Rationality of blocks of quasi-simple finite groups

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Declaration

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Abstract

The Morita Frobenius number of an algebra is the number of Morita equivalence classes of its Frobenius twists. Introduced by Kessar in 2004, these numbers are important in the context of Donovan's conjecture for blocks of finite group algebras. Let P be a finite ℓ-group. Donovan's conjecture states that there are finitely many Morita equivalence classes of blocks of finite group algebras with defect groups isomorphic to P. Kessar proved that Donovan's conjecture holds if and only if Weak Donovan's conjecture and the Rationality conjecture hold. Our thesis relates to the Rationality conjecture, which states that there exists a bound on the Morita Frobenius numbers of blocks of finite group algebras with defect groups isomorphic to P, which depends only on $|P|$. In this thesis we calculate the Morita Frobenius numbers, or produce a bound for the Morita Frobenius numbers, of many of the blocks of quasi-simple finite groups. We also discuss the issues faced in the outstanding blocks and outline some possible approaches to solving these cases.

Notation

Δ	Base of a root system
$\varepsilon_{\mathbf{G}}$	$(-1)^{\mathbb{F}_q\text{-rank of }\mathbf{G}}$
θ^G	Character of G induced from θ
$\Lambda(R)$	Fundamental group of R
$\sigma : k \to k$	Frobenius automorphism
$\sigma : kG \to kG$	Galois conjugation map
$\hat{\sigma} : K \to K$	Aut. of K fixed in Section 2.2.1
${}^{\hat{\sigma}}\chi(g)$	$\hat{\sigma}(\chi(g))$
$\sigma(b)$	$\sigma(b)$ if $R = k$, $\hat{\sigma}(\tilde{b})$ if $R = \mathcal{O}$
Φ	Root system
$\check{\Phi}$	Coroots
Φ^+	Positive roots in Φ
Φ^-	Negative roots in Φ
Φ_e	eth cyclotomic polynomial
ω_χ	Linear char. of $Z(KG)$ corr to χ
A, B	Finite dimensional algebras
A_0, B_0	Basic algebras
$A^{(\ell^m)}$	mth Frobenius twist of A
$A \sim_M B$	A and B are Morita equivalent
A_n	Alternating group on n letters
$\widetilde{A}_n$	Double cover of A_n
$A_{\mathbf{G}^*}(s)$	Group of components $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$
B, b	Blocks of kG or RG
$\tilde{b}$	Block of $\mathcal{O}G$
$B^2(G; F^\times)$	2-boundaries of G
$b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$	Block of $\mathbf{G}^F$ containing irreducible consitutents of $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\lambda)$
$\mathrm{Bl}(G)$	Set of blocks of G
Br_P	Brauer homomorphism
$d(b)$	Defect of b
$e_\ell(q)$	Order of q modulo ℓ
$e_s^{\mathbf{G}^F}$	Sum of block idempotents of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$
e_χ	Central primitive idempotent of KG corresponding to χ
$\mathcal{E}\left(\mathbf{G}^F, s\right)$	Lusztig series associated to $[s]$
$\mathcal{E}\left(\mathbf{G}^F, \ell'\right)$	Characters in $\mathcal{E}\left(\mathbf{G}^F, s\right)$ for any semisimple ℓ' element $s \in \mathbf{G}^{*F}$
$\mathcal{E}\left(\mathbf{G}^F, 1\right)$	Unipotent characters of $\mathbf{G}^F$
$\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$	Union of ℓ-blocks of $\mathbf{G}^F$ containing characters $\mathcal{E}(\mathbf{G}^F, s)$
F	Arbitrary field

$F : \mathbf{G} \to \mathbf{G}$	Frobenius morphism
FG	Group algebra of G over F
$frob(A)$	Frobenius number of A
$F_\alpha G$	Twisted group alg. of G over F
G	Finite group
$\mathbf{G}$	Algebraic group
$\mathbf{G}_a$	Additive group $\cong (F,+)$
$\mathbf{G}_m$	Multiplicative group $\cong GL_1(F)$
$\mathbf{G}_u$	Set of unipotent elements of $\mathbf{G}$
$\mathbf{G}_{\Phi_e}$	Sylow e-torus of $\mathbf{G}$
$\mathbf{G}^*$	Algebraic group dual to $\mathbf{G}$
$\mathbf{G}^\circ$	Connected component of $\mathbf{G}$ containing the identity element
$\mathbf{G}^F$	Finite group of Lie type
$H^2(G; F^\times)$	Second cohomology group of G
$i : \mathbf{G} \to \widetilde{\mathbf{G}}$	Regular embedding
$i^* : \widetilde{\mathbf{G}} \to \mathbf{G}$	Surjective morphism dual to i
$\mathrm{I}_G(b)$	Inertial group of b
$\mathrm{I}_G(\theta)$	Inertial group of θ
$\mathrm{Irr}(B)$	Set of irreducible characters in B
$\mathrm{Irr}(G)$	Set of irreducible characters of G
k	Field of characteristic ℓ
K	Field of characteristic zero
$(K, \mathcal{O}, k)$	ℓ-modular system
ℓ	A prime number
$\mathbf{L}$	Levi subgroup
$\mathcal{L} : \mathbf{G} \to \mathbf{G}$	Lang map $\mathcal{L}(g) = g^{-1}F(g)$
$mf(A)$	Morita Frobenius number of A

$N_{F^n/F}$	Norm map of F^n at F
$\mathcal{O}$	Complete discrete valuation ring
$\mathbf{P}$	Parabolic subgroup
(P, b)	Brauer pair
$P_{\mathbf{G}^F}(x)$	Order polynomial of $\mathbf{G}^F$ over $\mathbb{Z}$
$Q^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$	Green functions
R	$\mathcal{O}$ or k
$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$	Deligne Lusztig induction
${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$	Deligne Lusztig restriction
$R(\mathbf{G})$	Radical of $\mathbf{G}$
$R_u(\mathbf{G})$	Unipotent radical of $\mathbf{G}$
$\mathrm{rk}(\mathbf{G})$	Rank of $\mathbf{G}$
(s)	Geometric conjugacy class of s
$[s]$	Rational conjugacy class of s
s_α	Reflection corr. to $\alpha \in \Phi$
S_n	Symmetric group on n letters
$\widetilde{S}_n, \hat{S}_n$	Double covers of S_n
$\mathbf{T}$	Torus
$(\mathbf{T}, \theta) \overset{\mathbf{G}}{\leftrightarrow} (\mathbf{T}^*, s)$	$(\mathbf{T}, \theta)$ corresponds to $(\mathbf{T}^*, s)$ in bijection (4.2)
$W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$	Relative Weyl group of $(\mathbf{L}, \lambda)$
$(X, R, Y, \check{R})$	Abstract root datum
$X(\mathbf{T})$ / $Y(\mathbf{T})$	Character/Cocharacter group of $\mathbf{T}$
$Z^2(G; F^\times)$	2-cocycles of G
$\mathbb{Z}R$ / $\mathbb{Z}\check{R}$	Root/Coroot lattice

Chapter 1

Introduction

Modular representation theory is a branch of mathematics which began around the turn of the 20th century. Ordinary representation theory is concerned with the representations of finite groups over a field of characteristic zero. The term *modular representation theory* was first used by Dickson in 1907 [21] to describe the study of representations of finite groups over fields of positive characteristic. He showed that if the characteristic of the field divides the order of the group then the modular representation theory of a group is quite different to its ordinary representation theory. Important work of Brauer begun in the 1930s established modular representation theory as a mainstream area of mathematics. Today the field is very active with recent developments leading to breakthroughs for some fundamental conjectures which have been open in the area for many decades.

Let G be a finite group and let k be an algebraically closed field of positive characteristic ℓ. Our thesis is concerned with the indecomposable 2-sided ideals of the group algebra kG, known as blocks. For each block B of kG there exists an associated conjugacy class of finite ℓ-subgroups of G known as defect groups. The defect groups of B provide a measure of how far B is from being a semisimple algebra. One important open question in modular representation theory today is; can the blocks of finite group algebras be classified according to their defect groups? Our thesis relates to this question via Donovan's conjecture.

Conjecture (Donovan's Conjecture [1, Conjecture M])**.** *Let P be a finite ℓ-group. Then there are finitely many Morita equivalence classes of blocks of finite group algebras with defect groups isomorphic to P.*

Donovan's conjecture is open in the general case, but there are certain situations where it is known to hold. This is discussed in more detail in Section 2.3. There exists a weaker version of Donovan's conjecture, known as Weak Donovan's conjecture. This states that the entries of the Cartan matrices of blocks with defect groups isomorphic to P are bounded by some function which depends only on $|P|$. In 2004, Kessar showed in [46, Theorem 1.4] that the gap between Donovan's conjecture and Weak Donovan's conjecture is another finiteness condition usually referred to as the Rationality conjecture [46, Conjecture 1.3].

Conjecture (Rationality Conjecture)**.** *The Morita Frobenius numbers of blocks of finite group algebras with defect groups isomorphic to P are bounded by a function which depends only on $|P|$.*

The theorem of Kessar shows that Donovan's conjecture holds if and only if both Weak Donovan's conjecture and the Rationality conjecture hold. This gives us another way to approach Donovan's conjecture. The aim of our thesis is to calculate, or find bounds for, the Morita Frobenius numbers of the blocks of quasi-simple finite groups.

A Morita Frobenius number can be defined for any finite dimensional k-algebra A. For a positive integer m, the mth Frobenius twist of A is another finite dimensional k-algebra which is isomorphic to A as rings, endowed with a twisted scalar multiplication defined by $\lambda.x = \lambda^{\frac{1}{\ell^m}}x$ for all $x \in A$, $\lambda \in k$. The Morita Frobenius number of A is the least positive integer m such that A is Morita equivalent to its mth Frobenius twist as k-algebras, denoted by $mf(A)$. Since A is finite dimensional and k is algebraically closed, the Morita Frobenius number of A is finite.

Little is known in general about the values of the Morita Frobenius numbers. The Morita Frobenius number of a block of a finite group algebra can be greater than 1 – in 2007 Benson and Kessar gave a general method for producing examples of blocks of finite group algebras with Morita Frobenius number equal to 2 [2]. There exist algebras which have arbitrarily large Morita Frobenius numbers, for example some algebras in the family of algebras of quaternion type given in [28]. However, it is not known whether the algebras of quaternion type with arbitrarily large Morita Frobenius numbers arise as blocks of finite group algebras.

Our results are summarised in Theorems A, B, C and D below. An explanation of the notation can be found in Chapters 2 and 4. In many cases the Morita Frobenius number of B is 1 and in some cases we even find that B is isomorphic to its first Frobenius twist, not just Morita equivalent. There remain some open cases for the finite groups of Lie type in non-defining characteristic. Apart from these, which are discussed in Section 5.4, we prove that the Morita Frobenius number of B is at most 4.

Theorem A. *Let b be an ℓ-block of a quasi-simple finite group G. Let $\overline{G} = G/Z(G)$. Suppose that one of the following holds.*

(a) $\overline{G}$ is an alternating group

(b) $\overline{G}$ is a sporadic group

(c) $\overline{G}$ is a finite group of Lie type in characteristic ℓ

Then $mf(b) = 1$.

Theorem B. *Let ℓ and p be different primes and let q be a power of p. Let $\boldsymbol{G}$ be a simple, simply connected algebraic group defined over $\overline{\mathbb{F}}_p$ and let $F : \boldsymbol{G} \to \boldsymbol{G}$ be a not very twisted Frobenius morphism with respect to an $\mathbb{F}_q$-structure. Let s be a semisimple ℓ' element of $\boldsymbol{G}^{*F}$ and let $b \in \mathcal{E}_\ell(\boldsymbol{G}^F, s)$ be an ℓ-block of $\boldsymbol{G}^F$.*

(a) If b is a unipotent block not equal to one of the following blocks of E_8

- $b = b_{E_8}(\phi_1^2.E_6(q), E_6[\theta^i])$ $(i = 1, 2)$ *with $\ell = 2$ and q of order 1 modulo 4, or*
- $b = b_{E_8}(\phi_2^2.{}^2E_6(q), {}^2E_6[\theta^i])$ $(i = 1, 2)$ *with $\ell \equiv 2 \bmod 3$ and q of order 2 modulo ℓ,*

then $mf(b) = 1$. If b is one of the two blocks above then $mf(b) \leq 2$.

(b) If $s \neq 1$ is quasi-isolated in $\boldsymbol{G}^*$ then

- *if $\boldsymbol{G}$ is of type A or B then $mf(b) = 1$;*
- *if $\boldsymbol{G}$ is of type E_8 then $mf(b) \leq 4$; and*
- *otherwise $mf(b) \leq 2$.*

(c) If $s \neq 1$ is such that $C^{\circ}_{\boldsymbol{G}^*}(s)$ is a Levi subgroup of $\boldsymbol{G}^*$ and $A_{\boldsymbol{G}^*}(s)$ is cyclic, or if $C_{\boldsymbol{G}^*}(s)$ is connected and s is not isolated in $\boldsymbol{G}^*$, then

- *if $\boldsymbol{G}$ is of type E_7 or E_8 then $mf(b) \leq 2$,*
- *otherwise $mf(b) = 1$.*

One consequence of Theorem B is the following result for groups of type A.

Theorem C. *Let $\mathcal{G}_1 = \{SL_n(q) : n \in \mathbb{N}, q = p^a \text{ for some prime } p \neq \ell, a \in \mathbb{N}\}$, and let $\mathcal{G}_2 = \left\{SU_n(q) : n \in \mathbb{N}, q = p^a \text{ for some prime } p \neq \ell \text{ and some } a \in \mathbb{N} \text{ such that } \ell \nmid q^{2s+1} + 1 \ \forall \ s \in \mathbb{N}\right\}$. Then Donovan's conjecture holds for the ℓ-blocks of groups in $\mathcal{G}_1$ and $\mathcal{G}_2$.*

Theorem D. *Let $\boldsymbol{G}^F$ be a Suzuki or Ree group. Let b be an ℓ-block of $\boldsymbol{G}^F$. If b is a block of the large Ree group in non-defining characteristic, assume that b is unipotent. Then $mf(b) = 1$.*

We begin with an introduction to block theory in Section 2.1 followed by a summary of the methods used in Section 2.2. Results relating to groups not of Lie type are included in Section 3, including calculations of the Morita Frobenius numbers for the symmetric groups, the alternating groups and their double covers, and for the sporadic groups, their covers, and the exceptional covering groups. In Section 4 we introduce the finite groups of Lie type, and the calculations of the Morita Frobenius numbers of the blocks of the finite groups of Lie type are presented in Section 5. In Section 5.4 we discuss the outstanding open cases in the finite groups of Lie type and present progress made on these cases to date. Finally, the proofs of the Theorems A, B, C and D are given in Section 6.

Parts of results of this thesis have appeared in print in [29].

Chapter 2

Block theory, methods and the connection to Donovan's conjecture

2.1 Introduction to block theory

In this section we present all the necessary preliminaries from representation theory and block theory. The primary sources are [58], [44] and [57].

2.1.1 Algebras and modules

We start by recalling some definitions. Let F be an arbitrary field and let A be an F-algebra with unit element 1_A. We will assume for this section that all algebras are finite dimensional. The *Jacobson radical* of A, denoted by $J(A)$, is the maximal nilpotent ideal of A. If $J(A) = 0$ then A is *semisimple*, therefore by Wedderburn's theorem [58, Theorem 1.17], A is isomorphic to a direct product of matrix algebras.

A *left A-module* M is an F-vector space with an F-bilinear map $A \times M \to M$ given by $a \times m = am$ such that $(ab)m = a(bm)$ and $1_A m = m$ for all $a, b \in A$ and all $m \in M$. A *right A-module* is defined analogously with the product operation on the other side. An A-module is *irreducible* if its only submodules are 0 and itself. An *A-homomorphism* between two A-modules M and N is an F-linear map $\varphi : M \to N$ such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$. The *category of A-modules*, $\mathrm{Mod}(A)$, is the category with A-modules as objects, A-homomorphisms as morphisms, and with the composition of A-homomorphisms as composition of morphisms. We say that two finite dimensional F-algebras A and B are *Morita equivalent* if there is an F-linear equivalence between their module categories $\mathrm{Mod}(A)$ and $\mathrm{Mod}(B)$, and denote this by $A \sim_M B$.

For a finite group G, let FG denote the *group algebra* of G over F – a finite dimensional F-algebra with basis $\{g \in G\}$. The elements of FG are of the form $\sum_{g \in G} \alpha_g g$ with $\alpha_g \in F$ for every $g \in G$, and the product map is given by F-linear extension of the usual multiplication in G.

A non-zero element $a \in A$ is an *idempotent* if $a^2 = a$. Two idempotents a and b are *orthogonal* if $ab = 0 = ba$ and an idempotent is *primitive* if it cannot be expressed as the sum of two orthogonal idempotents. If a is an idempotent of A, a *primitive decomposition of* a is a finite set I of primitive orthogonal idempotents of A such that $\sum_{i \in I} i = a$. Since any two primitive decompositions of 1_A are conjugate, the primitive decomposition of 1_A in $Z(A)$ is unique [63, Corollary 4.2].

An F-algebra A is *basic* if for any primitive decomposition I of 1_A, the elements in I are mutually non-conjugate. Two basic algebras are Morita equivalent if and only if they are isomorphic [28, Lemma I.2.6]. Every F-algebra is Morita equivalent to a basic F-algebra. For a given algebra A, we can construct a basic algebra A_0 in the following way. Suppose that I is a primitive decomposition of 1_A. Let I_0 be a set of representatives of the conjugacy classes of elements in I and let $i_0 = \sum_{i \in I_0} i$. Set $A_0 := i_0 A i_0 = \{i_0 a i_0 \mid a \in A\}$. Then A and A_0 are Morita equivalent [28, Corollary I.2.7] . It therefore follows that if A and B are two finite dimensional F-algebras with basic algebras A_0 and B_0 respectively, then A and B are Morita equivalent if and only if A_0 and B_0 are isomorphic.

For the rest of this section let k be an algebraically closed field of characteristic ℓ and let A be a finite dimensional k-algebra. Recall that since algebraically closed fields are perfect, the

Frobenius homomorphism $\sigma : k \to k$ given by $\lambda \mapsto \lambda^{\ell}$ is an automorphism. We can therefore consider the inverse map $\sigma^{-1} : k \to k$ sending $\lambda \mapsto \lambda^{\frac{1}{\ell}}$.

Definition 2.1.1. For $m \in \mathbb{N}$, the *m-th Frobenius twist of A*, denoted by $A^{(\ell^m)}$, is a k-algebra with the same underlying ring structure as A, endowed with a new action of the scalars of k given by $\lambda . x = \lambda^{\frac{1}{\ell^m}} x$ for all $\lambda \in k$, $x \in A$.

Definition 2.1.2. The *Morita Frobenius number* of A, denoted by $mf(A)$, is the least integer m such that A is Morita equivalent to $A^{(\ell^m)}$.

Definition 2.1.3. The *Frobenius number* of A, denoted by $frob(A)$, is the the least integer m such that $A \cong A^{(\ell^m)}$ as k-algebras.

Note that the basic algebra of the m-th Frobenius twist of A is equal to the m-th Frobenius twist of its basic algebra, i.e. $\left(A^{(\ell^m)}\right)_0 = A_0^{(\ell^m)}$. Suppose that $\{a_1, \ldots, a_n\}$ is a basis for the k-vector space underlying a finite dimensional k-algebra A. The *structure constants* of A with respect to this basis are the scalars $c_{ijr} \in k$ such that

$$a_i a_j = \sum_{r=1}^{n} c_{ijr} a_r.$$

If the k-vector space underlying A has a basis such that all the structure constants c_{ijr} lie in $\mathbb{F}_{\ell^m}$, then A is said to *have an $\mathbb{F}_{\ell^m}$-form*. Note that a matrix algebra has a basis such that the structure constants are just 1 and 0, so every matrix algebra has an $\mathbb{F}_{\ell^m}$-form. It can be shown that a k-algebra A has an $\mathbb{F}_{\ell^m}$-form if and only if $A \cong A^{(\ell^m)}$ as k-algebras [46, Lemma 2.1]. In particular, A has an $\mathbb{F}_{\ell}$-form if and only if $frob(A)$= 1.

Lemma 2.1.4. *Let k be an algebraically closed field of characteristic ℓ and let A be a finite dimensional k-algebra. Then $mf\ (A)$ and $frob\ (A)$ are finite.*

Proof. Since k is an algebraically closed field of characteristic ℓ, every structure constant of A is contained in $\mathbb{F}_{\ell}^{r}$ for some positive integer r. As A is finite dimensional we can let t be the maximum such r, so $\mathbb{F}_{\ell}^{t}$ contains all the structure constants of A. Thus A has an $\mathbb{F}_{\ell}^{t}$-form, so $A \cong (A^{(\ell^t)})$ as k-algebras by [46, Lemma 2.1]. Therefore, $mf(A) \leq frob(A) \leq t < \infty$ as required. $\square$

Lemma 2.1.5. *Let A be a finite dimensional k-algebra and let A_0 be the basic algebra of A. Then*

$$1 \leq mf\,(A_0) = frob\,(A_0) = mf\,(A) \leq frob\,(A).$$

Proof. This follows directly from the fact that A and $A^{(\ell^m)}$ are Morita equivalent if and only if their basic algebras A_0 and $A_0^{(\ell^m)}$ are isomorphic. $\square$

2.1.2 The second cohomology group and twisted group algebras

Let G be a finite group and let F be an arbitrary field. A map $\alpha : G \times G \to F^\times$ is called a *2-cocycle* if $\alpha(xy, z)\alpha(x, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$. The set of all 2-cocycles of G with coefficients in $F^\times$ is denoted by $Z^2(G; F^\times)$, and $Z^2(G; F^\times)$ has the structure of an abelian group with multiplication defined by $(\alpha\alpha')(x, y) = \alpha(x, y)\alpha'(x, y)$ for all $\alpha, \alpha' \in Z^2(G; F^\times)$ and all $x, y \in G$. A 2-cocycle $\alpha \in Z^2(G; F^\times)$ is called a *2-boundary* if there exists a map $\gamma : G \to F^\times$ such that $\alpha(x, y) = \gamma(x)\gamma(y)\gamma(xy)^{-1}$ for all $x, y \in G$. The set of 2-boundaries of G with coefficients in $F^\times$ is a subgroup of $Z^2(G; F^\times)$ denoted by $B^2(G; F^\times)$.

The *second cohomology group of G with coefficients in $F^\times$* is the quotient group

$$H^2(G; F^\times) = Z^2(G; F^\times)/B^2(G; F^\times).$$

The *twisted group algebra of G by $\alpha \in Z^2(G; F^\times)$* is the F-algebra, $F_\alpha G$, which is equal to FG as an F-vector space, with a twisted multiplication operation $FG \times FG \to FG$ given by $x.y = \alpha(x, y)xy$ for all $x, y \in G$.

2.1.3 Representations

For this section, continue to let F be an arbitrary field and let A be a finite dimensional F-algebra. A *representation* of A is an algebra homomorphism $\mathbf{X} : A \to \mathrm{Mat}_n(F)$, for some natural number n called the *degree* of the representation. Two representations $\mathbf{X}$ and $\mathbf{X}'$ of an F-algebra A are *similar* if there exists an invertible $n \times n$ matrix T such that $T\mathbf{X}(a)T^{-1} = \mathbf{X}'(a)$ for every $a \in A$.

A representation $\mathbf{X}$ of A naturally defines an A-module, and vice versa. Consider the vector space V of n-dimensional column vectors over F. Given a representation $\mathbf{X}$ of A, we can give V the structure of a left A-module by defining multiplication by $av = \mathbf{X}(a)v$ for all $v \in V$ by $a \in A$. On the other hand, suppose we are given an A-module M. By picking a basis for M we can express any $a \in A$ as a matrix a_M in $\mathrm{Mat}_n(F)$ determined by the action of a on the basis elements. Then by defining $\mathbf{X}(a) = a_M$, we get a representation $\mathbf{X} : A \to \mathrm{Mat}_n(F)$ of A. Different choices of bases for M may give rise to different representations, but they will be similar representations. Conversely, two similar representations will determine isomorphic A-modules. A representation is *irreducible* if the module it determines is irreducible.

2.1.4 Characters

Let K be a field of characteristic 0 such that for any of the finite groups G considered below, K contains the $|G|$th roots of unity. By Brauer's Splitting Field Theorem, [57, Ch. 3, Theorem 4.11], K is then a splitting field for G. Let $\mathbf{X}$ be a representation of a group algebra KG. The *character of* G *afforded by* $\mathbf{X}$ is the function $\chi : G \to K$ given by $\chi(g) = \mathrm{tr}\ (\mathbf{X}(g))$, for all $g \in G$. The *degree* of χ is defined to be $\chi(1) = \mathrm{tr}\ (\mathbf{X}(1))$. Characters of degree 1 are called *linear characters*. A character is called *irreducible* if it is afforded by an irreducible representation.

Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $n \times n$ matrices A, B, similar representations $\mathbf{X}$ and $\mathbf{X}'$ afford equal characters: if $\mathbf{X}' = T\mathbf{X}T^{-1}$ for some invertible $n \times n$ matrix T, then $\mathrm{tr}(\mathbf{X}') = \mathrm{tr}(T\mathbf{X}T^{-1}) = \mathrm{tr}(T^{-1}T\mathbf{X}) = \mathrm{tr}(\mathbf{X})$. If we take one representative of each isomorphism class of irreducible A-modules, this determines a set of representations which afford the set of irreducible characters of G, denoted by $\mathrm{Irr}(G)$.

A *class function* on G is a function which is constant on conjugacy classes. For any $g, h \in G$, $\mathrm{tr}(\mathbf{X}(hgh^{-1})) = \mathrm{tr}(\mathbf{X}(h)\mathbf{X}(g)\mathbf{X}(h^{-1}))) = \mathrm{tr}(\mathbf{X}(g)\mathbf{X}(h^{-1})\mathbf{X}(h))) = \mathrm{tr}(\mathbf{X}(g))$ since $\mathbf{X}$ is a homomorphism. Therefore characters are class functions. In fact, with the sums of characters defined by

$$(\chi + \theta)(g) = \chi(g) + \theta(g),$$

for all $\theta, \chi \in \mathrm{Irr}(G)$, and for all $g \in G$, the irreducible characters of G form a basis for the space of all class functions on G [44, Theorem 2.8]. Therefore any character ψ of G can be expressed as a sum $\psi = \sum_{i=1}^{r} \lambda_i \chi_i$ where χ_i are irreducible characters of G and $\lambda_i, r \in \mathbb{N}$. If $\lambda_i \neq 0$ then we say that χ_i is an *irreducible constituent* of ψ.

We define an *inner product* on the space of class functions by

$$\langle \chi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\theta(g^{-1}),$$

for any two class functions χ and θ on G. The product of two characters χ and θ of G is defined by $\chi\theta(g) = \chi(g)\theta(g)$ for all $g \in G$. Then $\chi\theta$ is also a character of G [44, Corollary 4.2], although not necessarily irreducible. We say that a character χ is *rational valued* if $\chi(g) \in \mathbb{Q}$ for all $g \in G$.

The structure of the centre of the group algebra KG is related to the character theory of G in the following way. The set of primitive idempotents of $Z(KG)$ has the form [57, Ch. 3, Theorem 2.22]

$$\left\{ e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g \mid \chi \in \mathrm{Irr}(G) \right\},$$

and is a basis for $Z(KG)$. Thus any element $z \in Z(KG)$ can be expressed as $z = \sum_{\chi \in \mathrm{Irr}(G)} \omega_\chi(z) e_\chi$ for some $\omega_\chi(z) \in K$. The map $\omega_\chi : Z(KG) \to K$ defines a linear character of $Z(KG)$ known as the *linear character of Z(KG)* (or *central character of* KG) *corresponding to* χ. Let C be a conjugacy class of G and let $\hat{C} = \sum_{x \in C} x$. Then it can be shown that

$$\omega_\chi(\hat{C}) = \frac{|G|\chi(x)}{|C_G(x)|\chi(1)},$$

where x is a representative of the conjugacy class C (see [57, Ch. 3, Theorem 2.23]).

2.1.5 ℓ-modular systems

We are interested in studying the representation theory of finite groups over a field k of positive characteristic. A triple $(K, \mathcal{O}, k)$ is an *ℓ-modular system* if K is a field of characteristic zero with complete discrete valuation $\nu : K \to \mathbb{Z} \cup \{\infty\}$, $\mathcal{O}$ is the valuation ring of ν with unique maximal ideal $\mathfrak{m}$, and k is the residue field $\mathcal{O}/\mathfrak{m}$ of characteristic ℓ. By working over an ℓ-modular system it is possible to use the representation theory of G over K, and therefore the character theory of G, to determine properties of the representation theory of G over k.

From now on, let $(K, \mathcal{O}, k)$ be an ℓ-modular system in which k is an algebraically closed field and K contains a $|G|$th primitive root of unity. Let $\pi : \mathcal{O} \to k$ denote the quotient map and denote the induced map on the group algebras also by $\pi : \mathcal{O}G \to kG$. Many definitions and results apply to both $\mathcal{O}$ and k. If that is the case, then to simplify the notation we will state these results over R, where it is understood that R could denote either $\mathcal{O}$ or k.

2.1.6 Blocks

Blocks can be defined for a general finite dimensional algebra but here we focus on the case of group algebras. A *block idempotent* of RG is a primitive idempotent in $Z(RG)$. The sum of block idempotents $\sum_{i=1}^n b_i$ is the unique primitive decomposition of 1_{RG} in $Z(RG)$. For each block idempotent b_i, the algebra RGb_i is an indecomposable 2-sided ideal of RG called a *block algebra*. The decomposition of RG into its block algebras is the unique decomposition of RG

into a direct product of indecomposable factors [3, Lemma 1.8.2],

$$RG = \prod_{i=1}^{n} RGb_i.$$

The map $\pi : \mathcal{O}G \to kG$ induces a bijection between the set of block algebras of $\mathcal{O}G$ and the set of block algebras of kG [3, Section 6.1]. In general we will refer to a block algebra RGb as an 'ℓ-block' (or simply a 'block') of G and label it by its block idempotent b rather than writing the full block algebra RGb. We let $\mathrm{Bl}(G)$ denote the set of blocks of RG. Where we want to explicitly differentiate between blocks of $\mathcal{O}G$ and kG, if b is a block of kG then we denote the corresponding block of $\mathcal{O}G$ by $\tilde{b}$.

The block idempotent $\tilde{b}$ of a block of $\mathcal{O}G$ is a central idempotent in KG, but it may not be primitive in $Z(KG)$. Thus since $\{e_\chi \mid \chi \in \mathrm{Irr}(G)\}$ is the unique set of primitive idempotents in $Z(KG)$, as discussed in the previous section, there exists a subset Γ_b of $\mathrm{Irr}(G)$ such that

$$\tilde{b} = \sum_{\chi \in \Gamma_b} e_\chi,$$

where $\chi \in \Gamma_b$ if and only if $\tilde{b}e_\chi = e_\chi$. We say that the irreducible characters in Γ_b *belong* to $\tilde{b}$ and b (or are 'in' or 'of' $\tilde{b}$ and b), and write $\mathrm{Irr}(b) = \mathrm{Irr}\big(\tilde{b}\big) = \Gamma_b$. The *principal block* of RG is the block containing the trivial character. Let $G_{\ell'}$ denote the set of elements $g \in G$ such that $\ell \nmid o(g)$. Two irreducible characters $\chi, \chi' \in \mathrm{Irr}(G)$ lie in the same block of RG if and only if for every conjugacy class C of $G_{\ell'}$,

$$\pi\left(\omega_\chi\left(\hat{C}\right)\right) = \pi\left(\omega_{\chi'}\left(\hat{C}\right)\right),$$

(see [57, Ch. 3, Theorem 6.4]).

2.1.7 The Brauer homomorphism, defect groups and Brauer pairs

Let P be an ℓ-subgroup of G and consider the following map,

$$\begin{aligned} \mathrm{Br}_P : RG &\longrightarrow kC_G(P) \\ \sum_{g\in G} \alpha_g g &\longmapsto \sum_{g\in C_G(P)} \overline{\alpha_g} g, \end{aligned}$$

where $\overline{\alpha_g} = \pi(\alpha_g)$ if $R = \mathcal{O}$ and $\overline{\alpha_g} = \alpha_g$ if $R = k$. Let $(RG)^P$ denote the fixed points of RG under the R-linear extension of the conjugation action of P on G. It can be shown that $Br_P|_{(RG)^P}$ is a surjective homomorphism of R-algebras [47, Proposition 2.2]. This restriction is called the *Brauer homomorphism* and is usually also denoted by Br_P.

Let b be a block of RG. A *defect group* of b is a maximal ℓ-subgroup P of G such that $Br_P(b) \neq 0$. The defect groups of b form a G-conjugacy class of ℓ-subgroups of G [58, Theorem 4.3]. We define the *defect* of b to be the positive integer $d(b) \in \mathbb{N}$ such that $|P| = \ell^{d(b)}$ for any defect group P of b, and we say that b has *defect zero* or *trivial defect* if $d(b) = 0$. Blocks of defect zero are well understood as they are isomorphic to a matrix algebra $Mat_n(R)$ for some positive integer n [63, Theorem 39.1]. For an integer $x \in \mathbb{Z}$, let x_ℓ denote the ℓ-part of x. We have the following very useful collection of characterisations of blocks of defect zero.

Theorem 2.1.6 ([58, Theorem 3.18])**.** *Let b be a block of RG. The following are equivalent.*

- *b has defect zero*
- $|\mathrm{Irr}(b)| = 1$
- $\mathrm{Irr}(b)$ *contains a character χ such that* $\chi(1)_\ell = |G|_\ell$
- $\mathrm{Irr}(b)$ *contains a character χ such that* $\chi(g) = 0$ *for all elements $g \in G$ such that* $\ell \mid o(g)$

If a character χ satisfies $\chi(1)_\ell = |G|_\ell$ then we say that χ is of *ℓ-defect zero*.

A *Brauer pair* of G is a pair (P, b) where P is an ℓ-subgroup of G and b is a block of $RC_G(P)$. The group G acts on the set of Brauer pairs by conjugation. We write $(P_1, b_1) \trianglelefteq$

(P_2, b_2) if $P_1 \trianglelefteq P_2$, b_1 is P_2 stable, and $\mathrm{Br}_{P_2}(b_1)b_2 = b_2$. Then if there exists a sequence of Brauer pairs (S_i, c_i), $1 \le i \le n$, such that $(P_1, b_1) \trianglelefteq (S_1, c_1) \trianglelefteq (S_2, c_2) \trianglelefteq \cdots \trianglelefteq (S_n, c_n) \trianglelefteq (P_2, b_2)$ we write $(P_1, b_1) \le (P_2, b_2)$, and this defines a transitive order relation on the set of Brauer pairs of G.

If b is a block of RG then a Brauer pair (P, e) is called a *b-Brauer pair* if $(1, b) \le (P, e)$. A b-Brauer pair (P, b) is called *self centralising* or *centric* if $Z(P)$ is a defect group of b. For a given block b of RG, a group P is a defect group of b if and only if there exists a block e of $RC_G(P)$ such that $(1, b) \le (P, e)$. If $(1, b) \le (P, e)$ and (P, e) is maximal with respect to $\le$ then (P, e) is called a *maximal b-Brauer pair*.

2.1.8 Notation for restricted and induced characters

Let H be a subgroup of G. Suppose that $\mathbf{X}$ is a representation of KG affording a character χ. Then $\mathbf{X}_H$ is a representation of KH affording the character χ_H, the restriction of χ to H. More generally, for any class function χ of G we can consider the restriction χ_H of χ to H.

Suppose that θ is a class function of H. The class function of G induced from θ is denoted by θ^G and defined by

$$\theta^G(g) = \frac{1}{|H|} \sum_{h \in H} \theta'(hgh^{-1}),$$

where

$$\theta'(g) = \begin{cases} \theta(g) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

Frobenius Reciprocity holds for induced and restricted class functions: $\langle \theta, \chi_H \rangle = \langle \theta^G, \chi \rangle$, for all class functions χ on G and all class functions θ on H [44, Lemma 5.2].

Suppose now that θ is a character of H and let $\chi \in \mathrm{Irr}(G)$. Then since χ_H is a character of H, $\langle \theta, \chi_H \rangle$ is a non-negative integer. By Frobenius reciprocity it follows that $\langle \theta^G, \chi \rangle$ is a non-negative integer, so θ^G is a character of G. Note, however, that θ^G may not be irreducible, even if θ is an irreducible character of H.

2.1.9 Normal subgroups

Let N be a normal subgroup of G and let $\theta \in \mathrm{Irr}(N)$. There is a natural action of G on the irreducible characters of N. For any $g \in G$, define ${}^g\theta$ by

$${}^g\theta(n) = \theta(gng^{-1})$$

for all $n \in N$. It can be shown that ${}^g\theta$ is an irreducible character of N [44, Lemma 6.1]. Because N itself acts trivially, in fact this defines an action of G/N on $\mathrm{Irr}(N)$.

Let $\chi : G \to K$ be an irreducible character of G. The restriction $\chi_N : N \to K$ of χ to N is a character of N but it may not be irreducible. If $\theta \in \mathrm{Irr}(N)$ is an irreducible constituent of χ_N then we say that χ *covers* θ and we let $\mathrm{Irr}(G \mid \theta)$ denote the set of irreducible characters of G covering θ. A powerful result from Clifford allows us to work with restrictions of characters in a practical way. Let θ be an irreducible constituent of χ_N and let $\{\theta_i\}_{i=1}^r$ denote the set of G-conjugates of θ. Then

$$\chi_N = e\sum_{i=1}^{r}\theta_i,$$

where $e = \langle \chi_N, \theta \rangle$ is the multiplicity of θ in χ_N [44, Theorem 6.2].

The *inertial group* of an irreducible character θ of N is the stabilizer of θ under the action of G,

$$I_G(\theta) = \{g \in G \mid {}^g\theta = \theta\}.$$

By the Orbit-Stabilizer theorem, θ has $|G : I_G(\theta)|$ conjugates under G, so $r = |G : I_G(\theta_i)|$ in Clifford's result above. The irreducible characters of G covering θ are closely related to the irreducible characters of $I_G(\theta)$ covering θ: if $\psi \in \mathrm{Irr}(I_G(\theta) \mid \theta)$ then ψ^G is irreducible and also covers θ. In fact, the induction map defines a bijection known as the Clifford correspondence

$$\begin{array}{ccc} \mathrm{Irr}(I_G(\theta) \mid \theta) & \longrightarrow & \mathrm{Irr}(G \mid \theta) \\ \psi & \longmapsto & \psi^G, \end{array}$$

[44, Theorem 6.11]. In many situations, therefore, we can work over $I_G(\theta)$ instead of G and assume that the character θ is G-stable.

Let $\theta \in \mathrm{Irr}(N)$ and suppose that $\theta^G = \sum_{i=1}^t f_i \chi_i$ for some positive integers t and f_i. By Frobenius Reciprocity θ is an irreducible constituent of χ_{iN} for each i. We say that θ is *extendible* if there exists a $\chi \in \mathrm{Irr}(G)$ such that $\chi_N = \theta$. In block theory, many questions about blocks of G can be answered by looking at the blocks of a normal subgroup N, and at the connections between the character theory of N and G. If θ extends to χ then clearly θ is invariant under the action of G. It is not true in general that G-stable characters are extendible, however, but we will often use the following fact: if G/N is cyclic and $\theta \in \mathrm{Irr}(N)$ is invariant under the action of G, then θ is extendible to G [44, Corollary 11.22].

In the following Lemma for a character $\theta \in \mathrm{Irr}(G)$, let $\overline{\theta} \in \mathrm{Irr}(G)$ denote the character given by $\overline{\theta}(g) = \theta(g^{-1})$ for all $g \in G$.

Lemma 2.1.7. *Suppose $N \lhd G$ are finite groups such that G/N is abelian. Let $\chi \in \mathrm{Irr}(N)$ and $\theta_1, \theta_2 \in \mathrm{Irr}(G \mid \chi)$. Then there exists a linear character $\eta \in \mathrm{Irr}(G/N)$ such that $\theta_2 = \eta\theta_1$.*

Proof. The character $\overline{\chi}\chi$ is a constituent of $\left(\overline{\theta_1}\theta_2\right)_N$. Let 1_N denote the trivial character of N. Since $\langle \overline{\chi}\chi, 1_N \rangle = \langle \overline{\chi}, \overline{\chi} \rangle = 1$, 1_N is an irreducible constituent of $\overline{\chi}\chi$. It follows that there exists some irreducible constituent η of $\overline{\theta_1}\theta_2$ which covers 1_N. Thus $\eta \in \mathrm{Irr}(G/N)$ and $\langle \eta, \overline{\theta_1}\theta_2 \rangle \neq 0$. However, $\langle \eta, \overline{\theta_1}\theta_2 \rangle = \langle \eta\theta_1, \theta_2 \rangle$, so since $\eta\theta_1$ and θ_2 are both irreducible characters, it follows that $\theta_2 = \eta\theta_1$ as required. $\square$

2.1.10 Covering blocks

Let $N \lhd G$. The action of G on the irreducible characters of N is reflected in an action of G on the blocks of RN. If b is a block of RN then ${}^g b = gbg^{-1}$ is a *G-conjugate block* of b and it contains characters $\{{}^g\chi \mid \chi \in \mathrm{Irr}(b)\}$. Let $\{b_1, \ldots, b_n\}$ be the orbit of a block b of RN under the action of G, and let $f = \sum_{i=1}^n b_i$. Then f is an idempotent in RG such that

$$ {}^g f = \sum_{i=1}^n {}^g b_i = \sum_{i=1}^n b_i = f $$

for any $g \in G$, so in fact f is a central idempotent of RG. Let $f = \sum_{i=1}^m B_i$ be a primitive idempotent decomposition of f in $Z(RG)$. Each B_i is a central primitive idempotent of RG, that is, a block idempotent of RG. We say that the blocks B_i *cover* b, for $1 \le i \le m$, and we let $\mathrm{Bl}\,(G \mid b)$ denote the set of blocks in RG covering b.

Lemma 2.1.8. *Let $N \lhd G$. A block B of RG covers a block b of RN if and only if $Bb \neq 0$.*

Proof. Let b be a block of RN and let $\{b_1, \ldots, b_n\}$ be the orbit of b under the action of G. Let $f = \sum_{i=1}^n b_i$ and let $\sum_{i=1}^m B_i$ be a primitive idempotent decomposition of f in $Z(RG)$. A block B of RG covers b if and only if $B = B_j$ for some $1 \le j \le m$. This holds if and only if $Bf \neq 0$, in other words, $\sum_{i=1}^n Bb_i \neq 0$.

Note that if $b_r = gbg^{-1}$ is conjugate to b for some $g \in G$, then $Bb_r = B(gbg^{-1}) = g(Bb)g^{-1}$, so either $Bb = Bb_r = 0$, or both $Bb \neq 0$ and $Bb_r \neq 0$. It follows that $\sum_{i=1}^n Bb_i \neq 0$ if and only if $Bb \neq 0$ as required. $\square$

We note a few facts about covering blocks. Firstly, the set of blocks of RN covered by a block B of RG form a G-conjugacy class of $\mathrm{Bl}(N)$ [57, Ch. 5, Lemma 5.3]. If G/N is an ℓ-group, then a block b of RN is covered by a unique block B of RG [57, Ch. 5, Corollary 5.6], but in general, a block of RN can be covered by multiple blocks of RG. If B covers b then for every $\chi \in \mathrm{Irr}(B)$, each irreducible constituent of χ_N is contained in some G-conjugate block of b, and for every $\theta \in \mathrm{Irr}(b)$, there exists a $\chi \in \mathrm{Irr}(B)$ such that θ is an irreducible constituent of χ_N. On the other hand, if b is a block of RN containing an irreducible character θ, and if B is a block of RG containing a character χ such that θ is an irreducible contituent of χ_N, then B covers b. [57, Ch. 5, Lemma 5.7 and 5.8].

Like the stabilizer of a character, the stabilizer of a block b of RN under the action of G plays an important role in the theory of covering blocks. Let b be a block of RN. The *inertial group of* b is defined by

$$I_G(b) = \{g \in G \mid {}^g b = b\}.$$

By Fong-Reynolds [57, Ch. 5, Theorem 5.10] there exists a bijection

$$\mathrm{Bl}\ (I_G(b) \mid b) \longrightarrow \mathrm{Bl}\ (G \mid b)$$

such that if $B_0 \in \mathrm{Bl}\ (I_G(b) \mid b)$ corresponds to $B \in \mathrm{Bl}\ (G \mid b)$ under this bijection then $R\,(I_G(b))\,B_0$ is Morita equivalent to RGB [51, Theorem C]. Similar to the character theory situation where it is often possible to assume that a character θ of N is G-stable, we can often work over $I_G(b)$ instead of G and assume that b is a G-stable block.

Let b be a block of RN and suppose that $B \in \mathrm{Bl}\ (G \mid b)$. Choose a defect group P of B such that $P \leq I_G(b)$ (this is always possible by [3, Theorem 6.4.1 (ii)]). Then $P \cap N$ is a defect group of b by [57, Ch. 5, Theorem 5.16 (ii)]. If G/N is an ℓ'-group then $P \cap N = P$ so in particular, when G/N is an ℓ'-group any defect group of b is a defect group of B.

The linear characters of G form an abelian group with multiplication given by the usual product of characters – for two linear characters χ_1 and χ_2 of G, $\chi_1\chi_2(g) = \chi_1(g)\chi_2(g)$ for all $g \in G$. The trivial character acts as identity in this group. There is an action of the linear characters of G on $\mathrm{Irr}(G)$ which respects blocks. Let θ be a linear character of G and consider the following R-algebra isomorphism.

$$\begin{array}{cccc} \varphi_\theta : & RG & \longrightarrow & RG \\ & \sum_{g\in G}\alpha_g g & \longmapsto & \sum_{g\in G}\alpha_g\theta\left(g^{-1}\right)g. \end{array}$$

Suppose that B is a block of RG. Then $\varphi_\theta(B)$ is a block of RG which we denote by θB, containing irreducible characters $\{\theta\chi \mid \chi \in \operatorname{Irr}(B)\}$.

Lemma 2.1.9. *Let N be a normal subgroup of G such that G/N is abelian and suppose that B and B' are blocks of RG. Then B and B' cover the same block of RN if and only if $B' = \theta B$ for some linear character θ of G/N. In particular, if B and B' cover the same block of RN then $RGB \cong RGB'$ as R-algebras.*

Proof. First suppose that $B' = \theta B$ for some linear character θ of G/N. Then for every $\chi \in \operatorname{Irr}(B)$, $\chi_N = (\theta\chi)_N$ so the irreducible characters of B and of B' cover the same set of irreducible characters of N. Hence B and B' cover the same blocks of RN.

Now suppose that B and B' are blocks of RG which cover the same block b of RN. Let $\psi \in \operatorname{Irr}(b)$. Then there exists a $\chi \in \operatorname{Irr}(B)$ and a $\chi' \in \operatorname{Irr}(B')$ such that ψ is an irreducible constituent of χ_N and χ'_N. Therefore $\chi = \theta\chi'$ for some linear character $\theta \in \operatorname{Irr}(G/N)$ by Lemma 2.1.7. Hence $B' = \theta B$. □

2.1.11 Dominating blocks

Another approach to understanding the blocks of RG is to study the blocks of its quotient groups G/N for normal subgroups N. Let $N \lhd G$, let $\overline{G} = G/N$ and denote the quotient map by $\mu : G \to \overline{G}$. Extend μ linearly to an R-algebra homomorphism

$$\begin{array}{cccc} \mu : & RG & \longrightarrow & R\overline{G} \\ & \sum_{g\in G}\alpha_g g & \longmapsto & \sum_{g\in G}\alpha_g\mu(g), \end{array}$$

for all $\sum_{g\in G}\alpha_g g \in RG$. Since $\mu : RG \longrightarrow R\overline{G}$ is surjective, μ maps elements in $Z(RG)$ to elements in $Z(R\overline{G})$. It follows that for any block B of RG, $\mu(B)$ is either 0, or a central idempotent of $R\overline{G}$. If $\mu(B) \neq 0$ then let $\mu(B) = \sum_{i=1}^{n} \overline{B}_i$ denote a primitive idempotent decomposition of $\mu(B)$ in $Z(R\overline{G})$. We say that B *dominates* the blocks $\overline{B}_i$ of $R\overline{G}$, for $1 \leq i \leq n$.

The notion of dominance of blocks over $\mathcal{O}$ and dominance of blocks over k is compatible with the relation between $\mathcal{O}$ and k: a block b of kG dominates a block c of $k\overline{G}$ if and only if the corresponding block $\tilde{b}$ of $\mathcal{O}G$ dominates the corresponding block $\tilde{c}$ of $\mathcal{O}\overline{G}$.

Let $1_{R\overline{G}} = \sum_{b\in \mathrm{Bl}(\overline{G})} b$ be the unique primitive decomposition of $1_{R\overline{G}}$ in $Z(R\overline{G})$, and let $1_{RG} = \sum_{B\in \mathrm{Bl}(G)} B$ be the unique primitive decomposition of 1_{RG} in $Z(RG)$. By applying μ to 1_{RG} we see that

$$\sum_{b\in \mathrm{Bl}(\overline{G})} b = \sum_{B\in \mathrm{Bl}(G)} \mu(B).$$

Since each $b \in \mathrm{Bl}\ (G)$ appears precisely once in the sum it follows that $b\mu(B) = b$ for exactly one block B of RG. Therefore each block of $R\overline{G}$ is dominated by a unique block of RG. On the other hand, a block B of RG does not necessarily dominate any block of $R\overline{G}$, and if B does dominate some block of $R\overline{G}$, then in general B dominates multiple blocks of $R\overline{G}$.

The following Lemma collects together some other useful facts about block domination. Note that we consider $\mathrm{Irr}(\overline{G}) \subseteq \mathrm{Irr}(G)$ by identifying $\chi \in \mathrm{Irr}(\overline{G})$ with $\chi \circ \mu \in \mathrm{Irr}(G)$.

Lemma 2.1.10. *Let B be a block of RG, $N \lhd G$ and $\overline{G} = G/N$.*

(a) B dominates a block of $R\overline{G}$ if and only if it covers the principal block of RN

(b) If $N \leq Z(G)$ and B dominates some block of $R\overline{G}$, then B dominates a unique block of $R\overline{G}$

(c) If N is an ℓ'-group (not necessarily central) and B dominates some block of $R\overline{G}$, then B dominates a unique block $\overline{B}$ of $R\overline{G}$ and $RGB \cong R\overline{G}\overline{B}$ as R-algebras

Proof. By [57, Ch. 5, Lemma 8.6 (i)], B dominates a block of $R\overline{G}$ if and only if B contains a character χ such that $N \leq \ker \chi$. If this is the case then $\chi(n) = \chi(1)$ for all $n \in N$ so χ covers the trivial character of N. Since the trivial character of N is contained in the principal block of RN, it follows that B covers the principal block of RN. On the other hand, if B covers the principal block of RN then there exists a $\chi \in \mathrm{Irr}(B)$ covering the trivial character of N, so $N \leq \ker \chi$. Thus B dominates a block of $R\overline{G}$, showing part (a).

Part (b) is a special case of [57, Ch. 5 Theorem 8.11].

Finally for part (c), suppose that N is an ℓ'-subgroup of G and that B dominates a block $\overline{B}$ of $R\overline{G}$. By [57, Ch. 5, Theorem 8.8], $\overline{B}$ is the unique block of $R\overline{G}$ dominated by B and $\mathrm{Irr}(B) = \mathrm{Irr}(\overline{B})$. Therefore $\mu(B) = \overline{B}$, so μ restricts to the surjection

$$\begin{array}{cccc} \bar{\mu}: & RGB & \longrightarrow & R\overline{G}\overline{B} \\ & \left(\sum_{g\in G} \alpha_g g\right) B & \longmapsto & \left(\sum_{g\in G} \alpha_g \mu(g)\right) \overline{B}, \end{array}$$

for all $\sum_{g\in G} \alpha_g g \in RG$. Since $\dim_R(RGB) = \sum_{\chi\in\mathrm{Irr}(B)} \chi(1)^2$ and $\mathrm{Irr}(B) = \mathrm{Irr}(\overline{B})$, it follows that $\dim_R(RGB) = \dim_R(R\overline{G}\overline{B})$ so $\bar{\mu}: RGB \to R\overline{G}\overline{B}$ is injective. Therefore $RGB \cong R\overline{G}\overline{B}$ as R-algebras, showing part (c). □

2.2 Methods

In this section let G be a finite group and let $(K, \mathcal{O}, k)$ be an ℓ-modular system in which k is algebraically closed and K contains the $|G|$th roots of unity.

2.2.1 Galois conjugation

Let $\sigma : k \to k$ denote the Frobenius automorphism given by $\lambda \mapsto \lambda^\ell$ for all $\lambda \in k$. We also let σ denote the induced *Galois conjugation* map $\sigma : kG \to kG$ defined by

$$\sigma\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} \alpha_g^\ell g$$

for all $\sum_{g\in G} \alpha_g g \in kG$. Although not an isomorphism of k-algebras, Galois conjugation is a ring isomorphism so it permutes the blocks of kG. Let b be a block of kG. We call $\sigma(b)$ (or $kG\sigma(b)$) the *Galois conjugate* of b (respectively kGb), and we say that two blocks b and c of kG are *Galois conjugate* if $b = \sigma^n(c)$ for some positive integer n. Corresponding blocks $\tilde{b}$ and $\tilde{c}$ of $\mathcal{O}G$ are said to be *Galois conjugate* if b and c are Galois conjugate. Note that defect groups are preserved by Galois conjugation.

We fix an automorphism $\hat{\sigma} : K \to K$ such that $\hat{\sigma}(\zeta) = \zeta^\ell$ for any ℓ'-root of unity ζ in K. Then $\hat{\sigma}$ induces an ring automorphism of KG via

$$\hat{\sigma}\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} \hat{\sigma}(\alpha_g) g$$

for all $\sum_{g\in G} \alpha_g g \in KG$, and an action on $\mathrm{Irr}(G)$ via

$$^{\hat{\sigma}}\chi(g) = \hat{\sigma}(\chi(g))$$

for all $\chi \in \mathrm{Irr}(G)$, $g \in G$. Note that although $\hat{\sigma}$ may not preserve $\mathcal{O}$, it induces an action on the set of blocks of $\mathcal{O}G$ compatible with the action of σ on the blocks of kG [48, Lemma 3.1 (ii)]. The following Lemma shows this more precisely, and also shows that the irreducible characters of $\sigma(b)$ are the images of the irreducible characters of b under $\hat{\sigma}$.

Lemma 2.2.1. *Let b be a block of kG and $\tilde{b}$ be the corresponding block of $\mathcal{O}G$. Then*

(a) $\hat{\sigma}\left(\tilde{b}\right) = \widetilde{\sigma(b)}$, *and*

(b) $\mathrm{Irr}(\sigma(b)) = \left\{{}^{\hat{\sigma}}\chi \mid \chi \in \mathrm{Irr}(b)\right\}$.

Proof. For part (a), see [48, Lemma 3.1]. For part (b), we first note that the following holds for any $\chi \in \mathrm{Irr}(G)$.

$$\begin{aligned}
\hat{\sigma}(e_\chi) &= \hat{\sigma}\left(\frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g\right)\\
&= \frac{\chi(1)}{|G|}\sum_{g\in G}\hat{\sigma}\left(\chi(g^{-1})\right)g\\
&= \frac{{}^{\hat{\sigma}}\chi(1)}{|G|}\sum_{g\in G}{}^{\hat{\sigma}}\chi(g^{-1})g\\
&= e_{{}^{\hat{\sigma}}\chi}
\end{aligned}$$

Suppose that $\chi \in \mathrm{Irr}(b)$. Then

$$\widetilde{\sigma(b)}e_{{}^{\hat{\sigma}}\chi} = \hat{\sigma}\left(\tilde{b}\right)\hat{\sigma}(e_\chi) = \hat{\sigma}\left(\tilde{b}e_\chi\right) = \hat{\sigma}(e_\chi) = e_{{}^{\hat{\sigma}}\chi},$$

so ${}^{\hat{\sigma}}\chi \in \mathrm{Irr}(\sigma(b))$, showing that $\left\{{}^{\hat{\sigma}}\chi \mid \chi \in \mathrm{Irr}(b)\right\} \subseteq \mathrm{Irr}(\sigma(b))$.

On the other hand, for any $\psi \in \mathrm{Irr}(\sigma(b))$, since $\hat{\sigma}$ is an automorphism of K we can define a character $\chi \in \mathrm{Irr}(G)$ by $\chi(g) = \hat{\sigma}^{-1}(\psi(g))$ for all $g \in G$, so ${}^{\hat{\sigma}}\chi = \psi$. Since $\psi \in \mathrm{Irr}(\sigma(b))$, $\widetilde{\sigma(b)}e_\psi = e_\psi$, so

$$\hat{\sigma}\left(\tilde{b}e_\chi\right) = \hat{\sigma}(\tilde{b})\hat{\sigma}(e_\chi) = \widetilde{\sigma(b)}e_{{}^{\hat{\sigma}}\chi} = \widetilde{\sigma(b)}e_\psi = e_\psi = e_{{}^{\hat{\sigma}}\chi} = \hat{\sigma}(e_\chi).$$

Therefore $\tilde{b}e_\chi = e_\chi$ so $\chi \in \mathrm{Irr}(b)$, hence $\mathrm{Irr}(\sigma(b)) \subseteq \left\{{}^{\hat{\sigma}}\chi \mid \chi \in \mathrm{Irr}(b)\right\}$ and the result follows. □

When discussing blocks of RG, we will use $\sigma(b)$ to denote either $\sigma(b)$ if $R = k$ or $\hat{\sigma}(\tilde{b})$ if $R = \mathcal{O}$. The following Lemma is crucial when using Galois conjugation and the theory of covering and dominating blocks to calculate the Morita Frobenius numbers of blocks of kG.

Lemma 2.2.2. *Let $N \lhd G$. Let B be a block of RG, let b be a block of RN and let $\overline{B}$ be a block of $R(G/N)$. Then* *(a) B covers b if and only if $\sigma(B)$ covers $\sigma(b)$, and*

(b) B dominates $\overline{B}$ if and only if $\sigma(B)$ dominates $\sigma(\overline{B})$.

Proof. First recall that B covers b if and only if $Bb \neq 0$ by Lemma 2.1.8. This holds if and only if $\sigma(B)\sigma(b) = \sigma(Bb) \neq 0$ because σ is a ring isomorphism, thus if and only if $\sigma(B)$ covers $\sigma(b)$ showing part (a).

Now recall that by [57, Ch. 5, Lemma 8.6 (ii)], B dominates $\overline{B}$ if and only if $\mathrm{Irr}(\overline{B}) \subseteq \mathrm{Irr}(B)$. This holds if and only if we have the following.

$$\mathrm{Irr}\left(\sigma\left(\overline{B}\right)\right) = \left\{{}^{\hat{\sigma}}\chi \mid \chi \in \mathrm{Irr}\left(\overline{B}\right)\right\} \subseteq \left\{{}^{\hat{\sigma}}\chi \mid \chi \in \mathrm{Irr}(B)\right\} = \mathrm{Irr}(\sigma(B))$$

Therefore B dominates $\overline{B}$ if and only if $\sigma(B)$ dominates $\sigma\left(\overline{B}\right)$, as required for part (b). □

Lemma 2.2.3. *Let b be a block of RG and suppose that (P,e) is a (maximal) b-Brauer pair. Then $(P,\sigma(e))$ is a (maximal) $\sigma(b)$-Brauer pair and*

$$N_G(P,e)/PC_G(P) \cong N_G(P,\sigma(e))/PC_G(P).$$

Proof. Since (P,e) is a b-Brauer pair, $(1,b) \leq (P,e)$. Thus there exists a sequence of Brauer pairs (S_i,c_i), $1 \leq i \leq n$, such that $(1,b) \trianglelefteq (S_1,c_1) \trianglelefteq (S_2,c_2) \trianglelefteq \cdots \trianglelefteq (S_n,c_n) \trianglelefteq (P,e)$. Then $\mathrm{Br}_{S_1}(b)c_1 = c_1$, so $\sigma(\mathrm{Br}_{S_1}(b)c_1) = \sigma(c_1)$ and therefore $\mathrm{Br}_{S_1}(\sigma(b))\sigma(c_1) = \sigma(c_1)$. Hence $(1,\sigma(b)) \trianglelefteq (S_1,\sigma(c_1))$. Applying the same argument at each $\trianglelefteq$, it follows that $(1,\sigma(b)) \leq (P,\sigma(e))$, therefore $(P,\sigma(e))$ is a $\sigma(b)$-Brauer pair. Clearly $(P,\sigma(e))$ is maximal if (P,e) is maximal, and since $N_G(P,e) \cong N_G(P,\sigma(e))$, the last part of the statement also holds. □

2.2.2 Calculating Morita Frobenius numbers

The following important observation allows us to study the Galois conjugate of a block rather than its Frobenius twist.

Lemma 2.2.4. *There is a k-algebra isomorphism $kGb^{(\ell)} \cong kG\sigma(b)$ between the first Frobenius twist of kGb and the Galois conjugate of kGb.*

Proof. By definition, $kGb^{(\ell)}$ and $kG\sigma(b)$ are isomorphic as rings. As in Definition 2.1.1, denote scalar multiplication in $kGb^{(\ell)}$ by $\lambda.x$ for $\lambda \in k$ and $x \in kGb^{(\ell)}$. Let $\sum_{g\in G} \alpha_g g \in kGb^{(\ell)}$ and $\lambda \in k$. Then

$$\sigma\left(\lambda.\left(\sum_{g\in G} \alpha_g g\right)\right) = \sigma\left(\sum_{g\in G} \lambda^{\frac{1}{\ell}} \alpha_g g\right) = \left(\sum_{g\in G} \lambda \alpha_g^{\ell} g\right) = \lambda\sigma\left(\sum_{g\in G} \alpha_g g\right)$$

so $\sigma : kGb^{(\ell)} \to kG\sigma(b)$ is k-linear and therefore $kGb^{(\ell)} \cong kG\sigma(b)$ as k-algebras. □

The following proposition contains our most useful tools for calculating Morita Frobenius numbers.

Proposition 2.2.5. *Let b be a block of kG. Suppose that one of the following holds.*

(a) $\tilde{b} \in \mathbb{Q}G$

(b) $\mathrm{Irr}(b)$ *contains a subset of characters* $\{\chi_1, \dots, \chi_r\}$ *for some* $r \geq 1$*, such that for all* $g \in G$ $(\chi_1 + \dots + \chi_r)(g) \in \mathbb{Q}$

(c) There exists an irreducible character of ℓ-defect zero in b

(d) The defect groups of b are cyclic, dihedral or semi-dihedral

Then $mf(b) = 1$. Moreover, if (a), (b) or (c) holds then frob $(b) = 1$.

Proof. Let $\hat{\sigma} : K \to K$ be the automorphism fixed in Section 2.2.1. Clearly $\hat{\sigma}$ acts as the identity on $\mathbb{Q}$. Suppose that $\tilde{b} = \sum_{g\in G} \alpha_g g \in \mathbb{Q}G$. Then $\hat{\sigma}\left(\tilde{b}\right) = \sum_{g\in G} \hat{\sigma}(\alpha_g) g = \sum_{g\in G} \alpha_g g = \tilde{b}$, so $\hat{\sigma}$ stabilizes $\tilde{b}$. Since the action of $\hat{\sigma}$ on the blocks of $\mathcal{O}G$ is compatible with the action of σ on the blocks of kG as shown in Lemma 2.2.1, therefore $\sigma(b) = b$. It follows from Lemma 2.2.4 that $kGb^{(\ell)} \cong kG\sigma(b) \cong kGb$ as k-algebras. Therefore $frob(b) = 1$ and thus $mf(b) = 1$ by Lemma 2.1.5, showing part (a).

Suppose that there exists a set of characters of b, $\{\chi_1, \ldots, \chi_r\} \subset \mathrm{Irr}(b)$ for some $r \geq 1$, such that $(\chi_1 + \cdots + \chi_r)(g) \in \mathbb{Q}$ for all $g \in G$. Then

$$\left({}^{\hat{\sigma}}\chi_1 + \cdots + {}^{\hat{\sigma}}\chi_r\right)(g) = \hat{\sigma}\left((\chi_1 + \cdots + \chi_r)(g)\right) = (\chi_1 + \cdots + \chi_r)(g)$$

for all $g \in G$. It follows that $\{{}^{\hat{\sigma}}\chi_1, \ldots, {}^{\hat{\sigma}}\chi_r\}$ and $\{\chi_1, \ldots, \chi_r\}$ are equal as sets of irreducible characters, so $\sigma(b) = b$ by Lemma 2.2.1 (b). Therefore $frob(b) = mf(b) = 1$ following the same argument as in part (a).

By Theorem 2.1.6, if b contains a character of ℓ-defect zero then b has trivial defect, hence kGb is a matrix algebra. As discussed in Section 2.1.1, it follows that b has an $\mathbb{F}_\ell$-form and therefore $frob(b) = 1$. Hence $mf(b) = 1$ by Lemma 2.1.5 showing part (c).

If b has cyclic defect then its basic algebras are Brauer tree algebras, so they are defined over $\mathbb{F}_\ell$. By results of Erdmann given in [28, Tables starting page 294], if b has dihedral or semi-dihedral defect then its basic algebras are defined over $\mathbb{F}_2$. Thus if b has cyclic, dihedral or semi-dihedral defect then the Frobenius number of any basic algebra of b is 1, so $mf(b) = 1$ by Lemma 2.1.5. □

Lemma 2.2.6. *Let b be a block of kG. Suppose that there exists a group automorphism $\varphi \in Aut(G)$ such that for $R = \mathcal{O}$ or $R = k$, the induced R-algebra isomorphism $\varphi : RG \to RG$ satisfies $\varphi(b) = \sigma(b)$. Then $frob(b) = mf(b) = 1$.*

Proof. If $R = \mathcal{O}$ and $\varphi : \mathcal{O}G \to \mathcal{O}G$ is such that $\varphi(\tilde{b}) = \hat{\sigma}(\tilde{b})$ then $\varphi : \mathcal{O}G \to \mathcal{O}G$ induces a k-algebra isomorphism $\varphi : kG \to kG$ such that $\varphi(b) = \sigma(b)$. Thus when $R = \mathcal{O}$ or $R = k$, $\varphi|_{kGb} : kGb \to kG\sigma(b)$ is a k-algebra isomorphism so $kGb \cong kG\sigma(b)$ as k-algebras. It follows that $kGb \cong kGb^{(\ell)}$ as k-algebras by Lemma 2.2.4, so $frob(b) = 1$, whence $mf(b) = 1$ by Lemma 2.1.5. □

For the next Lemma we define a map $\phi : H^2(G; k^\times) \to H^2(G; k^\times)$ as follows. Let $\gamma \in H^2(G; k^\times)$ and let $\tilde{\gamma}$ be a 2-cocycle representing γ. Define $\phi(\gamma)$ to be the class in $H^2(G; k^\times)$

represented by the 2-cocycle given by

$$(g,h) \longmapsto \phi(\tilde{\gamma}(g,h)),$$

for all $g, h \in G$. It is easy to check that ϕ is a well-defined group homomorphism on $H^2(G;k^\times)$.

Lemma 2.2.7. *Let G be a finite group such that $H^2(G;k^\times) \cong C_2$ and let $\gamma \in H^2(G;k^\times)$. Then frob $(k_\gamma G)$ = mf $(k_\gamma G) = 1$.*

Proof. Let $\phi : H^2(G;k^\times) \to H^2(G;k^\times)$ be as defined above. If γ is non-trivial then is $\phi(\gamma)$ is also non-trivial since ϕ is a group homomorphism. Thus, since $H^2(G;k^\times) \cong C_2$, it follows that $k_\gamma G \cong k_{\phi(\gamma)}G$ as k-algebras.

Recall that $k_\gamma G^{(\ell)} \cong k_\gamma G$ as rings but not necessarily as k-algebras, and that scalar multiplication in $k_\gamma G^{(\ell)}$ is given by $\lambda.x = \lambda^{\frac{1}{\ell}}x$ for all $\lambda \in k, x \in k_\gamma G$. Let $\varphi : k_{\phi(\gamma)}G \to k_\gamma G^{(\ell)}$ be the map defined by

$$\varphi\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} \alpha_g^{\frac{1}{\ell}} g$$

for all $\sum_{g\in G} \alpha_g g \in k_{\phi(\gamma)}G$. This is a ring isomorphism, and

$$\varphi\left(\lambda \sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} (\lambda\alpha_g)^{\frac{1}{\ell}} g = \lambda^{\frac{1}{\ell}} \sum_{g\in G} \alpha_g^{\frac{1}{\ell}} g = \lambda.\varphi\left(\sum_{g\in G} \alpha_g g\right)$$

for all $\lambda \in k$ and $\sum_{g\in G} \alpha_g g \in k_{\phi(\gamma)}G$, so φ is in fact an isomorphism of k-algebras. Therefore $k_\gamma G \cong k_{\phi(\gamma)}G \cong k_\gamma G^{(\ell)}$ as k-algebras, so *frob*$(k_\gamma G) = 1$, hence *mf*$(k_\gamma G) = 1$. □

Lemma 2.2.8. *Suppose that there exists a finite group $\hat{G}$ such that $G \lhd \hat{G}$ and for all blocks $\hat{b}$ of $k\hat{G}$, either $\hat{b}$ has cyclic, dihedral or semi-dihedral defect or $\sigma(\hat{b}) = \hat{b}$. Then mf $(b) = 1$ for all blocks b of kG.*

Proof. First suppose that b is covered by some block $\hat{b}$ of $k\hat{G}$ which has cyclic, dihedral or semi-dihedral defect. Then, as discussed in Section 2.1.10, there exists some defect group P

of $\hat{b}$ such that $P \leq I_{\hat{G}}(b)$ and $P \cap G$ is a defect group of b. Thus the defect groups of b are also cyclic, dihedral or semi-dihedral. Therefore $mf(b) = 1$ by Proposition 2.2.5 (d).

Now suppose that b is covered by a block $\hat{b}$ of $k\hat{G}$ such that $\sigma(\hat{b}) = \hat{b}$. Then $\sigma(b)$ is also covered by $\hat{b}$, by Lemma 2.2.2 (a). Hence b and $\sigma(b)$ are in the same $\hat{G}$-orbit, so there is a group automorphism of G whose induced k-algebra automorphism of kG sends b to $\sigma(b)$. Therefore $frob(b) = mf(b) = 1$ by Lemma 2.2.6. □

Lemma 2.2.9. *Let b be a block of kG and suppose that b dominates a block $\bar{b}$ of $k(G/N)$ such that $\sigma\left(\bar{b}\right) = \bar{b}$. Then $frob(b) = mf(b) = 1$.*

Proof. By Lemma 2.2.2 (b), $\sigma(b)$ dominates $\sigma\left(\bar{b}\right)$, and by assumption, $\sigma(\bar{b}) = \bar{b}$. Therefore b and $\sigma(b)$ both dominate $\bar{b}$. Since every block of $k(G/N)$ is dominated by a unique block of kG, as discussed in Section 2.1.11, it follows that $b = \sigma(b)$. Therefore $frob(b) = mf(b) = 1$ by Lemmas 2.2.4 and 2.1.5. □

2.3 The connection to Donovan's conjecture

As discussed in the Introduction, Donovan's conjecture is one of the important open questions in modular representation theory today.

Conjecture 2.3.1 (Donovan's Conjecture [1, Conjecture M])**.** *Let P be a finite ℓ-group. Then there are finitely many Morita equivalence classes of blocks of finite group algebras with defect groups isomorphic to P.*

Donovan's conjecture dates from the 1960's. It is open in general but is known to hold in some specific cases. For example, if P is cyclic then results of Dade, Janusz and Kupisch show that there are finitely many Morita equivalence classes of blocks of finite group algebras with defect groups isomorphic to P. Erdman proved that Donovan's conjecture holds for dihedral and semi-dihedral P [28]. More recently, Eaton-Kessar-Külshammer-Sambale showed that Donovan's conjecture holds if P is an elementary abelian 2-group [25]. On the other hand, it is also known that if we only consider the blocks of finite group algebras for certain families of groups, then Donovan's conjecture holds for any P. For example, Külshammer showed in [51] that the finite group algebras of the family of ℓ-solvable groups only contribute finitely many Morita equivalence classes of blocks with defect groups isomorphic to P, for any given ℓ-group P.

Külshammer proved a reduction to Donovan's conjecture in [52, Section 5]. He showed that in order to prove Donovan's conjecture it is enough to show that for any finite ℓ-group P, there are only finitely many Morita equivalence classes of blocks of kG with defect groups isomorphic to P for finite groups of the form $G = \langle P^g \mid g \in G \rangle$, generated by conjugates of P.

The *Cartan matrix* of a finite dimensional k-algebra A is a square matrix (c_{ij}) where entry c_{ij} is the number of composition factors in a composition series of the jth projective indecomposable A-module which are isomorphic to the ith simple A-module. We can now state Weak Donovan's conjecture, which, as the name suggests, is implied by Donovan's conjecture.

Conjecture 2.3.2 (Weak Donovan's Conjecture)**.** *Let P be a finite ℓ-group. Then there are finitely many Cartan matrices of blocks of finite group algebras with defect groups isomorphic to P.*

In 1999, Düvel proved that Weak Donovan's conjecture can be reduced to blocks of quasi-simple finite groups [24]. Weak Donovan has also been proved in particular situations, for example the following two cases which we will use in Section 5.2.3.

Theorem 2.3.3 ([40, Theorem 8.6 (b) and Theorem 8.8 (c)])**.** *Let $\mathcal{G}_1 = \{SL_n(q) : n \in \mathbb{N}, q = p^a$ for some prime $p \neq \ell, a \in \mathbb{N}\}$, and let $\mathcal{G}_2 = \{SU_n(q) : n \in \mathbb{N}, q = p^a$ for some prime $p \neq \ell$ and some $a \in \mathbb{N}$ such that $\ell \nmid q^{2s+1} + 1 \;\forall\; s \in \mathbb{N}\Big\}$. Then Weak Donovan's conjecture holds for the ℓ-blocks of groups in $\mathcal{G}_1$ and $\mathcal{G}_2$.*

The connection between Donovan's conjecture and Morita Frobenius numbers was made by Kessar in 2004 [46]. Let P be a finite ℓ-group and consider a block B of a finite group algebra

kG with defect groups isomorphic to P. As mentioned in Section 2.2.1, Galois conjugation preserves defect groups. Since the Frobenius twist of B is isomorphic to its Galois conjugate by Lemma 2.2.4, B and its Frobenius twists have the same defect group. Hence if Donovan's conjecture holds, then there are only finitely many Morita equivalence classes of Frobenius twists of B. It follows that the Morita Frobenius number of B is bounded by some number which depends only on the defect group, P, of B. In particular, Donovan's conjecture implies the following conjecture.

Conjecture 2.3.4 (Rationality Conjecture [46, Conjecture 1.3])**.** *Let P be a finite ℓ-group. The Morita Frobenius numbers of blocks of finite group algebras with defect groups isomorphic to P are bounded by a function which depends only on $|P|$.*

It turns out that the Rationality conjecture is precisely the 'gap' between Donovan's conjecture and Weak Donovan's conjecture. This was proved by Kessar in [46], and it is this connection between Morita Frobenius numbers and Donovan's conjecture that provides the main motivation for our research.

Theorem 2.3.5 ([46, Theorem 1.4])**.** *Let P be a finite ℓ-group. Conjecture 2.3.1 holds if and only if both Conjecture 2.3.2 and Conjecture 2.3.4 hold.*

Remark 2.3.6. In the introduction we mentioned the family of algebras of quaternion type which are described in [26]. These algebras are known to satisfy Weak Donovan's conjecture, but there exist algebras in this family with arbitrarily large structure constants so they do not all satisfy the Rationality conjecture. Blocks with quaternion defect groups fall into this family of algebras. However, it is not known whether the algebras of quaternion type with arbitrarily large structure constants arise as blocks with quaternion defect. If they do then thanks to Theorem 2.3.5, these would be counter examples to Donovan's conjecture.

Although there is currently no reduction theorem for the Rationality conjecture, investigating the Morita Frobenius numbers of the quasi-simple finite groups is an important first step towards a proof of the conjecture.

Chapter 3

Morita Frobenius numbers of blocks of quasi-simple groups not of Lie type

3.1 The symmetric and alternating groups and their double covers

The ordinary representation theory of the symmetric group is very well understood. Here we present only the parts of this theory which are essential to our proofs. More information can be found in [56] and [45].

Let S_n denote the symmetric group on n letters for some positive integer n. A *partition* of n is a sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_m)$ called *parts*, such that $|\lambda| = \lambda_1 + \dots + \lambda_m = n$ and $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq m-1$. A partition is called *strict* if $\lambda_i > \lambda_{i+1}$ for all $1 \leq i \leq m-1$. The *parity* of λ is

$$\epsilon(\lambda) = \begin{cases} 1 & \text{if } n-m \text{ is even,} \\ -1 & \text{otherwise.} \end{cases}$$

An element $g \in S_n$ is a permutation so g can be expressed as a product of disjoint cycles. The *cycle type* of g is the sequence $\mu_g = (\mu_1, \dots, \mu_l)$ of the lengths of the disjoint cycles of g, and it is uniquely determined up to order. A permutation $g \in S_n$ is called *even* if it is the composition of an even number of transpositions, otherwise g is called *odd*. We say that g has *odd cycle type* if and only if all of its disjoint cycles are odd.

The irreducible characters of S_n are labelled by partitions of n. The character values can be calculated using the Murnaghan-Nakayama recursion formula [45, 2.4.7], and it is clear from this formula that $\chi(g) \in \mathbb{Q}$ for all $g \in G$, for any irreducible character χ of S_n.

Proposition 3.1.1. *Suppose that b is a block of kS_n. Then frob (b) = mf $(b) = 1$.*

Proof. Since all characters of S_n are rational valued the result follows immediately from Proposition 2.2.5 (b). □

Let A_n denote the alternating group on n letters – that is, the normal subgroup of S_n containing all even permutations. The irreducible characters of A_n arise as constituents of restrictions of irreducible characters of S_n and the blocks of kA_n are covered by the blocks if kS_n. More details about the block structure of kA_n can be found in [60, Section 12].

Proposition 3.1.2. *Suppose that b is a block of kA_n. Then frob (b) = mf $(b) = 1$.*

Proof. This follows immediately from the proof of Lemma 2.2.8 because $A_n \lhd S_n$ and $\sigma(\hat{b}) = \hat{b}$ for every block $\hat{b}$ of kS_n. □

Definition 3.1.3. Define $\widetilde{S}_n$ to be a *double cover of* S_n generated by elements $\{z, t_1, \dots, t_{n-1}\}$ such that $z^2 = 1$, and for all $1 \leq i, j \leq n-1$, $(t_i t_j)^{m_{ij}} = z$ where $m_{ii} = 1$ and

$$m_{ij} = \begin{cases} 3 & \text{if } i - j = \pm 1, \\ 2 & \text{otherwise.} \end{cases}$$

We define $\widetilde{S}_0 = \widetilde{S}_1 = \langle z \mid z^2 = 1 \rangle$ and let $\theta : \widetilde{S}_n \to S_n$ denote the surjective map sending $t_i \mapsto (i, i+1)$ with kernel $\langle z \rangle$.

This definition is just one of two possible ways to define a double cover of S_n. Let $\hat{S}_n$ denote the other double cover of S_n. If $n = 6$, $\widetilde{S}_n$ and $\hat{S}_n$ are isomorphic but for general n they are non-isomorphic [41, Theorem 2.12]. There is a one-to-one correspondence between the faithful irreducible characters of $\widetilde{S}_n$ and $\hat{S}_n$ for all n, however, and all results below still apply if we replace $\hat{S}_n$ by $\widetilde{S}_n$. For further information see [41, Chapter 2].

Suppose that $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a strict partition of n and let e be a positive integer. Then we say that a partition μ of $n - e$ is *obtained from λ by removing an e-bar* if one of the following holds:

- $\lambda_i = e$ for some $i \in \{1, \ldots, m\}$ and the parts of μ are $\{\lambda_1 \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m\}$;
- $\lambda_i > e$ for some $i \in \{1, \ldots, m\}$ such that $\lambda_i - e$ is not a part of λ, and the parts of μ are $\{\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - e, \lambda_{i+1}, \ldots, \lambda_m\}$; or
- $\lambda_i + \lambda_j = e$ for some $i \neq j$ in $\{1, \ldots, m\}$ and the parts of μ are $\{\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_m\}$.

Suppose that there exists a chain of strict partitions $\lambda = \lambda^{(0)}, \ldots, \lambda^{(t)} = \eta$ such that $\lambda^{(i)}$ is obtained from $\lambda^{(i-1)}$ by removing an e-bar for $1 \leq i \leq t$. Then if there does not exist a partition of $n - (t+1)e$ which can be obtained by removing an e-bar from η, η is called an *e-bar core* of λ.

The block theory of $\widetilde{S}_n$ is discussed in [13] and [59]. $\widetilde{S}_n$ has two types of characters – non-faithful characters with $\langle z \rangle$ in their kernel which correspond to the characters of S_n, and faithful characters known as *spin characters*. Spin characters are parametrized by the strict partitions of n with a strict partition λ labelling a unique spin character if $\epsilon(\lambda) = -1$ and labelling a pair of spin characters called *associates* if $\epsilon(\lambda) = 1$. The distribution of irreducible characters of $\widetilde{S}_n$ into ℓ-blocks for odd ℓ is given in [13, Theorems A and B].

The cycle type of $g \in \widetilde{S}_n$ is defined to be the cycle type of $\theta(g)$, its image in S_n. When $g \in \widetilde{S}_n$ has odd cycle type then by results of Schur and Morris (see [13, Theorems 3 (1) and 7]), the values of the spin characters on g can be calculated using an analogue of the Murnaghan

Nakayama formula. In particular, $\chi(g) \in \mathbb{Q}$ for every spin character χ of $\widetilde{S}_n$ if g has odd cycle type.

Proposition 3.1.4. *Suppose that b is a block of $k\widetilde{S}_n$. Then $mf(b) = 1$.*

Proof. Let b be a block of $k\widetilde{S}_n$. First we consider the case when $\ell = 2$. The 2-blocks of $\widetilde{S}_n$ are in one-to-one correspondence with the 2-blocks of S_n by [57, Ch. 5, Theorem 8.11]. Therefore each 2-block of $\widetilde{S}_n$ contains at least one rational valued character of S_n, so $\sigma(b) = b$ and $frob(b) = mf(b) = 1$ by Proposition 2.2.5 (b).

Assume now that ℓ is odd. Then S_n is a quotient of $\widetilde{S}_n$ by a central ℓ'-subgroup, so by Lemma 2.1.10 (c), $k\widetilde{S}_n$ has two types of blocks – blocks which dominate unique blocks of kS_n, and blocks which do not dominate any block of kS_n. Suppose first that b dominates a block $\bar{b}$ of kS_n and recall that then b is the unique block dominating $\bar{b}$. Then since $\sigma\left(\bar{b}\right) = \bar{b}$ it follows from Lemma 2.2.2 that $\sigma(b) = b$ and $frob(b) = mf(b) = 1$.

Now suppose that b does not dominate a block of kS_n. Then $\mathrm{Irr}(b)$ contains only spin characters. Let χ be one such character and suppose that χ is labelled by a strict partition λ of n. If $\epsilon(\lambda) = 1$ then χ is the unique character labelled by λ, and by [13, Theorem 3 (2)] $\chi(g) \neq 0$ only if g has odd cycle type. Thus as discussed above, $\chi(g) \in \mathbb{Q}$ for all $g \in G$, so $\sigma(b) = b$ and $frob(b) = mf(b) = 1$ by Proposition 2.2.5 (b).

If $\epsilon(\lambda) = -1$ then λ labels two spin characters, χ and its associate χ'. As shown in [13, Theorems A and B], there are two possibilities to consider. If λ is equal to its ℓ-bar core then χ and χ' lie in separate blocks of defect zero so $mf(b) = 1$ by Proposition 2.2.5 (d). If λ is not equal to its ℓ-bar core, then χ and χ' are both in $\mathrm{Irr}(b)$. In that case, if g has cycle type λ then by [13, Theorem 3 (3)], $\chi(g) = -\chi'(g)$, so $(\chi + \chi')(g) = 0$. If g has cycle type different to λ, then $\chi(g)$ and $\chi'(g)$ are non-zero only if g has odd cycle type, so $\chi(g)$ and $\chi'(g)$ are rational valued. Therefore $(\chi + \chi')(g) \in \mathbb{Q}$ for all $g \in G$, so again, $\sigma(b) = b$ and $frob(b) = mf(b) = 1$ by Proposition 2.2.5 (b). ∎

Definition 3.1.5. The *double cover of* A_n is $\widetilde{A}_n = \theta^{-1}(A_n)$ where $\theta : \widetilde{S}_n \to S_n$ is as given in Definition 3.1.3.

Note that the double cover of A_n is unique up to isomorphism – if $\hat{A}_n$ is defined analogously using $\hat{S}_n$ instead of $\widetilde{S}_n$, then $\hat{A}_n$ is isomorphic to $\widetilde{A}_n$. Again, see [41, Chapter 2] for more details.

Proposition 3.1.6. *Suppose that* b *is a block of* $k\widetilde{A}_n$. *Then* $mf(b) = 1$.

Proof. First we note that $\widetilde{A}_n \lhd \widetilde{S}_n$, and by the proof of Proposition 3.1.4, every block $\hat{b}$ of $\widetilde{S}_n$ either has defect zero or satisfies $\sigma(\hat{b}) = \hat{b}$. The result therefore follows from Lemma 2.2.8. □

3.2 The sporadic groups, their covers, and the exceptional covering groups

In this section we deal with the sporadic groups, the Tits group, the covers of the sporadic groups and the so-called ‘exceptional covering groups’, obtained from [37, Table 6.1.3].

The Sporadic groups and the Tits group [64, 1.2 (v)]

Mathieu groups	M_{11}, M_{12}, M_{22}, M_{23}, M_{24}
Leech lattice groups	Co_1, Co_2, Co_3, McL, HS, Suz, J_2
Fischer groups	Fi_{22}, Fi_{23}, Fi'_{24}
Monstrous groups	M, B, Th, HN, He
Pariahs	J_1, J_3, J_4, ON, Ly, Ru
The Tits group	${}^2F_4(2)'$

The covers of the sporadic groups [37, Table 6.1.3]

$2.M_{12}$, $12.M_{22}$, $2.J_2$, $3.J_3$, $2.Co_1$, $2.HS$, $6.Suz$, $3.McL$, $3.ON$, $2.Ru$, $6.Fi_{22}$, $3.Fi'_{24}$, $2.B$

The exceptional covering groups [37, Table 6.1.3]

$2.L_2(4)$, $2.L_3(2)$, $2.L_3(4)$, $4_1.L_3(4)$, $4_2.L_3(4)$, $6.L_3(4)$, $4^2.L_3(4)$, $12_1.L_3(4)$,
$12_2.L_3(4)$, $(4^2 \times 3).L_3(4)$, $2.L_4(2)$, $2.U_4(2)$, $2.U_6(2)$, $6.U_6(2)$, $2^2.U_6(2)$, $(2^2 \times 3).U_6(2)$,
$3.A_6$, $6.A_6$, $2.S_6(2)$, $2.Sz(8)$, $2^2.Sz(8)$, $2.O_8^+(2)$, $2^2.O_8^+(2)$, $2.G_2(4)$, $2.F_4(2)$,
$2.{}^2E_6(2)$, $6.{}^2E_6(2)$, $2^2.{}^2E_6(2)$, $(2^2 \times 3).{}^2E_6(2)$, $3_1.U_4(3)$, $3_2.U_4(3)$, $6_1.U_4(3)$, $6_2.U_4(3)$,
$3^2.U_4(3)$, $12_1.U_4(3)$, $12_2.U_4(3)$, $(3^2 \times 4).U_4(3)$, $3.O_7(3)$, $6.O_7(3)$, $3.G_2(3)$, $3.A_7$, $6.A_7$

Remark 3.2.1. [Exceptional Covering Groups] Let G be a finite simple group and let U be a universal central extension of G. The Schur multiplier $M(G)$ of G is the kernel of the surjection $U \to G$, contained in $Z(U)$ [37, Definition 5.1.6]. Suppose that the exceptional part of $M(G)$, as defined in [37, Definition 6.1.3], is non-trivial and let $\chi \in \mathrm{Irr}(U)$.

The *centre* of χ is defined to be $Z(\chi) = \{g \in U \mid |\chi(g)| = \chi(1)\}$. By [44, Lemma 2.27 (d)], if χ is faithful then $Z(\chi)$ is cyclic, and by [44, Corollary 2.28], $Z(U) \leq Z(\chi)$. Thus if $M(G)$ is not cyclic, then U has no faithful characters, and by Lemma 2.1.10 (b), every block of U dominates a unique block of U/Z for some $Z \leq Z(U)$ such that U/Z has cyclic centre. By the proof of Lemma 2.1.10 (c), it follows that if $M(G)$ is not cyclic then every block of U is isomorphic to a block of U/Z for some for some $Z \leq Z(U)$ such that U/Z has cyclic centre.

Thus, to find the Morita Frobenius numbers of the blocks of the exceptional covering groups of a simple group G, it is enough to conisder the blocks of U/Z where U is a universal central extension of G, and $Z \leq Z(U)$ is such that U/Z has cyclic centre.

Proposition 3.2.2. *Suppose that G is one of the groups listed in the three tables above. Then for any ℓ-block b of G, $mf(b) = 1$.*

Proof. By Remark 3.2.1, it is enough to consider the cases when G is a sporadic group, the Tits group, a cover of a sporadic group, or a quotient H/Z with cyclic centre where H is an exceptional covering group listed above, and $Z \leq Z(H)$. Let G be one of these groups and let ℓ be a prime dividing $|G|$.

By examination in GAP [34], if (G, ℓ) is not in Table 1 or 2 below, then there are no non-principal ℓ-blocks of G with equal, non-cyclic defect, which have the same number and degrees of characters, none of which are rational valued. It follows that every ℓ-block b of G either has cyclic defect or is stabilized by Galois conjugation. Thus by Proposition 2.2.5 (d) and Lemma 2.2.4, $mf(b) = 1$ for every ℓ-block b of G.

Suppose now that G is one of the groups listed in Table 1. Then by examination in GAP [34], there exist non-principal ℓ-blocks of G with equal, non-cyclic defect, which have the same number and degrees of characters, none of which are rational valued. However, there exists a finite group $\hat{G}$, listed in the third column, such that $G \lhd \hat{G}$ and by examination in GAP [34], for every ℓ-block $\hat{b}$ of $\hat{G}$, either $\hat{b}$ has cyclic defect or $\sigma(\hat{b}) = \hat{b}$. It follows that $mf(b) = 1$ for every block b of G by Lemma 2.2.8.

Table 1: $G \lhd \hat{G}$ and for every ℓ-block $\hat{b}$ of $\hat{G}$,
either $\hat{b}$ has cyclic defect or $\sigma(\hat{b}) = \hat{b}$

G	ℓ	$\hat{G}$
$12.M_{22}$	$2,3$	$12.M_{22}.2$
$3.J_3$	2	$3.J_3.2$
$6.Suz$	$2,5$	$6.Suz.2$
$3.McL$	$2,5$	$3.McL.2$
$3.ON$	$2,7$	$3.ON.2$
$6.Fi_{22}$	$2,5$	$6.Fi_{22}.2$
$3.Fi'_{24}$	$2,5,7$	$3.Fi'_{24}.2$
$4_1.L_3(4)$	3	$4_1.L_3(4).2_1$
$4_2.L_3(4)$	3	$4_2.L_3(4).2_1$
$6.L_3(4)$	2	$6.L_3(4).2_2$
$12_1.L_3(4)$	2	$12_1.L_3(4).2_2$
$12_1.L_3(4)$	3	$12_1.L_3(4).2_1$
$12_2.L_3(4)$	3	$12_2.L_3(4).2_2$

G	ℓ	$\hat{G}$
$12_2.L_3(4)$	3	$12_2.L_3(4).2_1$
$6.U_6(2)$	2	$6.U_6(2).2$
$6.A_6$	2	$6.A_6.2_1$
$6.{}^2E_6(2)$	$2,3,5,7$	$6.{}^2E_6(2).2$
$3.A_6$	2	$3.A_6.2_1$
$3_1.U_4(3)$	2	$3_1.U_4(3).2_1$
$3_2.U_4(3)$	2	$3_2.U_4(3).2_1$
$12_1.U_4(3)$	3	$12_1.U_4(3).2_2$
$12_2.U_4(3)$	3	$12_2.U_4(3).2_3$
$3.O_7(3)$	2	$3.O_7(3).2$
$3.G_2(3)$	2	$3.O_7(3).2$
$3.A_7$	2	$3.A_7.2$
$6.A_7$	2	$6.A_7.2$

Finally, let G be one of the groups listed in Table 2. By examination in GAP [34], there exist non-principal ℓ-blocks of G with equal, non-cyclic defect, which have the same number and degrees of characters, none of which are rational valued. However, for every pair of ℓ-blocks b_1, b_2 of G which are non-principal with equal, non-cyclic defect and the same number and degrees of characters, none of which are rational valued, there exists a single block $\hat{b}$ of a finite group $\hat{G}$ with $G \lhd \hat{G}$ such that $\hat{b}$ covers both b_1 and b_2. Therefore b_1 and b_2 are $\hat{G}$ conjugate so there is a group automorphism of G whose induced k-algebra automorphism of kG sends b_1 to b_2. It follows that $mf(b) = 1$ for every block b of G by Lemma 2.2.6.

Table 2: ℓ-blocks of G checked case by case*

G	ℓ	Defects of Bl (G)	$\hat{G}$	Defects of Bl $(\hat{G})$
$6_1.U_4(3)$	2	$[8,1,1,1,8,8,1,1]$	$6_1.U_4(3).2_1$	$[9,2,2,2,8,1]$
$6_2.U_4(3)$	2	$[8,1,1,1,8,8]$	$6_2.U_4(3).2_1$	$[9,2,2,2,8]$
$12_1.U_4(3)$	2	$[9,2,2,2,9,9,2,2]$	$12_1.U_4(3).2_1$	$[10,3,3,3,9,2]$
			$12_1.U_4(3).2_2$	$[10,2,3,10,10,3,3]$
$12_2.U_4(3)$	2	$[9,2,2,2,9,9]$	$12_2.U_4(3).2_1$	$[10,3,3,3,9]$
			$12_2.U_4(3).2_3$	$[10,2,3,10,10]$
$6.O_7(3)$	2	$[10,4,2,1,1,10,10,4,4]$	$6.O_7(3).2$	$[11,5,3,2,2,10,4]$

* The defects of the ℓ-blocks of G are listed in column 3. They are obtained from GAP [34] using the PrimeBlocks function. If two blocks b_1 and b_2 of G are non-principal with equal, non-cyclic defect and the same number and degrees of characters, none of which are rational valued, then their defects have the same colour in column 3. If a block $\hat{b}$ of $\hat{G}$ covers both b_1 and b_2 then the defect of $\hat{b}$ in column 5 has the same colour as that of b_1 and b_2. □

Chapter 4

Blocks of finite groups of Lie type: background

4.1 Finite groups of Lie type

4.1.1 Introduction to linear algebraic groups

We start this chapter with a very brief introduction to algebraic groups. More information can be found in [18], [55] and [23].

Let F be an algebraically closed field and let $I \lhd F[X_1, \ldots, X_n]$ be an ideal of the ring of polynomial functions in n variables over F. A subset of F^n annihilated by an ideal I is called an *algebraic set* and the *Zariski topology* on F^n is defined by taking complements of algebraic sets as open sets. An *affine algebraic variety* is an algebraic set $V(I)$, for a radical ideal I, together with the induced Zariski topology. Associated to an affine algebraic variety is a coordinate algebra $F[X_1, \ldots, X_n]/I$, the algebra of polynomial functions on $V(I)$. A map between two affine algebraic varieties is called a *morphism (of algebraic varieties)* if it is defined by polynomial functions in the coordinates. If $V(I)$ and $V(J)$ are two affine algebraic varieties with $I \lhd F[X_1, \ldots, X_n]$ and $J \lhd F[Y_1, \ldots, Y_m]$ for some integers n and m, then the cartesian product $V(I) \times V(J)$ equipped with the Zariski topology is an affine algebraic variety, realised as an algebraic set in F^{n+m}.

A *linear algebraic group*, often referred to just as an *algebraic group*, is an affine algebraic variety $\mathbf{G}$ with a group structure such that the group operation $m : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and the inverse map $i : \mathbf{G} \to \mathbf{G}$ are morphisms of varieties. The general linear group $GL_n(F) = \{(a_{ij}) \in F^{n \times n} : det(a_{ij}) \neq 0\}$ plays an important role in the theory of linear algebraic groups. Firstly, $GL_n(F)$ itself is a linear algebraic group. This is clearly illustrated if we re-express the definition as

$$GL_n(F) = \{(a_{11}, \dots, a_{nn}, b) \in F^{n^2+1} : b \ \det(a_{ij}) = 1\},$$

since the determinant map is a polynomial function. Any closed subgroup of $GL_n(F)$ is also a linear algebraic group. On the other hand, every linear algebraic group can be embedded as a closed subgroup into $GL_n(F)$ for some n [55, Theorem 1.7].

A linear algebraic group $\mathbf{G}$ is *connected* if it is irreducible as a topological space. The maximal irreducible varieties in $\mathbf{G}$ are called its *connected components* or just *components*. We denote the connected component of $\mathbf{G}$ containing the identity element by $\mathbf{G}^{\circ}$. This is a closed normal subgroup of finite index in $\mathbf{G}$ and the connected components of $\mathbf{G}$ are cosets of $\mathbf{G}^{\circ}$ [55, Proposition 1.13 (b)]. If $\mathbf{G}$ is connected, its *dimension* is defined to be the transcendence degree of the field of fractions of the coordinate algebra $F[X_1, \dots, X_m]/I$, over F, where I is the radical ideal defining $\mathbf{G}$. If $\mathbf{G}$ has more than one connected component then $\dim(\mathbf{G}) = \dim(\mathbf{G}^{\circ})$.

The *multiplicative group* $\mathbf{G}_m \cong GL_1(F)$ and the *additive group* $\mathbf{G}_a \cong (F, +)$ are algebraic groups of dimension 1. A *torus* of a linear algebraic group $\mathbf{G}$ is a subgroup $\mathbf{T}$ of $\mathbf{G}$ which is isomorphic to $\mathbf{G}_m \times \cdots \times \mathbf{G}_m$ for some number of copies of $\mathbf{G}_m$. Since all maximal tori of $\mathbf{G}$ are conjugate [55, Corollary 6.5], we can define the *rank* of $\mathbf{G}$ to be the dimension of the maximal tori, denoted by $\mathrm{rk}(\mathbf{G})$. A *Borel subgroup* of $\mathbf{G}$ is a maximal closed, connected, solvable subgroup of $\mathbf{G}$. The Borel subgroups of $\mathbf{G}$ are conjugate [55, Theorem 6.4 (a)].

Let $g \in \mathbf{G}$ and let $\rho : \mathbf{G} \to GL_n(F)$ be an embedding of algebraic groups. Then g is called *semisimple* if $\rho(g)$ is diagonalizable, and *unipotent* if $\rho(g) - 1$ is nilpotent, and it can be shown that these definitions are independent of the choice of ρ. By Jordan decomposition, every $g \in \mathbf{G}$ can be decomposed uniquely into $g = g_s g_u = g_u g_s$ where g_s is semisimple and g_u is unipotent [55, Theorem 2.5]. A group consisting entirely of unipotent elements is called a *unipotent group*, and we denote the subset of all unipotent elements of $\mathbf{G}$ by $\mathbf{G}_u$.

The *radical* of $\mathbf{G}$, $R(\mathbf{G})$, is the maximal closed connected solvable normal subgroup of $\mathbf{G}$. If $\mathbf{G}$ is connected and $R(\mathbf{G}) = 1$ then $\mathbf{G}$ is called *semisimple*. The *unipotent radical* of $\mathbf{G}$, $R_u(\mathbf{G}) = R(\mathbf{G})_u$, is the maximal closed connected normal unipotent subgroup of $\mathbf{G}$. If $R_u(\mathbf{G})$ is trivial then $\mathbf{G}$ is called *reductive*. The structure of connected reductive algebraic groups is well understood. If $\mathbf{G}$ is connected reductive then $R(\mathbf{G})$ is a torus equal to $Z(\mathbf{G})^\circ$ [55, Proposition 6.20 (a)], and $\mathbf{G} = Z(\mathbf{G})^\circ[\mathbf{G}, \mathbf{G}]$ [55, Corollary 8.22]. This product is almost direct, that is, the intersection $[\mathbf{G}, \mathbf{G}] \cap Z(\mathbf{G})^\circ$ is finite [23, Proposition 0.19].

Let $\mathbf{G}$ be a connected algebraic group. A closed subgroup of $\mathbf{G}$ containing a Borel subgroup is called a *parabolic subgroup* of $\mathbf{G}$. A parabolic subgroup of $\mathbf{G}$ admits a *Levi decomposition*. This means that $\mathbf{P} = R_u(\mathbf{P}) \rtimes \mathbf{L}$ for some closed subgroup $\mathbf{L}$ called a *Levi subgroup of* $\boldsymbol{P}$ [23, Proposition 1.15]. A *Levi subgroup of* $\boldsymbol{G}$ is a closed subgroup of $\mathbf{G}$ which is the Levi subgroup of some parabolic $\mathbf{P}$ of $\mathbf{G}$, and a Levi subgroup of $\mathbf{G}$ is a connected reductive algebraic group [55, Proposition 12.6]. For any maximal torus $\mathbf{T}$ contained in $\mathbf{P}$, there is a unique Levi subgroup of $\mathbf{P}$ containing $\mathbf{T}$, and any two Levi subgroups of $\mathbf{P}$ are conjugate by an element of $R_u(\mathbf{P})$ [23, Proposition 1.17, Corollary 1.18]. A Levi subgroup $\mathbf{L}$ is the centralizer of its central torus, $Z(\mathbf{L})^\circ$, and the centralizer of any torus $\mathbf{T}$ of $\mathbf{G}$ is a Levi subgroup of some parabolic subgroup of $\mathbf{G}$ [55, Proposition 12.6, Proposition 12.10].

We note that since a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ contains a maximal torus $\mathbf{T}$ of $\mathbf{G}$, and $\mathbf{T}$ is also a maximal torus of $\mathbf{L}$, since all maximal tori of $\mathbf{L}$ are conjugate, it follows that all the maximal tori of $\mathbf{L}$ are maximal tori of $\mathbf{G}$.

Lemma 4.1.1. *Let $\boldsymbol{G}$ be a connected algebraic group and let $\boldsymbol{L}$ be a Levi subgroup of $\boldsymbol{G}$. Then* $\dim\, Z^\circ(\boldsymbol{L}) = \mathrm{rk}(\boldsymbol{G}) - \mathrm{rk}\,([\boldsymbol{L},\boldsymbol{L}])$.

Proof. Since $\mathbf{L}$ is a connected reductive algebraic group, $\mathbf{L} = Z^\circ(\mathbf{L})[\mathbf{L},\mathbf{L}]$. Let $\mathbf{T}$ be a maximal torus of $[\mathbf{L},\mathbf{L}]$. Then $Z^\circ(\mathbf{L})\mathbf{T}$ is a torus of $\mathbf{L}$. We claim that $Z^\circ(\mathbf{L})\mathbf{T}$ is maximal in $\mathbf{L}$.

Suppose $\mathbf{S} \supseteq Z^\circ(\mathbf{L})\mathbf{T}$ is another torus of $\mathbf{L}$. Then since $\mathbf{L} = Z^\circ(\mathbf{L})[\mathbf{L},\mathbf{L}]$, $[\mathbf{L},\mathbf{L}] \trianglelefteq \mathbf{L}$ and $Z^\circ(\mathbf{L}) \leq \mathbf{S}$,

$$\mathbf{S} = Z^\circ(\mathbf{L})\,(\mathbf{S} \cap [\mathbf{L},\mathbf{L}])\,.$$

The intersection $\mathbf{S} \cap [\mathbf{L},\mathbf{L}]$ is a torus of $[\mathbf{L},\mathbf{L}]$ containing the maximal torus $\mathbf{T}$ of $[\mathbf{L},\mathbf{L}]$, therefore $\mathbf{S} \cap [\mathbf{L},\mathbf{L}] = \mathbf{T}$. Thus $\mathbf{S} = Z^\circ(\mathbf{L})\mathbf{T}$ so $Z^\circ(\mathbf{L})\mathbf{T}$ is a maximal torus of $\mathbf{L}$, as claimed. Since the maximal tori of $\mathbf{L}$ are maximal tori of $\mathbf{G}$, therefore $Z^\circ(\mathbf{L})\mathbf{T}$ is also a maximal torus of $\mathbf{G}$. Hence $\mathrm{rk}(\mathbf{G}) = \dim(Z^\circ(\mathbf{L})\mathbf{T})$.

Since $Z^\circ(\mathbf{L}) \cap [\mathbf{L},\mathbf{L}]$ is finite and $Z^\circ(\mathbf{L})$ and $\mathbf{T}$ are tori, $\mathrm{rk}(\mathbf{G}) = \dim\, Z^\circ(\mathbf{L}) + \dim\, \mathbf{T}$. As $\mathbf{T}$ is a maximal torus of $[\mathbf{L},\mathbf{L}]$, it follows that $\dim\, Z^\circ(\mathbf{L}) = \mathrm{rk}(\mathbf{G}) - \mathrm{rk}\,([\mathbf{L},\mathbf{L}])$. $\square$

Definition 4.1.2. Let s be a semisimple element of a connected reductive algebraic group $\mathbf{G}$. Then s is *quasi-isolated* if there does not exist a Levi subgroup $\mathbf{L}$ of a proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $C_{\mathbf{G}}(s) \subseteq \mathbf{L}$, and s is *isolated* if there does not exist a Levi subgroup $\mathbf{L}$ of a proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $C^\circ_{\mathbf{G}}(s) \subseteq \mathbf{L}$.

4.1.2 Root systems

Definition 4.1.3. Suppose that E is a finite dimensional real vector space with a positive definite inner product $\langle\cdot,\cdot\rangle$. A finite subset Φ of vectors of E is called an *(abstract) root system* in E if the following conditions hold.

- Φ is finite, non-empty, and spans E
- For any $\alpha \in \Phi$, if $c\alpha \in \Phi$ for some $c \in \mathbb{R}$, then $c = \pm 1$

- For each $\alpha \in \Phi$ there exists a reflection of $GL(E)$ denoted by s_α which sends α to $-\alpha$ and fixes Φ (as a set)

- (Crystallographic condition) For every $\alpha, \beta \in \Phi$, $s_\alpha.\beta - \beta = n\alpha$ for some integer $n \in \mathbb{Z}$

The subgroup $\langle s_\alpha \mid \alpha \in \Phi \rangle$ of $\mathrm{GL}(E)$ generated by the reflections s_α is called the *Weyl group* of Φ.

A *base* of a root system Φ is a subset $\Delta \subseteq \Phi$ such that for any $\beta \in \Phi$, $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with either $c_\alpha \leq 0$ for every $\alpha \in \Delta$ or $c_\alpha \geq 0$ for every $\alpha \in \Delta$. A *positive root* of Φ with respect to a base Δ is an element $\alpha \in \Phi$ that can be expressed as a non-negative linear combination of elements in Δ. We denote the set of positive roots by Φ^+ and make an analogous definition for negative roots and denote them by Φ^-.

A root system Φ with a base Δ is *indecomposable* if Δ cannot be partitioned into two mutually orthogonal, non-empty subsets. The indecomposable root systems can be classified according to their Dynkin diagrams. The *Dynkin diagram* of a root system Φ is a graph with one node for each element in the base Δ. Two nodes labelled by $\alpha, \beta \in \Delta$ are joined by a number of edges depending on the order of the product of the reflections s_α and s_β; they are not joined if $o(s_\alpha s_\beta) = 2$, they have one edge between them if $o(s_\alpha s_\beta) = 3$, two edges if $o(s_\alpha s_\beta) = 4$, and three edges if $o(s_\alpha s_\beta) = 6$. Arrows are drawn on edges between roots of different lengths, pointing from the longer root to the shorter one. It can be shown that up to isomorphism, an indecomposable Dynkin diagram has one of the following types [55, Theorem 9.6].

G	Dynkin diagram of **G**
A_n	$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \cdots - \overset{\alpha_{n-1}}{\circ} - \overset{\alpha_n}{\circ}$
B_n	$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \cdots - \overset{\alpha_{n-1}}{\circ} \Rightarrow \overset{\alpha_n}{\circ}$
C_n	$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \cdots - \overset{\alpha_{n-1}}{\circ} \Leftarrow \overset{\alpha_n}{\circ}$
D_n	$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \cdots - \underset{\substack{\vert \\ \circ\ \alpha_n}}{\overset{\alpha_{n-2}}{\circ}} - \overset{\alpha_{n-1}}{\circ}$
G_2	$\overset{\alpha_1}{\circ} \Rrightarrow \overset{\alpha_2}{\circ}$
F_4	$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} \Rightarrow \overset{\alpha_3}{\circ} - \overset{\alpha_4}{\circ}$
E_6	$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \underset{\substack{\vert \\ \circ\ \alpha_6}}{\overset{\alpha_3}{\circ}} - \overset{\alpha_4}{\circ} - \overset{\alpha_5}{\circ}$
E_7	$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \underset{\substack{\vert \\ \circ\ \alpha_7}}{\overset{\alpha_3}{\circ}} - \overset{\alpha_4}{\circ} - \overset{\alpha_5}{\circ} - \overset{\alpha_6}{\circ}$
E_8	$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \overset{\alpha_3}{\circ} - \overset{\alpha_4}{\circ} - \underset{\substack{\vert \\ \circ\ \alpha_8}}{\overset{\alpha_5}{\circ}} - \overset{\alpha_6}{\circ} - \overset{\alpha_7}{\circ}$

Definition 4.1.4. An *abstract root datum* is a quadruple $(X, R, Y, \check{R})$ such that the following hold.

- X and Y are free abelian groups of the same rank, and there is a non-degenerate inner product $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{Z}$ defining a perfect pairing betwen X and Y. That is, any homomorphism $X \to \mathbb{Z}$ has the form $\chi \mapsto \langle\chi,\gamma\rangle$ for some $\gamma \in Y$, and any homomorphism $Y \to \mathbb{Z}$ has the form $\gamma \mapsto \langle\chi,\gamma\rangle$ for some $\chi \in X$.

- $R \subseteq X$ and $\check{R} \subseteq Y$ are abstract root systems in $\mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{Z}\check{R} \otimes_{\mathbb{Z}} \mathbb{R}$ respectively

- There is a bijection between R and $\check{R}$ such that if α corresponds to $\check{\alpha}$ then $\langle\alpha,\check{\alpha}\rangle = 2$

- The reflections defined by the roots in R and $\check{R}$ are given by

$$\begin{aligned} s_\alpha\chi &= \chi - \langle\chi,\check{\alpha}\rangle\alpha \quad \text{for all } \chi \in X, \\ s_{\check{\alpha}}\gamma &= \gamma - \langle\alpha,\gamma\rangle\check{\alpha} \quad \text{for all } \gamma \in Y. \end{aligned}$$

Two root data $(X_1, R_1, Y_1, \check{R}_1)$ and $(X_2, R_2, Y_2, \check{R}_2)$ are said to be *isomorphic* if there exists a group isomorphism $\varphi : X_2 \to X_1$ with transpose map $\varphi' : Y_1 \to Y_2$ such that $\langle\varphi(\chi_2),\gamma\rangle = \langle\chi_2,\varphi'(\gamma)\rangle$ for all $\chi_2 \in X_2, \gamma \in Y_1$ such that $\varphi(R_2) = R_1$ and $\varphi'(\check{R}_1) = \check{R}_2$.

Let $\mathbf{G}$ be a connected reductive group with a maximal torus $\mathbf{T} \cong \mathbf{G}_m \times \cdots \times \mathbf{G}_m$. The *character group* of $\mathbf{T}$ is $X(\mathbf{T}) \coloneqq \mathrm{Hom}\,(\mathbf{T},\mathbf{G}_m)$ and the *cocharacter group* of $\mathbf{T}$ is $Y(\mathbf{T}) \coloneqq \mathrm{Hom}\,(\mathbf{G}_m,\mathbf{T})$. It is possible to construct an abstract root datum from $X(\mathbf{T})$ and $Y(\mathbf{T})$ in the following way.

Let $\chi \in X(\mathbf{T})$. Then for any $(t_1,\dots,t_n) \in \mathbf{T}$, $\chi(t_1,\dots,t_n) = t_1^{a_1}\dots t_n^{a_n}$ for some $a_1,\dots,a_n \in \mathbb{Z}$. Thus χ is determined by n integers $a_1,\dots,a_n$, and therefore $X(\mathbf{T}) \cong \mathbb{Z}^n$. Similarly any cocharacter $\gamma \in Y(\mathbf{T})$ is determined by n integers because for any $c \in \mathbf{G}_m$, $\gamma(c) = (c^{a_1},\dots,c^{a_n})$ for some $a_1,\dots,a_n \in \mathbb{Z}$, so $Y(\mathbf{T}) \cong \mathbb{Z}^n$. Note that $\chi \circ \gamma \in \mathrm{Hom}\,(\mathbf{G}_m,\mathbf{G}_m)$ and any homomorphism $\mathbf{G}_m \to \mathbf{G}_m$ is of the form $c \mapsto c^a$ for some integer a, for $c \in \mathbf{G}_m$. We can therefore define an inner product $\langle\cdot,\cdot\rangle : X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$ by setting $\langle\chi,\gamma\rangle \coloneqq a \in \mathbb{Z}$ such that $\chi \circ \gamma(c) = c^a$ for any $c \in \mathbf{G}_m$. This inner product defines a perfect pairing (as given in Definition 4.1.4) between $X(\mathbf{T})$ and $Y(\mathbf{T})$ [55, Proposition 3.6].

We now determine a subset $\Phi \subseteq X(\mathbf{T})$ of roots, following [18, Section 1.9]. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ and let $\mathbf{U} = R_u(\mathbf{B})$. Then $\mathbf{B}$ has a unique semidirect product decomposition $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$. Let $\mathbf{B}^-$ denote the unique Borel subgroup of $\mathbf{G}$ such that $\mathbf{B} \cap \mathbf{B}^- = \mathbf{T}$ and let $\mathbf{U}^- = R_u(\mathbf{B}^-)$. Then $\mathbf{U}$ and $\mathbf{U}^-$ are connected and normalized by $\mathbf{T}$, and are maximal unipotent subgroups of $\mathbf{G}$. Let $\{\mathbf{U}_i\}$ be the minimal nontrivial subgroups of $\mathbf{U}$ and $\mathbf{U}^-$ normalized by $\mathbf{T}$. Then $\mathbf{U}_i \cong \mathbf{G}_a$ for each i and $\mathbf{T}$ acts on these groups by conjugation, defining a homomorphism $\mathbf{T} \to \mathrm{Aut}\ \mathbf{G}_a$. Homomorphisms $\mathbf{G}_a \to \mathbf{G}_a$ are of the form $c \mapsto \lambda c$ for some $\lambda \in k^\times$, therefore $\mathrm{Aut}\ \mathbf{G}_a \cong \mathbf{G}_m$, so the action of $\mathbf{T}$ defines a homomorphism $\mathbf{T} \to \mathbf{G}_m$. In other words, for each $\mathbf{U}_i$ the action of $\mathbf{T}$ determines an element of $\mathrm{Hom}(\mathbf{T}, \mathbf{G}_m) = X(\mathbf{T})$. Let Φ be the set of elements of $X(\mathbf{T})$ determined in this way by the action of $\mathbf{T}$ on some $\mathbf{U}_i$.

The set of roots Φ is independent of the choice of the Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. The positive roots are those coming from subgroups $\mathbf{U}_i$ of $\mathbf{U}$ and the negative roots come from the subgroups $\mathbf{U}_i$ of $\mathbf{U}^-$. This fixes a base Δ for Φ: a root $\alpha \in \Phi$ is a base element if α cannot be expressed as a sum of two elements of Φ^+. We call the minimal subgroups $\mathbf{U}_i$ *root subgroups*, and label them according to the root they define, $\{\mathbf{U}_\alpha\}_{\alpha \in \Phi}$. For each $\alpha \in \Phi$ there exists a unique cocharacter $\check{\alpha} \in Y(\mathbf{T})$ such that $s_\alpha . \chi = \chi - \langle \chi, \check{\alpha} \rangle \alpha$ for all $\chi \in X(\mathbf{T})$ by [55, Lemma 8.19]. Define $\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi\}$ to be the coroots. Then we have the following important result.

Theorem 4.1.5 (Chevalley, [55, Proposition 9.11 and Theorem 9.13])**.**

(a) Let $\boldsymbol{G}$ be a connected reductive group and let $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{G}$. Let Φ and $\check{\Phi}$ be as defined in the last paragraph. Then $\big(X(\boldsymbol{T}), \Phi, Y(\boldsymbol{T}), \check{\Phi}\big)$ is an abstract root datum. A different choice of maximal torus gives rise to an abstract root datum isomorphic to $\big(X(\boldsymbol{T}), \Phi, Y(\boldsymbol{T}), \check{\Phi}\big)$.

(b) For any abstract root datum $(X, R, Y, \check{R})$ there exists a semisimple group with a maximal torus $\boldsymbol{T}$ such that its root datum with respect to $\boldsymbol{T}$ is isomorphic to $(X, R, Y, \check{R})$.

(c) Two semisimple groups are isomorphic if and only if their root data are isomorphic.

*(d) A semisimple group **G** corresponding to the abstract root datum $(X, R, Y, \check{R})$ is simple if and only if R is indecomposable.*

Let **G** be a simple algebraic group. We say that **G** is of *classical* type if the Dynkin diagram associated to the root datum of **G** is of type A_n, B_n, C_n or D_n, and say that **G** is of *exceptional* type otherwise. A Dynkin diagram does not uniquely determine the isomorphism class of a group. We say that two groups with the same Dynkin diagram are *isogenous*.

Given a root datum $(X, R, Y, \check{R})$, we call $\mathbb{Z}R \subseteq X$ the *root lattice* and $\mathbb{Z}\check{R} \subseteq Y$ the *coroot lattice*. The connected reductive groups determined by $(X, R, Y, \check{R})$ are semisimple if and only if rank $(\mathbb{Z}R)$ = rank X. Suppose that rank $(\mathbb{Z}R)$ = rank X. Let $\Omega = \text{Hom }(\mathbb{Z}\check{R}, \mathbb{Z})$. Because of the perfect pairing between X and Y, $X \cong \text{Hom }(Y, \mathbb{Z})$ [55, Proposition 3.6]. Thus there exists an injective restriction map

$$X \cong \text{ Hom }(Y, \mathbb{Z}) \longrightarrow \text{ Hom }(\mathbb{Z}\check{R}, \mathbb{Z}) = \Omega,$$

so we can view X, and therefore $\mathbb{Z}R \subseteq X$, as subgroups of Ω. The *fundamental group* of the root system R is defined to be $\Lambda(R) = \Omega/\mathbb{Z}R$. The fundamental group of R is independent of X, and each X satisfying $\mathbb{Z}R \subseteq X \subseteq \Omega$ determines a different root datum for a fixed root system R, up to automorphisms of Ω stabilising the roots R.

Each simple group **G** with a particular Dynkin diagram determines an X such that $\mathbb{Z}R \subseteq X \subseteq \Omega$. The isomorphism class of **G** is determined by the position of X between $\mathbb{Z}R$ and Ω. We say that **G** is *of adjoint type* if $X = \mathbb{Z}R$ and *of simply connected type* if $X = \Omega$. For a given isogeny class, there are only finitely many possibilities for X.

Definition 4.1.6. Let Φ be a root system. A prime number ℓ is said to be *good* for Φ if and only if $(\mathbb{Z}\Phi/\mathbb{Z}\beta)_\ell = \{0\}$ for every $\beta \in \Phi$. If **G** is a connected reductive group with root system Φ then ℓ is *good for **G*** if ℓ is good for Φ. If ℓ is not good for **G**, then we say that ℓ is *bad* for **G**.

Definition 4.1.7. Let $\mathbf{G}$ be a connected reductive group with maximal torus $\mathbf{T}$ and corresponding root datum $(X(\mathbf{T}), \Phi, Y(\mathbf{T}), \check{\Phi})$. A connected reductive group $\mathbf{G}^*$ is called a *dual group* of $\mathbf{G}$ if there exists a maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$ such that if $\left(X(\mathbf{T}^*), \Phi^*, Y(\mathbf{T}^*), \check{\Phi}^*\right)$ is the root datum of $\mathbf{G}^*$ with respect to $\mathbf{T}^*$, then there exists an isomorphism from $X(\mathbf{T})$ to $Y(\mathbf{T}^*)$ such that Φ is mapped isomorphically onto $\check{\Phi}$. In this case $\left(X(\mathbf{T}^*), \Phi^*, Y(\mathbf{T}^*), \check{\Phi}^*\right)$ is isomorphic to $(Y(\mathbf{T}), \check{\Phi}, X(\mathbf{T}), \Phi)$ and we say that the pair $(\mathbf{G}, \mathbf{T})$ is dual to the pair $(\mathbf{G}^*, \mathbf{T}^*)$.

4.1.3 Finite groups of Lie type

In this section let $\mathbf{G}$ be a linear algebraic group defined over $\overline{\mathbb{F}}_p$, an algebraic closure of the finite field of p elements. Let $i : \mathbf{G} \to GL_n\left(\overline{\mathbb{F}}_p\right)$ be an embedding of $\mathbf{G}$ into $GL_n\left(\overline{\mathbb{F}}_p\right)$ as discussed in Section 4.1.1. Let $q = p^a$ for some $a \in \mathbb{N}$ and let $F_q : GL_n\left(\overline{\mathbb{F}}_p\right) \to GL_n\left(\overline{\mathbb{F}}_p\right)$ denote the homomorphism given by $F_q\left((x_{ij})\right) = (x_{ij}^q)$ for every matrix $(x_{ij}) \in GL_n\left(\overline{\mathbb{F}}_p\right)$.

A homomorphism $F : \mathbf{G} \to \mathbf{G}$ is called a *standard Frobenius morphism (with respect to an* $\mathbb{F}_q$*-structure)* if there exists a q such that $(i \circ F)(g) = (F_q \circ i)(g)$ for every $g \in \mathbf{G}$. A homomorphism $F : \mathbf{G} \to \mathbf{G}$ is called a *Frobenius morphism (with respect to an* $\mathbb{F}_q$*-structure)* if there exists an $m \in \mathbb{N}$ such that F^m is a standard Frobenius morphism. Note that the terminology in [55] is different – what we call a standard Frobenius morphism is called a Frobenius morphism in [55], and what we call a Frobenius morphism is called a Steinberg morphism.

Although a Frobenius morphism is a bijective map, it is not an isomorphism of algebraic groups because the inverse map is not a polynomial function in the coordinates, and therefore is not a morphism of algebraic groups. The important point about Frobenius morphisms is that they are surjective homomorphisms such that the group of fixed points is finite. For a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$ we denote the group of fixed points by

$$\mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\},$$

and call groups of this form *finite groups of Lie type*.

Theorem 4.1.8 (Lang-Steinberg Theorem [55, Theorem 21.7]). *Let $\boldsymbol{G}$ be a connected linear algebraic group defined over $\overline{\mathbb{F}}_p$ and let $F : \boldsymbol{G} \to \boldsymbol{G}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure. The Lang map, $\mathcal{L} : \boldsymbol{G} \to \boldsymbol{G}$ given by $\mathcal{L}(g) = g^{-1}F(g)$, is surjective.*

For any Frobenius morphism F, $\mathbf{G}$ contains a pair of subgroups $\mathbf{T} \leq \mathbf{B}$ where $\mathbf{T}$ is an F-stable maximal torus and $\mathbf{B}$ is an F-stable Borel subgroup. For a given Frobenius morphism, any two such pairs $\mathbf{T} \leq \mathbf{B}$ are $\mathbf{G}^F$-conjugate [55, Corollary 21.12]. We call an F-stable maximal torus $\mathbf{T}$ contained in an F-stable Borel $\mathbf{B}$ a *maximally split* torus of $\mathbf{G}$.

A Frobenius morphism F is $\mathbb{F}_q$*-split* if there exists an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that $F(t) = t^q$ for all $t \in \mathbf{T}$. The *Chevalley groups* are the finite groups of Lie type which arise as the fixed points of $\mathbb{F}_q$-split Frobenius morphisms. If F is not split then it is called *twisted*. In this case F is the product of an $\mathbb{F}_q$-split endomorphism and an algebraic automorphism of $\mathbf{G}$ which induces a symmetry ρ of the Dynkin diagram of $\mathbf{G}$ (ignoring the arrows), see [55, Definition 22.4 and Theorem 11.11]. The fixed points of twisted Frobenius morphisms are called *twisted groups*. If F is twisted and ρ is the non-trivial symmetry of a Dynkin diagram of type B_2, G_2 or F_4 (still ignoring the arrows), then F is called *very twisted*. There are three types of very twisted finite groups of Lie type – the Suzuki groups ${}^2B_2(q^2)$ where $q^2 = 2^{2a+1}$, the small Ree groups ${}^2G_2(q^2)$ where $q^2 = 3^{2a+1}$ and the large Ree groups ${}^2F_4(q)$ with $q^2 = 2^{2a+1}$, for a some positive integer.

Definition 4.1.9. Suppose that $\mathbf{G}$ and $\mathbf{G}^*$ are connected reductive groups defined over $\mathbb{F}_q$ with maximal tori $\mathbf{T}$ and $\mathbf{T}^*$ such that $(\mathbf{G}, \mathbf{T})$ and $(\mathbf{G}^*, \mathbf{T}^*)$ are dual pairs as in Definition 4.1.7. Let $F : \mathbf{G} \to \mathbf{G}$ and $F^* : \mathbf{G}^* \to \mathbf{G}^*$ be Frobenius morphisms with respect to an $\mathbb{F}_q$-structure on $\mathbf{G}$ and $\mathbf{G}^*$ respectively. If $\mathbf{T}$ is F-stable, $\mathbf{T}^*$ is F^*-stable, and the isomorphism $X(\mathbf{T}) \cong Y(\mathbf{T}^*)$ is compatible with the action of the Frobenius morphisms, then we say that $(\mathbf{G}, F)$ is *dual* to $(\mathbf{G}^*, F^*)$.

Remark 4.1.10. If $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ are dual then since it will be clear which Frobenius we are referring to, we drop the * notation and just write F for both Frobenius maps.

4.1.4 The order of finite groups of Lie type, e-tori and $e_\ell(q)$

For any finite group of Lie type $\mathbf{G}^F$, there exists an *order polynomial* $P_{\mathbf{G}^F}(x) \in \mathbb{Z}[x]$. This is the unique polynomial such that $P_{\mathbf{G}^F}(q^m) = \left|\mathbf{G}^{F^m}\right|$ for infinitely many $m \in \mathbb{N}$ and it is independent of the isogeny type of $\mathbf{G}$.

Let e be a natural number. The *e-th cyclotomic polynomial* is the minimal polynomial of a primitive e-th root of unity over $\mathbb{Q}$, denoted by Φ_e. Many features of the representation theory of $\mathbf{G}^F$ over k do not in fact depend on ℓ, the characteristic of k, but only on the cyclotomic factor of $|\mathbf{G}^F|$ divisible by ℓ. For further discussion of this see, for example, [35]. It is thus often useful to factorise the order polynomial of $\mathbf{G}^F$ into a product of cyclotomic polynomials,

$$|\mathbf{G}^F| = q^{|\Phi^+|} \sum_{e \geq 1} \Phi_e(q)^{a(e)}$$

for some integers $a(e)$ [55, Section 25.1].

We define

$$e_\ell(q) := \begin{cases} \text{the order of } q \text{ modulo } \ell & \text{if } \ell \text{ is odd} \\ \text{the order of } q \text{ modulo } 4 & \text{if } \ell = 2. \end{cases}$$

Then for $\ell \neq 2$, $e_\ell(q)$ is the minimal e such that $\ell | \Phi_e(q)$. Note that $e_\ell(q) < \ell$, and if $\ell = 2$ or 5 then $e_\ell(q) \neq 3$.

An F-stable torus $\mathbf{T}$ of $\mathbf{G}$ is called an *e-torus* if $P_{\mathbf{T}^F}(x)$ is a power of the e-th cyclotomic polynomial, $\Phi_e(x)$. The e-tori of $\mathbf{G}$ satisfy a Sylow theory analogous to the theory of Sylow subgroups of finite groups. We say that an F-stable torus $\mathbf{T}$ is a *Sylow e-torus* of $\mathbf{G}$ if $P_{\mathbf{T}^F}(q) = \Phi_e(q)^{a(e)}$ where $a(e)$ is precisely the power of $\Phi_e(q)$ dividing $P_{\mathbf{G}^F}(q)$. By the Generic Sylow Theorems [55, Theorem 25.11], for every $e \geq 1$ there exists a Sylow e-torus of $\mathbf{G}$, any two Sylow e-tori are $\mathbf{G}^F$-conjugate, and any e-torus of $\mathbf{G}$ is contained in a Sylow e-torus. The original proofs of these results can be found in [10], as well as some further discussion on cyclotomic polynomials. We let $\mathbf{G}_{\Phi_e}$ denote a Sylow e-torus of $\mathbf{G}$. A Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is called *e-split* if it is the centralizer in $\mathbf{G}$ of an e-torus of $\mathbf{G}$.

Lemma 4.1.11. *Let $\boldsymbol{G}$ be a connected reductive algebraic group and let s be a semisimple element of $\boldsymbol{G}$ contained in an e-split Levi subgroup $\boldsymbol{L}$ of $\boldsymbol{G}$. Then $C^{\circ}_{\boldsymbol{L}}(s)$ is an e-split Levi subgroup of $C^{\circ}_{\boldsymbol{G}}(s)$.*

Proof. Since $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$, $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T})$ for some e-torus $\mathbf{T}$ of $\mathbf{G}$. Then $C_{\mathbf{L}}(s) = C_{C_{\mathbf{G}}(\mathbf{T})}(s) = C_{C_{\mathbf{G}}(s)}(\mathbf{T}) = C_{\mathbf{G}}(s) \cap C_{\mathbf{G}}(\mathbf{T})$ so $C^{\circ}_{\mathbf{L}}(s) = (C_{\mathbf{G}}(s) \cap C_{\mathbf{G}}(\mathbf{T}))^{\circ} \subseteq C^{\circ}_{\mathbf{G}}(s) \cap C^{\circ}_{\mathbf{G}}(\mathbf{T}) = C_{C^{\circ}_{\mathbf{G}}(s)}(\mathbf{T})$. On the other hand, $C_{C^{\circ}_{\mathbf{G}}(s)}(\mathbf{T})$ is the centralizer of an e-torus in a connected group, so it is a Levi subgroup of $C^{\circ}_{\mathbf{G}}(s)$. In particular, $C_{C^{\circ}_{\mathbf{G}}(s)}(\mathbf{T})$ is connected, so $C_{C^{\circ}_{\mathbf{G}}(s)}(\mathbf{T}) \subseteq C^{\circ}_{C_{\mathbf{G}}(s)}(\mathbf{T}) = C^{\circ}_{\mathbf{L}}(s)$. Therefore $C^{\circ}_{\mathbf{L}}(s) = C_{C^{\circ}_{\mathbf{G}}(s)}(\mathbf{T})$ so $C^{\circ}_{\mathbf{L}}(s)$ is the centralizer of an e-torus of $C^{\circ}_{\mathbf{G}}(s)$, hence an e-split Levi subgroup of $C^{\circ}_{\mathbf{G}}(s)$. ☐

Lemma 4.1.12. *Let $\boldsymbol{G}$ be a connected reductive algebraic group defined over $\overline{\mathbb{F}}_p$ and let $F : \boldsymbol{G} \to \boldsymbol{G}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure. Suppose ℓ is good for $\boldsymbol{G}$ and let $e = e_{\ell}(q)$. If Φ_e divides the order of $\boldsymbol{G}^F$ precisely once and if $\Phi_{e\ell^i}$ does not divide the order of $\boldsymbol{G}^F$ for any $i \geq 1$, then the blocks of $k\boldsymbol{G}^F$ have cyclic defect.*

Proof. By definition, defect groups of blocks of $k\mathbf{G}^F$ are finite ℓ-subgroups of $\mathbf{G}^F$. Since $\ell | \Phi_e(q)$ and $\Phi_{e\ell^i}(q)$ does not divide the order of $\mathbf{G}^F$ for any $i \geq 1$, $\Phi_e(q)$ is the only factor of $|\mathbf{G}^F|$ which is divisible by ℓ. Thus the ℓ-subgroups of $\mathbf{G}^F$ are contained in the fixed points of a Sylow e-torus of $\mathbf{G}^F$. Since Φ_e divides $|\mathbf{G}^F|$ precisely once, it follows from [10, Proposition 3.3] that the fixed point groups of the Sylow e-tori of $\mathbf{G}^F$ are cyclic of order $\Phi_e(q)$. Therefore all defect groups of blocks of $k\mathbf{G}^F$ are cyclic. ☐

The $\mathbb{F}_q$*-rank* of a torus $\mathbf{T}$ is the rank of a maximal subtorus $\mathbf{T}'$ of $\mathbf{T}$ such that there exists an isomorphism $\mathbf{T}' \cong (\mathbf{G}_m)^a$ defined over $\mathbb{F}_q$ where $a = \mathrm{rk}(\mathbf{T}')$. The $\mathbb{F}_q$*-rank* of an algebraic group $\mathbf{G}$ is the $\mathbb{F}_q$-rank of a maximally split torus of $\mathbf{G}$ [23, Definition 8.3] and we let $\varepsilon_{\mathbf{G}} := (-1)^{\mathbb{F}_q\text{-rank of } \mathbf{G}}$.

4.2 Character theory of finite groups of Lie type

From now until the end of Chapter 4, we fix the following setting. Let p be a prime different from ℓ and let $\mathbf{G}$ be a connected reductive algebraic group defined over $\overline{\mathbb{F}}_p$. Fix q, a power of p, and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure. Let $\mathbf{G}^F$ be the fixed points of $\mathbf{G}$ under F – a finite group of Lie type.

4.2.1 Harish-Chandra and Deligne-Lusztig induction and restriction

In order to apply inductive arguments to ℓ-blocks of finite reductive groups, we need a way to relate the characters of $\mathbf{G}$ to characters of subgroups of $\mathbf{G}$. The first maps introduced to do this are called *Harish-Chandra induction* and *restriction*. Let $\mathbf{P}$ be an F-stable parabolic subgroup of $\mathbf{G}$ and let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{P}$. Then Harish-Chandra induction is given by

$$\begin{aligned} R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : \quad \mathbb{C}\mathbf{L}^F\text{-mod} \quad &\longrightarrow \quad \mathbb{C}\mathbf{G}^F\text{-mod} \\ V \quad &\longmapsto \quad \mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{P}^F}\mathrm{Inf}^{\mathbf{P}^F}_{\mathbf{L}^F}(V), \end{aligned}$$

which is just just inflation from $\mathbf{L}^F$ to $\mathbf{P}^F$ followed by induction from $\mathbf{P}^F$ to $\mathbf{G}^F$. This is discussed in greater detail in [23, Example 4.6 (iii)]. Harish-Chandra restriction is given by

$$\begin{aligned} {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : \quad \mathbb{C}\mathbf{G}^F\text{-mod} \quad &\longrightarrow \quad \mathbb{C}\mathbf{L}^F\text{-mod} \\ V \quad &\longmapsto \quad \mathrm{Fix}_{R_u(\mathbf{P})^F}(V). \end{aligned}$$

A more powerful pair of adjoint maps can be defined using cohomology. These maps, known as *Deligne-Lusztig induction* and *restriction* can be defined even if the Levi subgroup $\mathbf{L}$ is not contained in an F-stable parabolic $\mathbf{P}$. Details about the construction of these maps can be found in [23, Chapters 10 and 11]. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ contained in a parabolic subgroup $\mathbf{P}$ which is not necessarily F-stable. Then Deligne-Lusztig induction and restriction are linear maps denoted by

$$\begin{aligned} R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}: &\quad \mathbb{Z}\mathrm{Irr}\left(\mathbf{L}^F\right) \longrightarrow \mathbb{Z}\mathrm{Irr}\left(\mathbf{G}^F\right), \\ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}: &\quad \mathbb{Z}\mathrm{Irr}\left(\mathbf{G}^F\right) \longrightarrow \mathbb{Z}\mathrm{Irr}\left(\mathbf{L}^F\right). \end{aligned}$$

If $\mathbf{P}$ is F-stable then Deligne-Lusztig induction reduces to Harish-Chandra induction (see [23, Chapter 11, page 81]) so there is no ambiguity in using the same notation for the Harish-Chandra and Deligne-Lusztig maps.

Let $\mathbf{P}$ and $\mathbf{P}'$ be parabolics of $\mathbf{G}$, and let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{P}$, $\mathbf{L}'$ an F-stable Levi subgroup of $\mathbf{P}'$. Let $Y = \{x \in \mathbf{G} \mid \mathbf{L} \cap {}^x\mathbf{L}'\}$ and let X be a set of representatives of the double cosets $\mathbf{L}^F \backslash Y / \mathbf{L}'^F$. Then the *Mackey formula* is

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{L}'\subset\mathbf{P}'} = \sum_{x\in X} R^{\mathbf{L}}_{(\mathbf{L}\cap{}^x\mathbf{L}')\subset(\mathbf{P}\cap{}^x\mathbf{P}')} \circ {}^*R^{{}^x\mathbf{L}'}_{(\mathbf{L}\cap{}^x\mathbf{L}')\subset(\mathbf{P}\cap{}^x\mathbf{L}')} \circ \text{ ad } x $$

where ad $x(\chi) = {}^x\chi$ for all $\chi \in \mathrm{Irr}\left(\mathbf{L}'^F\right)$, $x \in X$. When the Mackey formula holds, it is possible to show that $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is independent of the choice of $\mathbf{P}$, see [23, Proposition 6.1]. By [7, Theorem], the Mackey formula holds if $\mathbf{L}$ is an F-stable maximal torus of $\mathbf{G}$, and holds in all cases for Harish-Chandra induction and restriction. It also holds in all cases for Deligne-Lusztig induction and restriction except possibly when $\mathbf{G}^F$ contains a component of type ${}^2E_6(2), E_7(2)$ or $E_8(2)$. Where we know that Deligne-Lusztig induction and restriction do not depend on the choice of the parabolic we will drop the reference to $\mathbf{P}$ and just write $R^{\mathbf{G}}_{\mathbf{L}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}}$.

The properties of Deligne-Lusztig induction and restriction are particularly well understood when $\mathbf{L} = \mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$. In this case $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ is called a *Deligne-Lusztig character*, for any $\theta \in \mathrm{Irr}(\mathbf{T}^F)$.

Definition 4.2.1. A class function is called a *uniform function* if it is a linear combination of Deligne-Lusztig characters. The orthogonal projection of class functions onto the space of uniform functions is called *uniform projection* and is given by the operator

$$\frac{1}{|\mathbf{G}^F|}\sum_{\mathbf{T}}\left|\mathbf{T}^F\right| R_{\mathbf{T}}^{\mathbf{G}}\circ {}^*R_{\mathbf{T}}^{\mathbf{G}},$$

where the sum runs over all F-stable maximal tori $\mathbf{T}$ of $\mathbf{G}$. The uniform projection of an irreducible character $\chi\in\mathrm{Irr}\big(\mathbf{G}^F\big)$ uniquely determines a set of multiplicities of χ as an irreducible constituent of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for every F-stable maximal torus $\mathbf{T}$, and every $\theta\in\mathrm{Irr}\big(\mathbf{T}^F\big)$.

$$\left\{\langle\chi,R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle\mid \mathbf{T}\text{ is an }F\text{-stable maximal torus of }\mathbf{G}\text{ and }\theta\in\ \mathrm{Irr}\left(\mathbf{T}^F\right)\right\}$$

Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ contained in a parabolic subgroup $\mathbf{P}$. The *Green functions*, denoted by $Q_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$, send elements $(u,v)\in\mathbf{G}_u^F\times\mathbf{L}_u^F$ to values in $\mathbb{Z}$ [23, Definition 12.1]. Let $g\in\mathbf{G}$ and let $g=su$ be the Jordan decomposition of g. The *character formula* for Deligne-Lusztig induction is given by

$$\left(R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\chi)\right)(g)=\frac{1}{|\mathbf{L}^F||C_{\mathbf{G}}^{\circ}(s)^F|}\sum_{\{h\in\mathbf{G}^F|s\in{}^h\mathbf{L}\}}|C_{{}^h\mathbf{L}}^{\circ}(s)^F|\sum_{v\in C_{{}^h\mathbf{L}}^{\circ}(s)_u^F}Q_{C_{{}^h\mathbf{L}}^{\circ}(s)}^{C_{{}^h\mathbf{G}}^{\circ}(s)}(u,v^{-1})^h\chi(sv),\quad(4.1)$$

see [23, Proposition 12.2].

4.2.2 Lusztig series and Jordan decomposition

Many of the results of the following section can be found in [23] and [5]. We follow the notation of [5]. From now until the end of Chapter 4, we fix a connected reductive group $\mathbf{G}^*$ such that $(\mathbf{G},F)$ and $(\mathbf{G}^*,F)$ are dual pairs in the sense of Definition 4.1.9.

Let $\nabla(\mathbf{G},F)$ denote the set of all pairs $(\mathbf{T},\theta)$ such that $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and θ is a linear character of $\mathbf{T}^F$. Let $\nabla^*(\mathbf{G},F)$ denote the set of all pairs $(\mathbf{T}^*,s)$ such that $\mathbf{T}^*$ is an F-stable maximal torus of the dual group $\mathbf{G}^*$ and s is a semisimple element in $\mathbf{T}^{*F}$. By [23, Proposition 13.13] we can fix a bijection

$$\left\{\begin{array}{c}\mathbf{G}^F\text{-conjugacy classes of}\\ (\mathbf{T},\theta)\in\nabla(\mathbf{G},F)\end{array}\right\}\longleftrightarrow\left\{\begin{array}{c}\mathbf{G}^{*F}\text{-conjugacy classes of}\\ (\mathbf{T}^*,s)\in\nabla^*(\mathbf{G},F)\end{array}\right\}. \tag{4.2}$$

If $(\mathbf{T},\theta)$ corresponds to $(\mathbf{T}^*,s)$ via this bijection, then we write $(\mathbf{T},\theta)\overset{\mathbf{G}}{\longleftrightarrow}(\mathbf{T}^*,s)$, and we can denote $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ by $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$.

There are two different types of conjugacy for elements in the fixed points $\mathbf{G}^{*F}$. Let $s,s'\in\mathbf{G}^{*F}$ be semisimple. Then s and s' are *geometrically conjugate* if there exists $g\in\mathbf{G}^*$ such that $s={}^gs'$, and s and s' are *rationally conjugate* if there exists $g\in\mathbf{G}^{*F}$ such that $s={}^gs'$. We denote the geometric conjugacy class of $s\in\mathbf{G}^{*F}$ by (s) and the rational conjugacy class of s by $[s]$. Clearly rational conjugacy implies geometric conjugacy, but not the other way around. If $Z(\mathbf{G})$ is connected then geometric and rational conjugacy is the same in $\mathbf{G}^{*F}$, [5, Proposition 9.7].

Using bijection (4.2), we say that two pairs $(\mathbf{T},\theta)$, $(\mathbf{T}',\theta')$ are *geometrically conjugate* (respectively *rationally conjugate*) if $(\mathbf{T},\theta)\overset{\mathbf{G}}{\longleftrightarrow}(\mathbf{T}^*,s)$ and $(\mathbf{T}',\theta')\overset{\mathbf{G}}{\longleftrightarrow}(\mathbf{T}'^*,s')$ for some pairs $(\mathbf{T}^*,s)$, $(\mathbf{T}'^*,s')\in\nabla^*(\mathbf{G},F)$ such that s and s' are geometrically conjugate (respectively rationally conjugate). We can then fix the following notation.

$$\begin{aligned}
\nabla^*(\mathbf{G},F,(s)) &:= \{(\mathbf{T}^*,s')\in\nabla^*(\mathbf{G},F)\mid s'\in(s)\}\\
\nabla^*(\mathbf{G},F,[s]) &:= \{(\mathbf{T}^*,s')\in\nabla^*(\mathbf{G},F)\mid s'\in[s]\}\\
\nabla(\mathbf{G},F,(s)) &:= \left\{(\mathbf{T},\theta)\in\nabla(\mathbf{G},F)\mid(\mathbf{T},\theta)\overset{\mathbf{G}}{\longleftrightarrow}(\mathbf{T}^*,s')\text{ for some }(\mathbf{T}^*,s')\in\nabla^*(\mathbf{G},F,(s))\right\}\\
\nabla(\mathbf{G},F,[s]) &:= \left\{(\mathbf{T},\theta)\in\nabla(\mathbf{G},F)\mid(\mathbf{T},\theta)\overset{\mathbf{G}}{\longleftrightarrow}(\mathbf{T}^*,s')\text{ for some }(\mathbf{T}^*,s')\in\nabla^*(\mathbf{G},F,[s])\right\}
\end{aligned}$$

Let $\mathbf{T}$ be an F-stable torus of $\mathbf{G}$ and let $n\in\mathbb{N}$. We define the norm map of F^n at F by

$$\begin{array}{rccc} N_{F^n/F}: & \mathbf{T} & \longrightarrow & \mathbf{T}\\ & t & \longmapsto & t.F(t)\dots F^{n-1}(t).\end{array}$$

We will need the following property of $N_{F^n/F}$ in Section 5.2.2.

Lemma 4.2.2. *The norm map restricts to a surjection* $N_{F^n/F} : \boldsymbol{T}^{F^n} \to \boldsymbol{T}^F$.

Proof. First suppose that $t \in \mathbf{T}^{F^n}$. Then

$$
\begin{aligned}
F\left(N_{F^n/F}(t)\right) &= F(t.F(t)\dots F^{n-1}(t)) \\
&= F(t)F^2(t)\dots F^{n-1}(t)F^n(t) \\
&= F(t)F^2(t)\dots F^{n-1}(t)t \\
&= N_{F^n/F}(t)
\end{aligned}
$$

since $t \in \mathbf{T}$ is central. Therefore $N_{F^n/F}(\mathbf{T}^{F^n}) \subseteq \mathbf{T}^F$.

Now let $t \in \mathbf{T}$. By the Lang-Steinberg Theorem (Theorem 4.1.8), since F^n is a Frobenius morphism, there exists $u \in \mathbf{T}$ such that $t = uF^n(u^{-1})$. Then

$$
\begin{aligned}
N_{F^n/F}\left(uF(u^{-1})\right) &= uF(u^{-1})F(u)F^2(u^{-1})\dots F^{n-1}(u)F^n(u^{-1}) \\
&= uF^n(u^{-1}),
\end{aligned}
$$

so $N_{F^n/F}\left(uF(u^{-1})\right) = t$, showing that $N_{F^n/F}$ surjects onto $\mathbf{T}$.

Now suppose that $t \in \mathbf{T}^F$ and suppose that $s \in \mathbf{T}$ is such that $N_{F^n/F}(s) = t$. Then $F\left(N_{F^n/F}(s)\right) = N_{F^n/F}(s)$ therefore

$$
\begin{aligned}
F(s)F^2(s)\dots F^n(s) &= sF(s)\dots F^{n-1}(s) \\
F^n(s) &= s
\end{aligned}
$$

Therefore $s \in \mathbf{T}^{F^n}$ as required. □

The following result gives us a useful alternative formulation for geometric conjugacy of pairs in $\nabla(\mathbf{G}, F)$, see [5, Corollaire 9.5].

Lemma 4.2.3. *Two pairs $(\boldsymbol{T},\theta),(\boldsymbol{T'},\theta')\in\nabla(\boldsymbol{G},F)$ are geometrically conjugate if and only if there exists $n\in\mathbb{N}$ and $g\in\boldsymbol{G}^{F^n}$ such that ${}^g\boldsymbol{T}=\boldsymbol{T'}$ and for all $x\in\boldsymbol{T'}$,*

$$\theta'\left(N_{F^n/F}(x)\right)=\theta\left(N_{F^n/F}(gxg^{-1})\right).$$

Definition 4.2.4. Let $s\in\mathbf{G}^{*F}$ be semisimple. The *Lusztig series* associated to $[s]$, the rational conjugacy class of s, is

$$\mathcal{E}\left(\mathbf{G}^F,s\right)=\left\{\chi\in\ \mathrm{Irr}(\mathbf{G}^F)\mid\langle\chi,R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle\neq0\text{ for some }(\mathbf{T},\theta)\in\nabla\left(\mathbf{G},F,[s]\right)\right\}.$$

Note that some authors use the notation $\mathcal{E}(\mathbf{G}^F,(s))$ and $\mathcal{E}(\mathbf{G}^F,[s])$ to differentiate between Lusztig series corresponding to geometric and rational conjugacy classes of s. We will only use the Lusztig series corresponding to a rational conjugacy class of s so we drop the brackets in our notation. The irreducible characters in $\chi\in\mathcal{E}(\mathbf{G}^F,1)$ are called *unipotent* and a block containing a unipotent character is called a *unipotent block*.

By [23, Proposition 13.1], every irreducible character $\chi\in\mathrm{Irr}\big(\mathbf{G}^F\big)$ appears as a component of a Deligne-Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and some irreducible character $\theta\in\mathrm{Irr}\big(\mathbf{T}^F\big)$. The pair $(\mathbf{T},\theta)$ determines a $\mathbf{G}^{*F}$-conjugacy class of pairs $(\mathbf{T}^*,s)$ by bijection (4.2). More precisely, we have a partitioning of $\mathrm{Irr}\big(\mathbf{G}^F\big)$ according to Lusztig series, [23, Proposition 13.17].

Proposition 4.2.5. *The irreducible characters of $\boldsymbol{G}^F$ are partitioned by the Lusztig series for each rational conjugacy class of semisimple elements in $\boldsymbol{G}^{*F}$.*

$$\mathrm{Irr}\left(\boldsymbol{G}^F\right)=\coprod_{[s]}\mathcal{E}(\boldsymbol{G}^F,s)$$

We denote by $\mathcal{E}(\mathbf{G}^F, \ell')$ the set of all characters appearing in some $\mathcal{E}\left(\mathbf{G}^F, s\right)$ where s is a semisimple ℓ' element of $\mathbf{G}^{*F}$,

$$\mathcal{E}(\mathbf{G}^F, \ell') = \bigcup_{s \text{ an } \ell'\text{-element of } \mathbf{G}^{*F}} \mathcal{E}\left(\mathbf{G}^F, s\right)$$

Recall that $\mathbf{G}_\ell$ denotes the set of elements of $\mathbf{G}$ with order a power of ℓ.

Theorem 4.2.6 (Broué-Michel, Hiss)**.** *Let $\boldsymbol{G}$ be a connected reductive algebraic group and let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ' element. Then*

$$\mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right) \coloneqq \bigcup_{t \in C_{\boldsymbol{G}^*}(s)_\ell^F} \mathcal{E}(\boldsymbol{G}^F, st)$$

is a union of ℓ-blocks of $\boldsymbol{G}^F$. Moreover, each ℓ-block in $\mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$ contains an irreducible character of $\mathcal{E}\left(\boldsymbol{G}^F, s\right)$.

The original proof of the first part of this theorem is in [12]; for a full proof of both parts see [16, Theorem 9.12]. If b is an ℓ-block in $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ we write $b \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$, and the sum of block idempotents of blocks in $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ is denoted by $e_s^{\mathbf{G}^F}$.

Definition 4.2.7. Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ' element and suppose that $b \in \mathcal{E}_\ell(\mathbf{G}^F, s)$. Then b is a *(quasi-)isolated* block if s is (quasi-)isolated in $\mathbf{G}^*$.

In 1984 Lusztig proved a very important correspondence between the irreducible characters of a finite group of Lie type $\mathbf{G}^F$, and the unipotent characters of a subgroup of $\mathbf{G}^F$, [53].

Theorem 4.2.8 (Jordan decomposition for $Z(\mathbf{G})$ connected [16, Theorem 15.8] and [23, Remark 13.24])**.** *Let $\boldsymbol{G}$ be a connected reductive algebraic group and let $F : \boldsymbol{G} \to \boldsymbol{G}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure where $q = p^a$ for some prime p and some $a \in \mathbb{N}$. Assume that $\boldsymbol{G}$ has connected centre. Let $(\boldsymbol{G}^*, F)$ be dual to $(\boldsymbol{G}, F)$ and let $s \in \boldsymbol{G}^{*F}$ be a semisimple element. Then there exists a bijection*

$$\Psi_{\boldsymbol{G},s} : \mathcal{E}(C_{\boldsymbol{G}^*}(s)^F, 1) \longrightarrow \mathcal{E}\left(\boldsymbol{G}^F, s\right),$$

such that

$$\varepsilon_{\boldsymbol{G}}\langle \Psi_{\boldsymbol{G},s}(\chi), R^{\boldsymbol{G}}_{\boldsymbol{T}^*}(s)\rangle_{\boldsymbol{G}^F} = \varepsilon_{C_{\boldsymbol{G}^*}(s)}\langle \chi, R^{C_{\boldsymbol{G}^*}(s)}_{\boldsymbol{T}^*}(1)\rangle_{C_{\boldsymbol{G}^*}(s)^F}, \tag{4.3}$$

for all $\chi \in \mathcal{E}(C_{\boldsymbol{G}^*}(s)^F, 1)$, *where* $\varepsilon_{\boldsymbol{G}}$ *is as defined in Section 4.1.4. Moreover, for every* $\chi \in \mathcal{E}(C_{\boldsymbol{G}^*}(s)^F, 1)$,

$$\Psi_{\boldsymbol{G},s}(\chi)(1) = \frac{|\boldsymbol{G}^F|_{p'}}{|C_{\boldsymbol{G}^*}(s)^F|_{p'}}\chi(1).$$

Note that property (4.3) does not uniquely determine the bijection $\Psi_{\mathbf{G},s}$. It is, however, possible to add extra conditions to ensure that $\Psi_{\mathbf{G},s}$ is uniquely defined, as shown in [22, Part II]. If $Z(\mathbf{G})$ is not connected, then the centralizers of semisimple elements in $\mathbf{G}^*$ are not necessarily connected. If $C_{\mathbf{G}^*}(s)$ is not connected then we define a unipotent character of $C_{\mathbf{G}^*}(s)^F$ to be an irreducible character which covers a unipotent character of $C^\circ_{\mathbf{G}^*}(s)^F$. Jordan decomposition can be adapted to cater for this situation, but first we need some more theory.

4.2.3 Regular embeddings

Definition 4.2.9. Let $\mathbf{G}$ be a connected reductive group. A *regular embedding* of $\mathbf{G}$ is a homomorphism of algebraic groups $i : \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ where $\widetilde{\mathbf{G}}$ is a connected reductive algebraic group defined over $\mathbb{F}_q$ such that $Z(\widetilde{\mathbf{G}})$ is connected and $[\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}] \subseteq \mathbf{G}$. For example, let $\mathbf{G} = \mathrm{SL}_n$ and $\widetilde{\mathbf{G}} = \mathrm{GL}_n$. Then the inclusion map $i : \mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$ is a regular embedding.

Given a regular embedding $i : \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ and a maximal torus $\mathbf{T}$ of $\mathbf{G}$, let $\widetilde{\mathbf{T}} = i(\mathbf{T}).Z(\widetilde{\mathbf{G}})$. Then there exist pairs $(\mathbf{G}^*, \mathbf{T}^*)$ and $(\widetilde{\mathbf{G}}^*, \widetilde{\mathbf{T}}^*)$ which are dual to $(\mathbf{G}, \mathbf{T})$ and $(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$ respectively, and there exists a morphism $i^* : \widetilde{\mathbf{G}} \twoheadrightarrow \mathbf{G}$ dual to i. Note that i^* is surjective and restricts to a surjection on the fixed points, $i^* : \widetilde{\mathbf{G}}^{*F} \to \mathbf{G}^{*F}$ [5, Corollaire 2.7]. Then $\ker i^* \subseteq Z(\widetilde{\mathbf{G}}^*)$. The idea of a regular embedding is that we can first work in the "nicer" setting of $\widetilde{\mathbf{G}}$ where the centre is connected, and then use i^* to determine properties of $\mathbf{G}$. For a given connected reductive group $\mathbf{G}$ with maximal torus $\mathbf{T}$ it is possible to explicitly construct a suitable $\widetilde{\mathbf{G}}$, see [16, Section 15.1].

For the rest of this section, we fix a regular embedding $i : \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ and semisimple ℓ' elements $s \in \mathbf{G}^{*F}$ and $\tilde{s} \in \widetilde{\mathbf{G}}^{*F}$ such that $i^*(\tilde{s}) = s$. Since the centre of $\widetilde{\mathbf{G}}$ is connected, we have the following results.

Theorem 4.2.10 ([5, Théorème 3.5])**.** *The centralizers of semisimple elements in $\widetilde{\boldsymbol{G}}^*$ are connected.*

Proposition 4.2.11 ([5, Corollaire 4.4])**.** *For any Levi subgroup $\widetilde{\boldsymbol{L}}$ of $\widetilde{\boldsymbol{G}}$, $Z(\widetilde{\boldsymbol{L}})$ is connected.*

As discussed in [5, Section 9B and Corollaire 9.5], we can define two restriction maps $\mathfrak{Res}_{\mathbf{G}}^{\widetilde{\mathbf{G}}} : \nabla\left(\widetilde{\mathbf{G}}, F, (\tilde{s})\right) \to \nabla\left(\mathbf{G}, F, (s)\right)$ and ${}^*\mathfrak{Res}_{\mathbf{G}}^{\widetilde{\mathbf{G}}} : \nabla^*\left(\widetilde{\mathbf{G}}, F, (\tilde{s})\right) \to \nabla^*\left(\mathbf{G}, F, (s)\right)$ given by

$$\begin{aligned} \mathfrak{Res}_{\mathbf{G}}^{\widetilde{\mathbf{G}}}\left(\widetilde{\mathbf{T}}, \tilde{\theta}\right) &= \left(\widetilde{\mathbf{T}} \cap \mathbf{G},\ \mathrm{Res}_{\widetilde{\mathbf{T}}^F \cap \mathbf{G}^F}^{\widetilde{\mathbf{T}}^F} \tilde{\theta}\right) \\ {}^*\mathfrak{Res}_{\mathbf{G}}^{\widetilde{\mathbf{G}}}\left(\widetilde{\mathbf{T}}^*, \tilde{s}\right) &= \left(i^*\left(\widetilde{\mathbf{T}}^*\right), i^*\left(\tilde{s}\right)\right). \end{aligned}$$

These maps satisfy the following.

Lemma 4.2.12 ([5, Lemme 9.3])**.** *(a) Let $(\widetilde{\boldsymbol{T}}, \tilde{\theta}) \in \nabla(\widetilde{\boldsymbol{G}}, F)$ and $(\widetilde{\boldsymbol{T}}^*, \tilde{s}) \in \nabla^*(\widetilde{\boldsymbol{G}}, F)$ be such that $(\widetilde{\boldsymbol{T}}, \tilde{\theta}) \overset{\widetilde{G}}{\longleftrightarrow} (\widetilde{\boldsymbol{T}}^*, \tilde{s})$. Then $\mathfrak{Res}_{G}^{\widetilde{G}}(\widetilde{\boldsymbol{T}}, \tilde{\theta}) \overset{G}{\longleftrightarrow} {}^*\mathfrak{Res}_{G}^{\widetilde{G}}(\widetilde{\boldsymbol{T}}^*, \tilde{s})$.*

(b) Let $(\boldsymbol{T}, \theta) \in \nabla(\boldsymbol{G}, F)$ and $(\boldsymbol{T}^*, s) \in \nabla^*(\boldsymbol{G}, F)$ be such that $(\boldsymbol{T}, \theta) \overset{G}{\longleftrightarrow} (\boldsymbol{T}^*, s)$. Let $\widetilde{\boldsymbol{T}} = \boldsymbol{T}.Z(\widetilde{\boldsymbol{G}})$, $\widetilde{\boldsymbol{T}}^* = i^{*-1}(\boldsymbol{T}^*)$ and $\tilde{s} \in \widetilde{\boldsymbol{G}}^{*F}$ be such that $i^*(\tilde{s}) = s$. Then there exists an irreducible character $\tilde{\theta} \in \mathrm{Irr}\left(\widetilde{\boldsymbol{T}}^F\right)$ which extends θ such that $(\widetilde{\boldsymbol{T}}, \tilde{\theta}) \overset{\widetilde{G}}{\longleftrightarrow} (\widetilde{\boldsymbol{T}}^*, \tilde{s})$.

For semisimple elements in $\widetilde{\mathbf{G}}^{*F}$, geometric conjugacy and rational conjugacy coincide by [5, Proposition 9.7]. Therefore, two pairs $(\widetilde{\mathbf{T}}, \theta), (\widetilde{\mathbf{T}}', \theta') \in \nabla(\widetilde{\mathbf{G}}, F)$ are geometrically conjugate if and only if they are rationally conjugate – i.e.$\nabla\left(\widetilde{\mathbf{G}}, F, (\tilde{s})\right) = \nabla\left(\mathbf{G}, F, [\tilde{s}]\right)$. Two semisimple elements $s_1, s_2 \in \mathbf{G}$ are rationally conjugate if and only if there exist semisimple elements $\tilde{s}_1, \tilde{s}_2 \in \widetilde{\mathbf{G}}^{*F}$ such that $i^*(\tilde{s}_i) = s_i$ for $i = 1, 2$, and $\tilde{s}_1$ and $\tilde{s}_2$ are rationally (and therefore also geometrically) conjugate in $\widetilde{\mathbf{G}}^*$ [5, Proposition 9.9].

We denote the group of components by $A_{\mathbf{G}^*}(s) \coloneqq C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$. If $Z(\mathbf{G})$ is connected then $A_{\mathbf{G}^*}(s) = 1$ for all s.

Lemma 4.2.13 ([5, Lemme 8.3]).

$$A_{\boldsymbol{G}^*}(s) \cong \{z \in Ker\ i^* \mid \tilde{s} \text{ and } \tilde{s}z \text{ are conjugate in } \widetilde{\boldsymbol{G}}^*\}$$

We now return to Jordan decomposition for the case when $\mathbf{G}$ has non-connected centre. By [16, Theorem 15.13], there is an action of $\ker i^* \cap [s, \widetilde{\mathbf{G}}^{*F}]$ on $\mathcal{E}\left(\mathbf{G}^F, s\right)$ and of $A_{\mathbf{G}^*}(s)$ on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s), 1)$. Jordan decomposition generalizes to a bijection between orbits of characters under these actions. In particular, to each $\chi \in \mathcal{E}\left(\mathbf{G}^F, s\right)$ there corresponds an orbit of unipotent characters of $C^\circ_{\mathbf{G}^*}(s)^F$ under the action of $A_{\mathbf{G}^*}(s)$.

Theorem 4.2.14 (Jordan decomposition for $Z(\mathbf{G})$ not connected [16, Corollary 15.14]). *Let $\boldsymbol{G}$ be a connected reductive algebraic group and let $F : \boldsymbol{G} \to \boldsymbol{G}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure where $q = p^a$ for some prime p and some $a \in \mathbb{N}$. Let $i : \boldsymbol{G} \to \widetilde{\boldsymbol{G}}$ be a regular embedding, and let $\tilde{s} \in \widetilde{\boldsymbol{G}}^{*F}$ be a semisimple element such that $i^*(\tilde{s}) = s \in \boldsymbol{G}^{*F}$. Then there exists a bijection*

$$\left\{ \begin{array}{c} \text{Orbits of } \mathcal{E}\left(\boldsymbol{G}^F, s\right) \\ \text{under the action of} \\ \ker i^* \cap [s, \widetilde{\boldsymbol{G}}^{*F}] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Orbits of } \mathcal{E}(C^\circ_{\boldsymbol{G}^*}(s), 1) \\ \text{under the action of} \\ A_{\boldsymbol{G}^*}(s) \end{array} \right\}.$$

If $Z(\mathbf{G})$ is connected then the orbits on each side contain a unique character so this reduces to normal Jordan decomposition as in Theorem 4.2.8.

4.3 Block theory of finite groups of Lie type

We continue with the assumptions made at the beginning of section 4.2. In particular, p is a prime different to ℓ.

4.3.1 Generalized e-Harish-Chandra theory and the parametrisation of ℓ-blocks

Work towards a parametrisation of the ℓ-blocks of the finite groups of Lie type in non-defining characteristic began in the 1980s. It was completed in 2015 by Kessar and Malle [50]. The final parametrisation allows us to label an ℓ-block of an F-stable Levi subgroup of a simple simply connected algebraic group by what is called an e-Jordan quasi-central cuspidal pair. The e-Jordan quasi-central cuspidal pair labelling a block encodes many properties of the block, including information about the characters it contains. In many cases we can calculate the Morita Frobenius number of a block from an earlier (usually simpler but less general) edition of this parametrisation, so we now give a summary of the evolution of the theory, pointing out various results along the way which will be of particular use for our calculations.

In 1982, Fong and Srinivasan proved an explicit "Jordan-style" decomposition for the characters and blocks of the general linear and unitary groups [31]. This was possible because of the well-understood combinatorial nature of the character theory of these groups. Let $\mathbf{G}^F$ be the fixed points of a general linear or general unitary group under a Frobenius morphism F. Fong and Srinivasan first developed an analogue to the Nakayama conjecture (originally for the symmetric groups) for the unipotent characters and unipotent blocks of $\mathbf{G}^F$. They then extended this to a parametrisation of all characters and all blocks, giving explicit combinatorial methods for determining which characters are in which blocks.

In 1989 Fong and Srinivasan went on to show that if $\mathbf{G}^F$ is the fixed points of any classical group with connected centre under a Frobenius morphism, and p and ℓ are different from 2, then the ℓ-blocks of $\mathbf{G}^F$ can be parametrised by pairs (s, κ) where s is a representative of a conjugacy class of a semisimple ℓ' element of $\mathbf{G}^{*F}$, and κ is a unipotent block of $C_{\mathbf{G}^*}(s)^{*F}$ [33]. Again, the parametrisation is explicit and combinatorial because of the nature of the character theory of the classical groups.

For the next developments we first need a few more definitions. Let $e > 0$.

Definition 4.3.1. Let $\chi \in \mathrm{Irr}\big(\mathbf{G}^F\big)$. Then χ is *cuspidal* if ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\chi) = 0$ for any proper F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ contained in an F-stable parabolic $\mathbf{P}$ of $\mathbf{G}$, and χ is *e-cuspidal* if ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\chi) = 0$ for any proper e-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$. Note that χ is 1-cuspidal if and only if χ is cuspidal.

Definition 4.3.2. A *(unipotent) cuspidal pair* of $\mathbf{G}$ is a pair $(\mathbf{L}, \lambda)$ where $\mathbf{L}$ is an e-split F-stable Levi subgroup contained in an F-stable parabolic of $\mathbf{G}$, and λ is a (unipotent) cuspidal character of $\mathbf{L}^F$. A *(unipotent) e-cuspidal pair* is a pair $(\mathbf{L}, \lambda)$ where $\mathbf{L}$ is an e-split F-stable Levi subgroup contained in a parabolic of $\mathbf{G}$ which is not necessarily F-stable, and λ is a (unipotent) e-cuspidal character of $\mathbf{L}^F$.

Definition 4.3.3. We define a partial ordering on the set of all pairs $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ where $\mathbf{L}$ is an e-split F-stable Levi subgroup of $\mathbf{G}$ and λ is an irreducible character of $\mathbf{L}^F$, by setting $(\mathbf{L}_1, \lambda_1) \ll_e (\mathbf{L}_2, \lambda_2)$ if there exists a parabolic subgroup $\mathbf{P}$ of $\mathbf{L}_2$ such that $\mathbf{L}_1$ is an e-split Levi subgroup of $\mathbf{L}_2$ contained in $\mathbf{P}$, and λ_2 is an irreducible constituent of $R^{\mathbf{L}_2}_{\mathbf{L}_1\subseteq\mathbf{P}}(\lambda_1)$.

Definition 4.3.4. Let $(\mathbf{L}, \lambda)$ be an e-cuspidal pair of $\mathbf{G}$. The *e-Harish-Chandra series of $\boldsymbol{G}^F$ above $(\boldsymbol{L}, \lambda)$* denoted by $\mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right)$ is the set of irreducible characters which appear as constituents in $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\lambda)$, where $\mathbf{P}$ is a parabolic containing $\mathbf{L}$.

$$\mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right) = \left\{\chi \in \mathrm{Irr}\left(\mathbf{G}^F\right) \mid \langle \chi, R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\lambda)\rangle \neq 0 \text{ for some } \mathbf{P} \supseteq \mathbf{L}\right\}$$

A 1-Harish-Chandra series is usually just referred to as a Harish-Chandra series.

Definition 4.3.5. The *relative Weyl group* of an e-cuspidal pair $(\mathbf{L}, \lambda)$ is defined to be $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) = N_{\mathbf{G}^F}(\mathbf{L}, \lambda)/\mathbf{L}^F$.

To parametrise the blocks of the finite groups of Lie type of all types, we need the concept of a generalized e-Harish-Chandra theory. There are some variations in the literature as to what exactly constitutes a generalized e-Harish-Chandra theory. Since Deligne-Lusztig induction preserves Lusztig series (i.e. it restricts to a linear map $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : \mathbb{Z}\mathcal{E}\left(\mathbf{L}^F, s\right) \to \mathbb{Z}\mathcal{E}\left(\mathbf{G}^F, s\right)$ [50, Theorem 2.8 (b)]), it is possible to consider the e-Harish-Chandra series above each $(\mathbf{L}, \lambda)$ series by series. We will therefore use the following definition.

Definition 4.3.6. Let $s \in \mathbf{G}^{*F}$ be semisimple. A *generalized e-Harish-Chandra theory* holds in $\mathcal{E}\left(\mathbf{G}^F, s\right)$ if

(a) for every $\chi \in \mathcal{E}\left(\mathbf{G}^F, s\right)$ there exists, up to $\mathbf{G}^F$-conjugacy, a unique e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ such that $\chi \in \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right)$, and

(b) for every e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ there exists a collection of isometries

$$I^{\mathbf{M}}_{(\mathbf{L},\lambda)} : \mathbb{Z}\operatorname{Irr}\left(W_{\mathbf{M}^F}(\mathbf{L}, \lambda)\right) \longrightarrow \mathbb{Z}\mathcal{E}\left(\mathbf{M}^F, (\mathbf{L}, \lambda)\right)$$

where $\mathbf{M}$ runs over all of the e-split Levi subgroups of $\mathbf{G}$ and $(\mathbf{L}, \lambda)$ runs over all of the e-cuspidal pairs of $\mathbf{M}$, such that for all such $\mathbf{M}$ and $(\mathbf{L}, \lambda)$,

$$R^{\mathbf{G}}_{\mathbf{M}\subseteq\mathbf{P}} \circ I^{\mathbf{M}}_{(\mathbf{L},\lambda)} = I^{\mathbf{G}}_{(\mathbf{L},\lambda)} \circ \operatorname{Ind}^{W_{\mathbf{G}}(\mathbf{L},\lambda)}_{W_{\mathbf{M}}(\mathbf{L},\lambda)},$$

the collection $\left(I^{\mathbf{M}}_{(\mathbf{L},\lambda)}\right)_{\mathbf{M},(\mathbf{L},\lambda)}$ is stable under the conjugation action by $W_{\mathbf{G}^F}$, and $I_{(\mathbf{L},\lambda)}$ maps the trivial character of the trivial group $W_{\mathbf{L}^F}(\mathbf{L}, \lambda)$ to λ.

In some sources (for example [32] and [11]), the term generalized e-Harish-Chandra theory is used to refer just to unipotent characters – i.e. for every $\chi \in \mathcal{E}(\mathbf{G}^F, 1)$ there exists a unique *unipotent* e-cuspidal pair etc. When referring to these sources we will make this explicit by specifying that $s = 1$.

In 1986, Fong and Srinivasan showed that part (a) of a generalized e-Harish-Chandra theory holds for $\mathcal{E}(\mathbf{G}^F, 1)$ if $\mathbf{G}$ is a classical group, [32]. In 1993 Broué, Malle and Michel showed that parts (a) and (b) of a generalized e-Harish-Chandra theory hold for $\mathcal{E}(\mathbf{G}^F, 1)$ for all finite groups of Lie type of all types [11]. The proof in [11] is entirely done on a case by case basis and involves directly calculating the e-Harish-Chandra series in many situations.

This result paved the way for Cabanes and Enguehard who used the generalized e-Harish-Chandra theory of $\mathcal{E}(\mathbf{G}^F,1)$ to prove a parametrisation of the unipotent blocks of $\mathbf{G}^F$ when $\ell > 2$ is good for $\mathbf{G}$ and $\ell \neq 3$ if ${}^3D_4$ is a factor of $\mathbf{G}^F$ in 1994, [14]. Enguehard then extended the parametrisation to include bad ℓ in 2000 [26]. We state this parametrisation fully as we will use it to calculate the Morita Frobenius numbers of the unipotent blocks of the finite groups of Lie type in Section 5.2.1. First we need one more definition.

Definition 4.3.7. Let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$. Then χ is of *central ℓ-defect* if $\chi(1)_\ell |Z(\mathbf{G})^F|_\ell = |\mathbf{G}^F|_\ell$ and χ is of *quasi-central ℓ-defect* if χ covers a character of $[\mathbf{G},\mathbf{G}]^F$ which is of central ℓ-defect. An e-cuspidal pair $(\mathbf{L},\lambda)$ is of *(quasi-) central ℓ-defect* if λ is of (quasi-)central ℓ-defect.

We say that a group K is *involved* in a group G if there exists a surjective homomorphism from some subgroup H of G to K. If ℓ is odd, good for $\mathbf{G}$, and $\ell \neq 3$ if ${}^3D_4$ is involved in $\mathbf{G}$, then all unipotent e-cuspidal pairs are of central ℓ-defect [14, Theorem 4.3].

Theorem 4.3.8 (Parametrisation of unipotent blocks of finite groups of Lie type [14, Theorem 4.4] and [26, Théorème A])**.** *Let $\boldsymbol{G}$ be a connected reductive algebraic group with Frobenius morphism $F : \boldsymbol{G} \to \boldsymbol{G}$ defined with respect to an $\mathbb{F}_q$-structure for some $q = p^a$ where $p \neq \ell$ is a prime and $a \in \mathbb{N}$. Let $e = e_\ell(q)$.*

(a) Let $(\boldsymbol{L},\lambda)$ be a unipotent e-cuspidal pair of $\boldsymbol{G}$. Then all irreducible constituents of $R^{\boldsymbol{G}}_{\boldsymbol{L}\subseteq\boldsymbol{P}}(\lambda)$ lie in the same ℓ-block, $b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$, of $\boldsymbol{G}^F$.

(b) There exists a surjection

$$\Gamma : \left\{ \begin{array}{c} \boldsymbol{G}^F\text{-conjugacy classes of} \\ \text{unipotent } e\text{-cuspidal} \\ \text{pairs of } \boldsymbol{G} \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{c} \text{Unipotent} \\ \ell\text{-blocks of } \boldsymbol{G}^F \end{array} \right\}$$
$$(\boldsymbol{L},\lambda) \longmapsto b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$$

sending a unipotent e-cuspidal pair $(\boldsymbol{L}, \lambda)$ to the ℓ-block of $\boldsymbol{G}^F$ containing the irreducible constituents of $R^{\boldsymbol{G}}_{\boldsymbol{L} \subseteq \boldsymbol{P}}(\lambda)$, $b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$.

(c) *The map Γ restricts to a bijection Γ_C if we only consider $\boldsymbol{G}^F$-conjugacy classes of unipotent e-cuspidal pairs of central ℓ-defect.*

$$\Gamma_C : \left\{ \begin{array}{c} \boldsymbol{G}^F\textit{-conjugacy classes of} \\ \textit{unipotent e-cuspidal pairs of } \boldsymbol{G} \\ \textit{of central } \ell\textit{-defect} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Unipotent} \\ \ell\textit{-blocks of } \boldsymbol{G}^F \end{array} \right\}$$
$$(\boldsymbol{L}, \lambda) \longmapsto b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$$

(d) *When ℓ is odd, good for $\boldsymbol{G}$ and $\ell \neq 3$ if ${}^3D_4$ is involved in $\boldsymbol{G}$, then Γ is itself a bijection.*

Finally we come to the parametrisation of general blocks of finite groups of Lie type. Cabanes and Enguehard proved a general parametrisation for good ℓ in 1998 in [15]. This was extended to general ℓ by Kessar and Malle in 2015 in [50]. Before we can state the final parametrisation, again, we need a few more definitions. Let $e > 0$. Recall that $\mathbf{G}_{\Phi_e}$ denotes a Sylow e-torus of $\mathbf{G}$.

Definition 4.3.9. Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ' element and let $\chi \in \mathcal{E}\left(\mathbf{G}^F, s\right)$. We say that χ is *e-Jordan cuspidal* if $(\mathbf{G}, \chi)$ satisfies the *Jordan condition* (J) made of the following two parts:

(J_1) $Z^\circ(C^\circ_{\mathbf{G}^*}(s))_{\Phi_e} = Z^\circ(\mathbf{G}^*)_{\Phi_e}$

(J_2) χ corresponds to a $C_{\mathbf{G}^*}(s)^F$-orbit of e-cuspidal unipotent characters of $C^\circ_{\mathbf{G}^*}(s)^F$ via Jordan decomposition given in Theorem 4.2.14.

We say that χ is *e-Jordan quasi-central cuspidal* if it is e-Jordan cuspidal and if the $C_{\mathbf{G}^*}(s)^F$-orbit of e-cuspidal irreducible unipotent characters of $C^\circ_{\mathbf{G}^*}(s)^F$ to which it corresponds in (J_2) consists of characters of quasi-central ℓ-defect.

Definition 4.3.10. The pair $(\mathbf{L},\lambda)$ is an *e-Jordan (quasi-central) cuspidal pair* if $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$ (contained in a parabolic which is not necessarily F-stable) and λ is an e-Jordan (quasi-central) cuspidal character of $\mathbf{L}^F$.

Theorem 4.3.11 (Parametrisation of blocks of finite groups of Lie type [50, Theorem A])**.** *Let $\boldsymbol{H}$ be a simple, simply connected algebraic group and let $F:\boldsymbol{H}\to\boldsymbol{H}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure where $q=p^a$ for some prime $p\neq\ell$ and some $a\in\mathbb{N}$. Let $\boldsymbol{G}$ be an F-stable Levi subgroup of $\boldsymbol{H}$, and let $e=e_\ell(q)$.*

(a) Let $(\boldsymbol{L},\lambda)$ be an e-Jordan cuspidal pair of $\boldsymbol{G}$. Then all irreducible constituents of $R^{\boldsymbol{G}}_{\boldsymbol{L}\subseteq\boldsymbol{P}}(\lambda)$ lie in the same ℓ-block, $b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$, of $\boldsymbol{G}^F$.

(b) There exists a surjection

$$\Theta:\left\{\begin{array}{c}\boldsymbol{G}^F\text{-conjugacy classes of e-Jordan}\\ \text{cuspidal pairs } (\boldsymbol{L},\lambda) \text{ of } \boldsymbol{G}\\ \text{such that } \lambda\in\mathcal{E}(\boldsymbol{L}^F,\ell')\end{array}\right\}\twoheadrightarrow\left\{\ \ell\text{-blocks of } \boldsymbol{G}^F\ \right\}$$
$$(\boldsymbol{L},\lambda)\longmapsto b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda),$$

sending an e-Jordan cuspidal pair $(\boldsymbol{L},\lambda)$ to the ℓ-block of $\boldsymbol{G}^F$ containing the irreducible constituents of $R^{\boldsymbol{G}}_{\boldsymbol{L}\subseteq\boldsymbol{P}}(\lambda)$, $b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$.

(c) The map Θ restricts to another surjection Θ_{QC} if we only consider the $\boldsymbol{G}^F$-conjugacy classes of e-Jordan quasi-central cuspidal pairs.

$$\Theta_{QC}:\left\{\begin{array}{c}\boldsymbol{G}^F\text{-conjugacy classes of e-Jordan}\\ \text{quasi-central cuspidal pairs } (\boldsymbol{L},\lambda)\\ \text{of } \boldsymbol{G} \text{ such that } \lambda\in\mathcal{E}(\boldsymbol{L}^F,\ell')\end{array}\right\}\twoheadrightarrow\left\{\ \ell\text{-blocks of } \boldsymbol{G}^F\ \right\}$$
$$(\boldsymbol{L},\lambda)\longmapsto b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$$

(d) If $\ell\geq 3$ then Θ_{QC} is a bijection.

(e) If ℓ is good for $\boldsymbol{G}$, and $\ell \neq 3$ if ${}^3D_4$ is involved in $\boldsymbol{G}$, then Θ is a bijection.

It is clear from this theorem that the set of pairs $(\mathbf{L}, \lambda)$ needed to parametrise the blocks of $\mathbf{G}^F$ is determined by properties of the Jordan correspondent(s) of λ, which must be a unipotent, e-cuspidal, and of quasi-central ℓ-defect.

We make one final definition in relation to Jordan decomposition, using the final parametrisation of ℓ-blocks. Recall that for a semisimple ℓ' element $s \in \mathbf{G}^{*F}$, $\mathcal{E}_\ell(\mathbf{G}^F, s)$ is a union of ℓ-blocks of $\mathbf{G}^F$.

Definition 4.3.12. Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ' element and consider a block $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. By Theorem 4.3.11, $B = b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ for some e-Jordan quasi-central cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$. Note that $(\mathbf{L}, \lambda)$ may not be uniquely determined if ℓ is bad for $\mathbf{G}$. By condition (J_2), λ corresponds to a $C_{\mathbf{L}^*}(s)^F$-orbit of e-cuspidal unipotent characters of $C^\circ_{\mathbf{L}^*}(s)^F$ of ℓ-central defect via Jordan decomposition. Let α be character in this orbit. Then since $C^\circ_{\mathbf{L}^*}(s)$ is an e-split Levi subgroup of $C^\circ_{\mathbf{G}^*}(s)$ by Lemma 4.1.11, $\left(C^\circ_{\mathbf{L}^*}(s), \alpha\right)$ is a unipotent e-cuspidal pair for $C^\circ_{\mathbf{G}^*}(s)$ of ℓ-central defect. Therefore by Theorem 4.3.8 (b), $\left(C^\circ_{\mathbf{L}^*}(s), \alpha\right)$ labels a unipotent block of $C^\circ_{\mathbf{G}^*}(s)$, $b_{C^\circ_{\mathbf{G}^*}(s)^F}(C^\circ_{\mathbf{L}^*}(s), \alpha)$. We call the unipotent block $b_{C^\circ_{\mathbf{G}^*}(s)^F}(C^\circ_{\mathbf{L}^*}(s), \alpha)$ a *Jordan correspondent* of $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. Note that if $Z(\mathbf{G})$ is connected then λ corresponds to a unique unipotent e-cuspidal character of $C^\circ_{\mathbf{L}^*}(s)^F$, so the Jordan correspondent of $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ is unique.

4.3.2 Results from Bonnafé-Rouquier and Bonnafé-Dat-Rouquier

In this section we will give some important results of Bonnafé-Rouquier and Bonnafé-Dat-Rouquier which are used in Section 5.2. Recall that $e_s^{\mathbf{G}^F}$ denotes the sum of the block idempotents of the blocks in $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ for some semisimple ℓ' element $s \in \mathbf{G}^{*F}$.

Theorem 4.3.13 ([8, Théorème B']). *Let $\boldsymbol{G}$ be a connected reductive algebraic group. Let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ' element, and let $\boldsymbol{L}^*$ be an F-stable Levi subgroup of $\boldsymbol{G}^*$ such that*

$C_{\boldsymbol{G}^*}(s) \subseteq \boldsymbol{L}^*$. Let $\boldsymbol{L}$ be dual to $\boldsymbol{L}^*$. Then there exists a Morita equivalence

$$\mathcal{O}\boldsymbol{G}^F e_s^{\boldsymbol{G}^F} \sim_M \mathcal{O}\boldsymbol{L}^F e_s^{\boldsymbol{L}^F}, \tag{4.4}$$

so the blocks of $\mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$ are Morita equivalent to blocks in $\mathcal{E}_\ell(\boldsymbol{L}^F, s)$.

Let $\mathbf{L}^*$ be the smallest F-stable Levi subgroup of $\mathbf{G}^*$ containing $C_{\mathbf{G}^*}(s)$. Then Theorem 4.3.13 shows that blocks of $\mathbf{G}^F$ are Morita equivalent to quasi-isolated blocks of the fixed points of Levi subgroups of $\mathbf{G}$.

Remark 4.3.14. Note that it follows from Theorem 4.3.13 that in order to parametrise the blocks of the finite groups of Lie type up to Morita equivalence, it is enough to parametrise the quasi-isolated blocks. When Bonnafé- Rouquier published this result in 2003, the parametrisation of blocks of finite groups of Lie type for good ℓ had been completed by Cabanes and Enguehard. Thus for a full parametrisation up to Morita equivalence, it only remained to parametrise the quasi-isolated blocks of the finite groups of Lie type for bad ℓ. This was done by Kessar and Malle in [49] in 2013 with a case by case proof. We use their results extensively in Section 5.2.2.

We now give an important refinement of Theorem 4.3.13, also from [8]. Suppose that we are in the setting of Theorem 4.3.13 and that $C_{\mathbf{G}^*}(s) = \mathbf{L}^*$ is a Levi subgroup of $\mathbf{G}^*$. Then $\mathbf{L}^*$ is F stable and s is central in $\mathbf{L}^*$ so by [23, Proposition 13.30] there exists a linear character $\hat{s} \in \mathrm{Irr}\left(\mathbf{L}^F\right)$ such that tensoring by $\hat{s}$ defines a bijection

$$\mathcal{E}(\mathbf{L}^F, 1) \longrightarrow \mathcal{E}(\mathbf{L}^F, s).$$

As discussed in [8, Section 11.5], the map given by $l \mapsto \hat{s}(l)l$ for all $l \in \mathcal{O}\mathbf{L}^F$ is an automorphism of $\mathcal{O}\mathbf{L}^F$ and it restricts to an isomorphism of $\mathcal{O}$-algebras,

$$\mathcal{O}\mathbf{L}^F e_s^{\mathbf{L}^F} \xrightarrow{\sim} \mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}.$$

By [8, Théorème 11.8], this $\mathcal{O}$-algebra isomorphism can be combined with Equation (4.4) to give a Morita equivalence:

$$\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F} \sim_M \mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}. \tag{4.5}$$

Thus if $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ then every block of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ is Morita equivalent to a unipotent block of the fixed points of a Levi subgroup of $\mathbf{G}$ dual to $C_{\mathbf{G}^*}(s)$.

Bonnafé, Dat and Rouquier extended the results of [8] in [6]. We now set up some notation as in [6, Section 7]. Fix a semisimple ℓ' element $s \in \mathbf{G}^{*F}$. Let $\mathbf{L}^* = C_{\mathbf{G}^*}\left(Z^\circ\left(C^\circ_{\mathbf{G}^*}(s)\right)\right)$ be the minimal Levi subgroup of $\mathbf{G}^*$ containing $C^\circ_{\mathbf{G}^*}(s)$. Let $\mathbf{N}^* = C_{\mathbf{G}^*}(s)^F.\mathbf{L}^*$ and let $\mathbf{L}$ be dual to $\mathbf{L}^*$. Define $\mathbf{N}$ to be the subgroup of $N_{\mathbf{G}}(\mathbf{L})$ containing $\mathbf{L}$ such that $\mathbf{N}/\mathbf{L}$ corresponds to $\mathbf{N}^*/\mathbf{L}^*$ via the canonical isomorphism between $N_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$ and $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$.

Theorem 4.3.15 ([6, Theorem 7.7])**.** *Let $\boldsymbol{G}$ be a connected reductive algebraic group. Let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ' element, and let $\boldsymbol{L}$ and $\boldsymbol{N}$ be as defined above. Then there exists a Morita equivalence*

$$\mathcal{O}\boldsymbol{G}^F e_s^{\boldsymbol{G}^F} \sim_M \mathcal{O}\boldsymbol{N}^F e_s^{\boldsymbol{L}^F}. \tag{4.6}$$

Because of the minimality of $\mathbf{L}^*$, s is isolated in $\mathbf{L}^*$ so the blocks in $e_s^{\mathbf{L}^F}$ are isolated. Thus Theorem 4.3.15 shows that every block of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ is Morita equivalent to a block of a subgroup of $\mathbf{G}^F$ which covers an isolated block of the fixed points of a Levi subgroup of $\mathbf{G}$.

Recall the definition of the component group $A_{\mathbf{G}^*}(s) = C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$. By [6, Section 7A], $\mathbf{N}^F/\mathbf{L}^F = \mathbf{N}/\mathbf{L} \cong \mathbf{N}^*/\mathbf{L}^* = (\mathbf{N}^*/\mathbf{L}^*)^F$, and therefore by the third isomorphism theorem,

$$\mathbf{N}^F/\mathbf{L}^F = C_{\mathbf{G}^*}(s)^F/C^\circ_{\mathbf{G}^*}(s)^F \leq \left(C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)\right)^F = A_{\mathbf{G}^*}(s)^F. \tag{4.7}$$

Suppose that $A_{\mathbf{G}^*}(s)^F$ is cyclic and $C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$ is a Levi subgroup of $\mathbf{G}^*$. Then by [6, Example 7.9], $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F} \cong \mathcal{O}\mathbf{N}^F e_1^{\mathbf{L}^F}$, so Equation (4.6) gives a Morita equivalence;

$$\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F} \sim_M \mathcal{O}\mathbf{N}^F e_1^{\mathbf{L}^F}. \tag{4.8}$$

Therefore if $A_{\mathbf{G}^*}(s)^F$ is cyclic and $C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$ is a Levi subgroup of $\mathbf{G}^*$ then every block of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ is Morita equivalent to a block of a subgroup of $\mathbf{G}^F$, which covers a unipotent block of the fixed points of a Levi subgroup of $\mathbf{G}$.

Chapter 5

Morita Frobenius numbers of blocks of finite groups of Lie type

5.1 Defining characteristic

Let $\mathbf{G}$ be a simple, simply-connected algebraic group defined over $\overline{\mathbb{F}}_\ell$, an algebraic closure of the finite field of ℓ elements. Let q be a power of ℓ and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure with finite group of fixed points, $\mathbf{G}^F$. Note that we place no restriction on the type of Frobenius morphism here – F can be split, twisted, or very twisted as the methods of this section work in all cases. We start with the following result from [42, Chapter 8] which is fundamental to our arguments.

Theorem 5.1.1 (Humphreys). *$\boldsymbol{G}^F$ has $\left|Z\left(\boldsymbol{G}^F\right)\right|$ ℓ-blocks of maximal defect and a unique block of defect 0 containing the Steinberg character.*

From this theorem it follows that $k\mathbf{G}^F$ has $\left|Z\left(\mathbf{G}^F\right)\right| + 1$ blocks; the Steinberg block with trivial defect, and $\left|Z\left(\mathbf{G}^F\right)\right|$ many blocks whose defect groups are Sylow ℓ-subgroups of $\mathbf{G}^F$. We will use the next Theorem to calculate the Morita Frobenius numbers of these blocks.

Theorem 5.1.2. *Let B be a block of $k\boldsymbol{G}^F$ with Galois conjugate $\sigma(B)$. Then there exists a group automorphism $\varphi : \boldsymbol{G}^F \to \boldsymbol{G}^F$ such that for the induced k-algebra isomorphism $\varphi : k\boldsymbol{G}^F \to k\boldsymbol{G}^F$, $\varphi(B) = \sigma(B)$.*

Proof. First suppose that $\left|Z\left(\mathbf{G}^F\right)\right| \leq 2$. Then it follows from Theorem 5.1.1 that $\mathbf{G}^F$ has at most three ℓ-blocks. At least two of the blocks contain rational valued characters – the principal and Steinberg blocks – therefore $\sigma(B) = B$ for all ℓ-blocks B of $\mathbf{G}^F$ by Proposition 2.2.5 (b). Hence the result follows in this case by letting φ be the identity map.

Now suppose that $Z\left(\mathbf{G}^F\right) \cong C_m$ for some $m > 2$ coprime to ℓ. We will determine the block idempotents of the ℓ-blocks explicitly and then construct a suitable φ by defining its action on those block idempotents.

Let $Z\left(\mathbf{G}^F\right) = \langle g \rangle$. Then $Z\left(\mathbf{G}^F\right)$ has m irreducible characters $\chi_i : Z\left(\mathbf{G}^F\right) \to K$, and for each $0 \leq i \leq m-1$, χ_i has an associated central primitive idempotent of $KZ\left(\mathbf{G}^F\right)$,

$$e_i = \frac{1}{m} \sum_{0 \leq a \leq m-1} \chi_i(g^a) g^{-a}.$$

Since m is coprime to ℓ it is invertible in $\mathcal{O}$, so $e_i \in \mathcal{O}\mathbf{G}^F$. Let $\bar{e}_i$ be the image of e_i in $k\mathbf{G}^F$ under the canonical quotient mapping $\mathcal{O}\mathbf{G}^F \to k\mathbf{G}^F$. Then $\{\bar{e}_i\}_{i=0}^{m-1}$ are the m block idempotents for the ℓ-blocks of $kZ\left(\mathbf{G}^F\right)$.

Since $k\mathbf{G}^F$ has $m+1$ blocks, there are exactly $m+1$ primitive central idempotents in $k\mathbf{G}^F$. Note that $\{\bar{e}_i\}$ is a set of m central, but not necessarily primitive, idempotents of $k\mathbf{G}^F$. Thus since they are $\mathbf{G}^F$-stable, precisely one $\bar{e}_j$ is imprimitive in $Z(k\mathbf{G}^F)$. Since the trivial and Steinberg characters of $\mathbf{G}^F$ both restrict to the trivial character on $Z\left(\mathbf{G}^F\right)$, it follows that the block idempotent for the principal block of $kZ\left(\mathbf{G}^F\right)$ – $\bar{e}_0$, say – is imprimitive in $k\mathbf{G}^F$ and splits into the block idempotents for the principal and Steinberg blocks of $k\mathbf{G}^F$. The remaining $\{\bar{e}_i\}_{i=1}^{m-1}$ are block idempotents of $k\mathbf{G}^F$.

If B is either the principal or Steinberg block of $\mathbf{G}^F$ then, as mentioned above, $\sigma(B) = B$ so we can let φ be the identity map. For the other blocks, Galois conjugation acts on $\bar{e}_i$ by

$$\sigma(\bar{e}_i) = \frac{1}{m} \sum_{0 \le a \le m-1} \overline{\chi_i(g^a)}^{\ell} g^{-a}.$$

If $\ell \equiv 1 \bmod m$ then this action is trivial so again we can take $\varphi = id$. Thus assume from now on that B is an ℓ-block of $\mathbf{G}^F$ different from the principal and Steinberg blocks, with block idempotent $\bar{e}_i$ and that $\ell \not\equiv 1 \bmod m$. We will construct a suitable φ such that $\varphi(\bar{e}_i) = \sigma(\bar{e}_i)$.

Let $F_\ell : \mathbf{G} \to \mathbf{G}$ be an $\mathbb{F}_\ell$-split Frobenius morphism of $\mathbf{G}$ and let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ such that $F_\ell(t) = t^\ell$ for all $t \in \mathbf{T}$ (this is possible by [55, Theorem 16.5 and Example 22.6]). By [18, Proposition 3.6.8], $Z\left(\mathbf{G}^F\right) = Z\left(\mathbf{G}\right)^F$. Since $Z\left(\mathbf{G}\right)^F \subseteq Z\left(\mathbf{G}\right)$ and $Z\left(\mathbf{G}\right)$ is contained in every maximal torus of $\mathbf{G}$, it follows that $F_\ell(z) = z^\ell$ for every $z \in Z(\mathbf{G}^F)$. Therefore,

$$F_\ell(\bar{e}_i) = \frac{1}{m} \sum_{0 \le a \le m-1} \overline{\chi_i(g^a)} F_\ell(g^{-a}) = \frac{1}{m} \sum_{0 \le a \le m-1} \overline{\chi_i(g^a)} g^{-\ell a}.$$

Let ω be a primitive m-th root of unity such that $\chi_i(g^a) = \omega^{ia}$ for $1 \le a \le m$, let ϕ denote the Euler totient function, and define φ by $\varphi = (F_\ell)^{\phi(m)-1}$. Then

$$\varphi(\bar{e}_i) = \frac{1}{m} \sum_{0 \le a \le m-1} (\overline{\omega^{ia}}) g^{-\ell^{\phi(m)-1} a}$$

and letting $a' = \ell^{\phi(m)-1} a$ so that $\ell a' = \ell^{\phi(m)} a \equiv a \bmod m$, it follows that

$$\varphi(\bar{e}_i) = \frac{1}{m} \sum_{0 \le a' \le m-1} (\overline{\omega^{i\ell a'}}) g^{-a'} = \frac{1}{m} \sum_{0 \le a' \le m-1} \overline{\chi_i(g^{a'})}^{\ell} g^{-a'} = \sigma(\bar{e}_i),$$

as required.

Since q is a power of ℓ, by examination of [55, Table 24.2] it is clear that $\ell \nmid \left|Z\left(\mathbf{G}^F\right)\right|$ and the only case that remains to consider is when $\mathbf{G}^F = \mathrm{Spin}_{2n}^+(q)$, with $n \ge 4$ even, ℓ odd, and $Z\left(\mathbf{G}^F\right) \cong C_2 \times C_2$. The irreducible characters of $C_2 \times C_2$ are rational valued so the associated central primitive idempotents of $kZ\left(\mathbf{G}^F\right)$ are stabilized by Galois conjugation. It follows that the central primitive idempotents of $k\mathbf{G}^F$ are also stabilized by Galois conjugation, so again, we can let φ be the identity map. $\square$

Corollary 5.1.3. *Let Z be a central (possibly trivial) subgroup of $\boldsymbol{G}^F$ and suppose that B is an ℓ-block of $\boldsymbol{G}^F/Z$. Then frob (B) =mf $(B) = 1$.*

Proof. First suppose that Z is trivial. Then by Theorem 5.1.2 there exists a group automorphism $\varphi : \mathbf{G}^F \to \mathbf{G}^F$ such that for the induced k-algebra isomorphism $\varphi : k\mathbf{G}^F \to k\mathbf{G}^F$, $\varphi(B) = \sigma(B)$. Then *frob*(B) =*mf*$(B) = 1$ by Lemma 2.2.6.

Now suppose that Z is not trivial and B, an ℓ-block of $\mathbf{G}^F/Z$, is dominated by an ℓ-block $\hat{B}$ of $\mathbf{G}^F$. By above, *mf*$(\hat{B}) = 1$. Since $Z\left(\mathbf{G}^F\right)$ is an ℓ'-group, by Lemma 2.1.10 (c) $k\mathbf{G}^F\hat{B} \cong k\left(\mathbf{G}^F/Z\right)B$ as k-algebras. Therefore *frob*(B) =*frob*$(\hat{B}) = 1$ whence *mf*$(B) = 1$. $\square$

5.2 Non-defining characteristic

Throughout Section 5.2 let p and ℓ be different primes and let q be a power of p. Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\overline{\mathbb{F}}_p$, an algebraic closure of the finite field of p elements. Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure which is not very twisted, and let $\mathbf{G}^F$ denote the finite group of fixed points. Let $e = e_\ell(q)$, as defined in Section 4.1.4.

Let $\hat{\sigma}$ denote the automorphism of K defined in Section 2.2.1 and recall that $\hat{\sigma}$ acts on the irreducible characters of $\mathbf{G}^F$ by ${}^{\hat{\sigma}}\chi(g) = \hat{\sigma}(\chi(g))$ for all $g \in \mathbf{G}^F$, $\chi \in \mathrm{Irr}\left(\mathbf{G}^F\right)$.

Lemma 5.2.1. *Let $b = b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ be the ℓ-block of $\boldsymbol{G}^F$ containing the irreducible constituents of $R^{\boldsymbol{G}}_{\boldsymbol{L}\subseteq\boldsymbol{P}}(\lambda)$ for an e-Jordan quasi-central cuspidal pair $(\boldsymbol{L}, \lambda)$ of $\boldsymbol{G}$, as in Theorem 4.3.11 (a). Suppose that λ is rational valued. Then mf $\left(b_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)\right) = 1$.*

Proof. As discussed in Section 2.2.1 if λ is rational valued then ${}^{\hat{\sigma}}\lambda = \lambda$. By the Deligne Lusztig induction character formula (Equation (4.1)), since the Green functions are integer valued ([23, Definition 12.1]), it therefore follows that ${}^{\hat{\sigma}}\left(R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\lambda)\right) = R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}\left({}^{\hat{\sigma}}\lambda\right) = R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\lambda)$.

Let $\chi \in \mathrm{Irr}(b)$ be an irreducible constituent of $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\lambda)$. Then ${}^{\hat{\sigma}}\chi \in \mathrm{Irr}(\sigma(b))$, by Lemma 2.2.1 (b). On the other hand, ${}^{\hat{\sigma}}\chi$ is an irreducible constituent of ${}^{\hat{\sigma}}\left(R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\lambda)\right) = R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\lambda)$, by above, so ${}^{\hat{\sigma}}\chi$ is also an irreducible character in b. It follows that $\sigma(b) = b$ and therefore $k\mathbf{G}^F b \cong k\mathbf{G}^F b^{(\ell)}$ as k-algebras by Lemma 2.2.4, so $\mathit{frob}(b) = 1$, hence $\mathit{mf}(b) = 1$. □

5.2.1 Unipotent blocks

For Section 5.2.1 we will use the parametrisation of the unipotent blocks of finite groups of Lie type given in Theorem 4.3.8. As in the theorem, we let $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ denote the unipotent ℓ-block of $\mathbf{G}^F$ containing the irreducible constituents of $R^{\mathbf{G}}_{\mathbf{L}}(\lambda)$ for a unipotent e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ of central ℓ-defect.

Lemma 5.2.2. *Let b be a unipotent ℓ-block of $\boldsymbol{G}^F$ containing a unipotent e-cuspidal character $\chi \in \mathrm{Irr}\left(\boldsymbol{G}^F\right)$ of central ℓ-defect. Suppose that ℓ is good for $\boldsymbol{G}$ and $Z\left([\boldsymbol{G}, \boldsymbol{G}]^F\right)$ is an ℓ'-group. Then all characters in* $\mathrm{Irr}(b)$ *are e-cuspidal.*

Proof. Let $\chi_0 = \chi|_{[\mathbf{G},\mathbf{G}]^F}$. Since χ is unipotent, it follows from results of Lusztig that χ_0 is irreducible (see for example [14, Proposition 3.1]). We have $\left|\mathbf{G}^F\right|_\ell = \left|Z^\circ(\mathbf{G})^F\right|_\ell \left|[\mathbf{G},\mathbf{G}]^F\right|_\ell$ and since χ is of central ℓ-defect, $\left|\mathbf{G}^F\right|_\ell = \chi(1)_\ell \left|Z\left(\mathbf{G}^F\right)\right|_\ell = \chi_0(1)_\ell \left|Z\left(\mathbf{G}^F\right)\right|_\ell$, therefore

$$\chi_0(1)_\ell = \left|[\mathbf{G},\mathbf{G}]^F\right|_\ell \frac{\left|Z^\circ(\mathbf{G})^F\right|_\ell}{\left|Z\left(\mathbf{G}^F\right)\right|_\ell}. \tag{5.1}$$

The subgroup $Z^\circ(\mathbf{G})^F [\mathbf{G},\mathbf{G}]^F$ is of index $\left|Z^\circ(\mathbf{G})^F \cap [\mathbf{G},\mathbf{G}]^F\right|$ in $\mathbf{G}^F$. Since $Z^\circ(\mathbf{G})^F \cap [\mathbf{G},\mathbf{G}]^F \subseteq Z\left([\mathbf{G},\mathbf{G}]^F\right)$, and by assumption, $Z\left([\mathbf{G},\mathbf{G}]^F\right)$ is an ℓ'-group, it follows that $Z^\circ(\mathbf{G})^F [\mathbf{G},\mathbf{G}]^F$ is of ℓ' index in $\mathbf{G}^F$ and thus $\mathbf{G}^F / Z^\circ(\mathbf{G})^F [\mathbf{G},\mathbf{G}]^F$ is an ℓ'-group.

Consider the natural map

$$Z(\mathbf{G})^F \hookrightarrow \mathbf{G}^F \twoheadrightarrow \mathbf{G}^F / Z^\circ(\mathbf{G})^F [\mathbf{G},\mathbf{G}]^F.$$

This map has kernel $Z(\mathbf{G})^F \cap \left(Z^\circ(\mathbf{G})^F\,[\mathbf{G},\mathbf{G}]^F\right) = Z^\circ(\mathbf{G})^F \cap Z\left([\mathbf{G},\mathbf{G}]^F\right)$, therefore the quotient $Z(\mathbf{G})^F / \left(Z^\circ(\mathbf{G})^F \cap Z\left([\mathbf{G},\mathbf{G}]^F\right)\right)$ is ℓ', since it is isomorphic to a subgroup of the ℓ'-group $\mathbf{G}^F / Z^\circ(\mathbf{G})^F[\mathbf{G},\mathbf{G}]^F$. By the third isomorphism theorem,

$$Z(\mathbf{G})^F / \left(Z^\circ(\mathbf{G})^F \cap Z\left([\mathbf{G},\mathbf{G}]^F\right)\right) \cong \left(Z(\mathbf{G})^F / Z^\circ(\mathbf{G})^F\right) / \left(\left(Z^\circ(\mathbf{G})^F \cap Z\left([\mathbf{G},\mathbf{G}]^F\right)\right) / Z^\circ(\mathbf{G})^F\right).$$

Since $\left(Z^\circ(\mathbf{G})^F \cap Z\left([\mathbf{G},\mathbf{G}]^F\right)\right)$ is ℓ', the quotient $\left(Z^\circ(\mathbf{G})^F \cap Z\left([\mathbf{G},\mathbf{G}]^F\right)\right) / Z^\circ(\mathbf{G})^F$ is ℓ', and thus $Z(\mathbf{G})^F / Z^\circ(\mathbf{G})^F$ is also ℓ'. Therefore $\left|Z(\mathbf{G})^F\right|_\ell = \left|Z^\circ(\mathbf{G})^F\right|_\ell$ so it follows from equation (5.1) that

$$\chi_0(1)_\ell = \left|[\mathbf{G},\mathbf{G}]^F\right|_\ell.$$

Thus by Theorem 2.1.6, χ_0 is in a block $\bar{b}$ of $[\mathbf{G},\mathbf{G}]^F$ of defect 0.

Consider a character $\theta \in \mathrm{Irr}(b)$. Since b covers $\bar{b}$ and χ_0 is the only character in $\bar{b}$, θ covers χ_0. By [20, Corollary 11.7], therefore $\theta = \eta\chi$ for a uniquely determined character η of $\mathrm{Irr}\left(\mathbf{G}^F / [\mathbf{G},\mathbf{G}]^F\right)$. Since $\left[\mathbf{G}^F, \mathbf{G}^F\right] \subseteq [\mathbf{G},\mathbf{G}]^F$, $\mathbf{G}^F / [\mathbf{G},\mathbf{G}]^F$ is abelian, so η is a linear character.

By the e-cuspidality of χ (see Definition 4.3.1), $\langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\tau)\rangle = 0$ for any proper e-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and any $\tau \in \mathrm{Irr}\left(\mathbf{L}^F\right)$. Because η is linear, therefore $\langle \eta\chi, \eta R_{\mathbf{L}}^{\mathbf{G}}(\tau)\rangle = \langle \theta, R_{\mathbf{L}}^{\mathbf{G}}(\eta\tau)\rangle = 0$ for all $\tau \in \mathrm{Irr}\left(\mathbf{L}^F\right)$. Let $\tilde{\tau} = \eta\tau$. Then $\tilde{\tau}$ runs over $\mathrm{Irr}\left(\mathbf{L}^F\right)$ as τ does, so $\langle \theta, R_{\mathbf{L}}^{\mathbf{G}}(\tilde{\tau})\rangle = 0$ for all $\tilde{\tau} \in \mathrm{Irr}\left(\mathbf{L}^F\right)$. Therefore θ is e-cuspidal, as required. □

Proposition 5.2.3. *Let $b = b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$ be a unipotent block of $\boldsymbol{G}^F$ for a unipotent e-cuspidal pair $(\boldsymbol{L},\lambda)$ of $\boldsymbol{G}$ of central ℓ-defect. Suppose that ℓ is odd or $\boldsymbol{G}$ is of exceptional type. Then b has a defect group P such that $Z(\boldsymbol{L})_\ell^F \trianglelefteq P$ and $P/Z(\boldsymbol{L})_\ell^F$ is isomorphic to a Sylow ℓ-subgroup of $W_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$.*

Proof. By the proof of [49, Theorem 7.12], since ℓ is odd or $\mathbf{G}$ is of exceptional type, $C_{\mathbf{G}^F}\left(Z(\mathbf{L})_\ell^F\right) = \mathbf{L}^F$ and $\mathbf{G}^F$ has the following inclusion of Brauer pairs

$$(\{1\}, b) \trianglelefteq \left(Z(\mathbf{L})_\ell^F, b_{\mathbf{L}^F}(\lambda)\right) \trianglelefteq (P, f),$$

where $b_{\mathbf{L}^F}(\lambda)$ is the block of $\mathbf{L}^F$ containing λ, f is a block of $C_{\mathbf{G}^F}(P)$, $\left(Z(\mathbf{L})_\ell^F, b_{\mathbf{L}^F}(\lambda)\right)$ is self-centralizing and (P,f) is maximal. Then it follows from [49, Lemma 2.1] that $P/\left(P \cap Z(\mathbf{L})_\ell^F\right)$ $= P/Z(\mathbf{L})_\ell^F$ is isomorphic to a Sylow ℓ-subgroup of

$$N_{\mathbf{G}^F}\left(Z(\mathbf{L})_\ell^F, b_{\mathbf{L}^F}(\lambda)\right)/C_{\mathbf{G}^F}\left(Z(\mathbf{L})_\ell^F\right) = N_{\mathbf{G}^F}\left(Z(\mathbf{L})_\ell^F, b_{\mathbf{L}^F}(\lambda)\right)/\mathbf{L}^F = W_{\mathbf{G}^F}(\mathbf{L},\lambda),$$

as required. □

Similar results to Proposition 5.2.3 can be obtained if $\ell = 2$ and $\mathbf{G}$ is of classical type but these will not be needed here.

The next theorem shows that under certain conditions, we can apply a result of Puig [61, Theorem 5.5] to a unipotent block $b = b_{\mathbf{G}^F}(\mathbf{L},\lambda)$ to show that b is Morita equivalent to a specific block of $\mathcal{O}N_{\mathbf{G}^F}(\mathbf{L},\lambda)$. This result will be used to calculate the Morita Frobenius number of some unipotent blocks of $E_8(q)$.

First we recall the following. Suppose M is a finite group with a normal ℓ'-subgroup U, and suppose that $L \cong M/U$. Let $\mu : M \to L$ be the quotient map. Denote the $\mathcal{O}$-linear extension of μ also by $\mu : \mathcal{O}M \to \mathcal{O}L$. For $x \in \mathcal{O}L$, let $\tilde{x}$ denote an element of $\mathcal{O}M$ such that $\mu(\tilde{x}) = x$. If d is the principal block of $\mathcal{O}U$, then Fong reduction yields the following inverse $\mathcal{O}$-algebra isomorphisms,

$$\begin{aligned} \mathcal{O}L &\xrightarrow{\sim} \mathcal{O}Md \\ x &\longmapsto \tilde{x}d \\ \mu(y) &\longleftarrow\!\shortmid\ y, \end{aligned}$$

for all $x \in \mathcal{O}L$, $y \in \mathcal{O}Md$ [38, Proposition 3.5].

Theorem 5.2.4. *Let $b = b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$ be a unipotent block of $\boldsymbol{G}^F$ for a unipotent e-cuspidal pair $(\boldsymbol{L},\lambda)$ of $\boldsymbol{G}$ of central ℓ-defect. Suppose that $\ell \geq 5$, $\ell | q-1$, $\boldsymbol{L}$ is a proper Levi subgroup of $\boldsymbol{G}$ and $P = Z(\boldsymbol{L})_\ell^F$ is a defect group of b. Let $f = b_{\boldsymbol{L}^F}(\lambda)$ be the block of $\mathcal{O}\boldsymbol{L}^F$ containing*

λ. *Then f is a block of $\mathcal{O}N_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$ with defect group P such that $\mathcal{O}\boldsymbol{G}^F b$ and $\mathcal{O}N_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)f$ are Morita equivalent.*

Proof. Since $\mathbf{L}$ is a 1-split Levi subgroup, $\mathbf{L}$ is contained in an F-stable parabolic subgroup of $\mathbf{G}$, $\mathbf{M}$, say. Let $\mathbf{U} = R_u(\mathbf{M})$, so $\mathbf{M} = \mathbf{U} \rtimes \mathbf{L}$. We have $\mathbf{M}^F = \mathbf{U}^F \rtimes \mathbf{L}^F$. Then $\mathbf{M}^F/\mathbf{U}^F \cong \mathbf{L}^F$. Let $\mu : \mathbf{M}^F \to \mathbf{L}^F$ be the quotient map. Let $N = N_{\mathbf{G}^F}(\mathbf{L},\lambda)$ and let c be the block of $k\mathbf{M}^F$ that dominates f. We show that the hypotheses of [61, Theorem 5.5] are satisfied by $\mathbf{M}^F$, N, $\mathbf{L}^F$, c and f.

Because $\mathbf{U}^F$ is an ℓ'-group, c dominates a unique block of $\mathcal{O}\mathbf{L}^F$ by Lemma 2.1.10 (c), so $\mu(c) = f$. Let d be the principal block of $\mathcal{O}\mathbf{U}^F$. Then it follows from the isomorphisms due to Fong reduction mentioned above that $c = fd$. Since d is central in $\mathcal{O}\mathbf{M}^F$, therefore $cf = c$. Since λ is a 1-cuspidal unipotent character in f with central ℓ-defect, Lemma 5.2.2 shows that all the characters in f are 1-cuspidal. It then follows by arguments given in [61, 5.3] that $c(\mathcal{O}\mathbf{G}^F)c = c(\mathcal{O}N)c$.

Next, since $N_{\mathbf{G}^F}(\mathbf{L},\lambda) \subseteq N_{\mathbf{G}^F}(\mathbf{L}^F,\lambda)$, N normalizes $\mathbf{L}^F$ and therefore f. By the proof of [23, Corollary 1.18], $N_{\mathbf{G}}(\mathbf{L}) \cap \mathbf{U} = \{1\}$. Therefore $N_{\mathbf{M}^F}(\mathbf{L},\lambda) \cap \mathbf{U}^F = \{1\}$, so $N_{\mathbf{M}^F}(\mathbf{L},\lambda) \subseteq \mathbf{L}^F$ and thus $\mathbf{L}^F = N_{\mathbf{M}^F}(\mathbf{L},\lambda) = N \cap \mathbf{M}^F$. By [14, Proposition 2.2 (ii)], since $\ell \geq 5$ and $\mathbf{L}$ is a proper Levi subgroup of $\mathbf{G}$, $\mathbf{L}^F = C_{\mathbf{G}^F}\left(Z(\mathbf{L})_\ell^F\right) = C_{\mathbf{G}^F}(P)$. Therefore f is a block of $\mathcal{O}C_{\mathbf{G}^F}(P)$, so $Br_P(f) = f$ where Br_P denotes the Brauer homomorphism as defined in Section 2.1.6. It follows that $Br_P(c) = Br_P(df) = \frac{1}{|\mathbf{U}^F|}Br_P(f) \neq 0$, so all hypotheses of [61, Theorem 5.5] are satisfied.

Recall that we have the following inclusion of Brauer pairs $(1,b) \leq (P,f)$ from the proof of Proposition 5.2.3. Therefore $Br_P(b)f = f$. Since $N/\mathbf{L}^F$, the relative Weyl group of $(\mathbf{L}^F,\lambda)$ in $\mathbf{G}^F$, is an ℓ'-group, [61, 5.5.4] implies that f is a block of $\mathcal{O}N_{\mathbf{G}^F}(\mathbf{L},\lambda)$ with defect group P, and $\mathcal{O}Nf$ and $\mathcal{O}\mathbf{G}^F b$ are source algebra equivalent, and hence Morita equivalent by [63, Theorem 38.2]. □

Theorem 5.2.5. *Suppose that $\boldsymbol{G}$ is simple, simply-connected. Let $e = e_\ell(q)$. Let $b = b_{\boldsymbol{G}^F}(\boldsymbol{L},\lambda)$ be a unipotent block of $\boldsymbol{G}^F$ for a unipotent e-cuspidal pair $(\boldsymbol{L},\lambda)$ of $\boldsymbol{G}$ of central ℓ-defect. Then*

(a) $mf(b) \leq 2$ *and*

(b) $mf(b) = 1$, *except possibly in the following situations.*

- $\boldsymbol{G} = E_8$, $\boldsymbol{L} = \phi_1^2.E_6$, $\lambda = E_6[\theta^i]$ $(i = 1, 2)$, *with* $\ell = 2$ *and* $e = 1$
- $\boldsymbol{G} = E_8$, $\boldsymbol{L} = \phi_2^2.{}^2E_6$, $\lambda = {}^2E_6[\theta^i]$ $(i = 1, 2)$, *with* $\ell \equiv 2$ *mod 3 and* $e = 2$

Proof. By [36, Table 1 and Proposition 5.6], the unipotent characters of classical finite groups of Lie type (including ${}^3D_4(q)$) are rational valued. Thus if $\mathbf{G}$ is of classical type, if $\mathbf{L}$ has all classical components, or if λ is rational valued, then $mf(b) = 1$ by Proposition 2.2.5 (b) and Lemma 5.2.1. It therefore only remains to consider the cases where $\mathbf{G}$ is of exceptional type, $\mathbf{L}$ contains some component of exceptional type, and λ is not rational valued. The following table lists all the unipotent e-cuspidal pairs of central ℓ-defect for these cases, identified using [11, Appendix: Table 1], [26] and [18, Chapter 13]. The character labels are as in [18, Chapter 13].

$\mathbf{G}$	e	$(\mathbf{L}, \lambda)$	Is of ℓ-central defect for
G_2	$1, 2$	$(G_2, G_2[\theta^i])$	$\ell \neq 3$
F_4	$1, 2$	$(F_4, F_4[\theta^i])$	$\ell \neq 3$
F_4	$1, 2$	$(F_4, F_4[\pm i])^*$	$\ell \neq 2$
E_6	$1, 2$	$(E_6, E_6[\theta^i])$	$\ell \neq 3$
${}^2E_6$	$1, 2$	$({}^2E_6, {}^2E_6[\theta^i])$	$\ell \neq 3$
E_7	1	$(E_7, E_7[\pm\xi])^\dagger$	$\ell \neq 2$
E_7	2	$(E_7, \phi_{512,11}), (E_7, \phi_{512,12})$	$\ell \neq 2$
E_7	1	$(E_6, E_6[\theta^i])$	$\ell \neq 3$
E_7	2	$({}^2E_6, {}^2E_6[\theta^i])$	$\ell \neq 3$
E_8	$1, 4$	$(E_8, E_8[\pm\theta^i])$	$\ell \neq 2, 3$

E_8	$1,2$	$(E_8, E_8[\pm i])$	$\ell \neq 2$
E_8	$1,2,4$	$\left(E_8, E_8[\zeta^j]\right)$	$\ell \neq 5$
E_8	$2,4$	$\left(E_8, E_6[\theta^i], \phi_{2,1}\right), \left(E_8, E_6[\theta^i], \phi_{2,2}\right)$	$\ell \neq 5$
E_8	4	$\left(E_8, E_6[\theta^i], \phi_{1,0}\right), \left(E_8, E_6[\theta^i], \phi_{1,6}\right),$ $\left(E_8, E_6[\theta^i], \phi_{1,3'}\right), \left(E_8, E_6[\theta^i], \phi_{1,3''}\right),$ $(E_8, \phi_{4096,11}), (E_8, \phi_{4096,26}),$ $(E_8, \phi_{4096,12}), (E_8, \phi_{4096,27}),$ $(E_8, E_7[\pm\xi, 1]), (E_8, E_7[\pm\xi, \varepsilon])$	every ℓ
E_8	1	$(E_7, E_7[\pm\xi])$	$\ell \neq 2$
E_8	2	$(E_7, \phi_{512,11}), (E_7, \phi_{512,12})^{\ddagger}$	$\ell \neq 2$
E_8	1	$\left(E_6, E_6[\theta^i]\right)$	$\ell \neq 3$
E_8	2	$\left({}^2E_6, {}^2E_6[\theta^i]\right)$	$\ell \neq 3$

$\theta := \exp(2\pi i/3)$, $\zeta := \exp(2\pi i/5)$, $\xi := \sqrt{-q}$

*[26] omits this pair for $\ell = 3$, $e = 2$ †[26] writes $E_7[\pm\zeta]$ instead of $E_7[\pm\xi]$ for $\ell = 2, e = 1$

‡[26] writes $E_7[\pm\xi]$ instead of $\phi_{512,11}, \phi_{512,12}$ for $\ell = 5, e = 2$

Suppose that ℓ is good for $\mathbf{G}$. Then by inspection, the Sylow ℓ-subgroups of $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ are trivial so by Proposition 5.2.3, the defect groups of b are isomorphic to a Sylow ℓ-subgroup of $Z(\mathbf{L})^F$. If $\mathbf{L} = \mathbf{G}$, then the Sylow ℓ-subgroups of $Z(\mathbf{L}^F)$ are trivial by inspection of [55, Table 24.2]. By [18, Proposition 3.6.8], since $\mathbf{L}$ is connected reductive, $Z(\mathbf{L})^F = Z\left(\mathbf{L}^F\right)$, therefore b has trivial defect and $mf(b) = 1$ by Proposition 2.2.5 (d). If $\mathbf{L}$ and $\mathbf{G}$ are such that $\mathrm{rk}(\mathbf{G}) = \mathrm{rk}([\mathbf{L}, \mathbf{L}]) + 1$, then $\dim(Z^\circ(\mathbf{L})) = 1$ by Lemma 4.1.1. The Sylow ℓ-subgroups of $Z^\circ(\mathbf{L})^F$ are therefore isomorphic to subgroups of the multiplicative group $\mathbf{G}_m$, so they are cyclic. By [14, Proposition 2.2 (i)], since ℓ is good for $\mathbf{G}$, $Z(\mathbf{L})^F_\ell = Z^\circ(\mathbf{L})^F_\ell$, therefore b has cyclic defect so $mf(b) = 1$ by Proposition 2.2.5 (d).

Now suppose that ℓ is bad for $\mathbf{G}$, that $\mathbf{L} = \mathbf{G}$. By inspection of the character degrees given in [18, Chapter 13], we see that e-cuspidal characters λ of $\mathbf{G}^F$ satisfy $\lambda(1)_\ell = |\mathbf{G}^F|_\ell$, so $mf(b) = 1$ by Proposition 2.2.5 (c).

The remaining ℓ-blocks will be handled on a case-by-case basis. First, suppose that $\mathbf{G} = E_8$, $\mathbf{L} = \phi_1^2.E_6$ and $\lambda = E_6[\theta^i]$ $(i = 1, 2)$ with $\ell \geq 5$ and $e = 1$. Then by Theorem 5.2.4, $k\mathbf{G}^F b$ is Morita equivalent to kNf where $N = N_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ and $f = b_{\mathbf{L}^F}(\lambda)$ is the block of $k\mathbf{L}^F$ containing λ. Suppose that P is a defect group of $k\mathbf{L}^F f$. Then since ℓ is odd and $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) \cong D_{12}$ is an ℓ'-group, P is isomorphic to a Sylow ℓ-subgroup of $Z(\mathbf{L})^F$ by Proposition 5.2.3. Since N normalizes $\mathbf{L}$, $P \trianglelefteq N$ so kNf has normal defect. Then by [63, Theorem 45.12], kNf is Morita equivalent to a twisted group algebra $k_\alpha(P \rtimes D_{12})$, where $\alpha \in H^2(D_{12}; k^\times)$. Since $H^2(D_{12}; k^\times) \cong C_2$, it follows from Lemma 2.2.7 that $mf(k_\alpha(P \rtimes D_{12})) = 1$. Whence, $mf(b) = 1$.

Suppose now that $\mathbf{G} = E_8$. If $\ell = 3$ and $\mathbf{L} = \phi_1.E_7$, $\lambda = E_7[\pm\xi]$ and $e = 1$, or $\mathbf{L} = \phi_1.E_7$, $\lambda = \phi_{512,11}$ or $\phi_{512,12}$ and $e = 2$, then b has cyclic defect by [26, page 364]. If $\ell = 5$ and $\mathbf{L} = \phi_1.E_7$, $\lambda = E_7[\pm\xi]$ and $e = 1$, or $\mathbf{L} = \phi_1.E_7$, $\lambda = \phi_{512,11}$ or $\phi_{512,12}$ and $e = 2$, then the relative Weyl group $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) \cong S_2$ has trivial Sylow ℓ-subgroups, so by Proposition 5.2.3 the defect groups of b are isomorphic to a Sylow ℓ-subgroup of $Z(\mathbf{L})^F$. Note that $\mathrm{rk}(\mathbf{G}) = \mathrm{rk}([\mathbf{L}, \mathbf{L}]) + 1$, so $\dim(Z^\circ(\mathbf{L})) = 1$ and the Sylow ℓ-subgroups of $Z^\circ(\mathbf{L})^F$ are cyclic, as above. Again, using [14, Proposition 2.2], $Z(\mathbf{L})_\ell^F = Z^\circ(\mathbf{L})_\ell^F$, so b has cyclic defect and $mf(b) = 1$ by Proposition 2.2.5 (d).

Suppose that $\mathbf{G} = E_7$, $\mathbf{L} = \phi_1.E_6(q)$, $\lambda = E_6[\theta^i]$, $(i = 1, 2)$, with $\ell = 2$ and $e = 1$. Then b has dihedral defect by [26, page 357]. Therefore by Proposition 2.2.5 (d), $mf(b) = 1$.

Finally, suppose that we are in one of the following cases: $\mathbf{G} = E_8$, $\mathbf{L} = \phi_1^2.E_6$, $\lambda = E_6[\theta^i]$, $(i = 1, 2)$, with $\ell = 2$ and $e = 1$; or $\mathbf{G} = E_8$, $\mathbf{L} = \phi_2^2.{}^2E_6$, $\lambda = {}^2E_6[\theta^i]$, $(i = 1, 2)$, with $\ell \neq 3$ and $e = 2$. From [36, Table 1] we know that the character field of λ is $\mathbb{Q}(\theta)$ where $\theta = \exp(\frac{2\pi i}{3})$. Since $\ell \neq 3$, θ is an ℓ'-root of unity so $\hat{\sigma}(\theta) = \theta^\ell$ (see Section 5.2.4). If $\ell \equiv 1 \bmod 3$, then $\hat{\sigma}(\theta) = \theta$ so ${}^{\hat{\sigma}}\lambda = \lambda$. Therefore by the arguments of Lemma 5.2.1, $mf(b) = 1$. If $\ell \equiv 2 \bmod 3$, however, then $\hat{\sigma}(\theta) = \theta^2 \neq \theta$ so we cannot conclude that $mf(b) = 1$. Because $\hat{\sigma}^2(\theta) = \theta^4 = \theta$, however, it follows that ${}^{\hat{\sigma}^2}\lambda = \lambda$, so $mf(b)$ is at most 2. $\square$

Corollary 5.2.6. *Let $\boldsymbol{G}$ be as in Theorem 5.2.5 and suppose that $\boldsymbol{G}^F$ has non trivial centre. Let Z be a central subgroup of $\boldsymbol{G}^F$ and suppose that $\bar{b}$ is a block of $k(\boldsymbol{G}^F/Z)$ dominated by a unipotent block b of $k\boldsymbol{G}^F$. Then $mf\left(\bar{b}\right) = 1$.*

Proof. Since $\mathbf{G}^F$ has non trivial centre, $\mathbf{G} \neq E_8$, therefore by the proof of Theorem 5.2.5, either $\sigma(b) = b$, or b has trivial, cyclic or dihedral defect.

First suppose that $\bar{b}$ is dominated by a unipotent block b of $k\mathbf{G}^F$ such that $\sigma(b) = b$. Then by Lemma 2.2.2 (b), $\sigma\left(\bar{b}\right)$ is also dominated by b. Since Z is central, it then follows from parts (b) and (c) of Lemma 2.1.10 that $\sigma\left(\bar{b}\right) = \bar{b}$. Therefore $k\left(\mathbf{G}^F/Z\right)\bar{b} \cong k\left(\mathbf{G}^F/Z\right)\bar{b}^{(\ell)}$ as k-algebras by Lemma 2.2.4, so $frob\left(\bar{b}\right) = 1$, hence $mf\left(\bar{b}\right) = 1$.

Now suppose that $\bar{b}$ is dominated by a unipotent block b of $k\mathbf{G}^F$ which has either trivial, cyclic or dihedral defect. Then by [57, Ch.5, Theorem 8.7 (ii)], the defect groups of $\bar{b}$ are also either trivial, cyclic or dihedral. Therefore $mf\left(\bar{b}\right) = 1$ by Proposition 2.2.5 (d). $\square$

5.2.2 Quasi-isolated blocks

For the rest of Section 5.2 we fix a pair $(\mathbf{G}^*, F)$ which is dual to $(\mathbf{G}, F)$ as in Definition 4.1.9, and we fix a regular embedding $i : \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ as in Definition 4.2.9. Recall that an element $s \in \mathbf{G}^*$ is quasi-isolated if there does not exist a proper Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ such that $C_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$.

Lemma 5.2.7. *Let $s \in \boldsymbol{G}^{*F}$ be a semisimple element such that the components of $C_{\boldsymbol{G}^*}(s)$ are all of classical type and suppose that $Z(\boldsymbol{G})$ is connected. Then every $\chi \in \mathcal{E}\left(\boldsymbol{G}^F, s\right)$ is uniquely determined by its uniform projection.*

Proof. Since $\mathbf{G}$ is a reductive algebraic group with connected centre, $C_{\mathbf{G}^*}(s)$ is a connected group for any semisimple element $s \in \mathbf{G}^{*F}$ [23, Lemma 13.14 (iii)]. By assumption, $C_{\mathbf{G}^*}(s)$ has only classical components, so by results of Lusztig (see [15, Theorem 15.8]), the unipotent characters of $C_{\mathbf{G}^*}(s)^F$ are uniquely determined by their uniform projection. Thus by Jordan decomposition, Theorem 4.2.8, each $\chi \in \mathcal{E}\left(\mathbf{G}^F, s\right)$ is uniquely determined by its uniform projection. $\square$

The next two results use the notation of Section 4.2.2.

Lemma 5.2.8. *Suppose that* $\left(\widetilde{\boldsymbol{T}}_1, \widetilde{\theta}_1\right), \left(\widetilde{\boldsymbol{T}}_2, \widetilde{\theta}_2\right) \in \nabla\left(\widetilde{\boldsymbol{G}}, F\right)$ *are geometrically conjugate pairs. Let* $\varphi : K \to K$ *be a field automorphism, and for* $i \in \{1,2\}$ *define an action of* φ *on* $\widetilde{\theta}_i$ *by* ${}^{\varphi}\widetilde{\theta}_i(t) = \varphi(\widetilde{\theta}_i(t))$ *for all* $t \in \widetilde{\boldsymbol{T}}_i^F$. *Suppose that there exists a linear character* $\eta \in \mathrm{Irr}\left(\widetilde{\boldsymbol{G}}^F / \boldsymbol{G}^F\right)$ *such that* ${}^{\varphi}\widetilde{\theta}_1 = \widetilde{\theta}_1 \eta|_{\widetilde{\boldsymbol{T}}_1^F}$. *Then* ${}^{\varphi}\widetilde{\theta}_2 = \widetilde{\theta}_2 \eta|_{\widetilde{\boldsymbol{T}}_2^F}$.

Proof. By Lemma 4.2.3, since $\left(\widetilde{\mathbf{T}}_1, \widetilde{\theta}_1\right)$ and $\left(\widetilde{\mathbf{T}}_2, \widetilde{\theta}_2\right)$ are geometrically conjugate, there exists $n \in \mathbb{N}$ and $g \in \widetilde{\mathbf{G}}^{F^n}$ such that $\widetilde{\mathbf{T}}_1 = g\widetilde{\mathbf{T}}_2 g^{-1}$ and for all $x \in \widetilde{\mathbf{T}}_2$,

$$\widetilde{\theta}_2\left(N_{F^n/F}(x)\right) = \widetilde{\theta}_1\left(N_{F^n/F}(gxg^{-1})\right)$$

where, for $i \in \{1,2\}$, $N_{F^n/F} : \widetilde{\mathbf{T}}_i \to \widetilde{\mathbf{T}}_i$ is the norm map sending $t \mapsto tF(t)\dots F^{n-1}(t)$ for all $t \in \widetilde{\mathbf{T}}_i$.

The inclusion $\widetilde{\mathbf{G}}^F \hookrightarrow \widetilde{\mathbf{G}}^{F^n}$ induces an inclusion $\widetilde{\mathbf{G}}^F / \mathbf{G}^F \hookrightarrow \widetilde{\mathbf{G}}^{F^n} / \mathbf{G}^{F^n}$. Therefore, since $\widetilde{\mathbf{G}}^{F^n} / \mathbf{G}^{F^n}$ is an abelian group, $\eta : \widetilde{\mathbf{G}}^F / \mathbf{G}^F \to \mathbb{C}^\times$ extends to a linear character $\hat{\eta} : \widetilde{\mathbf{G}}^{F^n} / \mathbf{G}^{F^n} \to \mathbb{C}^\times$. We can view $\hat{\eta}$ as a character of $\widetilde{\mathbf{G}}^{F^n}$.

Let $x \in \widetilde{\mathbf{T}}_2^{F^n}$. Then for any $h \in \widetilde{\mathbf{G}}^F$,

$$
\begin{aligned}
\eta|_{\widetilde{\mathbf{T}}_1^F}\left(N_{F^n/F}(hxh^{-1})\right) &= \hat{\eta}\left(N_{F^n/F}(hxh^{-1})\right) \quad \text{by definition of } \hat{\eta}\\
&= \hat{\eta}\left(hxh^{-1}F(hxh^{-1})\dots F^n(hxh^{-1})\right) \quad \text{by definition of } N_{F^n/F}\\
&= \hat{\eta}(h)\hat{\eta}(x)\hat{\eta}(h^{-1})\hat{\eta}(F(h))\hat{\eta}(F(x))\hat{\eta}(F(h^{-1}))\dots\\
&\quad \hat{\eta}(F^{n-1}(h))\hat{\eta}(F^{n-1}(x))\hat{\eta}(F^{n-1}(h^{-1})) \quad \text{since } \hat{\eta} \text{ and } F \text{ are homomorphisms}\\
&= \hat{\eta}(x)\hat{\eta}(F(x))\dots\hat{\eta}(F^{n-1}(x))\\
&= \hat{\eta}\left(xF(x)\dots F^{n-1}(x)\right)\\
&= \hat{\eta}\left(N_{F^n/F}(x)\right) \quad \text{by definition of } N_{F^n/F}\\
&= \eta|_{\widetilde{\mathbf{T}}_2^F}\left(N_{F^n/F}(x)\right) \quad \text{since } N_{F^n/F}(x) \in \widetilde{\mathbf{T}}_2^F.
\end{aligned}
$$

Then for any $x \in \widetilde{\mathbf{T}}_2^{F^n}$,

$$
\begin{aligned}
{}^{\varphi}\widetilde{\theta}_2\left(N_{F^n/F}(x)\right) &= \varphi\left(\widetilde{\theta}_2\left(N_{F^n/F}(x)\right)\right)\\
&= \varphi\left(\widetilde{\theta}_1\left(N_{F^n/F}(gxg^{-1})\right)\right)\\
&= \widetilde{\theta}_1\eta|_{\widetilde{\mathbf{T}}_1^F}\left(N_{F^n/F}(gxg^{-1})\right) \quad \text{by assumption}\\
&= \widetilde{\theta}_1\left(N_{F^n/F}(gxg^{-1})\right)\eta|_{\widetilde{\mathbf{T}}_1^F}\left(N_{F^n/F}(gxg^{-1})\right)\\
&= \widetilde{\theta}_2\left(N_{F^n/F}(x)\right)\eta|_{\widetilde{\mathbf{T}}_2^F}\left(N_{F^n/F}(x)\right) \quad \text{by above}\\
&= \widetilde{\theta}_2\eta|_{\widetilde{\mathbf{T}}_2^F}\left(N_{F^n/F}(x)\right).
\end{aligned}
$$

Since the norm map restricts to a surjection $N_{F^n/F} : \widetilde{\mathbf{T}}_2^{F^n} \to \widetilde{\mathbf{T}}_2^F$ by Lemma 4.2.2, therefore ${}^{\varphi}\widetilde{\theta}_2 = \widetilde{\theta}_2\eta|_{\widetilde{\mathbf{T}}_2^F}$ as required. □

Proposition 5.2.9. *Let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ' element of order m such that the components of $C_{\boldsymbol{G}^*}(s)$ are all of classical type. Let $b \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$ be an ℓ-block of $\boldsymbol{G}^F$. Then $mf(b) \le \varphi(m)$ where φ is the Euler totient function.*

Proof. Let $\chi \in \mathrm{Irr}(b) \cap \mathcal{E}\left(\mathbf{G}^F, s\right)$ and let $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F, [s])$ be such that χ is an irreducible constituent of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. Let $\widetilde{\mathbf{T}} = \mathbf{T}.Z(\widetilde{\mathbf{G}})$ and let $\tilde{s} \in \widetilde{\mathbf{G}}^{*F}$ be such that $i^*(\tilde{s}) = s$. Then by Lemma 4.2.12 (b), there exists an irreducible character $\widetilde{\theta} \in \mathrm{Irr}\left(\widetilde{\mathbf{T}}^F\right)$ extending θ, so $\widetilde{\theta}|_{\mathbf{T}^F} = \theta$ and $(\widetilde{\mathbf{T}}, \widetilde{\theta}) \in \nabla(\widetilde{\mathbf{G}}, F, [\tilde{s}])$. It follows that there exists a character $\psi \in \mathrm{Irr}\left(\widetilde{\mathbf{G}}^F\right)$ which is an irreducible constituent of $R_{\widetilde{\mathbf{T}}}^{\widetilde{\mathbf{G}}}(\widetilde{\theta})$, such that χ is an irreducible constituent of $\psi|_{\mathbf{G}^F}$.

Let $n = \varphi(m)$. Since $o(s) = m$ and $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F, [s])$, it follows from [23, Proposition 13.11] that $o(\theta) = m$. Hence ${}^{\hat{\sigma}^n}\theta = \theta$. We claim that there exists a linear character $\eta \in \mathrm{Irr}\left(\widetilde{\mathbf{G}}^F\right)$ such that ${}^{\hat{\sigma}^n}\psi = \psi\eta$.

As ${}^{\hat{\sigma}^n}\theta = \theta$ is an irreducible component of ${}^{\hat{\sigma}^n}\widetilde{\theta}|_{\mathbf{T}^F}$, it follows from [43, Lemma 3.1] that ${}^{\hat{\sigma}^n}\widetilde{\theta} = \widetilde{\theta}\eta$ for some irreducible character $\eta \in \mathrm{Irr}\left(\widetilde{\mathbf{T}}^F/\mathbf{T}^F\right)$. Since $\widetilde{\mathbf{T}}^F/\mathbf{T}^F$ is abelian, η is a linear character, and since $\widetilde{\mathbf{T}}^F/\mathbf{T}^F \cong \widetilde{\mathbf{G}}^F/\mathbf{G}^F$ (see the proof of [5, Corollaire 2.7]) we can view η as a character of $\widetilde{\mathbf{G}}^F$. Thus ${}^{\hat{\sigma}^n}\widetilde{\theta} = \widetilde{\theta}\eta|_{\widetilde{\mathbf{T}}^F}$ for a linear character $\eta \in \mathrm{Irr}\left(\widetilde{\mathbf{G}}^F\right)$. It follows from Lemma 5.2.8 that for any pair $\left(\widetilde{\mathbf{S}}, \tilde{\gamma}\right) \in \nabla\left(\widetilde{\mathbf{G}}, F, [\tilde{s}]\right)$, ${}^{\hat{\sigma}^n}\widetilde{\gamma} = \widetilde{\gamma}\eta|_{\widetilde{\mathbf{S}}^F}$. Recall that since $Z(\widetilde{\mathbf{G}})$ is connected, geometric conjugacy is the same as rational conjugacy for $\widetilde{\mathbf{G}}$.

Let $\left(\widetilde{\mathbf{S}}, \tilde{\gamma}\right) \in \nabla\left(\widetilde{\mathbf{G}}, F\right)$. Then

$$\begin{aligned}
\langle \psi, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}) \rangle &= \begin{cases} 0 & \text{if } (\widetilde{\mathbf{S}}, \widetilde{\gamma}) \notin \nabla(\widetilde{\mathbf{G}}, F, [\tilde{s}]), \\ \langle {}^{\hat{\sigma}^n}\psi, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}({}^{\hat{\sigma}^n}\widetilde{\gamma}) \rangle & \text{otherwise,} \end{cases} \\
&= \begin{cases} 0 & \text{if } (\widetilde{\mathbf{S}}, \tilde{\gamma}) \notin \nabla(\widetilde{\mathbf{G}}, F, [\tilde{s}]), \\ \langle {}^{\hat{\sigma}^n}\psi, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}\eta|_{\widetilde{\mathbf{S}}^F}) \rangle & \text{otherwise.} \end{cases}
\end{aligned}$$

Since $\langle {}^{\hat{\sigma}^n}\psi, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}\eta|_{\widetilde{\mathbf{S}}^F}) \rangle = 0$ if ${}^{\hat{\sigma}^n}\psi \notin R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}\eta|_{\widetilde{\mathbf{S}}^F})$, and this holds if $(\widetilde{\mathbf{S}}, \tilde{\gamma}) \notin \nabla(\widetilde{\mathbf{G}}, F, [\tilde{s}])$, it follows that $\langle \psi, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}) \rangle = \langle {}^{\hat{\sigma}^n}\psi, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}\eta|_{\widetilde{\mathbf{S}}^F}) \rangle$ for every $(\widetilde{\mathbf{S}}, \tilde{\gamma}) \in \nabla(\widetilde{\mathbf{G}}, F)$.

On the other hand, η is a linear character so for every $\left(\widetilde{\mathbf{S}}, \tilde{\gamma}\right) \in \nabla\left(\widetilde{\mathbf{G}}, F\right)$,

$$\langle \psi, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}) \rangle = \langle \psi\eta, \left(R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma})\right)\eta \rangle = \langle \psi\eta, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}\eta|_{\widetilde{\mathbf{S}}^F}) \rangle,$$

and therefore

$$\langle {}^{\hat{\sigma}^n}\psi, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}\eta|_{\widetilde{\mathbf{S}}^F})\rangle = \langle \psi\eta, R_{\widetilde{\mathbf{S}}}^{\widetilde{\mathbf{G}}}(\widetilde{\gamma}\eta|_{\widetilde{\mathbf{S}}^F})\rangle.$$

Thus ${}^{\hat{\sigma}^n}\psi$ and $\psi\eta$ have the same uniform projection. Since $Z\left(\widetilde{\mathbf{G}}^F\right)$ is connected and all components of $C_{\widetilde{\mathbf{G}}^{*F}}(\tilde{s})$ are of classical type by assumption, it follows from Lemma 5.2.7 that ${}^{\hat{\sigma}^n}\psi = \psi\eta$.

Recall that χ is an irreducible component of $\psi|_{\mathbf{G}^F}$. Now since ${}^{\hat{\sigma}^n}\psi|_{\mathbf{G}^F} = (\psi\eta)|_{\mathbf{G}^F} = \psi|_{\mathbf{G}^F}$, it follows that ${}^{\hat{\sigma}^n}\chi$ is also an irreducible component of $\psi|_{\mathbf{G}^F}$. Thus by Clifford theory [44, Theorem 6.2], χ and ${}^{\hat{\sigma}^n}\chi$ are $\widetilde{\mathbf{G}}^F$-conjugate, and therefore b and $\hat{\sigma}^n(b)$ are $\widetilde{\mathbf{G}}^F$-conjugate. This action restricts to an automorphism of $\mathcal{O}\mathbf{G}^F$ sending b to $\hat{\sigma}^n(b)$, so $\mathcal{O}\mathbf{G}^F b \cong \mathcal{O}\mathbf{G}^F\hat{\sigma}^n(b)$. Therefore $mf(b) \leq n = \varphi(m)$. $\square$

Theorem 5.2.10. *Let $\boldsymbol{G}$ be simple and simply connected. Let $s \in \boldsymbol{G}^{*F}$ be a quasi-isolated semisimple ℓ' element such that the components of $C_{\boldsymbol{G}^*}(s)$ are all of classical type. Let $b \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$ be a block of $\boldsymbol{G}^F$. Then if $\mathbf{G}$ is of type B, $mf(b) = 1$; if $\mathbf{G}$ is of type C, D, G_2, F_4, E_6, ${}^2E_6$ or E_7, then $mf(b) \leq 2$; and if $\mathbf{G}$ is of type E_8 then $mf(b) \leq 4$.*

Proof. The orders and centralizers of the quasi-isolated elements of finite groups of Lie type are given in [4, Table 3] and [49, Table 1]. The result therefore follows directly from Proposition 5.2.9.

Type	$o(s) \in$	$\varphi(o(s)) \leq$
B	$\{1,2\}$	1
C	$\{1,2,4\}$	2
D	$\{1,2,4\}$	2
G_2	$\{1,2,3\}$	2
F_4	$\{1,2,3\}$	2
E_6	$\{1,2,3,6\}$	2
E_7	$\{1,2,3,6\}$	2
E_8	$\{1,2,3,5,6\}$	4

$\square$

The remaining quasi-isolated blocks not covered by Theorem 5.2.10 are blocks of E_7 and E_8. We need the following two results.

Lemma 5.2.11. *Suppose that s is a semisimple ℓ' element of $\boldsymbol{G}^{*F}$ of order m and $b \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$. Let $n = \varphi(m)$ where φ denotes the Euler totient function. Then $\sigma^n(b) \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$.*

Proof. Let $\chi \in \mathrm{Irr}(b) \cap \mathcal{E}\left(\mathbf{G}^F, s\right)$ where s is a semisimple ℓ' element of $\mathbf{G}^{*F}$ of order m and let $n = \varphi(m)$. Then ${}^{\hat{\sigma}^n}\chi \in \mathrm{Irr}(\sigma^n(b))$ and there exists a pair $(\mathbf{T}, \theta) \in \nabla(\mathbf{G}, F, (s))$ such that $\chi \in R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and ${}^{\hat{\sigma}^n}\chi \in R_{\mathbf{T}}^{\mathbf{G}}({}^{\hat{\sigma}^n}\theta)$. Since s has order m, so does θ, so $\mathbb{Q}(\theta) = \mathbb{Q}[\omega]$ for ω a primitive mth root of unity, and therefore ${}^{\hat{\sigma}^n}\theta = \theta$. Thus ${}^{\hat{\sigma}^n}\chi \in R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ so $\sigma^n(b) \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ as required. $\square$

In the next Lemma we use the general parametrisation of blocks of $\mathbf{G}^F$ from Theorem 4.3.11 (d), and let $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ denote the block of $\mathbf{G}^F$ containing the irreducible components of $R_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(\lambda)$ for an e-Jordan quasi-central cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$.

Lemma 5.2.12. *Let $\boldsymbol{G}$ be an exceptional group with connected centre. Let $\ell \geq 3$ and let s be a quasi-isolated semisimple ℓ' element of $\boldsymbol{G}^{*F}$. Let $B = B_{\boldsymbol{G}^F}(\boldsymbol{L}, \lambda)$ and $B' = B_{\boldsymbol{G}^F}(\boldsymbol{L}', \lambda')$ be blocks of $\mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$, and let α (respectively α') denote the e-cuspidal unipotent character of $C_{\boldsymbol{G}^*}(s)^F$ corresponding to λ (respectively λ') by Jordan decomposition, Theorem 4.2.8. Then if B is Galois conjugate to B', $\alpha(1) = \alpha'(1)$.*

Proof. By [15, Theorem 4.2] and [50] since $\mathbf{G}$ is of exceptional type and s is quasi-isolated, e-Jordan quasi-central cuspidality is equivalent to e-cuspidality for $(\mathbf{L}, \lambda)$ and $(\mathbf{L}', \lambda')$ (see [50, Remark 2.2]). Therefore, since $\ell \geq 3$, $(\mathbf{L}', \lambda')$ is the unique e-cuspidal pair of $\mathbf{G}$ up to $\mathbf{G}^F$-conjugacy such that B' contains the irreducible constituents of $R^{\mathbf{G}}_{\mathbf{L}' \subseteq \mathbf{P}'}(\lambda')$, by Theorem 4.3.11 (d).

Suppose that B and B' are Galois conjugate, so $B' = \sigma^n(B)$ for some positive integer n. Then B' contains the irreducible components of ${}^{\hat{\sigma}^n}\left(R^{\mathbf{G}}_{\mathbf{L} \subseteq \mathbf{P}}(\lambda)\right) = R^{\mathbf{G}}_{\mathbf{L} \subseteq \mathbf{P}}\left({}^{\hat{\sigma}^n}\lambda\right)$. As $(\mathbf{L}, \lambda)$ is an e-cuspidal pair, $(\mathbf{L}, {}^{\sigma^n}\lambda)$ is also an e-cuspidal pair. Thus $(\mathbf{L}, {}^{\hat{\sigma}^n}\lambda)$ is $\mathbf{G}^F$-conjugate to $(\mathbf{L}', \lambda')$. It follows that $\lambda'(1) = {}^{\hat{\sigma}^n}\lambda(1) = \lambda(1)$. By the degree formula for Jordan decomposition, Theorem 4.2.8, therefore $\alpha(1) = \alpha'(1)$. □

Remark 5.2.13. In the following propositions the Morita Frobenius number of certain blocks can be obtained by applying Ennola duality to other blocks. This is done by formally changing q to $-q$. See [11, Section 3A] for more details.

Proposition 5.2.14. *Let $\boldsymbol{G}$ be a simple, simply connected algebraic group of type E_8. Let $s \in \boldsymbol{G}^{*F}$ be a quasi-isolated semisimple ℓ' element such that the components of $C_{\boldsymbol{G}^*}(s)$ are not all of classical type. Let $B \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$ be an ℓ-block of $\boldsymbol{G}^F$. If $o(s) = 2$ then $mf(B) \leq 2$ and if $o(s) = 3$ then $mf(B) \leq 4$.*

Proof. By [49, Table 1], we need to consider the following cases:

$o(s)$	$C_{\mathbf{G}^*}(s)$
2	$E_7 \times A_1$
3	$E_6 \times A_2$
3	${}^2E_6 \times {}^2A_2$

Our strategy is the following. If a block B has cyclic defect, then $mf(B) = 1$ by Proposition 2.2.5 (d). If B has defect group different from the defect groups of all the other blocks of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$, then B is not Galois conjugate to any other block in $\mathcal{E}_{\mathbf{G}}(\mathbf{G}^F, s)$. We therefore restrict our attention to the blocks of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ of non-cyclic, non-unique defect within $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ and, for each $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$, we calculate the largest possible size of a collection of Galois conjugate blocks of non-cyclic defect in $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. Using Lemma 5.2.11, this allows us to determine upper bounds on the Morita Frobenius numbers of the blocks in $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$.

First suppose that ℓ is bad for $\mathbf{G}$ (i.e. $\ell \leq 5$). Then the blocks of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ are given in [49, Section 6] along with information about their defects. In the table below we list the blocks of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ which do not have cyclic defect, and do not have unique defect within $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. Each row contains blocks corresponding to the pairs $(\mathbf{L}, \lambda)$ in the bijection given in [49, Theorem 1.2 (a)]. As in the tables in [49], instead of giving λ, we give the unipotent character α of $C_{\mathbf{G}^*}(s)^F$ which corresponds to λ by Jordan decomposition, Theorem 4.2.8. Note that the α's given in rows 3, 4 and 5 of the table for $\ell = 5$ are identified using [11, Appendix: Table 1].

Row	ℓ	e	$C_{\mathbf{G}^*}(s)^F$	$\mathbf{L}^F$	α	$W_{\mathbf{G}^F}(\mathbf{L},\lambda)$
1	2	1	$E_6(q)A_2(q)$	$\Phi_1^2(q)E_6(q)$	$E_6[\theta^i]$ $(i=1,2)$	A_2
2	2	1	${}^2E_6(q)^2A_2(q)$	$\Phi_1(q)E_7(q)$	${}^2E_6[\theta^i]$ $(i=1,2)$	A_1
1	5	1	$E_6(q)A_2(q)$	$\Phi_1^2(q)E_6(q)$	$E_6[\theta^i]$ $(i=1,2)$	A_2
2	5	1	$E_7(q)A_1(q)$	$\Phi_1^2(q)E_6(q)$	$E_6[\theta^i]$ $(i=1,2)$	$A_1\times A_1$
3	5	4	$E_7(q)A_1(q)$	$\Phi_4^2(q)D_4(q)$	$1^3\otimes 1, 1^3\otimes\phi_{11}$ $\phi_{11}^3\otimes 1, \phi_{11}^3\otimes\phi_{11}$	G_8
4	5	4	$E_7(q)A_1(q)$	$\Phi_4^2(q)D_4(q)$	$1^2\phi_{11}\otimes 1, 1^2\phi_{11}\otimes\phi_{11}$ $1\phi_{11}^2\otimes 1, 1\phi_{11}^2\otimes\phi_{11}$	$G(4,1,2)$
5	5	4	$E_6(q)A_2(q)$	$\Phi_4^2(q)D_4(q)$	$1\otimes 1, 1\otimes\phi_{21}$ $1\otimes\phi_{111}$	G_8

When $\ell=2$, rows 1 and 2 contain pairs of blocks which could be Galois conjugate pairs. When $\ell=3$ all blocks have cyclic defect, or unique defect within $\mathcal{E}_\ell\big(\mathbf{G}^F,s\big)$. When $\ell=5$ rows 1 and 2 both contain pairs of blocks which could be Galois conjugate pairs. Rows 3 and 4 contain blocks of equal non-cyclic defect. Let B be a block from row 3 or 4 and let P denote the defect group of B. Since P is abelian, Lemma 2.2.3 shows that $N(P,B)/\mathbf{G}^F \cong N(P,\sigma(B))/\mathbf{G}^F$ and therefore B and $\sigma(B)$ have the same relative Weyl group. Thus blocks in row 3 are not Galois conjugate to blocks in row 4. Within row 3, all blocks correspond to an α of unique degree, therefore by Lemma 5.2.12 row 3 contains no Galois conjugate blocks. In row 4 there is one pair of blocks which could be Galois conjugate. In row 5 each α has unique degree so again by Lemma 5.2.12, there are no Galois conjugate blocks. The results for $e=2$ are analogous to $e=1$, obtained by applying Ennola duality. Therefore for bad ℓ, for any e, a block $B\in\mathcal{E}_\ell\big(\mathbf{G}^F,s\big)$ of non-cyclic defect is Galois conjugate to at most one other block in $\mathcal{E}_\ell\big(\mathbf{G}^F,s\big)$. Thus by Lemma 5.2.11, $mf(B)\le 2$ if $o(s)=2$ and $mf(B)\le 4$ if $o(s)=3$.

Now suppose that ℓ is good for $\mathbf{G}$. Since

$$|E_8(q)| = q^{120}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30},$$

we need to consider e = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18, 20, 24, 15 and 30. When e = 7, 9, 14, 18, 20, 24 or 30, Φ_e only divides $|\mathbf{G}^F|$ once, so by Lemma 4.1.12, the defect groups of all blocks are cyclic and therefore the Morita Frobenius number of every block is 1 by Proposition 2.2.5 (d). Hence it only remains to consider the cases when $e = 1$, 2, 3, 4, 5, 6, 8, 10 and 12.

When $e = 1, 2, 4$ the e-Jordan quasi-central cuspidal pairs labelling $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ are given in [49] and we calculate the defects using [15, Lemma 4.16]. The following table lists the blocks of $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ which do not have cyclic defect, and do not have unique defect within $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$.

Row	e	$C_{\mathbf{G}^*}(s)^F$	$\mathbf{L}^F$	α	$W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$
1	1	$E_6(q)A_2(q)$	$\Phi_1^2(q)E_6(q)$	$E_6[\theta^i]$ $(i = 1, 2)$	A_2
2	1	$E_7(q)A_1(q)$	$\Phi_1^2(q)E_6(q)$	$E_6[\theta^i]$ $(i = 1, 2)$	$A_1 \times A_1$
3	4	$E_6(q)A_2(q)$	$\Phi_4^2(q)D_4(q)$	$1 \otimes 1, 1 \otimes \phi_{21}$ $1 \otimes \phi_{111}$	G_8
4	4	$E_7(q)A_1(q)$	$\Phi_4^2(q)D_4(q)$	$1^3 \otimes 1, 1^3 \otimes \phi_{11}$ $\phi_{11}^3 \otimes 1, \phi_{11}^3 \otimes \phi_{11}$	G_8
5	4	$E_7(q)A_1(q)$	$\Phi_4^2(q)D_4(q)$	$1^2\phi_{11} \otimes 1, 1^2\phi_{11} \otimes \phi_{11}$ $1\phi_{11}^2 \otimes 1, 1\phi_{11}^2 \otimes \phi_{11}$	$G(4, 1, 2)$

The case $e = 2$ can again be obtained from $e = 1$ using Ennola duality. Following the same arguments as for bad ℓ, we find that for good ℓ, when $e = 1, 2$ or 4, a block $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ of non-cyclic defect is Galois conjugate to at most one other block in $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. Therefore by Lemma 5.2.11, $mf(B) \leq 2$ if $o(s) = 2$ and $mf(B) \leq 4$ if $o(s) = 3$.

For the remaining e's for good ℓ, the e-Jordan quasi-central cuspidal pairs corresponding to B are not explicitly listed in [49] so instead we look at properties of the Jordan correspondent block as defined in Definition 4.3.12. Let $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ and let $b = b_{C_{\mathbf{G}^*}(s)^F}(C^\circ_{\mathbf{L}^*}(s), \alpha)$ be the Jordan correspondent of B. By [15, Proposition 5.1], the defect groups of B and b are isomorphic. Suppose that $e = 5$, 8, 10 or 12. Then Φ_e divides the order of $C_{\mathbf{G}^*}(s)^F$ at most once so by Lemma 4.1.12, b has cyclic defect and hence so does B. Thus it only remains to consider when $e = 3$ or 6. Since these are Ennola dual, it is enough to just consider $e = 3$, and the results will hold analogously for $e = 6$.

The following table lists the unipotent 3-cuspidal pairs of $C_{\mathbf{G}^*}(s)^F$ for each of the three cases we need to consider. These are identified by finding the 3-cuspidal unipotent pairs for each of the components of $C_{\mathbf{G}^*}(s)^F$ (using [11, Appendix: Table 1] and [18, Section 13.9]), and then applying [27, Proposition 2.1.5 (b)]. We let $T_3(\mathbf{G})$ denote a 3-split maximal torus of $\mathbf{G}$. The defects are calculated using Proposition 5.2.3.

Row	$C_{\mathbf{G}^*}(s)^F$	$(C_{\mathbf{L}^*}(s), \alpha)$	$W_{\mathbf{L}^F}(C_{\mathbf{L}^*}(s), \alpha)$	Defect
1	$E_7(q)A_1(q)$	$\begin{cases} (\Phi_3^3.A_1(q), 1 \otimes 1) \\ (\Phi_3^3.A_1(q), 1 \otimes \phi_{11}) \end{cases}$	G_{26}	$\begin{cases} 1 + \log_7\left((\Phi_3^3)_7\right) & \text{if } \ell = 7 \\ \log_\ell\left((\Phi_3^3)_\ell\right) & \text{o/w} \end{cases}$
2	$E_7(q)A_1(q)$	$\begin{cases} (\Phi_1\Phi_3{}^3D_4(q).A_1(q), {}^3D_4[-1] \otimes 1) \\ (\Phi_1\Phi_3{}^3D_4(q).A_1(q), {}^3D_4[-1] \otimes \phi_{11}) \end{cases}$	Z_6	$\log_\ell\left((\Phi_3)_\ell\right)$
3	$E_7(q)A_1(q)$	$\begin{cases} (\Phi_3.A_5(q).A_1(q), \phi_{42} \otimes 1) \\ (\Phi_3.A_5(q).A_1(q), \phi_{42} \otimes \phi_{11}) \\ (\Phi_3.A_5(q).A_1(q), \phi_{2211} \otimes 1) \\ (\Phi_3.A_5(q).A_1(q), \phi_{2211} \otimes \phi_{11}) \end{cases}$	Z_6	$\log_\ell\left((\Phi_3)_\ell\right)$
4	$E_7(q)A_1(q)$	$\begin{cases} (E_7(q).A_1(q), 10 \text{ chars} \otimes 1) \\ (E_7(q).A_1(q), 10 \text{ chars} \otimes \phi_{11}) \end{cases}$	1	Trivial
5	$E_6(q)A_2(q)$	$(T_3(E_6)(q).T_3(A_2)(q), 1)$	$G_{25} \times Z_3$	$\log_\ell\left((\Phi_3^4)_\ell\right)$
6	$E_6(q)A_2(q)$	$\left(\Phi_3{}^3D_4(q).T_3(A_2)(q), {}^3D_4[-1]\right)$	$Z_3 \times Z_3$	$\log_\ell\left((\Phi_3^2)_\ell\right)$

7	$E_6(q)A_2(q)$	$\begin{cases} (E_6(q).T_3(A_2)(q), \phi_{81,6}) \\ (E_6(q).T_3(A_2)(q), \phi_{81,10}) \\ (E_6(q).T_3(A_2)(q), \phi_{90,8}) \end{cases}$	Z_3	$\log_\ell\left((\Phi_3)_\ell\right)$
8	${}^2E_6(q){}^2A_2(q)$	$\begin{cases} \left(T_3\left({}^2E_6\right)(q).{}^2A_2(q), 1 \otimes 1\right) \\ \left(T_3\left({}^2E_6\right)(q).{}^2A_2(q), 1 \otimes \phi_{21}\right) \\ \left(T_3\left({}^2E_6\right)(q).{}^2A_2(q), 1 \otimes \phi_{111}\right) \end{cases}$	G_{25}	$\log_\ell\left((\Phi_3^2)_\ell\right)$
9	${}^2E_6(q){}^2A_2(q)$	$\begin{cases} \left({}^2E_6(q).{}^2A_2(q), 9 \text{ chars} \otimes 1\right) \\ \left({}^2E_6(q).{}^2A_2(q), 9 \text{ chars} \otimes \phi_{21}\right) \\ \left({}^2E_6(q).{}^2A_2(q), 9 \text{ chars} \otimes \phi_{111}\right) \end{cases}$	1	Trivial

If $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ has a Jordan correspondent b contained in row 2, 3, 4, 7, or 9, then b has cyclic defect, so B has cyclic defect and thus $mf(B) = 1$. If b is in row 5 or 6 then b has unique defect within $\mathcal{E}_\ell(C_{\mathbf{G}^*}(s)^F, 1)$ so B has unique defect within $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ and therefore is not Galois conjugate to any other block in $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. Finally, if b is in row 1 or 8 then α has unique degree within the blocks of $\mathcal{E}_\ell(C_{\mathbf{G}^*}(s)^F, 1)$ of defect equal to the defect of b. Thus by Lemma 5.2.12, B is not Galois conjugate to any other block in $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. Therefore every block $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ of non-cyclic defect is fixed by Galois conjugation, so by Lemma 5.2.11, $mf(B) = 1$ if $o(s) = 2$ and $mf(B) \leq 2$ if $o(s) = 3$. □

Proposition 5.2.15. *Let $\boldsymbol{G}$ be a simple, simply connected algebraic group of type E_7. Let $s \in \boldsymbol{G}^{*F}$ be a quasi-isolated semisimple ℓ' element such that the components of $C_{\boldsymbol{G}^*}(s)$ are not all of classical type. Let $B \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$ be an ℓ-block of $\boldsymbol{G}^F$. Then $mf(B) \leq 2$.*

Proof. We follow the same strategy as in Proposition 5.2.14. The cases to consider are the following.

$o(s)$	Components of $C_{\mathbf{G}^*}(s)$
3	E_6
3	${}^2E_6$

Note that since $Z(\mathbf{G})$ is not connected, a block B of $\mathbf{G}^F$ has multiple Jordan correspondent blocks. However, [15, Proposition 5.1] still applies in this case, so the defect of B is equal to the defect of b where b is any Jordan correspondent of B.

First suppose ℓ is bad for $\mathbf{G}$. Then by [49, Table 4], every block $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ either has cyclic defect, or has unique defect within $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$.

Now suppose ℓ is good for $\mathbf{G}$. By checking which Φ_e's divide $|\mathbf{G}^F|$ and $|C_{\mathbf{G}^*}(s)^F|$ more than once, it follows that we only need to consider $e = 1$, 2, 3, 4 or 6. By [49, Table 4] and [15, Lemma 4.16], when $e = 1$ or 2, every block $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ either has cyclic defect, or has unique defect within $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. For $e = 3$ and 4, we identify the unipotent e-cuspidal pairs of $C_{\mathbf{G}^*}(s)^F$ explicitly from [11, Appendix: Table 1] and [18, Section 13.9]. Again, $T_e(\mathbf{G})$ denotes an e-split maximal torus of $\mathbf{G}$.

Row	$C_{\mathbf{G}^*}(s)^F$	e	$(C_{\mathbf{L}^*}(s), \alpha)$	$W_{\mathbf{L}^F}(C_{\mathbf{L}^*}(s), \alpha)$	Defect
1	$\Phi_1.E_6(q)$	3	$(\Phi_1.T_3\left(E_6\right)(q), 1)$	G_{25}	$\begin{cases} 1 + \log_5\left(\left(\Phi_3^3\right)_5\right) & \text{if } \ell = 5 \\ \log_\ell\left(\left(\Phi_3^3\right)_\ell\right) & \text{o/w} \end{cases}$
2	$\Phi_1.E_6(q)$	3	$\left(\Phi_1\Phi_3{}^3D_4(q), {}^3D_4[-1]\right)$	Z_3	$\log_\ell\left(\left(\Phi_3\right)_\ell\right)$
3	$\Phi_1.E_6(q)$	3	$\begin{cases} (\Phi_1.E_6(q), \phi_{81,6}) \\ (\Phi_1.E_6(q), \phi_{81,10}) \\ (\Phi_1.E_6(q), \phi_{90,8}) \end{cases}$	1	Trivial
1	$\Phi_1.E_6(q)$	4	$\left(\Phi_1^3\Phi_4^2, 1\right)$	G_8	$\begin{cases} 1 + \log_5\left(\left(\Phi_4^2\right)_5\right) & \text{if } \ell = 5 \\ \log_\ell\left(\left(\Phi_4^2\right)_\ell\right) & \text{o/w} \end{cases}$
2	$\Phi_1.E_6(q)$	4	$\left(\Phi_1^2\Phi_4.{}^2A_3(q), \phi_{22}\right)$	Z_4	$\log_\ell\left(\left(\Phi_4\right)_\ell\right)$
3	$\Phi_1.E_6(q)$	4	$(\Phi_1 E_6(q), 10 \text{ chars})$	1	Trivial
1	$\Phi_2.{}^2E_6(q)$	3	$\left(\Phi_2 T_3\left({}^2E_6\right)(q), 1\right)$	G_{25}	$\begin{cases} 1 + \log_5\left(\left(\Phi_3^2\right)_5\right) & \text{if } \ell = 5 \\ \log_\ell\left(\left(\Phi_3^2\right)_\ell\right) & \text{o/w} \end{cases}$
2	$\Phi_2.{}^2E_6(q)$	3	$\left(\Phi_2{}^2E_6(q), 9 \text{ chars}\right)$	1	Trivial

1	$\Phi_2.{}^2E_6(q)$	4	$\left(\Phi_2 T_4\left({}^2E_6\right)(q), 1\right)$	G_{25}	$\begin{cases} 1+\log_5\left(\left(\Phi_4^2\right)_5\right) & \text{if } \ell = 5 \\ \log_\ell\left(\left(\Phi_4^2\right)_\ell\right) & \text{o/w} \end{cases}$
2	$\Phi_2.{}^2E_6(q)$	4	$\left(\Phi_2^2\Phi_4 A_3(q), \phi_{22}\right)$	Z_4	$\log_\ell\left((\Phi_4)_\ell\right)$
3	$\Phi_2.{}^2E_6(q)$	4	$\left(\Phi_2{}^2E_6(q), 10 \text{ chars}\right)$	1	Trivial

By examination of the defects in column 6, when ℓ is good for $\mathbf{G}$ and $e = 3$ or 4, every block $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ either has cyclic defect, or has unique defect within $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. The results for $e = 6$ are analogous to those for $e = 3$ by Ennola duality.

Therefore for any ℓ and any e, every block $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ either has cyclic defect, or has unique defect within $\mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$, so since $o(s) = 3$, $mf(B) \leq 2$ by Lemma 5.2.11. □

5.2.3 $C^\circ_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ and $A_{\mathbf{G}^*}(s)^F$ is cyclic

Proposition 5.2.16. *Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ' element such that $C^\circ_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ and $A_{\mathbf{G}^*}(s)^F$ is cyclic. Let $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$ be an ℓ-block of $\mathbf{G}^F$. Let $m \geq 1$ be the minimum positive integer such that $\sigma^m(c') = c'$ for all unipotent blocks c' of $C^\circ_{\mathbf{G}^*}(s)$. Then $mf(B) \leq m$.*

Proof. Let $\mathbf{L}^* = C^\circ_{\mathbf{G}^*}(s)$. Since $\mathbf{L}^*$ is an F-stable Levi subgroup of $\mathbf{G}^*$, there exists an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ in duality with $\mathbf{L}^*$. By equivalence (4.6) there exists an ℓ-block b of a subgroup $\mathbf{N}^F$ of $\mathbf{G}^F$ such that b is Morita equivalent to B, and b covers an ℓ-block c of $\mathbf{L}^F$ in $\mathcal{E}_\ell(\mathbf{L}^F, s)$.

Since s is central in $\mathbf{L}^{*F}$, there exists an $\mathbf{N}^F$-stable linear character $\hat{s} \in \mathrm{Irr}\left(\mathbf{L}^F\right)$ such that for every $\chi \in \mathrm{Irr}(c)$, $\chi = \hat{s}\psi$ for a uniquely determined unipotent character $\psi \in \mathrm{Irr}\left(\mathbf{L}^F\right)$, by [23, Proposition 13.30, (ii)]. Therefore $c = \hat{s}c'$ for a uniquely determined unipotent block c' of $\mathbf{L}^F$. Now $\sigma(c) = {}^{\hat{\sigma}}(\hat{s})\sigma(c')$ and by assumption, $\sigma^m(c') = c'$. Therefore $\sigma^m(c) = {}^{\hat{\sigma}^m}(\hat{s})c'$. Let $\xi = {}^{\hat{\sigma}^m}(\hat{s})\hat{s}^{-1}$. Then ξ is a linear and $\mathbf{N}^F$-stable character of $\mathbf{L}^F$ and $\sigma^m(c) = \xi c$.

Since $\mathbf{N}^F/\mathbf{L}^F \leq A_{\mathbf{G}^*}(s)^F$ by equation (4.7), $A_{\mathbf{G}^*}(s)^F$ is cyclic by assumption, and ξ is $\mathbf{N}^F$-stable, it follows from [44, Corollary 11.22] that ξ extends to a linear character $\hat{\xi} \in \mathrm{Irr}\left(\mathbf{N}^F\right)$.

Therefore $\hat{\xi}b$ and $\sigma^m(b)$ both cover $\sigma^m(c)$. Hence, since $\mathbf{N}^F/\mathbf{L}^F$ is abelian, $\mathcal{O}\mathbf{N}^F(\sigma^m(b)) \cong \mathcal{O}\mathbf{N}^F(\hat{\xi}b) \cong \mathcal{O}\mathbf{N}^F b$ by Lemma 2.1.9. Therefore $mf(B) = mf(b) \leq m$. □

Theorem 5.2.17. *Let $\boldsymbol{G}$ be a simple, simply connected algebraic group. Let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ' element such that $C^{\circ}_{\boldsymbol{G}^*}(s)$ is a Levi subgroup of $\boldsymbol{G}^*$ and $A_{\boldsymbol{G}^*}(s)^F$ is cyclic. Let $B \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$ be an ℓ-block of $\boldsymbol{G}^F$. If $\boldsymbol{G}$ is of type E_7 or E_8 then $mf(B) \leq 2$ and in all other cases $mf(B) = 1$.*

Proof. First suppose that $\mathbf{G}$ is of type E_7 or E_8 and let $\mathbf{L}$ be a proper Levi subgroup of $\mathbf{G}$. Suppose $\chi \in \mathcal{E}(\mathbf{L}^F, 1)$. Then by examining the character fields of the unipotent characters in [36, Table 1], we see that ${}^{\hat{\sigma}^2}\chi(g) = \chi(g)$ for all $g \in \mathbf{L}^F$. Therefore $\sigma^2(c) = c$ for any unipotent block of $\mathbf{L}^F$ and thus $mf(B) \leq 2$ by Proposition 5.2.16. If $\mathbf{G}$ is of any other type, then any proper Levi subgroup of $\mathbf{G}$ has all classical components and therefore $\sigma(c) = c$ for all unipotent blocks c of proper Levi subgroups of $\mathbf{G}$. Therefore $mf(B) = 1$ by Proposition 5.2.16. □

Theorem 5.2.18. *The Rationality Conjecture (Conjecture 2.3.4) holds for the blocks of the finite groups of Lie type of type A.*

Proof. In type A, $C^{\circ}_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ [17] and $A_{\mathbf{G}^*}(s)^F$ is cyclic for any semisimple $s \in \mathbf{G}^{*F}$. The result follows therefore from Theorem 5.2.17. □

Theorem 5.2.19. *Let $\mathcal{G}_1 = \{SL_n(q) : n \in \mathbb{N}, q = p^a$ for some prime $p \neq \ell$ and some $a \in \mathbb{N}\}$, and let $\mathcal{G}_2 = \{SU_n(q) :\ n \in \mathbb{N}, q = p^a$ for some prime $p \neq \ell$* and some $a \in \mathbb{N}$ *such that $\ell \nmid q^{2s+1} + 1$ $\forall\ s \in \mathbb{N}\}$. Then Donovan's conjecture holds for theℓ-blocks of groups in $\mathcal{G}_1$ and $\mathcal{G}_2$.*

Proof. By Theorem 2.3.5, Donovan's conjecture holds if and only if both Weak Donovan's conjecture, Conjecture 2.3.2, and the Rationality conjecture, Conjecture 2.3.4, hold. The Rationality conjecture holds for the blocks of groups in $\mathcal{G}_1$ and $\mathcal{G}_2$ by Theorem 5.2.18. Weak Donovan's conjecture holds for the blocks of groups in $\mathcal{G}_1$ and $\mathcal{G}_2$ by Theorem 2.3.3. □

5.2.4 $C_{\mathbf{G}^*}(s)$ is connected, s not isolated

Proposition 5.2.20. *Let $\boldsymbol{G}$ be a simple, simply conected algebraic group. Let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ' element which is not isolated in $\boldsymbol{G}^*$ such that $C_{\boldsymbol{G}^*}(s)$ is connected and $C_{\boldsymbol{G}^*}(s)$ is not a Levi subgroup of $\boldsymbol{G}^*$. Let $B \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$. Let m be the maximal order of an isolated semisimple ℓ' element in a component of a proper Levi subgroup of $\boldsymbol{G}^*$ not of type A. Let φ be the Euler totient function. Then $mf(B) \leq \varphi(m)$.*

Proof. Let $\mathbf{L}^*$ be the minimal proper Levi subgroup of $\mathbf{G}^*$ such that $C_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$ and let $\mathbf{L}$ be the dual group of $\mathbf{L}^*$. By Theorem 4.3.13, B is Morita equivalent to an isolated block $b \in \mathcal{E}(\mathbf{L}^F, s)$. Suppose that all the components of $\mathbf{L}^*$ are of type A. Then s is central in $\mathbf{L}^*$ so $C_{\mathbf{G}^*}(s) = \mathbf{L}^*$, which contradicts our assumption that $C_{\mathbf{G}^*}(s)$ is not a Levi subgroup of $\mathbf{G}^*$. We can therefore assume that $\mathbf{L}^*$ contains precisely one component not of type A. Thus since $[\mathbf{L}, \mathbf{L}]$ is simply connected by [55, Proposition 12.14], without loss of generality we can let $[\mathbf{L}, \mathbf{L}] = \mathbf{M}_1 \times \cdots \times \mathbf{M}_r \times \mathbf{M}$ where $\mathbf{M}_i$ is of type A for $i = 1, \ldots, r$, and $\mathbf{M}$ is of the same type as $\mathbf{G}$ if $\mathbf{G}$ is classical, $\mathbf{M}$ is of type D if $\mathbf{G}$ is of type E_6 or ${}^2E_6$, $\mathbf{M}$ is of type D or E_6 if $\mathbf{G}$ is of type E_7, and $\mathbf{M}$ is of type D, E_6 or E_7 if $\mathbf{G}$ is of type E_8.

Let $i_1 : \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding as in Definition 4.2.9, and let $\widetilde{\mathbf{L}} = Z(\widetilde{\mathbf{G}})\mathbf{L}$. Define $i_2 : \mathbf{L} \hookrightarrow \widetilde{\mathbf{L}}$ to be the inclusion map, and $i_3 \coloneqq i_2|_{[\mathbf{L},\mathbf{L}]} : [\mathbf{L}, \mathbf{L}] \to \widetilde{\mathbf{L}}$ to be the restriction of i_2 to $[\mathbf{L}, \mathbf{L}]$. Then i_2 and i_3 are regular embeddings of $\mathbf{L}$ and $[\mathbf{L}, \mathbf{L}]$ respectively. Define a morphism $i^* : \mathbf{L}^* \to [\mathbf{L}, \mathbf{L}]^*$ by

$$i^*(x) = i_3^*(\tilde{x}),$$

for all $x \in \mathbf{L}^*$, where $\tilde{x}$ denotes an element of $\widetilde{\mathbf{L}}^*$ such that $i_2^*(\tilde{x}) = x$. Then i^* is well defined: suppose that $\tilde{x}_1$ and $\tilde{x}_2$ are elements in $\widetilde{\mathbf{L}}^*$such that $i_2^*(\tilde{x}_1) = i_2^*(\tilde{x}_2)$. Then there exists a $z \in \ker i_2^*$ such that $\tilde{x}_2 = z\tilde{x}_1$, and therefore $i_3^*(\tilde{x}_1) = i_3^*(\tilde{x}_2)$.

Let $\bar{s} = i^*(s)$ denote the image of s in $[\mathbf{L}, \mathbf{L}]^*$. Then $\bar{s} = (1, \ldots, 1, t)$ where t is an element of $\mathbf{M}^*$, a group dual to $\mathbf{M}$ and $o(\bar{s}) = o(t)$. It follows from [4, Proposition 2.3] that since s

is isolated, $\bar{s}$, and therefore t, are also isolated. Let $c \in \mathcal{E}_\ell\left([\mathbf{L},\mathbf{L}]^F, \bar{s}\right)$ be a block of $[\mathbf{L},\mathbf{L}]^F$ covered by b.

The components of $C_{\mathbf{M}^*}(t)$ are all of classical type by [4, Section 5B]. Therefore $C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})$ has all classical components, so by the proof of Proposition 5.2.9, $\sigma^n(c) = {}^x c$ for some $x \in \widetilde{\mathbf{L}}$. Then ${}^x b$ and $\sigma^n(b)$ both cover $\sigma^n(c)$ so, since $\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F$ is abelian, it follows from Lemma 2.1.9 that $\mathcal{O}\mathbf{L}^F b \cong \mathcal{O}\mathbf{L}^F {}^x b \cong \mathcal{O}\mathbf{L}^F \sigma^n(b)$. Therefore $mf(b) \le \varphi(m)$, whence $mf(B) \le \varphi(m)$ as required. $\square$

Theorem 5.2.21. *Let* $\boldsymbol{G}$ *be a simple, simply connected algebraic group. Let* $s \in \boldsymbol{G}^{*F}$ *be a semisimple* ℓ' *element which is not isolated in* $\boldsymbol{G}^*$*, such that* $C_{\boldsymbol{G}^*}(s)$ *is connected. Let* $B \in \mathcal{E}_\ell\left(\boldsymbol{G}^F, s\right)$*. Then if* $\mathbf{G}$ *is type* E_7 *or* E_8*,* $mf\ (B) \le 2$ *and if* $\boldsymbol{G}$ *is of any other type,* $mf\ (B) = 1$*.*

Proof. First suppose that $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$. By assumption, s is not isolated so $C_{\mathbf{G}^*}(s)$ is a proper subgroup of $\mathbf{G}^*$. By equivalence (4.5), B is Morita equivalent to a unipotent block $b \in \mathcal{E}_\ell(C_{\mathbf{G}^*}(s)^F, 1)$. It then follows from examination of the character fields of unipotent characters given in [36, Table 1] that $\sigma^2(b) = b$ if $\mathbf{G}$ is of type E_7 or E_8, and $\sigma(b) = b$ if $\mathbf{G}$ is of any other type. Thus if $\mathbf{G}$ is of type E_7 or E_8, $mf(b) \le 2$ so $mf(B) \le 2$ and if $\mathbf{G}$ is of any other type, $mf(b) = mf(B) = 1$.

Now suppose that $C_{\mathbf{G}^*}(s)$ is not a Levi subgroup of $\mathbf{G}^*$. By Proposition 5.2.20, therefore $mf(B) \le \varphi(m)$ where m is the maximal order of an isolated semisimple ℓ' element in a component of a proper Levi subgroup of $\mathbf{G}^*$ not of type A, and φ is the Euler totient function.

By [4, Section 5B], if $\mathbf{G}$ is of type E_7 or E_8 then the maximal order of an isolated semisimple ℓ' element in a component of a proper Levi subgroup of $\mathbf{G}^*$ not of type A is 4. Therefore $mf(B) \le \varphi(m) \le 2$. If $\mathbf{G}$ is not of type E_7 or E_8 then the maximal order of an isolated semisimple ℓ' element in a component of a proper Levi subgroup of $\mathbf{G}^*$ not of type A is 2. Therefore $mf(B) \le \varphi(m) = 1$. $\square$

5.3 Ree and Suzuki groups

Theorem 5.3.1. *Let $\boldsymbol{G}$ be a simple, simply connected algebraic group defined over $\overline{\mathbb{F}}_p$. Let $F : \boldsymbol{G} \to \boldsymbol{G}$ be a very twisted Frobenius morphism with respect to an $\mathbb{F}_q$-structure for q some power of p, and let $\boldsymbol{G}^F$ be the associated finite Suzuki or Ree group. Let b be an ℓ-block of $\boldsymbol{G}^F$. If $p \neq \ell$ and $\boldsymbol{G}^F$ is a large Ree group, then assume that b is unipotent. Then $mf(b) = 1$.*

Proof. First let $\mathbf{G}^F = {}^2B_2(q^2)$ $(q^2 = 2^{2m+1})$ be a Suzuki group. If $\ell = 2$ then $mf(b) = 1$ by Corollary 5.1.3. Suppose that $\ell \neq 2$. The subgroups of $\mathbf{G}^F$ of odd order are cyclic [62, Theorem 9], so b has cyclic defect and therefore $mf(b) = 1$ by Proposition 2.2.5 (d).

Next let $\mathbf{G}^F = {}^2G_2(q^2)$ $(q^2 = 3^{2m+1})$ be a small Ree group and let b be a 2-block of $\mathbf{G}^F$. The Sylow 2-subgroups of $\mathbf{G}^F$ are elementary abelian of order 8 and [64, I. 8] shows that the only 2-block of $\mathbf{G}^F$ of full defect is the principal block, which contains the rational valued trivial character. If b is not the principal block, then the defect groups of b are proper subgroups of an elementary abelian group of order 8, so b either has Klein-4 or cyclic defect. Therefore $mf(b) = 1$ by Proposition 2.2.5 (b) and (d). If $\ell = 3$ then $mf(b) = 1$ by Corollary 5.1.3.

Now let $\mathbf{G}^F$ be a small Ree group and $\ell \geq 5$, and let b be an ℓ-block of $\mathbf{G}^F$. The order of $\mathbf{G}^F$ is $|\mathbf{G}^F| = q^6\Phi_1\Phi_2\Phi_4\Phi_{12}$. Since ℓ divides only one Φ_i for some $i \in \{1, 2, 4, 12\}$, and each Φ_i divides $|\mathbf{G}^F|$ exactly once, by Lemma 4.1.12 b has cyclic defect and thus $mf(b) = 1$ by Proposition 2.2.5 (d).

Finally, let $\mathbf{G}^F = {}^2F_4(q^2)$ $(q^2 = 2^{2m+1})$ be a large Ree group, and let b be a unipotent ℓ-block of $\mathbf{G}^F$. If $\ell = 2$ then $mf(b) = 1$ by Corollary 5.1.3. Suppose that $\ell \neq 2$. By [54], there are two cases to consider. In the first case we suppose that $\ell \nmid (q^2 - 1)$. Then b is either the principal block of $\mathbf{G}^F$, or b has trivial defect and therefore $mf(b) = 1$ by Proposition 2.2.5 (b) and (d). In the second case, suppose that $\ell \mid (q^2 - 1)$. Then there are three unipotent blocks – the principal block, and two blocks of cyclic defect [39, Appendix D], so $mf(b) = 1$ by Proposition 2.2.5 (b) and (d). $\square$

5.4 Next steps

The original goal of this project was to find a bound for the Morita Frobenius numbers of all blocks of quasi-simple finite groups. The following cases are outstanding.

- Non-unipotent blocks of the large Ree group in non-defining characteristic

  With the notation and setup of Section 5.2:

- $B \in \mathcal{E}_\ell(\mathbf{G}^F, s)$ such that $C^\circ_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ and $A_{\mathbf{G}^*}(s)^F$ is not cyclic
- $B \in \mathcal{E}_\ell(\mathbf{G}^F, s)$ such that s is not quasi-isolated, $C_{\mathbf{G}^*}(s)$ is not connected, and $C^\circ_{\mathbf{G}^*}(s)$ is not a Levi subgroup of $\mathbf{G}^*$

Recall that we also have two cases of unipotent blocks of E_8 where we know that the Morita Frobenius number of the blocks is at most 2, but we don't know whether or not it is equal to 1. In this section we first discuss a strategy which may enable us to prove that all unipotent blocks of E_8 have Morita Frobenius number equal to 1. We then discuss progress made to date on the third case listed above where $B \in \mathcal{E}_\ell(\mathbf{G}^F, s)$ such that s is not quasi-isolated, $C_{\mathbf{G}^*}(s)$ is not connected, and $C^\circ_{\mathbf{G}^*}(s)$ is not a Levi subgroup of $\mathbf{G}^*$.

Let p and ℓ be different primes and let q be a power of p. Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\overline{\mathbb{F}}_p$ and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius morphism with respect to an $\mathbb{F}_q$-structure which is not very twisted. As usual, we fix a connected reductive group $\mathbf{G}^*$ such that $(\mathbf{G}, F)$ is dual to $(\mathbf{G}^*, F)$. Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ' element.

Unipotent blocks of E_8 with $\boldsymbol{mf}(B) \leq 2$

Recall the following case from Theorem 5.2.5. Let $\mathbf{G} = E_8$. Suppose that $s = 1$, $\ell \neq 2$, $\ell \equiv 2 \bmod 3$ and $e = e_\ell(q) = 2$. Let b be the unipotent block of $\mathbf{G}^F = E_8(q)$ containing the irreducible constituents of the unipotent e-cuspidal pair $(\mathbf{L}, \lambda) = \left(\phi_2^2.{}^2E_6, {}^2E_6[\theta^i]\right)$ for $i = 1$ or 2.

Since ℓ is odd, it follows from Proposition 5.2.3 that b has a defect group P such that $Z(\mathbf{L})_\ell^F \trianglelefteq P$ and $P/Z(\mathbf{L})_\ell^F$ is isomorphic to a Sylow ℓ-subgroup of $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$. Since $W_{\mathbf{G}^F}(\mathbf{L},\lambda) \cong D_{12}$ (by [11, Appendix: Table 1]), $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ is an ℓ' group. Therefore $P \cong Z(\phi_2^2.{}^2E_6)_\ell^F$, and this is isomorphic to $C_{\ell^a} \times C_{\ell^a}$, where ℓ^a is the power of ℓ dividing $q+1$, since $e = 2$ and $\ell \nmid |Z\left({}^2E_6(q)\right)|$, by [10, Proposition 3.3 (iii)]. Hence b has abelian defect.

A recent result of Kessar and Chuang, [19], relates the Morita Frobenius numbers of perversely equivalent blocks. Let A be a k-algebra. For a field automorphism $\sigma : k \to k$, a σ-*twist* of A is the k-algebra A^σ which is equal to A as a ring, endowed with scalar multiplication given by $\lambda.x = \sigma^{-1}(\lambda)x$ for all $x \in A$, $\lambda \in k$. Note that the mth Frobenius twist of a k-algebra A (as defined in Section 2.1.1) is a specific case of a σ-twist of A, with $\sigma : k \to k$ equal to the Frobenius automorphism $\lambda \to \lambda^\ell$.

Definition 5.4.1. A k-linear equivalence $\mathcal{E} :\ \mathrm{Mod}(A) \to\ \mathrm{Mod}(A^\sigma)$ is a σ-*Morita equivalence* if $\mathcal{E}(V) \cong V^\sigma$ for all simple A-modules V. We say that A and A^σ are σ-*Morita equivalent* if there is a σ-Morita equivalence between them. The σ-*Morita Frobenius number* of A is the least positive number m such that A and A^{σ^m} are σ^m-Morita equivalent.

For the rest of this section, let σ denote the Frobenius automorphism. Note that the Morita Frobenius number of A is always less than or equal to the σ-Morita Frobenius number of A. The following result relates the σ-Morita Frobenius number of two perversely equivalent k-algebras.

Theorem 5.4.2 ([19, Corollary 5])**.** *Let A and B be finite dimensional k-algebras. If A and B are perversely equivalent, then the σ-Morita Frobenius number of A is equal to the σ-Morita Frobenius number of B.*

Recall Broué's abelian defect group conjecture. For further discussion of Brauer correspondent blocks, see [30, Ch. III, Section 9].

Conjecture (Broué's Abelian Defect Group Conjecture [9])**.** *Let G be a finite group and let B be a block of kG. Let P be a defect group of B and let C be the block of $kN_G(P)$ in Brauer correspondence with B. If P is abelian then B and C are derived equivalent.*

As discussed in [19], results of Chuang, Rouquier and Craven suggest that for the finite groups of Lie type in non-defining characteristic, the derived equivalence between a block and its Brauer correspondent predicted by Broué is in fact a perverse equivalence.

We expect that for $b = b_{E_8(q)}\left(\phi_2^2.{}^2E_6, {}^2E_6[\theta^i]\right)$, $mf(b) = 1$. We hope that it will be possible to prove this by first showing that Broué's abelian defect group conjecture holds for b, then showing that b is in fact perversely equivalent to its Brauer correspondent c, and finally by showing that the σ-Morita Frobenius number of c is 1 (where σ is the usual Frobenius automorphism) and applying Theorem 5.4.2.

General blocks of finite groups of Lie type

Suppose that $\mathbf{G}$ is simple, simply connected, and let $B \in \mathcal{E}_\ell\left(\mathbf{G}^F, s\right)$. Suppose that s is not quasi-isolated in $\mathbf{G}^*$, that $C_{\mathbf{G}^*}(s)$ is not connected and that $A_{\mathbf{G}^*}(s)^F$ is cyclic of prime order. In this section we will outline the progress made on this general situation to date. Theorem 4.3.15 of Bonnafé-Dat-Rouquier can be applied to B in many cases to give bounds for $mf(B)$. However, our method runs into problems that we have not yet been able to resolve in all situations.

First, recall the notation of [6, Section 7]. Let $\mathbf{L}^* = C_{\mathbf{G}^*}\left(Z^\circ\left(C^\circ_{\mathbf{G}^*}(s)\right)\right)$ be the minimal Levi subgroup of $\mathbf{G}^*$ containing $C^\circ_{\mathbf{G}^*}(s)$ and let $\mathbf{L}$ be dual to $\mathbf{L}^*$. Let $\mathbf{N}^* = C_{\mathbf{G}^*}(s)^F.\mathbf{L}^*$ and define $\mathbf{N}$ to be the subgroup of $N_{\mathbf{G}}(\mathbf{L})$ containing $\mathbf{L}$ such that $\mathbf{N}/\mathbf{L}$ corresponds to $\mathbf{N}^*/\mathbf{L}^*$ via the canonical isomorphism between $N_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$ and $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$. Theorem 4.3.15 shows that B is Morita equivalent to an ℓ-block b of $\mathbf{N}^F$ covering a block $c \in \mathcal{E}_\ell\left(\mathbf{L}^F, s\right)$. By the minimality of $\mathbf{L}^*$, c is an isolated block.

Let $i_1 : \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding and let $\widetilde{\mathbf{L}} = Z(\widetilde{\mathbf{G}})\mathbf{L}$. Define $i_2 : \mathbf{L} \hookrightarrow \widetilde{\mathbf{L}}$, $i_3 : [\mathbf{L},\mathbf{L}] \hookrightarrow \widetilde{\mathbf{L}}$, $i^* : \mathbf{L}^* \to [\mathbf{L},\mathbf{L}]^*$ and $\bar{s} = i^*(s)$ as in Proposition 5.2.20, so i_2 and i_3 are regular embeddings, i^* is a surjection, and $\bar{s}$ is a semisimple element of $[\mathbf{L},\mathbf{L}]^{*F}$. Let $d \in \mathcal{E}_\ell\left([\mathbf{L},\mathbf{L}]^F, \bar{s}\right)$ denote a block of $[\mathbf{L},\mathbf{L}]^F$ covered by c and note that by [4, Proposition 2.3], d is an isolated block.

As in Proposition 5.2.20, we can assume without loss of generality that $[\mathbf{L},\mathbf{L}] = \mathbf{M}_1 \times \cdots \times \mathbf{M}_r \times \mathbf{M}$ where $\mathbf{M}_i$ is of type A for $i = 1,\dots,r$, and $\mathbf{M}$ is not of type A or E_8. Let $\mathbf{M}^*$ denote a group dual to $\mathbf{M}$. It follows that $\bar{s} = (1,\dots,1,t)$ for some isolated element $t \in \mathbf{M}^{*F}$. Let $m = o(t)$ and $n = \varphi(m)$. Since s is isolated and $\mathbf{M}^*$ is simple, not of type E_8, $C_{\mathbf{M}^*}(s)$ has only classical components [4, Section 5]. Thus by the proof of Proposition 5.2.9, $\sigma^n(d)$ is $\widetilde{\mathbf{L}}^F$-conjugate to d. Let $x \in \widetilde{\mathbf{L}}^F$ be such that $\sigma^n(d) = {}^x d$.

Since $\sigma^n(c)$ and ${}^x c$ both cover $\sigma^n(d) = {}^x d$, and $\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F$ is abelian, it follows from Lemma 2.1.9 that $\sigma^n(c) = \theta {}^x c$ for some linear character $\theta \in \mathrm{Irr}\big(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F\big)$ and $\mathcal{O}\mathbf{L}^F\, {}^x c \cong \mathcal{O}\mathbf{L}^F \sigma^n(c)$. In particular, $mf(c) \leq n$.

We claim that $\widetilde{\mathbf{L}}^F$ acts on $\mathbf{N}^F$. Suppose that $\tilde{a_1} \in \widetilde{\mathbf{L}}^F$ and $a_2 \in \mathbf{N}^F$. Then clearly $\tilde{a_1} a_2 \tilde{a_1}^{-1} \in \mathbf{G}^F$ so it only remains to show that $\tilde{a_1} a_2 \tilde{a_1}^{-1} \in \mathbf{N}$. Since $\widetilde{\mathbf{L}} = Z(\widetilde{\mathbf{G}})\mathbf{L}$, let $\tilde{a_1} = z a_1$ where $z \in Z(\widetilde{\mathbf{G}}), a_1 \in \mathbf{L}$. Then $\tilde{a_1} a_2 \tilde{a_1}^{-1} = z a_1 n a_1^{-1} z^{-1} = a_1 n a_1^{-1} \in \mathbf{N}$ since $\mathbf{L} \subset \mathbf{N}$, thus $\tilde{a_1} a_2 \tilde{a_1}^{-1} \in \mathbf{N}^F$ as claimed. We can therefore consider the block ${}^y b$ for any $y \in \widetilde{\mathbf{L}}^F$. In particular, we can consider the block ${}^x b$ where $x \in \widetilde{\mathbf{L}}^F$ is such that $\sigma^n(d) = {}^x d$.

Proposition 5.4.3. *Let $\boldsymbol{G}$ be a simple, simply connected algebraic group. Let $s \in \boldsymbol{G}^{*F}$ be a semisimple ℓ' element. Suppose that s is not quasi-isolated in $\boldsymbol{G}^*$, $C^*_{\boldsymbol{G}}(s)$ is not connected, and $A_{\boldsymbol{G}^*}(s)^F$ is cyclic of prime order. Let $B \in \mathcal{E}_\ell(\boldsymbol{G}^F, s)$ and let $\boldsymbol{L}^*$, $\boldsymbol{L}$, $\boldsymbol{N}^*$, $\boldsymbol{N}$, i^*, $\bar{s}$, t, m, n, b, c, d, x and θ be as defined above. Suppose that one of the following holds.*

(a) $\boldsymbol{N}^F/[\boldsymbol{L},\boldsymbol{L}]^F$ is abelian

(b) θ is $\boldsymbol{N}^F$-stable

(c) c is not stable in $\boldsymbol{N}^F$

(d) $C_{[\boldsymbol{L},\boldsymbol{L}]^*}(\bar{s})$ is connected

Then $mf(B) = mf(b) \leq n$.

Proof. First suppose that $\mathbf{N}^F/[\mathbf{L},\mathbf{L}]^F$ is abelian. Then since $\sigma^n(b)$ and ${}^x b$ both cover $\sigma^n(d) = {}^x d$, it follows from Lemma 2.1.9 that $\sigma^n(b) = \eta\, {}^x b$ for some linear character $\eta \in \operatorname{Irr}\big(\mathbf{N}^F/[\mathbf{L},\mathbf{L}]^F\big)$, and $\mathcal{O}\mathbf{N}^F\, {}^x b \cong \mathcal{O}\mathbf{N}^F \sigma^n(b)$. Therefore $mf(b) \le n$.

If θ is $\mathbf{N}^F$-stable then since $\mathbf{N}^F/\mathbf{L}^F \le A_{\mathbf{G}^*}(s)^F$ by equation (4.7) and since $A_{\mathbf{G}^*}(s)^F$ is cyclic by assumption, θ extends to some $\hat{\theta} \in \operatorname{Irr}\big(\mathbf{N}^F\big)$ by [44, Lemma 11.22]. Therefore $\hat{\theta}\, {}^x b$ and $\sigma^n(b)$ both cover $\sigma^n(c) = \theta\, {}^x c$, so $\sigma^n(b) = \lambda \hat{\theta} b$ for some linear character $\lambda \in \operatorname{Irr}\big(\mathbf{N}^F/\mathbf{L}^F\big)$ by Lemma 2.1.9 and $\mathcal{O}\mathbf{N}^F\, {}^x b \cong \mathcal{O}\mathbf{N}^F \sigma^n(b)$. Thus $mf(b) \le n$ showing part (b).

Now suppose that c is not stable in $\mathbf{N}^F$. By assumption, $A_{\mathbf{G}^*}(s)^F$ is cyclic of prime order. Therefore $\operatorname{Stab}_{\mathbf{N}^F}(c) = \mathbf{L}^F$ so c is Morita equivalent to b by [51, Theorem C]. Hence $mf(b) = mf(c) \le n$.

Finally, suppose that $C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})$ is connected. If any of conditions (a), (b) or (c) hold then we are done, so assume that $\mathbf{N}^F/[\mathbf{L},\mathbf{L}]^F$ is not abelian, θ is not $\mathbf{N}^F$-stable, and c is $\mathbf{N}^F$-stable. Then for any $y \in \mathbf{N}^F$, ${}^y(\sigma^n(c)) = \sigma^n({}^y c) = \sigma^n(c)$ so $\sigma^n(c)$ is also $\mathbf{N}^F$-stable.

Suppose that ${}^x c$ is not $\mathbf{N}^F$-stable. Then there exists a $y \in \mathbf{N}^F$ such that ${}^y({}^x c) \neq {}^x c$. This holds if and only if $c \neq x^{-1}y^{-1}xcx^{-1}yx = {}^{x^{-1}yx}c$. Since $x \in \widetilde{\mathbf{L}}^F$, $y \in \mathbf{N}^F$ and $\widetilde{\mathbf{L}}^F$ acts on $\mathbf{N}^F$, it follows that $x^{-1}yx \in \mathbf{N}^F$. Thus $c \neq {}^{x^{-1}yx}c$ contradicts the fact that c is $\mathbf{N}^F$-stable, so ${}^x c$ is $\mathbf{N}^F$-stable.

Let $y \in \mathbf{N}^F$ be such that ${}^y\theta \neq \theta$. Then $\sigma^n(c) = {}^y(\sigma^n(c)) = {}^y(\theta\, {}^x c) = {}^y\theta\, {}^x c$. Therefore ${}^y\theta\, {}^x c = \theta\, {}^x c$, so ${}^x c = ({}^y\theta)^{-1}\theta\, {}^x c$. Let $\eta = ({}^y\theta)^{-1}\theta$, a linear character of $\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F$, so ${}^x c = \eta\, {}^x c$. We claim that since $C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})$ is connected, η is trivial. If the claim holds, then ${}^y\theta = \theta$ for every $y \in \mathbf{N}^F$, which contradicts the assumption that θ is not $\mathbf{N}^F$-stable. Thus if $C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})$ is connected, $mf(b) \le n$, as required.

We now prove the claim. Since $\mathbf{G}$ is simply connected, $\mathbf{G}^*$ is of adjoint type so $Z(\mathbf{G}^*)$ is trivial, therefore $Z(\mathbf{L}^*)$ is connected by [5, Corollaire 4.4]. Thus $\ker i^* = Z(\mathbf{L}^*)$. Any element $s' \in \mathbf{L}^{*F}$ such that $i^*(s') = \bar{s}$ is of the form $s' = zs$ for some $z \in \ker i^{*F} = Z(\mathbf{L}^*)^F$. In other words, there are $|Z(\mathbf{L}^*)^F|$ lifts of $\bar{s}$ in $\mathbf{L}^{*F}$.

Two lifts s and s' are $\mathbf{L}^{*F}$-conjugate if and only if there exists a $g \in \mathbf{L}^{*F}$ such that $s = gs'g^{-1} = gzsg^{-1}$. This holds if and only if $z = s^{-1}gsg^{-1}$ for some $g \in \mathbf{L}^{*F}$. Let $i^*(g) = \bar{g}$. By applying i^* to the equation $s = gs'g^{-1}$, it follows that s and s' are $\mathbf{L}^{*F}$-conjugate if and only if $z = s^{-1}gsg^{-1}$ for some $g \in \mathbf{L}^{*F}$ such that $\bar{s} = \bar{g}\bar{s}\bar{g}^{-1}$, i.e. such that g is a lift of an element $\bar{g} \in C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F$.

Let $\ker {'i^*} = [\mathbf{L},\mathbf{L}] \cap \ker i^*$. Define a map,

$$\begin{array}{cccc} \varphi_s : & C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F & \longrightarrow & \ker {'i^{*F}} \\ & \bar{g} & \longmapsto & [g,s] \end{array}$$

where $g \in \mathbf{L}^F$ is such that $i^*(g) = \bar{g}$. This map does not depend on the choice of lift g for $\bar{g}$. Then s and s' are $\mathbf{L}^{*F}$-conjugate if and only if $s' = sz$ for some $z \in \operatorname{Im} \varphi_s$. The kernel of φ_s is the set of all elements $\bar{g} \in C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F$ such that the lifts of $\bar{g}$ to $\mathbf{L}^{*F}$ commute with s, therefore $\ker \varphi_s = i^*\left(C_{\mathbf{L}^*}(s)^F\right)$. Since $i^*\left(C_{\mathbf{L}^*}(s)^F\right) = C^\circ_{[\mathbf{L},\mathbf{L}]^*}(s)^F$, and $C_{[\mathbf{L},\mathbf{L}]^*}(s)$ is connected by assumption, it follows that $\operatorname{Im} \varphi_s \cong C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F / C^\circ_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F$ is trivial. Thus no two lifts of $\bar{s}$ are $\mathbf{L}^{*F}$-conjugate.

Let $\psi \in \operatorname{Irr}({}^x d) \cap \mathcal{E}([\mathbf{L},\mathbf{L}], \bar{s})$. Then $\operatorname{Irr}\big(\mathbf{L}^F \mid \psi\big)$ fall into $|\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F|$ conjugacy classes. There is a transitive action of $\operatorname{Irr}\big(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F\big)$ on $\operatorname{Irr}\big(\mathbf{L}^F \mid \psi\big)$. Recall that two irreducible characters χ and χ' of $\mathbf{L}^F$ are in the same block if and only if

$$\pi(\omega_\chi(\hat{C})) = \pi(\omega_{\chi'}(\hat{C}))$$

for every conjugacy class C of $\mathbf{L}^F$, where ω_χ is the linear character of χ as defined in Section 2.1.4. The action of $\operatorname{Irr}\big(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F\big)$ on $\operatorname{Irr}\big(\mathbf{L}^F \mid \psi\big)$ therefore induces a transitive action of $\operatorname{Irr}\big(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F\big)_{\ell'}$ on the blocks $\operatorname{Bl}(\mathbf{L}^F \mid {}^x d)$.

There are $|\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F|_{\ell'}$ ℓ' Lusztig series of characters covering ψ and each corresponds to a different block covering ${}^x d$, so $|\operatorname{Bl}\big(\mathbf{L}^F \mid {}^x d\big)| = |\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F|_{\ell'}$. In other words, the action of

$\mathrm{Irr}\big(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F\big)_{\ell'}$ on $\mathrm{Bl}\big(\mathbf{L}^F \mid {}^x d\big)$ is regular (has no fixed points). Thus, since ${}^x c \in \mathrm{Bl}\big(\mathbf{L}^F \mid {}^x d\big)$, $\eta \in \mathrm{Irr}\big(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F\big)_{\ell'}$ and $\eta\, {}^x c = {}^x c$, it follows that η is trivial, as claimed. $\square$

Remark 5.4.4. In the proof of Proposition 5.4.3, if $C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})$ is not connected then

$$\mathrm{Im}\ \varphi_s \cong C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F / C^\circ_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F = A_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F,$$

so the lifts of $\bar{s}$ fall into $|Z(\mathbf{L}^*)^F|/|A_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F|$ conjugacy classes of $\mathbf{L}^{*F}$. Thus the action of $\mathrm{Irr}\big(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F\big)_{\ell'}$ on $\mathrm{Bl}\big(\mathbf{L}^F \mid {}^x d\big)$ has a kernel of size $|A_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})^F|$ and so η could be non-trivial. If η is non-trivial, then $({}^y\theta)^{-1}\theta$ for some $y \in \mathbf{N}^F$, so θ is not $\mathbf{N}^F$-stable. Therefore θ doesn't extend to $\mathrm{Irr}\big(\mathbf{N}^F\big)$ and so the method of Proposition 5.4.3 falls down if $C_{[\mathbf{L},\mathbf{L}]^*}(\bar{s})$ is not connected. At present we have not found any way around this issue.

Chapter 6

Proof of Theorems A, B, C and D

Theorem A. *Let b be an ℓ-block of a quasi-simple finite group G. Let $\overline{G} = G/Z(G)$. Suppose that one of the following holds.*

(a) $\overline{G}$ is an alternating group

(b) $\overline{G}$ is a sporadic group

(c) $\overline{G}$ is a finite group of Lie type in characteristic ℓ

Then $mf(b) = 1$.

Proof. Part (a) is proved in Proposition 3.1.6, part (b) is proved in Proposition 3.2.2 and part (c) is proved in Corollary 5.1.3. □

Theorem B. *Let ℓ and p be different primes and let q be a power of p. Let $\boldsymbol{G}$ be a simple, simply connected algebraic group defined over $\overline{\mathbb{F}}_p$ and let $F : \boldsymbol{G} \to \boldsymbol{G}$ be a not very twisted Frobenius morphism with respect to an $\mathbb{F}_q$-structure. Let s be a semisimple ℓ' element of $\boldsymbol{G}^{*F}$ and let $b \in \mathcal{E}_\ell(\boldsymbol{G}^F, s)$ be an ℓ-block of $\boldsymbol{G}^F$.*

(a) If b is a unipotent block not equal to one of the following blocks of E_8

- $b = b_{E_8}(\phi_1^2.E_6(q), E_6[\theta^i])$ $(i = 1, 2)$ *with $\ell = 2$ and q of order 1 modulo 4, or*
- $b = b_{E_8}(\phi_2^2.{}^2E_6(q), {}^2E_6[\theta^i])$ $(i = 1, 2)$ *with $\ell \equiv 2$ mod 3 and q of order 2 modulo ℓ,*

then $mf(b) = 1$. If b is one of the two blocks above then $mf(b) \leq 2$.

(b) If $s \neq 1$ is quasi-isolated in $\boldsymbol{G}^*$ then

- *if $\boldsymbol{G}$ is of type A or B then $mf(b) = 1$;*
- *if $\boldsymbol{G}$ is of type E_8 then $mf(b) \leq 4$; and*
- *otherwise $mf(b) \leq 2$.*

(c) If $s \neq 1$ is such that $C^{\circ}_{\boldsymbol{G}^*}(s)$ is a Levi subgroup of $\boldsymbol{G}^*$ and $A_{\boldsymbol{G}^*}(s)$ is cyclic, or if $C_{\boldsymbol{G}^*}(s)$ is connected and s is not isolated in $\boldsymbol{G}^*$, then

- *if $\boldsymbol{G}$ is of type E_7 or E_8 then $mf(b) \leq 2$,*
- *otherwise $mf(b) = 1$.*

Proof. Part (a) is proved in Theorem 5.2.5 and Corollary 5.2.6, part (b) follows from Theorem 5.2.10 and Propositions 5.2.14 and 5.2.15. Part (c) follows for $C^{\circ}_{\mathbf{G}^*}(s)$ a Levi subgroup of $\mathbf{G}^*$ and $A_{\mathbf{G}^*}(s)$ cyclic by Theorem 5.2.17 and for $C_{\mathbf{G}^*}(s)$ connected and s not isolated in $\mathbf{G}^*$ by Theorem 5.2.21.

□

Theorem C. *Let $\mathcal{G}_1 = \{SL_n(q) : n \in \mathbb{N}, q = p^a \text{ for some prime } p \neq \ell, a \in \mathbb{N}\}$, and let $\mathcal{G}_2 = \left\{SU_n(q) : n \in \mathbb{N}, q = p^a \text{ for some prime } p \neq \ell \text{ and some } a \in \mathbb{N} \text{ such that } \ell \nmid q^{2s+1} + 1 \ \forall \ s \in \mathbb{N}\right\}$. Then Donovan's conjecture holds for the ℓ-blocks of groups in $\mathcal{G}_1$ and $\mathcal{G}_2$.*

Proof. This is Theorem 5.2.19. □

Theorem D. *Let G be a Suzuki or Ree group. Let b be an ℓ-block of G. If b is a block of the large Ree group in non-defining characteristic, assume that b is unipotent. Then $mf(b) = 1$.*

Proof. This is Theorem 5.3.1. □

Appendix A

Calculations for Section 3.2

In this Appendix we give an idea of the GAP code used to calculate the Morita Frobenius numbers of the sporadic groups as discussed in Section 3.2.

For each group G and each $\ell||G|$, we check the following. Note that if the answer is no at any stage, we exit the loop and move on to the next group.

- Check if there are more than 2 blocks in kG
- If yes, check if there is some non-principal block of cyclic defect
- If yes, check if there are two non-principal blocks with equal, non-cyclic defect
- If yes, check if there are two non-principal blocks with equal non-cyclic defect and no rational valued characters
- If yes, check if there are two non-principal blocks with equal non-cyclic defect and no rational valued characters that have an equal numbers of characters
- If yes, check if there are two non-principal blocks with equal non-cyclic defect and no rational valued characters, with an equal numbers of characters whose degrees add up to the same number
- If yes, list all the blocks for this ℓ, indicating
  - if the block is the principal block
  - if not, if the block has cyclic defect

– if not, if the block has unique defect amongst non-principal blocks
– if not, if the block has a rational valued character (name the character)
– if not, if the block has unique number of characters amongst the non-principal blocks of equal defect containing no rational valued characters
– if not, if the block has unique sum of degrees of characters amongst the non-principal blocks of equal defect containing no rational valued characters with the same number of characters
– if not, call the block a Problem Block

If there are any "Problem Blocks" then we have to deal with these separately, as illustrated in the following examples.

A.1 Example: J_3 and $3.J_3$

The output for $G = J_3$ is as follows.

1. G = J_3

Group Order = 50232960
Prime factors of $|G|$ = [2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 5, 17, 19]
Number of unique prime factors = 5

Prime 1: $\ell = 2$
Number of blocks: 5
Defects: [7, 0, 0, 0, 0]
More than two blocks, but all non-principal blocks have cyclic defect.

- -

Prime 2: $\ell = 3$
Number of blocks: 4
Defects: [5, 1, 0, 0]
More than two blocks, but all non-principal blocks have cyclic defect.

- -

Prime 3: ℓ = 5
Number of blocks: 12
Defects: [1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0]
More than two blocks, but ℓ divides $|G|$ just once so all blocks are cyclic.

- -

Prime 4: ℓ = 17
Number of blocks: 12
Defects: [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
More than two blocks, but ℓ divides $|G|$ just once so all blocks are cyclic.

- -

Prime 5: ℓ = 19
Number of blocks: 11
Defects: [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
More than two blocks, but ℓ divides $|G|$ just once so all blocks are cyclic.

We find that for $G = J_3$, there do not exist collections of non-principal ℓ-blocks of G with equal, non-cyclic defect. Therefore every block B either has cyclic defect, or is not Galois conjugate to any other block, so $mf(B) = 1$.

The output for $G = 3.J_3$ is as follows.

2. G = $3.J_3$

Group Order = 150698880
Prime factors of $|G|$ = [2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 5, 17, 19]
Number of unique prime factors = 5

Prime 1: ℓ = 2
Number of blocks: 7
Defects: [7, 0, 0, 0, 0, 7, 7]

More than two blocks and ℓ divides $|G|$ more than once, so can't conclude that all blocks have cyclic defect, and there is some pair of non-principal, non-cyclic blocks which have equal defect.

Block 1:	principal block
Block 2:	cyclic defect
Block 3:	cyclic defect
Block 4:	cyclic defect
Block 5:	cyclic defect
Block 6:	**Problem Block**
Block 7:	**Problem Block**

Prime 2: $\ell = 3$
Number of blocks: 4
Defects: [6, 2, 1, 1]
More than two blocks and ℓ divides $|G|$ more than once, so can't conclude that all blocks have cyclic defect, but all non-principal, non-cyclic blocks have unique defect.

Prime 3: $\ell = 5$
Number of blocks: 28
Defects: [1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
More than two blocks, but ℓ divides $|G|$ just once so all blocks are cyclic.

Prime 4: $\ell = 17$
Number of blocks: 28
Defects: [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
More than two blocks, but ℓ divides $|G|$ just once so all blocks are cyclic.

Prime 5: $\ell = 19$
Number of blocks: 25
Defects: [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
More than two blocks, but ℓ divides $|G|$ just once so all blocks are cyclic.

When $G = 3.J_3$ and $\ell \neq 2$, there do not exist collections of non-principal ℓ-blocks of G with equal, non-cyclic defect. Therefore every block B either has cyclic defect, or is not Galois conjugate to any other block, so $mf(B) = 1$.

When $G = 3.J_3$ and $\ell = 2$, however, there are two non principal, non-cyclic blocks with the same number and degrees of characters, none of which are rational valued. We therefore let $\hat{G} = 3.J_3.2$ and check what happens for the blocks of $\hat{G}$ when $\ell = 2$.

3. G = $3.J_3.2$

Prime: $\ell = 2$
Number of blocks: 6
Defects: [8, 1, 1, 1, 1, 7]
More than two blocks and ℓ divides $|G|$ more than once, so can't conclude that all blocks have cyclic defect, but all non-principal, non-cyclic blocks have unique defect so any non-cyclic block is stabilized by Galois conjugation.

When $\ell = 2$, every block $\hat{B}$ of $\hat{G} = 3.J_3.2$ either has cyclic defect, or $\sigma(\hat{B}) = \hat{B}$. Therefore by Lemma 2.2.8, $mf(B) = 1$ for every block B of $3.J_3$ when $\ell = 2$.

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