Double Chain Ladder, Claims Development Inflation and Zero Claims

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Abstract

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Double Chain Ladder demonstrated how the classical chain ladder technique can be broken down into separate components. It was shown that, under certain model assumptions and via one particular estimation technique, it is possible to interpret the classical chain ladder method as a model of the observed number of counts with a built-in delay function from when a claim is reported until it is paid. In this paper, we investigate the double chain ladder model further and consider the case when other knowledge is available, focusing on two specific types of prior knowledge namely prior knowledge on the number of zero claims for each underwriting year and prior knowledge about the relationship between the development of the claim and its mean severity. Both types of prior knowledge readily lend themselves to be included in the double chain ladder framework.

Keywords: Prior Knowledge; Claims Reserves; Reserve Risk; Over-dispersed Poisson Model; Cash flow; Bootstrap.

1 Introduction

In a recent series of papers Verrall, Nielsen and Jessen (2010), Martínez-Miranda, Nielsen, Nielsen and Verrall (2011) and Martínez-Miranda, Nielsen and Verrall (2012a) have analyzed the claims generating process and used this to understand, visualize and estimate the underlying components implicit in the classical multiplicative chain ladder structure. One of the basic requirements of the approach taken in these papers is that there are two triangles of data available: a triangle of paid data together with a corresponding triangle of the number of reported claims. By using these two sets of information, it is possible to gain a much deeper understanding of the fundamental drivers of the claims development than is possible with the basic chain ladder technique. The paper Martínez-Miranda et al. (2012a) was divided into two parts. One was concerned with predicting the best estimate of the reserve only, or the mean of the outstanding claims only, and the other part considering the distribution. It turned out the framework of double chain ladder works under very general conditions when only the mean is predicted. In the second part of Martínez-Miranda et al. (2012a) more specific assumptions were given to access the distributional properties of the underlying model. For example, when considering the best estimate only, the model of Martínez-Miranda et al. (2012a) works under a wide array of stochastic assumptions on the nature and dependency structure of payments. There can, for example, be multiple payments on each claim with complicated correlation patterns. When analysing the stochastic nature of the simplest possible version of double chain ladder, Martínez-Miranda et al. (2012a) made a number of simplifying assumptions including one payment per claim and constant mean severity of claims. These additional assumptions are needed to understand the full predicted distribution, but they are not needed to understand the mean. In this paper we add insight to this discussion. We show that if prior knowledge was available about the future number of zero claims and future severity inflation (depending on payment development delay), then while this information does not change the best estimates, it does affect the predicted distribution of outstanding claims. Therefore, if the issue is to qualify or improve best estimates, prior knowledge of zero claims and development year severity inflation is not important. If the focus is the best estimate of outstanding claims, then one should (for example) consider underwriting year severity inflation as in Martínez-Miranda et al. (2012b) or adjusting the calendar effect as in Kuang, Nielsen and Nielsen (2008a, 2008b, 2011) and Jessen and Rietdorf (2011).

In this paper we show that prior knowledge on the nature of future zero claims, see also Erhardt and Czado (2012), and on future severity development inflation are surprisingly simple to include into a double chain ladder framework. We also show how such information can be extracted from data when one extra triangle is available on the number of payments. Our approach is different, but related to that taken by Martínez-Miranda et al. (2012c) which uses the general Poisson cluster approach of Jessen, Mikosch and Samorodnitsky (2011). That paper is based on the same type of data as in this paper, in the sense that it considers the two triangles used in double chain ladder and combines these with the third triangle of the number of payments. In this paper we model the extra information via a prior knowledge approach, while Martínez-Miranda et al. (2012c) goes through the full mathematical statistical modelling of the entire system behind the three triangles. It could argued that modelling the entire system over-complicates the approach, since the added knowledge does not change the best estimate of the reserve, but only makes a correction to the distributional properties.

We believe that it is essential to consider all available prior knowledge when exploring the underlying characteristics and not just rely on inference and projection based on a single triangle of aggregated data. The issue may not be that the predicted values from the basic chain ladder technique are inappropriate. However, this basic method may be too limited to address the challenges of setting reserves and assessing risk, when other information is available. This reflects the fact that many actuaries make adjustments to the parameters of the chain ladder technique before setting reserves. The difficulties become much more acute when considering issues such as the distribution around the chain ladder prediction or when different assumptions about the future evolution of claims need to be considered. This paper addresses this latter issue directly, and illustrates how external information could be used more precisely since the parameters in the model now relate directly to real quantities. This is in contrast to the parameters of the basic chain ladder technique (and other similar approaches) where the parameters can be affected by a range of different factors. Specifically in this paper, we show how to include external information about the relationship between the mean claim severity and the development year and also the proportion of claims which are settled without payment (known as "zero claims").

The methods are applied to a real set of data, which consists of triangles each with 14 rows and columns corresponding to incremental yearly-aggregated run-off triangles. The first two triangles (in tables 8 and 9) contain the information required by, for example, the double chain ladder method of Martínez-Miranda et al. (2012a). The third triangle (table 10) contains the extra information required to estimate the number of zero claims.

The paper is set out as follows. Section 2 contains the assumptions at the level of individual claims and summarizes the model for aggregated paid claims. Section 3 contains a description of the intuitively appealing and simple estimation method known as double chain ladder. Section 4 describes how prior knowledge can be incorporated into this framework. Section 5 gives an outline of how bootstrapping can be used to derive estimates of predictive distributions in this context. Section 6 makes some suggestions about the sources that could be used for the prior knowledge. Note that it is possible to use other sources of external information to formulate the prior knowledge. Finally section 7 provides some concluding remarks.

2 Model formulation

This section sets out the model assumptions, which can be considered as a strategic extension of the model assumptions of the second part of the double chain ladder paper Martínez-Miranda et al. (2012a). If we were just interested in the mean or the best estimate, the model assumptions could be much more general than those below. However, since we are interested in the distributional properties, we generalize below the original assumptions of the second part of the double chain ladder so that the added prior knowledge available allows us to identify the model parameters which enter. We assume, without loss of generality, that the data are available in triangular form. We denote this by $\mathcal{I}_m = \{(i,j) : i = 1,\dots,m, j = 0,\dots,m-1;\ i+j \leq m\}$, with i denoting the accident or underwriting year, j the development year and m the last observed accident year. We consider the following stochastic components for all (i,j), both observed and future data. Thus, both here and in the assumptions below, we consider $\mathcal{I} = \{(i,j) : i = 1,\dots,m, j = 0,1,\dots\}$.

Number of reported claims. Let N_{ij} denote the total number of claims with accident year i which are reported in year $i+j$ (i.e. reporting delay of j years). Note that each of these N_{ij} reported claims is assumed to generate a number of payments i.e. a claims payment cash flow.

Number of payments. Let N_{ijl}^{paid} denote the number of claim payments originating from the N_{ij} reported claims, which are paid with a payment delay of l years, with $l = 0,\dots,m-1$.

Individual severity claims. Let $Y_{ijl}^{(k)}$ denote the individual settled payments which arise from N_{ijl}^{paid} $(k = 1,\dots,N_{ijl}^{paid}$, $(i,j) \in \mathcal{I}, l = 0,\dots,m-1)$.

It is often the case that individual claims payment data are not available at this level of detail and it is therefore important to consider models for more aggregated data. Note that the models for the aggregated data are built using assumptions at the level of individual claims, and thereby enable us to consider quantities which have a real interpretation. Hence, we define the following aggregated claims payment information:

Total payments in accounting year $i+j$ generated by all claims which were incurred in year i,

$$X_{ij} = \sum_{l=0}^{j} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}. \tag{1}$$

These are usually presented in the form of a run-off triangle, which we denote by $\Delta_m = \{X_{ij} : (i,j) \in \mathcal{I}_m\}$. A triangle of the number of reported claims denoted by $\aleph_m = \{N_{ij} : (i,j) \in \mathcal{I}_m\}$.

Thus, it is assumed that a triangle of payments, X_{ij} , and a triangle of reported numbers of claims, N_{ij} , are available. We make the following assumptions about these data.

D1. The numbers of reported claims, N_{ij}, are independent random variables for all (i, j) and have a Poisson distribution with cross-classified mean $\mathrm{E}[N_{ij}] = \alpha_i \beta_j$ and identification (Mack 1991), $\sum_{j=0}^{m-1} \beta_j = 1$.

D2. Given N_{ij}, the numbers of paid claims follow a multinomial distribution, so that the random vector $(N_{i,j,0}^{paid}, \ldots, N_{i,j,m-1}^{paid}) \sim \mathrm{Multi}(N_{ij}; p_0, \ldots, p_{m-1})$, for each (i, j), where $m-1$ is the assumed maximum delay. Let $(p_0, \ldots, p_{m-1})$ denote the delay probabilities such that $\sum_{l=0}^{m-1} p_l = 1$ and $0 \leq p_l \leq 1, \forall l = 0, \ldots, m-1$.

D3. The individual payments $Y_{i,j-l,l}^{(k)}$ are independent and have a mixed type distribution with Q_i being the probability of a "zero-claim" i.e. $\mathrm{P}\left\{Y_{i,j-l,l}^{(k)} = 0\right\} = Q_i$. We assume that $Y_{i,j-l,l}^{(k)} | Y_{i,j-l,l}^{(k)} > 0$ has a distribution with conditional mean μ_{ij} and conditional variance σ_{ij}^2, for each $i = 1, \ldots, m, j = 0, \ldots, m-1$. We also assume that the mean depends on the accident year and payment year such that $\mu_{ij} = \mu \gamma_i \delta_j$. Here, μ a common mean factor and δ_j and γ_i can be interpreted as being the inflation in the payment year and the accident year, respectively. The variance follows a similar structure, with $\sigma_{ij}^2 = \sigma^2 \gamma_i^2 \delta_j^2$, where σ^2 is a common variance factor.

D4. *Independence*: We assume that settled payments, $Y_{ijl}^{(k)}$ are independent of the numbers of reported claims, N_{ij}.

This is a more general situation than Martínez-Miranda et al. (2012a) since it assumes that the distribution depends on the accident year and the development year and also allows for zero-claims. Under these assumptions, the first two moments of the unconditional distribution of $Y_{i,j-l,l}^{(k)}$ are given by:

$$\mathrm{E}[Y_{i,j-l,l}^{(k)}] = \gamma_i \delta_j (1 - Q_i) \mu \tag{2}$$

$$\mathrm{V}(Y_{i,j-l,l}^{(k)}) = \gamma_i^2 \delta_j^2 (1 - Q_i) \left(\sigma^2 + Q_i \mu^2\right) \tag{3}$$

Following the similar calculations as Martínez-Miranda et al. (2012a), it can be shown that under the above assumptions the unconditional mean of X_{ij} can be written as

$$\mathrm{E}[X_{ij}] = \gamma_i (1 - Q_i) \mu \alpha_i \delta_j \sum_{l=0}^{j} \beta_{j-l} p_l = \widetilde{\alpha}_i \widetilde{\beta}_j, \tag{4}$$

where

$$\widetilde{\alpha}_i = \gamma_i(1 - Q_i)\mu\alpha_i$$

and

$$\widetilde{\beta}_j = \delta_j \sum_{l=0}^{j} \beta_{j-l} p_l.$$

Note that when Q_i is identical zero and $\delta_j = 1$ for all $j = 0, \ldots, m-1$, the situation reverts back to the double chain ladder model of Martínez-Miranda et al. (2012a).

3 The Double Chain Ladder method.

The double chain ladder (DCL) estimation method was proposed by Martínez-Miranda et al. (2012a) to provide simple and intuitive estimators for the parameters $\{p_l, \mu, \sigma^2, \gamma_i : i = 1, \ldots, m; l = 0, \ldots, m-1\}$. Below we quickly go through this double chain ladder approach that is a special case of the approach suggested in this paper. The assumptions in Martínez-Miranda et al. (2012a) are identical to D1-D4 except that it is assumed that $Q_i = 0$ and $\delta_j = 1$, for all $i = 1, \ldots, m; j = 0, \ldots, m-1$. It is therefore assumed that the individual payments $Y^{(k)}_{i,j-l,l}$ have means $\mu_{ij} \equiv \mu_i = \gamma_i \mu$ and variances $\sigma^2_{ij} \equiv \sigma^2_i = \gamma^2_i \sigma^2$ for all $i = 1, \ldots, m$ and $j = 0, \ldots, m-1$. In this section, we briefly summarize the key steps in the DCL method.

The DCL estimation method applies the chain ladder algorithm twice, using the data in the two run-off triangles $(\aleph_m, \Delta_m)$. As the same method is repeated on each triangle, we illustrate it just for the triangle of the number of reported claims $\aleph_m$ and the parameters α_i and β_j. A distribution-free approach is used and hence we use the method of moments to obtain the estimators. Aggregating over the rows and columns, we obtain the first moment equalities

$$\begin{aligned}
\sum_{k=0}^{m-i} \mathrm{E}[N_{ik}] &= \alpha_i \sum_{k=0}^{m-i} \beta_k \quad \text{for } i = 1, \ldots, m, \\
\sum_{k=0}^{m-j} \mathrm{E}[N_{kj}] &= \beta_j \sum_{k=1}^{m-j} \alpha_k \quad \text{for } j = 0, \ldots, m-1.
\end{aligned}$$

Unbiased estimators for the parameters on the right-hand side of these equalities can be obtained by replacing the moments $\mathrm{E}[N_{ij}]$ by their observed values N_{ij} for $(i,j) \in \mathcal{I}_m$. Then the resulting system of linear equations can be solved for α_i

and β_j which provides the corresponding estimators for these parameters. This is the spirit of the "total marginals" method of Bailey (1963) and Jung (1968). Kremer (1985) and Mack (1991) have shown that in the case of triangular data $\aleph_m$ this leads to the chain ladder estimators that can easily calculated. Thus $\aleph_m$ provides the chain ladder estimators $\widehat{\alpha}_i$ and $\widehat{\beta}_j$ for α_i and β_j, respectively; and Δ_m provides the chain ladder estimators $\widehat{\widetilde{\alpha}}_i$ and $\widehat{\widetilde{\beta}}_j$ for $\widetilde{\alpha}_i$ and $\widetilde{\beta}_j$, respectively. Once these parameter estimates have been calculated, estimates of $\{p_0, \ldots, p_{m-1}\}$ can be obtained by solving and afterwards adjusting the solution of the linear system

$$\widetilde{\beta}_j = \sum_{l=0}^{j} \beta_{j-l} p_l \quad \text{for } j = 0, \ldots, m-1. \tag{5}$$

We denote by $\{\widehat{\pi}_0, \ldots, \widehat{\pi}_{m-1}\}$ the solution of the above system. Since the solution has been derived with no restrictions, in order to provide suitable estimates $\{\widehat{p}_0, \ldots, \widehat{p}_{m-1}\}$ for the probability delay parameters in the model (D2), which satisfy that $0 \leq \widehat{p}_l \leq 1$ for all $l = 0, \ldots, m-1$ and $\sum_{l=0}^{m-1} \widehat{p}_l = 1$, the initial general estimates $\widehat{\pi}_l$ have to be adjusted. Such an adjustment can be done in different ways but note that a suitable adjustment should not alter substantially the RBNS delay described by the general estimates $\{\widehat{\pi}_0, \ldots, \widehat{\pi}_{m-1}\}$. As was proved in Martínez-Miranda et al. (2012a), if we used the general estimates, $\widehat{\pi}_0, \ldots, \widehat{\pi}_{m-1}$, we could obtain exactly the same estimate of the mean of future payments as the standard chain ladder technique would give. However, the estimated probabilities $\{\widehat{p}_0, \ldots, \widehat{p}_{m-1}\}$ yield a slightly different estimated mean and therefore predicted reserve. The effect of using general and adjusted delay parameters, and also how to carry out the adjustments, will be illustrated in the next sections.

The mean of the distribution of individual payments, including the parameters which measure the inflation in the accident years, can be obtained using

$$\widehat{\gamma}_i = \frac{\widehat{\widetilde{\alpha}}_i}{\widehat{\alpha}_i \mu} \quad i = 1, \ldots, m, \tag{6}$$

and

$$\widehat{\mu} = \frac{\widehat{\widetilde{\alpha}}_1}{\widehat{\alpha}_1}, \tag{7}$$

where to ensure identifiability γ_1 is set to one. It only remains to correct the final $\widehat{\mu}$ according to the estimates $\widehat{p}_l$ and in order to ensure Mack's identification. This is done by replacing the estimate $\widehat{\mu}$ by the corrected $\widehat{\mu}/\kappa$, with $\kappa = \sum_{j=0}^{m-1} \sum_{l=0}^{j} \widehat{\beta}_{j-l} \widehat{p}_l$. Hereafter, in a slight abuse of notation, we will denote by $\widehat{\mu}$ the corrected estimator of μ which is in agreement with the estimated probabilities $\widehat{p}_l$ $(l = 0, \ldots, m-1)$ in the model.

The estimate of outstanding claims is obtained by substituting in the above estimates into the expression for the unconditional mean. In doing this, it is useful to split it into the Reported But Not Settled (RBNS) and Incurred But Not Reported (IBNR) components by considering payments on already reported claims and claims which will be reported in the future. For $i + j > m$, we define

$$\widehat{X}_{ij}^{rbns} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\widehat{p}_l\widehat{\mu}\widehat{\gamma}_i \tag{8}$$

and

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=\max(0,j-m+1)}^{i-m+j-1} \widehat{N}_{i,j-l}\widehat{p}_l\widehat{\mu}\widehat{\gamma}_i, \tag{9}$$

respectively, where $\widehat{N}_{ij} = \widehat{\alpha}_i\widehat{\beta}_j$.

The estimate of total outstanding claims is calculated by adding the RBNS and IBNR components i.e. $\widehat{X}_{ij}^{DCL} = \widehat{X}_{ij}^{rbns} + \widehat{X}_{ij}^{ibnr}$. This is equivalent to the aim of the standard chain ladder in just the lower triangle (ignoring any tail effects), i.e. for $(i, j) \in \mathcal{J}_1 = \{i = 2, \ldots, m; j = 0, \ldots, m-1 \text{ so } i + j = m + 1, \ldots, 2m - 1\}$. For the DCL, the predictions can spread out to provide the tail by considering $i = 1, \ldots, m$ and $j = m, \ldots, 2m - 1$.

Finally to provide the full cash flow the predictive distribution can be approximated using parametric bootstrap methods as Martínez-Miranda et al. (2011) described. In order to do this, it is necessary to estimate the variances, σ_i^2 $(i = 1, \ldots, m)$. Verrall et al. (2010) showed that assumptions similar to D1–D4 can be used to show that the conditional variance of X_{ij} is approximately proportional to its mean. Using this result, it is straightforward to estimate the variance using over-dispersed Poisson distributions. More specifically, the over-dispersion parameter φ can be estimated by

$$\widehat{\varphi} = \frac{1}{n-m} \sum_{i,j \in \mathcal{I}_m} \frac{(X_{ij} - \widehat{X}_{ij}^{DCL})^2}{\widehat{X}_{ij}^{DCL}\widehat{\gamma}_i},$$

with $n = m(m + 1)/2$ and $\widehat{X}_{ij}^{DCL} = \sum_{l=0}^{j} \widehat{N}_{i,j-l}\widehat{p}_l\widehat{\mu}\widehat{\gamma}_i$. And therefore the variance estimators are defined by

$$\widehat{\sigma}_i^2 = \widehat{\sigma}^2\widehat{\gamma}_i^2$$

for each $i = 1, \ldots, m$, where $\widehat{\sigma}^2 = \widehat{\mu}\widehat{\varphi} - \widehat{\mu}^2$.

We now provide an illustration of the DCL method considering the dataset of dimension $m = 14$ shown in tables 8, 9 and 10. These triangles consist of yearly aggregated data of number of reported claims, payments and number of payments, respectively. We have assumed a maximum delay of 13 years and provided point forecasts for the reserves from the expression (4), with $Q_i = 0$ and $\delta_j = 1$ ($i = 1, \ldots, m; j = 0, \ldots, m-1$). We have considered two variations when calculating predictions. First, we use the estimated delay parameters $\widehat{\pi}_l$ resulting from solving (5) without any adjustments, which provides exactly the classical chain ladder reserve (ignoring the tail). And second, we calculate a slightly modified reserve by using the adjusted delay probabilities $\widehat{p}_l$. Figure 1 shows both versions of the delay parameters (general and adjusted parameters) in the top panel. Also the lower panel shows the estimated DCL inflation parameters in the underwriting year direction using expression (6). In this example the estimates of the mean and variance components of the individual payments are $\widehat{\mu} = 824.456$ and $\sigma^2 = 97130427$. The point forecasts in the lower triangle (where the standard chain ladder technique would provide estimates) can be separated into the RBNS and the IBNR reserve using the expressions (8) and (9). The resulting forecasts are shown in table 1 together with the standard chain ladder results for comparison. Note that, as mentioned above, the DCL method allows us to separate out the RBNS and IBNR components but still provides the same chain ladder mean in the lower triangle. Note that when we consider an adjustment of the delay parameters as shown in figure 1, the mean remains almost the same but with a slight deviation mainly due to rounding errors. The adjustment considered in this case was the simple procedure suggested in Martínez-Miranda et al. (2012a), which is defined as follows. First count the number $d+1 \leq m-1$ of successive $\widehat{\pi}_l \geq 0$ such that $\sum_{l=0}^{d-1} \widehat{\pi}_l < 1 \leq \sum_{l=0}^{d} \widehat{\pi}_l$. Then the estimated delay probabilities are defined as $\widehat{p}_l = \widehat{\pi}_l$, $l = 0, \ldots, d-1$, $\widehat{p}_d = 1 - \sum_{l=0}^{d-1} \widehat{p}_l$ and $\widehat{p}_{d+1} = \cdots = \widehat{p}_{m-1} = 0$. Note that other adjustments can be done as for example those suggested in the close (but more complex) model of Martínez-Miranda et al. (2012c). However such adjustment should be chosen carefully in order to not alter the original pattern of the general delay parameters, $\{\widehat{\pi}_0, \ldots, \widehat{\pi}_{m-1}\}$. To find an approximation to the delay function that is a multinomial distribution is sometimes a non-trivial exercise, as it was with this data set. We have not found a general approximation method that always work. The approximation chosen might depend on the situation and on the specific application of the model. In this case we can assess the suitability of this adjustment first from figure 1 in that both general and adjusted delay parameters almost coincide for all the years. Also table 1 illustrates that the point predictions from adjusted delay probabilities and general parameters are almost the same for each calendar year.

[Table 1 should be here]

[Figure 1 should be here]

4 Incorporating prior knowledge into Double Chain Ladder

In this section, we take the DCL method as set out in section 3 and consider how to incorporate prior knowledge about the severity of individual claims and on the number of zero claims. The first two subsections deal with each of these separately, and the final subsection considers how to do them both together.

4.1 Incorporating prior knowledge on claims development inflation

In this subsection, we first consider the case where the prior information of Q_i is that it is identically equal to zero for all $i = 1, \dots, m$, but our prior knowledge on the δ_j's is unrestricted. It turns out to be surprisingly simple to include this type of prior knowledge in the double chain ladder framework. Observed payments are divided by the prior severity inflation, double chain ladder is then carried out on these adjusted payments and in the final step we multiply back in the prior severity inflation. This is indeed very simple, both when it comes to computations and intuitive understanding. It is illustrative to compare this simple approach to including severity inflation to the more complicated and complex approach taken in Martínez-Miranda et al. (2012c) where the modelling complexity increases exponentially with the added information. Considering the same problem in our way as just adding prior knowledge to double chain ladder simplifies these complicated issues for the practical actuary making it more easy for the practitioner to understand what is going on and to manipulate the model. Let $\widetilde{X}_{ij} = X_{ij}/\delta_j$. It is easy to verify that the triangle $\{\widetilde{X}_{ij}; (i,j) \in \mathcal{I}_m\}$ together with the counts triangle $\aleph_m$ follow model assumptions D1-D4 in section 2 with Q_i identical zero and δ_j identical one ($i = 1, \dots, m; j = 0, \dots, m-1$). Therefore, the DCL method can be applied to $\widetilde{X}_{ij}$. Let $\widetilde{X}_{ij}^{DCL}$ be the predicted value of $\widetilde{X}_{ij}$ by using the DCL method. Then the predicted value of X_{ij} including the prior information will be given by $\widetilde{X}_{ij}^{DCLP}$ $=\delta_j \widetilde{X}_{ij}^{DCL}$, for$(i,j) \in \mathcal{J}_1$. In this way it is possible to generate the distribution of future values incorporating the prior information.

To illustrate this approach we calculate again the predictions in table 1 but using as prior development inflation shown in figure 2 (bottom panel). The results are shown in table 3 considering general delay parameters and also adjusted probabilities. The delay parameters from this approximation are shown in the top panel of figure 2 and reported in table 2. In this case we have considered an adjustment of the delay parameters different from that described in section 3. The reason is because of the special pattern of delay parameter estimates obtained by solving the linear system (5), which is shown in table 2. Since some negatives values arise in the general estimates in the first years the simple adjustment used in the previous section seems to be inadequate. In fact, if we consider that method the delay pattern will be modified dramatically providing wrong point forecasts. For this kind of pattern we suggest the following alternative adjustment of the delay parameters $\widehat{\pi}_l$. First we define $\widehat{\pi}'_0, \ldots, \widehat{\pi}'_{m-1}$ being the same as $\widehat{\pi}_0, \ldots, \widehat{\pi}_{m-1}$ but replacing the negatives values by zeros. And second we increase (or decrease) the strictly positive values $\widehat{\pi}'_l$ by calculating the adjusted parameters, $\widehat{p}_l = \widehat{\pi}'_l + (1-\tau)\widehat{\pi}'_l/\tau$, where $\tau = \sum_{l=0}^{m-1} \widehat{\pi}'_l$. With this adjustment we can assess in table 3 that the point predictions are very close when calculated with both general parameters and adjusted probabilities for all the calendar years.

In table 3 we can see that when considering prior information about the development inflation the mean of the total reserves (RBNS+IBNR) is almost unaltered compared with the mean predictions from DCL without any prior (table 1). The prior knowledge only provides a slight reduction in the overall total from the value 13352 (given by DCL with no prior) to the value 13322. However the split between RBNS and IBNR claims is indeed altered. Note that the overall total of RBNS claims when the prior knowledge is ignored is 11751, which is reduced to 9630 when the development inflation information is incorporated. This reduction is therefore compensated with an increase in the IBNR reserve from 1601 to 3692. These numbers correspond to the case of using the general delay parameters, but a similar effect can be observed in the predictions calculated with adjusted delay probabilities.

[Table 2 should be here]

[Table 3 should be here]

[Figure 2 should be here]

4.2 Incorporating prior knowledge on the number of zero claims

In this subsection, we consider the case where we have prior knowledge on the future number of zero claims. While this gives us more information - or rather one extra freely varying parameter - to handle the predicted distribution, it does not affect the best estimate of the reserves. It takes a little more effort to include this type of prior knowledge into the double chain ladder framework than it took to include development severity inflation, but it is still quite straightforward and computationally tractable. We first consider the case where the prior information of δ_j is that it is equal to one for all $j = 0, \ldots, m-1$, while the Q_i's $(i = 1, \ldots, m)$ are unrestricted between zero and one. In this case, there are two adjustments to the double chain ladder method. First note that the conditional variance is approximated by:

$$
\begin{aligned}
\mathrm{V}[X_{ij}|\aleph_m] &= \gamma_i^2\delta_j^2(1-Q_i)(\sigma^2+Q_i\mu^2)\sum_{l=0}^{j} N_{i,j-l}p_l \\
&\quad +\gamma_i^2\delta_j^2(1-Q_i)^2\mu^2\sum_{l=0}^{j} N_{i,j-l}p_l(1-p_l) \\
&\approx \gamma_i^2\delta_j^2(1-Q_i)(\sigma^2+\mu^2)\sum_{l=0}^{j} N_{i,j-l}p_l \\
&= \gamma_i\delta_j\frac{\sigma^2+\mu^2}{\mu}\mathrm{E}[X_{ij}|\aleph_m] \\
&= \varphi_{ij}\mathrm{E}[X_{ij}|\aleph_m].
\end{aligned}
$$

where $\varphi_{ij} = \gamma_i\delta_j\varphi$ and $\varphi = \frac{\sigma^2+\mu^2}{\mu}$. This means that an over-dispersed Poisson model can be used to approximate the parameters, as in Martinez-Miranda et al. (2012a). In order to consider the sensitivity of this approximation with respect to the values of Q_i and p_j, we have carried out the following exercise. Firstly we consider the estimated values calculated in Section 6 from the data example. Using these estimates we evaluate and compare (by taking the ratio) the actual expression of $\mathrm{V}[X_{ij}|\aleph_m]$, with the approximation given by the term $\gamma_i^2\delta_j^2(1-Q_i)(\sigma^2+\mu^2)\sum_{l=0}^{j} N_{i,j-l}p_l$. The summary of the resulting ratios reveals values very close to 1. In fact, these ratios vary between 0.9960 (minimum) and 0.9992 (maximum). Secondly we replace the parameters Q_i and p_j by random values and evaluate again the ratios. For Q_i generated form a Uniform distribution between 0.2 and 0.8 and p_j generated from a Uniform distribution between 0 and 1 (rescaled to be a probability vector) the summary still gives values very close to 1. This simple exercise gives us confidence about the approximation used. From the arguments above the only difference from estimating the parameters in this model and the DCL model is that we have to adjust the estimated row parameters with the known Q_i's.

Using the information from the zero claims contained in the data as shown in the lower panel in figure 4 we can again derive the point forecasts using both the general delay parameters and the adjusted delay probabilities. Here we have considered the same adjustment described in subsection 4.1. The resulting predictions are shown in table 4. Note that now we have a decomposition of the inflation in the underwriting year direction which is shown in the top panel of figure 4. Specifically, the DCL inflation from original payments X_{ij} is equal to $\gamma_i^{DCL}(1-Q_i)/(1-Q_1)$ where $(1-Q_i)/(1-Q_1)$ is the zero-claims effect, and γ_i^{DCL} is the DCL inflation from the triangle removing the zero claims effect namely $\widetilde{X}_{ij} = X_{ij}/(1-Q_i)$. The middle panel in this figure shows the zero-claims effect and top panel compares the estimates of the inflation adjusted by the zero-claims, γ_i^{DCL}, with the DCL inflation in the underwriting year.

As happened when introducing prior knowledge about development inflation in subsection 4.1, we can see in table 4 that the information about zero claims has almost no impact on the point predictions for the total reserves, which remain very close to those from DCL without any prior (table 1). Note that the new total reserve is 13344 compared with the value 13352 given by DCL with no prior. Also, and opposite to the case of considering prior knowledge about the severity inflation, the zero-claims knowledge does not alter the split between RBNS and IBNR claims, which remains almost the same as in table 1.

[Table 4 should be here]

4.3 Incorporating prior knowledge on both the severity development inflation and zero claims

In this section, we show that the above approaches we can combined and information on both prior information on severity inflation and future number of zero claims included. If both the probabilities Q_i's $(i = 1, \ldots, m)$ are different from zero and the severity inflation parameters δ_j's $(i = 0, \ldots, m-1)$ are different from one then we can estimate this broader model combining the procedures of subsections 4.1 and 4.2. First we adjust for δ_j's values which then gives the situation of subsection 4.2.

Using this procedure we can calculate the predictions for the data set presented in previous sections incorporating the prior knowledge about the proportion of non-zero claims plotted in figure 4 and the severity development inflation shown in the bottom panel of figure 2. Again we calculate the predictions and the split between RBNS and IBNR considering general delay parameters $\widehat{\pi}_l$, which provides exactly the classical chain ladder reserve, and also the adjusted delay probabilities $\widehat{p}_l$ $(l = 0, \ldots, m-1)$. The results are shown in table 5. The adjusted probabilities have been calculated using the same method described in subsection 4.1. Note that once again we assess that the prior knowledge introduced does not alter the point predictions compared with those derived from DCL without any prior (table 1). Here the predicted total reserve is 13314 compared with the value 13352 given by DCL with no prior. Again the development inflation has a notable effect on the split between RBNS and IBNR as in subsection 4.1. Note that the reduction in the overall RBNS reserve and the increase in the IBNR reserve is quite remarkable but analogous to the provided when only the development inflation knowledge is considered.

[Table 5 should be here]

5 Bootstrap methods

In this section we outline how the bootstrap methods described by Martínez-Miranda et al. (2011) and Martínez-Miranda et al. (2012a), can be used to provide the predictive distribution of the reserve. In doing this, we use the prior knowledge about development year inflation and/or zero-claims in a similar way as the section 4. In other words, we first adjust the payments triangle by removing the effect of the prior knowledge and we then apply a parametric bootstrap from the DCL distributional model to simulate the RBNS and IBNR distributions. Finally, we replace the inflation effects which were removed. In the DCL framework, Martínez-Miranda et al. (2012a) used a parametric bootstrap method to describe the possible fluctuations of the true outstanding loss liability cash flows. By exploiting the distributional assumptions in the DCL model (see assumptions in Section 5 in Martínez-Miranda et al. 2012a) two different resampling schemes can be defined to simulate separately the predictive distribution of the RBNS and IBNR cash flows, using Monte Carlo methods. For completeness, we provide more details and an explicit algorithm in Appendix A.

For each situation and type of prior knowledge resampling schemes can be applied as described by Martínez-Miranda et al. (2011). There are two alternative methods, the first of which ignores the uncertainty of the parameters $\{p_l, \mu, \sigma^2, \gamma_i : i = 1, \ldots, m; l = 0, \ldots, m-1\}$ estimated from the input data $(\aleph_m, \Delta_m)$. The second incorporates the uncertainty of these parameters. When the severity inflation and the probability of zero claims are also estimated from data a further extension can be defined which takes also into account the uncertainty of these parameters. In this paper we assume as prior knowledge the severity inflation in the development year plotted in figure 2 and/or the zero claims effect which is assumed to be as was plotted in figure 4.

In the previous section, it was observed that the prior knowledge does not alter the point predictions in the total reserves. In fact only slight deviations from the predictions by the DCL method with no prior information were observed. Only the knowledge about the development inflation has a noticeable effect on the split between RBNS and IBNR claims. The question now is whether the prior knowledge modifies the predictive distribution. A summary of the distribution for the total outstanding claims is shown in table 6. The same table also shows the DCL bootstrap distribution with and without the prior information about development inflation and/or zero-claims. The derived cash flows are compared with the results from the bootstrap method of England and Verrall (2002) for the CLM as implemented in the package ChainLadder in R (Gesmann, Murphy and Zhang 2012). "Prior A" denotes when only severity inflation is considered, "Prior B" when considering only zero-claims and "Prior C" when considering both severity inflation and zero-claims. Also "Boot I" and "Boot II" denote the bootstrapping ignoring and taking into account the uncertainty of the parameters, respectively. The distribution with and without prior knowledge is indeed altered especially when prior knowledge about development inflation is incorporated. From the upper quantiles reported in table 6 it can be seen that the prior knowledge on severity inflation provides a longer tailed distribution. This effect can be visualized more clearly by plotting the bootstrap distribution as in figure 3. This shows histograms of the predictive distribution of the total reserve as well as the split between RBNS and IBNR claims with and without prior knowledge. From visual inspection of these histograms we can confirm that the introduced prior knowledge on development inflation induces a longer tailed distribution but also it alters the split between RBNS and IBNR claims. On the other hand, prior knowledge about zero claims has almost no influence in the distribution. Finally note that these plots correspond to the bootstrap method which does not take into account the uncertainty of the parameters (labelled as "Boot I"). When we take into account the uncertainty of the parameters as "Boot II" does, the shape and the main properties of the distribution remain the same but with a wider range.

[Table 6 should be here]

[Figure 3 should be here]

6 An example showing how other data can be used to provide prior information in practice

The methods described above assume some extra information is available. It has been shown how the DCL method can be easily applied with simple adjustments to allow for prior information about development inflation and zero claims. Here we show how this prior information could be easily obtained by observing a new run-off triangle. Specifically we observe the total number of non-zero payments in accounting year $i+j$ from claims with accident year i and denote this by R_{ij}. The corresponding triangle is denoted by $\mathcal{R}_m = \{R_{ij} : (i,j) \in \mathcal{I}\}$. Note that R_{ij} is the number of claims from the $\sum_{l=0}^{j} N_{i,j-l,l}^{paid}$ which yields non-zero payments. Also each cell in the new triangle can be decomposed into delay-dependent components, $R_{ij} = \sum_{l=0}^{j} R_{i,j-l,l}$, with R_{ijl} being the number of non-zero payments from the N_{ij} reported which were paid with l periods delay. In the next subsection we will prove that the variables R_{ij} have cross-classified (unconditional) mean $\mathrm{E}[R_{ij}] = \alpha_i^R \beta_j^R$ for all (i,j). Thus here we propose to use simultaneously the three triangles $(\aleph_m, \mathcal{R}_m, \Delta_m)$ to provide prior information about Q_i and δ_j. To do this, we apply the chain ladder algorithm three times:

$$\aleph_m \text{ provides the chain ladder estimators } \widehat{\alpha}_i \text{ and } \widehat{\beta}_j \text{ for } \alpha_i \text{ and } \beta_j,$$
$$\mathcal{R}_m \text{ provides the chain ladder estimators } \widehat{\alpha}_i^R \text{ and } \widehat{\beta}_j^R \text{ for } \alpha_i^R \text{ and } \beta_j^R,$$
$$\Delta_m \text{ provides the chain ladder estimators } \widehat{\widetilde{\alpha}}_i \text{ and } \widehat{\widetilde{\beta}}_j \text{ for } \widetilde{\alpha}_i \text{ and } \widetilde{\beta}_j.$$

From the chain ladder estimates $\{(\widehat{\alpha}_i, \widehat{\beta}_j), (\widehat{\alpha}_i^R, \widehat{\beta}_j^R), (\widehat{\widetilde{\alpha}}_i, \widehat{\widetilde{\beta}}_j) : i = 1, \ldots, m, j = 0, \ldots, m-1\}$, we describe in the following how the DCL estimation method can be applied twice to provide the required prior information.

6.1 Estimation of the zero-claims probability

Using the above definitions, the required information about the probability of zero-claims, Q_i, can be extracted using the triangles $\aleph_m$ and $\mathcal{R}_m$. Note first that using the assumed independence in D4 between the severity and the IBNR delay, we can calculate the first moment of each variable R_{ij}. This gives the conditional mean

$$\mathrm{E}[R_{ij}|\aleph_m] = \sum_{l=0}^{j} N_{i,j-l}(1-Q_i)p_l$$

and the unconditional mean

$$\mathrm{E}[R_{ij}] = \alpha_i(1-Q_i)\sum_{l=0}^{j} \beta_{j-l}p_l := \alpha_i^R\beta_j^R. \tag{10}$$

Thus, the pair of triangles $(\aleph_m, \mathcal{R}_m)$ follows the model described by Martínez-Miranda et al. (2012a) and therefore the DCL method can be applied to these triangles in order to estimate the target parameters Q_i $(i = 1, \ldots, m)$. Specifically from the chain ladder estimates, $\widehat{\alpha}_i$ and $\widehat{\alpha}_i^R$, of the underwriting year parameters, α_i and α_i^R $(i = 1, \ldots, m)$, respectively, the probability of zero-claims in the underwriting year can be estimated from the expression

$$\widehat{Q}_i = 1 - \frac{\widehat{\alpha}_i^R}{\widehat{\alpha}_i}. \tag{11}$$

Using the data in tables 8 and 10 the zero-claims probabilities are estimated by the values plotted in the bottom panel of figure 4.

6.2 Estimation of the severity development inflation

Now we consider the situation defined by assumption D3 where the severity depends on the underwriting year but also on the development year. Specifically, we assume that it has a development inflation component δ_j which is not considered in the DCL model of Martínez-Miranda et al. (2012a). In this case the usual input of the DCL method, namely the paid and incurred counts triangles $(\aleph_m, \Delta_m)$ are not enough to solve the over-parametrization problem of the chain ladder mean described by Martínez-Miranda et al. (2012a). However it can be easily solved by considering the extra information provided by the triangle $\mathcal{R}_m$ introduced above. From the expression (10) for the unconditional mean of R_{ij} it can be seen that

$$\sum_{l=0}^{j} \beta_{j-l}\pi_l = \beta_j^R.$$

By substituting this into the expression for the unconditional mean of X_{ij} (with $Q_i = 0$ for all $i = 1, \ldots, m$), it can be seen that

$$\mathrm{E}[X_{ij}] = \gamma_i \mu \alpha_i \delta_j \sum_{l=0}^{j} \beta_{j-l} p_l = \gamma_i \mu \alpha_i \delta_j \beta_j^R = \widetilde{\alpha}_i \widetilde{\beta}_j.$$

Thus, the new inflation parameters can be estimated by

$$\widehat{\delta}_j = \frac{\widehat{\widetilde{\beta}}_j}{\sum_{l=0}^{j} \widehat{\beta}_{j-l} \widehat{\pi}_l} = \frac{\widehat{\widetilde{\beta}}_j}{\widehat{\beta}_j^R}. \tag{12}$$

Therefore the prior $\widehat{\delta}_j$ can be obtained just from the chain ladder estimates $\widehat{\widetilde{\beta}}_j$ and $\widehat{\beta}_j^R$ of the parameters $\widetilde{\beta}_j$ and β_j^R, respectively. But also the estimated delay parameters, $\{\widehat{p}_0, \ldots, \widehat{p}_{m-1}\}$, in the model (D2) can be estimated by a suitable adjustment of the solutions of the linear system

$$\beta_j^R = \sum_{l=0}^{j} \beta_{j-l} p_l \quad \text{for } j = 0, \ldots, m-1. \tag{13}$$

Note that since the parameters are only derived in the observation triangle $\mathcal{I}_m$, it is only possible to predict outstanding claims in the lower triangle $\mathcal{J}_1$. Hence, to extend the forecasts to provide the tail any suitable model should be fitted to such inflation parameters.

The estimated development inflation $\widehat{\delta}_j$ $(j = 0, \ldots, m-1)$, and also the general delay parameters derived by solving the linear system (13), for the data set presented in previous sections are shown in table 7. Also, the implied severity development inflation reported in this table has been plotted in bottom panel of figure 3. It can be seen that the estimated severity development inflation shows an increasing trend in the development year as was expected. However some slight deviations from this trend indicates that the mean of the payments could also depend on other directions such as the settlement delay or the calendar year.

[Table 7 should be here]

[Figure 4 should be here]

7 Conclusions

This paper has illustrated how prior knowledge of severity inflation and future zero claims can be included quite simply in the framework of double chain ladder. While this added knowledge does not significantly affect the predicted mean reserve, it does add to the understanding of the underlying distributional properties of the reserve. In the data study, the two effects have similar implications: the prior knowledge of zero claims make the final distribution more long-tailed and prior knowledge of severity claims does the same but also with a change in the split between RBNS and IBNR claims. Adding the two types of prior knowledge at the same time does not provide further effects to this long-tailness or separation between RBNS and IBNR claims. Other data sets might give other conclusions. Our final conclusion is that it is surprisingly easy to add complicated model structures of zero claims and severity inflation to double chain ladder. The double chain ladder model and its extensions considered in this paper gives a granular model of a single claim, even though the original data is aggregated; see also Antonio and Plat (2012). This is interesting and might have a number of applications beyond the simple distributional application of this paper.

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A Bootstrap algorithms

Consider the distributional model described in D1-D4 (Section 2) with development inflation $\delta_j, j = 0, \dots, m-1$ and probability of zero-claims $Q_i, i = 1, \dots, m$. Assume that δ_j and Q_i are known and denote by $\theta = \{p_l, \mu_{ij} = \gamma_i \delta_j \mu, \sigma^2_{ij} = \sigma^2 \gamma_i^2 \delta_j^2;\ l = 0, \dots, m-1, i = 1, \dots, m\}$ the set of parameters in the model. Consider the adjusted aggregated payments $\widetilde{\Delta}_m = \{\widetilde{X}_{ij} : (i,j) \in \mathcal{I}_m\}$, with $\widetilde{X}_{ij} = X_{ij}/[\delta_j(1 - Q_i)]$. Following arguments given in the paper, the triangles $(\aleph_m, \widetilde{\Delta}_m)$ follow the DCL model described in Martínez-Miranda et al. (2012a) with delay parameters p_l^{DCL}, underwriting year inflation γ_i^{DCL} and severity parameters $(\mu^{DCL}, \sigma^2_{DCL})$. Using the expressions summarized in section 3 these parameters can be estimated by $\widehat{p}_l, \widehat{\gamma}_i$, $\widehat{\mu}$ and $\widehat{\sigma}^2$. The desired parameter θ can be estimated by $\widehat{\theta} = \{\widehat{p}_l, \widehat{\mu}_{ij} = \widehat{\gamma}_i \delta_j \widehat{\mu}, \widehat{\sigma}^2_{ij} = \widehat{\sigma}_i^2 \widehat{\gamma}_i^2 \delta_j^2;\ l = 0, \dots, m-1, i = 1, \dots, m\}$, where $\widehat{\sigma}_i^2 = [(1 - Q_i)\widehat{\sigma}^2 - Q_i \widehat{\mu}]$.

The predictive distribution of the RBNS cash flow (taking into account the uncertainty of the unknown parameters) can be simulated by running the following algorithm:

Algorithm RBNS

Step 1. *Estimation of the parameters and distributions.* From the observed (adjusted) data $(\aleph_m, \widetilde{\Delta}_m)$ estimate the model parameters θ by the estimator $\widehat{\theta}$, as described above. The payment delay distribution is estimated by a Multinomial distribution with estimated parameter, i.e. $(N^{paid}_{i,j,0}, \dots, N^{paid}_{i,j,m-1}) \sim \text{Multi}(N_{ij}; \widehat{p}_0, \dots, \widehat{p}_{m-1})$, for each (i,j), where $m-1$ is the assumed maximum delay. The distribution of the non-zero individual payments $(Y^{(1)}_{i,j,l} > 0, l = 0, \dots, m-1)$ is estimated by a gamma distribution with mean $\mu_i = \widehat{\gamma}_i \widehat{\mu}$ and variance σ_i^2, this is, with shape parameter $\widehat{\lambda}_i = \widehat{\gamma}_i^2 \widehat{\mu}^2 / \widehat{\sigma}_i^2$ and scale parameter $\widehat{\kappa}_i = \widehat{\sigma}_i^2 / \widehat{\gamma}_i \widehat{\mu}$.

Step 2. *Bootstrapping the data.* Conditional on the observed number of reported claims $\aleph_m$ generate new bootstrapped triangles $\Delta^*_m = \{X^*_{ij}; (i,j) \in \mathcal{I}_{\Updownarrow}\}$ as follows:

(i) Simulate the payment delay: from each N_{ij}, $(i,j) \in \mathcal{I}_{\Updownarrow}$, generate the number of payments, $N^{paid*}_{i,j,l}$ from the Multinomial distribution estimated in Step 1.

(ii) Simulate the number of "non-zero" payments N^{paid*}_{ij}, at each $(i,j) \in \mathcal{I}_m$, from a Binomial with size parameter $\sum_{l=0}^{j} N^{paid*}_{i,j-l,l}$ and probability $1 - Q_i$.

(iii) Simulate the bootstrapped aggregated payments X_{ij}^* from the gamma distribution with shape parameter $N_{ij}^{paid*}\widehat{\lambda}_i$ and scale parameter $\widehat{\kappa}_i$ (estimated in Step 1).

Step 3. *Bootstrapping the parameters to include the parameters uncertainty.* From the (adjusted) bootstrapped data $\widetilde{\Delta}_m^* = \{\widetilde{X}_{ij}^* : (i,j) \in \mathcal{I}_m\}$, with $\widetilde{X}_{ij}^* = X_{ij}^*/[(1-Q_i)]$ and the original $\aleph_m$, estimate again the parameter θ and get a bootstrapped parameter θ^*.

Step 4. *Bootstrapping the RBNS predictions.* Simulate the RBNS cash flow, $\widetilde{X}_{ij}^{rbns*}$, for $i+j>m$, using similar specifications to (i)–(iii) in Step 2, but with bootstrapped parameter θ^*. Incorporate again the development severity inflation (removed for estimation purposes in Step 1) by $X_{ij}^{rbns*} = \delta_j \widetilde{X}_m^{rbns*}$.

Step 5. *Monte Carlo approximation.* Repeat Steps 2-4 B times and get the empirical bootstrap distribution of the RBNS cash flows $\{X_{ij}^{rbns*,b}; i=1,\ldots,m, j=0,\ldots,m-1, i+j>m, b=1,\ldots,B\}$.

Note that, in Step 2-(iii) above, we can simulate directly the aggregated payments at each cell (i,j) because of the convolution property of the gamma distribution, together with the expression (1). In fact, the sum of independent individual payments, which are gamma distributed with the same scale parameter, is also gamma distributed with such scale and shape parameter being the sum of the individual shapes.

A simpler algorithm, which does not take into account the uncertainty of the estimated parameters, consists of steps 1, 4 and 5, replacing the bootstrapped parameter θ^* by the $\widehat{\theta}$, estimated from the original data in Step 1.

The algorithm to simulate the IBNR cash flows (taking into account the uncertainty of parameters) follows the same steps as the algorithm RBNS but, in addition, it involves the estimation and the simulation of the number of reported claims N_{ij} in the lower triangle, this is, $\{(i,j); i=2,\ldots,m, j=0,\ldots,m-1, m<i+j<2m\}$. In this case, to include the uncertainty of these extra parameters, we should also simulate (bootstrapped) reported-counts upper-triangles, $\aleph_m^* = \{N_{ij}^*;\ (i,j)\in\mathcal{I}_m\}$. Using assumption (D1), this can be done by simulating from a Poisson distribution with estimated chain ladder parameters $\{\widehat{\alpha}_i, \widehat{\beta}_j; i=1,\ldots,m, j=0,\ldots,m-1\}$ (see Step 2 in Algorithm IBNR by Martínez-Miranda et al. 2011 for more details). Again the simpler version which does not take into account the uncertainty of parameters would not require the simulation of such counts.

[Table 8 should be here]

[Table 9 should be here]

[Table 10 should be here]

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Legends of tables

Table 1. Point forecasts from DCL without prior information (the numbers are given in thousands). Columns 2-4 show the forecasts (RBNS, IBNR and total= IBNR+RBNS) using the general delay parameters estimated by solving the linear system (5). Columns 5-7 show the same forecasts but using adjusted delay probabilities $\{\widehat{p}_0, \ldots, \widehat{p}_{m-1}\}$. The last column shows the classical chain ladder forecasts which are reproduced by DCL using the general delay.

Table 2. Estimated delay parameters considering prior information about the severity development inflation.

Table 3. Point forecasts considering prior information about the severity inflation (the numbers are given in thousands).

Table 4. Point forecasts considering prior information about the zero claims (the numbers are given in thousands).

Table 5. Point forecasts considering prior information about the zero claims and severity inflation (the numbers are given in thousands).

Table 6. Summary of the bootstrap predictive distribution for the total reserve. The DCL distribution with no prior is showed in columns 2–3. The third column shows the results from the chain ladder bootstrapping of England and Verrall (2002). The DCL distribution using prior information about the development inflation (prior A), the zero-claims (prior B) and also both at the same time (prior C) are showed in columns 5–10. Bootstrap methods ignoring or taking into account the uncertainty of the parameters are showed in columns labelled as “Boot I” and “Boot II”, respectively. The numbers are given in thousands.

Table 7. Estimated parameters from DCL applied to the three triangles $\aleph_m, \Delta_m$ and $\mathcal{R}_m$. The first column reports the general delay parameters calculated by solving system (13). The second column shows the estimated proportion of zero-claims estimated from (11). The last column shows the severity development inflation estimated from equation (12).

Table 8. Incremental incurred counts: $\aleph_m = \{N_{ij} : (i, j) \in \mathcal{I}_m\}$.

Table 9. Incremental paid data: $\Delta_m = \{X_{ij} : (i, j) \in \mathcal{I}_m\}$.

Table 10. Incremental number of non-zero payments: $\mathcal{R}_m = \{R_{ij} : (i, j) \in \mathcal{I}_m\}$.

Legends of figures

Figure 1. Estimated DCL parameters assuming a maximum delay of 13 years. The top panel shows the delay parameters: the solid blue curve corresponds to the adjusted delay probabilities and the discontinuous green curve shows the general parameters which provide the classical chain ladder reserve. The last panel shows the DCL inflation parameters in the underwriting year direction.

Figure 2. Delay parameters considering prior knowledge about the severity development inflation (δ_j). The first panel shows the delay parameters from DCL on the adjusted triangle $\widetilde{X}_{ij} = X_{ij}/\delta_j$. The general delay parameters (solid blue curve) without any restriction are compared with the adjusted delay probabilities (discontinuous green curve). The prior severity development inflation is showed in the bottom panel.

Figure 3. Bootstrap predictive distribution. The first row shows the distribution of the total reserves. The second and third rows show the RBNS and IBNR distributions, respectively. The DCL distribution when no prior is incorporated is shown in the first column. Columns 2–4 show the derived distribution considering prior knowledge as in table 6. The histograms show the bootstrap distribution which ignores the uncertainty of the parameters.

Figure 4. Inflation in the underwriting year. The top panel shows the inflation removing the zero-claims effect and compares it with the inflation estimated using DCL and ignoring the zero-claims knowledge. The second panel shows the zero-claims effect and the last panel shows the probability of zero-claims for each underwriting year.

Tables

Table 1: Point forecasts from DCL without prior information (the numbers are given in thousands). Columns 2-4 show the forecasts (RBNS, IBNR and total= IBNR+RBNS) using the general delay parameters estimated by solving the linear system (5). Columns 5-7 show the same forecasts but using adjusted delay probabilities $\{\widehat{p}_0, \ldots, \widehat{p}_{m-1}\}$. The last column shows the classical chain ladder forecasts which are reproduced by DCL using the general delay.

	General delay			Adjusted delay			CLM
Future	RBNS	IBNR	Total	RBNS	IBNR	Total	Total
1	4799	891	5691	4799	891	5691	5691
2	1781	429	2210	1780	429	2209	2210
3	1465	69	1535	1466	69	1535	1535
4	1052	61	1113	1052	61	1112	1113
5	737	43	780	740	43	782	780
6	566	25	592	566	25	592	592
7	471	14	485	472	14	486	485
8	367	15	383	367	15	383	383
9	262	16	277	262	16	277	277
10	171	14	185	170	14	184	185
11	90	11	101	90	11	101	101
12	-12	14	1	0	12	12	1
13	1	-1	0	0	1	1	0
Total	11751	1601	13352	11764	1601	13365	13352

Figures

Table 2: Estimated delay parameters considering prior information about the severity development inflation.

l	$\widehat{\pi}_l$	$\widehat{p}_l$
0	0.8037	0.7956
1	0.1981	0.1961
2	-0.0101	0.0000
3	0.0045	0.0045
4	0.0011	0.0011
5	0.0008	0.0008
6	0.0005	0.0005
7	0.0004	0.0004
8	0.0003	0.0003
9	0.0003	0.0003
10	0.0003	0.0003
11	0.0002	0.0002
12	0.0000	0.0000
13	0.0000	0.0000

Table 3: Point forecasts considering prior information about the severity inflation (the numbers are given in thousands).

	General delay			Adjusted delay			CLM
Future	RBNS	IBNR	Total	RBNS	IBNR	Total	Total
1	4116	1567	5683	4472	1551	6023	5683
2	940	1263	2203	1328	1250	2579	2203
3	1310	228	1538	1297	335	1632	1538
4	858	249	1107	849	258	1107	1107
5	656	125	781	649	127	777	781
6	507	80	587	502	81	583	587
7	421	63	484	416	64	480	484
8	326	54	379	322	54	376	379
9	248	27	276	246	28	274	276
10	166	18	184	164	18	182	184
11	82	18	100	81	18	99	100
12	1	1	2	1	1	1	2
13	0	0	0	0	0	0	0
Total	9630	3692	13322	10328	3785	14113	13322

Table 4: Point forecasts considering prior information about the zero claims (the numbers are given in thousands).

	General delay			Adjusted delay			CLM
Future	RBNS	IBNR	Total	RBNS	IBNR	Total	Total
1	4796	891	5688	4810	890	5700	5688
2	1783	428	2212	1800	428	2228	2212
3	1465	69	1535	1483	69	1553	1535
4	1056	60	1116	1073	60	1134	1116
5	733	44	776	751	44	794	776
6	563	25	588	578	25	603	588
7	469	14	483	484	14	497	483
8	366	15	382	379	15	395	382
9	262	15	278	275	15	290	278
10	171	14	185	183	14	197	185
11	90	11	101	103	11	114	101
12	-12	14	1	1	14	14	1
13	1	-1	0	1	1	2	0
Total	11743	1601	13344	11921	1600	13521	13344

Table 5: Point forecasts considering prior information about the zero claims and severity inflation (the numbers are given in thousands).

	General delay			Adjusted delay			CLM
Future	RBNS	IBNR	Total	RBNS	IBNR	Total	Total
1	4113	1567	5680	4465	1551	6017	5680
2	942	1261	2204	1327	1249	2576	2204
3	1309	229	1538	1296	335	1631	1538
4	862	248	1110	853	257	1110	1110
5	651	126	777	644	129	773	777
6	504	80	584	499	81	580	584
7	419	63	482	415	63	478	482
8	325	54	378	322	54	375	378
9	249	27	276	247	28	274	276
10	166	18	183	164	18	182	183
11	82	18	100	81	18	99	100
12	1	1	1	1	1	1	1
13	0	0	0	0	0	0	0
Total	9623	3691	13314	10314	3783	14097	13314

Table 6: Summary of the bootstrap predictive distribution for the total reserve. The DCL distribution with no prior is showed in columns 2–3. The third column shows the results from the chain ladder bootstrapping of England and Verrall (2002). The DCL distribution using prior information about the development inflation (prior A), the zero-claims (prior B) and also both at the same time (prior C) are showed in columns 5–10. Bootstrap methods ignoring or taking into account the uncertainty of the parameters are showed in columns labelled as "Boot I" and "Boot II", respectively. The numbers are given in thousands.

	DCL		CLM	Prior A		Prior B		Prior C	
	Boot I	Boot II	EV-2002	Boot I	Boot II	Boot I	Boot II	Boot I	Boot II
mean	13087	13446	13376	13510	13584	13423	13576	13574	13864
pe	1271	2045	2313	1945	2995	1401	1998	2057	3064
50%	13080	13342	13246	13394	13150	13383	13399	13335	13397
90%	14764	16084	16286	16058	17248	15222	16176	16377	18079
95%	15235	16972	17259	16859	19103	15763	17219	17269	19671
99%	16024	18266	19408	18395	22413	16884	18883	19073	23990

Table 7: Estimated parameters from DCL applied to the three triangles $\aleph_m$, Δ_m and $\mathcal{R}_m$. The first column reports the general delay parameters calculated by solving system (12). The second column shows the estimated proportion of zero-claims estimated from (10). The last column shows the severity development inflation estimated from equation (11).

	$\widehat{\pi}_l$	$\widehat{Q}_i$	$\widehat{\delta}_j$
1	0.8037	0.207	0.751
2	0.1981	0.220	1.100
3	-0.0101	0.236	2.833
4	0.0045	0.228	7.081
5	0.0011	0.234	12.501
6	0.0008	0.248	14.474
7	0.0005	0.280	12.865
8	0.0004	0.306	17.349
9	0.0003	0.327	26.193
10	0.0003	0.347	24.391
11	0.0003	0.352	23.660
12	0.0002	0.339	40.284
13	0.0000	0.320	2.095
14	0.0000	0.346	

Table 8: Incremental incurred counts: $\aleph_m = \{N_{ij} : (i,j) \in \mathcal{I}_m\}$.

i j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	18247	3083	124	22	5	5	3	1	0	1	1	0	0	0
2	17098	2567	98	25	6	1	1	3	0	1	0	0	0	
3	16110	2700	107	18	7	5	4	1	4	0	0	0		
4	14426	2253	103	17	10	3	2	1	1	1	0			
5	14142	2173	62	11	7	4	0	1	1	0				
6	14275	1850	86	25	6	2	0	0	1					
7	14019	1797	97	19	5	1	1	1						
8	13933	1602	84	24	6	3	1							
9	12962	1503	65	11	2	2								
10	12226	1352	74	18	7									
11	11124	1347	57	12										
12	10360	1307	56											
13	10371	1141												
14	10435													

Table 9: Incremental paid data: $\Delta_m = \{X_{ij} : (i,j) \in \mathcal{I}_m\}$.

i j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	9829717	5690608	874882	420112	154884	55497	46239	313960	290204	12936	6218	18755	4678	0
2	9263718	5004173	971523	660324	208000	531391	495368	48367	566099	49905	362747	388190	0	
3	9402126	5625116	805027	322263	325505	101469	160747	310837	30754	69395	8123	51756		
4	8650875	5150702	752354	802485	209590	466859	197654	41763	25349	367750	123091			
5	8848118	4748516	1390699	1140610	412090	359991	20169	220227	54395	240967				
6	9070691	5890678	519808	539202	127701	86472	122060	83853	6660					
7	8763254	4293444	1339396	292330	1515615	155402	28210	36709						
8	7777082	4145234	642816	504127	92030	101250	6620							
9	7212984	3498230	778132	354855	626442	342182								
10	6265457	3737631	546644	182490	297995									
11	5737447	3281469	748102	456983										
12	5612232	3495586	593774											
13	6386024	3289703												
14	6110750													

Table 10: Incremental number of non-zero payments: $\mathcal{R}_m = \{R_{ij} : (i,j) \in \mathcal{I}_m\}$.

i j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	11761	4800	324	71	39	14	10	6	3	5	5	2	2	0
2	10927	4077	303	60	28	12	13	5	8	4	5	5	0	
3	9856	4168	294	71	23	23	16	10	9	4	4	3		
4	8915	3682	246	70	27	16	7	7	4	7	4			
5	8854	3340	265	46	33	9	4	6	2	5				
6	8881	3000	199	70	22	15	8	8	4					
7	8170	2983	221	46	18	8	5	6						
8	7827	2741	184	55	22	15	3							
9	6999	2540	166	44	18	7								
10	6240	2420	184	45	18									
11	5652	2210	184	45										
12	5223	2317	148											
13	5627	2024												
14	5483													

Figure 1: Estimated DCL parameters assuming a maximum delay of 13 years. The top panel shows the delay parameters: the solid blue curve corresponds to the adjusted delay probabilities and the discontinuous green curve shows the general parameters which provide the classical chain ladder reserve. The last panel shows the DCL inflation parameters in the underwriting year direction.

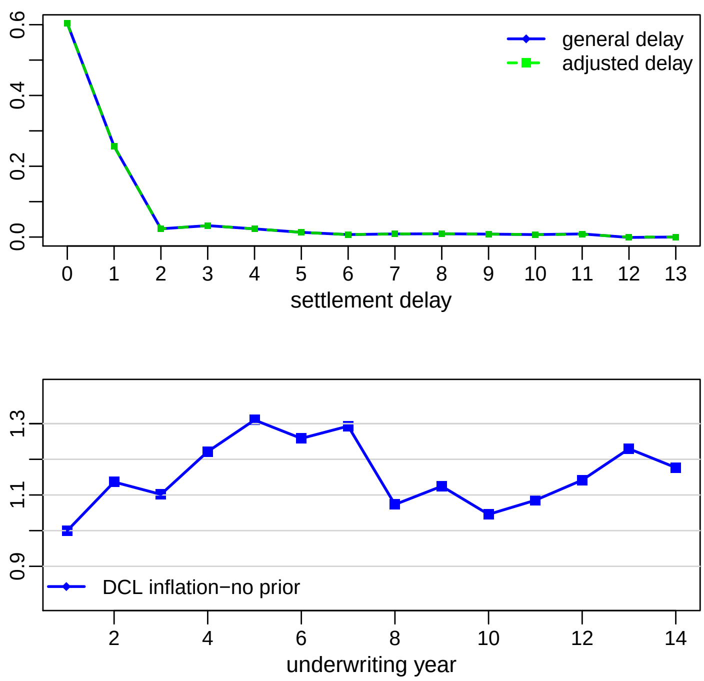

Figure 2: Delay parameters considering prior knowledge about the severity development inflation (δ_j). The first panel shows the delay parameters from DCL on the adjusted triangle $\widetilde{X}_{ij} = X_{ij}/\delta_j$. The general delay parameters (solid blue curve) without any restriction are compared with the adjusted delay probabilities (discontinuous green curve). The prior severity development inflation is showed in the bottom panel.

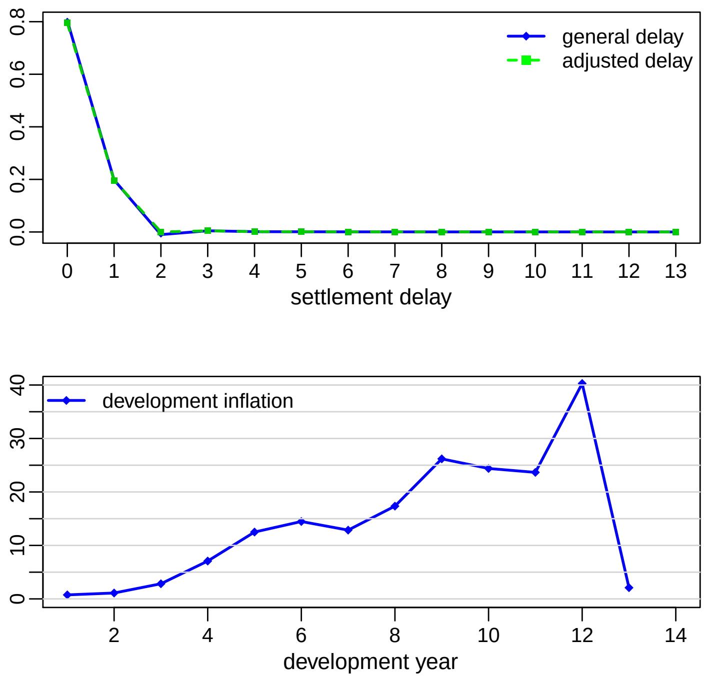

Figure 3: Bootstrap predictive distribution. The first row shows the distribution of the total reserves. The second and third rows show the RBNS and IBNR distributions, respectively. The DCL distribution when no prior is incorporated is shown in the first column. Columns 2–4 show the derived distribution considering prior knowledge as in table 6. The histograms show the bootstrap distribution which ignores the uncertainty of the parameters.

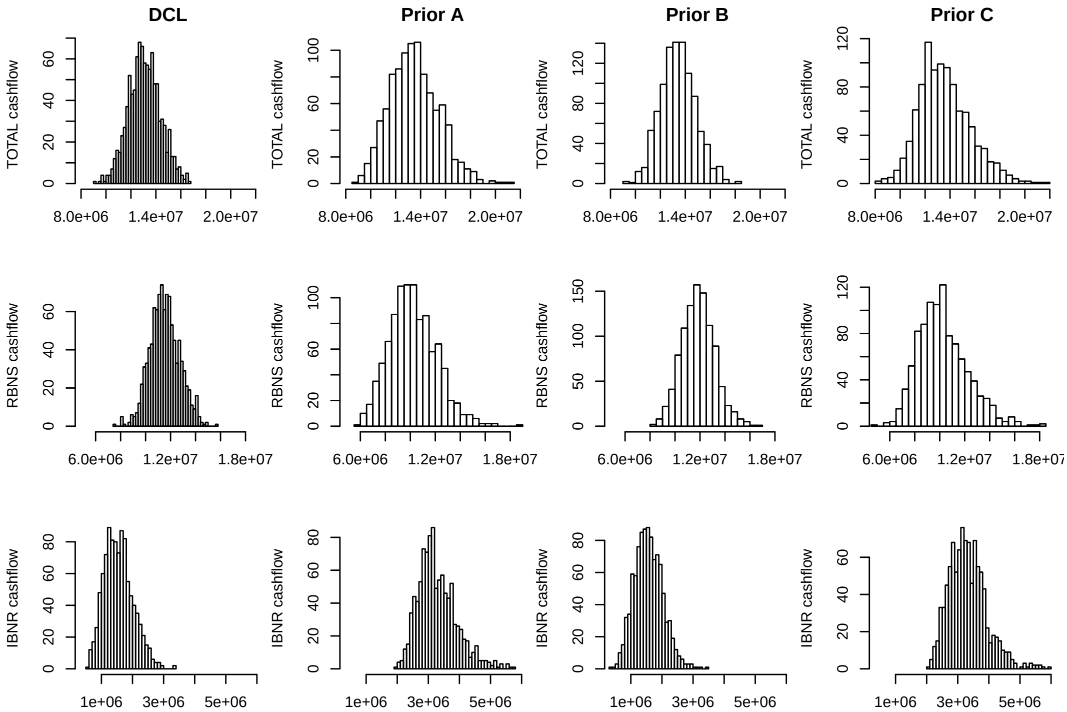

Figure 4: Inflation in the underwriting year. The top panel shows the inflation removing the zero-claims effect and compares it with the inflation estimated using DCL and ignoring the zero-claims knowledge. The second panel shows the zero-claims effect and the last panel shows the probability of zero-claims for each underwriting year.