0.34	0.09	0.47	0.24	1.16	504	505	505	505	0.1	
sim_v5-2.pl	288	0.23	0.05	0.15	0.07	0.32	455	455	455	457	0.4	0.4
peval.pl	332	0.17	0.05	0.23	0.18	0.39	27	27	27	27		
aircraft.pl	395	0.55	0.11	0.21	0.14	0.55	687	687	687	687		
essln.pl	595	0.48	0.12	2.70	0.19	0.93	158	162	162	162	2.4	
chat_80.pl	883	1.53	0.38	8.17	0.76	4.53	852	855	855	855	0.3	
aqua_c.pl	3928	3.47	1.70	–	4.26	144.62	1222	–	1285	1285	4.9	

Table 7. *Groundness Results: Larger Programs*

smaller programs, with *EPos* outperforming non-canonical *Def* on some of the larger benchmarks. This is unsurprising given the much better theoretical behaviour of *EPos*, indeed it is much in the favour of non-canonical *Def* that it is competitive with *EPos*. The DBCF analyser suffers from the problems discussed in section 4. The cost in meet of maintaining the canonical form often becomes significant. In cases (such as in music.pl) where the number of variables, the number of body atoms and the size of the representation are all large, the exponential nature of reducing to canonical form leads to a massive blowup in analysis time. Hence the DBCF analyser fails to produce a result for several examples and gives poor scalability. Also, the analysis appears to lack robustness – the sensitivity of the meet to the form of the program clauses leads to widely varying results. *Pos* performs well on most programs, but is still consistently several times slower than non-canonical *Def*. *Pos* performs particularly poorly on parity.pl (a program designed to be problematic for BDD-based *Pos* analysers) and aqua_c.pl. Again, since the *Pos* analyser uses BDDs (essentially a canonical form) there is a cost in maintaining the representation. This can lead to a lack of robustness. It should be pointed out that the *Pos* analyser is not state of the art and that one using the GER representation (Bagnara & Schachte, 1999) would probably give improved results. Of course, widening could be used to give improved times for *Pos*, but at the cost of precision.

6.2 Iteration Strategy: Timings and Updates

Table 8 gives timing results for non-canonical *Def* analysis when driven by various iteration strategies. The column headers are abbreviations as follows: ord stand for ordered induced magic; eim stands for eager induced magic; bom stands for Bourdoncle magic; scm stands for SCC magic; scc stands for SCC induced magic; dyd stands for dynamic dependency. The timings are split into two sections. The overhead time is the preprocessing overhead incurred in calculating the SCCs required to drive the analyses. For bom and scm, SCCs are calculated on the call and answer graph of the magic program. For scc, SCCs are calculated on the call graph of the

	overhead			strategy					
file	bom	scm	scc	ord	eim	bom	scm	scc	dyd
dbqas.pl	0.02	0.02	0.01	0.03	0.03	0.03	0.06	0.03	0.07
ann.pl	0.05	0.04	0.01	0.09	0.14	0.18	0.22	0.09	0.19
asm.pl	0.06	0.06	0.02	0.04	0.08	0.09	0.13	0.05	0.15
nand.pl	0.10	0.08	0.02	0.05	0.06	0.21	0.13	0.05	0.17
lnprolog.pl	0.07	0.06	0.03	0.07	0.10	0.23	0.19	0.07	0.22
ili.pl	0.06	0.04	0.02	0.17	0.29	0.73	0.38	0.16	0.68
strips.pl	0.10	0.08	0.03	0.03	0.01	0.10	0.06	0.03	0.07
sim.pl	0.10	0.07	0.02	0.18	0.35	0.38	0.29	0.19	0.37
rubik.pl	0.29	0.15	0.04	0.16	0.19	1.12	0.33	0.15	0.34
chat_parser.pl	0.19	0.08	0.05	0.24	0.44	2.31	0.67	0.24	1.89
sim_v5-2.pl	0.25	0.12	0.04	0.07	0.07	0.57	0.18	0.07	0.22
peval.pl	0.06	0.06	0.04	0.18	0.30	0.31	0.29	0.17	0.38
aircraft.pl	0.73	0.26	0.14	0.14	0.23	1.13	0.53	0.13	0.44
essln.pl	0.40	0.19	0.11	0.19	0.27	1.58	0.61	0.18	0.46
chat_80.pl	0.96	0.34	0.15	0.76	1.36	21.22	2.59	0.73	3.30
aqua_c.pl	17.91	1.59	0.84	4.26	10.69	454.22	20.52	4.30	15.74

Table 8. *Timing Results for Iteration Strategies*

original program. The strategies ord, eim and dyd do not require any preprocessing, hence have no overhead. The strategy times are the times for analysing each program (that is, the time taken for the fixpoint calculation, not including the preprocessing overhead). Table 9 gives a second measure of the cost of each iteration strategy; this time in terms of the number of updates (writes to database/extension table) required to reach the fixpoint.

One important measure of the success of an iteration strategy is the number of updates required in the analysis. This impacts directly on the number of calls to abstract operations and hence the amount of work (speed) of the analysis. Table 9 indicates that ord, scc and dyd give the best behaviour over a large number of programs. However, all of the other strategies give the best result for some programs, indicating that each has its merits. Observe that, as predicted in section 5, ord and scc give very similar results.

In measuring performance of a particular analysis, the overall time taken is also of importance. Table 8 indicates that the methods based on SCCs in the call graph of the magic program have problems. Firstly, they require SCCs to be calculated – the cost of this (in particular for Bourdoncle magic) is significant. Secondly, the fixpoint times for bom and scm are much greater than would be expected from the results in Table 9. This is partly because the bom and scm strategies cannot be integrated with induced magic, which impacts heavily on speed. The bom strategy also has a third drawback – the proportion of re-evaluations not resulting in an update rises dramatically for larger programs. Larger programs often give rise to deeply nested SCCs. Suppose an SCC, say A, nests a subSCC, say B. In detecting the stability of A, the stability of the head of B needs to be established. This in turn requires a single pass over B. If n passes over A are required to reach stability, then n passes over B are also needed (even if B is already stable). Extrapolating, the number of times an SCC is passed over is determined

	strategy							strategy					
file	ord	eim	bom	scm	scc	dyd	file	ord	eim	bom	scm	scc	dyd
append	3	3	3	3	3	3	cs_r	66	66	66	66	66	66
rotate	7	7	7	7	7	6	tictactoe	60	56	56	57	60	55
mortgage	6	6	6	6	6	6	flatten	81	95	80	107	81	71
qsort	8	7	8	8	8	7	mastermind	86	84	82	85	86	82
rev	11	11	11	11	11	11	dialog	82	95	79	82	82	77
queens	12	12	12	12	12	12	neural	83	78	78	102	83	78
zebra	12	12	12	12	12	12	bridge	13	13	13	13	13	13
laplace	12	12	12	12	12	12	conman	14	14	14	14	14	14
shape	12	10	10	10	12	10	unify	92	114	92	97	92	83
parity	38	47	38	38	38	37	kalah	91	93	92	93	91	92
treeorder	17	18	17	18	17	14	nbody	125	173	124	162	125	122
fastcolor	18	19	18	18	18	18	peep	61	61	62	61	61	58
music	13	13	13	12	13	13	sdda	91	105	96	100	93	94
serialize	16	18	16	16	16	10	bryant	202	210	189	161	202	214
crypt_wamcc	23	23	23	23	23	23	boyer	99	107	102	101	99	105
option	30	35	30	30	30	29	read	119	127	90	114	119	91
circuit	32	31	30	34	32	29	qplan	95	95	95	94	95	93
air	32	35	32	36	32	29	trs	86	92	88	96	88	69
dnf	8	8	8	8	8	8	press	224	222	221	217	224	241
dcg	31	30	30	30	31	30	reducer	118	173	173	158	118	163
hamiltonian	28	28	28	28	28	28	parser_dcg	170	170	157	168	169	160
nandc	49	51	44	51	49	49	simple_analyzer	200	242	200	321	201	189
semi	53	51	51	54	53	48	dbqas	105	105	94	109	105	98
life	30	30	30	31	30	30	ann	207	233	229	281	207	192
poly10	24	24	24	24	24	24	asm	169	237	174	217	169	181
meta	46	29	40	40	46	40	nand	188	188	186	187	188	186
rings-on-pegs	37	37	37	37	37	37	lnprolog	253	300	279	281	253	264
browse	43	43	43	43	43	43	ili	209	318	318	330	209	312
gabriel	48	48	48	50	48	47	strips	108	101	111	106	108	99
tsp	66	66	65	73	66	65	sim	280	310	269	277	281	266
map	68	68	68	68	68	68	rubik	372	369	375	383	372	373
csg	12	12	12	12	12	12	chat_parser	445	682	659	652	445	621
disj_r	58	58	58	58	58	58	sim_v5-2	256	256	254	254	256	256
ga	60	60	59	60	60	59	peval	280	331	312	309	281	285
critical	42	39	44	44	42	36	aircraft	506	506	506	506	506	506
robot	28	28	28	28	28	28	essln	485	547	473	516	485	450
scc1	51	50	50	50	51	50	chat_80	1322	1657	1494	1579	1323	1454
ime_v2-2-1	77	74	72	77	77	70	aqua_c	4751	5779	5667	6106	4842	4611

Table 9. *Number of Updates for Iteration Strategies*

by the sum of the number of passes over each SCC containing it. If the SCC is deeply nested and large this involves a large number of re-evaluations producing no updates. As the scm strategy does not involve nested SCCs, this problem does not arise. It appears that Bourdoncle's recursive strategy is not well suited for driving groundness analyses of logic programs. Table 8 also indicates that whilst SCCs on the call graph give comparable analysis times to ordered induced magic, they too come with an overhead of precomputation. Sophisticated dynamic dependency graphs do not pay for themselves in a groundness analysis involving lightweight domain operations, as reflected by the timings for dyd. However, they are more amenable to optimisation than ordered induced magic (which is itself essentially an optimisation of induced magic) and in an analysis where the cost of the abstract operations is higher it is to be expected that this strategy would be more effective. Also, by using a different programming paradigm, the dynamic changes to the

	ord									dyd							
file	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8
dbqas.pl	55	21	1	0	1					60	16	2					
ann.pl	88	39	11	2						100	32	4	4				
asm.pl	140	13	1							130	21	3					
nand.pl	172	5	2							173	5	1					
lnprolog.pl	168	32	7							155	47	5					
ili.pl	89	24	20	3						41	54	23	5	5	7	1	
strips.pl	82	10	2							89	5						
sim.pl	144	43	12	1	2					152	38	10	2				
rubik.pl	264	54								264	53	1					
chat_parser.pl	207	78	14	7	1	0	1			101	144	37	14	7	5		
sim_v5-2.pl	248	4								248	4						
peval.pl	114	45	13	4	3	1				111	52	9	3	2	1	1	1
aircraft.pl	468	19								468	19						
essln.pl	321	59	9	2	1	1				341	48	3	1				
chat_80.pl	537	224	70	22	4	2	1			466	261	93	22	10	7	1	
aqua_c	2135	742	205	64	28	12	2	1	3	2170	781	151	48	26	11	2	3

Table 10. *Chain Length Distributions*

dependency graph could be made more efficiently (for example, (Fecht & Seidl, 1999) use SML).

6.3 Chain Length

Table 10 gives further details of the number of updates required in program analysis with non-canonical *Def*. This table gives the distribution of the number of updates required to reach the fixpoint for the various program predicates. Results are given for ord and dyd as it is clear from Table 9 that these are the most competitive strategies. Each column gives the number of predicates requiring that number of updates. Entries beyond the maximum number of updates have been left blank to highlight the maximum chain length.

Chain length gives a good indication of the robustness of the iteration strategies. Whilst it is always possible to construct programs exhibiting worst case behaviour (Codish, 1999b; Genaim *et al.*, 2001), Table 10 shows that for both ord and dyd, very few chains are longer than 4 and that at worst chains have length 9. It also again indicates that different strategies can give significantly different behaviour for the analysis.

6.4 Optimisations

A number of optimisations have been discussed in this paper. Table 11 details the effect of these, singly and in combination. The five optimisations considered have each been abbreviated by a single letter: e denotes filtering by entailment checking; g denotes the use of a GEP factorisation; p denotes filtering projection; r denotes the use of redundancy removal; t denotes the maintenance of a true factorisation. The column headers describe which optimisations have been switched on; for example, gpr denotes the situation where the analysis uses a GEP factorisation, where

	switches									
file	egpr	egprt	egp	epr	gpr	egr	pr	er	gr	r
dbqas.pl	0.02	0.02	0.02	0.03	0.03	0.02	0.03	0.03	0.03	0.03
ann.pl	0.09	0.09	0.09	0.10	0.10	0.09	0.11	0.10	0.10	0.11
asm.pl	0.04	0.04	0.04	0.05	0.04	0.04	0.05	0.05	0.04	0.06
nand.pl	0.05	0.05	0.05	0.07	0.06	0.05	0.09	0.07	0.06	0.09
lnprolog.pl	0.06	0.07	0.07	0.08	0.07	0.07	0.09	0.08	0.07	0.10
ili.pl	0.16	0.16	0.16	0.17	0.16	0.16	0.18	0.17	0.18	0.19
strips.pl	0.02	0.02	0.02	0.03	0.03	0.02	0.04	0.03	0.03	0.03
sim.pl	0.18	0.18	0.18	0.20	0.20	0.20	0.23	0.23	0.22	0.25
rubik.pl	0.15	0.16	0.16	0.16	0.17	0.19	0.18	0.20	0.19	0.21
chat_parser.pl	0.24	0.24	0.24	0.29	0.27	0.25	0.33	0.30	0.28	0.34
sim_v5-2.pl	0.06	0.07	0.06	0.08	0.07	0.07	0.10	0.08	0.07	0.10
peval.pl	0.17	0.17	0.17	0.18	0.17	0.18	0.19	0.19	0.18	0.20
aircraft.pl	0.14	0.14	0.14	0.17	0.16	0.14	0.21	0.17	0.16	0.21
essln.pl	0.18	0.19	0.19	0.22	0.20	0.19	0.24	0.22	0.21	0.25
chat_80.pl	0.73	0.74	0.73	0.84	0.81	0.76	0.96	0.89	0.84	0.99
aqua_c.pl	4.25	4.20	4.28	4.74	4.73	4.81	5.34	5.36	5.29	5.99

Table 11. *Timing Results for Combinations of Optimisations*

projection is filtered and where redundancy removal is used, but the factorisation is not true and the entailment checking filter for join is not applied. Note that the switch for the entailment checking does not entirely turn off the entailment check filter for join, as the *Def* analysers enforce termination using the same entailment check which filters join. In Proposition 2, the filtering of join has three cases; the entailment check switch turns the first (most lightweight) case on and off. The default for the non-canonical *Def* analyser which has been used for other timings in this paper is egpr, since this gives the best result for most programs.

The first three columns of Table 11 all give very similar times, indicating that true factorisation and redundancy removal have little effect on analysis times, essentially paying for themselves. The next three columns give times for the situation with one of e, g, p switches off (relative to the default case). It is clear that turning off any of these optimisation gives a slow down of, perhaps, 10%. The next three columns give results for switching off optimisations in pairs. Again there is a clear slowdown from the previous three results (although notice that the epr and gr results are very similar), a slowdown of 15-20% from the default case. Finally, the last column shows that switching off all the optimisations results in a slowdown of approximately 25% in most programs.

One conclusion to be drawn from Table 11 is that the non-canonical *Def* analysis is extremely robust. By turning off all the optimisations for both the size of representation and the efficiency of the abstract operations, the analysis is still fast. It is expected that the effect of turning off these optimisations would be bigger when using a less effective iteration strategy or a less suitable (orthogonal) representation.

7 Related Work

Van Hentenryck *et al.* (Van Hentenryck *et al.*, 1995) is an early work which laid a foundation for BDD-based *Pos* analysis. Corsini *et al.* (Corsini *et al.*, 1993) describe how variants of *Pos* can be implemented using Toupie, a constraint language based on the μ-calculus. If this analyser was extended with, say, magic sets, it might lead to a very respectable goal-dependent analysis. More recently, Bagnara and Schachte (Bagnara & Schachte, 1999) have developed the idea (Bagnara, 1996) that a factorised implementation of ROBDDs which keeps definite information separately from dependency information is more efficient than keeping the two together. This hybrid representation can significantly decrease the size of an ROBDD and thus is a useful implementation tactic.

Heaton *et al.* (Heaton *et al.*, 2000) propose *EPos*, a sub-domain of *Def*, that can only propagate dependencies of the form $(x_1 \leftrightarrow x_2) \wedge x_3$ across procedure boundaries. This information is precisely that contained in one of the fields of the GEP factorised domain. The main finding of (Heaton *et al.*, 2000) is that this sub-domain performs reasonably well for goal-dependent analysis.

Armstrong *et al.* (Armstrong *et al.*, 1998) study a number of different representations of Boolean functions for both *Def* and *Pos*. An empirical evaluation on 15 programs suggests that specialising Dual Blake Canonical Form (DBCF) for *Def* leads to the fastest analysis overall. Armstrong *et al.* (Armstrong *et al.*, 1998) also perform interesting precision experiments. *Def* and *Pos* are compared, however, in a bottom-up framework that is based on condensing and is therefore biased towards *Pos*. The authors point out that a top-down analyser would improve the precision of *Def* relative to *Pos*.

García de la Banda *et al.* (García de la Banda *et al.*, 1996) describe a Prolog implementation of *Def* that is also based on an orthogonal DBCF representation (though this is not explicitly stated) and show that it is viable for some medium sized benchmarks. Fecht and Seidl (Fecht, 1997; Fecht & Seidl, 1999) describe another groundness analyser for *Pos* that is not coded in C. They adopt SML as a coding medium in order to build an analyser that is declarative and easy to maintain. Their analyser employs a widening.

Codish and Demoen (Codish & Demoen, 1995) describe a non-ground model based implementation technique for *Pos* that would encode $x_1 \leftrightarrow (x_2 \wedge x_3)$ as three tuples $\langle true, true, true\rangle$, $\langle false, \_, false\rangle$, $\langle false, false, \_\rangle$. King *et al.* show how, for *Def*, meet, join and projection can be implemented with quadratic operations based on a *Sharing* quotient (King *et al.*, 1999). *Def* functions are essentially represented as a set of models and widening is thus required to keep the size of the representation manageable. Ideally, however, it would be better to avoid widening by, say, using a more compact representation.

Most recently, Genaim and Codish (Genaim & Codish, 2001) propose a dual representation for *Def*. For function f, the models of $coneg(f)$ are named and f is represented by a tuple recording for each variable of f which of these models the variable is in. For example, the models of $coneg(x \rightarrow y)$ are $\{\{x, y\}, \{x\}, \emptyset\}$. Naming the three models a, b, c respectively, f is represented by $\langle ab, a\rangle$. This representation cleverly allows the well known ACI1 unification theory to be used for the domain operations. (Genaim & Codish, 2001) report promising experimental results, but still need a widening to analyse the aqua_c benchmark.

8 Conclusion

By considering the way in which goal-dependent groundness analyses proceed, an intelligent choice can be made as to how to represent the abstract domain and how the cost of the domain operations should be balanced. Analysing the relative frequencies of the domain operations leads to a representation which is compact, and where the most commonly called domain operations are the most lightweight. Filters for the more expensive domain operations are described which allow these operations to be calculated by inexpensive special cases. Ways in which a non-ground representation for Boolean functions may exploit the language features of Prolog to obtain an efficient implementation are described. The iteration strategy for driving an analysis is also extremely important. Several strategies are discussed and compared. It is concluded that for groundness analysis the fastest implementation uses a simple strategy avoiding precomputation and sophisticated data-structures. An implementor might find some or all of the issues discussed and ideas raised in this paper useful in designing a program analysis and in implementing it in Prolog.

The end product of this work is a highly principled goal-dependent groundness analyser combining the techniques described. It is written in Prolog and is small and easily maintained. The analyser is a robust, fast, precise and scalable and does not require widening for the largest program in the benchmark suite. Experimental results show that the speed of the fixpoint calculation is very close to that of reading, parsing and normalising the input file. Results also suggest that the performance of the analyser compares well with other groundness analysers, including BDD-based analysers written in C.

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