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Doctoral Thesis

On the use of Micro Models for Claims Reserving based on aggregate data

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Declaration of Authorship

I, Carolin MARGRAF, declare that this thesis titled, 'On the use of Micro Models for Claims Reserving based on aggregate data' and the work presented in it are my own. I confirm that:

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Co-authors declaration

Carolin Margraf was the driving force behind the papers in this thesis, while the practical usefulness, the computational accuracy and the theoretical insights were closely supervised. In the papers "The Link Between Classical Reserving and Granular Reserving Through Double Chain Ladder and its Extensions" and "Cash flow generalisations of general insurance expert systems estimating outstanding liabilities", Jens Perch Nielsen supervised the practical side, María Dolores Martínez Miranda supervised the computational aspects and Munir Hiabu gave some support developing the theory. The paper "A likelihood approach to Bornhuetter-Ferguson analysis" was supervised by Bent Nielsen, while Carolin executed the computations as well as most of the writing. In the paper "Micro models for reinsurance reserving based on aggregate data", the practical insights were given by Valandis Elpidorou, and Carolin developed the rest of the paper, supervised by Richard Verrall. While Carolin did most of the work presented in this thesis, the help and supervision from everybody involved was important for the final product.

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CITY UNIVERSITY LONDON

Abstract

Faculty of Actuarial Science and Insurance

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Doctor of Philosophy

On the use of Micro Models for Claims Reserving based on aggregate data

by Carolin Margraf

In most developed economies, the insurance sector earns premiums that amount to around eight percent of their GNP. In order to protect both the financial market and the real economy, this results in strict regulations, such as the Solvency II Directive, which has monitored the EU insurance sector since early 2016. The largest item on general insurers' balance sheets is often liabilities, which consist of future costs for reported claims that have not yet been settled, as well as incurred claims that have not yet been reported. The best estimate of these liabilities, the so-called reserve, is given attention to in Article 77 of the Solvency II Directive. However, the guidelines in this article are quite vague, so it is not surprising that modern statistics has not been used to a great extent in the reserving departments of insurance companies.

This thesis aims to combine some theoretical results with the practical world of claims reserving. All results are motivated by the chain ladder method, and provide different reserving methods that will be introduced thoughout four separate papers.

The first two papers show how claim estimates can be embedded into a full statistical reserving model based on the double chain ladder method. The new methods introduced incorporate available incurred data into the outstanding liability cash flow model. In the third paper a new Bornhuetter-Ferguson method is suggested, that enables the actuary to adjust the relative ultimates. Adjusted cash flow estimates are obtained as constrained maximum likelihood estimates. The last paper addresses how to consider reserving issues when there is excess-of-loss reinsurance. It provides a practical example as well as an alternative approach using recent developments in stochastic claims reserving.

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Contents

1

Introduction

Insurance is a social good, since it allows individuals to pool and protect against financial risks that they would otherwise be forced to bear on their own.

Even Winston Churchill recognised the importance of insurance to society, saying "*Insurance brought the miracle of averages to the rescue of the masses*" when talking about the creation of unemployment and health insurance by the Liberal government of 1906 to 1916. Recognising the importance of property insurance, he also said "*Had I the powers of a dictator I would cause the word 'insure' to be inscribed on the lintel of every house in the land.*"

The miracle of averages is known in mathematics as the law of large numbers on which insurance is based. It states that the average of the results of an experiment will be close to the expected value, if independent experiments are performed a large number of times. In a more illustrative way, think of many cars driving on the streets and accidents happening; it can't be predicted who is going to have an accident. But based on the data collected in the past years, it is possible to forecast the amount of accidents that will happen in the next year. Based on this forecast the insurer charges premiums enabling him to cover the cost in the event of an accident (insurance claim). Therefore, the individuals will carry no risks of having to pay the whole cost in case of an accident.

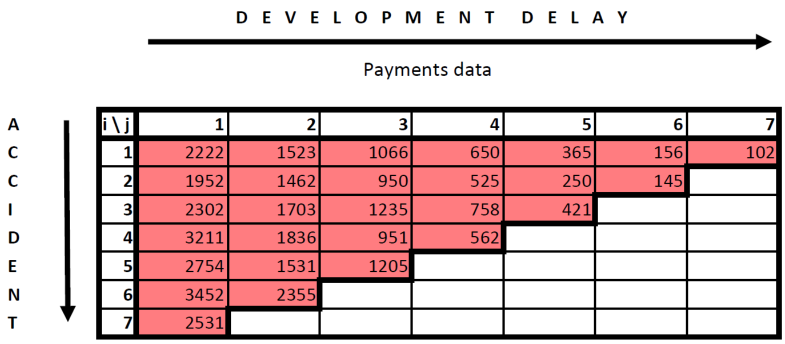
This thesis is motivated by the claims reserving problem in general insurance, which aims to provide a best estimate for outstanding loss liabilities, known as the reserve. General insurance (as it is called in the U.K., also known as 'non-life insurance' in Europe or 'property and casualty insurance' in the U.S.) includes all forms of insurance except for life insurance. Examples of general insurance include motor/car insurance, health insurance, property insurance, travel insurance, liability insurance and marine insurance.

In order to be able to settle the expected future and ongoing cost of claims arising from policies written in the past, a general insurer will set aside sufficient assets, known as a 'claims reserve'. Therefore, it is necessary to forecast the value of claims which have been underwritten, but are yet to be settled. Usually, there is a delay between the occurrence of a claim and its final settlement by the insurer, called development delay. These delays may be caused by the time taken to establish the insurer's liability, the size of the claim amount and whether multiple payments or the reopening of previously closed claims is required. This delay can be divided into the reporting delay (the time between the claim occurrence and when it is reported to the insurer) and the settlement delay (the time between the reporting date and the final settlement of the claim).

Claims are typically aggregated by date of the claim occurrence (the accident date) in years, the development delay and sometimes the year the policy was written. It is important that the data is fully understood, in terms of the nature of the business being written and the profile of the policyholders, in order to be able to apply the most appropriate reserving method.

Liabilities is usually the largest item on the balance sheet of a general insurance company. Therefore, it is very important to estimate it accurately, especially in order to avoid either carrying excessive reserves and to avoid insolvency. Consequently, it is of major importance to be able to validate existing models for calculating the claims reserve and to be able to extend and improve them to obtain more accurate estimates. These considerations are even more important when considering the regularity requirements of the Solvency II regulations in the EU, in force for financial periods starting after 1st January 2016. The EU Directive requires several statistical standards for the quality of the models used to quantify the technical provisions. In particular, Article 77, 'Calculations of technical provisions', states that "*the calculation of the best estimate shall be based upon up-to-date and credible information and realistic assumptions and be performed using adequate, applicable and relevant actuarial and statistical methods.*" However, those methods are not prescribed in detail, which gives insurers the flexibility to adopt the most appropriate method for their specific liabilities.

One of the most popular reserving methods used in practice is the chain ladder method (CLM). It operates on a run-off payments triangle which uses the value of historic paid claims, aggregated by accident year and development delay. Since the data is historic, this forms a triangle when tabulated (see Figure 1.1).



i \ j	1	2	3	4	5	6	7
1	2222	1523	1066	650	365	156	102
2	1952	1462	950	525	250	145	
3	2302	1703	1235	758	421		
4	3211	1836	951	562			
5	2754	1531	1205				
6	3452	2355					
7	2531						

FIGURE 1.1: Example payments triangle, aggregated by accident year and development delay.

The CLM produces estimates for the value of those claims which have been incurred but are not yet settled by extrapolating the data into the lower triangle. The method was developed before the advent of widespread and inexpensive computers, when it was important to have relatively simple procedures that could be implemented by hand. Nevertheless, it is still one of the most commonly used methods today because it is a simple and robust technique that is intuitively appealing and which often gives reasonable results.

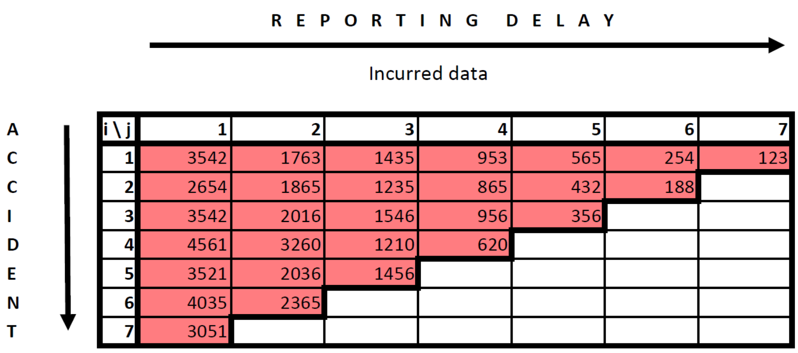
Originally, the CLM devised merely as a clever algorithm for calculating outstanding liabilities rather than a well-defined model based on sound mathematical statistics.

Over the course of time, developments in actuarial science helped to clarify the statistical foundations of the CLM (see Kremer (1982) or Mack (1993)). Having an underlying statistical model enables the model user to include the uncertainties of predicting the future liabilities. However, the intention of these developments was to keep the original intuition and simplicity of the CLM, and to maintain the same reserve estimates. Reformulating the CLM also allowed practitioners to make adjustments or add extensions to the CLM, for example to incorporate claims inflation, that might be useful in different contexts.

Mack (1991), Verrall (1991) and recently Kuang, Nielsen, and Nielsen (2009) have all identified the CLM forecasts as classical maximum likelihood estimates under a Poisson model for the claims. This framework plays an important role for the work in Chapter 4. For comprehensive reviews of stochastic reserving extending the CLM see England and Verrall (2002) and Wüthrich and Merz (2008). Furthermore, Mack (1993) introduced a distribution free CLM, which was further developed in Gisler and Wüthrich (2008) and Peters, Wüthrich, and Shevchenko (2010).

Unfortunately, all CLM based models have the drawback that there is considerable uncertainty in the estimate of the total claims arising from the most recent accident year, since this has the least data available, but accounts for a significant proportion of the outstanding loss liability.

One intuitively appealing idea to solve this problem is to incorporate more data on the nature of the claims, in order to refine our estimates of the outstanding claim amounts. One common method is the incurred chain ladder (ICL) method, which is applied to the triangle of incurred claims. This consists of case estimates of already reported claims (which are provided by expert opinion) in addition to the value of already settled claims. For this incurred triangle, the development parameter, unlike for the payments triangle, denotes the reporting delay, which is the time between the accident date of a claim and its report. Therefore, it is not possible to compare the incurred triangle and the payments triangle directly, which is the main problem of this framework (see Figure 1.2).



i \ j	1	2	3	4	5	6	7
1	3542	1763	1435	953	565	254	123
2	2654	1865	1235	865	432	188	
3	3542	2016	1546	956	356		
4	4561	3260	1210	620			
5	3521	2036	1456				
6	4035	2365					
7	3051						

FIGURE 1.2: Example incurred triangle, aggregated by accident year and reporting delay.

In practice the CLM and the ICL are often used independently and the resulting reserves can be substantially different between the two approaches. The ICL procedure is criticised in Quarg and Mack (2004), where the authors propose a mixture of CLM and ICL based on the dependence between the delays of the two triangles, but the paper does not include a statistical model, i.e. the data generating process of the claims is not considered. For more paid-incurred chain ladder (PIC) literature see for example Merz and Wüthrich (2010), where the authors use a Bayesian method in a log-normal PIC model to predict the outstanding liabilities (see also Happ and Wüthrich (2013) and Peters, Dong, and Kohn (2014)).

An alternative is the Bornhuetter-Ferguson (BF) method (see Bornhuetter and Ferguson (1972)), which combines data in a run-off triangle with external knowledge on for instance the total written premium value or estimates for the ultimate loss reserves for each accident year. This external knowledge could for example be provided by the incurred triangle.

In this thesis, we propose various different methods based on the classical chain ladder method as well as the double chain ladder (DCL) framework introduced in Martínez-Miranda, Nielsen, and Verrall (2012), which may be suitable in different contexts. In the DCL framework, the authors build on the CLM by incorporating information on the reported number of claims as well as the claims amounts. With DCL, it is possible to produce the same results as the CLM, but it also provides information about the distribution of the outstanding liabilities. With the additional information provided by the counts data the authors are able to achieve a complete statistical model framework, which not only allows to replicate exactly the same results as the CLM but also all of the previous methods of combining incurred and paid data discussed above.

By using more data, it is expected that the DCL method will be less volatile than the CLM. Not only do the authors derive a surprisingly simple method for forecasting the outstanding liabilities, but they are also able to estimate the value of reported but not settled (RBNS) and incurred but not reported (IBNR) claims separately, which is necessary for the clear attribution of reserves. This separation is made possible by the connection between the counts and the payments triangles in the DCL, which enables the estimation of the settlement delay (reporting delay+settlement delay=development delay). This is of major importance since it is a requirement of Solvency II, which EU insurers must now comply with. Another advantage regarding Solvency II is that due to the micro structure of all reserving methods introduced in this thesis, it is possible to define a parametric bootstrap, which offers an alternative to the bootstrap of England and Verrall (1999).

One big advantage of the DCL framework is that every method based on the incurred triangle and the payments triangle can now be compared and validated, via the connection with the settlement delay. The validation, introduced in Agbeko et al. (2014), is based on backtesting data previously omitted while estimating the parameters for each method. It is the first approach which enabled comparisons of results on paid data versus incurred data. This is one reason why the research in Chapters 2 and 3 takes advantage of the DCL framework and develops new methods that improve the reserving techniques, which may be compared to each other for any specific dataset.

The aim of this thesis is to develop the connections between the reserving methods used in practice to fundamental mathematical statistics, and therefore be able to explain and extend the practical results more completely. The thesis itself is composed of four self-contained chapters stemming from four separate research papers.

The purpose of each paper is to improve reserving for general insurance companies. The papers are all either based on the DCL framework or are able to reproduce the same results as given by a DCL-based reserving method. Therefore, the classical CLM is re-invented via the DCL and its extensions in order to introduce statistically solid approaches of combining paid and incurred data. Being self-contained, each chapter has its own introduction, notation, conclusions and references. However, they form part of a single, unified research project.

A brief description of the contributions of each chapter follows.

Chapter 2: The Link Between Classical Reserving and Granular Reserving Through Double Chain Ladder and its Extensions

Assuming the existence of additional knowledge, for instance in form of incurred data, the paper introduces the RBNS-preserving double chain ladder (PDCL) method. In PDCL, the aim is to preserve the RBNS values given by the case estimates included in the incurred triangle in form of expert knowledge. As mentioned above, the incurred triangle is considered as expert knowledge since it consists of the paid data as well as the RBNS case estimates estimated by the claims department of the insurance companies.

Also included in this paper is the validation, which enables us to compare all the reserving methods in the DCL model and choose the best method for one particular dataset.

Chapter 3: Cash flow generalisations of non-life insurance expert systems estimating outstanding liabilities

This paper is based on the innovations of Chapter 2 and extends them by introducing two new methods; the expert double chain ladder (EDCL) method as well as the RBNS-preserving expert double chain ladder (PEDCL) method.

As in Chapter 2, we want to take advantage of the expert knowledge in form of the RBNS case estimates given via the incurred claims data. The EDCL method uses that expert knowledge in form of incurred data, RBNS case estimates included, as pseudo data. It replicates the steps of the PDCL method, but iterates them until the process converges to a homogeneous solution, which combines the data on both incurred and paid claims to a single reserve.

Based on this iterative procedure, the PEDCL method uses the estimated EDCL parameters, but preserves the RBNS case estimates, as with the PDCL method.

Chapter 4: A likelihood approach to Bornhuetter-Ferguson analysis

In this paper we develop a likelihood approach to the BF analysis. Recent research has analysed the case where the mean of the ultimate reserves is known, see Mack (2006), Mack (2008) and Alai, Merz, and Wüthrich (2009), Alai, Merz, and Wüthrich (2010), see also Verrall (2004).

This paper considers a similar approach to the previous studies, where we assume that all relative increments of the ultimate reserves for each underwriting year are known. We show that this situation lends itself to a simple maximum-likelihood analysis under Poisson assumptions for the claims. Our analysed situation results in an analysis based on distributions for the claim amounts from the exponential familiy, which has the advantage of including a simple likelihood equation with a unique solution.

The extended BF approach of this paper can also be used to improve on the recent DCL approach that uses data for both claim counts and amounts in the analysis, as in for example Martínez-Miranda, Nielsen, and Verrall (2012) and Martínez-Miranda, Nielsen, and Verrall (2013). The difference between this paper and the DCL framework is that this approach can be applied using only a single triangle, usually the payments triangle, but therefore it is not able to predict the settlement delay, which is necessary to distinguish between the RBNS and IBNR reserve.

Chapter 5: Micro models for reinsurance reserving based on aggregate data

This paper considers a situation where an insurer has covered excess-of-losses of their of individual claims via their reinsurer. The insurer now wants to split the reserve in two parts; one net reserve of the insurance companies liabilities and the other part of the gross reserve covered by the reinsurance company. While classical mean-linear reserving techniques do not provide a solution to this type of problem, this paper will advocate that a relatively simple extension of double chain ladder does.

The model based on this idea is introduced in Martínez-Miranda et al. (2015), which is the DCL model that includes a development inflation parameter representing the relationship between the development of the claim and its mean severity. Furthermore, we expand this model and include again the expert knowledge in form of the via incurred data estimated severity inflation, just like in BDCL (see Martínez-Miranda, Nielsen, and Verrall (2013)).

The split is now done by simulating each claim individually with a Gamma distribution and comparing their values to a given retention. This method is compared to the practical approach that is used in reserving departments of insurance companies.

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2

The Link Between Classical Reserving and Granular Reserving Through Double Chain Ladder and its Extensions

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The Link Between Classical Reserving and Granular Reserving Through Double Chain Ladder and its Extensions

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Abstract

The relationship of the chain ladder method to mathematical statistics has long been debated in actuarial science. During the nineties, it became clear that the originally deterministic chain ladder can be seen as an autoregressive time series or as a multiplicative Poisson model. This paper draws on recent research and concludes that chain ladder can be seen as a structured histogram. This gives a direct link between classical aggregate methods and continuous granular methods. When the histogram is replaced by a smooth counter part, we have a continuous chain ladder model. Re-inventing classical chain ladder via double chain ladder and its extensions introduces statistically solid approaches of combining paid and incurred data with direct link to granular data approaches. This paper goes through some of the extensions of double chain ladder and introduces new approaches to incorporating and modelling incurred data.

Keywords: Stochastic Reserving; General Insurance; Solvency II; Chain Ladder; Reserve Risk; Claims Inflation; Incurred Data; Model Validation; Granular Data.

2.1 Introduction

Double chain ladder is a bridge between the chain ladder method (CLM) and mathematical statistics. Double chain ladder is modelling the full system of reported claims, their delay and the resulting claims. Bootstrapping it with or without parameter uncertainty is easy. Double chain ladder bootstrapping does not face the stability problems resulting when bootstrapping the CLM. The full model structure is the key here: bootstrapping a well defined statistical model is simple and straightforward.

The reason it is difficult to bootstrap the CLM is that only one part of the system is modelled: the aggregated paid or incurred claims. The full data generation process is not known in classical chain ladder, and approximations have to be introduced to come up with some sort of bootstrapping. The typical assumption taken is that all adjusted residuals arise from the same distribution. But adjusted residuals on the aggregated paid data or incurred data models do not follow the same distribution. These residuals can be very close to the normal distribution and very right skewed depending on the underlying number of claims leading to this residual. Instability occurs if an unimportant right skewed residual of little weight is reshuffled as a very important residual in the bootstrap.

Double chain ladder is estimated from the exact same data structure as chain ladder. It uses triangle type of data on frequencies, paid and incurred data. Communicating the implementation and structure of double chain ladder to actuaries is therefore a simple exercise. Furthermore, double chain ladder gives - almost - the exact same reserve as chain ladder. One can therefore see double chain ladder as a more stable, better understood version of CLM with the clear advantage of being easy to generalize. When generalizing or developing double chain ladder, the actuary can see any development as moving away from chain ladder. The vast amount of experience and tacit knowledge actuaries have invested in the chain ladder model is therefore directly useful when working with and interpreting double chain ladder and its extensions.

In this paper we will consider double chain ladder, double chain ladder and Bornhuetter-Ferguson, incurred double chain ladder and RBNS-preserving double chain ladder and we will give these four methods the acronyms DCL, BDCL, IDCL and PDCL. BDCL was the first published extension of DCL. It was verified that the severity inflation

(inflation in cost per claim) in the underwriting year direction is the key to many of the hardest challenges of chain ladder and it was shown that this severity inflation could be extracted from incurred data via a simple estimation trick. Replacing the paid data's severity inflation in DCL with the incurred data's severity inflation is the definition of BDCL. Incurred double chain ladder is simply defined as that severity inflation (cost per claim in the underwriting year direction) resulting exactly in the same reserves for every underwriting year as the reserve resulting from the chain ladder method applied to incurred data. The advantage of having IDCL instead of the incurred chain ladder is similar to the advantages of having DCL instead of chain ladder given above. Finally PDCL is one version of double chain ladder that does not change the RBNS values.

DCL was published via the three Astin Bulletin papers, Verrall, Nielsen, and Jessen (2010), Martínez-Miranda et al. (2011), and Martínez-Miranda, Nielsen, and Verrall (2012). BDCL was published in North American Actuarial Journal in Martínez-Miranda, Nielsen, and Verrall (2013a), PDCL is introduced in this British Actuarial Journal paper and IDCL was introduced in the Variance paper Agbeko et al. (2014). One could have that point of view that developments of double chain ladder might become redundant, when full granular reserving based on micro models enter actuarial practice. While this might be true, then we believe that granular reserving should be developed in the exact same way as double chain ladder was developed: one should be able to follow step by step how an aggregate chain ladder is changed into a granular model and developed. When progressing this way, one makes sure that the tacit knowledge and experience of actuaries, built via the CLM, is carried over to the granular data approach. We call this "the bathwater approach" to developing reserving techniques, because we do not want to throw the baby out with the bathwater and develop new methods missing important features and properties of classical methods.

In Section 2.6, a preliminary first approach to granular chain ladder called continuous chain ladder is described. Continuous chain ladder is a smooth structured density reflecting the fact that chain ladder could be viewed as a structured histogram. The difference between a structured smooth density and a structured histogram is just which nonparametric estimation procedure is applied. The histogram approach reproducing chain ladder or a smooth version of it called continuous chain ladder. Since chain ladder itself is a granular method based on a suboptimal histogram approach, everything we develop via double chain ladder and it's extensions can indeed be viewed as granular methods with smooth continuous counter parts waiting to be formally defined.

The rest of the paper is structured as follows. Section 2.2 describes the data and the expert knowledge, introduces the notation and defines the model assumptions. Section 2.3 discusses the outstanding loss liabilities point estimates. Section 2.4 describes four methods to estimate the parameters in the model: DCL, BDCL, PDCL and IDCL. The validation of these four methods is considered in Section 2.5 through a back-testing procedure. Section 2.6 describes the link between classical reserving and granular reserving. Section 2.7 provides some concluding remarks.

2.2 Data and first moment assumptions and some comments on granular data

This section describes the classical aggregated data used in most general insurance companies. However, in Section 2.6 below we make it clear that working with this kind of aggregated data indeed is very closely connected to working with granular data. The resulting estimators of aggregated data are piecewise constant or structured histograms, while the resulting estimators of continuous data are continuous and easier to optimize. Because the classical chain ladder method is closely related to the continuous chain ladder method, every single extension of double chain ladder is also a contribution to granular methodology. One can - so to speak - develop the practical ideas on aggregated data and develop the continuous versions later.

This paper will work on aggregated data and contribute to the understanding and validation of chain ladder, but it will in particular introduce new ways of considering incurred data and expert opinion. We start by describing the data and expert knowledge extracted from incurred data, that we are going to work with. Data are aggregated incurred counts (data), aggregated payments (data) and aggregated incurred payments (expert knowledge). All of those three objects have the same structural form, i.e. they live on the upper triangle

$$\mathcal{I} = \{(i,j) : i = 1, \dots, m, j = 0, \dots, m-1;\ i+j \leq m\},$$

$m > 0$. Here, m is the number of underwriting years observed. It will be assumed that the reporting delay, that is the time from underwriting of a claim until it is reported, as well as the settlement delay, that is the delay between the report of a claim and its settlement, are bounded by m. This, in contrast to the classical CLM, will make it possible to also get estimates in the tail, that is when reporting delay plus settlement delay is greater than m. Our data can now be described as follows.
The data:

Aggregated incurred counts: $N_{\mathcal{I}} = \{N_{ik} : (i,k) \in \mathcal{I}\}$, with N_{ik} being the total number of claims of insurance incurred in year i which have been reported in year $i+k$, i.e. with k periods delay from year i.

Aggregated payments: $X_{\mathcal{I}} = \{X_{ij} : (i,j) \in \mathcal{I}\}$, with X_{ij} being the total payments from claims incurred in year i and paid with j periods delay from year i.

Note that the meaning of the second coordinate of triangle $\mathcal{I}$ varies between the two different data. While in the counts triangle it represents the reporting delay, in the payments triangle it represents the development delay, that is reporting delay plus settlement delay.
To describe the aggregated incurred payments, we need some theoretical micro-structural descriptions. These micro-structural descriptions follow the line of Martínez-Miranda, Nielsen, and Verrall (2012) and also build the base of the forthcoming DCL assumptions. By N_{ikl}^{paid}, we will denote the number of the future payments originating from the N_{ik} reported claims, which were finally paid with a delay of $k+l$, where $l = 0, \ldots, m-1$.

Also, let $X_{ikl}^{(h)}$ denote the individual settled payments which arise from N_{ikl}^{paid}, $h = 1, \ldots, N_{ikl}^{paid}$. Finally, we define

$$X_{ikl} = \sum_{h=1}^{N_{ikl}^{paid}} X_{ikl}^{(h)}, \quad (i,k) \in \mathcal{I}, \; l = 0, \ldots, m-1,$$

i.e. those payments originating from underwriting year i, which are reported after a delay of k and paid with an overall delay of $k+l$.
The aggregated incurred payments are then considered as unbiased estimators of $\sum_{l=0}^{m-1} X_{ikl}$. Technically, we model the expert knowledge as follows.

Expert knowledge:

Aggregated incurred payments: $I_{\mathcal{I}} = \{I_{ik} : (i,k) \in \mathcal{I}\}$, with I_{ik} being

$$I_{ik} = \sum_{s=0}^{k} \sum_{l=0}^{m-1} E[X_{isl}|\ \mathcal{F}_{(i+k)}] - \sum_{s=0}^{k-1} \sum_{l=0}^{m-1} E[X_{isl}|\ \mathcal{F}_{(i+k-1)}],$$

where $\mathcal{F}_h$ is an increasing filtration illustrating the expert knowledge at time point h.

In this manuscript, we will only consider best estimates (or pointwise estimates) and for this we can define the DCL model just under first-order moment assumptions, i.e. assumptions on the mean. We show that the classical chain ladder multiplicative structure holds under very general underlying dependencies on the mean. For fixed $i = 0, \ldots, m; \quad k, l = 0, \ldots, m-1,$ and $h = 1, \ldots, N_{ikl}^{paid}$, the first-order moment conditions of the DCL model are formulated as follows.

A1. The counts, N_{ik}, are random variables with mean having a multiplicative parametrization $\mathrm{E}[N_{ik}] = \alpha_i \beta_k$, for given parameters α_i, β_j, under the identification $\sum_{k=0}^{m-1} \beta_k = 1$.

A2. The number of payments, N_{ikl}^{paid}, representing the RBNS delay, are random variables with conditional mean $\mathrm{E}[N_{ikl}^{paid}|N_{\mathcal{I}}] = N_{ik}\widetilde{\pi}_l$, for given parameters $\widetilde{\pi}_l$.

A3. The individual payments sizes $X_{ikl}^{(h)}$ are random variables whose mean conditional on the number of payments and the counts is given by $\mathrm{E}[X_{ikl}^{(h)}|N_{ikl}^{paid}, N_{\mathcal{I}}] = \widetilde{\mu}_l \gamma_i$, for given parameters $\widetilde{\mu}_l, \gamma_i$.

Assumption A1 is the classical chain ladder assumption applied on the counts triangle. See also Mack (1991). The main point hereby is the multiplicativity between underwriting year and reporting delay. Assumptions A2 and A3 are necessary to connect reporting delay, settlement delay and development delay - the main idea of DCL. See also Verrall, Nielsen, and Jessen (2010), Martínez-Miranda et al. (2011) and Martínez-Miranda, Nielsen, and Verrall (2012).

Note that the observed aggregated payments can be written as

$$X_{ij} = \sum_{l=0}^{j} X_{i,j-l,l} = \sum_{l=0}^{j} \sum_{h=1}^{N^{paid}_{i,j-l,l}} X^{(h)}_{i,j-l,l}.$$

And then, using assumptions A1 to A3, we can derive the mean of the aggregated payments conditional to the counts as follows:

$$\begin{aligned}
\mathrm{E}[X_{ij}|N_{\mathcal{I}}] &= \mathrm{E}\left[\sum_{l=0}^{j} \sum_{h=1}^{N^{paid}_{i,j-l,l}} X^{(h)}_{i,j-l,l}|N_{\mathcal{I}}\right] \\
&= \sum_{l=0}^{j} \mathrm{E}\left[\sum_{h=1}^{N^{paid}_{i,j-l,l}} \mathrm{E}[X^{(h)}_{i,j-l,l}|N_{\mathcal{I}}, N^{paid}_{i,j-l,l}]|N_{\mathcal{I}}\right] \\
&= \sum_{l=0}^{j} \mathrm{E}[N^{paid}_{i,j-l,l}\widetilde{\mu}_l\gamma_i|N_{\mathcal{I}}] \\
&= \gamma_i \sum_{l=0}^{j} N_{i,j-l}\widetilde{\pi}_l\widetilde{\mu}_l.
\end{aligned}$$

Thus, the unconditional mean is given by

$$\mathrm{E}[X_{ij}] = \alpha_i\gamma_i \sum_{l=0}^{j} \beta_{j-l}\widetilde{\mu}_l\widetilde{\pi}_l. \tag{2.1}$$

Inspecting equation (2.1), we can reduce the amount of parameters by setting $\mu = \sum_{l=0}^{j} \widetilde{\pi}_l\widetilde{\mu}_l$ and $\pi_l = \widetilde{\pi}_l\widetilde{\mu}_l\mu^{-1}$, so that $\mu\pi_l = \widetilde{\mu}_l\widetilde{\pi}_l$ and therefore the unconditional mean of the payments becomes

$$\mathrm{E}[X_{ij}] = \alpha_i\gamma_i\mu \sum_{l=0}^{j} \beta_{j-l}\pi_l. \tag{2.2}$$

Equation (2.2) is the key in deriving the outstanding loss liabilities. These are the values of X_{ij} in the lower triangle and the tail (that is for $i = 1, \ldots, m$; $j = 0, \ldots, 2m-1$; $i+j \geq m+1$). In the sequel we will write all the DCL parameters, i.e. the parameters involved in the DCL model, as

$$(\alpha, \beta, \pi, \gamma, \mu) = (\alpha_1, \ldots, \alpha_m, \beta_0, \ldots, \beta_{m-1}, \pi_0, \ldots, \pi_{m-1}, \gamma_1, \ldots, \gamma_m, \mu).$$

In the next section, we will see that in a very natural way, we are able to distinguish between RBNS and IBNR claims. This is possible due to the separation of the development delay into the reporting delay, β, and the settlement delay, π.

2.3 Forecasting outstanding claims: the RBNS and IBNR reserves

To produce outstanding claims forecasts under the DCL model we need to estimate the DCL parameters. Section 2.4 below is devoted to this issue. In this section, we assume that the DCL parameters $(\alpha, \beta, \pi, \gamma, \mu)$ have been already estimated by $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}, \widehat{\mu})$, and show how easily point forecasts of the RBNS and IBNR components of the reserve can be calculated. Using the notation of Verrall, Nielsen, and Jessen (2010) and Martínez-Miranda et al. (2011), we consider predictions over the triangles illustrated in Figure 2.1.

$$\mathcal{J}_1 = \{i = 2, \ldots, m; j = 0, \ldots, m-1 \text{ with } i + j \geq m+1\},$$
$$\mathcal{J}_2 = \{i = 1, \ldots, m; j = m, \ldots, 2m-1 \text{ with } i + j \leq 2m-1\},$$
$$\mathcal{J}_3 = \{i = 2, \ldots, m; j = m, \ldots 2m-1 \text{ with } i + j \geq 2m\}.$$

The classical CLM produces forecasts over only $\mathcal{J}_1$. So, if the CLM is being used, it is necessary to construct tail factors in some way. For example, this is sometimes done by assuming that the run-off will follow a set shape, thereby making it possible to extrapolate the development factors. In contrast, under the DCL model it is possible to provide also the tail over $\mathcal{J}_2 \cup \mathcal{J}_3$, just by using the underlying assumptions about the development.

Following Martínez-Miranda, Nielsen, and Verrall (2012), we calculate the forecasts using the expression for the mean of the aggregated payments derived in (2.2) and replacing the unknown DCL parameters by their estimates. Note that the RBNS component arises from claims reported in the past and therefore, as Martínez-Miranda, Nielsen, and Verrall (2012) discuss, it is possible to calculate the forecasts using the true observed value N_{ik} instead of their chain ladder estimates, $\widehat{\alpha}_i, \widehat{\beta}_k$, which are involved in the formulae (2.2).

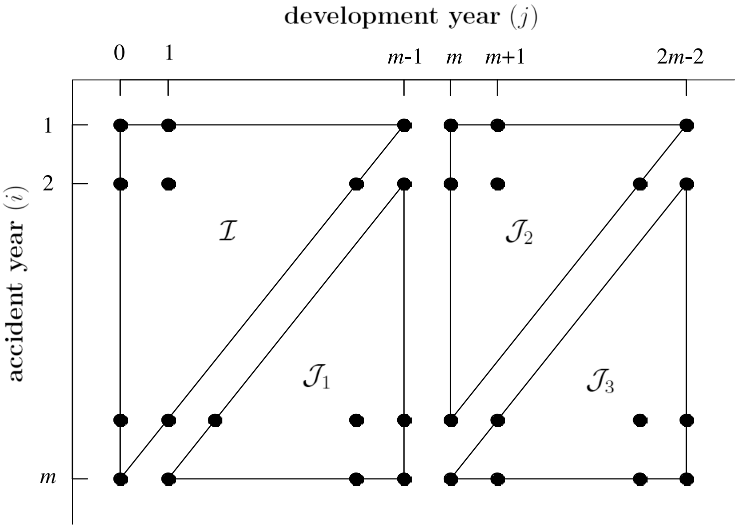

FIGURE 2.1: Index sets for aggregate claims data, assuming a maximum delay $m-1$.

However, for the IBNR reserves, this is not possible since those values arise from claims reported in the future and then it is necessary to use all DCL parameters.

From these comments we define the RBNS component as follows, where we consider two possibilities depending on whether the estimates of N_{ik} are used or not.

$$\widehat{X}_{ij}^{rbns(1)} = \sum_{l=i-m+j}^{j} N_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i, \quad (i,j) \in \mathcal{J}_1 \cup \mathcal{J}_2, \tag{2.3}$$

and

$$\widehat{X}_{ij}^{rbns(2)} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i, \quad (i,j) \in \mathcal{J}_1 \cup \mathcal{J}_2, \tag{2.4}$$

where $\widehat{N}_{ik} = \widehat{\alpha}_i\widehat{\beta}_k$. In most cases, to shorten the notation, we will simply write $\widehat{X}_{ij}^{rbns}$ for the RBNS estimates. However, whenever it is necessary, we will state which version is taken. The IBNR component always needs all DCL parameters and it is calculated always as follows:

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i, \quad (i,j) \in \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3. \tag{2.5}$$

By adding up the RBNS and IBNR components we have the outstanding loss liabilities pointwise forecasts, which spread out on the forecasting sets $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$ as follows.

$$\widehat{X}_{ij} = \begin{cases} \widehat{X}_{ij}^{rbns} + \widehat{X}_{ij}^{ibnr} & \text{if } (i,j) \in \mathcal{J}_1 \cup \mathcal{J}_2, \\ \widehat{X}_{ij}^{ibnr} & \text{if } (i,j) \in \mathcal{J}_3. \end{cases} \tag{2.6}$$

The outstanding liabilities per accident year are the row sums of forecasts $\widehat{X}_{ij}$ above. For a fixed i, we write $\mathcal{J}_a(i) = \{j : (i,j) \in \mathcal{J}_a\}, a = 1,2,3$. Then the outstanding liabilities per accident year $i = 1,\ldots,m$ are

$$\widehat{R}_i = \sum_{j \in \mathcal{J}_1(i) \cup \mathcal{J}_2(i)} \widehat{X}_{ij}^{rbns} + \sum_{j \in \mathcal{J}_1(i) \cup \mathcal{J}_2(i) \cup \mathcal{J}_3(i)} \widehat{X}_{ij}^{ibnr}.$$

2.4 Estimation of the parameters in the double chain ladder model

In the previous section we have described how to estimate the outstanding claims and thereby construct RBNS and IBNR reserves once the DCL parameters have been estimated. Now we describe how to get suitable estimators for the DCL parameters. Specifically we are going to explore four different estimations methods, all of them based on the chain-ladder algorithm.

2.4.1 The DCL method

The DCL method is the most simple method to derive the parameters in the DCL model. It is the original method proposed by Martínez-Miranda, Nielsen, and Verrall (2012) which makes the following additional assumption on the payments triangle $X_{\mathcal{I}}$:

B1 The payments X_{ij}, with $i = 1,\ldots,m$, and $j = 0,\ldots,m-1$, are random variables with mean having a multiplicative parametrization:

$$E[X_{ij}] = \widetilde{\alpha}_i \widetilde{\beta}_j, \qquad \sum_{j=0}^{m-1} \widetilde{\beta}_j = 1. \tag{2.7}$$

We use the identification of Mack (1991). Any other identification could be used here, but this one allows the $\widetilde{\beta}_j$ to have an interpretation as probabilities. Then, merging the previously derived expression (2.2) and the above (2.7), we have that

$$\alpha_i \gamma_i \mu \sum_{l=0}^{j} \beta_{j-l} \pi_l = \widetilde{\alpha}_i \widetilde{\beta}_k,$$

and then the DCL parameters can be identified from the chain ladder parameters, $\widetilde{\alpha}_i, \widetilde{\beta}_k$, using the following equations:

$$\alpha_i \mu \gamma_i = \widetilde{\alpha}_i, \tag{2.8}$$

$$\sum_{l=0}^{j} \beta_{j-l} \pi_l = \widetilde{\beta}_j. \tag{2.9}$$

Even though many other micro-structure formulations might exist, the above model can be considered as a detailed specification of the classical chain ladder. Martínez-Miranda, Nielsen, and Verrall (2012) discuss that if the RBNS component is estimated using (2.4), DCL completely replicates the results of CLM applied to the aggregated payments triangle. Thus, from the above two equations we can see how the underwriting and development chain ladder components are decomposed into separate components which capture the separate sources of delay inherent in the way claims emerge and the severity specification.

Now, the main idea to derive the DCL parameters is to estimate the chain ladder parameters $(\widehat{\alpha}, \widehat{\beta})$ and $(\widehat{\widehat{\alpha}}, \widehat{\widehat{\beta}})$ (cf. A1, B1) by applying the classical chain ladder algorithm on the counts triangle $N_{\mathcal{I}}$ and the payments triangle $X_{\mathcal{I}}$, respectively. Afterwards, the remaining DCL parameters, this is $(\widehat{\gamma}, \widehat{\mu}, \widehat{\pi})$, can be calculated by simple algebra using (2.8) and (2.9).

For illustration of the chain ladder algorithm, we assume an incremental triangle (C_{ij}) (in our case this would be $N_{\mathcal{I}}$ or $X_{\mathcal{I}}$), and that we want to estimate its chain ladder parameters $(\widehat{\widehat{\alpha}}, \widehat{\widehat{\beta}})$. To apply the chain ladder algorithm, one has to transform the triangle (C_{ij}) into a cumulative triangle (D_{ij}):

$$D_{ij} = \sum_{k=1}^{j} C_{ik}.$$

Then, the chain ladder algorithm can be applied on (D_{ij}). It will produce estimates of development factors, λ_j, $j = 1, 2, \ldots, m-1$ which can be described by

$$\widehat{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} D_{ij}}{\sum_{i=1}^{n-j+1} D_{i,j-1}}.$$

These development factors can be converted into estimates of $(\overline{\alpha}, \overline{\beta})$ using the following identities which were derived in Verrall (1991).

$$\widehat{\overline{\beta}}_0 = \frac{1}{\prod_{l=1}^{m-1} \widehat{\lambda}_l},$$
$$\widehat{\overline{\beta}}_j = \frac{\widehat{\lambda}_j - 1}{\prod_{l=j}^{m-1} \widehat{\lambda}_l},$$
$$\widehat{\overline{\alpha}}_i = \sum_{j=0}^{m-i} C_{ij} \prod_{j=m-i+1}^{m-1} \widehat{\lambda}_j.$$

Alternatively, analytical expressions for the estimators can also be derived directly (rather than using the chain ladder algorithm), and further details can be found in Kuang, Nielsen, and Nielsen (2009).

Once the chain ladder parameters $(\widehat{\alpha}, \widehat{\beta})$ and $(\widehat{\widetilde{\alpha}}, \widehat{\widetilde{\beta}})$ are derived, the settlement delay parameter, π, can be estimated just by solving the following linear system.

$$\begin{pmatrix} \widehat{\widetilde{\beta}}_0 \\ \vdots \\ \vdots \\ \widehat{\widetilde{\beta}}_{m-1} \end{pmatrix} = \begin{pmatrix} \widehat{\beta}_0 & 0 & \cdots & 0 \\ \widehat{\beta}_1 & \widehat{\beta}_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \widehat{\beta}_{m-1} & \cdots & \widehat{\beta}_1 & \widehat{\beta}_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \vdots \\ \pi_{m-1} \end{pmatrix}. \tag{2.10}$$

Let $\widehat{\pi}$ denote the solution of (2.10).

Now we consider the estimation of the parameters involved in the means of individual payments. The model is technically over-parametrised since there are too many inflation parameters in (2.8). The simplest way to ensure identifiability is to set $\gamma_1 = 1$, which means that the inflation effect of all accident years are compared to the first year. Then the estimate of μ, $\widehat{\mu}$, can be obtained from

$$\widehat{\mu} = \frac{\widehat{\widetilde{\alpha}}_1}{\widehat{\alpha}_1}.$$

Using $\widehat{\mu}$, the remaining estimates for $\gamma_i,\ i = 2, \ldots, m$, are directly derived from (2.8). The DCL estimation procedure described above has been implemented in the R-package DCL created by Martínez-Miranda, Nielsen, and Verrall (2013b). Using this software, we have derived Table 2.1, which shows the values of $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\pi}$ and $\widehat{\gamma}$, calculated from a real dataset included also in the DCL package. The data is real motor data from a major insurer, containing information during $m = 19$ years.

i	k,l	$\widehat{\alpha}(i)$	$\widehat{\beta}(k)$	$\widehat{\pi}(l)$	$\widehat{\gamma}(i)$
1	0	1078	0.7599	0.0592	1.0000
2	1	1890	0.2097	0.3098	1.1173
3	2	2066	0.0189	0.2032	1.4947
4	3	2353	0.0064	0.1996	1.7461
5	4	3015	0.0016	0.1388	2.1075
6	5	3727	0.0010	0.0440	2.0936
7	6	5057	0.0009	0.0227	2.2495
8	7	6483	0.0007	0.0095	2.1250
9	8	7727	0.0003	0.0018	1.9028
10	9	7134	0.0001	0.0029	2.0197
11	10	7319	0.0001	0.0002	2.0704
12	11	6152	0.0000	0.0026	2.2666
13	12	5242	0.0001	0.0019	2.3157
14	13	6150	0.0000	0.0032	2.4747
15	14	7028	0.0001	-0.0002	2.3829
16	15	6725	0.0000	0.0013	2.8391
17	16	5260	0.0000	-0.0004	3.1815
18	17	5869	0.0000	0.0000	4.1747
19	18	5953	0.0000	0.0000	6.7501
		$\widehat{\mu} = 2579$			

TABLE 2.1: DCL parameter estimates derived by the DCL method

2.4.2 Bornhuetter-Ferguson and double chain ladder: the BDCL method

The chain ladder and Bornhuetter-Ferguson (BF) methods are among the easiest claim reserving methods and due to their simplicity, they are two of the most commonly used techniques in practice. Some recent papers on the BF method include Verrall (2004), Mack (2008), Schmidt and Zocher (2008), Alai, Merz, and Wüthrich (2009) and Alai, Merz, and Wüthrich (2010). The BF method introduced by Bornhuetter and Ferguson (1972) aims to address one of the well known weaknesses of CLM, which is the effect outliers can have on the estimates of outstanding claims. Especially the most recent underwriting years are the years with nearly no data and thus very sensitive to outliers.

However, these recent underwriting years build the very major part of the outstanding claims. Hence, the CLM estimates of the outstanding liabilities might differ fatally from the true (unknown) values.

Acknowledging this problem, the BF method incorporates prior knowledge from experts and is therefore more robust than the CLM method, which relies completely on the data contained in the run-off triangle $X_{\mathcal{I}}$.

In this section, we briefly summarize the Bornhuetter-Ferguson double chain ladder (BDCL) method introduced in Martínez-Miranda, Nielsen, and Verrall (2013a), which mimics BF in the framework of DCL. The BDCL method starts with identical steps as DCL but instead of using the estimate of the inflation parameters, γ and μ, from the triangle of paid claims, $X_{\mathcal{I}}$, it deploys expert knowledge in the form of the incurred triangle, $I_{\mathcal{I}}$, to adjust the estimation of the sensitive inflation parameter, γ. This is done as follows. From assumptions $A2, A3$ and equation (2.8), we easily deduce that

$$E[I_{ik}] = \alpha_i \mu \gamma_i \beta_k = \widetilde{\alpha}_i \beta_k. \tag{2.11}$$

Hence, the incurred triangle, $I_{\mathcal{I}}$, has multiplicative mean and its underwriting year factor, $\widetilde{\alpha}$, is identical to the one of the payments triangle, $X_{\mathcal{I}}$ (cf. (2.7)). However, its estimation is less sensitive to outliers since it incorporates all incurred claims via expert knowledge. We conclude that we can replace the payments triangle by the incurred payments triangle when we calculate estimates of the inflation parameters, γ, μ, in (2.8). Note that the severity mean, μ, is going to remain the same since the first rows of $X_{\mathcal{I}}$ and $I_{\mathcal{I}}$ are identical.

Summarised, the BDCL-method can be carried out as follows.

- *Step 1: Parameter estimation.*
  Estimate the DCL parameters $(\alpha, \beta, \pi, \gamma, \mu)$ using the DCL method of Section 2.4.1 with the data in the triangles $N_{\mathcal{I}}$ and $X_{\mathcal{I}}$ and denote the parameter estimates by $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}, \widehat{\mu})$.
  Repeat this estimation using the DCL method but replacing the triangle of paid claims, $X_{\mathcal{I}}$, by the triangle of incurred data, $I_{\mathcal{I}}$. Keep only the resulting estimated inflation parameters, denoted by $\widehat{\gamma}^{BDCL}$.

- *Step 2: BF adjustment.*

  Replace the inflation parameters $\widehat{\gamma}$ from the paid data by the estimate from the incurred triangle, $\widehat{\gamma}^{BDCL}$.

From these two steps, the final BDCL estimates of the DCL parameters are $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}^{BDCL}, \widehat{\mu})$.

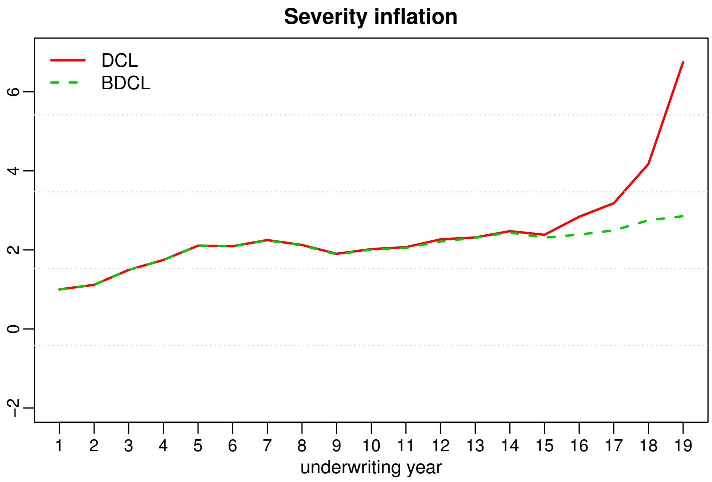

FIGURE 2.2: Plot of severity inflation estimates. DCL: $\widehat{\gamma}_i$ (red), BDCL: $\widehat{\gamma}_i^{BDCL}$ (green).

Again, using the R-package DCL, we can derive the Figure 2.2 that shows the severity inflation estimates derived by DCL and BDCL. BDCL, with the incorporated expert knowledge, seems to stabilize the severity inflation in the most recent underwriting years while keeping the values in the other years. The result is a more realistic estimate correcting the DCL parameter $\widehat{\gamma}_i$ exactly in its weakest point, that is in those years where the payments triangle, $X_{\mathcal{I}}$, has nearly no data. Again, those recent underwriting years contain the very major part of the outstanding liabilities.

2.4.3 The PDCL method

In the last section, we have described a method which incorporates expert knowledge in form of the incurred triangle, $I_{\mathcal{I}}$. The values in $I_{\mathcal{I}}$ arise from case estimates for RBNS claims, developed in the case department of the insurance company, and claims which are already paid. Thus, if one subtracts these already paid claims (which are given via the payments triangle $X_{\mathcal{I}}$) from the incurred triangle, one can reconstruct the RBNS case estimates. However, as soon as this is done, it is obvious that these RBNS case estimates do not match with the RBNS estimates (2.3) and (2.4), using any DCL method (including BDCL).

We conclude that the reserve department, using double chain ladder (and also chain ladder), calculates different RBNS estimates than those given by the case department. If this difference is huge, consultation between the case department and reserve department is necessary. The case department possesses expert knowledge on every single claim that is reported and they can use that knowledge of the claims in conjunction with their expertise to improve estimation. Below we introduce an alternative reserving method preserving the RBNS estimates given by the case department. We call this method RBNS-preserving double chain ladder (PDCL).

The first step is to construct a preliminary square $(S_{ij}), i = 1, \dots, m, \quad j = 0, \dots, m-1$, which will yield new estimates for the DCL parameters. The upper triangle of the square (i.e. $(i, j) \in \mathcal{I}$) should have the same entries as the payments triangle (X_{ij}). The lower triangle (i.e. $(i, j) \in \mathcal{J}_1$) should consist of preliminary estimates of the outstanding loss liabilities. The outstanding loss liabilites comprise an RBNS and an IBNR part (cf. (2.6)). However, we only want to estimate the IBNR component of these outstanding loss liabilities while taking the RBNS case estimates as the RBNS component. More precisely, we do the following. We take the BDCL parameter estimates $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}^{BDCL}, \widehat{\mu})$ and use these parameters to estimate the RBNS component $(\widehat{X}_{ij}^{rbns})$ and IBNR component $(\widehat{X}_{ij}^{ibnr})$ using (2.4) and (2.5). As mentioned above, we want the RBNS estimate to be equal to the RBNS case estimates, which can only be reconstructed per accident year. For $i = 1, \dots, m$, they can be described as

$$X_i^{rbns.case.estimate} = \sum_{j=0}^{m-i} I_{ij} - \sum_{j \in \mathcal{J}_2(i)} \widehat{X}_{ij}^{rbns} - \sum_{j=0}^{m-i} X_{ij}.$$

Hence, we define the RBNS preserving components

$$\widehat{X}_{ij}^{rbns.pres} = \frac{\sum_{j=0}^{m-i} I_{ij} - \sum_{j \in \mathcal{J}_2(i)} \widehat{X}_{ij}^{rbns} - \sum_{j=0}^{m-i} X_{ij}}{\sum_{j \in \mathcal{J}_1(i)} \widehat{X}_{ij}^{rbns}} \widehat{X}_{ij}^{rbns}.$$

Note that

$$\sum_{j \in \mathcal{J}_1(i)} \widehat{X}_{ij}^{rbns.pres} = X_i^{rbns.case.estimate}.$$

Thus we define the preliminary square (S_{ij}) as

$$S_{ij} = \begin{cases} X_{ij}, & \text{if } (i,j) \in \mathcal{I}, \\ \widehat{X}_{ij}^{rbns.pres} + \widehat{X}_{ij}^{ibnr}, & \text{if } (i,j) \in \mathcal{J}_1. \end{cases}$$

We easily see that payments square (S_{ij}) has multiplicative mean $E[S_{ij}] = \widetilde{\alpha}_i \widetilde{\beta}_j$. Therefore, we can use (S_{ij}) to completely replace $X_{\mathcal{I}}$ to estimate the DCL parameters (cf. (2.7)).

Note that in the BDCL method we were only able to balance the estimator of the inflation parameter $\widetilde{\gamma}_i$ (cf. (2.11)). Again, while in the BDCL method, we use the expert knowledge to only adjust the inflation parameters. Here, we can take full advantage of the triangle $I_{\mathcal{I}}$ and also equalize the delay parameters.

Since (S_{ij}) has a multiplicative structure, we use the CLM idea to estimate $\widetilde{\alpha}_i$ and $\widetilde{\beta}_j$. We define

$$\widehat{\widetilde{\alpha}}_i^{PDCL} = \sum_{j=0}^{m-1} S_{ij}, \qquad \widehat{\widetilde{\beta}}_j^{PDCL} = \frac{\sum_{i=1}^{m} S_{ij}}{\sum_{(i,j)\in\mathcal{I}\cup\mathcal{J}_1} S_{ij}}.$$

Exactly as in the previous sections, we can now apply (2.8) and (2.9) to derive the PDCL parameters $(\widehat{\alpha}_i, \widehat{\beta}_j, \widehat{\pi}^{PDCL}, \widehat{\gamma}^{PDCL^*}, \widehat{\mu}^{PDCL})$. Since this approach is still not RBNS preserving, we balance $\widehat{\gamma}^{PDCL^*}$ by defining a new scaled inflation factor estimate $\widehat{\gamma}^{PDCL}$ such that

$$\widehat{\gamma}^{PDCL} = \frac{\sum_{j=0}^{m-i} I_{ij} - \sum_{j\in\mathcal{J}_2(i)} \widehat{X}_{ij}^{rbns} - \sum_{j=0}^{m-i} X_{ij}}{\widehat{X}_{ij}^{rbns}},$$

where $\widehat{X}_{ij}^{rbns}$ is calculated with the parameters $(\widehat{\alpha}_i, \widehat{\beta}_j, \widehat{\pi}^{PDCL}, \widehat{\gamma}^{PDCL^*}, \widehat{\mu}^{PDCL})$ using (2.4).

2.4.4 The IDCL method

One could look at the methods BDCL and PDCL as belonging to the tradition of reserving literature using paid-incurred information, see Happ and Wüthrich (2013), Merz and Wüthrich (2013) and Peters, Dong, and Kohn (2014). In the BDCL definition, we incorporate an additional triangle of incurred claims in order to produce a more stable estimate of the underwriting inflation parameter γ_i. The derived BDCL method becomes a variant of the Bornhuetter-Ferguson technique using prior knowledge contained in the incurred triangle. In the PDCL method, we use the additional information to get better IBNR estimates while preserving the RBNS estimates given by the claims department. But now, one natural question is whether one of those derived reserve estimates is the classical incurred chain ladder. However, this is not the case and neither the BDCL nor the PDCL method is replicating the results obtained by applying the classical CLM to the incurred triangle.

Among practitioners, the incurred reserve seems to be more realistic for many datasets compared to the classical paid chain ladder reserve. From this motivation Agbeko et al. (2014) have introduced a new method to estimate the DCL parameters which completely replicates the chain ladder reserve from incurred data. The method is called incurred double chain ladder (IDCL) and it is easily defined just by rescaling the underwriting inflation parameter estimated from the DCL method. Specifically, a new scaled inflation factor estimate $\widehat{\gamma}^{IDCL}$ is defined by

$$\widehat{\gamma}_i^{IDCL} = \frac{\widehat{R}_i^*}{\widehat{R}_i}\widehat{\gamma}_i,$$

where R_i^* are the outstanding loss liabilities per underwriting year as predicted by applying the classical CLM on the incurred data, $\widehat{\gamma}_i$ are the inflation parameters estimated using the DCL method and R_i are the outstanding loss liabilities per accident year calculated using the parameters estimated by the DCL method (see Section 2.4.1).

The final IDCL estimates of the DCL parameters are then $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}^{IDCL}, \widehat{\mu})$. With the new inflation parameter estimate, $\widehat{\gamma}^{IDCL}$, the outstanding liabilities derived by the IDCL estimates of the parameters completely replicate the CLM forecasts on the incurred triangle.

Figure 2.3 shows a plot of the four severity inflation parameters derived by DCL, BDCL, PDCL and IDCL. The impression is that the rather rough adjustment of the PDCL and IDCL method leads to fluctuations in the estimate. These fluctuations are stronger in the less important and older underwriting years. It coincides with the following intuition. CLM on incurred triangle relies on the RBNS case estimates which are too small in earlier underwriting years. Thus, they lead to volatile estimates of the severity inflation in those years. However, the important most recent underwriting year estimates match the one from BDCL. In the most recent years one gets the impression that IDCL might underestimate the severity inflation.

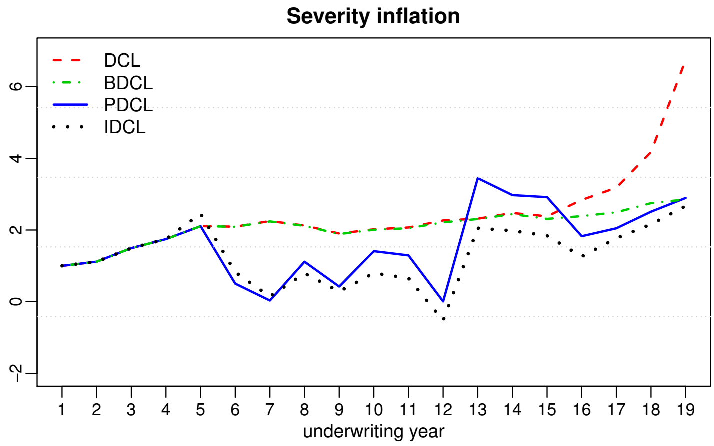

FIGURE 2.3: Plot of severity inflation estimates. DCL: $\widehat{\gamma}_i$ (red), BDCL: $\widehat{\gamma}_i^{BDCL}$ (green), PDCL: $\widehat{\gamma}_i^{PDCL}$ (yellow), IDCL: $\widehat{\gamma}_i^{IDCL}$ (blue).

Table 2.2 shows the reserve estimates per underwriting year derived with the four different methods. In Figure 2.3, it is visualized that the underwriting inflation parameters of PDCL and IDCL might be too volatile in the first five years. However, these first five years have nearly no impact and account for far less than 0.1% of the total loss liabilities estimates. The very most recent years on the other hand account for the very major part of the outstanding liabilities. The unrealistic severity inflation of the DCL method in the most recent underwriting year nearly doubles the ultimate estimates. More realistic results are derived when incorporating the expert knowledge in form of the incurred triangle, $I_{\mathcal{I}}$, using BDCL, PDCL or IDCL.

i	CLM	DCL	BDCL	IDCL	PDCL
1	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0005	0.0003	0.0005	0.0003
3	0.0000	0.0001	-0.0003	0.0001	0.0037
4	0.0000	0.0007	-0.0014	0.0007	-0.0096
5	0.0173	0.0039	0.0151	0.0045	0.0365
6	0.0346	0.0309	0.0313	0.0122	0.0067
7	0.1381	0.1407	0.1408	0.0090	0.0039
8	0.2449	0.2485	0.2483	0.0927	0.0970
9	0.3522	0.3582	0.3563	0.0536	0.0642
10	0.3943	0.3818	0.3800	0.1507	0.2024
11	0.5524	0.5246	0.5206	0.1664	0.2597
12	0.6839	0.6309	0.6169	-0.1458	0.0299
13	1.0504	0.9764	0.9733	0.8648	1.2578
14	2.5361	2.5483	2.5164	2.0388	2.8155
15	5.7370	5.4483	5.2846	4.2095	6.2722
16	14.0889	15.4373	12.9824	6.8542	9.4736
17	21.0057	21.7407	17.0455	12.0924	13.5066
18	44.6877	44.4580	29.2840	23.0002	26.1823
19	98.9723	98.9722	41.8444	39.1522	42.6500
SUM	190.4957	191.9021	112.2385	88.5565	102.8528

TABLE 2.2: Outstanding loss liabilities per underwriting year in million

2.5 Model validation

This section describes the validation process for the four methods DCL, BDCL, IDCL and PDCL discussed in Section 2.4.

The validation process is based on back-testing data previously omitted while estimating the parameters for each method. More precisely, we cut off the most recent diagonals of the data triangles, which are the calendar years, in order to get smaller triangles to which we can apply the different reserving methods. Then we compare the forecasts for these diagonals to the original data. This validation technique is described in detail by Agbeko et al. (2014).

Below, we have omitted the most recent calendar year and the four most recent calendar years, respectively (in all three available triangles). Therefore, since our dataset consists of $m = 19$ years, there are 18 and 60 cells, respectively, to be compared with the true values.

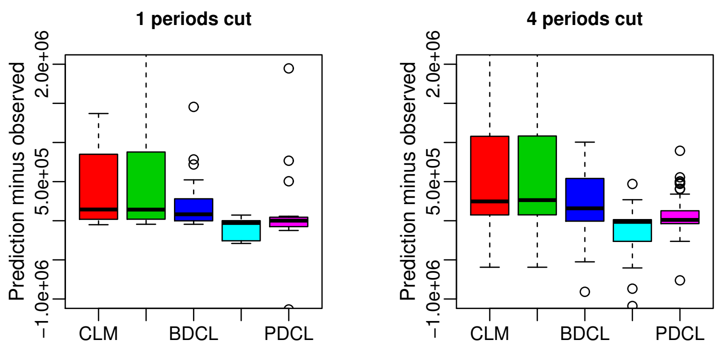

FIGURE 2.4: Box plot of the cell errors (CLM, DCL, BDCL, IDCL, PDCL from left to right)

Figure 2.4 shows two box plots of the respectively 18 and 60 errors calculated by taking the difference between estimated and true value. While we have also tried to omit different amounts of calendar years, the results were all similar and quiet clear. The three methods incorporating expert knowledge, that is BDCL, IDCL and PDCL, outperform the CLM and DCL method which only work with real data.

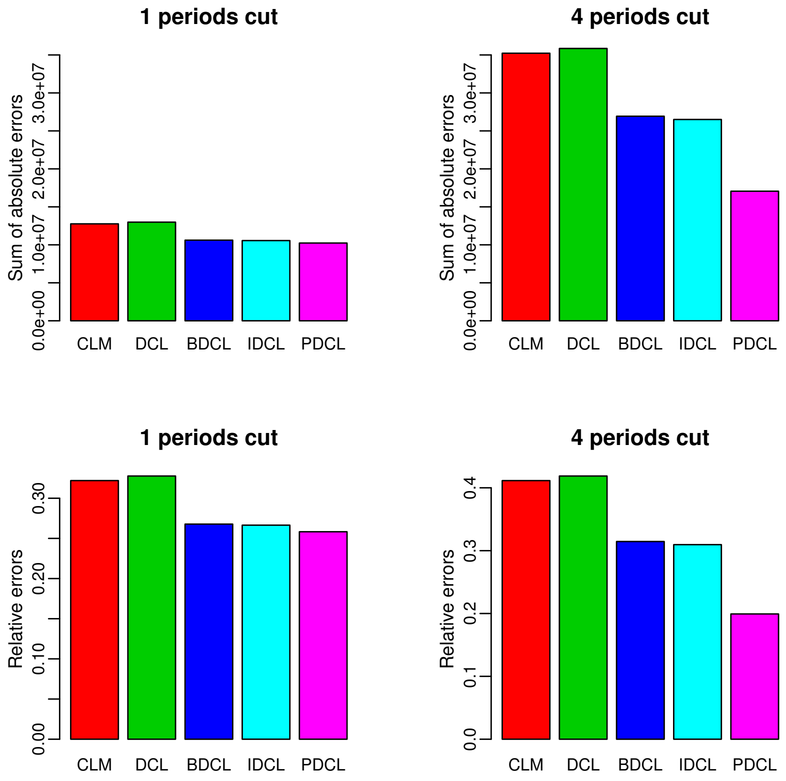

FIGURE 2.5: Bar plot for the sum of absolute cell errors and the relative errors.

In the top panels of Figure 2.5, we have plotted the sum of the absolute cell errors (ℓ_1 error). That is,

$$\text{Sum of absolute cell errors} = \sum_{(i,j)\in\mathcal{B}} |\widehat{X}_{ij} - X_{ij}|,$$

$$\mathcal{B} = \{(i,j)|\ i = 2,\dots,m-c;\ j = 0,\dots,m-c-1;\ \ i+j = m-c+1,\dots,m\},$$

where c is the number of recent calendar years omitted for back testing (here: 1 and 4).

The relative errors, that is

$$\frac{\text{Sum of absolute cell errors}}{\text{Sum of absolute true values}} = \frac{\sum_{(i,j)\in\mathcal{B}} |\widehat{X}_{ij} - X_{ij}|}{\sum_{(i,j)\in\mathcal{B}} |X_{ij}|},$$

are shown in the bottom panels of Figure 2.5. The conclusion is the same as in the box plots. The estimates of BDCL, IDCL, PDCL are more accurate, while no great distinction can be made in between those winners.

2.6 Continuous Chain Ladder

This section is a motivating section. The message of this section is: when double chain ladder is extended, then it is also a contribution to granular reserving. This section gives a very short introduction of recent research interpreting the chain ladder model as a structured histogram. We do not provide theory here. We just give a taste of this new interpretation of chain ladder and its potential.

Continuous chain ladder was first published in Martínez-Miranda et al. (2013), where it is verified that the classical reserving problem really is a multivariate density estimation problem and that the classical chain ladder technique is a structured histogram version of this density estimator. While histograms are a good choice, it is well known from smoothing theory that one can do better by introducing more smoothing. Also, many actuaries use the chain ladder method without realising that when they choose weekly, monthly, quarterly or yearly data, they are really picking a smoothing parameter which could be optimized via validation methodology.

Natural extension of classical chain ladder methodology would be to smooth it via kernel smoother or some other smoothers. Hereby, one takes advantage on the vast literature of mathematical statistics, when deciding the amount of smoothing (week, month, quarter, year or something completely different) and perhaps allow one-self - in full consistency with the literature - to vary the smoothing according the difference of information at different underwriting years. Martínez-Miranda et al. (2013) introduces these ideas and call the approach continuous chain ladder. In its simplest version, continuous chain ladder is based on simple kernel smoothers providing intuitive and natural improvement to histograms.

Martínez-Miranda et al. (2013) and Mammen, Martínez-Miranda, and Nielsen (2015) consider the multiplicative density model $f(x, y) = f_1(x)f_2(y)$, where f_1 is the density in underwriting direction (corresponding to α) and f_2 the density in development direction (corresponding to β). This is analogue to the chain ladder method where the multiplicative assumption $\mathrm{E}[N_{ik}] = \alpha_i\beta_k$ also implies independence of underwriting date effect and reporting delay effect. They estimate these densities via a least-squares or maximum likelihood criterion. Notice that one hereby estimates one-dimensional functions, not parameters. The aim is to estimate the density components $f_1(x)$ and $f_2(y)$ from observations of the two-dimensional density provided in the triangle $\mathcal{I}$ (see definition in Section 2.2).

Classical CLM considers histogram smoothers (with bins corresponding to the accident and delay periods) to estimate both f_1 and f_2. One can use a local linear kernel when estimating the density on the triangle. This will automatically correct for the boundaries. See also Fan and Gijbels (1996) for an explanation of local polynomial estimation in the regression case. The density on the square can then be derived by projecting the triangular density onto the multiplicative space, $f(x, y) = f_1(x)f_2(y)$.

The natural context for continuous chain ladder is of course micro claims data or granular data, however it can still be applied to aggregated data - the data traditionally used in reserving. Now, we illustrate how the continuous chain ladder method can be applied to the paid data described in the previous sections and compared with the classical chain ladder histogram. The input data for both approaches are quarterly-aggregated triangles for 76 quarters (this is 19 years).

Figure 2.6 shows a histogram of the observed payments considering bins of 4 quarters (a year). Such a histogram is the first step in classical CLM which leads to the predicted cash-flow plotted in Figure 2.7. Continuous chain ladder replaces this yearly histogram with a more efficient local linear kernel density estimator shown in the left panel of Figure 2.8. A functional projection of this two-dimensional density down on a multiplicative space derives the smooth cash-flow shown in the right panel of Figure 2.8. While the two approaches are quite similar, however, the chain ladder histogram approach results in piece-wise constant functions as the shown in Figure 2.9, while continuous chain ladder indeed results in the continuous functions shown also in Figure 2.9.

2.7 Conclusions

This paper has developed a new method called PDCL, which combines classical chain ladder methodology with expert knowledge via the double chain ladder methodology. While the preceding IDCL method is able to replicate the incurred chain ladder reserves, which are most commonly used in practice, the new PDCL method replicates the exact expert knowledge of the claims handling department via the estimated RBNS reserves.

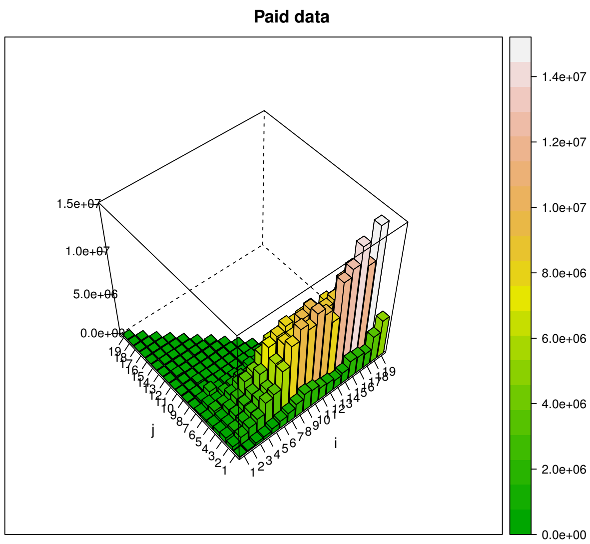

FIGURE 2.6: Histogram of the paid data using yearly bins: the starting point for classical CLM.

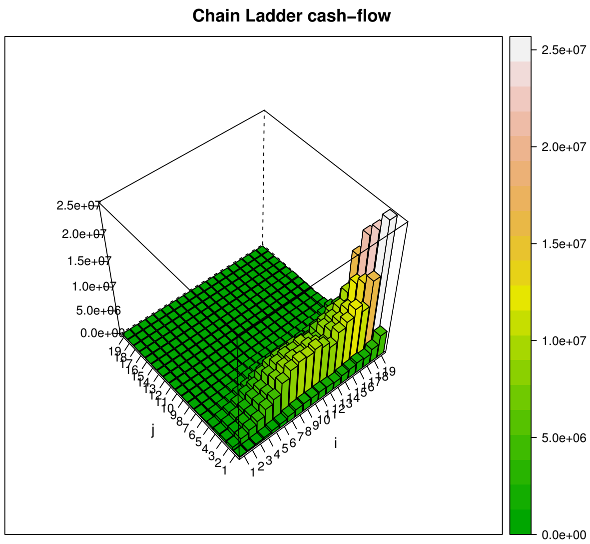

FIGURE 2.7: Classical chain ladder forecasts.

Among a number of advantages, both PDCL and IDCL methods inherit the good mathematical statistical properties of the double chain ladder methodology including a full statistical model and a stochastic cash flow interpretation. This in turn allows for a validation procedure cutting of recent payments and forecasting them. Such a validation procedure between paid chain ladder (or DCL) and incurred chain ladder (or IDCL) have hitherto not been available.

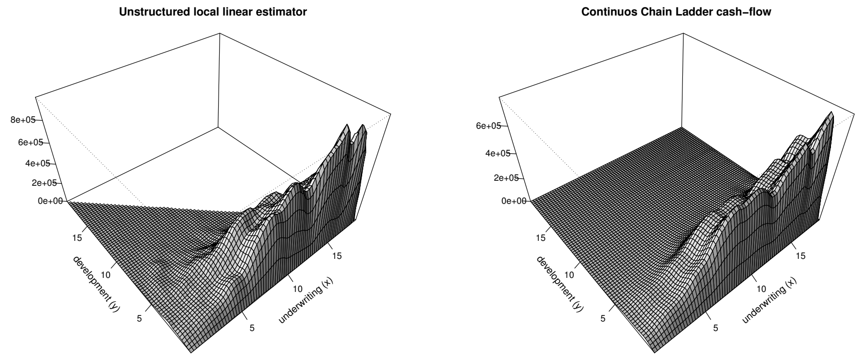

FIGURE 2.8: The Continuous chain ladder approach. Left panel shows the local linear kernel density estimator based on the observed data. Right panel shows the forecasts calculated assuming a multiplicative structure.

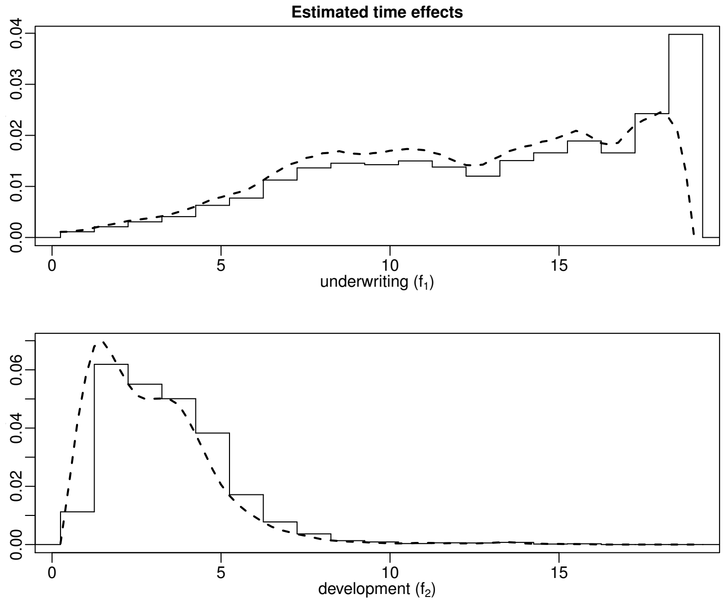

FIGURE 2.9: Estimated density components. Top panel shows the underwriting component and bottom panel the development component. The smooth kernel estimates derived by continuous chain ladder are compared with the histograms provided by classical CLM.

We believe that our new results can upgrade the scientific quality of model selection in the perhaps most important single modelling process of a general insurance company. Now a scientifically based validation exist between DCL, BDCL, IDCL and PDCL, where the three latter are various version of combining expert knowledge with observed payment data. Finally, we have pointed out the close link between our methodology and granular reserving indicating that the insights of this paper could be transferred to granular reserving. Another recent trend is to use so called granular data or micro data for reserving, see Antonio and Plat (2014) for one of the most interesting recent contributions in that area.

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3

Cash flow generalisations of general insurance expert systems estimating outstanding liabilities

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Cash flow generalisations of general insurance expert systems estimating outstanding liabilities

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Abstract

For as long as anyone remembers general insurance companies have used the so called chain ladder method to reserve for outstanding liabilities. When historical payments of claims are used as observations then chain ladder can be understood as estimating a multiplicative model. In most general insurance companies a mixture of paid data and expert knowledge, incurred data, is used as observations instead of just payments. This paper considers recent statistical cash flow models for asset-liability hedging, capital allocation and other management decision tools, and develops two new such methods incorporating available incurred data expert knowledge into the outstanding liability cash flow model. These two new methods unbundle the incurred data to aggregates of estimates of the future cash flow. By a re-distribution to the right algorithm, the estimated future cash flow is incorporated in the overall estimation process and considered as data. A statistical validation technique is developed for these two new methods and they are compared to the other recent cash flow methods. The two methods show to have a very good performance on the real-life data set considered.

Keywords: Stochastic Reserving; General Insurance; Chain Ladder; Claims Inflation; Incurred Data; Model Validation.

3.1 Introduction

The general insurance business is an important part of the economy for most developed countries with market revenues amounting to around five percent of GNP's. The best estimate of outstanding liabilities - often called the reserve - is perhaps the single most important number on the balance sheet of most general insurance companies. Insufficient reserves is one common reason for general insurance companies to go broke. A mismanaged reserving process can also lead to spurious and volatile yearly results leading to uninformed management decisions. Finally, a smoother and more transparent reserving process leads to significant cost savings in almost any general insurance company.

Therefore, it is perhaps surprising that statistical models for the most often used data set, the incurred run off triangle, are rarely considered in the literature. Incurred data is a mixture of historical payments of already settled claims and predicted severities of reported but not settled claims. The predicted severities are based on all available expert opinion in the company and are called case estimates. In that spirit, incurred data has added information about reported claims which are not available when only historical payments are considered.

Actuaries often prefer the incurred triangle to the triangle of historical payments since its predicted reserve seems in many cases more reliable. In practice, the chain ladder model developed for historical paid data is hereby used directly on incurred data. Nonetheless the maybe most used methodology for reserve estimation - the chain ladder method on incurred data - has not been considered in the literature. While chain ladder probably makes good sense in a deterministic framework, the stochastic nature of the expert opinion type of incurred data is not been taken into account through such a practise. There is a little literature acknowledging the added value of incurred data: the probably most famous Munich chain ladder approach by Quarg and Mack (2004), regression approaches by Halliwell (1997), Halliwell (2009), Venter (2008), and a paid-incurred chain reserving method by Posthuma et al. (2008), Merz and Wüthrich (2010), Happ, Merz, and Wüthrich (2012) and Happ and Wüthrich (2013). The aim in those papers is to combine payment data and incurred data into one statistical model which then results in one reserve estimate. The idea is that all available information, that is historical payments and incurred data triangles, is used in a consistent and reproducible way which then also enables to asses prediction uncertainty. They all, however, do not model the structural relationship between these both data sets even though both data sets aggregate the same historical payments in one way or another. More precisely, the difference between those two triangles is that they are aggregated in different time scales. Incurred data counts the delay of every claim until the report is done while the paid data is counting the time until the final payment.

To model this relationship one needs micro-structural assumptions about the underlying claim process. Pigeon, Antonio, and Denuit (2014) does exactly this. It is based on a recent trend to use so called granular data or micro data for reserving, see Antonio and Plat (2014) for one of the most interesting recent contributions in that area. See also Martínez-Miranda et al. (2013) for a continuous interpretation of the classical chain ladder methodology. While these approaches indeed seem to be favorable, they are not well established yet and rely on data and granular information that is most often not available at hand.

Double chain ladder, introduced in Martínez-Miranda, Nielsen, and Verrall (2012), builds on micro-structural assumptions but does not need granular data in the estimation procedure. It is based on the methods of Verrall, Nielsen, and Jessen (2010) and Martínez-Miranda et al. (2011) where the objective was to only rely on data that is already available in most reserving departments. It uses additional information of claim counts (or often called frequencies) which is another triangle most often available in the data portfolio of a reserving department. The result is a full statistical model based on historical payments and counts capable of incorporating the information of the incurred data in a natural way.

Agbeko et al. (2014) recently introduced a model reproducing the deterministic results of the earlier mentioned chain ladder method on incurred data by incorporating the incurred data expert opinion into the well-defined full stochastic model of double chain ladder. One direct advantage of this approach is that the chain ladder model based on paid data can be validated against the chain ladder model based on incurred data. While the paid chain ladder and incurred chain ladder methods have been available for a long time as part of almost any general actuary's tool kit, it has never before been possible to compare them against each other when only the typical aggregated data were available.

Two other methods of incorporating expert knowledge of incurred data into these full cash flow models have been introduced in Martínez-Miranda, Nielsen, and Verrall (2013a) and Hiabu et al. (2016). The first of these two methods is extracting the inflation of the cost of a single claim from the incurred data and then incorporates that information in the double chain ladder model of Martínez-Miranda, Nielsen, and Verrall (2012). The second of these two methods suggests to incorporate a RBNS-preserving property. RBNS stands for Reported But Not yet Settled, and it can be estimated by the sum of all case estimates. This estimate is the best the claims department of an insurance company (with all the expert knowledge on the nature and severity of each claim available in such a department) is able to do. Hiabu et al. (2016) therefore produced a version of double chain ladder reproducing exactly the expert judgement of the RBNS reserves. Note that other non granular methods which do not exclusively rely on incurred data, are not able to separate the RBNS part from the reserve.

All those mentioned double chain ladder extensions take advantage of the underlying structure of the incurred data, extract the relevant information from it and plug it into the original double chain ladder method. The advantage of this approach is that the simplicity and intuition of the simple chain ladder method is preserved and that the full statistical interpretation and stochastic cash flow formulation is inherited from double chain ladder.

The two new stochastic cash flow methods developed in this paper both build on the ideas and techniques of Hiabu et al. (2016). The first treats the expert knowledge of the incurred data as real data and incorporates it in the model; the second builds a second RBNS preserving cash flow model on top of this method. The idea is to unbundle the incurred data to aggregates of estimates of the future cash flow, that is the so called aggregated reported but not settled claims. These aggregated numbers are re-distributed according to the estimated delay such that the resulting algorithm takes both historical data and expert data into account in the final estimation. We therefore let the estimated future cash flow be incorporated in the overall estimation process by considering it as data. Note that both these two new methods are cash flow models of the same nature as the models considered in Agbeko et al. (2014), and they can therefore be validated and compared to the models considered there. In the applied data example, this validation indicates, that the two new methods seem to take better advantage of the incurred expert data than previous methods did.

Recent years have seen a growing interest in expert systems related to general insurance, see for example Belles-Sampera, Guillén, and Santolino (2014) and Abbasi and Guillén (2013), who consider ways of understanding risk in general insurance. Guelman and Guillén (2014) work with pricing of insurance claims and the customers sensitivity to that price and Guillén et al. (2012) and Kaishev, Nielsen, and Thuring (2013) transfer knowledge from one business line to another to optimize cross-selling. Human judgement is important in all these insurance applications. When prices are set, there is a business intelligence department evaluating how much weight to put on the model at hand and how much weight to put on market prices as such. When risk is evaluated, human judgement calibrates the entering parameters. And when RBNS claims reserves are set, then there is an element of human judgement in the settlement of every single claim. It is also a human judgement when it is decided to use model based claims reserves for some subset of the claims, for example the smaller ones.

We conclude this introduction by noting that Martínez-Miranda, Nielsen, and Verrall (2012) has two versions of double chain ladder; one version where the delay is not adjusted and another where it is adjusted. In this paper only the unadjusted version of double chain ladder is considered. One reason for the adjustment of the delay in Martínez-Miranda, Nielsen, and Verrall (2012) was to improve the performance of estimating the out-of-sample tail reserve. While this is a very important issue, it is beyond the scope of this paper to consider the out-of-sample tail reserve.

The rest of the paper is structured as follows. Section 3.2 describes the data and the expert knowledge, introduces the notation and defines the model assumptions. Section 3.3 discusses the outstanding loss liabilities point estimates. Section 3.4 describes four methods to estimate the parameters in the model: DCL, BDCL, PDCL, IDCL, EDCL and PEDCL. An application is considered in Section 3.5 and the validation of the six methods against each other is gone through in Section 3.6. Finally, Section 3.7 provides some concluding remarks.

3.2 Data and first moment assumptions

This chapter introduces the data used in maybe every insurance reserving department to calculate their outstanding liabilities. Also the methods described in this paper rely on these data sets. They are often shortly called run-off triangles. These run-off triangles are the aggregated incurred counts (data), aggregated payments (data) and aggregated incurred payments (mixture of data and expert knowledge). All of those three objects have the same structural form, i.e., they live on the upper triangle

$$\mathcal{I} = \{(i, j) : i = 1, \dots, m, j = 0, \dots, m-1;\ i + j \leq m\}, \quad m > 0,$$

where m is the number of underwriting years observed. The parameter m also has another crucial role. If no tail-factors are considered, which will be assumed throughout this paper, then $m-1$ is the maximum delay, that is the time from the underwriting of the policy a claim is based on until its payment. This assumption is called being run-off, hence the name run-off triangles.
Let us first introduce the two data triangles.

Aggregated incremental incurred counts: $N_{\mathcal{I}} = \{N_{ik} : (i,k) \in \mathcal{I}\}$, with N_{ik} being the total number of claims of insurance incurred in year i which have been reported in year $i+k$, i.e. with k periods delay from year i.

Aggregated incremental payments: $X_{\mathcal{I}} = \{X_{ij} : (i,j) \in \mathcal{I}\}$, with X_{ij} being the total payments from claims incurred in year i and paid with j periods delay from year i.

A often confusing point is that the meaning of the second coordinate of the triangle $\mathcal{I}$ varies between the two different data. While in the counts triangle it represents the reporting delay, in the payments triangle it represents the development delay, that is reporting delay plus settlement delay.

The definition of the incurred payments triangles is not that straight forward. To allow for a exact description, we first introduce micro structural variables of the claims process. We hereby follow the line of Martínez-Miranda, Nielsen, and Verrall (2012), since those variables will also play a key role in the underlying assumption of the double chain ladder model.

By N_{ikl}^{paid}, we denote the number of the future payments originating from the N_{ik} reported claims, which were finally paid with a delay of $k+l$, where $l = 0, \dots, m-1$. Also, let $X_{ikl}^{(h)}$ denote the individual settled payments which arise from N_{ikl}^{paid}, $h = 1, \dots, N_{ikl}^{paid}$.

Finally, we define

$$X_{ikl} = \sum_{h=1}^{N_{ikl}^{paid}} X_{ikl}^{(h)}, \quad (i,k) \in \mathcal{I},\ l = 0, \dots, m-1,$$

i.e., those payments originating from underwriting year i, which are reported after a delay of k and paid with an overall delay of $k+l$.

The aggregated incurred payments are then considered as unbiased estimators of $\sum_{l=0}^{m-1} X_{ikl}$. Technically, we model the expert knowledge as follows.

Aggregated incurred payments: $I_{\mathcal{I}} = \{I_{ik} : (i,k) \in \mathcal{I}\}$, with I_{ik} being

$$I_{ik} = \sum_{s=0}^{k} \sum_{l=0}^{m-1} E[X_{isl}|\ \mathcal{F}_{(i+k)}] - \sum_{s=0}^{k-1} \sum_{l=0}^{m-1} E[X_{isl}|\ \mathcal{F}_{(i+k-1)}],$$

where $\mathcal{F}_h$ is an increasing filtration illustrating the expert knowledge at time point h.

In this manuscript, we will only consider best estimates and therefore only need assumptions on the mean. We show that the classical CLM multiplicative structure holds under very general underlying dependencies on the mean. The first moment conditions of the DCL model are formulated below.

For fixed $i = 0, \dots, m; \quad k, l = 0, \dots, m-1,$ and $h = 1, \dots, N_{ikl}^{paid}$, it holds that

A1. The counts N_{ik} are random variables with mean having a multiplicative parametrization $\mathrm{E}[N_{ik}] = \alpha_i \beta_k$, and identification $\sum_{k=0}^{m-1} \beta_k = 1$.

A2. The mean of the RBNS delay variables is $\mathrm{E}[N_{ikl}^{paid} | N_{\mathcal{I}}] = N_{ik} \widetilde{\pi}_l$.

A3. The mean of the individual payments size conditional on the number of payments and the counts is given by $\mathrm{E}[X_{ikl}^{(h)} | N_{ikl}^{paid}, N_{\mathcal{I}}] = \widetilde{\mu}_l \gamma_i$.

Assumption A1 is the classical chain ladder assumption applied on the counts triangle, see also Mack (1991). The main point hereby is the multiplicativity between underwriting year and reporting delay. Assumptions A2 and A3 are necessary to connect reporting delay, settlement delay and development delay - the main idea of DCL. See also Verrall, Nielsen, and Jessen (2010), Martínez-Miranda et al. (2011) and Martínez-Miranda, Nielsen, and Verrall (2012).

Using A1 to A3, we have that

$$\begin{aligned} \mathrm{E}\left[\sum_{h=1}^{N^{paid}_{i,j-l}} X^{(h)}_{i,j-l,l} | N_{\mathcal{I}}\right] &= \mathrm{E}\left[\sum_{h=1}^{N^{paid}_{i,j-l,l}} \mathrm{E}[X^{(h)}_{i,j-l,l}|N_{\mathcal{I}}, N^{paid}_{i,j-l,l}] | N_{\mathcal{I}}\right] \\ &= \mathrm{E}[N^{paid}_{i,j-l,l}\widetilde{\mu}_l\gamma_i|N_{\mathcal{I}}] = N_{i,j-l}\widetilde{\pi}_l\widetilde{\mu}_l\gamma_i. \end{aligned}$$

Note that the observed aggregated payments can be written as

$$X_{ij} = \sum_{l=0}^{j} X_{i,j-l,l} = \sum_{l=0}^{j}\sum_{h=1}^{N^{paid}_{i,j-l,l}} X^{(h)}_{i,j-l,l}.$$

With the previous consideration, we derive

$$\mathrm{E}[X_{ij}|N_{\mathcal{I}}] = \gamma_i \sum_{l=0}^{j} N_{i,j-l}\widetilde{\pi}_l\widetilde{\mu}_l,$$

and the unconditional mean is

$$\mathrm{E}[X_{ij}] = \alpha_i\gamma_i \sum_{l=0}^{j} \beta_{j-l}\widetilde{\mu}_l\widetilde{\pi}_l. \tag{3.1}$$

Inspecting equation (3.1), we can reduce the amount of parameters by simply setting $\mu = \sum_{l=0}^{j} \widetilde{\pi}_l\widetilde{\mu}_l$ and $\pi_l = \widetilde{\pi}_l\widetilde{\mu}_l\mu^{-1}$, so that $\mu\pi_l = \widetilde{\mu}_l\widetilde{\pi}_l$ and therefore the unconditional mean of the payments becomes

$$\mathrm{E}[X_{ij}] = \alpha_i\gamma_i\mu \sum_{l=0}^{j} \beta_{j-l}\pi_l. \tag{3.2}$$

Equation (3.2) is the key in deriving the outstanding loss liabilities. These are the values of (X_{ij}) in the lower triangle. Consequently in the sequel,

$$(\alpha, \beta, \pi, \gamma, \mu) = (\alpha_1, \ldots, \alpha_m, \beta_0, \ldots, \beta_{m-1}, \pi_0, \ldots, \pi_{m-1}, \gamma_1, \ldots, \gamma_m, \mu)$$

are called the DCL parameters. In the next section, we will see that in a very natural way, we are able to distinguish between RBNS and IBNR claims. This is possible due to the separation of the development delay into the reporting delay β and the settlement delay π.

3.3 Forecast outstanding claims: the RBNS and IBNR reserves and predictive distributions

In this section, we assume that the DCL parameters $(\alpha, \beta, \pi, \gamma, \mu)$ are already derived and show how easily point forecasts of the RBNS and IBNR components of the reserve can be calculated. Note that when calculating the RBNS part, it is possible to replace the parameter (α_i, β_k) by the true value N_{ik}, since the claims are already reported and thus N_{ik} is observed. However, for the IBNR reserves, it is obviously necessary to use all DCL parameters, including the estimates of future numbers of incurred claims $\alpha_i \beta_k$. Using the notation of Verrall et al. (2010) and Martínez-Miranda et al. (2011), we consider predictions over the triangle, $\mathcal{J} = \mathcal{J} = \{i = 2, \ldots, m; j = 0, \ldots, m-1 \text{ with } i + j \geq m+1\}$, illustrated in Figure 3.1.

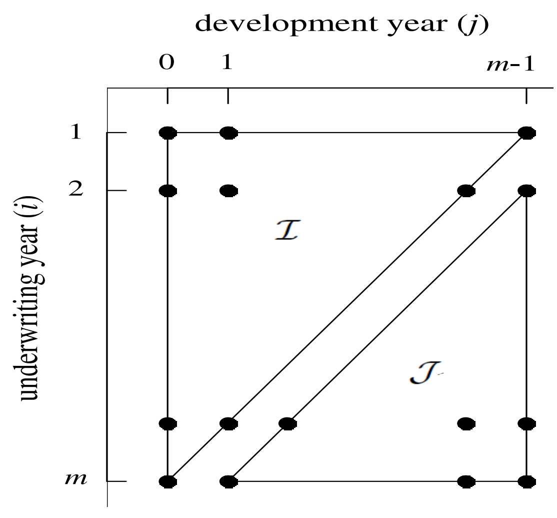

FIGURE 3.1: Index sets for aggregate claims data, assuming a maximum delay of $m-1$.

We define the RBNS component as follows, where we consider two possibilities depending on whether the estimates of N_{ik} are used or not.

$$\widehat{X}_{ij}^{rbns(1)} = \sum_{l=i-m+j}^{j} N_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i, \quad (i,j) \in \mathcal{J}. \tag{3.3}$$

and

$$\widehat{X}_{ij}^{rbns(2)} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i, \quad (i,j) \in \mathcal{J}, \tag{3.4}$$

where $\widehat{N}_{ij} = \widehat{\alpha}_i\widehat{\beta}_j$. In most cases, to shorten the notation, we will simply write $\widehat{X}_{ij}^{rbns}$ for the RBNS estimates. However, whenever it is necessary, we will state which version is taken. The IBNR component always needs all DCL parameters:

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l}\widehat{\pi}_l\widehat{\mu}\widehat{\gamma}_i, \quad (i,j) \in \mathcal{J}. \tag{3.5}$$

The outstanding loss liabilities point estimates are then,

$$\widehat{X}_{ij} = \widehat{X}_{ij}^{rbns} + \widehat{X}_{ij}^{ibnr} \tag{3.6}$$

The outstanding liabilities per accident year are the row sums of IBNR and RBNS estimates. For a fixed i, we write $\mathcal{J}(i) = \{j : (i,j) \in \mathcal{J}\}$. Then the outstanding liabilities per accident year $i = 1, \ldots, m$ are

$$\widehat{R}_i = \sum_{j \in \mathcal{J}(i)} \widehat{X}_{ij}^{rbns} + \widehat{X}_{ij}^{ibnr}.$$

In the next section, we describe several methods to derive the DCL parameters.

3.4 Estimation of the parameters in the Double Chain Ladder model

To estimate the outstanding claims and thereby construct RBNS and IBNR reserves, we need to estimate the parameters involved in (3.2). In this section, we explore six different estimators.

3.4.1 The DCL method

The DCL method is the original and maybe most simple method to derive the parameters introduced in the previous section. It was introduced in Martínez-Miranda, Nielsen, and Verrall (2012). To estimate the DCL parameters in (3.2), assumptions on the payments triangle $X_{\mathcal{I}}$ are needed. DCL assumes the assumptions underlying the CLM method.

B1 The payments X_{ij}, with $i = 1, \dots, m$, and $j = 0, \dots, m-1$, are random variables with mean having a multiplicative parametrization:

$$E[X_{ij}] = \widetilde{\alpha}_i \widetilde{\beta}_j, \qquad \sum_{j=0}^{m-1} \widetilde{\beta}_j = 1. \tag{3.7}$$

Finally, merging (3.2) and (3.7), we conclude

$$\alpha_i \gamma_i \mu \sum_{l=0}^{j} \beta_{j-l} \pi_l = \widetilde{\alpha}_i \widetilde{\beta}_k,$$

and identify the parameters by

$$\alpha_i \mu \gamma_i = \widetilde{\alpha}_i, \tag{3.8}$$

$$\sum_{l=0}^{j} \beta_{j-l} \pi_l = \widetilde{\beta}_j. \tag{3.9}$$

Again, many other micro-structure formulations might exist, thus the one specified by (3.8) and (3.9) is only one of several possible. However, the above model can be considered as a detailed specification of the CLM. In Martínez-Miranda, Nielsen, and Verrall (2013a) it is shown that if the RBNS component is calculated by (3.4), DCL completely replicates the results of CLM.

Now, the main idea to derive the DCL parameters is to estimate the chain ladder parameters $(\widehat{\alpha}, \widehat{\beta})$ and $(\widehat{\widetilde{\alpha}}, \widehat{\widetilde{\beta}})$ (cf. A1, B1) by applying the classical chain ladder algorithm on the payments triangle $X_{\mathcal{I}}$ and the counts triangle $N_{\mathcal{I}}$. Afterwards, the parameters left in (3.2) (this is $(\widehat{\gamma}, \widehat{\mu}, \widehat{\pi})$) can be calculated by simple algebra using (3.8) and (3.9). For illustration of the chain ladder algorithm, we assume an incremental triangle (C_{ij})

(in our case this would be $N_{\mathcal{I}}$ or $X_{\mathcal{I}}$), and that we want to estimate its chain ladder parameters $(\widehat{\widehat{\alpha}}, \widehat{\widehat{\beta}})$. To apply the chain ladder algorithm, one has to transform the triangle (C_{ij}) into a cumulative triangle (D_{ij}):

$$D_{ij} = \sum_{k=1}^{j} C_{ik}.$$

Then, the chain ladder algorithm can be applied on (D_{ij}). It will produce estimates of development factors, λ_j, $j = 1, 2, \ldots, m-1$, which can be described by

$$\widehat{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} D_{ij}}{\sum_{i=1}^{n-j+1} D_{i,j-1}}.$$

These development factors can be converted into estimates of $(\overline{\alpha}, \overline{\beta})$ using the following identities which were derived in Verrall (1991).

$$\widehat{\overline{\beta}}_0 = \frac{1}{\prod_{l=1}^{m-1} \widehat{\lambda}_l}$$
$$\widehat{\overline{\beta}}_j = \frac{\widehat{\lambda}_j - 1}{\prod_{l=j}^{m-1} \widehat{\lambda}_l}$$
$$\widehat{\overline{\alpha}}_i = \sum_{j=0}^{m-i} C_{ij} \prod_{j=m-i+1}^{m-1} \widehat{\lambda}_j$$

Alternatively, analytical expressions for the estimators can also be derived directly (rather than using the chain ladder algorithm) and further details can be found in Kuang, Nielsen, and Nielsen (2009).

Once the chain ladder parameters $(\widehat{\alpha}, \widehat{\beta})$ and $(\widehat{\widehat{\alpha}}, \widehat{\widehat{\beta}})$ are derived, the settlement delay parameter π can be estimated just by solving the following linear system.

$$\begin{pmatrix} \widehat{\widehat{\beta}}_0 \\ \vdots \\ \vdots \\ \widehat{\widehat{\beta}}_{m-1} \end{pmatrix} = \begin{pmatrix} \widehat{\beta}_0 & 0 & \cdots & 0 \\ \widehat{\beta}_1 & \widehat{\beta}_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \widehat{\beta}_{m-1} & \cdots & \widehat{\beta}_1 & \widehat{\beta}_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \vdots \\ \pi_{m-1} \end{pmatrix}. \tag{3.10}$$

Let $\widehat{\pi}$ denote the solution of (3.10).

Now we consider the estimation of the parameters involved in the means of individual payments. Of course, the model is technically over-parametrised since there are too many inflation parameters in (3.8). The simplest way to ensure identifiability is to set $\gamma_1 = 1$, and then the estimate of μ, $\widehat{\mu}$, can be obtained from

$$\widehat{\mu} = \frac{\widehat{\widetilde{\alpha}}_1}{\widehat{\alpha}_1}.$$

Using $\widehat{\mu}$, the remaining estimates for γ_i, $i = 2, \ldots, m$, are directly derived from (3.8). The estimation procedure of double chain ladder is already programmed with the language R. We have used the R-package DCL Martínez-Miranda, Nielsen, and Verrall (2013b) to derive Table 3.1, which shows the values of $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\pi}$ and $\widehat{\gamma}$ calculated from real data included in the DCL package.

i	k,l	$\widehat{\alpha}(i)$	$\widehat{\beta}(k)$	$\widehat{\pi}(l)$	$\widehat{\gamma}(i)$
1	0	1078	0.7599	0.0592	1.0000
2	1	1890	0.2097	0.3098	1.1173
3	2	2066	0.0189	0.2032	1.4947
4	3	2353	0.0064	0.1996	1.7461
5	4	3015	0.0016	0.1388	2.1075
6	5	3727	0.0010	0.0440	2.0936
7	6	5057	0.0009	0.0227	2.2495
8	7	6483	0.0007	0.0095	2.1250
9	8	7727	0.0003	0.0018	1.9028
10	9	7134	0.0001	0.0029	2.0197
11	10	7319	0.0001	0.0002	2.0704
12	11	6152	0.0000	0.0026	2.2666
13	12	5242	0.0001	0.0019	2.3157
14	13	6150	0.0000	0.0032	2.4747
15	14	7028	0.0001	-0.0002	2.3829
16	15	6725	0.0000	0.0013	2.8391
17	16	5260	0.0000	-0.0004	3.1815
18	17	5869	0.0000	0.0000	4.1747
19	18	5953	0.0000	0.0000	6.7501
			$\widehat{\mu} = 2579$		

TABLE 3.1: DCL parameter estimates derived by the DCL method

3.4.2 The BDCL method

The CLM and Bornhuetter-Ferguson (BF) methods are among the easiest claim reserving methods and, due to their simplicity, they are two of the most commonly used techniques in practice. Some recent papers on the BF method include Verrall (2004), Mack (2008), Schmidt and Zocher (2008), Alai, Merz, and Wüthrich (2009) and Alai, Merz, and Wüthrich (2010). The BF method introduced by Bornhuetter and Ferguson (1972) aims to address one of the well known weaknesses of CLM, which is the effect outliers can have on the estimates of outstanding claims. Especially the most recent underwriting years are the years with nearly no data and thus very sensitive to outliers. However, these recent underwriting years build the very major part of the outstanding claims. Hence, the CLM estimates of the outstanding liabilities might differ fatally from the true (unknown) values.

Acknowledging this problem, the BF method incorporates prior knowledge from experts and is therefore more robust than the CLM method, which relies completely on the data contained in the run-off triangle $X_{\mathcal{I}}$.

In this section, we briefly summarize the Bornhutter-Ferguson double chain ladder (BDCL) method introduced in Martínez-Miranda, Nielsen, and Verrall (2013a), which mimics BF in the framework of DCL. The BDCL method starts with identical steps as DCL but instead of using the estimate of the inflation parameters, γ and μ, from the triangle of paid claims $X_{\mathcal{I}}$, it deploys expert knowledge in the form of the incurred triangle $I_{\mathcal{I}}$ to adjust the estimation of the sensitive inflation parameter γ. This is done as follows. From assumptions $A2, A3$ and equation (3.8), we easily deduce that

$$E[I_{ik}] = \alpha_i \mu \gamma_i \beta_k = \widetilde{\alpha}_i \beta_k. \tag{3.11}$$

Hence, the incurred triangle $I_{\mathcal{I}}$ has multiplicative mean and its underwriting year factor, $\widetilde{\alpha}$, is identical to the one of the payments triangle $X_{\mathcal{I}}$ (cf. (3.7)). However, its estimation is less sensitive to outliers since it incorporates all incurred claims via expert knowledge. We conclude that we can replace the payments triangle by the incurred payments triangle when we calculate estimates of the inflation parameters, γ, μ, in (3.8). Note that the severity mean μ is going to remain the same since the first rows of $X_{\mathcal{I}}$ and $I_{\mathcal{I}}$ are identical.

Summarized, the BDCL-method can be carried out as follows.

- *Step 1: Parameter estimation.*

  Estimate the DCL parameters $(\alpha, \beta, \pi, \gamma, \mu)$ using the DCL method of Section 3.4.1

with the data in the triangles $N_{\mathcal{I}}$ and $X_{\mathcal{I}}$ and denote the parameter estimates by $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}, \widehat{\mu})$.

Repeat this estimation using the DCL method but replacing the triangle of paid claims $X_{\mathcal{I}}$ by the triangle of incurred data $I_{\mathcal{I}}$. Keep only the resulting estimated inflation parameters, denoted by $\widehat{\gamma}^{BDCL}$.

- *Step 2: BF adjustment.*

  Replace the inflation parameters $\widehat{\gamma}$ from the paid data by the estimate from the incurred triangle, $\widehat{\gamma}^{BDCL}$.

From Step 1 and Step 2, the final BDCL estimates of the DCL parameters are $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}^{BDCL}, \widehat{\mu})$.

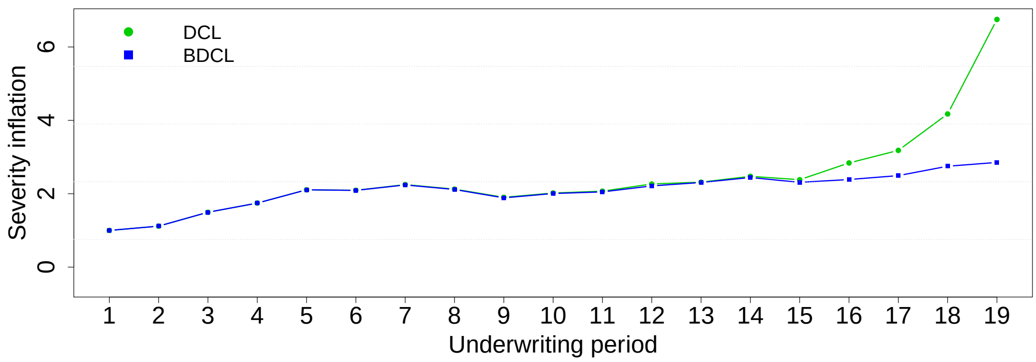

FIGURE 3.2: Plot of severity inflation estimates. DCL: $\widehat{\gamma}_i$ (green), BDCL: $\widehat{\gamma}_i^{BDCL}$ (blue).

Using the R-package DCL Martínez-Miranda, Nielsen, and Verrall (2013b), we derive Figure 3.2 which shows the severity inflation estimates derived by DCL and BDCL. BDCL, with the incorporated expert knowledge, seems to stabilize the severity inflation in the most recent underwriting years while keeping the values in the other years. The result is a more realistic estimate correcting the DCL parameter $\widehat{\gamma}_i$ exactly in its weakest point, that is in those years where the payments triangle $X_{\mathcal{I}}$ has nearly no data. Again, those recent underwriting years contain the very major part of the outstanding liabilities.

3.4.3 The IDCL method

This section gives a brief theoretical introduction to the IDCL method of Agbeko et al. (2014). In the BDCL definition, we incorporated an additional triangle of incurred claims in order to produce a more stable estimate of the underwriting inflation parameter γ_i. The derived BDCL method becomes a variant of the Bornhuetter-Ferguson technique using prior knowledge contained in the incurred triangle. One natural question is whether the derived reserve estimate is the classical incurred chain ladder. The answer is that this not the case; the BDCL method does not replicate the results obtained by applying the classical chain ladder method to the incurred triangle. Among practitioners, the incurred reserve seems to be more realistic for many datasets compared to the classical paid chain ladder reserve. IDCL mimics the reserve estimate of chain ladder on the incurred triangles in the DCL framework. It is defined just by rescaling the underwriting inflation parameter estimate from the DCL method. Specifically, we define a new scaled inflation factor estimate $\widehat{\gamma}^{IDCL}$ such that

$$\widehat{\gamma}_i^{IDCL} = \frac{\widehat{R}_i^*}{\widehat{R}_i}\widehat{\gamma}_i,$$

where R_i^* is the outstanding loss liabilities per accident year as predicted by applying the traditional CLM on incurred data, and $(R_i,\widehat{\gamma}_i)$ are the outstanding loss liabilities per accident year and the inflation parameter respectively, using the DCL method (cf. Section 3.4.1).

The final IDCL estimates of the DCL parameters are then $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}^{IDCL}, \widehat{\mu})$. With the new inflation parameter estimate $\widehat{\gamma}^{IDCL}$, the outstanding liabilities derived by the DCL parameters completely replicate the CLM estimates on the incurred triangle.

3.4.4 The PDCL method

This section gives a brief theoretical introduction to the PDCL method introduced in Hiabu et al. (2016). See also Nielsen (2016). In the last section, we have described a method which incorporates expert knowledge in form of the incurred triangle $I_{\mathcal{I}}$. The values in $I_{\mathcal{I}}$ arise from case estimates for RBNS claims, developed in the case department of the insurance company, and claims which are already paid. Thus, if one subtracts these already paid claims (which are given via the payments triangle $X_{\mathcal{I}}$) from the incurred triangle, one can reconstruct the RBNS case estimates. However, as soon as this is done, it is obvious that these RBNS case estimates do not match with the RBNS estimates (3.3) and (3.4), using any DCL method (including BDCL). We conclude that the reserve department, using double chain ladder (and also chain ladder), calculates different RBNS estimates than those given by the case department.

PDCL replaces the calculated RBNS estimates with the case estimates. The method, however, does not only preserve the RBNS case estimates by adding the IBNR estimates to conclude the reserve, it also takes the RBNS case estimates to correct the other DCL parameters. Therefore, also the total IBNR size will change.

The first step is to construct a preliminary square (S_{ij}), $i = 1, \ldots, m$, $\quad j = 0, \ldots, m-1$, which yields new estimators for the DCL parameters. The upper triangle of the square (i.e., $(i, j) \in \mathcal{I}$) should have the same entries as the payments triangle (X_{ij}). The lower triangle (i.e., $(i, j) \in \mathcal{J}$) should consist of preliminary estimates of the outstanding loss liabilities. The outstanding loss liabilites comprise an RBNS and an IBNR part (cf. (3.6)). However, only the IBNR part of these outstanding loss liabilities is estimated. For the RBNS component, the RBNS case estimates are taken. More precisely, one takes the DCL parameter estimates $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}, \widehat{\mu})$ and use these parameters to estimate the RBNS component $(\widehat{X}_{ij}^{rbns})$ and IBNR component $(\widehat{X}_{ij}^{ibnr})$ using (3.4) and (3.5). As mentioned above, the RBNS estimate should be be equal to the RBNS case estimates, which can only be reconstructed per accident year. For $i = 1, \ldots, m$, they can be described as

$$X_i^{rbns.case.estimate} = \sum_{j=0}^{m-i} I_{ij} - \sum_{j=0}^{m-i} X_{ij}. \tag{3.12}$$

Hence, we define the RBNS preserving components

$$\widehat{X}_{ij}^{rbns.pres} = \frac{\sum_{j=0}^{m-i} I_{ij} - \sum_{j=0}^{m-i} X_{ij}}{\sum_{j \in \mathcal{J}(i)} \widehat{X}_{ij}^{rbns}} \widehat{X}_{ij}^{rbns}.$$

Note that

$$\sum_{j \in \mathcal{J}(i)} \widehat{X}_{ij}^{rbns.pres} = X_i^{rbns.case.estimate}.$$

Thus we define the preliminary square (S_{ij}) as

$$S_{ij} = \begin{cases} X_{ij}, & \text{if } (i, j) \in \mathcal{I}, \\ \widehat{X}_{ij}^{rbns.pres} + \widehat{X}_{ij}^{ibnr}, & \text{if } (i, j) \in \mathcal{J}. \end{cases}$$

One can easily see that payments square (S_{ij}) has approximately multiplicative mean $E[S_{ij}] \approx \widetilde{\alpha}_i \widetilde{\beta}_j$. Therefore, we can use (S_{ij}) to completely replace $X_{\mathcal{I}}$ to estimate the DCL parameters (cf. (3.7)). Note that in the BDCL method we were only able to balance the estimator of the inflation parameter $\widetilde{\gamma}_i$ (cf. (3.11)). Again, while in the BDCL method, one uses the expert knowledge to only adjust the inflation parameters, here, one can take full advantage of the triangle $I_{\mathcal{I}}$ and also correct the delay parameters.

Since (S_{ij}) has a multiplicative structure, the CLM idea is used to estimate $\widetilde{\alpha}_i$ and $\widetilde{\beta}_j$. We define

$$\widehat{\widetilde{\alpha}}_i^{PDCL} = \sum_{j=0}^{m-1} S_{ij}, \qquad \widehat{\widetilde{\beta}}_j^{PDCL} = \frac{\sum_{i=1}^{m} S_{ij}}{\sum_{(i,j)\in\mathcal{I}\cup\mathcal{J}} S_{ij}}. \tag{3.13}$$

Exactly as in the previous sections, one can now apply (3.8) and (3.9) to derive the PDCL parameters $(\widehat{\alpha}_i, \widehat{\beta}_j, \widehat{\pi}^{PDCL}, \widehat{\gamma}^{PDCL^*}, \widehat{\mu}^{PDCL})$. Since this approach is still not RBNS preserving, $\widehat{\gamma}^{PDCL^*}$ is balanced by defining a new scaled inflation factor estimate $\widehat{\gamma}^{PDCL}$ such that

$$\widehat{\gamma}^{PDCL} = \frac{\sum_{j=0}^{m-i} I_{ij} - \sum_{j=0}^{m-i} X_{ij}}{\widehat{X}_{ij}^{rbns}}, \tag{3.14}$$

where $\widehat{X}_{ij}^{rbns}$ is calculated with the parameters $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}^{PDCL}, \widehat{\gamma}^{PDCL^*}, \widehat{\mu}^{PDCL})$ using (3.4).

3.4.5 The EDCL method

In this chapter, we introduce a new method called expert double chain ladder (EDCL), indicating that expert knowledge in form of incurred data and RBNS's are incorporated into the system as pseudo data. The idea of the EDCL method is to replicate the basic steps of the previously introduced PDCL method (3.12)-(3.13), but without the adjustment of the severity inflation in (3.14). Instead those steps are iterated until convergence. The iteration forces a homogeneous solution which incorporates the incurred triangle, I, and the payment triangle, X, to one reserve. The discrepancy between estimated RBNS and RBNS provided by the case estimates can be explained by variation of the observations around their mean. Note that given the model assumptions, we are estimating the mean of the RBNS.

The EDCL estimation can be described as follows. In the first step of the iteration, we start with the DCL parameter estimates $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}, \widehat{\gamma}, \widehat{\mu})$, which we denote by $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}^{EDCL,(0)}, \widehat{\gamma}^{EDCL,(0)}, \widehat{\mu}^{EDCL,(0)})$.

In the $k-$th step of the iteration, we take the EDCL parameter estimates we obtained in the $(k-1)-$th step $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}^{EDCL,(k-1)}, \widehat{\gamma}^{EDCL,(k-1)}, \widehat{\mu}^{EDCL,(k-1)})$ and use these parameters to estimate the RBNS component $(\widehat{X}_{ij}^{rbns,(k-1)})$, and IBNR component $(\widehat{X}_{ij}^{ibnr,(k-1)})$, using (3.4) and (3.5).

Then, we calculate the $k-$th RBNS preserving components

$$\widehat{X}_{ij}^{rbns.pres,(k)} = \frac{\sum_{j=0}^{m-i} I_{ij} - \sum_{j=0}^{m-i} X_{ij}}{\sum_{j\in\mathcal{J}(i)} \widehat{X}_{ij}^{rbns,(k-1)}} \widehat{X}_{ij}^{rbns,(k-1)}.$$

Now, we are able to define the $k-$th preliminary square $(S_{ij}^{(k)})$ as

$$S_{ij}^{(k)} = \begin{cases} X_{ij}, & \text{if } (i,j) \in \mathcal{I}, \\ \widehat{X}_{ij}^{rbns.pres,(k)} + \widehat{X}_{ij}^{ibnr,(k-1)}, & \text{if } (i,j) \in \mathcal{J}. \end{cases}$$

Since $(S_{ij}^{(k)})$ has an approximately multiplicative structure, we use the CLM idea to estimate $\widetilde{\alpha}_i$ and $\widetilde{\beta}_j$. We define

$$\widehat{\widetilde{\alpha}}_i^{EDCL,(k)} = \sum_{j=0}^{m-1} S_{ij}^{(k)}, \qquad \widehat{\widetilde{\beta}}_j^{EDCL,(k)} = \frac{\sum_{i=1}^{m} S_{ij}^{(k)}}{\sum_{(i,j)\in\mathcal{I}\cup\mathcal{J}} S_{ij}^{(k)}}.$$

Exactly as in the previous sections, we can now apply (3.8) and (3.9) to derive the EDCL parameters $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}^{EDCL,(k)}, \widehat{\gamma}^{EDCL,(k)}, \widehat{\mu}^{EDCL,(k)})$.

After iterating until convergence, we derive the final EDCL parameters denoted by $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}^{EDCL}, \widehat{\gamma}^{EDCL}, \widehat{\mu}^{EDCL})$.

3.4.6 The PEDCL method

In this section, we introduce the RBNS-preserving expert double chain ladder (PEDCL). As the name suggests, this method builds on the PDCL method by using the EDCL parameters $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}^{EDCL}, \widehat{\gamma}^{EDCL}, \widehat{\mu}^{EDCL})$. As mentioned in the previous section, EDCL does not preserve the RBNS case estimates. The reason was mentioned in the beginning of the previous section and has a parallelism to the two different RBNS estimates in (3.3) and (3.4).

If as in PDCL, one decides not to change the case estimates, one can thus replace the RBNS part by the RBNS derived from the case estimates. As this information is only available per underwriting year, it can be incorporated by changing the inflation parameter accordingly. More precisely, we define a new scaled inflation factor estimate as

$$\widehat{\gamma}^{PEDCL} = \frac{\sum_{j=0}^{m-i} I_{ij} - \sum_{j=0}^{m-i} X_{ij}}{\widehat{X}_{ij}^{rbns}},$$

where $\widehat{X}_{ij}^{rbns}$ is calculated with the parameters $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}^{EDCL}, \widehat{\gamma}^{EDCL}, \widehat{\mu}^{EDCL})$ using (3.4). This new inflation parameter $\widehat{\gamma}^{PEDCL}$ is used to replace $\widehat{\gamma}^{EDCL}$ in the parameter set $(\widehat{\alpha}, \widehat{\beta}, \widehat{\pi}^{EDCL}, \widehat{\gamma}^{EDCL}, \widehat{\mu}^{EDCL})$ when calculating only the RBNS part which preserves the case estimates.

Therefore, in contrast to the previous methods, PEDCL possesses two inflation parameters. Firstly, $\widehat{\gamma}^{PEDCL}$ for the RBNS part, that is when calculating (3.4) and secondly $\widehat{\gamma}^{EDCL}$ for the IBNR part, that is when calculating (3.5). Note that since we are just using the EDCL parameters to calculate the IBNR part, this is exactly the same as in the EDCL method.

3.5 Real Data Application

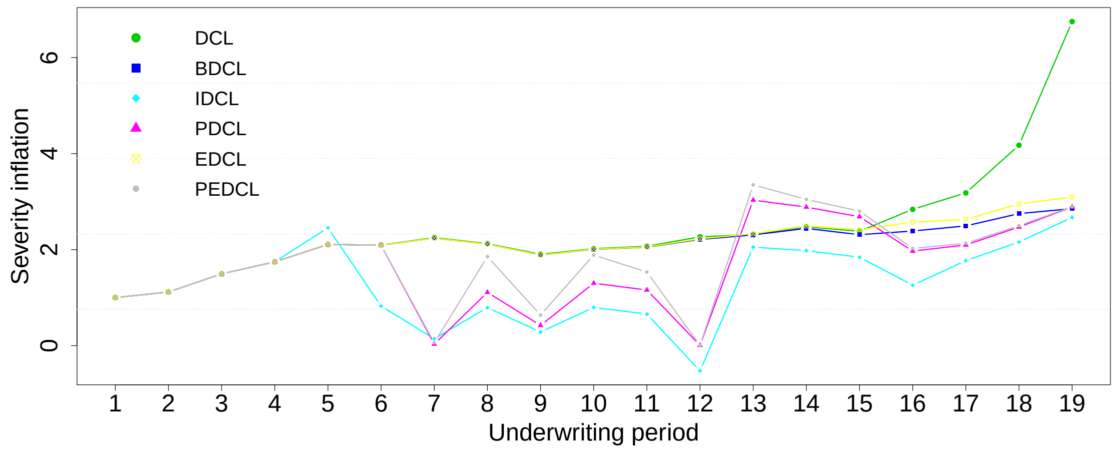

FIGURE 3.3: Plot of severity inflation estimates.

In this section, we apply the methods to a real data set obtained from a UK motor business. The data is also available via the DCL R package Martínez-Miranda, Nielsen, and Verrall (2013b). Figure 3.3 shows a plot of the six severity inflation parameters derived by DCL, BDCL, IDCL, PDCL, EDCL and PEDCL. There were some rather rough corrections going on for IDCL, PDCL and PEDCL in the first five years. These five years represent less than 0.1% of the total loss liabilities estimates (cf. Table 3.2-3.4) and they are not really important. We have therefore taken the first five years out for these three estimation methods. Otherwise, these unimportant first five years would have dominated the graph and perhaps have confused the reader.

Figure 3.3 shows that IDCL, PDCL and PEDCL still are quite volatile with a tendency to have lower severity inflations than DCL, BDCL and EDCL. This is because IDCL, PDCL and PEDCL adjust to high or low RBNS's in certain years, while in particular BDCL and EDCL take a more balanced point of view. From a modelling perspective the severity inflations resulting from BDCL and EDCL seem more attractive, because they are more stable and therefore seem more realistic. This graph illustrates very well why the PEDCL method uses the EDCL parameters while estimating the IBNR reserve. These parameters seem more realistic than the RBNS-preserving parameters. However, the RBNS-preserving parameters might be more realistic when estimating the somewhat realised RBNS reserve. If one believes the RBNS estimates are of good quality or are of the best possible quality one can do with the data, then one should of course use a method preserving these RBNS estimates such as PDCL and PEDCL.

However, for the IBNR's it is another matter. There is no expert knowledge available for the IBNR estimates and therefore, one should use the methodology with the most credible parameters. We believe that EDCL is the method with the most credible parameters and PEDCL is therefore - in our opinion - the optimal methodology if one wishes to preserve the RBNS estimates. If one is looking for a more balanced view, where the RBNS's are allowed to impact the parameters, but where the observed data also should play a role in some sort of validation of the RBNS as data, then one should use the EDCL method. We believe that the DCL method, the EDCL method and the PEDCL method are the three best methods to consider for practising actuaries.

To visualise these arguments, Table 3.2 shows the outstanding loss liabilities per underwriting year for all six methods mentioned in chapter 4 as well as the classical chain ladder method. Furthermore, the splitted values of RBNS and IBNR per underwriting year are illustrated in Table 3.3 and Table 3.4.

i	CLM	DCL	BDCL	IDCL	PDCL	EDCL	PEDCL
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0005	0.0003	0.0005	0.0000	0.0000	0.0000
3	0.0000	0.0001	-0.0003	0.0001	0.0040	0.0001	0.0040
4	0.0000	0.0007	-0.0014	0.0007	-0.0095	0.0003	-0.0095
5	0.0173	0.0039	0.0151	0.0045	0.0365	0.0076	0.0365
6	0.0346	0.0309	0.0313	0.0122	0.0054	0.0135	0.0068
7	0.1381	0.1407	0.1408	0.0090	0.0014	0.0541	0.0040
8	0.2449	0.2485	0.2483	0.0927	0.0949	0.1102	0.0972
9	0.3522	0.3582	0.3563	0.0536	0.0591	0.1774	0.0643
10	0.3943	0.3818	0.3800	0.1507	0.1981	0.2148	0.2024
11	0.5524	0.5246	0.5206	0.1664	0.2515	0.3415	0.2595
12	0.6839	0.6309	0.6169	-0.1458	0.0015	0.3415	0.0300
13	1.0504	0.9764	0.9733	0.8648	1.2718	0.8896	1.2576
14	2.5361	2.5483	2.5164	2.0388	2.8302	2.3205	2.8153
15	5.7370	5.4483	5.2846	4.2095	6.2887	5.4225	6.2721
16	14.0889	15.4373	12.9824	6.8542	9.4207	11.9785	9.4735
17	21.0057	21.7407	17.0455	12.0924	13.4243	16.5608	13.5068
18	44.6877	44.4580	29.2840	23.0002	25.9634	30.6858	26.1851
19	98.9723	98.9722	41.8444	39.1522	42.1009	45.0575	42.8640
SUM	190.4957	191.9021	112.2385	88.5565	101.9427	114.3021	103.0696

Table 3.2: Outstanding loss liabilities per underwriting year in million

i	CLM	DCL	BDCL	IDCL	PDCL	EDCL	PEDCL
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0005	0.0003	0.0005	0.0000	0.0000	0.0000
3	0.0000	0.0001	-0.0003	0.0001	0.0040	0.0001	0.0040
4	0.0000	0.0007	-0.0014	0.0007	-0.0095	0.0003	-0.0095
5	0.0173	0.0039	0.0151	0.0045	0.0365	0.0076	0.0365
6	0.0329	0.0291	0.0295	0.0115	0.0050	0.0118	0.0050
7	0.1355	0.1382	0.1382	0.0088	0.0014	0.0516	0.0014
8	0.2399	0.2436	0.2434	0.0909	0.0923	0.1053	0.0923
9	0.3455	0.3515	0.3496	0.0526	0.0576	0.1707	0.0576
10	0.3822	0.3697	0.3680	0.1459	0.1903	0.2027	0.1903
11	0.5339	0.5062	0.5023	0.1606	0.2411	0.3231	0.2411
12	0.6550	0.6020	0.5887	-0.1392	0.0014	0.4387	0.0014
13	1.0034	0.9294	0.9265	0.8231	1.2101	0.8420	1.2101
14	2.4415	2.4537	2.4229	1.9631	2.7197	2.2249	2.7197
15	5.5906	5.3020	5.1427	4.0964	6.1235	5.2739	6.1235
16	13.8419	15.1902	12.7746	6.7444	9.2492	11.7542	9.2492
17	20.5131	21.2482	16.6593	11.8184	13.0995	16.1534	13.0995
18	42.7693	42.5397	28.0204	22.0078	24.8281	29.3288	24.8281
19	74.0936	74.0942	31.3260	29.3108	31.4544	33.6479	31.4544
SUM	162.5956	164.0027	99.5059	77.1009	89.3045	100.5369	89.3045

Table 3.3: RBNS per underwriting year in million

i	CLM	DCL	BDCL	IDCL	PDCL	EDCL	PEDCL
1-5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0018	0.0018	0.0018	0.0007	0.0004	0.0018	0.0018
7	0.0026	0.0026	0.0026	0.0002	0.0000	0.0026	0.0026
8	0.0049	0.0049	0.0049	0.0018	0.0026	0.0049	0.0049
9	0.0067	0.0067	0.0067	0.0010	0.0015	0.0067	0.0067
10	0.0121	0.0121	0.0120	0.0048	0.0078	0.0121	0.0121
11	0.0184	0.0184	0.0183	0.0058	0.0103	0.0184	0.0184
12	0.0289	0.0289	0.0282	-0.0067	0.0001	0.0285	0.0285
13	0.0471	0.0471	0.0469	0.0417	0.0617	0.0476	0.0476
14	0.0946	0.0946	0.0934	0.0757	0.1105	0.0957	0.0957
15	0.1464	0.1464	0.1419	0.1131	0.1652	0.1486	0.1486
16	0.2470	0.2471	0.2077	0.1097	0.1715	0.2243	0.2243
17	0.4926	0.4925	0.3862	0.2740	0.3248	0.4073	0.4073
18	1.9183	1.9183	1.2636	0.9924	1.1353	1.3570	1.3570
19	24.8787	24.8779	10.5185	9.8414	10.6465	11.4096	11.4096
SUM	27.9001	27.8993	99.5059	11.4556	12.6382	13.7651	13.7651

Table 3.4: IBNR per underwriting year in million

3.6 Model validation

This section describes the validation process for the six methods DCL, BDCL, IDCL, PDCL, EDCL and IPDCL discussed in Section 3.4.

The following validation was introduced in Agbeko et al. (2014) and builds on back-testing. More precisely, we cut off the most recent diagonals of the data triangles, which are the calendar years, in order to get smaller triangles to which we can apply the different reserving methods. Then we compare the forecasts for these diagonals to the original data. To our knowledge this is the only developed method which is able to validate reserving estimates based on incurred data with estimates based on paid data. That is to validate DCL against IDCL in our terminology. However, this validation methodology is sufficiently general to allow all these six procedures to be validated against each other.

The validation process is based on the fact that all introduced DCL methods provide reserve estimates by predicting into the same paid triangle, X. Therefore, all methods can be compared on the same scale so to speak. Below, we have omitted the most recent calendar year and the four most recent calendar years, respectively (in all three available triangles). Therefore, since our dataset consists of $m = 19$ years, there are 18 and 60 cells, respectively, to be compared with the true values.

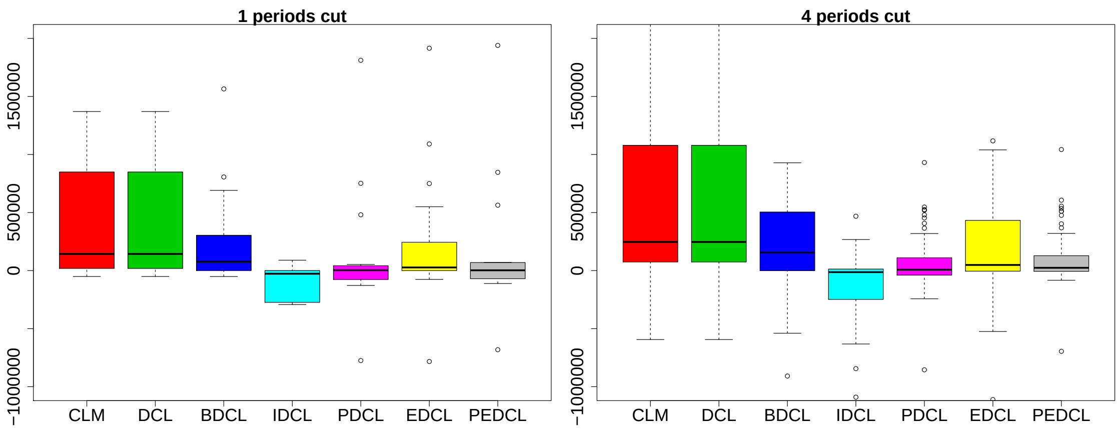

FIGURE 3.4: Box plot of the cell errors

Figure 3.4 shows two box plots of the respectively 18 and 60 errors calculated by taking the difference between estimated and true values. One conclusion is that DCL and CLM seem to be inferior to the five methods that take advantage of expert knowledge.

Between these five methods, it seems that BDCL and EDCL have similar performance with a slight advantage to the new EDCL method. Also PDCL and PEDCL have similar performance, which was to be expected because they only differ in the IBNR reserves while having identical RBNS reserves.

The omnipresent IDCL method does not provide convincing performance in this data example. Earlier studies based on a number of data sets have shown that IDCL sometimes have very good performance, but equally often fails badly. Of the three methods available at the time, DCL, BDCL and IDCL, only the BDCL method had a stable performance. Sometimes DCL was winning convincingly, other times IDCL was the winner. In the long run, the eternal runner-up was the BDCL method, and most of the time it did almost as well - but not quite - as the best of the two other methods.

We do, however, find EDCL a more convincing method than BDCL. While they have some similarities and while both have stable underwriting year severities as we saw in the application, EDCL seems more theoretically correct in its way of exploring the expert knowledge and we believe it will be replacing BDCL in the long run. Based on long-term considerations, we think DCL, EDCL and PEDCL should be sufficient in the practical actuaries tool box. BDCL, IDCL and PDCL are - in our view - less convincing methods.

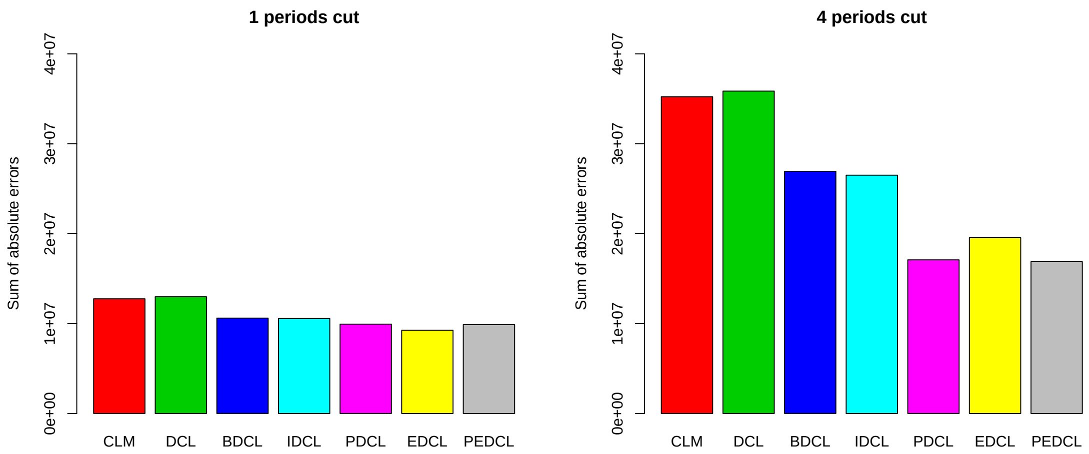

FIGURE 3.5: Bar plot for the sum of absolute cell errors

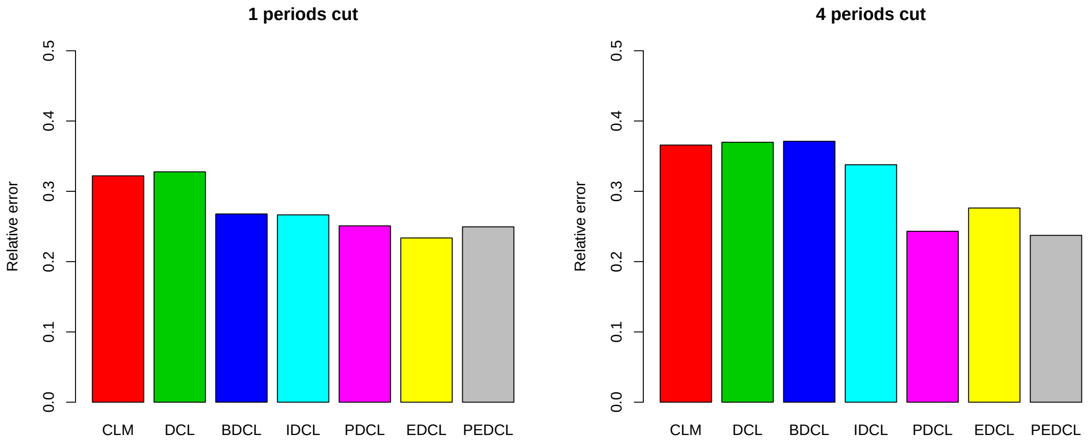

FIGURE 3.6: Bar plot for the relative errors

In the top panels of Figure 3.5, we have plotted the sum of the absolute cell errors (ℓ_1 error). That is

$$\text{Sum of absolute cell errors} = \sum_{(i,j)\in\mathcal{B}} |\widehat{X}_{ij} - X_{ij}|,$$

$$\mathcal{B} = \{(i,j)|\ i = 2,\dots,m-c;\ j = 0,\dots,m-c-1;\ \ i+j = m-c+1,\dots,m\},$$

where c is the number of recent calendar years omitted for back testing (here: 1 and 4).

The relative errors, that is

$$\frac{\text{Sum of absolute cell errors}}{\text{Sum of absolute true values}} = \frac{\sum_{(i,j)\in\mathcal{B}} |\widehat{X}_{ij} - X_{ij}|}{\sum_{(i,j)\in\mathcal{B}} |X_{ij}|},$$

is shown in the bottom panels of Figure 3.5. The conclusion Figure 5 is similar to the conclusion of Figure 4.

3.7 Conclusions

This paper has developed two new methods combining classical chain ladder methodology with expert knowledge via the double chain ladder methodology. The new EDCL introduces RBNS's as pseudo data and uses an iterative procedure to improve the originally estimated DCL parameters that did not take RBNS information into account.

It shows to have very good performance. The new PEDCL method preserves RBNS estimates also after the reserve has been estimated. Validation is introduced for these two new methods and they are compared to the previous methods DCL, BDCL, IDCL and PDCL.

Our conclusion is that while DCL always will be some kind of benchmark in the actuaries tool box, then EDCL and PEDCL seem to have sufficient quality to replace BDCL, IDCL and PDCL. That also means that EDCL and PEDCL seem to have sufficient quality to replace incurred chain ladder as the actuaries preferred method of incorporating incurred data expert knowledge. This is important, because most reserves in the actuarial practise use incurred chain ladder as the basis of estimation. This incurred chain ladder might be manually manipulated according to expert knowledge and the values of paid chain ladder. However, it is the most common basis of estimation in actuarial practise.

Only the future will be able to show whether EDCL and PEDCL or other similar innovations will be able to replace actuaries habit of using the - in our view outdated - incurred chain ladder approach.

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4

A likelihood approach to Bornhuetter-Ferguson analysis

This chapter is a working paper, which has been submitted to a journal.

It is joint work with Bent Nielsen.

A likelihood approach to Bornhuetter-Ferguson analysis

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Abstract

A new Bornhuetter-Ferguson method is suggested. This is a variant of the traditional chain ladder method. The actuary can adjust the relative ultimates. These correspond to linear constraints on the Poisson likelihood underpinning the chain ladder method. Adjusted cash flow estimates are then obtained as constrained maximum likelihood estimates.

Keywords: chain ladder, Bornhuetter-Ferguson, maximum likelihood, exponential families, canonical parameters, prior knowledge.

4.1 Introduction

The chain ladder method is the basic actuarial tool for reserving in general insurance. This method is based on the paid run-off triangle and provides estimates for the ultimate reserve along with development factors that are used for determining the cash flow. In practice, the actuary usually adjusts the ultimates using additionally available information. With the Bornhuetter and Ferguson (1972) method the ultimates are adjusted while the adjusted cash flow is proportional to the original chain ladder cash flow. Mack (2000) gave a credibility interpretation of the Bornhuetter-Ferguson method.

The adjustment of the ultimates can be done in two ways; either by correcting the levels of the ultimates or the relative levels of the ultimates. By this, we distinguish between the situation where the actuary has an estimate for the ultimate for a given policy year and the situation where the actuary is more comfortable with the prediction that the ultimate for a given policy year is 10% higher, say, than in the previous year. Such an estimate could for instance come from chain ladder analysis of incurred data. Indeed, we provide an empirical illustration where this is the case. The levels approach is most common in the literature, see for instance Mack (2000), Mack (2006), Taylor (2000), Verrall (2004), Wüthrich and Merz (2008). The relative levels approach is more recent, see Martínez-Miranda, Nielsen, and Verrall (2013) and Martínez-Miranda et al. (2015).

There are potentially two concerns with the traditional Bornhuetter-Ferguson correction. It may move the reserves too much and the cash flow distribution is not adjusted in light of the external information. Verrall (2004) addressed this in a Bayesian setup while Mack (2006) proposed an alternative approach where new weights are computed by combining actual payments and the externally estimated reserves.

Our proposal is related to that of Mack (2006), but with weights derived from a likelihood function. Adjusting relative ultimates as opposed to level ultimates is natural when working with the likelihood function in the same way as traditional chain ladder development factors are concerned with relative effects. A feature of our approach is therefore that external information is linked directly to the parameters of the underlying Poisson model and it is possible to express the Bornhuetter-Ferguson adjustment in terms of adjustments to the development factors. Another feature of this approach is that we can evaluate how much the adjustment moves the reserves and establish inequalities relating our approach and the traditional Bornhuetter-Ferguson adjustments.

A fundamental interpretation of the Bornhuetter-Ferguson method arises from the credibility formula derived by Mack (2000). This shows that an adjustment of the ultimates yields a partial adjustment of the reserves. He then continues to show that the iterations of the credibility formula leads to the Benktander (1976) approach. These ideas are taken a step further by Gigante, Picech, and Sigalotti (2013), whereas Taylor (2000) and Wüthrich and Merz (2008) give general overviews of the Bornhuetter-Ferguson method. Our first contribution is to show that the credibility formula also applies when adjusting the relative levels of the ultimates.

It is useful to recall that the chain ladder method has the nice interpretation as maximum likelihood in Poisson model. This result was proved by Kremer (1985). In particular, it is possible to show that the development factors have interpretation as maximum likelihood estimators, see Kuang, Nielsen, and Nielsen (2009). The maximum likelihood result mean that it is possible to compute the chain ladder estimates using generalized linear model methods. In practice the Poisson assumption is not realistic as the paid data typically have considerable over-dispersion, see for instance England and Verrall (2002). Nonetheless, the chain ladder method provides good reserve estimates that are, at least, anchored in a quasi-likelihood. An alternative approach would be to use Poisson-Tweedie models, see Tweedie (1984), Smyth and Jørgensen (2002), Wüthrich (2003) and Peters, Shevchenko, and Wüthrich (2009).

The main idea of our approach is to impose the externally estimated relative ultimates on the Poisson likelihood. Initially, it is useful to work with the standard parametrization of the generalized linear model as opposed to the development factors. We can then formulate the relative ultimates constraint as a linear constraint on the parameters and derive maximum likelihood estimators. Subsequently, we translate these estimators into adjusted development factors.

The constrained maximum likelihood approach satisfies a monotonicity result. If, for instance, all the relative ultimates are increased relative to the chain ladder ultimates, then it follows that the reserves are increased. However, these new reserves are increasing less that the traditional Bornhuetter-Ferguson reserves that would arise by combining the adjusted relative ultimates with the chain ladder development factors.

We apply the methods to a motor portfolio from a Greek insurer. These data include both paid and incurred triangles. In addition, an external estimate of the reserve is available so that this example nicely illustrates the practical issues that lead to the use of the Bornhuetter-Ferguson method.

4.2 The Bornhuetter-Ferguson problem

We present two standard Bornhuetter-Ferguson approaches. For now we will not formulate a statistical model, but just use the standard chain ladder formulas.

4.2.1 Data

Consider a standard run-off triangle of paid amounts. The dimension is denoted k and we use the incremental form of the triangle. Each entry is denoted Y_{ij} so that i is the accident year index and j is the development year index. The indices vary in the upper triangle with indices $1 \leq i, j \leq k$ and $i + j - 1 \leq k$. This is the area I in Figure 4.1. The objective is to forecast values of Y_{ij} in the lower triangle with indices $1 \leq i, j \leq k$ and $k + 1 \leq i + j - 1 \leq 2k - 1$. This is the area J in Figure 4.1.

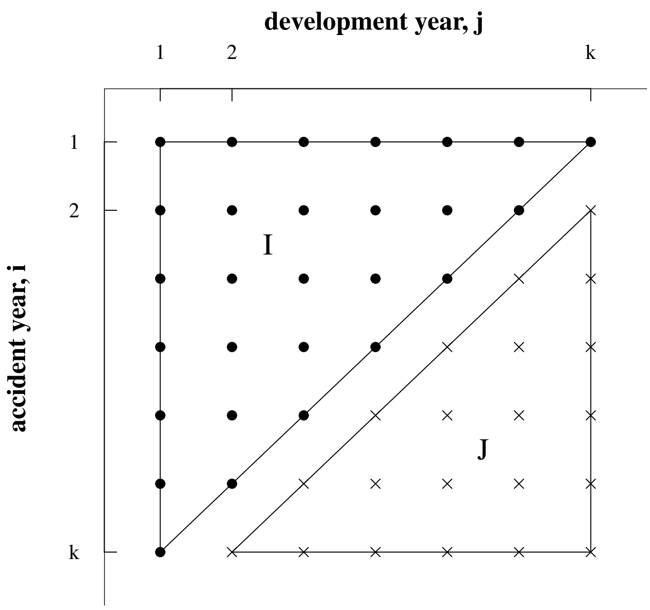

FIGURE 4.1: Illustration of data layout

4.2.2 The chain ladder method

The objective is to forecast values of Y_{ij} in the lower triangle with indices $1 \leq i, j \leq k$ and $k+1 \leq i+j-1 \leq 2k-1$. For this purpose we compute row sums or cumulative payments R_i and development factors F_j defined as

$$R_i = \sum_{j=1}^{k+1-i} Y_{ij}, \qquad F_j = \frac{\sum_{i=1}^{k+1-j} \sum_{\ell=1}^{j} Y_{i\ell}}{\sum_{i=1}^{k+1-j} \sum_{\ell=1}^{j-1} Y_{i\ell}} \quad \text{for } j = 2, \ldots, k. \tag{4.1}$$

We can use this to predict amounts in the lower triangle by

$$\widetilde{Y}_{ij} = R_i(F_j - 1) \prod_{\ell=k+2-i}^{j-1} F_\ell. \tag{4.2}$$

From this we compute the reserve for accident year i, for $i = 2, \ldots, k$, as

$$V_i = \sum_{j=k+2-i}^{k} \widetilde{Y}_{ij} = R_i(F_i^{prod} - 1) \qquad \text{where} \qquad F_i^{prod} = \prod_{\ell=k+2-i}^{k} F_\ell, \tag{4.3}$$

and the predicted ultimate payment as

$$U_i = R_i + V_i = R_i F_i^{prod} \quad \text{for } i = 2, \ldots, k. \tag{4.4}$$

If we use the convention that empty products are unity this matches with $U_1 = R_1$ and $V_1 = 0$, so that the in-sample prediction of the sum of the payments for accident year one equals the observation.

It will be convenient to express the above formulas in terms of certain weights. Thus, define weights, for $i = 2, \ldots, k$, $j = k+2-i, \ldots, k$,

$$W_{ij} = (F_j - 1) \frac{\prod_{\ell=k+2-i}^{j-1} F_\ell}{F_i^{prod}} = \frac{F_j - 1}{\prod_{\ell=j}^{k} F_\ell} \tag{4.5}$$

$$W_i = \frac{F_i^{prod} - 1}{F_i^{prod}} = \sum_{j=k+2-i}^{k} W_{ij}. \tag{4.6}$$

These are numbers between zero and unity. We have $W_i = 0$ if and only if one of the development factors is zero, whereas W_i approaches unity if and only if the product of the development factors approaches infinity. We can then write the predictions for each cell and each row in the lower triangle as

$$\widetilde{Y}_{ij} = U_i W_{ij}, \qquad V_i = U_i W_i. \tag{4.7}$$

These formulas show how the reserve V_i can be found as a fraction of the predicted ultimate U_i, while Y_{ij} indicates how the cashflow is distributed.

The Chain Ladder is maximum likelihood in a Poisson model that will be presented in Chapter 4.3. A feature of the model equation (4.12) is that it is symmetric in the indices for accident year i and development year j. This observation leads to a new expression for the forecast of the reserve. Traditionally, we forecast by computing row sums R_i of the data and multiply by the column wise forward factors F_j as in (4.2). Alternatively, we can compute columns sums C_j and row-wise forward factors G_i

$$C_j = \sum_{i=1}^{k+1-j} Y_{ij}, \qquad G_i = \frac{\sum_{j=1}^{k+1-i} \sum_{\ell=1}^{i} Y_{\ell j}}{\sum_{j=1}^{k+1-i} \sum_{\ell=1}^{i-1} Y_{\ell j}} \quad \text{for } i = 2, \ldots, k \tag{4.8}$$

and combine these to get the forecasts

$$\widetilde{Y}_{ij} = R_i(F_j - 1) \prod_{\ell=k+2-i}^{j-1} F_\ell = C_j(G_i - 1) \prod_{\ell=k+2-j}^{i-1} G_\ell. \tag{4.9}$$

4.2.3 Bornhuetter-Ferguson using levels of ultimates

This follows the interpretation offered by Mack (2000) see also England and Verrall (2002), Verrall (2004), Mack (2006) and Alai, Merz, and Wüthrich (2009).

England and Verrall present the Bornhuetter-Ferguson idea as follows. Suppose we replace the chain ladder ultimate U_i by an externally estimated reserve U_i^{level} in the formula (4.7). Then we get the level-based Bornhuetter-Ferguson reserve

$$V_i^{BF,level} = U_i^{level} W_i \quad \text{for } i = 2, \ldots, k.$$

Thus, the Bornhuetter-Ferguson reserve is the proportion W_i of the externally estimated level of the ultimate. In a similar fashion the Bornhuetter-Ferguson cash flow is given by

$$\widetilde{Y}_{ij}^{BF,level} = U_i^{level} W_{ij} \quad \text{for } i = 2, \ldots, k, j = k+2-i, \ldots, k.$$

The predicted ultimate payout turns out to be a convex combination of the chain ladder reserve U_i and the externally generated number U_i^{level}. To see this use the formulas (4.4), (4.6) to write the cumulated payments as $R_i = U_i/F_i^{prod} = U_i(1-W_i)$. It then follows that

$$U_i^{BF,level} = R_i + V_i^{BF,level} = U_i(1-W_i) + U_i^{level}W_i.$$

Mack (2000) refers to this a credibility formula and traces it back to Benktander (1976). Mack points out that it can be iterated by replacing U_i^{level} by $U_i^{BF,level}$. Another consequence is the following ordering, assuming $0 < W_i < 1$,

$$U_i < U_i^{level} \qquad \Rightarrow \qquad U_i < U_i^{BF,level} < U_i^{level}.$$

4.2.4 Bornhuetter-Ferguson using relative ultimates

This approach has been suggested by Martínez-Miranda, Nielsen, and Verrall (2013). The idea is now to replace the relative ultimates rather than levels of ultimates. We then rewrite (4.7) as

$$V_i = R_1 \frac{U_i}{U_1} W_i \quad \text{for } i = 2, \ldots, k, \tag{4.10}$$

recalling that $U_1 = R_1$. We now replace U_i/U_1 by some external measure U_i^{rel}/U_1^{rel}, which only provides information about the relative ultimates, such as the figure for year i is 10% higher than that for year $i-1$. This results in the relative level-based Bornhuetter-Ferguson reserve

$$V_i^{BF,rel} = R_1 \frac{U_i^{rel}}{U_1^{rel}} W_i \quad \text{for } i = 2, \ldots, k.$$

The corresponding cashflow is then

$$\widetilde{Y}_{ij}^{BF,rel} = R_1 \frac{U_i^{rel}}{U_1^{rel}} W_{ij} \quad \text{for } i = 2, \ldots, k, j = k+2-i, \ldots, k. \tag{4.11}$$

The relative Bornhuetter-Ferguson reserve also satisfies an actuarial credibility formula. To see this recall $U_1 = R_1$, write $R_i = R_1(R_i/U_1)$ and combine with $R_i = U_i(1-W_i)$ as before, to get

$$U_i^{BF,rel} = R_i + V_i^{BF,rel} = R_1 \left\{ \frac{U_i}{U_1}(1-W_i) + \frac{U_i^{rel}}{U_1^{rel}} W_i \right\}.$$

Once again, we have the ordering assuming $0 < W_i < 1$,

$$\frac{U_i}{U_1} < \frac{U_i^{rel}}{U_1^{rel}} \qquad \Rightarrow \qquad U_i < U_i^{BF,rel}.$$

Martínez-Miranda, Nielsen, and Verrall (2013) suggest that the relative external numbers could be computed from an incurred triangle. They extend this further to allow for reporting delays using a double chain ladder method. However, in the present paper we focus on the consequences of a Bornhuetter-Ferguson correction rather than how the external numbers are generated.

4.2.5 Proposed Bornhuetter-Ferguson reserves

With the above approaches the future cash flow is determined by the chain ladder method through the weights W_{ij} and not influenced by the external information. As argued by Verrall (2004) and Mack (2006) it may be desirable that the cash flow is also influenced by the external information. Our proposal allows the cash flow to be determined by a Poisson likelihood, constrained by the external information. Before we give the derivation it is useful to give a brief overview of the results.

The proposed Bornhuetter-Ferguson approach evolves around the chain ladder reserving formula (4.9) involving column sums C_j and row-wise forward factors G_i. Suppose we have externally given relative ultimates U_i^{rel}/U_1^{rel} for $i = 2, \ldots, k$, with the convention that $U_i^{rel}/U_1^{rel} = 1$ for $i = 1$. We then construct Bornhuetter-Ferguson row-wise forward factors

$$\Gamma_i^{rel} = \frac{\sum_{\ell=1}^{i}(U_\ell^{rel}/U_1^{rel})}{\sum_{\ell=1}^{i-1}(U_\ell^{rel}/U_1^{rel})} \qquad \text{for } i = 2, \ldots, k.$$

The Bornhuetter-Ferguson forecasts of individual payments and of reserves are then

$$\widetilde{Y}_{ij} = C_j(\Gamma_i^{rel} - 1) \prod_{\ell=k+2-j}^{i-1} \Gamma_\ell^{rel}, \qquad V_i^{rel} = \sum_{j=k+2-i}^{k} \widetilde{Y}_{ij}.$$

In the following section we will derive these forecasts as restricted maximum likelihood estimators. Since they are cast in terms of row-wise development factors we will also derive an equivalent expression involving new column-wise development factors. These development factors will be different for different accident years. For this reason we refer to them as pseudo development factors. Finally, we will compare them with the traditional Bornhuetter-Ferguson reserves with chain ladder weights and show that the proposed Bornhuetter-Ferguson reserves move the chain ladder reserves less than the traditional Bornhuetter-Ferguson reserves. In that way they may give a better approximation to the adjustment desired by the actuary.

4.3 Generalized Linear Model framework

We present a Generalized Linear Model framework for Bornhuetter-Ferguson analysis. The usual chain ladder estimators are maximum likelihood in a Poisson model, see Kremer (1985). In practice, reserving data have considerable over-dispersion, see England and Verrall (2002), so that Poisson likelihood becomes a quasi likelihood. In the present paper this distinction is not so important as we will only be concerned with point forecasts. Now, if we maximise the likelihood while imposing constraints from external relative levels of ultimates we get a closed form cash flow prediction that adapts to both data and the imposed constraints.

4.3.1 Statistical model

We assume that the incremental observations Y_{ij} are independent Poisson with log expectation $\mathsf{E}Y_{ij} = \exp(\mu_{ij})$, where the predictor is given by

$$\mu_{ij} = \alpha_i + \beta_j + \delta. \tag{4.12}$$

Here α_i is the level of the accident year effect, β_j is the level of the development year effect and δ is an overall level. The parametrisation presented in (4.12) does not identify the distribution, so we switch to the invariant parametrisation of Kuang, Nielsen, and Nielsen (2009), that is

$$\mu_{ij} = \mu_{11} + \sum_{\ell=2}^{i} \Delta\alpha_\ell + \sum_{\ell=2}^{j} \Delta\beta_\ell, \tag{4.13}$$

with the convention that empty sums are zero. Here $\Delta\alpha_i = \alpha_i - \alpha_{i-1}$ is the relative accident year effect and $\Delta\beta_j = \beta_j - \beta_{j-1}$ is the relative development year effect, while the overall level is determined by μ_{11}. The Poisson log likelihood function is

$$\ell(\mu_{11}, \Delta\alpha_i, \Delta\beta_j) = \sum_{1\le i,j,i+j-1\le k} \{\mu_{ij}Y_{ij} - \exp(\mu_{ij}) - \log(Y_{ij}!)\}. \tag{4.14}$$

This is a regular exponential family with canonical parameters $\mu_{11}, \Delta\alpha_i, \Delta\beta_j$.

4.3.2 The chain ladder

The chain ladder arises by maximizing the unconstrained likelihood. **Theorem 3**; Kuang, Nielsen, and Nielsen (2009) show that the maximum likelihood estimators are

$$\begin{aligned}
\Delta\widehat{\alpha}_i &= \Delta \log R_i + \log F_{k+2-i} \quad \text{for } i = 2, \dots k, \\
\Delta\widehat{\beta}_j &= \Delta \log C_j + \log G_{k+2-j} \quad \text{for } j = 2, \dots k, \\
\widehat{\mu}_{11} &= \log R_1 - \sum_{j=2}^{k} \log F_j.
\end{aligned}$$

while the relative ultimates are estimated by

$$\frac{U_i}{U_1} = \frac{\sum_{j=1}^{k} \exp(\widehat{\mu}_{ij})}{\sum_{j=1}^{k} \exp(\widehat{\mu}_{1j})} = \exp\left(\sum_{\ell=2}^{i} \Delta\widehat{\alpha}_\ell\right) \quad \text{for } i = 2, \dots k,$$

which are the relative ultimates entering in equation (4.10). Moreover, we get that $R_1 = U_1$ is the maximum likelihood estimator for the expected ultimates $\mathsf{E}R_1$ for the first accident year. Therefore the maximum likelihood estimators for the ultimates satisfy

$$U_i = U_1 \frac{U_i}{U_1} = U_1 \exp\left(\sum_{\ell=2}^{i} \Delta\widehat{\alpha}_\ell\right) \quad \text{for } i = 2, \dots k,$$

which are the ultimates in (4.7). Thus, in both cases the ultimate formulates are closely linked to the estimated relative accident year effects $\Delta\widehat{\alpha}_i$.

An additional result from Kuang, Nielsen and Nielsen (2009, Theorem 3) is that the forward factors F_j and G_i can be viewed as maximum likelihood estimators for combinations of the canonical parameters $\Delta\beta_j$ and $\Delta\alpha_i$, respectively. That is $F_j = \widehat{\Phi}_j$ and $G_i = \widehat{\Gamma}_i$ are maximum likelihood estimators for, for $i, j = 2, \dots, k$,

$$\Phi_j = \frac{\sum_{\ell=1}^{j} \exp(\sum_{h=2}^{\ell} \Delta\beta_h)}{\sum_{\ell=1}^{j-1} \exp(\sum_{h=2}^{\ell} \Delta\beta_h)}, \qquad \Gamma_i = \frac{\sum_{\ell=1}^{i} \exp(\sum_{h=2}^{\ell} \Delta\alpha_h)}{\sum_{\ell=1}^{i-1} \exp(\sum_{h=2}^{\ell} \Delta\alpha_h)}, \tag{4.18}$$

with the convention that empty sums are zero.

4.3.3 Imposing external information on the relative ultimates

Suppose some external values are available for the relative ultimates, U_i^{rel}/U_1^{rel}. Equivalently, we could have external values for the relative accident year effects $\Delta\alpha_i^\dagger$. We could impose these as a constraint on the likelihood (4.14). The constraint is linear and the likelihood remains that of a regular exponential family.

The constrained maximum likelihood estimators have a simple analytic form. In line with the parameters Γ_i defined in (4.18) define

$$\Gamma_i^\dagger = \frac{\sum_{\ell=1}^{i} \exp(\sum_{h=2}^{\ell} \Delta\alpha_h^\dagger)}{\sum_{\ell=1}^{i-1} \exp(\sum_{h=2}^{\ell} \Delta\alpha_h^\dagger)}.$$

We then have the following result, which is proved in the Appendix.

Theorem 4.1. *Consider the Poisson likelihood* (4.14) *with known* $\Delta\alpha_i = \Delta\alpha_i^\dagger$ *for* $i = 2, \dots, k$ *and define* $\Gamma_i^\dagger$ *as (4.18) computed using* $\Delta\alpha_i^\dagger$. *The constrained maximum likelihood estimator is unique if and only if* $C_j > 0$ *for all* $j = 1, \dots k$ *and given by*

$$\Delta\widehat{\beta}_j^\dagger = \Delta\log C_j + \log\Gamma_{k+2-j}^\dagger \qquad \text{for } j = 2, \dots, k, \tag{4.19}$$

$$\widehat{\mu}_{11}^\dagger = \log C_1 - \log\left\{1 + \sum_{i=2}^{k} \exp\left(\sum_{\ell=2}^{i} \Delta\alpha_\ell^\dagger\right)\right\} = \log C_1 - \sum_{\ell=2}^{k} \log\Gamma_\ell^\dagger. \tag{4.20}$$

As a consequence the out-of-sample forecast from the constrained chain ladder has a simple explicit form. This resembles the forecast in the unrestricted chain ladder computed from column sums and row-wise development factors as described in (4.9).

Theorem 4.2. *Consider the setup in Theorem 4.1. Point forecasts for the lower triangle are given by*

$$\widetilde{Y}_{ij}^\dagger = C_j(\Gamma_i^\dagger - 1) \prod_{\ell=k+2-j}^{i-1} \Gamma_\ell^\dagger. \tag{4.21}$$

We can now compute a Bornhuetter-Ferguson reserve based on Theorem 4.2. For each accident year we get

$$V_i^\dagger = \sum_{j=k+2-i}^{k} \widetilde{Y}_{ij}^\dagger. \tag{4.22}$$

4.3.4 Implementation in GLM software

The constrained model can also be estimated using ready-made algorithms for Generalized Linear Models. The analysis presented above shows that the constrained model is a regular exponential family so the algorithms should perform well.

For the implementation we organise the triangle Y as a vector $\mathbf{Y}$, say, of dimension $k(k+1)/2$. A design matrix $\mathbf{X}$ can be constructed from the formula (4.13). It has dimension $\{k(k+1)/2\} \times (2k-1)$ and the row corresponding to entry i, j is given by

$$X'_{ij} = \{1, 1_{(2\leq i)}, \ldots, 1_{(k\leq i)}, 1_{(2\leq j)}, \ldots, 1_{(k\leq j)}\},$$

where the indicator function $1_{(m\leq i)}$ takes the value unity if $m \leq i$ and zero otherwise. The unrestricted model is then estimated through a Generalized Linear Model regression of $\mathbf{Y}$ on $\mathbf{X}$ using the Poisson distribution with a log-link function.

In the constrained model the parameters $\theta_{known} = (\Delta\alpha_2, \ldots, \Delta\alpha_k)'$ are known. Deleting the corresponding columns from $\mathbf{X}$ gives a design matrix $\mathbf{X}_{reduced}$ with k columns. The deleted columns are collected as $\mathbf{X}_{known}$ say. The model is then estimated as a Generalized Linear Model regression of $\mathbf{Y}$ on $\mathbf{X}_{reduced}$ using the Poisson distribution with a log-link function and offset given by $\mathbf{X}_{known}\theta_{known}$.

4.3.5 A mixed approach

Let us first summarise the results we obtained so far in terms of the log likelihood. In the classical chain ladder approach, we maximize the unrestricted likelihood in (4.14), which leads to the unrestricted estimator

$$\widehat{\xi} = \max_{\xi} \ell(\xi) = (\widehat{\mu}_{11}, \Delta\widehat{\alpha}_i, \Delta\widehat{\beta}_j)'.$$

The restricted likelihood from Chapter 4.3.3 with restriction $\Delta\alpha_i = \Delta\alpha_i^\dagger$ has restricted likelihood maximum likelihood estimator given by

$$\widehat{\xi}^\dagger = \max_{\xi:\Delta\alpha=\Delta\alpha^\dagger} \ell(\xi) = (\widehat{\mu}_{11}^\dagger, \Delta\alpha_i^\dagger, \Delta\widehat{\beta}_j^\dagger)'.$$

Notice, that if $\Delta\alpha_i^\dagger = \Delta\widehat{\alpha}_i$, then $\widehat{\mu}_{11}^\dagger = \widehat{\mu}_{11}$ and $\Delta\widehat{\beta}_j^\dagger = \Delta\widehat{\beta}_j$.

A third estimator is achieved by mixing the above estimators. This combines the unrestricted estimators for μ_{11} and β_j with the given $\Delta\alpha_i^\dagger$, such that

$$\widehat{\xi}^\ddagger = (\widehat{\mu}_{11}, \Delta\alpha_i^\dagger, \Delta\widehat{\beta}_j)'.$$

In the following, parameters resulting from this mixed approach will be marked with the index "‡", just as parameters resulting from the constrained method will be marked with "†". The reserve computed from $\widehat{\xi}^\ddagger$ is

$$\widetilde{Y}_{ij}^\ddagger = \exp\left(\widehat{\mu}_{11} + \sum_{h=2}^{i} \Delta\alpha_h^\dagger + \sum_{h=2}^{j} \Delta\widehat{\beta}_h\right) \tag{4.23}$$

In the appendix we prove the identities

$$\widetilde{Y}_{ij}^\ddagger = \widetilde{Y}_{ij} \frac{\exp(\sum_{h=2}^{i} \Delta\alpha_h^\dagger)}{\exp(\sum_{h=2}^{i} \Delta\widehat{\alpha}_h)} = \widetilde{Y}_{ij}^\dagger \frac{\sum_{\ell=2}^{k+1-j} \exp(\sum_{h=2}^{\ell} \Delta\alpha_h^\dagger)}{\sum_{\ell=2}^{k+1-j} \exp(\sum_{h=2}^{\ell} \Delta\widehat{\alpha}_h)} \tag{4.24}$$

In the case when the known accident parameters are derived by applying chain ladder on the incurred data, such that $\Delta\alpha_i^\dagger = \Delta\widehat{\alpha}_i^{inc}$, this method gives exactly the same results as the Bornhuetter-Ferguson double chain ladder (BDCL) method in Martínez-Miranda, Nielsen, and Verrall (2013).

The log likelihood function evaluated in the three points satisfies

$$\ell(\widehat{\xi}) \geq \ell(\widehat{\xi}^\dagger) \geq \ell(\widehat{\xi}^\ddagger).$$

The first inequality holds since $\widehat{\xi}$ is maximum likelihood, while $\widehat{\xi}^\dagger$ is restricted maximum likelihood. The second inequality holds since $\widehat{\xi}^\ddagger$ satisfies the restriction, but it is not maximum likelihood.

4.3.6 Pseudo development factors

It is common practice to think about the classical chain ladder method in terms of row sums R_i and column wise development factors F_j given in (4.1). The forecasts for the lower triangle are then computed using the formula (4.9) by forwarding the row sums R_i using the factors F_j. However, in this classical setting the predicted value for the row sum equals the row sum. In the likelihood analysis this stems from a likelihood equation of the type $R_i = \mathsf{E}(R_i)$, see equation 20 in Kuang, Nielsen, and Nielsen (2009). Thus, we can also interpret the chain ladder forecast as forwarding the predicted row sums.

Once we have imposed external information on the relative ultimates then the forecast changes and we break the link to the original row sums and development factors. We can, however, construct pseudo predictions of the row sums and pseudo forward factors that satisfy a relationship like (4.9) but with the new forecasts.

Under the constraint that $\Delta\alpha = \Delta\alpha^\dagger$ we compute estimates $\widehat{\mu}_{11}^\dagger$ and $\Delta\widehat{\beta}_j^\dagger$ using (4.19), (4.20) in Theorem 4.1. From these we compute pseudo forward factors from (4.18), that is

$$F_j^\dagger = \frac{\sum_{\ell=1}^{j} \exp(\sum_{h=2}^{\ell} \Delta\widehat{\beta}_h^\dagger)}{\sum_{\ell=1}^{j-1} \exp(\sum_{h=2}^{\ell} \Delta\widehat{\beta}_h^\dagger)}, \tag{4.25}$$

and a pseudo first row sum from (4.17) as

$$\log R_1^\dagger = \widehat{\mu}_{11}^\dagger + \sum_{j=2}^{k} \log F_j^\dagger, \tag{4.26}$$

and then the remaining pseudo row sums from (4.15) as

$$\Delta \log R_i^\dagger = \Delta\alpha_i^\dagger - \log F_{k+2-i}^\dagger. \tag{4.27}$$

We show in the appendix that the forecast from (4.21) can be computed as

$$\widetilde{Y}_{ij}^\dagger = R_i^\dagger (F_j^\dagger - 1) \prod_{\ell=k+2-i}^{j-1} F_\ell^\dagger. \tag{4.28}$$

The above formulas for predicted reserve and the cash flow can also be written in the credibility format we saw in (4.11). To see this introduce the weights

$$W_{ij}^\dagger = (F_j^\dagger - 1)\frac{\prod_{\ell=k+2-i}^{j-1} F_\ell^\dagger}{F_i^{prod\dagger} - 1}, \qquad W_i^\dagger = \frac{F_i^{prod\dagger} - 1}{F_i^{prod\dagger}},$$

where, as before, $F_i^{prod\dagger} = \prod_{\ell=k+2-i}^{k} F_\ell^\dagger$. Introducing the ultimates and relative ultimates

$$U_i^\dagger = R_i^\dagger F_i^{prod\dagger}, \qquad \frac{U_i^\dagger}{U_{i-1}^\dagger} = \frac{R_i^\dagger}{R_{i-1}^\dagger} F_{k+2-i}^\dagger = \exp(\Delta\alpha_i^\dagger)$$

we can then write the predicted reserve and cash flow as

$$\widetilde{Y}^{\dagger}_{ij} = U^{\dagger}_i W^{\dagger}_{ij}, \qquad V^{\dagger}_i = U^{\dagger}_i W^{\dagger}_i.$$

4.3.7 Chain ladder prediction with the mixed approach

In the mixed approach we follow a similar procedure to satisfy a relationship like (4.9) in order to obtain the new forecasts. The difference to the constrained method is that we can keep the forward factors from the unconstrained chain ladder model, F_j. However, we need to construct pseudo row sums $R^{\ddagger}_i$ as follows.

We fix the pseudo first row sum as

$$\log R^{\ddagger}_1 = \log R_1, \tag{4.29}$$

and then compute the remaining pseudo row sums from (4.15) as

$$\Delta \log R^{\ddagger}_i = \Delta\alpha^{\dagger}_i - \log F_{k+2-i}. \tag{4.30}$$

We show in the appendix that the forecast from (4.23) can be computed as

$$\widetilde{Y}^{\ddagger}_{ij} = R^{\ddagger}_i (F_j - 1) \prod_{\ell=k+2-i}^{j-1} F_\ell. \tag{4.31}$$

The forecast can be written in terms of weights as before. Since the cash flow is derived from the chain ladder development factors the weights are as defined in (4.5), (4.6). In particular we have the ultimates and relative ultimates

$$U^{\ddagger}_i = R^{\ddagger}_i F^{prod}_i, \qquad \frac{U^{\ddagger}_i}{U^{\ddagger}_{i-1}} = \frac{R^{\ddagger}_i}{R^{\ddagger}_{i-1}} F_{k+2-i} = \exp(\Delta\alpha^{\dagger}_i)$$

we can then write the predicted reserve and cashflow as

$$\widetilde{Y}^{\ddagger}_{ij} = U^{\ddagger}_i W_{ij}, \qquad V^{\ddagger}_i = U^{\ddagger}_i W_i. \tag{4.32}$$

4.3.8 Monotonicity

Let us consider the case when the known accident parameters, $\Delta\alpha_i^\dagger$, are bigger than the accident parameters we obtain from the chain ladder method on paid data, $\Delta\widehat{\alpha}_i$. The following theorem shows monotonicity results regarding the remaining parameters given in Theorem 4.1 and the resulting predictions we obtain from the constrained approach in Chapter 4.3.3, $\widetilde{Y}_{ij}^\dagger$, as well as the mixed approach in Chapter 4.3.5, $\widetilde{Y}_{ij}^\ddagger$.

Theorem 4.3. *Suppose* $\Delta\alpha_i^\dagger > \Delta\widehat{\alpha}_i$ *for all* $2 \leq i \leq k$. *Then*

(a) $\Gamma_i^\dagger > G_i$ *for all* $2 \leq i \leq k$;

(b) $\Delta\widehat{\beta}_j^\dagger > \Delta\widehat{\beta}_j$ *for all* $2 \leq j \leq k$;

(c) $\widehat{\mu}_{11}^\dagger < \widehat{\mu}_{11}$;

(d) $\widetilde{Y}_{ij}^\ddagger > \widetilde{Y}_{ij}^\dagger > \widetilde{Y}_{ij}$ *for all* i, j *so that* $k < i + j - 1 < 2k$;

(e) $F_j^\dagger > F_j$ *for all* $2 \leq j \leq k$.

(f) $R_i^\ddagger > R_i^\dagger$ *for all* $2 \leq i \leq k$.

(g) $R_i^\ddagger > R_i$ *for all* $2 \leq i \leq k$.

4.4 Empirical illustration

We illustrate the new methods by an example where the external knowledge comes from incurred payments. In practice the external knowledge may also come from incurred counts, from other business lines or from other sources.

We use data from a Greek non-life insurer for motor third party liability, aggregated over bodily injury and property damage. The data are presented as cumulative run-off triangles for accident years from 2005 to 2013. Table 4.1 shows payments, while Table 4.2 shows incurred amounts.

2005	34492471	47124007	55244404	59817460	62550940	66042036	69311560	70992659	72265079
2006	39467733	54003286	61349336	69986825	76412887	81768759	86684598	90726054	
2007	38928855	57087550	65905902	77128507	84158380	92436441	97838371		
2008	34202332	50932726	60560484	68566905	76409739	82082804			
2009	35657409	52397264	59849582	66698806	72724524				
2010	25404394	37040589	42371049	50709319					
2011	21268516	31311410	35973015						
2012	17404447	27786399							
2013	17676374								

TABLE 4.1: Payments in Euro.

2005	54018141	56699807	60273204	61112600	63729660	67142341	69733859	71980196	72738376
2006	68706483	70534436	70254136	75919965	77900147	83401774	88690144	92171660	
2007	64613205	72600950	76163387	82388057	87424383	96246891	102854340		
2008	58071632	66701421	69420629	75280537	81978240	89923269			
2009	60368719	67868349	72528239	80726223	85339588				
2010	47282519	56488940	60896832	65900623					
2011	49905225	54801141	60026903						
2012	48425940	52652928							
2013	47449977								

Table 4.2: Incurred amounts in Euro.

$\Delta\widehat{\alpha}_i$	$\Delta\alpha_i^\dagger$	$\Delta\widehat{\beta}_j$	$\Delta\widehat{\beta}_j^\dagger$
0.24526809	0.247261682	-0.80044252	-0.76965582
0.11149938	0.145178053	-0.68857388	-0.65777806
-0.12057425	-0.077312634	0.02370846	0.06137844
-0.04769497	0.027019249	-0.32208939	-0.29855013
-0.27637689	-0.204202408	-0.05908884	-0.03399479
-0.21412347	-0.018592530	-0.22363447	-0.20684905
-0.11353717	-0.078902778	-0.37786842	-0.36440835
-0.08135422	-0.005083078	-0.68021278	-0.67909386
$\widehat{\mu}_{11} = 17.18463300$		$\widehat{\mu}_{11}^\dagger = 17.00538277$	

Table 4.3: Estimates

Table 4.3 shows parameter estimates computed by applying the chain ladder and the Bornhuetter-Ferguson constrained method to the paid data. For the moment we focus on the canonical parameters $\Delta\alpha_i$ for the relative accident year effect, $\Delta\beta_j$ for the relative development year effect and μ_{11} for the overall level. First, the chain ladder estimates are reported as $\Delta\widehat{\alpha}_i$, $\Delta\widehat{\beta}_j$, $\Delta\widehat{\mu}_{11}$. Second, for the constrained model we first apply chain ladder to the incurred data. The estimates for the relative accident year effect are reported as $\Delta\alpha_i^\dagger$. The estimates $\Delta\widehat{\beta}_j^\dagger$, $\Delta\widehat{\mu}_{11}^\dagger$ are then computed from the paid data using Theorem 4.1. We note that the ordering $\Delta\alpha_i^\dagger > \Delta\widehat{\alpha}_i$ applies for these data for all $i = 2, ..., k = 9$. Thus, the monotonicity results from Theorem 4.3 apply. In particular we see that $\Delta\widehat{\beta}_j^\dagger > \Delta\widehat{\beta}_j$ for all $j = 2, ..., k = 9$ and $\widehat{\mu}_{11}^\dagger < \widehat{\mu}_{11}$ in Table 4.3.

A third approach is to use the mixed approach outlined in Chapter 4.3.5. Here we use the external estimate $\Delta\alpha_i^\dagger$ for the relative accident year effects along with the chain ladder estimates $\Delta\widehat{\beta}_j$, $\Delta\widehat{\mu}_{11}$. When the external estimate is based on the incurred data, as here, this is the same as the Bornhuetter-Ferguson Double chain ladder (BDCL) approach of Martínez-Miranda, Nielsen, and Verrall (2013).

Table 4.4 presents the estimated (pseudo) forward factors and the (pseudo) row sums. For the chain ladder we have the observed row sums R_i and the traditional forward factors F_j computed by (4.1). For the Bornhuetter-Ferguson constrained model we have

i,j	R_i	$R_i^\dagger$	$R_i^\ddagger$	F_j	$F_j^\dagger$
1	72265079	63989145	72265079		
2	90726054	80309654	90907105	1.449130	1.463172
3	97838371	80309654	101391484	1.155676	1.163975
4	82082804	77559430	88824492	1.137937	1.149793
5	72724524	73428364	84802647	1.087838	1.096652
6	50709319	54589726	63556691	1.076112	1.085188
7	35973015	46603309	54823701	1.056555	1.063832
8	27786399	37000367	43839471	1.036684	1.041678
9	17676374	25159556	30098881	1.017923	1.020288

TABLE 4.4: Row sums and forward factors

the pseudo row sums $R_i^\dagger$ and the pseudo forward factors $F_j^\dagger$ computed by (4.25)-(4.27). For the mixed approach we have the pseudo row sums $R_i^\ddagger$ computed by (4.29),(4.30) and the traditional forward factors F_j. Once again we see that the monotonicity results from Theorem 4.3 apply so that $R_i^\ddagger > R_i^\dagger$ and $F_j^\dagger > F_j$.

$\sum_{i=1}^k V_i$	external valuation	$\sum_{i=1}^k V_i^\dagger$	$\sum_{i=1}^k V_i^\ddagger$
110.1	137	149.1	156.6

TABLE 4.5: Reserves in million Euro

Table 4.5 shows the reserves resulting from the classical chain ladder method, $\sum_{i=2}^k V_i$ from (4.3), the constrained approach, $\sum_{i=2}^k V_i^\dagger$ from (4.22), as well as the mixed approach, $\sum_{i=2}^k V_i^\ddagger$ from (4.32). We see that the ordering from Theorem 4.3 applies. For comparison we note that this portfolio was evaluated at 137M by an external actuary, with the comment that this figure may be slightly too low. This valuation is based on the information that since 2009 the incurred case reserves have been gradually increased, but the gap between incurred and paid reserves has not been fully closed by 2014. In light of this, the Bornhuetter-Ferguson constrained method appears to apply rather well in this situation.

4.5 Conclusions

The paper introduced a Bornhuetter-Ferguson approach that replaces the relative ultimates rather than levels of ultimates. This approach has been suggested in the BDCL method in Martínez-Miranda, Nielsen, and Verrall (2013). The traditional Bornhuetter-Ferguson method uses Chain Ladder weights, whereas we have estimated weights.

We make use of the fact that the Chain Ladder method has the nice interpretation as maximum likelihood in a Poisson model and we formulate the relative ultimates constraint as a linear constraint on the parameters and derive maximum likelihood estimators. Furthermore, we follow this approach to reproduce the results of the BDCL method in a mixed approach, combining the constrained method with the classical Chain Ladder.

Monotonicity results compare the constrained method, the mixed approach and the original Chain Ladder results. Finally, an example illustrates the mentioned results with data from a Greek general insurer. The example shows that, when comparing all methods mentioned above, including Chain Ladder, the reserve given by the constrained method is in fact the closest estimate to the number given by an external expert.

The advantage of this approach is that it can, unlike the BDCL method, be applied using only one triangle, usually the payments triangle. However, while the model is able to produce satisfying reserves, it is not able to distinguish between IBNR and RBNS.

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4.A Appendix: Proofs of Theorems

Proof of Theorem 4.1. *The likelihood* When the $\Delta\alpha_i$s are known the likelihood is

$$\ell(\mu_{11}, \Delta\beta) = \mu_{11} \sum_{j=1}^{k} C_j + \sum_{j=2}^{k} \Delta\beta_j \sum_{\ell=j}^{k} C_j + g(\mu_{11}, \Delta\beta) + h(data),$$

where g is a function of the unknown parameters, not depending on the data and h is a function of the data, not depending on the unknown parameters.

Uniqueness of the estimator. For a full exponential family the maximum likelihood estimator is unique if and only if the natural statistic is interior to its convex support (**Theorem 9.13**; Barndorff-Nielsen (1978).

The natural statistic $T_k^\dagger = \sum_{i,j\in\mathcal{I}}(Y_{ij}, C_2, \ldots, C_k)'$ arises through a bijective, linear mapping of $(C_1, \ldots, C_k)'$. Since $Y_{ij} \geq 0$ by the Poisson assumption then $C_j \geq 0$, with $C_j = 0$ as a possible outcome. Since $C_1, \ldots, C_k$ are based on unrelated observations then the interior of the convex support is given by the condition that $C_j > 0$ for all $j = 1, \ldots, k$.

Likelihood equations. Since the exponential family is regular the k likelihood equations are $T_k^\dagger = \mathsf{E}T_k^\dagger$ **Corollary 9.6**; Barndorff-Nielsen (1978).
Since $\sum_{i=1}^{k}\sum_{j=1}^{k+1-i} Y_{ij} = \sum_{j=1}^{k} C_j$ this in turn implies the equations

$$C_j = \mathsf{E}C_j, \qquad \text{for } j = 1, \ldots k.$$

Estimating the level. The expression for $\widehat{\mu}_{11}^\dagger$ arises from the first likelihood equation

$$C_1 = \mathsf{E}C_1 = \exp(\mu_{11}) \sum_{i=1}^{k} \exp(\alpha_i - \alpha_1),$$

since the parameters $\alpha_i - \alpha_1 = \sum_{\ell=2}^{i} \Delta\alpha_\ell$ are known.

Estimating the development parameters. The expression for $\Delta\widehat{\beta}_j^\dagger$ arises by combining the $(j-1)$th and jth likelihood equations

$$\frac{C_j}{C_{j-1}} = \frac{\mathsf{E}C_j}{\mathsf{E}C_{j-1}} = \frac{\exp(\mu_{11} + \beta_j - \beta_1)\sum_{i=1}^{k+1-j}\exp(\alpha_i - \alpha_1)}{\exp(\mu_{11} + \beta_{j-1} - \beta_1)\sum_{i=1}^{k+2-j}\exp(\alpha_i - \alpha_1)}.$$

Recalling the expression for Γ_i in (4.8) this reduces to

$$\frac{C_j}{C_{j-1}} = \frac{\exp(\Delta\beta_j)}{\Gamma_{k+2-j}},$$

which has the desired solution. □

Proof of Theorem 4.2. Use the expressions from Theorem 4.1 to get

$$\begin{aligned}
\widetilde{Y}_{ij}^{\dagger} &= \exp(\widehat{\mu}_{11}^{\dagger} + \alpha_i^{\dagger} - \alpha_1^{\dagger} + \widehat{\beta}_j^{\dagger} - \widehat{\beta}_1^{\dagger}) \\
&= \frac{C_1}{\prod_{\ell=2}^{k}\Gamma_\ell^{\dagger}}(\Gamma_i^{\dagger}-1)\left(\prod_{\ell=2}^{i-1}\Gamma_\ell^{\dagger}\right)\frac{C_j}{C_1}\prod_{\ell=2}^{j}\Gamma_{k+2-\ell}^{\dagger} \\
&= C_j(\Gamma_i^{\dagger}-1)\frac{\prod_{\ell=k+2-j}^{k}\Gamma_\ell^{\dagger}}{\prod_{\ell=i}^{k}\Gamma_\ell^{\dagger}}.
\end{aligned}$$

We get the desired result by simplifying the last fraction using that $i > k+2-j$. □

Proof of equation (4.24). *First identity.* Combine the forecasts, see (4.23),

$$\widetilde{Y}_{ij}^{\ddagger} = \exp(\widehat{\mu}_{11} + \sum_{h=2}^{i}\Delta\alpha_h^{\dagger} + \sum_{h=2}^{j}\Delta\widehat{\beta}_h), \qquad \widetilde{Y}_{ij} = \exp(\widehat{\mu}_{11} + \sum_{h=2}^{i}\Delta\widehat{\alpha}_h + \sum_{h=2}^{j}\Delta\widehat{\beta}_h).$$

Second identity. From (4.9) we have $\widetilde{Y}_{ij} = C_j(G_i - 1)\prod_{\ell=k+2-j}^{i-1}G_\ell$. Write $G_i = \widehat{N}_i/\widehat{N}_{i-1}$ where $\widehat{N}_i = \sum_{\ell=1}^{i}\exp(\sum_{h=2}^{\ell}\Delta\widehat{\alpha}_h)$ and $\widehat{N}_i - \widehat{N}_{i-1} = \exp(\sum_{h=2}^{i}\Delta\widehat{\alpha}_h)$. Then, we get

$$\widetilde{Y}_{ij} = C_j\frac{\widehat{N}_i - \widehat{N}_{i-1}}{\widehat{N}_{i-1}}\prod_{\ell=k+2-j}^{i-1}\frac{\widehat{N}_\ell}{\widehat{N}_{\ell-1}} = C_j\frac{\widehat{N}_i - \widehat{N}_{i-1}}{\widehat{N}_{k+1-j}} = C_j\frac{\exp(\sum_{h=2}^{i}\Delta\widehat{\alpha}_h)}{\sum_{\ell=1}^{k+1-j}\exp(\sum_{h=2}^{\ell}\Delta\widehat{\alpha}_h)}.$$

Correspondingly, we get from (4.21) that

$$\widetilde{Y}_{ij}^{\dagger} = C_j\frac{\exp(\sum_{h=2}^{i}\Delta\alpha_h^{\dagger})}{\sum_{\ell=1}^{k+1-j}\exp(\sum_{h=2}^{\ell}\Delta\alpha_h^{\dagger})}.$$

Now, combine the expressions for $\widetilde{Y}_{ij}^{\dagger}$ and $\widetilde{Y}_{ij}^{\dagger}$. □

Proof of equation (4.28). The point forecast is given by $\widetilde{Y}_{ij}^{\dagger} = \exp(\widehat{\mu}_{11}^{\dagger} + \sum_{h=2}^{i}\Delta\alpha_h^{\dagger} + \sum_{h=2}^{j}\Delta\widehat{\beta}_h^{\dagger})$. Insert the expression for $\widehat{\mu}_{11}^{\dagger}$ from (4.26) and for $\Delta\alpha_i^{\dagger}$ from (4.27), as well as $\exp(\sum_{h=2}^{j}\Delta\widehat{\beta}_h^{\dagger}) = (F_j^{\dagger}-1)\prod_{\ell=2}^{j-1}F_\ell^{\dagger}$, which follows from

(4.25), to get

$$\widetilde{Y}_{ij}^{\dagger} \;=\; \frac{R_1^{\dagger}}{\prod_{\ell=2}^{k} F_{\ell}^{\dagger}} \left(\frac{R_i^{\dagger}}{R_1^{\dagger}} \prod_{\ell=2}^{i} F_{k+2-\ell}^{\dagger} \right) (F_j^{\dagger} - 1) \prod_{\ell=2}^{j-1} F_{\ell}^{\dagger}.$$

Equation (4.28) follows by reducing common factors and noting that $j > k + 2 - i$. □

Proof of equation (4.31). The point forecast is $\widetilde{Y}_{ij}^{\ddagger} = \exp(\widehat{\mu}_{11} + \sum_{h=2}^{i} \Delta\alpha_h^{\dagger} + \sum_{h=2}^{j} \Delta\widehat{\beta}_h)$ as given in (4.23). Insert the expression for $\widehat{\mu}_{11}$ from (4.17), the expression for $\Delta\alpha_i^{\dagger}$ from (4.30), as well as $\exp(\sum_{h=2}^{j} \Delta\widehat{\beta}_h) = (F_j - 1)\prod_{\ell=2}^{j-1} F_{\ell}$, which follows from (4.18) noting that $F_j = \widehat{\Phi}_j$, to get

$$\widetilde{Y}_{ij}^{\ddagger} \;=\; \frac{R_1}{\prod_{\ell=2}^{k} F_{\ell}} \left(\frac{R_i^{\ddagger}}{R_1} \prod_{\ell=2}^{i} F_{k+2-\ell} \right) (F_j - 1) \prod_{\ell=2}^{j-1} F_{\ell}.$$

Equation (4.31) follows by reducing common factors and noting that $j > k + 2 - i$. □

Proof of Theorem 4.3. (a) We show that Γ_i defined in (4.18) is increasing in the $\Delta\alpha_i$'s. Write $\Gamma_i = N_i/N_{i-1}$ where $N_i = \sum_{\ell=1}^{i} \exp(\sum_{h=2}^{\ell} \Delta\alpha_h)$. Thus, we must show that the derivative of Γ_i with respect to $\Delta\alpha_n$ is positive for all $n \leq i$ and zero otherwise. It suffices to consider the numerator of that derivative, which is $\dot{N}_i N_{i-1} - N_i \dot{N}_{i-1}$. Now,

$$\dot{N}_i = \frac{\partial N_i}{\partial \Delta\alpha_n} = \sum_{\ell=n}^{i} \exp(\sum_{h=2}^{\ell} \Delta\alpha_h) = N_i - N_{n-1},$$

for $n \leq i$ and zero otherwise. This implies $\dot{N}_i N_{i-1} - N_i \dot{N}_{i-1} = N_{n-1}(N_i - N_{i-1})$, noting that the cases where $n < i$ and $n = i$ have to be checked separately. The desired result now follows by noting that N_{n-1} and $N_i - N_{i-1}$ are both positive.

(b) Using (4.19) and (a) we get

$$\Delta\widehat{\beta}_j^{\dagger} = \Delta \log C_j + \log \Gamma_{k+2-j}^{\dagger} > \Delta \log C_j + \log G_{k+2-j} = \Delta\widehat{\beta}_j,$$

where the last equality is of a similar type as (4.19) and comes from **Theorem 3**; Kuang, Nielsen, and Nielsen (2009).

(c) Using (4.20) and (a) we get

$$\widehat{\mu}_{11}^{\dagger} = \log C_1 - \sum_{\ell=2}^{k} \log \Gamma_{\ell}^{\dagger} < \log C_1 - \sum_{\ell=2}^{k} \log G_{\ell} = \widehat{\mu}_{11},$$

where the last equality comes from **Theorem 3**; Kuang, Nielsen, and Nielsen (2009).

(d) First, we compare the new reserve $\widetilde{Y}_{ij}^{\dagger}$ with $\Delta\alpha_i^{\dagger}$ known to the old reserve $\widetilde{Y}_{ij}$ from CL. Since $1 \leq G_i < \Gamma_i^{\dagger}$ for $2 \leq i \leq k$ then, by (4.9), (a), (4.21),

$$\widetilde{Y}_{ij} = C_j(G_i - 1) \prod_{\ell=k+2-j}^{i-1} G_{\ell} < C_j(\Gamma_i^{\dagger} - 1) \prod_{\ell=k+2-j}^{i-1} \Gamma_{\ell}^{\dagger} = \widetilde{Y}_{ij}^{\dagger}.$$

Second, we compare the new reserve $\widetilde{Y}_{ij}^{\dagger}$, using $\Delta\alpha_i^{\dagger}$, $\widehat{\mu}_{11}^{\dagger}$, $\Delta\widehat{\beta}_j^{\dagger}$, to the mixed reserve $\widetilde{Y}_{ij}^{\ddagger}$, using $\Delta\alpha_i^{\dagger}$, $\widehat{\mu}_{11}$, $\Delta\widehat{\beta}_j$. From (4.24) we have

$$\widetilde{Y}_{ij}^{\ddagger} = \widetilde{Y}_{ij}^{\dagger} \frac{\sum_{\ell=2}^{k+1-j} \exp(\sum_{h=2}^{\ell} \Delta\alpha_h^{\dagger})}{\sum_{\ell=2}^{k+1-j} \exp(\sum_{h=2}^{\ell} \Delta\widehat{\alpha}_h)}$$

Since $\Delta\alpha_i^{\dagger} > \Delta\widehat{\alpha}_i$ for all $2 \leq i \leq k$ it follows that $\widetilde{Y}_{ij}^{\ddagger} > \widetilde{Y}_{ij}^{\dagger}$.

(e) Similar to the argument in (a), but using the ordering for $\Delta\widehat{\beta}$ derived in (b).

(f) Equations (4.28), (4.31) applied for any $k + 2 - i \leq j \leq k$ show that

$$\frac{R_i^{\ddagger}}{R_i^{\dagger}} = \frac{Y_{ij}^{\ddagger}}{Y_{ij}^{\dagger}} \frac{(F_j^{\dagger} - 1) \prod_{\ell=k+2-i}^{j-1} F_{\ell}^{\dagger}}{(F_j - 1) \prod_{\ell=k+2-i}^{j-1} F_{\ell}}$$

Then apply the orderings $\widetilde{Y}_{ij}^{\ddagger} > \widetilde{Y}_{ij}^{\dagger}$ and $F_j^{\dagger} > F_j$ from (d), (e).

(g) Use (4.18), (4.31) to get $R_i^{\ddagger}/R_{ij} = \widetilde{Y}_i^{\ddagger}/\widetilde{Y}_{ij}$ for all $k + 2 - i \leq j \leq k$. Apply (f). □

5

Micro models for reinsurance reserving based on aggregate data

This chapter is a working paper, which has been submitted to a journal.
It is joint work with Richard Verrall and Valandis Elpidorou.

Micro models for reinsurance reserving based on aggregate data

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Abstract

This paper addresses a new problem in the literature, which is how to consider reserving issues for a portfolio of general insurance policies when there is excess-of-loss reinsurance. This is very important for pricing considerations and for decision making regarding capital issues. The paper sets out how this is currently often tackled in practice and provides an alternative approach using recent developments in stochastic claims reserving. These alternative approaches are illustrated and compared in an example using real data. The stochastic modelling framework used in this paper is double chain ladder, but other approaches would also be possible. The paper sets out an approach which could be explored further and built on in future research.

5.1 Introduction

The subject of this paper is the distribution of outstanding claims for a portfolio of general insurance policies in the presence of excess-of-loss reinsurance protection. This is a subject area which, to the best knowledge of the authors, has not previously appeared in the actuarial literature. It is a very important subject from a practical point of view, and there have been many papers on the estimation of outstanding claims and reinsurance separately to develop both the theory and practical tools for actuaries. To date, they have not been considered together.

This is perhaps surprising because the estimation of outstanding claims net of reinsurance for such a portfolio is very commonly needed, for example when a reinsurance underwriter is pricing either a retrospective loss portfolio transfer treaty or a prospective proportional quota share on the retention. These are both actively used for solvency capital management and it would therefore be desirable to have estimates of both the expected net outstanding claims and the uncertainty around these. Better still would be estimates of the distribution of net outstanding claims. This paper develops methods to address all of these issues and compares the results with what is often done in the practical context using existing reserving methods.

With the advances in stochastic reserving methodology, it is now possible to develop coherent theoretical frameworks for the estimation of the distribution of outstanding claims net of excess-of-loss reinsurance. It is important to note that the most commonly used stochastic claims reserving methods, such as bootstrapping the over-dispersed Poisson model (England and Verrall (1999), and England and Verrall (2002)), will be of limited value in this context. The fundamental issue that needs to be addressed is how to consider the net outstanding claims such that the effect of the excess-of-loss reinsurance contract can be accurately taken into account. The only way to do this is to use a model which considers individual claims, or at least one which simulates future claims individually rather than aggregated.

Individual claims reserving, or reserving based on granular data, has been the subject of increased attention in actuarial literature. See for example Antonio and Plat (2014). The majority of the methods which have been developed operate entirely at the level of individual claims and this can perhaps make them appear to be overly complex to implement and use in a practical context. In contrast, a series of papers beginning with Verrall, Nielsen, and Jessen (2010) and continuing with Martínez-Miranda et al. (2011) and Martínez-Miranda, Nielsen, and Verrall (2012) has developed a hybrid approach which uses data aggregated in the standard way into triangles in order to estimate models for claims at the individual level. We believe that this makes it easier to apply the fundamental advantages of stochastic reserving for individual claims using the theory which has recently been developed to more complex practical issues such as excess-of-loss reinsurance. Of course, it would be possible to investigate these practical issues using other individual claims reserving methods, and we anticipate that this may be done in the future by other authors.

In this paper, we use the methodology of the double chain ladder (DCL) method introduced in Martínez-Miranda, Nielsen, and Verrall (2012). Building on the experience of these papers, it is clear that it is very important to have stable estimates of all parameters if practically useful simulations of future claims are to be generated. In particular, the parameters which measure the results can be particularly sensitive to the way claims increase with development period. For this reason we propose in this paper a new modification to the existing methodology which is likely to improve the stability of the results.

The paper is set out as follows. Section 5.2 outlines the approach which is commonly used in practice when considering reserves with reinsurance. Section 5.3 summarises the theoretical model which we will use in this paper, DCL. This section also includes the new modification to DCL, which we call Bornhuetter-Ferguson double chain ladder prior (BDCL prior). In section 5.4, we describe how the data are usually prepared in practice in order to analyse the claims net of reinsurance (and the reinsurers claims). In section 5.5, we show how this can be done in a more coherent way within the framework of DCL and BDCL prior. Sections 5.4 and 5.5 also continue illustrations and comparison of the practical approach and the new approach. Section 5.6 contains the conclusions.

5.2 The practical approach

In general insurance or casualty portfolios (including general third party liability, motor third party liability, employer's liability, medical malpractice) insurance companies commonly seek excess-of-loss reinsurance protection on an occurrence year basis. This means that the insurer's exposure to any individual loss occurring in any given year is limited to a predefined amount called the retention or priority. The retention is usually chosen taking into account the volatility of claims which are likely to arise from the portfolio, the insurer's risk appetite and solvency position. And in practice it is also driven by past experience of claims from the portfolio and the available price in the market. Typically, these reinsurance treaties have a one year duration and are renegotiated every year so that the retention level may change from year to year. There may be clauses in the treaties which affect the actual retention on claims each year: for example, an indexation clause. Thus, whenever data are considered over a period of years for such portfolios, the insurer's retained amounts for any individual loss will be dependent on the year in which the loss occurred.

The estimation of the ultimate net incurred claims in order to set the net total unpaid reserve for such a portfolio is a common actuarial task for reinsurance underwriters when asked to price either a retrospective loss portfolio transfer (LPT) treaty or a prospective proportional quota share (QS) on the retention. In the case of a LPT, the cession to the LPT reinsurer can be either on a gross basis, in which case the cedant will transfer the right of recoveries from excess-of-loss reinsurers to the LPT reinsurer, or a net basis, which means that only the retained loss portfolio is ceded. However, irrespective of the cession basis (assuming an acceptable counterparty rating of the excess-of-loss reinsurers) the evaluation of the reserves is to be done on the loss portfolio net of historical inuring excess-of-loss recoveries. In the case of the prospective QS, the actual historical excess-of-loss retentions are ignored as the estimation of the net outstanding claims is carried out on an 'as-if' basis using a common historical retention equal to that of the prospective excess-of-loss treaty. From a theoretical point of view, QS is a simpler subcase of what would be the more generalized case of the LPT where instead of one common excess-of-loss retention for all years there can be different historical retentions depending on the conditions of each year's excess-of-loss treaty. In this paper, we will consider the QS case, thereby assuming one common retention for all occurrence years.

Typically, the kind of data the reinsurer receives for the purpose of pricing these treaties may come in various formats. If the systems of the insurer are set to account for the existence of excess-of-loss reinsurance, it is possible to receive triangular data with incurred losses already capped at the historical retention. In short, these are known as net triangles. In addition to this, most insurers should be able to query their databases to produce net triangles at a given common retention. In practice, however, the insurance company will either submit gross triangles plus the recoveries triangles, or in the case of QS, gross triangles plus the triangulations of large individual claims, for example with incurred amount at 50% of the prospective retention or above. This is the typical threshold that an excess-of-loss reinsurer sets for the claims data requirement. Receiving triangles of large individual claims means that the QS reinsurer will be able construct the recoveries triangles and price the treaty at different levels of prospective excess-of-loss retention.

In practice, the reinsurance underwriter or actuary will estimate the net outstanding claims (in the case of an LPT) or the ultimate claims (in the case of QS) for each accident year by applying traditional actuarial reserving methods on the net triangles which result from subtracting the reinsurance recovery triangle from the gross triangle.

The problem with this approach is that although actuarial reserving methods can be applied on the resulting net triangle in the same way as they are applied gross triangles, reinsurance recoveries for potential future development of individual claims or newly reported claims are not taken into account because the recoveries triangle construction is limited to the development period already observed. In other words, the recoveries triangle is constructed on the basis of the incurred value at the given valuation date and not on the basis of the ultimate cost of each claim. This presents many issues for the reinsurer to consider. Not only is the ultimate incurred value of a claim unknown, just the incurred value at the particular development point in time, but also the observation period for each of these claims depends on when they were reported. This typically results in there being no recoveries observed in recent accident years. In addition to this, different accident years may have different reporting lags. Ultimately this is a problem of incorrect sampling of the recoveries triangle and this leads to problems with the net triangle to which actuarial reserving methods are applied. As a result of this, it is not clear whether estimating the net reserve using the net triangles constructed this way leads to reasonable point estimates. It may also give a biased distribution of net reserves. The investigation of these issues is the contral purpose of this paper.

As reinsurance is a very competitive business, price is the principal factor for an insurer in deciding whether to cede the portfolio to one particular reinsurer or another. In the case of capital motivated reinsurance transactions, reinsurance competes with other forms of capital such as subordinated debt, and the pricing implications of the estimation of net outstanding claims can also lead to a decision not to cede at all if the cost of reinsurance is directly compared to the cost of the capital relief such a transaction achieves. For these reasons, having more information about the accuracy of the estimation would be very desirable.

The ideal solution to this problem would be to estimate the ultimate incurred for each individual claim. This could be done by modelling the individual aspects of each claim, which could include (for example) loss of income, dependants, future inflation, medical expenses etc. This is the aim of claims adjusters, and it has to be recognised that their estimates can be quite volatile. An actuarial approach would be to simulate from the individual claims so that to estimate the ultimate recoveries per accident year. While there have been considerable advances in the consideration of individual claims data in recent years, the application of the methods would probably still present challenges in practical settings. For this reason, the approach in this paper is to use methods which use aggregated data for the estimation but which are designed in order to allow inferences and simulation to be carried out at the level of individual claims. The framework we use is Double Chain Ladder and its extensions, which are set out in the next section.

5.3 Double Chain Ladder

This section summarises the double chain ladder method (DCL) developed in Verrall, Nielsen, and Jessen (2010) and Martínez-Miranda et al. (2011) and Martínez-Miranda, Nielsen, and Verrall (2012). The formulation of DCL and related models in Martínez-Miranda et al. (2011) and Martínez-Miranda, Nielsen, and Verrall (2012) allows us to estimate the settlement delay and therefore to predict Reported But Not Settled (RBNS) and Incurred But Not Reported (IBNR) reserves separately. In contrast with other approaches (for example Antonio and Plat (2014)) which are also based on individual claims (micro models), our aim is not to perform the estimation using the individual claims data. It would be possible to use such an approach, but we believe that DCL offers a simpler procedure which should be easier to use in the practical context described in Section 5.2 since DCL and the related models are estimated using only data in the aggregated triangles which are usually available in practice. We begin by summarising DCL in Subsections 5.3.1 and 5.3.2 and then define the new modification in Subsection 5.3.3.

5.3.1 Model formulation

The approach of DCL is based on a model defined at the level of individual claims (a micro model) but estimated using data in aggregated triangles. We first describe the micro model, and then show how this can be estimated using conventional triangles of data. The micro model is constructed from three components: the settlement delay, the individual payments and the reported counts.

This section sets out the model assumptions from Martínez-Miranda, Nielsen, and Verrall (2012). If we were just interested in the mean or the best estimate, the model assumptions could be much more general than those below. However, since we are interested in the distributional properties, we generalize below the original assumptions of DCL so that in the following section it is possible to add prior knowledge.

DCL makes use of both the data and expert knowledge extracted from incurred data. The information required to apply DCL are the aggregated incurred counts (data), aggregated payments (data) and aggregated incurred payments (expert knowledge). All of these three objects will have the same structural form, and without loss of generality they are assumed to consist of usual triangles defined on $\mathcal{I}$, where

$$\mathcal{I} = \{(i,j) : i = 1,\dots,m, j = 0,\dots,m-1;\; i+j \leq m\}.$$

Here, $m > 0$ is the number of underwriting or accident years observed. It will be assumed that the reporting delay (the time from the occurance of a claim until it is reported), and the settlement delay (the time between the report of a claim and its settlement) are both bounded by m. This, in contrast to the classical CLM, will make it possible to also get estimates in the "tail" where the reporting delay plus the settlement delay is greater than m. The information required is as follows.

Aggregated incurred counts: $N_{\mathcal{I}} = \{N_{ik} : (i,k) \in \mathcal{I}\}$, with N_{ik} being the total number of claims which were incurred in year i and reported in year $i+k$ (i.e. a reporting delay of k). Note that each of these N_{ik} reported claims is assumed to generate a number of payments, i.e. a claims payment cash flow.

Aggregated payments: $X_{\mathcal{I}} = \{X_{ij} : (i,j) \in \mathcal{I}\}$, with X_{ij} being the total payments from claims incurred in year i and paid with j periods delay from year i.

Note that the meaning of the second suffix of triangle $\mathcal{I}$ varies between the two different sets of data. In the counts triangle it represents the reporting delay and in the payments triangle it represents the development delay, which is reporting delay plus settlement delay.

For the aggregated incurred payments, some theory at the level of individual claims is required.

Let N_{ikl}^{paid} denote the number of the future payments originating from the N_{ik} reported claims, which are paid with a delay of $k+l$, where $l = 0, \dots, m-1$.

Also, let $Y_{ikl}^{(h)}$ denote the individual settled payments which arise from N_{ikl}^{paid}, $h = 1, \dots, N_{ikl}^{paid}$.

Finally, define X_{ikl} to be the aggregate claims originating from underwriting year i, which are reported after a delay of k and paid with an overall delay of $k+l$. Then

$$X_{ikl} = \sum_{h=0}^{N_{ikl}^{paid}} Y_{ikl}^{(h)}, \quad (i,k) \in \mathcal{I},\ l = 0, \dots, m-1,$$

The observed aggregated payments can be written as

$$X_{ij} = \sum_{l=0}^{j} X_{i,j-l,l} = \sum_{l=0}^{j} \sum_{h=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(h)}.$$

The aggregated incurred payments are then considered as unbiased estimators of $\sum_{l=0}^{m-1} X_{ikl}$.

With these definitions, we make the following distributional assumptions.

A1. The numbers of reported claims, N_{ik}, are independent random variables for all (i,k) and have a Poisson distribution with cross-classified mean $\mathrm{E}[N_{ik}] = \alpha_i\beta_k$ and identification $\sum_{k=0}^{m-1}\beta_k = 1$.

A2. Given N_{ik}, the numbers of paid claims follow a multinomial distribution, so that the random vector $(N^{paid}_{i,k,0},\dots,N^{paid}_{i,k,m-1}) \sim \mathrm{Multi}(N_{ik}; p_0,\dots,p_{m-1})$, for each (i,k), where $m-1$ is the assumed maximum delay and $(p_0,\dots,p_{m-1})$ denote the delay probabilities such that $\sum_{l=0}^{m-1} p_l = 1$ and $0 \le p_l \le 1, \forall l = 0,\dots,m-1$.

A3. The individual payments $Y^{(h)}_{i,j-l,l}$ are independent and have a mixed type distribution with Q_i being the probability of a "zero-claim" i.e. $\mathrm{P}\left\{Y^{(h)}_{i,j-l,l} = 0\right\} = Q_i$. We assume that $Y^{(h)}_{i,j-l,l}|Y^{(h)}_{i,j-l,l} > 0$ has a distribution with conditional mean μ_{ij} and conditional variance σ^2_{ij}, for each $i = 1,\dots,m, j = 0,\dots,m-1$. We also assume that the mean depends on the accident year and payment year such that $\mu_{ij} = \mu\gamma_i\delta_j$. Here, μ a common mean factor and δ_j and γ_i can be interpreted as being the inflation in the payment year and the accident year, respectively. The variance follows a similar structure, with $\sigma^2_{ij} = \sigma^2\gamma_i^2\delta_j^2$, where σ^2 is a common variance factor.

A4. *Independence*: We assume that settled payments, $Y^{(h)}_{ikl}$ are independent of the numbers of reported claims, N_{ik}, and also the RBNS and IBNR delays.

Assumption A1 is, apart from the distribution, the classical chain ladder assumption applied to the counts triangle, with the main point being the multiplicativity between underwriting year and reporting delay. Assumptions A2-A4 are necessary to connect reporting delay, settlement delay and development delay - the main idea of DCL. A3 also acknowledges the fact that reported claims can be closed with a payment being made - the so-called zero-claims.

This is a more general situation than Martínez-Miranda, Nielsen, and Verrall (2012) since it assumes that the distribution depends on the accident year and the development year and also allows for zero-claims. Under these assumptions, the first two moments of the unconditional distribution of $Y^{(h)}_{i,j-l,l}$ are given by:

$$
\begin{aligned}
\mathrm{E}[Y^{(h)}_{i,j-l,l}] &= \gamma_i \delta_j (1-Q_i)\mu \\
\mathrm{V}(Y^{(h)}_{i,j-l,l}) &= \gamma_i^2 \delta_j^2 (1-Q_i)\left(\sigma^2 + Q_i \mu^2\right)
\end{aligned}
$$

Following the similar calculations as Martínez-Miranda, Nielsen, and Verrall (2012), it can be shown that under the above assumptions the unconditional mean of X_{ij} can be written as

$$
\mathrm{E}[X_{ij}] = \gamma_i (1-Q_i)\mu\alpha_i \delta_j \sum_{l=0}^{j} \beta_{j-l} p_l = \widetilde{\alpha}_i \widetilde{\beta}_j, \tag{5.3}
$$

where

$$
\widetilde{\alpha}_i = \gamma_i (1-Q_i)\mu\alpha_i
$$

and

$$
\widetilde{\beta}_j = \delta_j \sum_{l=0}^{j} \beta_{j-l} p_l.
$$

Equation (5.3) is the key in deriving the outstanding loss liabilities.

Note that when $Q_i = 0 \; \forall i = 1, ..., m$ and $\delta_j = 1 \; \forall j = 0, \ldots, m-1$, the situation reverts back to the DCL model as set out in Martínez-Miranda, Nielsen, and Verrall (2012).

5.3.2 Parameter estimation for the DCL method

In this section we first set $\delta_j = 1 \quad \forall j$ and $Q_i = 0 \quad \forall i$ and show how to estimate the remaining parameters in DCL. The approach to incorporating the development inflation and the zero-claims probability will be described in the next section.

The case when $\delta_j = 1 \quad \forall j$ and $Q_i = 0 \quad \forall i$ was proposed in Martínez-Miranda, Nielsen, and Verrall (2012) which developed the DCL method to estimate the parameters and a summary of this is provided in this section. The DCL method considers the simple chain-ladder algorithm applied to the triangles of paid claims, $X_{\mathcal{I}}$, and incurred counts, $N_{\mathcal{I}}$. Therefore, as implied by the name Double Chain Ladder, the classical chain-ladder model (CLM) is applied twice and from this everything needed to estimate the outstanding claims is available. It was also shown that this estimation procedure can give identical results as the CLM for paid data when the observed counts are replaced by their fitted values.

An appealing feature of the DCL estimation method is that it uses the estimates of the chain ladder parameters from the triangle of counts and the triangle of payments. Assumption A2 in Section 5.3.1 defined a standard chain-ladder model for the counts data, N_{ij}. A similar model can be defined for the triangle of paid data, X_{ij}, with parameters $\widetilde{\alpha}_i$ and $\widetilde{\beta}_j$. We denote the estimates of the parameters, using the chain-ladder model on each triangle, by $(\widehat{\alpha}_i, \widehat{\beta}_j)$ and $(\widehat{\widetilde{\alpha}}_i, \widehat{\widetilde{\beta}}_j)$, respectively, for $i = 1, \ldots, m, j = 0, \ldots, m-1$. Note that it is straightforward to obtain these estimates using the development factors provided by the chain ladder algorithm, as follows.

Consider the counts triangle (a similar approach can be used for the parameters of the paid triangle) and denote by $\widehat{\lambda}_j$, $j = 1, 2, \ldots, m-1$, the corresponding estimated development factors, where

$$D_{ij} = \sum_{k=1}^{j} N_{ik}.$$

and

$$\widehat{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} D_{ij}}{\sum_{i=1}^{n-j+1} D_{i,j-1}}.$$

Then the estimates of β_j for $j = 0, \ldots, m-1$ can be calculated by

$$\widehat{\beta}_0 = \frac{1}{\prod_{l=1}^{m-1} \widehat{\lambda}_l}$$

and

$$\widehat{\beta}_j = \frac{\widehat{\lambda}_j - 1}{\prod_{l=j}^{m-1} \widehat{\lambda}_l}$$

for $j = 1, \ldots, m-1$. The estimates of the parameters for the accident years can be derived from the latest cumulative entry in each row through the formula:

$$\widehat{\alpha}_i = \sum_{j=0}^{m-i} N_{ij} \prod_{j=m-i+1}^{m-1} \widehat{\lambda}_j.$$

The same procedure can be used to produce $(\widehat{\widetilde{\alpha}}_i, \widehat{\widetilde{\beta}}_j)$ from the triangle of paid data, and the DCL method estimates the rest of the parameters in the model formulated in A1-A4 using just the above estimates. Specifically, the reporting delay probabilities $\{p_0, \ldots, p_{m-1}\}$ can be estimated by solving the linear system given below to obtain estimates of $\{\pi_0, \ldots, \pi_{m-1}\}$.

$$\begin{pmatrix} \widetilde{\beta}_0 \\ \vdots \\ \vdots \\ \widetilde{\beta}_{m-1} \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 & \cdots & 0 \\ \beta_1 & \beta_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \beta_{m-1} & \cdots & \beta_1 & \beta_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \vdots \\ \pi_{m-1} \end{pmatrix}.$$

Once the solution $\{\widehat{\pi}_0, \ldots, \widehat{\pi}_{m-1}\}$ is obtained, these preliminary delay parameters are adjusted to have the desired real probability vector, $(\widehat{p}_0, \ldots, \widehat{p}_{m-1})$ which satisfies the restrictions that $0 \leq \widehat{p}_l < 1$ and $\sum_{l=0}^{m-1} \widehat{p}_l = 1$. For more details of this estimation procedure, see Martínez-Miranda, Nielsen, and Verrall (2012).

For the mean and variance of the distribution of individual payments DCL estimates the inflation parameters, $\gamma = \{\gamma_i : i = 1, \ldots, m\}$, and the mean factor, μ, through the expression:

$$\widehat{\gamma}_i = \frac{\widehat{\widehat{\alpha}}_i}{\widehat{\alpha}_i \widehat{\mu}} \quad i = 1, \ldots, m. \tag{5.4}$$

To ensure identifiability DCL sets $\gamma_1 = 1$, so that μ can be estimated by

$$\widehat{\mu} = \frac{\widehat{\widehat{\alpha}}_1}{\widehat{\alpha}_1}.$$

The inflation parameters, $\widehat{\gamma}_i$, are estimated by substituting $\widehat{\mu}$ into (5.4). It only remains to adjust the final $\widehat{\mu}$ according to the estimates $\widehat{p}_l$ and in order to ensure that $\sum_{k=0}^{m-1} \beta_k = 1$. This is done by dividing $\widehat{\mu}$ by κ, where $\kappa = \sum_{j=0}^{m-1} \sum_{l=0}^{j} \widehat{\beta}_{j-l} \widehat{p}_l$. Hereafter, in a slight abuse of notation, we will retain the notation $\widehat{\mu}$ for the corrected estimator of μ.

The estimate of outstanding claims is obtained by substituting the above estimates into the expression for the unconditional mean. In doing this, it is useful to split it into the Reported But Not Settled (RBNS) and Incurred But Not Reported (IBNR) components by considering payments on already reported claims and claims which will be reported in the future. For $i + j > m$, we define

$$\widehat{X}_{ij}^{rbns} = \sum_{l=i-m+j}^{j} \widehat{N}_{i,j-l} \widehat{p}_l \widehat{\mu} \widehat{\gamma}_i \tag{5.5}$$

and

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=\max(0,j-m+1)}^{i-m+j-1} \widehat{N}_{i,j-l}\widehat{p}_l\widehat{\mu}\widehat{\gamma}_i, \tag{5.6}$$

respectively, where $\widehat{N}_{ij} = \widehat{\alpha}_i\widehat{\beta}_j$.

The estimate of total outstanding claims is calculated by adding the RBNS and IBNR components i.e. $\widehat{X}_{ij}^{DCL} = \widehat{X}_{ij}^{rbns} + \widehat{X}_{ij}^{ibnr}$. This is equivalent to the aim of the standard CLM in just the lower triangle (ignoring any tail effects), i.e. for $(i,j) \in \mathcal{J}_1 = \{i = 2,\ldots,m; j = 0,\ldots,m-1 \text{ so } i+j = m+1,\ldots,2m-1\}$. For the DCL, the estimates of outstanding claims extend further to provide tail estimates by considering $i = 1,\ldots,m$ and $j = m,\ldots,2m-1$.

Finally to provide the full cash flow the predictive distribution can be approximated using parametric bootstrap methods as Martínez-Miranda et al. (2011) described. In order to do this, it is necessary to estimate the variances, σ_i^2 $(i = 1,\ldots,m)$. Verrall, Nielsen, and Jessen (2010) showed that assumptions similar to A1–A4 can be used to show that the conditional variance of X_{ij} is approximately proportional to its mean. Using this result, it is straightforward to estimate the variance using over-dispersed Poisson distributions.

More specifically, the over-dispersion parameter φ (defined in Section 5.3.1) can be estimated by

$$\widehat{\varphi} = \frac{1}{n-m}\sum_{i,j\in\mathcal{I}} \frac{(X_{ij} - \widehat{X}_{ij}^{DCL})^2}{\widehat{X}_{ij}^{DCL}\widehat{\gamma}_i},$$

where $n = m(m+1)/2$ and $\widehat{X}_{ij}^{DCL} = \sum_{l=0}^{m-1} N_{i,j-l}\widehat{p}_l\widehat{\mu}\widehat{\gamma}_i$. Then the variance factor of individual payment can be estimated by

$$\widehat{\sigma}_i^2 = \widehat{\sigma}^2\widehat{\gamma}_i^2$$

for each $i = 1,\ldots,m$, where $\widehat{\sigma}^2 = \widehat{\mu}\widehat{\varphi} - \widehat{\mu}^2$.

5.3.3 The BDCL prior method

In this section, we take the DCL method as set out in Section 5.3.2 and show how to incorporate information about inflation in the severity of individual claims and the number of zero claims. We first assume that the parameters for this are known, and note that they cannot be estimated using the triangles $X_{\mathcal{I}}$ and $N_{\mathcal{I}}$. The estimation of these parameters requires some extra data, which is described at the end of this section.

As in Martínez-Miranda et al. (2015), the payments triangle X_{ij} is first adjusted by dividing by the development inflation δ_j and the zero-claims probability Q_i. This gives a triangle of adjusted payments:

$$\widetilde{X}_{ij} = \frac{X_{ij}}{\delta_j(1-Q_i)}.$$

It is easy to verify that the triangle $\{\widetilde{X}_{ij}; (i,j) \in \mathcal{I}\}$ together with the counts triangle $N_{\mathcal{I}}$ follow model assumptions A1-A4 with $Q_i = 0 \quad \forall i$ and $\delta_j = 1 \quad \forall j$. The DCL method is applied to the adjusted payments triangle and the reported counts triangle as usual and all the DCL parameters are estimated.

Since estimating the underwriting year inflation, γ_i, in DCL is a weak point because it might be estimated with significant uncertainty (see Martínez-Miranda, Nielsen, and Verrall (2013a)), we estimate the underwriting year inflation from the less volatile incurred data.

The model for the incurred triangle, which is technically based on expert knowledge and not actual data is as follows.

Aggregated incurred payments: $I_{\mathcal{I}} = \{I_{ik} : (i,k) \in \mathcal{I}\}$, where

$$I_{ik} = \sum_{s=0}^{k}\sum_{l=0}^{m-1} E[X_{isl}|\ \mathcal{F}_{(i+k)}] - \sum_{s=0}^{k-1}\sum_{l=0}^{m-1} E[X_{isl}|\ \mathcal{F}_{(i+k-1)}],$$

and $\mathcal{F}_h$ is an increasing filtration illustrating the expert knowledge at time point h.

The incurred data are also adjusted in the following way

$$\widetilde{I}_{ik} = \frac{I_{ik}}{(1 - Q_i)}.$$

so that the adjusted payments triangle $\widetilde{X}_{ij}$ and the adjusted incurred triangle $\widetilde{I}_{ik}$ have the same underwriting year inflation.

The next step is based on the so called BDCL method in Martínez-Miranda, Nielsen, and Verrall (2013a), which allows us to use the underwriting year inflation of the incurred data. We apply DCL to the triangles of reported counts, N_{ik}, and the adjusted aggregated incurred claims, $\widetilde{I}_{ik}$. Through the adjustment of the incurred triangle it fulfills the model assumptions A1-A4 with $Q_i = 0$. The parameters $\{\gamma_i : i = 1, \ldots, m\}$ are estimated exactly as described in Section 5.3.2, except that the triangle of aggregate paid claims is replaced by the triangle of adjusted aggregated incurred claims. Then we can replace the DCL underwriting year inflation estimates by those obtained from the adjusted incurred data.

The last step is now to multiply the estimates of the outstanding liabilities we obtained in this procedure by the development inflation δ_j and the zero-claims probability $(1-Q_i)$ again. Let $\widetilde{X}_{ij}^{BDCL}$ be the predicted value of $\widetilde{X}_{ij}$ by using the above described BDCL method. This is obtained exactly as described in the DCL method above by adding the RBNS and IBNR, but with the replaced underwriting year inflation using the adjusted incurred triangle. Then the predicted value of X_{ij} including the prior information will be given by $\widetilde{X}_{ij}^{new} = \delta_j(1-Q_i)\widetilde{X}_{ij}^{BDCL}$, for $(i, j) \in \mathcal{J}_1$. This way it is possible to generate the distribution of future values incorporating the prior information.

The new method then consists of the following six-step procedure:

- *Step 1: Payments triangle adjustment.*
  Devide the payments triangle by the development inflation δ_j and the zero-claims probability $(1-Q_i)$ to get the adjusted payments triangle $\widetilde{X}_{ij} = \frac{X_{ij}}{\delta_j(1-Q_i)}$ to attain to the DCL framework.

- *Step 2: Incurred data adjustment.*
  Devide the aggregated incurred data by the zero-claims probability $(1-Q_i)$ to get the adjusted incurred triangle $\widetilde{I}_{ik} = \frac{I_{ik}}{(1-Q_i)}$ so that the estimate of the underwriting

year inflation doesn't change. Note that I_{ik} are incurred claims for accident year i and development period k.

- *Step 3: Parameter estimation.*

  Estimate the model parameters using DCL for the data in the triangles $N_{\mathcal{I}}$ and $\widetilde{X}_{\mathcal{I}}$ and denote the parameter estimates by $(\widehat{p}_0, \dots, \widehat{p}_{m-1})$, $\widehat{\mu}$, $\widehat{\sigma}^2$ and $\{\widehat{\gamma}_i : i = 1, \dots, m\}$.

  Repeat this estimation using DCL but replacing the adjusted triangle of paid claims by the adjusted triangle of incurred data: $\widetilde{I}_{\mathcal{I}} = \{\widetilde{I}_{ik} : (i,k) \in \mathcal{I}\}$. Keep only the resulting estimated inflation parameters, denoted by $\{\widehat{\gamma}_i^I : i = 1, \dots, m\}$.

- *Step 4: Bornhuetter-Ferguson adjustment.*

  Replace the inflation parameters $\{\widehat{\gamma}_i : i = 1, \dots, m\}$ from the adjusted paid data by the estimates from the adjusted incurred triangle, $\{\widehat{\gamma}_i^I : i = 1, \dots, m\}$.

- *Step 5: DCL prediction.*

  Get the prediction of the outstanding liabilities using DCL, more precisely the RBNS and IBNR estimates as in (5.5) and (5.6).

- *Step 6: Prediction readjustment.*

  Readjust the RBNS and IBNR estimates by multiplying them with the development inflation δ_j and the zero-claims probability $(1 - Q_i)$ and sum them up to get the final estimate of the total outstanding claims in our original framework.

While Martínez-Miranda et al. (2015) did assume prior knowledge on severity development inflation and zero-claims, it did not take advantage of prior knowledge of accident year inflation that often could be extracted from incurred data, see Martínez-Miranda, Nielsen, and Verrall (2013a). We provide a theoretical proof in the appendix for that one can indeed extract the incurred accident inflation and stabilise estimation of the model considered in this paper. The appendix shows that

$$E\left[\widetilde{I}_{ik}\right] = E\left[\frac{I_{ik}}{(1 - Q_i)}\right] = \alpha_i \gamma_i \mu \beta_k \delta_k, \tag{5.7}$$

which shows that $\widetilde{X}_{ij}$ and $\widetilde{I}_{ik}$ have the same underwriting parameters. This justifies that we replace $\widehat{\gamma}_i$ which we obtained by applying DCL on $\widetilde{X}_{ij}$ by the accident year inflation $\widehat{\gamma}_i{}^I$ we got by applying DCL on the adjusted incurred triangle $\widetilde{I}_{ik}$.

Finally, we give an example showing how additional data can be used to provide prior information in practice. As in Martínez-Miranda et al. (2015), the development inflation and the zero-claims probability can be estimated by using a new run-off triangle. Specifically we observe the total number of non-zero payments in accounting year $i+j$ from claims with accident year i and denote this by R_{ij}. The corresponding triangle is denoted by $\mathcal{R}_{\mathcal{I}} = \{R_{ij} : (i,j) \in \mathcal{I}\}$. Note that the variables R_{ij} have cross-classified mean $\mathrm{E}[R_{ij}] = \alpha_i^R \beta_j^R$ for all (i,j). Therefore, we can use the three triangles $(N_I, \mathcal{R}_I, X_I)$ simultaneously and simply apply the chain ladder algorithm three times:

$$\begin{aligned}
&N_{\mathcal{I}} \ \text{ provides the chain ladder estimators } \ \widehat{\alpha}_i \text{ and } \widehat{\beta}_j \text{ for } \alpha_i \text{ and } \beta_j,\\
&\mathcal{R}_{\mathcal{I}} \ \text{ provides the chain ladder estimators } \ \widehat{\alpha}_i^R \text{ and } \widehat{\beta}_j^R \text{ for } \alpha_i^R \text{ and } \beta_j^R,\\
&X_{\mathcal{I}} \ \text{ provides the chain ladder estimators } \ \widehat{\widetilde{\alpha}}_i \text{ and } \widehat{\widetilde{\beta}}_j \text{ for } \widetilde{\alpha}_i \text{ and } \widetilde{\beta}_j.
\end{aligned}$$

Now the probability of zero-claims in the underwriting year, Q_i, can be estimated from the expression

$$\widehat{Q}_i = 1 - \frac{\widehat{\alpha}_i^R}{\widehat{\alpha}_i}.$$

Furthermore, the development inflation parameters can be estimated by

$$\widehat{\delta}_j = \frac{\widehat{\widetilde{\beta}}_j}{\sum_{l=0}^{j} \widehat{\beta}_{j-l} \widehat{\pi}_l} = \frac{\widehat{\widetilde{\beta}}_j}{\widehat{\beta}_j^R}.$$

5.4 Classical chain ladder split

This section explains how the reinsurance split is done in practice. For the purposes of this paper we have used motor third party liability bodily injury loss portfolio data from a medium-sized Greek insurer. The triangular data provided were on an accident year basis with yearly development periods for accident periods from 2000 to 2014 and included all gross bodily injury claims incurred during the period and reported by 31 December 2014, which is the valuation date.

The total portfolio exposure measured as earned vehicle years was by 2014 around 400,000 in comparison to around 100,000 in 2000. In addition to the incurred and paid triangles, we also received the corresponding reported (non-zero) counts triangle and the open claims counts triangles.

Individual claim triangulations were received for claims above EUR 200,000. For this illustration, we have applied a common priority of EUR 500,000 to every individual claim. In order to create the recoveries triangles, we first identify all individual claims for which the excess-of-loss reinsurer has a participation, i.e. for losses whose incurred value based on cumulative payments and the case reserve as at the valuation date exceeds the priority threshold of EUR 500,000. Then the reinsurance recoveries triangles for those individual claims is constructed and aggregated on an accident year basis to match the gross triangles. Finally to construct the net triangles we subtracted the recoveries triangles from the gross triangles.

Tables 5.1, 5.2 and 5.3 in Section 5.B in the appendix show the gross payments triangle as well as the net payments triangle and the recoveries triangle after the split.

We can now apply different reserving methods to all three of these triangles and compare the results. First, for the gross payments triangle, the classical CLM as well as DCL with the appropriate adjustments give a reserve of EUR 138,952,059. We think this is an appropriate estimate and are therefore trying to get a similar result. While the original BDCL method (EUR 168,124,495) as well as the DCL prior method (EUR 160,468,905) give a much higher reserve, the new BDCL prior method has a reserve of EUR 139,434,204, which is very close to the CLM result and may provide some justification for the new modification.

Applied to the net payments triangle, the CLM and DCL provide a reserve of EUR 133,200,160. Similar to the results for the gross payments, the BDCL (EUR 161,432,377) and the DCL prior method (EUR 153,728,832) have a much higher reserve, whereas the new BDCL prior method comes to a reserve of EUR 133,724,859, which is again very close to the CLM reserve.

The results for the recoveries triangle are a bit different. As predicted in section 2, the CLM and DCL reserves are slightly overestimated at EUR 3,667,605. While the BDCL method gives an even bigger reserve of EUR 4,010,271, the DCL prior reserve is appears better estimated with a value of EUR 2,437,866. Again, the new BDCL prior method calculates a reserve which appears to be more appropriate at EUR 2,502,285.

Given these results, the conclusion for this practical approach is that we should apply a bootstrap method based on the new BDCL prior method introduced in Chapter 5.3.3. The bootstrap method is based on the method of Martínez-Miranda et al. (2011), which can be found in the DCL R-package of Martínez-Miranda, Nielsen, and Verrall (2013b). The authors simulate the counts from a Poisson distribution, the individual claims from a gamma distribution and the delay from a Multinomial distribution. This is followed by a Monte Carlo approximation. In addition to this, we include the development inflation and the severity inflation of the incurred triangle, as defined in Chapter 5.3.3.

For simplicity we did not include the option for zero-claims probability in the bootstrap results. Figure 5.1 shows the results of the BDCL prior bootstrap method applied to the gross payments triangle. This shows the cash flow on the left hand side, and the reserve on the right hand side, split into IBNR and RBNS as well as the total reserve, all in Euros. The corresponding estimates, such as the mean for the total reserve of EUR 135,826,096, can be found in Table 5.4 in the appendix.

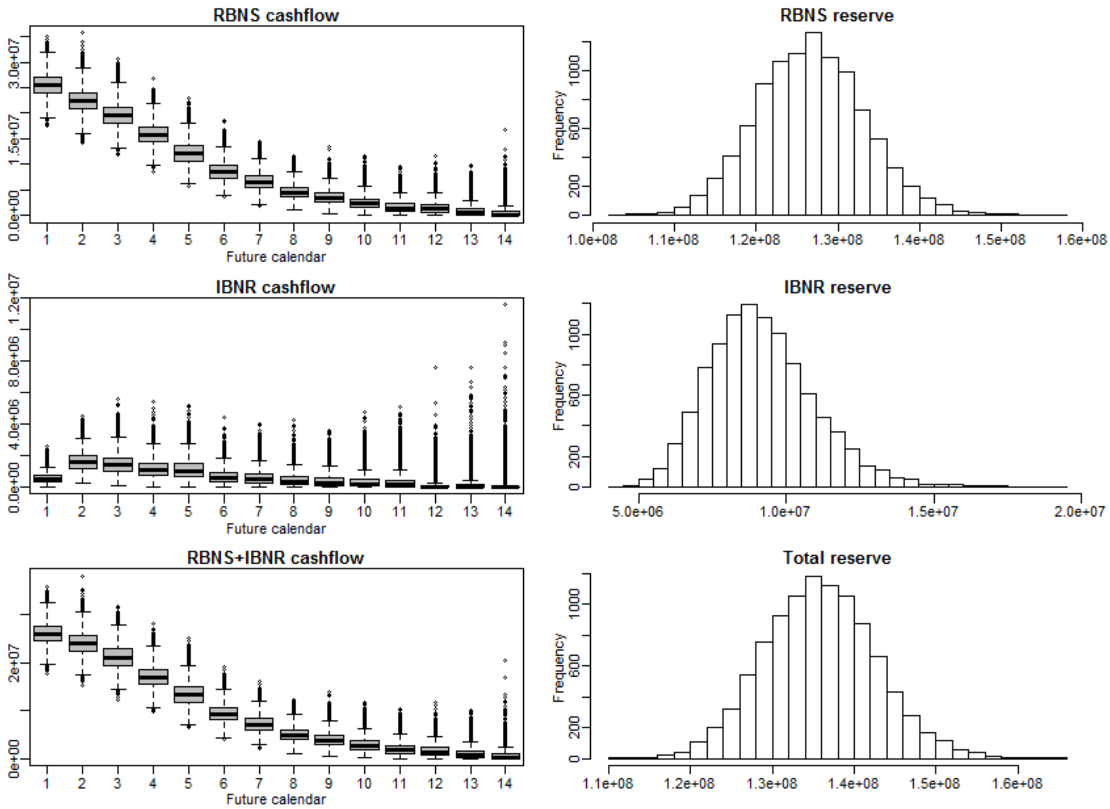

FIGURE 5.1: BDCL prior bootstrap applied on the gross payments triangle, 10,000 times. The cash flow as well as the reserves in Euros. The results are given for the IBNR, RBNS and the total.

The more interesting results for this bootstrap can be found in Figure 5.2, which shows the reserves for the net triangle on the left side together with the recoveries triangle on the right side. These results will be compared to the split done using the new method and simulation of individual claims in the following section. The corresponding estimates can be found in Table 5.5 for the net triangle (mean total reserve = EUR 130,115,416) as well as Table 5.6 for the recoveries triangle (mean total reserve = EUR 2,346,690).

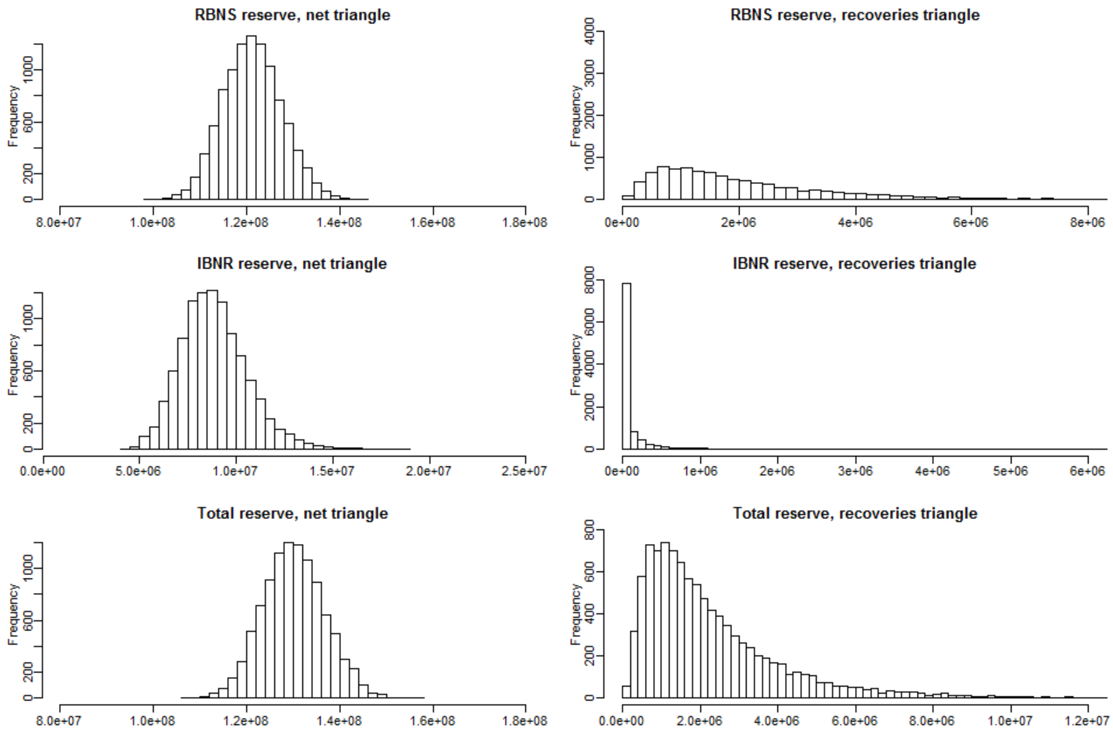

FIGURE 5.2: BDCL prior bootstrap results for the reserves given in Euros, separated into IBNR and RBNS together with the Total reserves. On the left hand side the bootstrap method was applied on the net payments triangle and on the right hand side on the recoveries triangle. The bootstrap was done 10,000 times.

5.5 The prior knowledge double chain ladder split

In this section we consider an alternative method for the split into net payments triangle and recoveries triangle as it is done in practice. This method will simulate individual claims using the method in Section 5.3 which requires just the aggregated data and then split the simulated individual claims using a given retention. When these individual claims are split into a net and a recoveries part, they will be aggregated again and a bootstrap method will be applied. In the following, this simulation method will be explained in detail.

First, the development inflation needs to be extracted using the approach of Martínez-Miranda et al. (2015). Then, following the approach of this paper, the development inflation is divided out of the payments, so that we are in the original DCL model. The BDCL method is applied and we obtain the corresponding parameters, which are used to calculate the mean and variance which we need for the simulation of individual claims. The individual claims are simulated with a Gamma distribution, using the counts which are simulated with a Poisson distribution in line with the DCL framework.

In contrast with the practical approach outlined in the previous section, the split between insurance and reinsurance can be done for each individual claim, where it is simply decided whether the value of the claim is smaller than a predetermined value given by the reinsurance assumptions. If the claim is smaller, it is added to the insurance triangle. If it is bigger, the predetermined value is added to the insurance triangle and the excess is added to the reinsurance triangle.

This process is done separately for IBNR and RBNS claims. Then, the individual claims can be added up to the usual aggregated IBNR and RBNS triangles. This procedure is repeated multiple times. Finally, we multiply the development inflation back to the results and calculate any required statistics such as the mean, quantiles etc.

The new simulation method then consists of the following procedure, following along the steps of Section 5.3.3:

- *Step 1: Payments triangle adjustment.*

  Devide the payments triangle by the development inflation δ_j and the zero-claims probability $(1 - Q_i)$ to get the adjusted payments triangle $\widetilde{X}_{ij} = \frac{X_{ij}}{\delta_j(1-Q_i)}$ to attain to the DCL framework.

- *Step 2: Incurred data adjustment.*

  Devide the aggregated incurred data by the zero-claims probability $(1 - Q_i)$ to get the adjusted incurred triangle $\widetilde{I}_{ik} = \frac{I_{ik}}{(1-Q_i)}$ so that the estimate of the underwriting year inflation doesn't change. Note that I_{ik} are incurred claims for accident year i and development period k.

- *Step 3: Parameter estimation.*

  Estimate the model parameters using DCL for the data in the triangles $N_{\mathcal{I}}$ and $\widetilde{X}_{\mathcal{I}}$ and denote the parameter estimates by $(\widehat{p}_0, \ldots, \widehat{p}_{m-1})$, $\widehat{\mu}$, $\widehat{\sigma}^2$ and $\{\widehat{\gamma}_i : i = 1, \ldots, m\}$.

  Repeat this estimation using DCL but replacing the adjusted triangle of paid claims by the adjusted triangle of incurred data: $\widetilde{I}_{\mathcal{I}} = \{\widetilde{I}_{ik} : (i,k) \in \mathcal{I}\}$. Keep only the resulting estimated inflation parameters, denoted by $\{\widehat{\gamma}_i^I : i = 1, \ldots, m\}$.

- *Step 4: Bornhuetter-Ferguson adjustment.*

  Replace the inflation parameters $\{\widehat{\gamma}_i : i = 1, \ldots, m\}$ from the adjusted paid data by the estimates from the adjusted incurred triangle, $\{\widehat{\gamma}_i^I : i = 1, \ldots, m\}$.

- *Step 5: Mean and variance.*

  Calculate $E[Y_{ij}] = \gamma_i \delta_j \mu$ and $\text{Var}[Y_{ij}] = \gamma_i^2 \delta_j^2 \sigma^2$.

- *Step 6: Simulate counts.*

  Simulate IBNR counts, N_{IBNR}, and RBNS counts, N_{RBNS}, separately using $pois(\alpha_i \beta_j)$.

- *Step 7: Simulate individual payments.*

  Simulate individual payments (IBNR and RBNS separately) using $gamma(E^2/Var, Var/E)$ N_{IBNR}-times and N_{RBNS}-times, respectively.

- *Step 8: Split.*

  The individual claims are split into a net part for the insurance company as well as a recoveries part for the reinsurance company. The split is done by comparing the values of the individual claims to a given retention value for each accident year.

- *Step 9: Aggregate data.*

  The split claims are aggregated into triangles.

- *Step 10: Repeat multiple times.*

  The process is repeated multiple times.

- *Step 11: Prediction readjustment.*

  Readjust the RBNS and IBNR estimates by multiplying them with the development inflation δ_j and the zero-claims probability $(1 - Q_i)$ and sum them up to get the final estimate of the total outstanding claims in our original framework.

- *Step 12: Calculate mean, quantiles etc.*

  Finally, statistical values such as the mean and variance can be calculated.

Figure 5.3 shows this simulation applied on the gross payments data triangle given in Table 5.1 with 10,000 repetitions. This gives the reserves in Euros, split into IBNR and RBNs together with the total reserves. On the left hand side we have the plots for the net triangle resulting from the simulated split, and on the right hand side the results for the recoveries triangle.

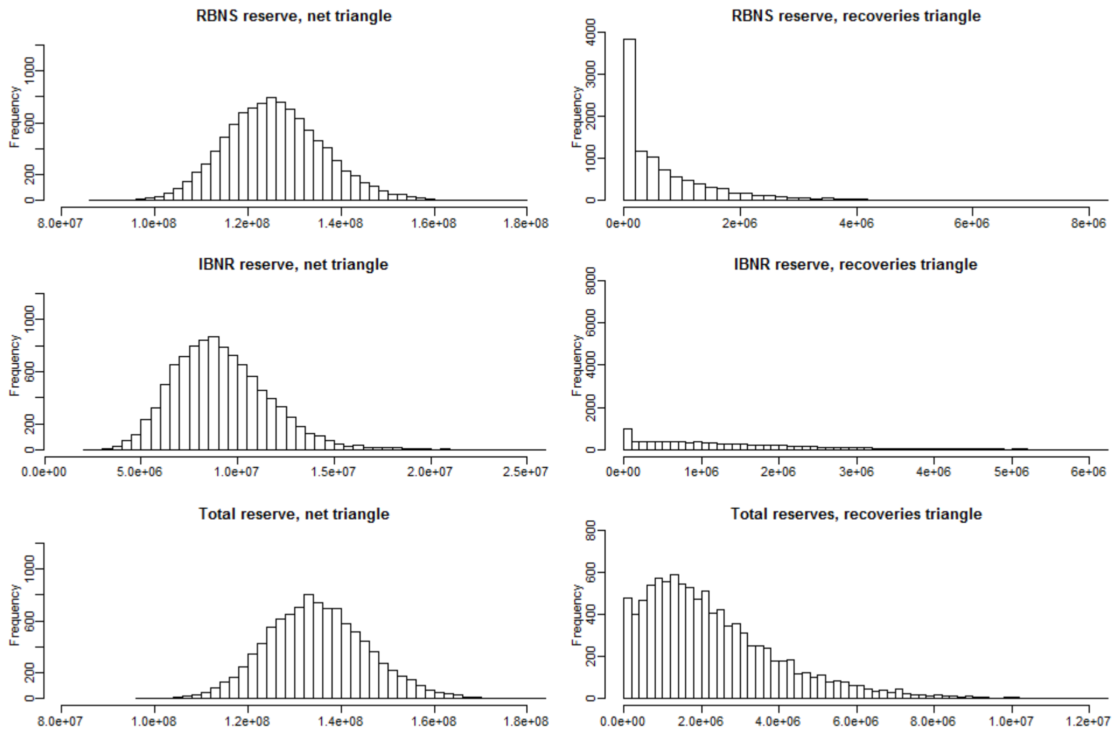

FIGURE 5.3: Results of the simulations for the reserves given in Euros and seperated into IBNR and RBNS, together with the Total reserves. On the left hand side the net payments triangle is shown, where the split was done for the individually simulated claims. On the right hand side the recoveries triangle is presented. The simulation was done 10,000 times.

The results of the new method can be applied to those of the practical approach in the previous section. This is summarised in Figure 5.4.

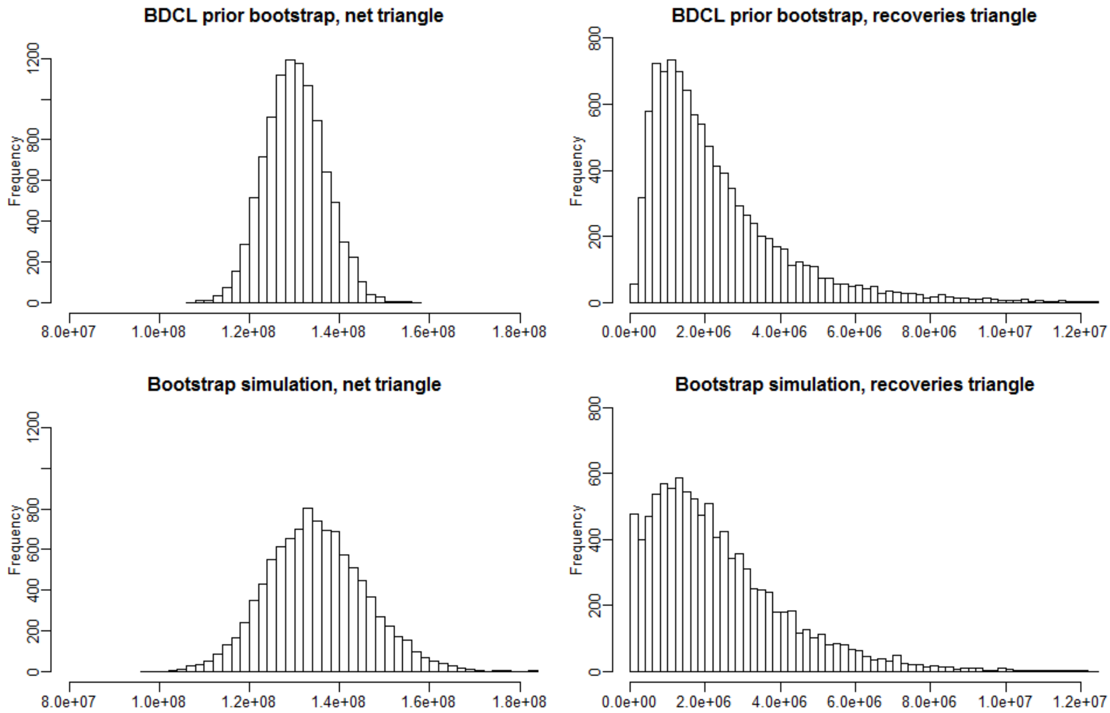

FIGURE 5.4: Total reserves for the BDCL prior bootstrap, applied on the split data from Chapter 5.4, in comparison to the Total reserves from the bootstrap simulation from Chapter 5.5

The left side shows the Total reserves for the net triangles. While the graph for the BDCL prior bootstrap in clearly higher, the tail for the simulation method is clearly bigger on both sides. The results for the Total reserves for the recoveries triangles are similar. The graph for the BDCL prior bootstrap starts lower around zero, but the rest of the two graphs is very similar. However, the distributions of the two different approaches have some similarities but need to have greater consideration in a practial context.

5.6 Conclusions

As presented in Figure 5.4, the results of our new simulation split method can be compared to the results given by the method used in practice. For the new method, the split via simulation presented in Chapter 5.5 is carried out by estimating individual claims from aggregate data and applying a bootstrap method afterwards. This gives a more credible, coherent and flexible framework and it would be very interesting to test this in a practical context where the retention level varies by year. It would also be useful to assess the usefulness of this approach by considering a range of different types of data and to compare with other individual reserving methods. We believe that the framework of DCL is probably easier to use in practice and yet still has enough flexibility for a more coherent consideration of the effects of reinsurance on reserves.

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5.A Appendix: Proof of (5.7)

To justify our new method introduced in Chapter 5.3.3, we need to calculate the expectation of the adjusted incurred triangle.

First, we use the definition of I_{ik}, the tower property and the definition of X_{ikl} to get

$$\begin{aligned}
E[I_{ik}] &= E\left[\sum_{s=0}^{k}\sum_{l=0}^{m-1} E[X_{isl}|\ \mathcal{F}_{(i+k)}] - \sum_{s=0}^{k-1}\sum_{l=0}^{m-1} E[X_{isl}|\ \mathcal{F}_{(i+k-1)}]\right] \\
&= \sum_{s=0}^{k}\sum_{l=0}^{m-1} E[X_{isl}] - \sum_{s=0}^{k-1}\sum_{l=0}^{m-1} E[X_{isl}] \\
&= \sum_{l=0}^{m-1} E[X_{ikl}] \\
&= \sum_{l=0}^{m-1} E\left[\sum_{h=0}^{N_{ikl}^{paid}} Y_{ikl}^{(h)}\right].
\end{aligned}$$

Now, using (A4), we can apply Wald's equation and use the tower property again. Hence, we obtain

$$\begin{aligned}
E[I_{ik}] &= \sum_{l=0}^{m-1} E\left[N_{ikl}^{paid}\right] E\left[Y_{ikl}^{(h)}\right] \\
&= \sum_{l=0}^{m-1} E\left[E\left[N_{ikl}^{paid}|N_{ik}\right]\right] E\left[Y_{ikl}^{(h)}\right].
\end{aligned}$$

Therefore, using (A2), (A1), (A3), and (5.1), we conclude the following unconditional mean for the incurred claims

$$\begin{aligned}
E[I_{ik}] &= \sum_{l=0}^{m-1} E[N_{ik}p_l] E\left[Y_{ikl}^{(h)}\right] \\
&= \sum_{l=0}^{m-1} \alpha_i\beta_k p_l \gamma_i \delta_k (1-Q_i)\mu \\
&= \alpha_i\gamma_i(1-Q_i)\mu\beta_k\delta_k \sum_{l=0}^{m-1} p_l \\
&= \alpha_i\gamma_i(1-Q_i)\mu\beta_k\delta_k.
\end{aligned}$$

Finally, we consider the expectation of $\widetilde{I}_{ik}$, which is the incurred claims triangle where we devided out the zero-claims probability

$$E\left[\widetilde{I}_{ik}\right] = E\left[\frac{I_{ik}}{(1-Q_i)}\right] = \alpha_i \gamma_i \mu \beta_k \delta_k,$$

which proves (5.7). This means that $\widetilde{X}_{ij}$ and $\widetilde{I}_{ik}$ have the same underwriting parameters. Therefore, using either one of these triangles, we estimate the same DCL parameters, including the accident year inflation parameter γ_i. That justifies that we replace $\widehat{\gamma_i}$ which we obtained by applying DCL on $\widetilde{X}_{ij}$ by the accident year inflation $\widehat{\gamma_i}^I$ we got by applying DCL on the adjusted incurred triangle $\widetilde{I}_{ik}$.

5.B Appendix: Data

345115	550155	1165973	1703483	1890192	1165754	1112124	1172613	424153	401842	147602	171353	79772	281345	266548
350771	1307179	1843726	2746214	2587467	1844121	768873	307287	600012	37169	955466	209065	273838	25515	
495836	1698146	2426735	4139259	3010904	3245849	2424486	2040852	722079	1836805	885848	226836	619320		
618012	1711657	2560425	3288447	2973974	2834974	1397271	766589	1327613	692792	525813	159328			
948012	2809693	3788681	3993582	2964559	3255971	812468	1953830	810337	624285	128553				
953333	2325650	4075835	5574008	5441890	4169716	2838634	2334831	1134118	542453					
1266559	3436062	3840499	5820558	5346293	5101668	1912277	2327070	1806402						
1590238	3794768	4470900	4917158	5652933	4753303	3542325	2247495							
1484722	2726065	3958068	4602047	4100856	3243477	1490074								
1832755	2174965	4105838	3803571	3609703	4316400									
1829793	2603989	3768836	4089502	2826510										
1569902	2820043	4116023	2578524											
1391136	3576140	1997271												
2260336	3695117													
981728														

Table 5.1: Gross payments in Euro.

345115	550155	1165973	1703483	1890192	1165754	1112124	1172613	269840	397567	142807	161728	79772	281345	266548
350771	1307179	1843726	2746214	2566190	1844121	768873	307287	600012	37170	917356	209065	273838	25515	
495836	1698146	2425159	4051087	3010904	3131364	2212115	1751729	717586	1558083	612638	217397	499180		
618012	1711657	2560425	3258378	2903194	2741042	1392506	761189	1125816	607190	523861	158426			
948012	2744998	3782286	3954751	2872013	3096314	805336	1945795	789209	604091	96978				
953333	2325650	4075835	5444170	5235030	3869351	2793398	2224029	1040856	516568					
1266559	3436062	3840500	5669342	5345112	4861694	1807798	2244928	1802414						
1590238	3794768	4470899	4917158	5613373	4488985	3438274	2246031							
1484722	2726066	3958068	4590839	4100856	3243477	1490074								
1832755	2174965	4105838	3785096	3605612	4314345									
1829793	2603989	3768836	4037457	2416537										
1569902	2820044	4116022	2578524											
1391136	3576141	1997270												
2260336	3695117													
981728														

TABLE 5.2: Net payments in Euro.

0	0	0	0	0	0	0	0	154313	4275	4795	9625	0	0	0
0	0	0	0	21277	0	0	0	0	0	38110	0	0	0	
0	0	1576	88172	0	114485	212371	289123	4493	278722	273209	9440	120140		
0	0	0	30069	70780	93931	4765	5400	201797	85601	1953	902			
0	64695	6396	38830	92545	159658	7132	8035	21128	20196	31574				
0	0	0	129838	206860	300365	45235	110803	93262	25886					
0	0	0	151216	1180	239975	104479	82142	3988						
0	0	0	0	39561	264317	104051	1465							
0	0	0	11209	0	0	0								
0	0	0	18475	4091	2055									
0	0	0	52045	409973										
0	0	0	0.00											
0	0	0												
0	0													
0														

TABLE 5.3: Recoveries in Euro.

5.C Appendix: Results

	mean.total	sd.total	Q1.total	Q5.total	Q50.total	Q95.total	Q99.total
1	26097618.95	2392931.78	20886771.49	22324140.69	26032394.84	30174384.25	32079872.39
2	24158342.00	2521628.99	18748036.29	20192400.26	24091427.19	28492022.04	30484948.59
3	21139107.92	2530340.00	15801701.38	17228688.02	21014715.81	25525694.20	27528586.39
4	17091987.23	2379023.40	12055839.23	13347033.36	16983395.41	21181064.02	23252390.09
5	13372286.22	2323742.40	8653948.82	9808879.04	13216264.91	17435683.34	19528445.78
6	9353233.12	1938764.78	5505334.88	6405189.05	9234879.15	12725025.87	14589337.48
7	7270948.52	1782991.56	3830245.99	4640998.13	7126495.58	10452121.69	12142565.81
8	5092876.23	1565939.62	2152866.47	2815683.11	4930607.05	7959667.88	9439412.66
9	3962208.40	1526824.45	1300943.34	1856061.17	3763333.34	6731605.89	8528455.08
10	2834724.63	1403185.60	604038.64	986673.29	2617255.21	5409836.47	7127125.04
11	1966353.27	1273631.19	183392.94	412683.67	1713562.13	4430254.95	6110966.12
12	1665933.68	1263099.74	55024.65	211665.11	1369893.70	4069701.90	5879908.19
13	1081440.15	1211645.98	0.00	3929.37	693181.94	3515330.99	5457305.15
14	739036.15	1258684.81	0.00	0.00	197604.85	3293106.90	5580981.11
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00	0.00	0.00
22	0.00	0.00	0.00	0.00	0.00	0.00	0.00
23	0.00	0.00	0.00	0.00	0.00	0.00	0.00
24	0.00	0.00	0.00	0.00	0.00	0.00	0.00
25	0.00	0.00	0.00	0.00	0.00	0.00	0.00
26	0.00	0.00	0.00	0.00	0.00	0.00	0.00
27	0.00	0.00	0.00	0.00	0.00	0.00	0.00
28	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Tot.	135826096.47	6838341.10	120590726.18	124742962.72	135733838.59	147245419.76	152650623.32

Table 5.4: Results from the BDCL prior bootstrap of Chapter 5.4 for the gross payments triangle in Euro. These are the total reserves, so the sum of the IBNR and RBNS. The bootstrap was done 10,000 times.

	mean.total	sd.total	Q1.total	Q5.total	Q50.total	Q95.total	Q99.total
1	25375456.64	2309316.00	20410202.99	21812817.43	25272033.15	29356647.09	31145553.62
2	23367871.35	2391471.13	18264214.15	19600152.45	23304682.05	27451256.58	29394697.61
3	20374818.94	2435560.18	15175869.33	16509581.78	20282202.40	24608492.86	26523551.82
4	16413241.22	2287170.69	11487626.17	12906855.53	16295219.87	20434716.77	22162714.95
5	12717529.98	2223454.61	8145667.34	9350835.16	12539153.03	16613696.28	18518427.48
6	8838253.04	1808325.93	5274691.37	6130174.30	8692292.16	12008506.92	13727742.86
7	6778492.54	1711673.76	3445007.42	4273133.05	6610866.19	9837135.77	11560686.36
8	4737099.84	1455547.35	2077801.13	2629966.98	4581030.90	7373478.63	8819808.31
9	3615560.12	1405362.20	1198578.44	1685708.15	3406266.55	6214069.17	7716290.78
10	2627966.69	1307648.92	552692.38	926132.07	2411894.66	5038921.62	6740394.75
11	1895212.46	1236013.10	154656.55	378060.29	1651657.82	4237719.26	5827700.68
12	1580611.59	1196904.29	51467.78	211778.91	1291175.73	3880311.90	5580858.46
13	1074318.33	1203949.11	0.00	4932.29	679491.51	3577721.41	5309385.83
14	718983.45	1174810.48	0.00	0.00	197268.01	3101279.38	5554770.14
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00	0.00	0.00
22	0.00	0.00	0.00	0.00	0.00	0.00	0.00
23	0.00	0.00	0.00	0.00	0.00	0.00	0.00
24	0.00	0.00	0.00	0.00	0.00	0.00	0.00
25	0.00	0.00	0.00	0.00	0.00	0.00	0.00
26	0.00	0.00	0.00	0.00	0.00	0.00	0.00
27	0.00	0.00	0.00	0.00	0.00	0.00	0.00
28	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Tot.	130115416.19	6562266.35	115379290.79	119627422.38	129969539.83	141239104.59	145533609.00

TABLE 5.5: Results from the BDCL prior bootstrap of Chapter 5.4 for the net payments triangle in Euro. These are the total reserves, so the sum of the IBNR and RBNS. The bootstrap was done 10,000 times.

	mean.total	sd.total	Q1.total	Q5.total	Q50.total	Q95.total	Q99.total
1	585872.24	941073.48	3137.43	9692.16	234273.07	2324506.56	4640738.44
2	293379.56	516258.50	282.59	1565.84	99755.80	1255333.65	2582023.85
3	327232.59	652097.26	325.07	3114.14	108482.10	1371200.64	3107224.35
4	295013.89	623676.31	52.34	903.10	82677.89	1370850.44	3250473.11
5	285731.58	693697.94	0.09	22.14	50689.04	1307049.38	3516983.09
6	143036.48	593527.17	0.00	0.09	6699.26	590149.50	2798207.32
7	82530.96	242632.80	0.00	0.00	3369.64	437610.08	1128328.50
8	216566.46	976588.86	0.00	0.00	539.53	996581.57	4397546.27
9	35267.81	188761.48	0.00	0.00	0.00	144058.40	948118.52
10	33907.97	239799.97	0.00	0.00	0.00	86469.40	1014379.78
11	6000.67	47366.28	0.00	0.00	0.00	6537.46	168103.36
12	42150.26	277582.67	0.00	0.00	0.00	135258.32	1185302.35
13	0.00	0.00	0.00	0.00	0.00	0.00	0.00
14	0.00	0.00	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00	0.00	0.00
22	0.00	0.00	0.00	0.00	0.00	0.00	0.00
23	0.00	0.00	0.00	0.00	0.00	0.00	0.00
24	0.00	0.00	0.00	0.00	0.00	0.00	0.00
25	0.00	0.00	0.00	0.00	0.00	0.00	0.00
26	0.00	0.00	0.00	0.00	0.00	0.00	0.00
27	0.00	0.00	0.00	0.00	0.00	0.00	0.00
28	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Tot.	2346690.46	1988645.51	243455.77	447488.14	1791080.87	6151492.72	9678591.63

Table 5.6: Results from the BDCL prior bootstrap of Chapter 5.4 for the recoveries triangle in Euro. These are the total reserves, so the sum of the IBNR and RBNS. The bootstrap was done 10,000 times.

	mean.total_net	sd.total_net	Q1.total_net	Q5.total_net	Q50.total_net	Q95.total_net	Q99.total_net
1	25905055.39	3727872.86	18100170.70	20132476.50	25711255.14	32201375.33	35508312.24
2	23932228.36	3965650.38	15820465.78	17946494.20	23679016.53	30885342.55	34376799.12
3	20917492.41	3926735.56	12793883.25	14877169.70	20690675.61	27788991.01	31208214.31
4	17049799.96	3786929.48	9634375.59	11338190.75	16765247.54	23598688.19	27331568.40
5	13240383.34	3585513.43	6547856.46	8031280.04	12866692.70	19726623.28	23318292.49
6	9248605.85	2926938.34	3933173.83	5117001.41	8882062.82	14567816.43	17615450.70
7	7219015.05	2807373.23	2490929.63	3427338.94	6835029.89	12397890.85	15656106.74
8	5114767.79	2446660.80	1175107.61	1911161.06	4723691.15	9712334.84	12595022.88
9	3928749.50	2327558.23	561106.50	1058269.55	3510452.03	8218138.60	11571565.67
10	2908164.48	2210888.76	179192.80	463185.85	2362591.36	7073530.70	10415142.51
11	1947509.09	1916662.98	12757.13	95797.52	1397202.77	5713283.57	9201440.39
12	1665008.85	1940642.31	1420.44	22588.46	1015936.26	5613686.03	9156401.85
13	1098807.18	1928443.07	0.00	0.81	294000.20	4877308.49	9461457.99
14	758933.18	1923947.33	0.00	0.00	13448.57	4233172.23	9580323.43
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00	0.00	0.00
22	0.00	0.00	0.00	0.00	0.00	0.00	0.00
23	0.00	0.00	0.00	0.00	0.00	0.00	0.00
24	0.00	0.00	0.00	0.00	0.00	0.00	0.00
25	0.00	0.00	0.00	0.00	0.00	0.00	0.00
26	0.00	0.00	0.00	0.00	0.00	0.00	0.00
27	0.00	0.00	0.00	0.00	0.00	0.00	0.00
28	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Tot.	134934520.44	10884256.09	111265162.53	117982934.21	134411921.90	153765855.98	162619948.01

TABLE 5.7: Results for split via the simulation of Chapter 5.5. Presented are the resulting net payments in Euro. These are the total reserves, so the sum of the IBNR and RBNS. The simulation was done 10,000 times.

	mean.total_Ex	sd.total_Ex	Q1.total_Ex	Q5.total_Ex	Q50.total_Ex	Q95.total_Ex	Q99.total_Ex
1	263869.44	490083.99	0.00	0.00	0.00	1262238.30	2279729.73
2	305291.26	572326.32	0.00	0.00	12446.89	1395779.60	2519679.49
3	316386.93	602782.84	0.00	0.00	0.00	1519499.66	2759573.25
4	301615.74	621647.67	0.00	0.00	0.00	1527947.84	2883860.62
5	283558.21	643016.92	0.00	0.00	0.00	1479181.86	2963271.38
6	198625.05	516634.95	0.00	0.00	0.00	1183692.09	2368634.91
7	161994.67	495942.87	0.00	0.00	0.00	1034987.24	2399881.13
8	124559.56	443331.70	0.00	0.00	0.00	855657.13	2190795.45
9	121863.37	519594.03	0.00	0.00	0.00	793234.00	2456611.84
10	92518.12	436308.94	0.00	0.00	0.00	582338.07	2281324.28
11	78236.23	493011.67	0.00	0.00	0.00	222062.24	2175314.15
12	50266.04	412240.24	0.00	0.00	0.00	0.00	1572289.27
13	43666.00	414266.48	0.00	0.00	0.00	0.00	1526001.39
14	33882.15	460701.52	0.00	0.00	0.00	0.00	164083.49
15	0.00	0.00	0.00	0.00	0.00	0.00	0.00
16	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00
21	0.00	0.00	0.00	0.00	0.00	0.00	0.00
22	0.00	0.00	0.00	0.00	0.00	0.00	0.00
23	0.00	0.00	0.00	0.00	0.00	0.00	0.00
24	0.00	0.00	0.00	0.00	0.00	0.00	0.00
25	0.00	0.00	0.00	0.00	0.00	0.00	0.00
26	0.00	0.00	0.00	0.00	0.00	0.00	0.00
27	0.00	0.00	0.00	0.00	0.00	0.00	0.00
28	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Tot.	2376332.76	1933288.21	0.00	213633.08	1947229.02	5954247.89	8892712.70

Table 5.8: Results for split via the simulation of Chapter 5.5. Presented are the resulting recoveries in Euro. These are the total reserves, so the sum of the IBNR and RBNS. The simulation was done 10,000 times.