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Implicit Network Descriptions of RLC Networks and the Problem of Re-engineering

by

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A thesis submitted for the degree of
Doctor of Philosophy (Ph.D.) in Control Theory

in the
Systems and Control Research Centre
School of Mathematics, Computer Science and Engineering

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Abstract

This thesis introduces the general problem of Systems Re-engineering and focuses to the special case of passive electrical networks. Re-engineering differs from classical control problems and involves the adjustment of systems to new requirements by intervening in an early stage of system design, affecting various aspects of the underlined system structure that affect the final control design problem. Addressing problems of re-engineering requires the development of a system representation able to embody these structural changes. In the case of Re-engineering in passive electrical networks, certain types of re-engineering transformations involve alterations of values or nature of existing elements, modification of network's topology and possible evolution of the network. We resort to the Implicit Network Description $W(s)$ as a unifying representation, which stems from the Impedance/Admittance integral-differential models, since it enables the representation of such parametric and structural changes of the system as perturbations on it. By using tools and results from classical network theory and algebraic systems theory, the thesis deals with the development and study of fundamental system aspects of this new description in terms of McMillan degree, regularity and other system properties of the implicit network description. The thesis also examines the effect of transformations that preserve network cardinality on the Implicit Network Description and particularly in the natural frequencies of the network. This leads to the formulation of Determinantal Frequency Assignment Problems for natural frequency improvements. Using the exterior algebra, algebraic geometry framework we prove sufficient conditions for complex frequency assignability for a special case of network transformations and we examine whether real solutions to the problem exist. Additionally, transformations linked to the variation of network cardinality, are represented as augmentation or reduction in terms of dimension of the Implicit Network Description and by identifying those that remain intact we are in position to define fixed dynamics, enabling the formulation of partial

structure assignment problems. The results derived in this thesis provide the means for addressing the general systems re-engineering problem in a rather structured setup.

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Declaration of Authorship

I, Maria Livada, hereby declare that this thesis titled, ‘Implicit Network Descriptions of *RLC* Networks and the Problem of Re-engineering’ and the work presented in it are my own. I also confirm that:

- The work presented in this Thesis is my own unless stated and referenced in the text accordingly.
- Where I have consulted the published work of others, this is always clearly attributed.
- I have acknowledged all main sources of help.
- I grant powers of discretion to the Librarian of City University London to allow single copies of this Thesis for study purposes, subject to normal conditions of acknowledgement.

Signed:

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Abbreviations

cmi	column m inimal i ndices
DAP	D eterminantal A ssignment P roblem
FAP	F requency A ssignment P roblem
IAI	I mpedance- A dmittance I mplicit
MFD	M atrix F raction D escription
QPR	Q uadratic P lücker R elations
PPM	P ole P lacement M ap
RLC	R esistance, I nductance, C apacitance
rmi	row m inimal i ndices
SES	S tructure E volving S ystems
SoS	S ystem o f S ystems
fed	F inite E lementary D ivisors
ied	I nfinite E lementary D ivisors

Notation

$W(s)$	Implicit Network Description (Operator)
$P(s)$	Implicit Network Pencil (Loop / Nodal)
$\mathbf{L}, \mathbf{R}, \mathbf{C}$	Matrices of Inductors, Resistors and Capacitors
δ_m	Implicit McMillan degree
\mathbb{C}^n	Complex n -space
G	Graph of a network
$\rho(G)$	Rank of a graph G
$\mu(G)$	Nullity of a graph G
ξ	Implicit vector of the state space description
p	derivative operator
$G(s)$	Transfer function
$N_r(s)$	Right polynomial matrix numerator
$D_r(s)$	Right Polynomial matrix denominator
$\mathbb{R}(s)$	Field of rational functions
$\mathbb{R}[s]$	Ring of polynomials
$\mathbf{T}(\mathbf{s}), \mathbf{H}(\mathbf{s}), \mathbf{M}(\mathbf{s})$	Polynomial matrix
$f_i(s)$	Invariant polynomial matrix
I_c	Column minimal indices of rational space
I_r	Row minimal indices of rational space
\mathcal{N}_l	Left-rational vector space
\mathcal{N}_r	Right-rational vector space
C_p	p^{th} compound matrix
$Q_{p,n}$	Set of strictly increasing sequences of p integers
$Q_{p,n}(t)$	Element of the ordered set
$M_{m,n}(\mathbb{F})$	Set of $(m \times n)$ matrices with elements from the field \mathbb{F}

$P^n(\mathbb{C})$	n -Projective space
\mathcal{X}	Topological space
\mathcal{V}	Smooth variety
$\mathcal{A}^*\mathcal{V}$	Intersection ring
$H^*(\mathcal{V}, \Lambda)$	Cohomology Ring
χ	Polynomial map
$Z(s)$	Network impedance matrix / operator
$Y(s)$	Network admittance matrix / operator
G_v	Natural vertex graph of a network
B	Matrix of A-type elements
C	Matrix of T-type elements and
D	Matrix of D-type elements
P (p)	Rosenbrock system matrix pencil
P_t	Frequency Assignment Map
DP_t	Differential of the frequency assignment map
a	Cohomology class
φ_i	Roots of polynomial $t(s)$
\oplus	direct sum
\otimes	tensor product
$\mathcal{E}(x)$	Equivalence class
Adj_p	p-th Adjoint
$p(s)$	target polynomial
$W^{-1}(s)$	Implicit transfer function

Chapter 1

Introduction

The thesis deals with aspects of Systems Re-engineering specialised to the case of passive electrical networks. Re-engineering is a problem different from traditional control problems and this emerges when it is realised that the systems designed in the past cannot perform according to the new performance requirements and such performance cannot be improved by traditional control activities. Re-engineering implies that we intervene in early stages of system design involving sub-processes, values of physical elements, interconnection topology, selection of systems of inputs and outputs and of course retuning of control structures. This is a very challenging problem which has not been addressed before in a systematic way and needs fundamental new thinking, based on understanding of structure evolution during the stages of integrated design [Kar08]. A major challenge in the study of this problem is to have a system representation that allows study of evolution of system properties as well as structural invariants [Mor73, KM80]. For linear systems the traditional system representations, such as transfer functions, state space models and polynomial type models do not provide a suitable framework for study structure and property evolutions, since for every change we need to compute again these models and the transformations we have used do not appear in an explicit form in such models. It is for this reason, for a general system, such system representations are not suitable for study of system representations on re-engineering.

It has been recognized [KL06, Kar11, SR61] that for the special family of systems defined by the passive electrical networks (RLC), there exists a representation introduced by the loop/ nodal analysis, expressed by the impedance/admittance integral-differential models, which have the property of re-engineering transformations of the following type:

1. Changing the values or possible nature of existing elements without changing the network topology,
2. Modifying the network topology without changing network cardinality, that is number of independent loops or nodes,
3. Augmenting or reducing the network by addition or deletion of sub-networks,
4. Combination of all the above transformations.

These kinds of transformations may be represented as perturbations on the original impedance/admittance models. The above indicates that impedance/admittance integral-differential models, which from now on will be referred to as Implicit Network Descriptions is the natural vehicle for studying re-engineering on electrical networks. Although issues related to realisation of impedance/admittance transfer functions within *RLC* topologies, has been the topic of classical network synthesis [BD49, HS14], the system aspects of such descriptions have not been properly considered. Addressing problems of network re-engineering requires the development of the fundamental system aspects of such new descriptions in terms of McMillan degree, regularity and a number of other properties. Certain problems of evolution (of system properties) are linked to Frequency Assignment, as far as natural frequencies under re-engineering and this requires use of techniques developed within control theory for Frequency Assignment Problems [KG84, KLG88, LK95b, LK09].

Thesis Objectives

The main objectives of this research are summarised below:

- (i) Development of system properties for the Implicit Network Descriptions.
- (ii) Defining network transformations under re-engineering and express them as transformations on the Implicit Network Operator $W(s)$.
- (iii) Study of Frequency Assignment under re-engineering.

Approach

Realising the above objectives requires use of various tools and results from classical network theory [SR61, AV73, KM71], especially those related to system modelling, graph

theoretic results, algebraic systems theory and finally the framework for studying Determinantal Assignment Problem (DAP) from control theory, particularly tools from exterior algebra [MM64, KG84], algebraic geometry and intersection theory [Mum76, Ful84, Bor91].

Main Achievements

The main achievements of this thesis are in the area of:

1. System Properties of Implicit Network Descriptions in terms of characterising the property of regularity, McMillan degree, existence of infinite frequencies.
2. Transformations preserving the network cardinality are defined and represented as additive transformations on the Implicit Network Description and this naturally leads to formulation of Determinantal Natural Frequency Assignment problems.
3. Transformations linked to the variation of network cardinality, that is augmentation or deletion of sub-networks are represented as augmentation or reduction (in terms of dimension) of the Implicit Network Description. This leads in a natural way into the identification of fixed dynamics under such transformations and the formulation of partial structure assignment problems.
4. The exterior algebra, algebraic geometry, intersection theory framework [MM64, KG84], [Mum76, Ful84, Bor91] has been specialised to Natural Frequency Assignment of networks under re-engineering. Sufficient conditions for complex frequency assignability have been proven for a certain case of network transformations, the existence of real solutions to the problem has been investigated and necessary conditions for natural frequencies improvements have been established.
5. The new framework for re-engineering is based on autonomous system descriptions, that is they are implicit, without inputs and outputs. Such descriptions provide the means for studying system structure assignment problems by the selection of input-output, however such a problem has been considered for future research.

Thesis Outline

The structure of the thesis is as follows:

In Chapter 2, a summary of the background methodologies, basic definitions and fundamental concepts, that are deployed as background in this thesis, are presented. Fundamental concepts from graph theory and basic results for polynomial matrices and matrix pencils are also provided. Additionally, an abstract version of the Determinantal Assignment Problem (DAP) is stated along with the basic notions from exterior algebra which are essential in the study of this problem. Finally, a summary of notions from algebraic geometry/topology and intersection theory are provided.

The motivation for the study of *RLC* Network Re-engineering problems, as part of the general problem of Systems Re-engineering, is given in Chapter 3. Apart from that, the complexity of the overall problem is explained and different aspects of Re-engineering are presented. Several different aspects of the network theory which are related to this problem and those regarding the Determinantal Assignment Problem (DAP) are reviewed and these results lead to the development of a research agenda for the thesis.

In Chapter 4, the first part is concerned about the two fundamental types of systems modeling in *RLC* networks, i.e. the Admittance/Impedance models, and their corresponding natural topologies. The above analysis leads to the development of the *Implicit Network description* $W(s)$ which is a unifying description of an *RLC* network and its associated *Implicit Network Pencil* $P(s)$. These two descriptions consist a unifying framework for the analysis of the network re-engineering problem and the study of their properties, which is essential for tackling this problem, is considered. Specifically, we restrict ourselves in examining the regularity property of the *Implicit Network Operator* $W(s)$, where a result is derived linked with the connectivity of the network and in studying regularity issues and zero structure of the *Implicit Network Pencil* $P(s)$. The latter one is accomplished by using results derived for the characterization of infinite elementary divisors and cmi, utilizing Toeplitz matrices based on the triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$.

The problem of determining the *Implicit McMillan degree* δ_m of $W(s)^{-1}$, which defines the maximum number of independent dynamical elements required to describe the network fully, is addressed in Chapter 5 and is related with the rank properties of the matrices of the dynamical elements (capacitors and inductances) that characterize an *RLC* network. Furthermore, necessary and sufficient conditions for the *Implicit McMillan degree* to attain its maximum value are developed and links are established between the associated network pencil $P(s)$ and the McMillan degree δ_m of the network.

In Chapter 6, we investigate the effect of certain types of re-engineering transformations on the structure of the *Implicit Network Operator* $W(s)$, or equivalently on the structure of the a triple of matrices $\mathbf{L}, \mathbf{C}, \mathbf{R}$ that characterise the network, through various examples. It is shown that these types of transformations may or may not affect the cardinality (and/or the Implicit McMillan degree δ_m) of the *RLC* network. Finally, the identification of *fixed dynamics* of an *RLC* network, under such transformations, is examined and the main result is derived.

In Chapter 7, the network re-engineering problem under cardinality preserving transformations is examined as a Frequency Assignment Problem. We restrict ourselves in a special case of DAP, that is the Zero Assignment via Diagonal Perturbations and we consider the case where non-dynamical elements (resistors) are added to the network, in order to assign the desired natural frequencies. The zeros of the *Implicit Network Operator* $W(s)$ describe the natural frequencies of the network, which can be tuned to achieve the desired properties. Since we are interested in the generic solvability of the problem we allow complex solutions and we investigate the surjectivity property of the Frequency Assignment Map of the problem, which is linked with the rank of its differential. Then we provide a generic solution by using the Dominant Morphism theorem and we prove that the sufficient conditions hold true. Furthermore, after compactifying \mathbb{C}^n we use the *cohomology ring* of the compactified space $(P^1(C))^n$ to compute the number of solutions of the problem (for a known polynomial with desired frequencies). We distinguish two cases and for each one, we count the number of solutions in terms of the maximum value of the *Implicit McMillan degree* δ_m . Finally, in the last section we examine the frequency assignment problem via diagonal perturbations in an *RLC* network (where resistors are added), for natural frequency improvements. We establish the necessary conditions for the natural frequencies to be assigned in a certain area of the stability region.

Finally, Chapter 8 provides a summary of all the results that are derived in this thesis and issues that are still open and need further research are highlighted. From the open topics that emerge we propose a future research work scheme addressing both the network re-engineering problem and the more general systems re-engineering.

Publications

During the PhD studies the following publications have been made:

Conferences

- N. Karcianas, J. Leventides, and M. Livada. Matrix Pencil Representation of Structural Transformations of Passive Electrical Networks. In *International Symposium on Communications, Control, and Signal Processing*, Athens, Greece, 2014
- N. Karcianas, J. Leventides, and M. Livada. Multi-parameter structural transformations of passive electrical networks and natural frequency assignment. In *2014 22nd Mediterranean Conference on Control and Automation, MED 2014*, pages 984–989, 2014
- J. Leventides, M. Livada, and N. Karcianas. McMillan Degree of Impedance, Admittance Functions of RLC Networks. In *21st International Symposium on Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, 2014
- J. Leventides, M. Livada, and N. Karcianas. Zero assignment problem in rlc networks. *IFAC-Papers On Line*, 49(9):92 – 98, 2016
- N. Karcianas, M. Livada, and J. Leventides. System Properties of Implicit Passive Electrical Networks Descriptions. In *20th IFAC World Congress, IFAC 2017*, Toulouse, France, 2017

Journals

- T. Berger, N. Karcianas, and M. Livada. The mcmillan degree of implicit transfer functions of rlc networks. *Circuits, Systems & Signal Processing (CSSP)*, (Under review)

Chapter 2

Systems and Mathematics Background

2.1 Introduction

The aim of this chapter is to present a summary of the background methodologies, theoretical control results, basic definitions, fundamental concepts and properties that are used as background in this thesis. The various topics presented in this chapter may be found in more detail in the list of references.

The structure of this chapter is as follows: In the section 2.2 we present fundamental definitions and notions from graph theory, which consists the basis for the classical network theory as well as the matrix representation of graphs in terms of fundamental matrices. Next, in section 2.3, basic results for polynomial matrices and matrix pencils are summarized and various invariants are given under strict equivalence of matrix pencils. In section 2.4 the *Abstract Determinantal Assignment problem* is formulated, which is a unifying framework for studying problems of certain nature and the Pole Placement Map (PPM) of the problem is defined, whose onto properties are related with the solvability of the problem. In section 2.5 basic tools from exterior algebra such as the compound matrix and its properties are defined. In the next section (2.6) the Laplace expansion technique is introduced in a simple manner, which will be used extensively in Chapter 5. In section 2.7 a brief description on basic definitions for real and complex varieties is given and the notion of a morphism (for real and complex varieties) is explained. Furthermore, the *Dominant Morphism theorem* is stated, which will be used for the derivation of the sufficient condition for arbitrary frequency assignment in Chapter 7.

Finally, the last section (2.8) is concerned with central aspects in *Intersection Theory* of complex algebraic varieties. A brief discussion about the process of *compactification* is made and how this process affects the intersection problem under consideration. This is also illustrated by means of examples. Furthermore, the *intersection ring* of a variety is introduced, which in turn sets the grounds for defining the *cohomology ring* of a topological space, which in the context of algebraic geometry is an intersection ring. The *cohomology ring* will be utilized in Chapter 7, in order to compute the number of solutions of a system of polynomial equations, defining the Zero Assignment Problem in *RLC* networks.

2.2 Background of Graph Theory and Properties

2.2.1 Linear Graphs

This subsection is concerned with those aspects of electrical network theory that rely on graph theory. Initially, some basic definitions on Linear Graphs [SR61] are presented:

Definition 2.1. Edge or Element: An edge (or element) of a graph is a line segment including its distinct end-points. □

Definition 2.2. Vertex or Node: The endpoint of an edge is called a vertex (or node). □

After introducing the notions of a vertex and an edge, we can easily define a linear graph.

Definition 2.3. : Linear Graph: A linear graph is a collection of edges with the property that the only point in common which two of them have is a vertex (or node). □

It should be stated here that only finite graphs are considered here, i.e. graphs containing finite number of edges and vertices. Some examples of basic linear graphs are shown in figure 2.2.1.

At this point some basic definitions that are essential background material are presented:

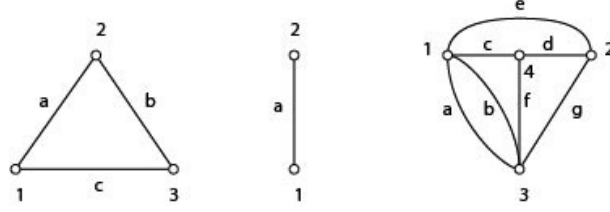


FIGURE 2.1: Examples of Linear Graphs

Definition 2.4. Sub-graph: A subset of the edges of a graph is a sub-graph. Thus, a sub-graph is itself a graph. A sub-graph is called proper if it does not contain all the edges of the graph. \square

Definition 2.5. Initial, final and terminal vertices: An initial vertex is the vertex of the first edge that is not shared by the second edge. Likewise, a final vertex is the vertex of the last edge that is not common to the previous edge. Both the initial and final vertices are called the terminal vertices of an edge sequence. \square

Definition 2.6. Degree of a vertex: The number of edges that are incident to a vertex is called the degree of a vertex. \square

Next, we introduce the notion of a path and of a circuit or loop:

Definition 2.7. Path: A sequence of edges that all appear only once in the sequence is called a path if the degree of each non-terminal or internal vertex of the sequence is 2 and the degree of each terminal vertex is 1. \square

Definition 2.8. Circuit or loop: An sequence of edges as defined in the above definition is called a circuit or a loop if it is closed and all vertices are of degree 2. \square

Definition 2.9. Connected graph: A graph G is connected if there exists a path between any two vertices of the graph. \square

The next figure is an example of a connected and an unconnected graph respectively.

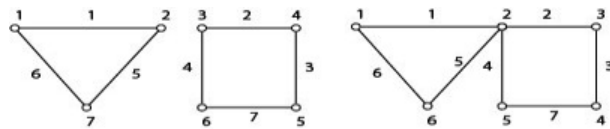


FIGURE 2.2: Unconnected and connected graphs

Finally, we introduce the notion of a complement of a graph G :

Definition 2.10. Complement of a graph G : The complement of a simple linear graph G , where v is the number of vertices of G and E is the number of edges of G is the graph: G' , where its edges are exactly the edges not in G . \square

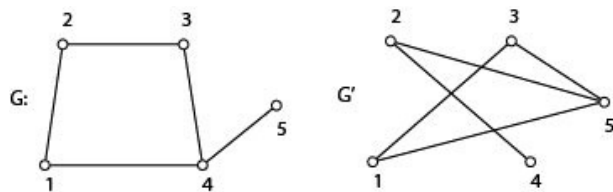


FIGURE 2.3: Graph G and its complement G'

Definition 2.11. Cut vertex of a graph G : A vertex of a graph G is a cut vertex of G if the graph $G - v$ resides of a greater number of components than G . \square

We shall demonstrate this with the following example:

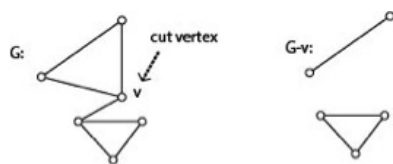


FIGURE 2.4: cut vertex of a graph G

Next, the notion of a *separable graph* is given:

Definition 2.12. Separable graph G : A graph G is separable if either is not connected or there exists at least one cut vertex in the graph. Else, the graph G is non-separable (i.e. if every subgraph of G has at least two vertices in common with its complement.) \square

Remark 2.1. [SR61]

- (a) A connected separable graph G must contain at least one subgraph, which has only one vertex in common with its complement.
- (b) A necessary and sufficient condition that a connected graph be non-separable is that it contains no cut-vertex.

\square

Apart from these, another fundamental issue of graphs are its trees and co-spanning trees. There are necessary for the development of the independent loops as we shall see next.

Definition 2.13. Forest- Sub-forest: A graph G that does not contain any circuits (circuitless) is called a forest. A subgraph of a forest is called a sub-forest. \square

Definition 2.14. Tree- Subtree: A tree is a connected forest. A connected subgraph of a tree is called subtree respectively. \square

Thus, a more formal definition of a tree is that is a connected subgraph of a connected graph, which contains all the vertices of the graph but does not contain any circuits.

Definition 2.15. Spanning tree: A subtree of a connected graph G is called spanning tree if it includes all the vertices of the graph G . \square

Definition 2.16. Cospinning tree: The cospinning tree of a graph G is defined by:

$$G - T \triangleq T * .$$

\square

Definition 2.17. Branches: Branches are called the edges of a spanning tree. \square

Definition 2.18. Links (Chords): Links or chords are called the edges of a co-spanning tree respectively. \square

The above definitions are demonstrated in figure (2.5). Next we give the definition of

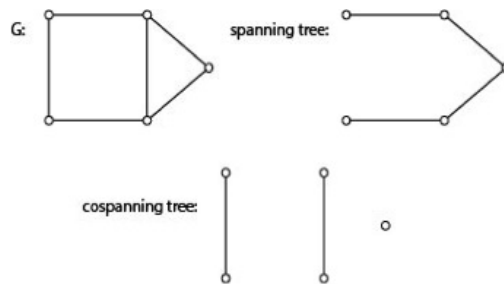


FIGURE 2.5: A graph G , its spanning tree and cospinning tree respectively

fundamental circuits:

Definition 2.19. f-circuits (fundamental circuits): f-circuits of a connected graph G for a tree T are the $e - v + 1$ circuits formed by each chord and its unique tree path. \square

To provide the next definition it is essential to state the *rank* and *nullity* of a graph G .

Definition 2.20. Rank of a graph G : The rank of the graph G is equal to $\rho(G) = n - k$, where n is the number of vertices of the graph and k is the number of maximal connected subgraphs of the graph. \square

Definition 2.21. Nullity of a graph G : We denote the nullity of the graph G as $\mu(G) = m - n + k$, where m denotes the number of edges, n is the number of vertices and k is the number of maximal connected subgraphs of the graph. \square

It is important to note that $\rho(G) \geq 0$ and that $\mu(G) + \rho(G) = m$, where m is the number of edges of the graph.

Definition 2.22. Cut- set of a graph G : A cut-set is a set of edges of a connected graph G such that the removal of these edges from the graph reduces the rank of G by one, provided that no proper subset of this set reduces the rank of G by one when it is removed from G . \square

Thus, it follows that removing the cut-set of edges without their vertices it separates the graph into two pieces, hence the graph is unconnected.

Definition 2.23. f-cut set (fundamental system of cut sets): The fundamental system of cut-sets with respect to a tree T is the set of $v - 1$ cut-sets, one for each branch, in which each cut-set includes exactly one branch of T . \square

Finally, before we establish the notion of an *electrical network*, we describe the notion of *planar* and *directed graphs*.

Definition 2.24. Planar Graphs: A graph is called *planar* if it can be mapped onto a plane and there are no two edges with a common point that is not a vertex. \square

Definition 2.25. Directed Graph: A *directed graph* or *digraph* is a pair (V, E) where V denotes the set of vertices of the graph and E is the set of pair of vertices. The main difference between the usual graphs and the directed graphs is that the elements of E

are ordered pairs, that is the arc from vertex U to vertex V is expressed as (u, v) and the other pair (v, u) is the opposite direction arc. We also have to keep track of the multiplicity of the arc \square

Electrical network theory is formulated in terms of two variables, current and voltage, associated with each network element. We now state the definition of an electrical network [SR61]:

Definition 2.26. Electrical Network: An electrical network is a directed (oriented) linear graph consisting of two real-valued functions $v(t)$, $i(t)$ associated with each edge and which satisfy the vertex and path laws [SR61]. \square

The Vertex and Path laws as well as the development of independent loops are demonstrated in Chapter 4, where an extensive description is given.

2.2.2 Graphs and Matrix Representation

In this subsection we describe the matrix representation of graphs. We restrict the presentation in terms of the following matrices; the *vertex incidence* matrix, the *incidence* matrix of a graph and the *circuit matrix*, as these are related with some of the results in this thesis. An extensive presentation of matrix representations of linear graphs can be found in [SR61].

Vertex Incidence Matrix

For a non empty *directed* graph $G = (V, E)$ that contains no-loops, the *vertex incidence* matrix is a matrix $\mathbf{A} = (a_{ij})$ of dimension $n \times m$, where n denotes the number of vertices, m the number of edges in the graph and each a_{ij} is:

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the initial vertex of } e_j \\ -1, & \text{if } v_i \text{ is the terminal vertex of } e_j \\ 0, & \text{otherwise} \end{cases}$$

Incidence Matrix

We can construct the *incidence matrix* of a graph by eliminating a row from the all vertex incidence matrix and hence the *incidence matrix* of a graph is not unique, as there exist n possible rows that can be removed. The vertex corresponding to the eliminated row is known as the reference vertex.

Circuit Matrix

Let $G = (V, E)$ a directed graph that contains circuits (or loops). The circuits in the directed graph have an orientation, i.e. every circuit is given an arbitrary direction. Then, the entries of the circuit matrix $\mathbf{B} = (b_{ij})$ of the directed graph G are given by:

$$b_{ij} = \begin{cases} 1, & \text{if the arc } e_j \in C_i \text{ and they are in the same direction} \\ -1, & \text{if the arc } e_j \in C_i \text{ and they are in opposite directions} \\ 0, & \text{otherwise} \end{cases}$$

where C_1, \dots, C_l correspond to the circuits of the graph G .

2.3 Polynomial Matrices and Matrix Pencils

[KV02b] In this section we will introduce some fundamental results on *polynomial matrices* and *matrix pencils*, which are essential for the study of properties of the *Implicit Network Operator* and the zero structure of linear systems [Kar09].

State Space and Transfer Function Representations

The most general state-space representation of a linear time invariant multivariable system with p inputs, m outputs and n state variables is given by the following model:

$$S(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) : \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \quad (2.1)$$

where \mathbf{x} is an n - vector describing the *state variables*, \mathbf{u} is a p - vector of inputs and finally, \mathbf{y} is an m - vector of outputs. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are of dimension $n \times n$,

$n \times p$, $m \times n$ and $m \times p$ respectively.

The implicit (autonomous) form of description (2.1) is given by:

$$S(\Phi, \Omega) : \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} x \\ u \\ y \end{bmatrix} \quad (2.2)$$

$\begin{matrix} \triangleq_{\Phi} & \triangleq_{\xi} & \triangleq_{\Omega} & \triangleq_{\xi} \end{matrix}$

where Φ, Ω denote the coefficient matrices and $\xi = [x^t, u^t, y^t]^t$ is the implicit vector of the state space description, which contains the state, input and output vectors and makes no distinction between them. The above description is a generalized autonomous differential description of the form:

$$S(\mathbf{F}, \mathbf{G}) : \mathbf{F}\dot{z} = \mathbf{G}z \quad (2.3)$$

In equation (2.3), \mathbf{F}, \mathbf{G} are matrices of dimension $r \times k$ and z is a k - vector.

The *matrix pencil* $p\mathbf{F} - \mathbf{G}$ is referred as the *implicit system pencil* and characterizes completely the state-space description and the above system. The implicit description (2.2) may be also expressed as:

$$S(\Gamma, \Delta) : \begin{bmatrix} p\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ -\mathbf{C} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}, \quad \mathbf{P}(p) = \begin{bmatrix} p\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ -\mathbf{C} & -\mathbf{D} \end{bmatrix} \quad (2.4)$$

where $\mathbf{P}(p)$ is the matrix pencil, p denotes the derivative operator and it is known as the *Rosenbrock system matrix pencil* [Ros70].

Time domain descriptions may be expressed in the s -domain by introducing Laplace transforms. Thus, the matrix pencils are expressed as *polynomial matrices* in s .

Linear systems can also be expressed in terms of a transfer function model $\mathbf{G}(s)$ as:

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s) \quad (2.5)$$

where $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ denote the Laplace transforms of the output and input vectors respectively and $\mathbf{G}(s)$ is an $m \times p$ rational matrix. Additionally, the *matrix fraction*

description of $\mathbf{G}(s)$ is given by the following form:

$$\mathbf{G}(s) = \mathbf{N}_r(s)\mathbf{D}_r(s)^{-1} = \mathbf{D}_l(s)^{-1}\mathbf{N}_l(s) \quad (2.6)$$

where $\mathbf{N}_r(s)$, $\mathbf{N}_l(s)$ are the $(m \times p)$ right, left polynomial matrix numerators respectively and $\mathbf{D}_r(s)$, $\mathbf{D}_l(s)$ correspond to $p \times p$ and $(m \times m)$ polynomial matrix denominators, where $\mathbf{D}_r(s), \mathbf{N}_r(s)$, and $\mathbf{D}_l(s), \mathbf{N}_l(s)$ assumed right and left coprime respectively.

Proposition 2.1. *For an $m \times p$ rational matrix $\mathbf{G}(s)$ consider the matrix fraction description $\mathbf{G}(s) = \mathbf{N}_r(s)\mathbf{D}_r(s)^{-1} = \mathbf{D}_l(s)^{-1}\mathbf{N}_l(s)$ where $\mathbf{N}_r(s)$, $\mathbf{N}_l(s)$ are the $m \times p$ right, left polynomial matrix numerators respectively and $\mathbf{D}_r(s)$, $\mathbf{D}_l(s)$ are the corresponding $p \times p$, $m \times m$ polynomial matrix denominators. Then,*

a) *The pair $\mathbf{D}_r(s)$, $\mathbf{N}_r(s)$ is right coprime, if and only if the composite matrix*

$$\mathbf{T}_r(s) = \begin{bmatrix} \mathbf{N}_r(s)^t, \mathbf{D}_r(s)^t \end{bmatrix}^t$$

has full rank and no zeros.

b) *The pair $\mathbf{D}_l(s)$, $\mathbf{N}_l(s)$ is left coprime, if and only if the composite matrix*

$$\mathbf{T}_l(s) = [\mathbf{D}_l(s), \mathbf{N}_l(s)]$$

has full rank and no zeros.

□

Polynomial Matrices and Matrix Pencils [Kar09]

Definition 2.27. A $(q \times r)$ matrix $\mathbf{T}(s)$ with elements from the field of rational functions $\mathbb{F} = \mathbb{R}(s)$ is called *rational*, whereas if the elements of the matrix are from the ring of polynomials $\mathbb{R}[s]$ is called *polynomial*. \square

Next, we present the *rank* and the *zeros* of a polynomial matrix.

- The rank of $\mathbf{T}(s)$ over $\mathbb{R}(s)$ is denoted by $\rho = \text{rank}(T(s))$ and is called the *normal rank* of $\mathbf{T}(s)$.
- $\mathbf{T}(s)$ may be viewed as a function of the complex variable s . The *zeros* of $\mathbf{T}(s)$ are the values $s = z$, such that $\text{rank}(T(s)) = \rho_z < \rho$. ρ_z is called *local rank* of $\mathbf{T}(s)$.

The structure of zeros of $\mathbf{T}(s)$ is linked to study of certain form of equivalence defined on such matrices, which reveals the zeros as roots of invariant polynomials [Kar09].

Definition 2.28. Let $\mathbf{T}_1(s), \mathbf{T}_2(s)$ be $q \times r$ polynomial matrices. These matrices are called $\mathbb{R}[s]$ -unimodular equivalent, or simply $\mathbb{R}[s]$ -equivalent, if there exist $q \times q$ and $r \times r$ polynomial matrices $\mathbf{U}_l(s), \mathbf{U}_r(s)$ respectively with the property $|\mathbf{U}_r(s)| = c_1 \neq 0$, $|\mathbf{U}_l(s)| = c_2 \neq 0$ and called $\mathbb{R}[s]$ -unimodular such that:

$$\mathbf{T}_1(s) = \mathbf{U}_l(s)\mathbf{T}_2(s)\mathbf{U}_r(s)$$

This relation reveals an equivalence and for any matrix $\mathbf{T}(s)$ there is an equivalence class and associated invariants. \square

Before we proceed we will introduce the notions of equivalence and invariants.

Definition 2.29. [KV02b] For a set \mathcal{X} , we denote by \mathcal{E} an equivalence relation on \mathcal{X} and let $x \in \mathcal{X}$; the equivalence class, or orbit of x under \mathcal{E} is defined as:

$$\mathcal{E}(x) = \{y : y \in \mathcal{X} : x\mathcal{E}y\}$$

Quotient or *orbit* set is called the set of all equivalence classes and is denoted by \mathcal{X}/\mathcal{E} . \square

Definition 2.30. [KV02b] Let, \mathcal{X}, \mathcal{T} be sets, \mathcal{E} an equivalence relation defined on \mathcal{X} .

We define:

- (i) A function $f : \mathcal{X} \rightarrow \mathcal{T}$ is called an *invariant* of \mathcal{E} , when $\forall x, y \in \mathcal{X} : x\mathcal{E}y$ implies $f(x) = f(y)$,
- (ii) $f : \mathcal{X} \rightarrow \mathcal{T}$ is called a *complete invariant* of \mathcal{E} , when $f(x) = f(y)$ implies $x\mathcal{E}y$,
- (iii) A set of invariants $\{f_i : f_i : \mathcal{X} \rightarrow \mathcal{T}_i, i = 1, 2, \dots, k\}$ is a complete set for \mathcal{E} , if the map defined by $f : \mathcal{X} \rightarrow \mathcal{T}_1 \times \dots \times \mathcal{T}_k$, where $x \rightarrow (f_1(x), \dots, f_k(x))$ is a complete invariant for \mathcal{E} on \mathcal{X} .

□

A complete invariant defines a one to one correspondence between the equivalence classes $\mathcal{E}(x)$ and the image of f . If $f : \mathcal{X} \rightarrow \mathcal{T}_1 \times \dots \times \mathcal{T}_k$ where $x \rightarrow (f_1(x), \dots, f_k(x))$ is a complete invariant for \mathcal{E} on \mathcal{X} , then the set $(f_1(x), \dots, f_k(x))$ characterizes uniquely $\mathcal{E}(x)$. The values $f_i(x)$ are often called invariants [KV02b].

Definition 2.31. [KV02b] A set of *canonical forms* for \mathcal{E} equivalence on \mathcal{X} is a subset \mathcal{C} of \mathcal{X} such that $\forall x \in \mathcal{X}$ there is a unique $c \in \mathcal{C}$ for which $x\mathcal{E}c$.

□

Theorem 2.1. Smith Form [Kar09] If $\mathbf{T}(s)$ is a $q \times r$ polynomial matrix with normal rank $\rho \leq \min(q, r)$ there exist unimodular matrices $\mathbf{U}_l(s), \mathbf{U}_r(s)$ such that:

$$\mathbf{U}_l(s)\mathbf{T}(s)\mathbf{U}_r(s) = \begin{bmatrix} f_1(s) & & & 0 \\ & \ddots & & 0 \\ & & f_\rho(s) & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = \mathbf{S}(s)$$

where $\mathbf{S}(s)$ is $q \times r$ polynomial matrix $f_1(s), \dots, f_\rho(s)$ are uniquely defined and $f_1(s)/f_2(s) \cdots /f_\rho(s)$.

□

The polynomials $f_i(s)$ are called *invariant polynomials* of $\mathbf{T}(s)$ and the set $f_i(s), i = 1, \dots, \rho$ is a *complete invariant* under $\mathbb{R}[s]$ -equivalence. The *finite zeros* of $\mathbf{T}(s)$ are defined by the roots of $f_i(s)$ (including multiplicities). By factorizing the $f_i(s)$ into irreducible factors over the real or complex numbers the structure of these zeros can be defined, i.e.

multiplicities and groupings. The set of z - *elementary divisors* is defined for every zero z by grouping all factors with root at z . The set of all elementary divisors is a *complete invariant* under $\mathbb{R}[s]$ - equivalence [Kar09].

Below we present the definition of a *matrix pencil*.

Definition 2.32. [Kar09] A matrix pencil $s\mathbf{F} - \mathbf{G}$ is a special case of a polynomial matrix, where \mathbf{F} , \mathbf{G} are $q \times r$ real (or complex) matrices and s is an independent complex variable taking values on the compactified complex plane (including points at infinity). \square

Definition 2.33. [Kar09] Two pencils $s\mathbf{F} - \mathbf{G}$, $s\mathbf{F}' - \mathbf{G}'$ of dimension $q \times r$ are *strict equivalent*, if there exist real matrices \mathbf{Q} , \mathbf{R} of dimension $q \times q$, $r \times r$ respectively such that:

$$s\mathbf{F}' - \mathbf{G}' = \mathbf{Q}(s\mathbf{F} - \mathbf{G})\mathbf{R}, \quad |\mathbf{Q}|, |\mathbf{R}| \neq 0$$

\square

Pencils may be represented in a homogeneous form as $s\mathbf{F}' - \hat{s}\mathbf{G}'$, with s, \hat{s} independent complex variables. An ordered pair (α, β) where at least one of the $\alpha, \beta \neq 0$ describes the frequencies on the compactified complex plane. Finite frequencies correspond to $(\alpha, \beta) : \beta \neq 0$. Two single variable pencils may be linked to the homogeneous pencil $s\mathbf{F} - \hat{s}\mathbf{G}$. These are $s\mathbf{F} - \mathbf{G}$ and $s\mathbf{F} - \hat{s}\mathbf{G}$ and some sets of invariants may be defined [Kar09].

Strict Equivalence Invariants of Matrix Pencils

Here we present sets of invariants under strict equivalence of matrix pencils.

Elementary Divisors: [Kar09] The Smith form of the homogeneous pencil $s\mathbf{F} - \mathbf{G}$ defines a set of *elementary divisors* of the following type: s^p , $(s - a\hat{s}^\tau)$, \hat{s}^q . The set of elementary divisors s^p , $(s - a\hat{s}^\tau)$ are called *zero and non-zero finite elementary divisors* (fed) respectively of $s\mathbf{F} - \mathbf{G}$, whereas those of the \hat{s}^q type are called *infinite elementary divisors* (ied) of $s\mathbf{F} - \mathbf{G}$.

Minimal Indices: [Kar09] A matrix pencil $s\mathbf{F} - \mathbf{G}$, where at least one of $\mathcal{N}_r(s\mathbf{F} - \mathbf{G})$, or $\mathcal{N}_l(s\mathbf{F} - \mathbf{G})$ are non trivial, i.e. $\neq 0$ are called *singular*, otherwise they are called *regular*. By $\mathcal{N}_r(s\mathbf{F} - \mathbf{G})$ we define:

$$\mathcal{N}_r(\mathbf{F}, \mathbf{G}) = \{ \underline{x}(s) : (s\mathbf{F} - \mathbf{G})\underline{x}(s) = \underline{0}, \underline{x}(s) \text{ } r \times 1 \text{ vectors} \}$$

and is the right- rational vector space with dimension $\dim \mathcal{N}_r(s\mathbf{F} - \mathbf{G}) = r - \rho$ and by $\mathcal{N}_l(\mathbf{F}, \mathbf{G})$ the left- rational vector space with $\dim \mathcal{N}_l(s\mathbf{F} - \mathbf{G}) = q - \rho$

$$\mathcal{N}_l(\mathbf{F}, \mathbf{G}) = \{ \underline{y}^t(s) : \underline{y}^t(s)(s\mathbf{F} - \mathbf{G}) = \underline{0}, \underline{y}^t(s) \text{ } 1 \times q \text{ vectors} \}$$

If $\mathcal{N}_r(s\mathbf{F} - \mathbf{G}) \neq 0$, then the minimal indices of this rational space are denoted $\mathcal{I}_c(\mathbf{F}, \mathbf{G}) = \{\epsilon_i, i = 1, \dots, \mu\}$ and referred to as *column minimal indices* (cmi) of the pencil. Similarly, if $\mathcal{N}_l(s\mathbf{F} - \mathbf{G}) \neq 0$ then the minimal indices of this rational vector space are denoted by $\mathcal{I}_r(\mathbf{F}, \mathbf{G}) = \{\eta_j, j = 1, \dots, \nu\}$ and referred to as *row minimal indices* (rmi).

In general, if $\mathbf{X}(s)$ is an $r \times (r - \rho)$ polynomial basis for $\mathcal{N}_r(\mathbf{T})$, or any rational vector space \mathcal{X} with $\dim \mathcal{X} = r - \rho$, then it is called *least degree* if it has no zeros. A polynomial basis $\mathbf{X}(s) = (x_1(s), \dots, x_{r-\rho}(s))$ with column degrees $d_1, \dots, d_{r-\rho}$ is said to be of least complexity, if $\sum d_i = \delta(\mathbf{X})$ where $\delta(\mathbf{X})$ stands for the degree of $\mathbf{X}(s)$, which is defined as the maximal of the degrees of all maximal order minors of $\mathbf{X}(s)$. A *minimal basis* is a least degree and least complexity polynomial basis of $\mathcal{N}_r(\mathbf{T})$ and the ordered set of degrees $d_1, \dots, d_{r-\rho}$ are called *right minimal indices* and $\delta_r(T) = \sum d_i$ as the right-order of $\mathbf{T}(s)$. Equivalently, *left minimal indices* and left order are defined on $\mathcal{N}_l(\mathbf{T})$ [Kar09].

2.4 Determinantal Assignment Problem

The *Determinantal Assignment Problem* (DAP) [KG84, KLG88] is fundamental in many areas of classical control theory. DAP approach emerges first and foremost in control system design, when controllers of fixed structure are used to place the poles/ zeros of a system to specific locations [KG84, Wan94]. This approach was firstly introduced by Karcanas and Giannakoulouos [KG84, KG89, KLG88] and has been developed for determinantal problems which are of multilinear nature and thus may be naturally split into

a linear and multilinear problem (decomposability of multivectors), or an intersection of a linear variety with a nonlinear projective variety.

The *Abstract DAP* has been defined as the problem of solving the following equation with respect to polynomial matrix $H(s)$:

$$\det\{H(s) \cdot M(s)\} = f(s) \quad (2.7)$$

where, $f(s)$ is a polynomial of an appropriate degree d and $M(s)$ a given polynomial matrix. It has been proven in [Kar13a], that all dynamics can be shifted from $H(s)$ to $M(s)$. Thus, the problem is transformed to a *constant DAP*. An equivalent formulation of the problem is described below:

Problem 2.1 (Abstract DAP). Given a polynomial matrix $M(s) \in \mathbb{R}^{(m+p) \times p}[s]$, investigate the solvability of the equation:

$$f_M(s, H) = \det\{H \cdot M(s)\} = f(s) \quad (2.8)$$

with respect to $H \in \mathbb{R}^{(p \times (p+m))}[s]$, where $f(s)$ is an arbitrary polynomial of degree equal to the degree¹ of $M(s)$. □

Using the Binet-Cauchy Theorem [MM64] the *constant DAP* can be formulated as follows:

$$C_p(H) \cdot C_p(M(s)) = f(s) \quad (2.9)$$

Then the problem can be factored as a:

- Linear problem: Solve the following equation with respect to \underline{x} :

$$\underline{x} \cdot P = \underline{f} \quad (2.10)$$

- Multi-linear problem: For a given \underline{x} find a matrix such that:

$$\underline{x} = C_p(H) \quad (2.11)$$

¹the maximum polynomial degree of all $p \times p$ minors of $M(s)$.

which is an intersection of a linear variety, with the Grassmann set of all decomposable vectors [KG84].

If H is of the form $H = \begin{bmatrix} I & \Lambda \end{bmatrix}$ and $M(s) = [D(s)^t, N(s)^t]^t$ the composite matrix of a coprime MFD of a strictly proper system, then we can define a map [Lev07]:

$$F : \mathbb{C}^{p \times m} \rightarrow \mathbb{C}^n \quad (2.12)$$

such that:

$$F(\Lambda) = [f_{n-1}, \dots, f_0]$$

where the determinant $\det(D(s) + \Lambda N(s)) = s^n + f_{n-1}s^{n-1} + \dots + f_0$. The map F is defined as the *pole placement map* of the problem, which in turn can be factored in a linear and a multilinear map as illustrated below:

$$F : \mathbb{C}^{p \times m} \xrightarrow{T_1} C^{\sigma_1} \xrightarrow{P_1} \mathbb{C}^n$$

The multilinear map of the problem is:

$$\begin{aligned} T_1(\Lambda) &= C_p([I_p, \Lambda]) \\ F(H) &= C_p([I_p, \Lambda]) P_1 \end{aligned}$$

where $\sigma_1 = \frac{(m+p)!}{m!p!}$, whereas the linear map is represented by the coefficient matrix P_1 of the p -th compound C_p of $M(s)$, i.e.

$$C_p(M(s)^t) = [1, s, \dots, s^n] P_1^t$$

The two central aspects of DAP concern the solvability conditions of the problem and whenever the problem is solvable, to provide methods for constructing solutions which may be distinguished into *exact* and *generic* solutions.

The derivation of solutions in this class of determinantal problems relies on degenerate controllers ². Specifically, the solvability of the problem relies on the surjectivity properties of the related map and especially on the rank of its differential at the degenerate

²more about degenerate controllers may be found in [LK95b, BB81]

controller. That is, when the rank of the differential (of the map) is full at the degenerate controller then the problem is solvable [LK95b]. Generically, this condition is satisfied when the number of controller parameters exceeds the number of independent equations and thus numerical procedures can be utilized for the construction of solutions [Lev07].

The complex solvability of the determinantal problem may be tackled by applying the Dominant Morphism theorem [Bor91, Hum75, MH78] for complex varieties, which relates to the onto properties of a complex rational or polynomial map. In fact, such a map is almost onto when there exists a point in the domain of the map, such that the differential at this point (a linear map) is onto. The surjectivity of the related map constitutes a sufficient condition for arbitrary pole assignment.

Some fundamental results has been developed so far. For a generic system with transfer function $G(s) = \frac{N(s)}{D(s)}$, such that $mp > n$, the PPM F is onto. This case is still open for a non-generic system. The surjectivity property of F was proved by the computation of the differential $D(F)_{\Lambda_0}$ at the degenerate controller Λ_0 . Whenever the $D(F)_{\Lambda_0}$ has full rank, F is onto (for complex and real PPM F). This has been dealt in [LK95b]. Furthermore, the case where $mp = n$ has been examined in [HM77, BB81], which prove that F is generically (almost) onto and is still open for a non-generic system.

2.5 Tools from Exterior Algebra

In this section we present the main tools from exterior algebra and algebraic geometry such as the compound matrices which are very useful and are encountered in several applications.

2.5.1 Lexicographic Ordering [Kar87]

- a. $Q_{p,n}$ denotes the set of strictly increasing sequences of p integers ($1 \leq p \leq n$) chosen from $1, \dots, n$, e.g. $Q_{2,4} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$. The number of sequences that belong to $Q_{p,n}$ is $\binom{n}{p}$. If $\alpha, \beta \in Q_{p,n}$, then α precedes β , i.e. $\alpha \prec \beta$, if there exists an integer t ($1 \leq t \leq p$) for which $\alpha_1 = \beta_1, \dots, \alpha_{t-1} = \beta_{t-1}, \alpha_t < \beta_t$, where α_i, β_i denote the elements of α, β respectively. For example, in the set $Q_{3,8}$,

$(3, 5, 8) < (4, 5, 6)$. That is the *lexicographic ordering* of the elements in $Q_{p,n}$. The set of sequences $Q_{p,n}$ will be assumed with its sequences lexicographically ordered and the elements of the ordered set $Q_{p,n}$ will be denoted by $Q_{p,n}(t)$, $t = 1, 2, \dots, \binom{n}{p}$ or simply by ω .

- b.** The subset of $Q_{p,n}$ whose sequences do not contain any of the indices of a given $\alpha \in Q_{p,n}$ will be denoted by $Q_{p,n}^\alpha$, e.g. $Q_{2,4}^\alpha = \{(1, 4)\}$, if $\alpha = (2, 3)$. The number of elements in this set is equal to $\binom{n-p}{p}$. The elements of $Q_{p,n}^\alpha$ will be denoted by $Q_{p,n}^\alpha(t)$ or by ω_α .
- c.** If k_1, \dots, k_n are elements of the field \mathbb{F} and $\omega = (i_1, \dots, i_p)$ is a sequence in $Q_{p,n}$, ($1 \leq p \leq n$), then the product k_{i_1}, \dots, k_{i_p} will be denoted by k_ω .
- d.** Assume that $\mathbf{A} = [a_{ij}] \in \mathbf{M}_{m,n}(\mathbb{F})$, where $\mathbf{M}_{m,n}(\mathbb{F})$ denotes the set of $(m \times n)$ matrices with elements from the field \mathbb{F} ; let k, p be positive integers that satisfy $1 \leq k \leq m$, $1 \leq p \leq n$ and let $\alpha = (i_1, \dots, i_k) \in Q_{k,m}$ and $\beta = (j_1, \dots, j_p) \in Q_{p,n}$. Then $\mathbf{A}[\alpha | \beta] \in \mathbf{M}_{k,p}(\mathbb{F})$ denotes the submatrix of \mathbf{A} which contains rows i_1, \dots, i_k and columns j_1, \dots, j_p .

2.5.2 Compound Matrices

In mathematics and particularly in the field of exterior algebra, the p -th compound matrix (or the p -th adjugate) of an $m \times p$ matrix $\mathbf{A} \in \mathbb{F}^{m \times n}$ is the $\binom{m}{p} \times \binom{n}{p}$ matrix formed from the determinants of all $p \times p$ sub-matrices of \mathbf{A} , i.e. $p \times p$ minors, whose matrix entries are arranged in lexicographic order as it was demonstrated in subsection (2.5.1).

For the case of 2-vectors, if $\{\underline{e}_i \otimes \underline{e}_j\}_{(i,j) \in \{1,2,\dots,n\}, i \neq j}$, is a basis of $\mathcal{V} \times \mathcal{V}$, $\dim \mathcal{V} = n$, then

$$\begin{aligned}
 \underline{x} \wedge \underline{y} &= (x_i \underline{e}_i) \wedge (y_j \underline{e}_j) = (x_i \underline{e}_i) \otimes (y_j \underline{e}_j) - (y_j \underline{e}_j) \otimes (x_i \underline{e}_i) \\
 &= x_i y_j \underline{e}_i \otimes \underline{e}_j - y_j x_i \underline{e}_j \otimes \underline{e}_i = x_i y_j \underline{e}_i \wedge \underline{e}_j \\
 &= x_i y_j \underline{e}_i \wedge \underline{e}_j + x_j y_i \underline{e}_j \wedge \underline{e}_i, \quad i < j \\
 &= (x_i y_j - x_j y_i) \underline{e}_i \wedge \underline{e}_j, \quad i < j \\
 &= \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \underline{e}_i \wedge \underline{e}_j, \quad i < j
 \end{aligned}$$

Thus a decomposable 2-vector may be derived by the 2-minors of a matrix. Next, follows an extensive definition of the compound matrix, sometimes called the p -th exterior power of \mathbf{A} .

Definition 2.34 (Compound Matrix [MM64]). The p - compound matrix of a matrix $\mathbf{A} \in \mathbb{F}^{m \times n}$, $1 \leq p \leq \min\{m, n\}$ is a $\binom{m}{p} \times \binom{n}{p}$ matrix whose entries are $\det(\mathbf{A}[\alpha|\beta])$, $\alpha \in Q_{p,m}$, $\beta \in Q_{p,n}$ arranged lexicographically in α and β . This matrix will be designated by $C_p(\mathbf{A})$. To demonstrate this, we present the following example:

If $\mathbf{A} \in \mathbb{F}^{3 \times 3}$ and $p = 2$, the $Q_{2,3} = \{(1, 2), (1, 3), (2, 3)\}$ and

$$C_2(\mathbf{A}) = \begin{bmatrix} \det \{\mathbf{A}(1, 2) | (1, 2)\} & \det \{\mathbf{A}(1, 2) | (1, 3)\} & \det \{\mathbf{A}(1, 2) | (2, 3)\} \\ \det \{\mathbf{A}(1, 3) | (1, 2)\} & \det \{\mathbf{A}(1, 3) | (1, 3)\} & \det \{\mathbf{A}(1, 3) | (2, 3)\} \\ \det \{\mathbf{A}(2, 3) | (1, 2)\} & \det \{\mathbf{A}(2, 3) | (1, 3)\} & \det \{\mathbf{A}(2, 3) | (2, 3)\} \end{bmatrix}$$

□

It is clear that, the special case $p = \binom{n}{m}$ implies an $\binom{n}{p}$ - dimensional column-vector $C_p(\mathbf{A})$, which is decomposable. Hence, if $\mathbf{A} = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k) \in \mathbb{F}^{n \times p}$, $1 \leq p \leq n$ then

$$C_p(\mathbf{A}) = \underline{a}_1 \wedge \underline{a}_2 \wedge \dots \wedge \underline{a}_p \quad (2.13)$$

and the entries of the p -th compound of matrix \mathbf{A} , i.e. $C_p(\mathbf{A})$ are the Plücker coordinates.

The following fundamental theorem is essential for the development of several parts in this thesis.

Theorem 2.2 (Binet-Cauchy Theorem [MM64]). *If $\mathbf{A} \in \mathbb{F}^{m \times n}$, $\mathbf{B} \in \mathbb{F}^{n \times k}$ and $1 \leq p \leq \min\{m, n, k\}$ then the following equality holds*

$$C_p(\mathbf{A} \cdot \mathbf{B}) = C_p(\mathbf{A}) \cdot C_p(\mathbf{B}) \quad (2.14)$$

which expresses in a form of compound matrices the composition law of the exterior powers of linear maps when matrix representations are considered. \square

Remark 2.2. Properties of Compound Matrices [MM64]

- i) $(C_p(\mathbf{A}))^t = C_p(\mathbf{A}^t)$, where \mathbf{A}^t is the transpose of \mathbf{A} .
- ii) $C_p(\lambda \mathbf{A}) = \lambda^p C_p(\mathbf{A})$, $\lambda \in \mathbb{F}$.
- iii) $C_p(I_n) = I_{\binom{n}{p}}$, where I_p is the $p \times p$ identity matrix.
- iv) $(C_p(\mathbf{A}))^{-1} = C_p(\mathbf{A})^{-1}$
- v) $C_p(\mathbf{A})^* = (C_p(\mathbf{A}))^*$, where \mathbf{A}^* is the conjugate transpose of \mathbf{A} $\mathbb{F} = \mathbb{C}$.
- vii) $C_p(\overline{\mathbf{A}}) = \overline{C_p(\mathbf{A})}$, where $\overline{\mathbf{A}}$ is the conjugate of \mathbf{A} .
- viii) **Sylvester - Franke Theorem:** $\det(C_p(\mathbf{A})) = (\det \mathbf{A})^{\binom{n-1}{p-1}}$ \square

2.6 Laplace Expansion Technique

[Mey00] In this section the generalized *Laplace Expansion* technique is introduced and demonstrated how it can be utilized for the computation of determinants. The technique is revisited in more detail in the context of the cofactor. This technique is essential as it in the derivation later results.

For an $n \times n$ matrix \mathbf{A} , let

$$\mathbf{A} (i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k)$$

the $k \times k$ submatrix of \mathbf{A} that lies on the intersection of i_1, i_2, \dots, i_k rows and j_1, j_2, \dots, j_k columns, and

$$M(i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k)$$

the $(n - k) \times (n - k)$ minor determinant obtained by deleting the i_1, i_2, \dots, i_k rows and j_1, j_2, \dots, j_k columns respectively from the matrix \mathbf{A} .

The *cofactor* of $\mathbf{A}(i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k)$ is defined as the signed minor:

$$\widehat{A}(i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k) = (-1)^{i_1 + i_2 + \cdots + i_k + j_1 + j_2 + \cdots + j_k} M(i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k)$$

Equivalently, for each fixed set of column indices $1 \leq j_1 \leq \cdots \leq j_k \leq n$ the determinant of \mathbf{A} may be expressed as:

$$\det(\mathbf{A}) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \det \mathbf{A}(i_1 \cdots i_k | j_1 \cdots j_k) \widehat{A}(i_1 \cdots i_k | j_1 \cdots j_k) \quad (2.15)$$

where each of the sums in equation (2.15) contains $\binom{n}{k}$ terms.

2.7 Complex, Real Varieties and Morphisms

In this section the basic notions of real and complex varieties are introduced [Mum76, Lev93, Hum75].

Fundamental Notions on Varieties

Initially, we introduce the notions of *projective* and *affine varieties*.

Definition 2.35. Affine variety: A set \mathcal{X} of \mathbb{F}^n whose coordinates, i.e. $x = (x_1, x_2, \dots, x_n)$ satisfy the polynomial equations $f_i(x) = 0$, $1 \leq i \leq p$ is called an *affine variety* and will be denoted as \mathcal{V} . □

If we define a *projective space* $\mathbb{P}^n(\mathbb{F})$ over a field \mathbb{F} , then the *projective variety* is defined as follows:

Definition 2.36. Projective variety: The set of all points of $\mathbb{P}^n(\mathbb{F})$ whose coordinates satisfy the following homogenous polynomial equations $f_i(x_0, x_1, x_2, \dots, x_n) = 0$, $1 \leq i \leq p$ is a *projective variety* $\bar{\mathcal{X}}$. \square

We shall note here that every affine variety \mathcal{X} in \mathbb{F}^n can be compactified to a projective variety $\bar{\mathcal{X}}$ in $\mathbb{P}^n(\mathbb{F})$ and vice versa.

A subset of a variety \mathcal{V} that satisfies an additional set of equations is called *subvariety* of \mathcal{V} . If a variety \mathcal{V} cannot be expressed as a sum of two proper subvarieties is called *irreducible*, otherwise is called *reducible*.

The topology that stems from defining all closed sets of a variety \mathcal{V} as its subvarieties is a *Zarisky topology* and the open sets of this topology are called *Zarisky open sets*.

In general the *dimension* of a variety \mathcal{V} is the minimum number of independent parameters that define the variety. in other words, the *dimension* of an irreducible variety \mathcal{V} is the dimension of the tangent space (for tangent space see [Mum76]) of a smooth point of \mathcal{V} . Computationally, the *dimension* of a variety is given by $n - \text{rank}(J)$, where J denotes the *Jacobian*, i.e. $J = \frac{\partial f_i}{\partial x_j}$ calculated on a smooth point of the variety and n is the dimension of the underlying space [Lev93].

If $\mathcal{V}_1, \mathcal{V}_2$ two projective varieties in \mathbb{P}^n defined by the equations

$$\begin{aligned} f_i(x_0, x_1, x_2, \dots, x_n) &= 0, \quad 1 \leq i \leq p_1 \\ h_j(x_0, x_1, x_2, \dots, x_n) &= 0, \quad 1 \leq j \leq p_2 \end{aligned} \tag{2.16}$$

then the *intersection of varieties* $\mathcal{V}_1, \mathcal{V}_2$ is defined by the points of \mathbb{P}^n which satisfy both equations simultaneously and will be denoted by $\mathcal{V}_1 \cap \mathcal{V}_2$.

The union $\mathcal{V}_1 \cup \mathcal{V}_2$ of two projective varieties $\mathcal{V}_1, \mathcal{V}_2$ in \mathbb{P}^n is defined by the points of \mathbb{P}^n that satisfy the equations:

$$f_i(x_0, x_1, x_2, \dots, x_n)h_j(x_0, x_1, x_2, \dots, x_n) = 0 \text{ for } 1 \leq i \leq p_1 \text{ and } 1 \leq j \leq p_2 \tag{2.17}$$

For two projective varieties $\mathcal{V}_1, \mathcal{V}_2$ to intersect the following condition should hold:

Lemma 2.1. [Lev93] Let $\mathcal{V}_1, \mathcal{V}_2$ two projective varieties in $\mathbb{P}^n(\mathbb{C})$. The variety $\mathcal{V}_1 \cap \mathcal{V}_2$ is nonvoid and $\dim(\mathcal{V}_1 \cap \mathcal{V}_2) \geq \dim \mathcal{V}_1 + \dim \mathcal{V}_2 - n$ if $\dim \mathcal{V}_1 + \dim \mathcal{V}_2 \geq n$.

The variety $\mathcal{V}_1 \cap \mathcal{V}_2$ is generically empty if $\dim \mathcal{V}_1 + \dim \mathcal{V}_2 < n$. \square

Equivalently, for two affine varieties $\mathcal{V}_1, \mathcal{V}_2$ in \mathbb{C}^n we have the following lemma.

Lemma 2.2. [Lev93] Let two irreducible affine varieties $\mathcal{V}_1, \mathcal{V}_2$ in \mathbb{C}^n . Then either

a. $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, or

b. $\dim(\mathcal{V}_1 \cap \mathcal{V}_2) \geq \dim \mathcal{V}_1 + \dim \mathcal{V}_2 - n$.

\square

If \mathcal{V}_1 and \mathcal{V}_2 are Zarisky open subsets of the projective varieties, then their intersections can be analyzed by using their closures $\bar{\mathcal{V}}_1, \bar{\mathcal{V}}_2$ and lemma 2.1 [Lev93].

Morphisms of Complex and Real Varieties

At this point we will present the notion of a *morphism* for both complex and real varieties and we will introduce the *Dominant Morphism theorem* for complex varieties, which is essential for establishing some of the results in this thesis.

Morphisms of Complex Varieties

If \mathcal{X}, \mathcal{Y} two *affine* varieties, then a **morphism** $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is a map defined by $\phi = (\phi_1, \dots, \phi_n)$, where ϕ_1, \dots, ϕ_n are polynomial functions.

In the case where \mathcal{X}, \mathcal{Y} two *projective* varieties, then then a **morphism** $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is a map defined by $\phi = (\phi_1, \dots, \phi_n)$, where ϕ_1, \dots, ϕ_n are homogeneous polynomial functions of the same degree [Lev93, Hum75].

Next, we state when a morphism is called *dominant*.

Definition 2.37. Let \mathcal{X}, \mathcal{Y} , two irreducible affine varieties. A *morphism* $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is called **dominant** if the image is dense in \mathcal{Y} , i.e. $\phi(\bar{\mathcal{X}}) = \mathcal{Y}$ \square

A **dominant morphism** is very close to be onto, i.e. there is a Zarisky open subset of \mathcal{Y} , \mathcal{U} , such that $\mathcal{U} \subset \phi(\mathcal{X})$. To check whether a morphism $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is *dominant* it is sufficient to find a point $x \in \mathcal{X}$ where ϕ is locally onto. This can be achieved by calculating the differential $(D\phi)_x$ at the point $x \in \mathcal{X}$; if the differential is onto then ϕ is locally onto at $x \in \mathcal{X}$ [Lev93].

Corollary 2.1. *If $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ a morphism of varieties and $\exists x \in \mathcal{X}$ such that the differential $(D\phi)_x$ is onto, then ϕ is almost onto.* \square

Finally, we present the *Dominant Morphism* theorem.

Theorem 2.3. Dominant Morphism Theorem [Hum75]

If ϕ is an algebraic map between two complex varieties \mathcal{X} and \mathcal{Y} such that $\dim \mathcal{X} \geq \dim \mathcal{Y}$ then $\exists x \in \mathcal{X}$: $\text{rank } D\phi_x = \dim \mathcal{Y}$ if and only if ϕ is (almost) onto. \square

Morphisms of Real Varieties

A morphism can be described similarly for the case of real affine varieties and projective real varieties. Unlike the case of complex varieties, where the image of a projective variety through a morphism is always a variety, in the case of real varieties the image of a morphism is a semialgebraic set.

Next, we will state the notion of a *dominant morphism* for the case of real varieties.

If $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ a morphism of two irreducible varieties \mathcal{X}, \mathcal{Y} , the ϕ is called **dominant** if and only if $\phi(\bar{\mathcal{X}}) = \mathcal{Y}$. We can test whether the morphism is dominant via the rank of its differential at some point $x \in \mathcal{X}$. The difference between the complex and the real case is that in the real case, if the morphism is dominant it is not implied that $\phi(\mathcal{X})$ covers almost the whole \mathcal{Y} . In fact, the image $\phi(\mathcal{X})$ has dimension equal to the dimension of \mathcal{Y} and is defined by inequalities [Lev93, Hum75].

2.8 Intersection Theory of Complex Algebraic Varieties

2.8.1 Compactification

[Lev93] The Zero Assignment Problem which is a subproblem of the Determinantal

Assignment Problem (DAP) and is examined in Chapter 7 of this thesis, is a problem that involves the solution of algebraic equations, which is a problem of *intersection of varieties* [Ful84]. This intersection problem consists the parametrization of one set of varieties by another set and this can be visualized by a certain element of an intersection ring of a variety.

Complex numbers $\mathbb{F} = \mathbb{C}$ consist the natural field for the intersection theory of varieties, which is algebraically closed. That is, every polynomial equation of one complex variable can always be solved and the number of solutions (when their multiplicities are considered as well) is equal to the degree of the polynomial. However, there are cases where this does not always apply, i.e the system to be solvable, and the equations might intersect at infinity, where infinity describes the infinity space of the projectivisation. Projectivisation is a method which associates a non- zero vector space \mathcal{V} with a projective space $\mathbb{P}(\mathcal{V})$, whose elements are one- dimensional subspaces of \mathcal{V} . For example the system of equations $xy = 1$ and $xy = -1$ is not solvable and the two equations intersect at infinity, i.e after projectivising them into $xy = z^2$ and $xy = -z^2$, then their intersection occurs only if $z = 0$, which describes the infinity space of the projectivisation. We know that two projective varieties $\mathcal{X}, \mathcal{Y} \subset \mathbb{P}^n(\mathbb{C})$ always intersect given that $\dim \mathcal{X} + \dim \mathcal{Y} \geq n$ (lemma 2.1) and the intersection is *proper* if every irreducible component of $\mathcal{X} \cap \mathcal{Y}$ has dimension equal to $\dim \mathcal{X} + \dim \mathcal{Y} - n$. Also of great interest is the fact that in the case of projective varieties as spaces of parametrized intersections, the number of points of intersection, given that are finite, remains the same as parameters vary. This may not happen in the case of parametrized intersections on affine varieties, as some of the points of the intersection may disappear at infinity as parameters vary and that consists a great disadvantage.

As a result, it is convenient to utilize projective varieties rather than affine ones. We call the projective variety that stems from the affine one *compactification*. We can create this new projective variety by combining a negligible set of points of the affine variety, i.e. the *points at infinity*. We shall note here that there is not a unique way of compactifying \mathbb{C}^n into a projective variety and in general depends each time on the intersection problem under consideration. There exist cases, where the number of equations is equal to the number of unknowns and therefore we would expect finite number of solutions but the existence of solutions at infinity might not allow the correct calculation of finite solutions. Ideally, a good compactification would have to be *smooth* and the variety of

finite solutions would be of greater dimension than the variety of solutions at infinity. In this case whenever the intersection is nonvoid on the compactified space, it should contain a finite point [Lev93].

The above are demonstrated in the following examples:

Example 2.1. [Lev93] Let

$$\begin{cases} a_1 + b_1x + c_1y + d_1xy = 0 \\ a_2 + b_2x + c_2y + d_2xy = 0 \end{cases}$$

a set of algebraic equations in \mathbb{C}^2 , with $d_1, d_2 \neq 0$. The above set of equations will either have points as solutions or no solutions at all, depending upon the coefficients. By compactifying \mathbb{C}^2 into $\mathbb{P}^2(\mathbb{C})$ this corresponds to homogenizing these equations as:

$$\begin{cases} a_1 + b_1\frac{x}{\lambda} + c_1\frac{y}{\lambda} + d_1\frac{x}{\lambda}\frac{y}{\lambda} = 0 \\ a_2 + b_2\frac{x}{\lambda} + c_2\frac{y}{\lambda} + d_2\frac{x}{\lambda}\frac{y}{\lambda} = 0 \end{cases}$$

or equivalently

$$\begin{cases} \lambda^2 a_1 + \lambda b_1 x + \lambda c_1 y + d_1 xy = 0 \\ \lambda^2 a_2 + \lambda b_2 x + \lambda c_2 y + d_2 xy = 0 \end{cases}$$

To find solutions at infinity, we set $\lambda = 0$ and so $xy = 0$. Hence the solutions of this system are: $(1, 0, 0)$ and $(0, 1, 0)$. Both of them correspond to solutions at infinity, since $\lambda = 0$. What we observe is that the new solution set is not smaller than the finite solution set since it is zero dimensional. Since the new set of equations will always have a solution and dimensional arguments cannot be used to conclude whether the set will contain a finite solution or not, it is necessary to compute the number of finite solutions in another way. The total number of solutions (i.e. finite and infinite) can be computed by utilizing Bezout's theorem, which can be applied in the projective space $\mathbb{P}^n(\mathbb{C})$. Bezout's theorem states that the number of common points of two algebraic curves, that do not have infinitely many common points, is at most equal to the product of their degrees, with equality if points at infinity, points with complex coordinates are considered and if each point is counted with its intersection multiplicity [Ful84]. Thus,

the number of finite solutions can be calculated by subtracting the number of infinite solutions from the total number of solutions. In this case, the total number of solutions is equal to $2 \cdot 2 = 4$ and the finite solutions are equal to $4 - 2 = 2$. Whenever the infinity solutions set contains a variety of excess dimension, the computation of them is an issue. The problem can be resolved by considering another compactification, where solutions at infinity won't exist. This will be demonstrated in subsection 2.8.2 where another compactification will be introduced.

□

Example 2.2. Consider the set of algebraic curves in the affine space \mathbb{C}^2 :

$$\begin{cases} xy + 2x^2 = 1 \\ x^2 - y = 0 \end{cases}$$

The compactification of \mathbb{C}^2 into $\mathbb{P}^2(\mathbb{C})$ corresponds to the homogenisation of the system of equations as:

$$\begin{cases} \frac{x}{\lambda} \frac{y}{\lambda} + 2 \frac{x^2}{\lambda^2} = 1 \\ \frac{x^2}{\lambda^2} - \frac{y}{\lambda} = 0 \end{cases}$$

or equivalently:

$$\begin{cases} xy + 2x^2 = \lambda^2 \\ x^2 - \lambda y = 0 \end{cases}$$

The total number of solutions (finite and infinite) is given by Bezout's theorem, which holds for the projective space $\mathbb{P}^n(\mathbb{C})$. Hence, the total number of solutions is equal to $2 \cdot 2 = 4$.

Solutions at infinite can be determined for $\lambda = 0$. Then, the systems becomes:

$$\begin{cases} xy + 2x^2 = 0 \\ x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} 0 = 0 \\ x = 0 \end{cases}$$

and the system one solution at infinity, i.e. $(0, 1, 0)$. Thus, the number of solutions at infinity is equal to: $4 - 1 = 3$.

□

2.8.2 Cohomology Ring as an Intersection Ring

[Lev93] For the purposes of this thesis, in particular to tackle the *Zero Assignment Problem* in *RLC* networks (Chapter 7), we utilize a topological intersection theory, called *cohomology theory*. The following subsection introduces a brief description in the notions of an *intersection ring* and subsequently of the *cohomology ring* of a topological space \mathcal{X} . The approach adopted in this thesis utilizes the cohomology ring to find the total number of solutions for the Zero Assignment Problem via diagonal perturbations, in a rather simple and numerical manner. Thus, the purpose of this subsection is to familiarize the reader with the main idea rather than present the mathematical formalism that depicts this theory.

Intersection Ring

The intersection ring of a smooth variety $\mathcal{V} \in \mathbb{P}^n(\mathbb{C})$ can be denoted by $\mathcal{A}^*\mathcal{V}$. Aside from being an additive group it is also enriched with the structure of a graded ring and has the structure of \mathbb{Z} module. In this ring, every subvariety of co-dimension k corresponds to an equivalence class $\langle \mathcal{X} \rangle$, which belongs to the $\mathcal{A}^k\mathcal{V}$, i.e. the k -th graded component of the intersection ring. The *cup* product, which is the dual of the intersection product, serves the multiplication in the ring.

The intersection ring stems from the fact that every subvariety $\mathcal{X} \subset \mathcal{V}$ of a smooth variety $\mathcal{V} \in \mathbb{P}^n(\mathbb{C})$ may be described by an equivalence class $\langle \mathcal{X} \rangle$ of a suitable equivalence relation defined on the set of all formal sums $\sum k_i \mathcal{X}_i$ of irreducible subvarieties of \mathcal{X} . The dual of the intersection ring, denoted by $\mathcal{A}_*\mathcal{V}$, is the additive group of all equivalence classes on \mathcal{V} . The intersection of varieties corresponds to the product operation in $\mathcal{A}_*\mathcal{V}$. That is, if $\langle \mathcal{X}_1 \rangle, \langle \mathcal{X}_2 \rangle$ two equivalence classes such that the intersection $\mathcal{X}_1 \cap \mathcal{X}_2$ is proper, then the product of $\langle \mathcal{X}_1 \rangle \cdot \langle \mathcal{X}_2 \rangle$ forms a linear combination of the irreducible components of the intersection $\mathcal{X}_1 \cap \mathcal{X}_2$, whose coefficients are the intersection multiplicities. For a finitely generated intersection ring, with a finite basis $e_{ij} = \langle \mathcal{V}_j^i \rangle$ for every graded component $\mathcal{A}^j\mathcal{V}$, the multiplication of the ring can be established by detecting how the elements of the basis intersect with each other.

Cohomology Ring

[Lev93] The cohomology ring defined by $H^*(\mathcal{V}, \Lambda)$ with coefficients in Λ , is a graded ring, which can be assigned to every topological space \mathcal{X} . Λ is a commutative ring, i.e. $\Lambda = \mathbb{R}$ or \mathbb{C} or \mathbb{Z} or \mathbb{Z}_n or \mathbb{Q} ; the cohomology ring is a positively graded ring up to the dimension of \mathcal{X} , thus for an m - dimensional topological space \mathcal{X} we have that:

$$H^*(\mathcal{X}, \Lambda) = \bigoplus_{j=0}^m H^j(\mathcal{X}, \Lambda)$$

where $H^j(\mathcal{X}, \Lambda)$ is the j -th cohomology module of \mathcal{X} with coefficients in Λ and the grading is called cup product.

In the context of algebraic geometry, $H^*(\mathcal{V}, \mathbb{Z})$ is an intersection ring (graded ring) like the intersection ring [Ful84] $\mathcal{A}^*(\mathcal{V})$, that multiplication corresponds to intersection of varieties and addition corresponds to union of varieties. Finally, every sub-variety coincides to a cycle, i.e. an element of the cohomology ring or in other words, each algebraic subset of a variety is assigned a cohomology class. Continuously varying the subset, yields another subset with the same cohomology class.

The cup and cross product of Topological spaces

Consider two subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ of a topological space \mathcal{X} . The *cup product* is defined as the following operation:

$$H^k(\mathcal{X}, \mathcal{A}) \otimes H^n(\mathcal{X}, \mathcal{B}) \rightarrow H^{k+n}(\mathcal{X}, \mathcal{A} \cup \mathcal{B})$$

On cohomology level the cup product operation commutes up to a sign determined by the grading. Specifically, for $a \in H^k(\mathcal{X})$ and $b \in H^n(\mathcal{X})$, we have that $ba = (-1)^{kn}ab$. Hence, as mentioned before the cohomology ring $H^*(\mathcal{X})$ is a *commutative graded ring*.

Next, we will present the cross product and the cohomology of the products of two topological spaces.

Let two cohomology classes $a \in H^k(\mathcal{X}, \mathcal{A})$ and $b \in H^n(\mathcal{Y}, \mathcal{B})$, where \mathcal{A} and \mathcal{B} are open subsets of \mathcal{X} and \mathcal{Y} respectively. Then the *cross product*, denoted by $a \times b$, is defined as

the cohomology class:

$$(p_1^*a) \cup (p_2^*b) \in H^{k+n}(\mathcal{X} \times \mathcal{Y}, (\mathcal{A} \times \mathcal{Y}) \cup (\mathcal{X} \times \mathcal{B}))$$

where p_1, p_2 are the projection maps [Lev93]:

$$\begin{aligned} p_1 : (\mathcal{X} \times \mathcal{Y}, \mathcal{A} \times \mathcal{Y}) &\rightarrow (\mathcal{X}, \mathcal{A}) \\ p_2 : (\mathcal{X} \times \mathcal{Y}, \mathcal{X} \times \mathcal{B}) &\rightarrow (\mathcal{Y}, \mathcal{B}) \end{aligned}$$

For two topological spaces \mathcal{X} and \mathcal{Y} the cross product operation gave rise to the structured-preserving map:

$$x : \bigoplus_{i+j=m} H^i(X) \otimes H^j(X) \rightarrow H^m(X \times Y)$$

In other words, there is a cross product operation by which an i -cycle on \mathcal{X} and a j -cycle on \mathcal{Y} may be combined to create an $(i + j)$ - cycle on $\mathcal{X} \times \mathcal{Y}$; so that there is an explicit linear mapping defined from the direct sum to $H^m(X \times Y)$. The above decomposition, known as Künneth decomposition, is a statement relating the homology of 2 objects to the homology of their product and can be performed for spaces if certain requirements are satisfied.

The number of j - dimensional holes in a topological space is measured by the torsion free part of $H^j(\mathcal{V}, \mathbb{Z})$, $j > 0$, while the number of connected components in \mathcal{V} is measured via $H^0(\mathcal{V}, \mathbb{Z})$. Certain connected spaces without holes (like \mathbb{C}^n) have trivial cohomology rings $H^*(\mathcal{V}, \mathbb{Z}) = H^0(\mathcal{V}, \mathbb{Z}) = \mathbb{Z}$ and their use do not generate results. Hence, it is more suitable the intersection problem under consideration each time, to be examined in the compactified space \mathbb{C}^n . The compactification of \mathbb{C}^n creates certain holes whose dimension and number depends upon the way that points at infinity are joined together. Thus, the new compactified space is richer and the corresponding cohomology ring is more ideal for calculations [Lev93].

In the previous setting and considering the above, a system of polynomial equations can be assigned to a cycle in the cohomology ring. The number of solutions may be calculated via the cup product of the cohomology ring $H^*(\mathcal{X}, \mathbb{Z})$. The equations are defined on a non compact space \mathcal{X} and this space can be compactified to $\bar{\mathcal{X}}$. Then the calculations may be done in the cohomology ring of $\bar{\mathcal{X}}$. The solutions in \mathcal{X} can be found by subtracting the solutions at infinity in: $\bar{\mathcal{X}} - \mathcal{X}$.

To illustrate the above let us consider the example (2.1).

Example 2.3. [Lev93] If we consider another compactification, with no solutions at infinity, then the solution of the problem is straightforward. Indeed if we introduce two new parameters λ_1, λ_2 then the initial system of equations becomes:

$$\begin{cases} a_1 + b_1 \frac{x}{\lambda_1} + c_1 \frac{y}{\lambda_2} + d_1 \frac{x}{\lambda_1} \frac{y}{\lambda_2} = 0 \\ a_2 + b_2 \frac{x}{\lambda_1} + c_2 \frac{y}{\lambda_2} + d_2 \frac{x}{\lambda_1} \frac{y}{\lambda_2} = 0 \end{cases}$$

or equivalently,

$$\begin{cases} \lambda_1 \lambda_2 a_1 + b_1 \lambda_2 x + c_1 \lambda_1 y + d_1 = 0 \\ \lambda_1 \lambda_2 a_2 + b_2 \lambda_2 x + c_2 \lambda_1 y + d_2 = 0 \end{cases}$$

and the compactification considered is $P(\mathbb{C}) \times P(\mathbb{C})$. Solutions at infinity are determined when $\lambda_1 \lambda_2 = 0$. Thus, such solutions do not exist for almost all $(a_i, b_i, c_i, d_i)_{i=0}^2$. In this case, Bezout's theorem cannot be applied as it holds for the case of $P^2(\mathbb{C})$ (in general for $P^2(\mathbb{C})$) and hence to derive the total number of solutions we need to introduce another approach, utilizing the intersection ring of $P(\mathbb{C}) \times P(\mathbb{C})$ as follows:

The intersection ring of $P(\mathbb{C}) \times P(\mathbb{C})$ is defined as $\mathcal{A}^*(P(\mathbb{C}) \times P(\mathbb{C})) = \frac{Z[a]}{\langle a^2=0 \rangle} \otimes \frac{Z[b]}{\langle b^2=0 \rangle}$. Each one of the equations can be expressed by an element $(a+b) \in \mathcal{A}^*(P(\mathbb{C}) \times P(\mathbb{C}))$, where a, b are linear hypersurfaces in $P^1(\mathbb{C})$. Their intersection is given by $z = (a+b)^2$, and if expanded this yields to $z = 2ab$, because of the relations $a^2 = 0$, $b^2 = 0$ that characterize the ring. Thus, the equations contain only two common solutions. \square

It can be observed that the compactification we use to tackle each problem is important. The number of solutions may be determined easily using a nice compactification, which in turn converts the problem into an algebraic one. This is due to the fact that certain elements of the intersection ring of the compactification has to be examined.

2.9 Conclusions

The purpose of this chapter was to review the mathematical tools which underpin the nature of the research problems and are used for the development of this thesis. Certain

sections of this chapter were presented in a rather brief and simple way, avoiding the mathematical formalism, in order to be comprehensive for the reader and to highlight the basic aspects. Extensive literature may be found in various books provided in the bibliography.

Chapter 3

Systems Re-engineering and Networks: Problem Statement, Literature Review and Research Agenda

3.1 Introduction

This Chapter provides the motivation for the study of *RLC* Network Re-engineering problems as part of the general problem of Systems Re-engineering. The different aspects of Re-engineering are considered and the complexity of the overall problem is explained. It emerges that Re-engineering has a system model evolutionary role and that the study of such problems requires an appropriate representation of the re-engineering process, which in turn needs an appropriate model representation. State space and transfer function models are not appropriate system representations for studying re-engineering of a general system. The reason is that such representations do not permit the explicit representation of re-engineering transformations as design parameters. For the special family of *RLC* networks the Impedance-Admittance Implicit model [Kar11] provides a suitable framework for the representation of re-engineering transformations as design parameters, which in turn provide the means for the study of the evolution of system properties as functions of the re-engineering transformations. The study of re-engineering of *RLC* networks is a simple family on which we can study such problems. The Chapter also reviews the different aspects of the network theory which are related

to our problem and those regarding the Determinantal Assignment Problem (DAP) and these results lead to the development of a research agenda for the thesis.

3.2 The Re-engineering Problem

3.2.1 The Integrated Nature of Engineering Systems

Engineering systems are the results of integration of different design stages which they define a natural evolution of the system and the associated models. This is frequently referred as Systems Integration [Rij91, Kar95, Kar96] and it is a multi-dimensional complex engineering problem. This problem has a multidisciplinary character and has the following major aspects:

1. Business aspects
2. Process Operations
3. Engineering Design Stages

In this thesis we focus on the physical process dimension of the system that deals predominantly with issues of design-redesign of the engineering system. The general features of the technological stages linked to the Engineering Design Stages of the overall system design are defined by:

- **STAGE (0):** Problem Definition, Requirements
- **STAGE (I):** Process Synthesis
- **STAGE (II):** Overall System Instrumentation (Global Instrumentation)
- **STAGE (III):** Control Design

and are described by the diagram of Figure 3.1 [Kar95]. The process synthesis – global instrumentation – control design stages have a cascade nature with feedback loops between the various sub-stages and have an iterative nature. The cascade nature of design is underlying the evolutionary process of model shaping, that drives the integrated

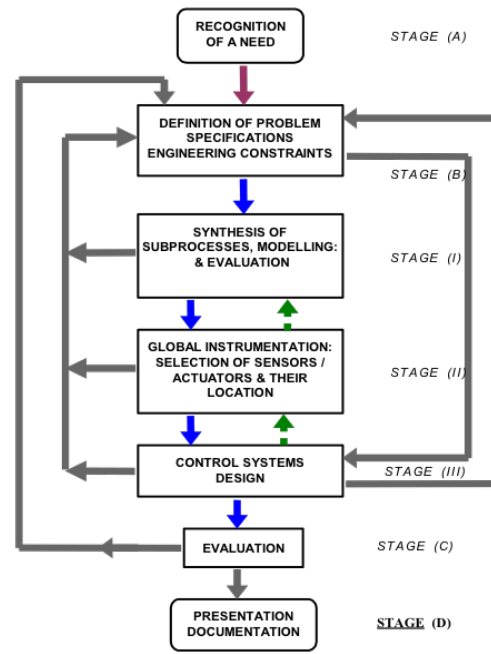


FIGURE 3.1: Simplified form of Engineering Design Process

design paradigm [Kar95], [Kar96, Kar08]. The cascade design process is dynamic in the sense that what it is feasible to achieve at a given stage is influenced by the decisions taken at the previous design stages. The overall process of design has an evolutionary character and this has motivated the definition of a new family of systems referred to as Structure Evolving Systems (SES) [Kar08, Kar11]. The main design stages are [Kar96, Kar08]:

Process Synthesis: This is an act of determining the optimal interconnection of processing units, as well as the optimal type and design of the units within a process system.

Global, or System Instrumentation: This deals with the classification of system variables and the selection of the set and the distribution of inputs and outputs and its study revolves around the investigation of a number of fundamental system type problems. This is contrary to traditional instrumentation of a process that deals with the measurement, or implementation of action upon given physical variables.

Control System Design: This is the last stage of system design that assumes that the system structure is already fixed by decisions in the previous stages. The task involves the design of a new system that when it is connected in a feedback configuration shapes the composite system behavior and achieves the overall design objectives.

The potential of a fixed system to achieve certain performance depends on structural characteristics formed during the process of system formation (evolution during the process synthesis and system instrumentation) and involves the system interconnection topology and the system structural invariants [Ros70, Kar13b, Kal71], [Pop72, Mor73, Wol74, For75, MK76, KK79, Won85, Kai80, KM80, CD82, KG84, KK89, LÖMK91, Kar98]. The formation of structural characteristics of the overall process is reminiscent of an evolution process. The first stage, the process synthesis, acts as the parent gene and thus predetermines a possible range of key characteristics of the final process. Structural properties evolve, but not in a simple manner. Ideally, we would like to have them assigned in order to guarantee certain desirable characteristics and properties. The assignment of desirable structural characteristics in a system may not be possible and thus a more feasible design philosophy, is to direct the model evolution process towards final designs that may possess some desirable properties and avoid the formation of undesirable features that may penalize the final control design.

3.2.2 System Re-engineering and its Complexity

Within the ever-increasing complexity of a large engineering system, solutions to partial problems must guarantee the optimal functioning of the system as a whole, in terms of cost and energy efficiency, safety, reliability etc. This problem becomes more difficult for systems which have been designed in the past with specifications inadequate to satisfy the current needs and that have evolved through time by upgrading components and functionalities. Re-engineering of a complex system emerges as the task of changing the system itself aiming to achieve desirable system structural properties or the avoidance of undesirable properties. Re-engineering is a highly complex problem and addresses all aspects of the system that go beyond those of the mainstream engineering. The types of systems complexity are intimately linked to the notion of the Integration of the different aspects of the engineering system, which are represented by the diagrams of Figure 3.2 [KK89, LÖMK91]. The multi-facet nature of the system the lack of boundaries between the different functionalities and the strong interaction between the subsystems and components gives to the overall system the character forms of complexity frequently referred to as System of Systems (SoS) complexity [Kar95]. It is worth noting that the interaction of the physical, communications/information layer, operational functionalities and

management aspects makes the study of crucial emergent properties difficult. The engineering system nature may be represented by the conceptual diagram in figure 3.2. This form of complexity represented by the above diagrams and the strong linking be-

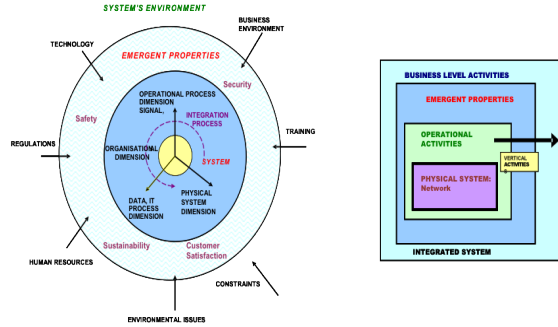


FIGURE 3.2: Complex Engineering System

tween the different aspects of the integrated system, makes the problem of re-engineering an extremely challenging problem. In fact, redesigning the system should be based on global performance criteria, but acting on the subsystems will impact on many other physical, information, or functional parts and thus achieving the global re-engineering objectives, as a task of shaping critical emergent properties becomes an extremely hard task to achieve. The complexity of the re-engineering problem is expressed as the nesting of structural invariants and system properties illustrated by diagram 3.3 [Kar08], where the linking between the different aspects is not well understood. It is this lack of deep understanding between graph structure, systems invariants, primary properties and emergent properties which makes re-engineering an extremely challenging problem. We may address the general re-engineering problem by identifying the following three

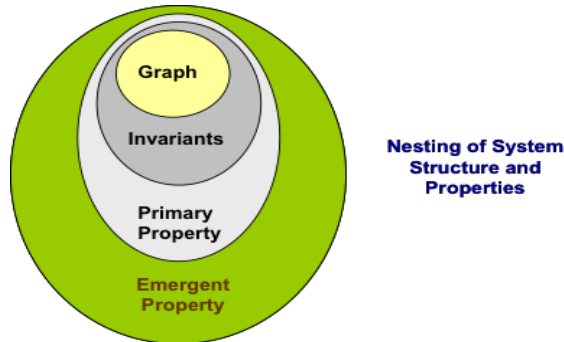


FIGURE 3.3: System Properties Nesting

aspects:

- Business Processes Re-engineering

- Process Operations Re-engineering
- Re-engineering of Engineering Design Stages

In the thesis we are dealing with the last of the above areas, which itself is a complex problem linked to the complexity of the overall system. In fact we may distinguish the following aspects of the *Re-engineering of Engineering Design*:

- i. Re-engineering of Process Synthesis
- ii. Re-engineering of the System Instrumentation
- iii. Re-engineering of the Control Design

Re-engineering of the Control Design is a main stream Control activity and it is not considered here. The area of Re-engineering the System Instrumentation is a theme that has already been dealt with within the area of selection of effective systems of inputs and outputs [Kar08, RR70, KG89, DLM88, SS90, Kar94, LMZZ98, KV02a, LK08, LK09]; this area will be partially addressed within the topic on RLC network. The research is focused on the Re-engineering of Process Synthesis which is a problem that has not been considered in a systematic way so far with the exception of some results linked to the representation of composite systems [LÖMK91]. The study of this problem requires investigating the following issues:

1. Effect of changing values and nature of physical elements within a given interconnection topology.
2. Effect of changing the interconnection topology of a given system.
3. Effect of adding, or removing subsystems, or components.
4. Effect of any combination of the above three transformations

Studying such transformations requires a suitable modeling framework allowing representation of the above transformations. This is needed in order to be able to setup design problems such that we can study the evolution of a number of important system properties. State space, or input-output models are not appropriate system representations for studying re-engineering design problems on a general system. Such transformations

require starting modeling from first principles whenever we use such transformations and thus they do not allow the study of evolution of system properties when the re-engineering transformations become the design parameters. A special family of systems which provides an appropriate framework for studying re-engineering as an evolutionary process with the re-engineering transformations becoming design parameters is the family of *RLC* networks [Kar11, BHK12]. The impedance-admittance models provide a natural tool and this motivates the study of the system aspects of such models and the corresponding re-engineering problems undertaken in this thesis.

3.3 Review of Network Research

The development of this research involves examining the state of the art in areas such as classical network analysis and synthesis [AV73, Bel68, BD49, Bru31, Dar99, Gui77] as far as issues related to links of topology and natural frequencies and examining the methodologies for determinantal assignment [KG84, LK95b, KG89, LK09] as far as their suitability for natural frequency assignment under network transformations. Thus, this section provides an insight in some fundamental results concerning the aforementioned areas.

3.3.1 Origins and Topological Aspects of Classical Network Theory

The **Electrical Network Problem** dates to G. Kirchhoffs famous article in 1847 [Kir47], where he formulated the three fundamental laws that govern any electrical network and he developed the framework of modeling electrical circuits using methods from graph theory. An attempt for extending Kirchhoffs work was made by J. C. Maxwell some years later who studied the duality problem under the more generalized RLC circuits, where impedances had been introduced [Max73]. Later, after the topological theorem that correlates the determinant of the node admittance matrix with the admittance products, H. Poincare, in 1900, generalized the innovative idea of using the incidence matrix to represent a graph [Poi00] and as Veblen mentions in one of his works the use of that matrix was firstly introduced by Kirchhoff [Kir47]. This formed the basis for the enhancement of the development of the field of algebraic topology in network theory.

Since then, topology and matrix algebra have been used widely for the in-depth analysis of electrical networks, as they form an enormously valuable mathematical language for network theory as they constitute an essential tool for studying such networks from a controls system perspective. In the first textbook on topology, written by O. Veblen [Veb31], the topological manifold, the fundamental group and the topological classification problem were firstly defined, constituting the beginnings of algebraic topology. The theories of homology and cohomology consist more advanced methods of topological study of electrical networks.

Homology, in its general form, is a way to link a sequence of algebraic objects, i.e. modules or abelian groups to topological spaces. Furthermore, cohomology is considered as a method of assigning richer algebraic invariants to a topological space. One of the most influential works that applies the algebraic topology to numerical analysis with emphasis to electrical networks is that of J. P. Roth [Rot55]. In this article, Roth proves the existence and uniqueness of solution to the network problem, by examining this problem in a purely algebraic-topological way.

In 1959, J. P. Roth published an article [Rot59] in which he stated that every system that can be described via linear equations may be represented as a network problems. This, did not necessarily imply that a suitable way existed for efficient tearing to apply to this representation. In this work, Roth utilized Krons method of tearing to construct the solution to the network problem and described K-partitioning, an efficient method for solving a linear system. Krons main contribution [Kro33, Kro34]) was the utilization of tensor analysis, an extension of vector calculus to tensor fields, to embed within a topological framework the notion of impedance for both stationary and non-stationary networks. Even though the concept of tearing is based on Krons insight of the network problem, the interconnection of solutions appeared in Roths work, where an algebraic-topological framework of this problem and a proof of the validity of the method were provided.

To summarize the above, the use of algebraic topology and matrix algebra in the study of network theory, even if was motivated from and directly applied to electrical networks, has been also applied in mechanical and structural systems. This stems from the fact that once the properties of topological and algebraic structure are identified, then it is feasible to establish network analogies. One of the most indicative example of that, is the work of F. H. Branin Jr. [BJ66], in which Maxwells equations for the electromagnetic

field were interpreted topologically and the network representations of two large classes of partial differential equations were validated.

All these years electrical networks have been considered as a topic of great interest for many authors. P.R. Bryant, amongst others, has published a series of articles indicating the links between the algebra and the topology in electrical networks. In one of his works [Bry58] he utilizes simple topological methods to associate polynomial structures with functions that describe the network. In more detail, he considers RLC networks (i.e. networks that consist solely from resistors, inductors and capacitors without containing transformers or mutual inductances) for which he establishes and proves a richer expression for the determinant of the admittance matrix of a connected RLC network. This result was based upon the well-known Maxwells rule, that stated that the determinant of the admittance matrix can be written as the sum of tree - products in the graph of an RLC network [Max73]. A dual result is also obtained in Bryants work for the determinant of the loop-impedance matrix, known as Kirchhoffs rule, by providing an extension to other network functions (i.e. driving point admittance). Kirchhoffs and Maxwells rule have been discussed in an extensive depth by many authors [BSST09, Cau58, Fra25, Ku52, MS57, Oka55a, Oka55b, Rez58, Per53, Tal55, Wei58]. Bryants result is equivalent with those suggested by Reza [Rez55], who in 1955 suggested an expression for what is known as the "order of complexity" of a network. Furthermore, Otterman [Ott57] proposed a procedure for the determination of the order of the DE describing the network. Both these numbers, i.e. the "order of complexity" and the "order of the DE" of the network are equal with the degree of the numerator polynomial in the expression of the networks matrix determinant.

Following Reza's publication in 1955, P. R. Bryant published a monograph in the Institution of Electrical Engineers [Bry59] considering the "order of complexity" of electrical networks. In his work, he expresses the natural frequencies of an RLC network and he defines as the "order of complexity" of the network their number i.e. the number of roots of the determinant polynomial of the operator matrix. Furthermore, he associates this number with the number of inductors, number of nodes and the number of separate parts of the network (connectivities), its subgraphs that include only capacitors and those that contain capacitors and resistors. He also extends his results for non-connected networks. Finally, he shows that the order of complexity is equal to the number of integration constants that result from the general solution of the differential

equations that describe the network and the number of variables, which are independent in a dynamic sense.

3.3.2 Classical Network Synthesis

Network synthesis, considered by many to be the most useful method for designing filters, has been applied extensively to the design of those that belong in the general class of linear passive analog filters, i.e. networks that consist only by passive elements. The network synthesis, which is the inverse process of network analysis, is the latest method in filter design field and poses many advantages comparing with previous ones, like the image method.

One of the most influential results on the field is the proof of the necessary and sufficient conditions for an impedance to be realized by a passive network, which was conducted by O. Brune in 1931 [Bru31]. In that work that was based on his PhD dissertation, O. Brune made use of the positive-real (PR) analytic functions, the so-called Brune functions that are rational, real when s is real and with positive real part functions, to facilitate his proof. He also concluded to the fact that for the case of scalar PR functions the realization of the network it is not necessary contain ideal transformers since it can be based only on passive elements. An extension to that was made some years later by R. Duffin and R. Bott and led to the fundamental theorem of filter design, the Bott-Duffin theorem [BD49]. Through this, they give a similar synthesis method of arbitrary impedances by utilizing serial or parallel combinations of inductors, resistors and capacitors, with the proof being relied on the rank, i.e. the sum of the degrees of polynomials in the numerator and denominator of the Brune function (without having any common factor).

Since its appearance in 1949, the *Bott-Duffin* procedure has concerned both circuit and system researchers as the networks produced contained seemingly an excessive number of elements, which exceeded its McMillan degree. M.C. Smith [Smi02] has extended the analogy between electrical and mechanical domains by introducing a new mechanical element, the *inertor*, that allows the use of electrical network synthesis for the design of mechanical networks, thus opening up a new field of applications for the classical network synthesis. With these fundamental results being still in use in electronic system

design, a lot of contemporary work has been conducted mainly concentrated on minimal realizations and boundary interpolations. The non-minimal representation that resulted from realization procedures in RLC networks intrigued T. H. Hughes and M. C. Smith [HS14]. In the paper, they considered a class of networks for the realization process of PR functions that are based on a simplification of *Bott-Duffin* networks and proved that they contain the least possible number of energy storage elements and resistors.

These works consider the McMillan degree of the functions, the degree of the characteristic polynomial obtained as the least common denominator of all minors, aiming to characterize them in terms of sizing. In the case where a positive real function is at stake, the Foster procedure, a preliminary procedure applied at every stage, can be used for its conversion into a minimum function without losing the positive realness. The *Bott-Duffin* procedure along with its simplification, which is called the *Reza-Pantell-Fialkow-Gerst* procedure, are identified as the most indicative methods for obtaining minimal realizations. In the minimal realization application in bi-quadratic minimum functions, which have McMillan degree two, it is shown that such functions can be realized with fewer than seven under some conditions that define a large class of impedance functions.

Additionally, in the work [CWZ16] the authors deal with the generalized theorem of Reichert for bi-quadratic minimum functions and show by validating some of the cases that such functions when they can be realized using networks of a precise number of reactive elements and an arbitrary number of resistors they can also be represented by a minimal structure with respect to resistors.

3.3.3 The Determinantal Assignment Problem (DAP)

The *Determinantal Assignment Problem (DAP)* [KG84] is considered to be a unifying approach for the analysis and study of problems of linear multi-variable systems. DAP was introduced by Karcianas and Giannakopoulos in 1984 and because of its determinantal nature is appropriate for tackling problems of pole and zero assignment. Before we present any background results for DAP it is evident to formulate it.

The *Abstract Determinantal Assignment Problem* has been defined as the problem of solving the following determinantal equation with respect to the polynomial matrix

$H(s)$:

$$\det(H(s) \cdot M(s)) = p(s) \quad (3.1)$$

where $p(s)$ is a polynomial of an appropriate degree d and $M(s)$ a given polynomial matrix. However, as shown in by the analysis in [Kar13a] all dynamics from $H(s)$ can be shifted to $M(s)$, which, in turns transforms the problem to a constant DAP.

A sub-problem of the Abstract Determinantal Assignment Problem is the *Constant DAP* and this can be formulated as follows: Let $M(s) \in \mathbb{R}^{(p+r) \times p} [s]$ such that $\text{rank}(M(s)) = p$ and let \mathcal{H} be a family of full rank constant $p \times (p+r)$ matrices having a certain structure; also let $p(s)$ be an arbitrary polynomial of an appropriate degree d . To obtain a solution for the constant DAP, solve the following determinantal equation with respect to $H \in \mathcal{H}$:

$$\det(H \cdot M(s)) = p(s) \quad (3.2)$$

In general, DAP approaches can be categorised based on the techniques [Lev93] used and thus we distinguish the two main classes of them as:

1. The algebraic and conventional state space techniques
2. Geometric techniques

Although the use of algebraic and state space techniques is restricted due to their inability to resolve fundamental features of DAP, where non-linearities occur, they have been extensively used for the output feedback pole placement problem as they offer a straightforward and algorithmic approach, they are simple and suitable for construction of solutions. On the other hand, the geometric techniques for approaching DAP are more suitable for understanding the nature of the problem. Geometric techniques are more suitable for proving the existence of solutions rather than developing algorithms for the construction of them. The word geometric stems from the fact that DAP is reduced to the study of relations (intersections, maps) of auxiliary geometrical objects such as algebraic varieties, linear spaces and manifolds. These objects must be located in the underlying space in a canonical way, this is called transversality property of the objects or general position (i.e. a notion that describes how spaces can intersect). This property can be measured, most of the times, via the rank of a matrix formed by the systems parameters [Lev93].

Both approaches have been extensively used and have been developed in parallel throughout these years with our focus lying mainly in works published in 1970s and beyond.

The general zero-assignment problem was initiated by Rosenbrock in 1970 [RR70] and [Ros70]. State space techniques were first considered in this work considering possible zero structure Smith forms that could be assigned to a controllable pair (\mathbf{A}, \mathbf{B}) by selecting the matrix \mathbf{C} of the resulting square system. The study of zero assignment via squaring down [KM76] lies within the framework of state space techniques and is a sub-problem of the general zero assignment problem. Although the authors in the aforementioned work did not derive any solvability conditions for the problem, they suggested methods for assigning part of the zero structure. The study of zero assignment via constant squaring down was also studied in [KK79] where sufficient conditions for partial assignability of zeros were stated. The authors proposed an algorithm, which was based on eigenvector assignment techniques, such that the resulting square $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ triplet has a given structure. Also, of extreme interest are the solvability conditions presented in [KG89], where for the constant squaring down case, a general approach for computing solutions has been stated using methods from exterior algebra and algebraic geometry. An extension to this approach towards the decentralised pole-zero assignment problem has been made in the work of [KLG88] where a framework for studying such problems has been defined and the existence of necessary conditions has been investigated. Necessary and sufficient conditions that do not depend to the system graph were also stated in a general form for both the generic and exact pole-zero assignment. As far as the dynamic case is concerned, the work of [SS90] examines the problem of zero assignment by static and dynamic compensators.

A different approach for tackling such type of problems was initiated by [WH78] for the case of the output pole placement problem. This approach belongs to the geometric techniques, which can be further classified into Infinitesimal techniques, Topological Intersection techniques, Combinatorial Geometric techniques, Projective techniques and Enumerative Geometry techniques [Lev93]. Using geometric techniques DAP was examined by considering the following polynomial map:

$$\chi : F^\mu \rightarrow F^d \quad (3.3)$$

where F corresponds to a field, i.e. $F = \mathbb{R}$ or $F = \mathbb{C}$ and μ are the degrees of freedom of H in the determinantal equation. This map χ , maps H to the coefficient vector of the polynomial $p(s)$ in equations 3.2.

This map was introduced in [WH78] for the case of the output pole placement problem. In that work, the authors related the solvability conditions of this problem with the onto properties of the map (the solvability was reduced by checking if the map was onto) by using the dominant morphism theorem for complex algebraic varieties (suitable for examining the onto properties of a polynomial map), which can be found in Chapter 2. In the case examined in [WH78] for $F = \mathbb{C}$ the *differential* and the generic rank of the matrix were computed and by utilising the dominant morphism theorem a necessary and sufficient condition for generic pole assignability was derived. Following that publication, Martin and Herman also derived necessary and sufficient conditions for complex system transformations considering a generic class of systems using tools from algebraic geometry [HM77], [MH77] and [MH78].

Exterior algebra is considered to be a suitable framework for studying DAP due to the problems multi-linear skew symmetric nature, a concept that was firstly introduced by Karcianas & Giannakopoulos in [KG84]. DAPs property of allowing the problem to be scaled down to two subproblems, a linear and multi-linear problem namely, was also proved in the aforementioned work. Based on that distinction, DAPs solvability depends on the solvability of the linear subproblem, where under the existence of a solution, a linear space is defined. For the characterization of the linear space's decomposability property the set of Quadratic Plücker Relations (QPR), a set that also defines its Grassmann variety, is used. To find the intersections, the real ones, between the Grassmann and the linear variety of that linear space can be considered as a final reduction for the solvability problem of DAP. This approach, which is also used in [KLG88] and [LK95b], differs from the one used in [BB81] and in [MH78], where the usefulness of applying tools and techniques of algebraic geometry to problems of control theory has been demonstrated. The difference can be identified in the fact that the latter study the problem in an affine space setting while the former in a projective one.

More recent works involve that of Leventides and Karcianas [LK93]. The authors examine the properties of PPM (dimensionality of the image) under real and complex output feedback and relate them to system invariants. They also establish a new expression

for the rank of the differential of the PPM relating the Markov parameters and the Plucker matrix of the system. Hence, new conditions for pole assignability were derived. Moreover, in 1996 the authors examined the assignability properties of a system with two outputs, relating the controllability indices of the system and the ranks of the multilinear maps. Based on the ranks of the Plucker matrices, bounds for low complexity were considered [LK].

A restricted version of the standard squaring down problem was introduced in [LK08]. The authors consider systems oftenly met in applications, where not all outputs are free parameters. For the study of the problem a new blow-up methodology is used, the so-called Global Linearisation, introducing the notion of degenerate solutions [LK95b]. By utilizing this methodology, it is proved that the problem can be solved generically if certain conditions are met.

A different approach for investigating DAP was published in 2013 [LPK14]. The paper with title Approximate DAP concerns a relaxed version of DAP. In this work, the computation of the approximate solution was reduced to a distance problem between a point in the projective space from the Grassmann variety. Furthermore, two special cases were examined and a new algorithm for computing the approximate solution, whenever exact solutions did not exist, was proposed.

Following that publication, Karcianas and Leventides [KL15] present a new approach for the computation of both exact and approximate solutions of DAP. In this paper, new criteria for existence of solutions are developed, which are based on the properties of the Grassmann matrices. New tests for decomposability of multi-vectors in terms of the rank properties of the Grassmann matrix are provided. This provides a different characterization of the decomposability problem and of the Grassmann variety to that defined so far providing the means for the development of a new computational method.

3.3.4 The Problem of Tuning the Natural Frequencies of a Network

One of the fundamental control problems, that is mostly treated nowadays from a synthesis aspect of view, is that of tuning the natural frequencies of a network. It is widely known that the natural frequencies are strictly related with the nature of the elements and the topology of the given network. The general network synthesis problem [Vla83],

[KM71], [Van60] involves the assignment of the characteristic frequencies when both the elements and the topology of the networks are free design parameters and that gives the opportunity to the designer to exploit all available degrees of freedom. In specific, this problem is equivalent with determining the conditions under which a rational matrix could be realized as an RLC network.

A different approach to that of the general synthesis problem was introduced in [KL06], [LK09]. The problem of redesigning passive autonomous electrical networks for natural frequencies improvements was considered by the authors. This problem differs significantly from the synthesis problem, since it involves modification in the topology that would possibly lead to evolution of the given network (by increase, or reduction of elements, or branches) and/ or alteration of the values of dynamic (inductances, capacitances) and non-dynamic elements (resistors) to achieve the desirable natural frequencies. By utilizing the admittance and impedance methodologies [Vla83] two natural topologies emerge from the system graph, i.e. the admittance and impedance graphs [KL06], which are suitable for the investigation of such structured transformations.

Within this framework two classes of problems were considered. The first concerned the effect on the natural frequencies of the network of a single dynamic, or non-dynamic element change and the latter one the robustness of the natural frequencies under dynamic or non- dynamic element bounded perturbations. Hence, the issues that were naturally raised where connected to the movement of the natural frequencies, which differs from the problem of frequency assignment, that was discussed before [LK09], [KG84].

Hence, when the topology and the nature of elements are not free parameters for the designer, the degrees of freedom are reduced and the problem becomes a general problem of assignment of impedance or admittance matrices. To achieve the desirable frequencies, the designer can exploit the different degrees of freedom as follows.

The first case with restricted degrees of freedom is when the nature and the topology of the network are known, but the values of the elements are to be determined. A restricted version of that problem, i.e. the topology, nature of elements and some of the values of the elements are given, but the rest need to be determined, has been examined in the work of [LK09]. The authors considered two special cases for RLC networks i.e. the RL (resistor-inductances) or RC (resistors-capacitors) networks, where the admittance or impedance operator becomes a matrix pencil. For these two special cases of networks,

the authors investigated the problem of zero assignment under structured additive transformations, which in this specific problem, may be described as diagonal perturbations of the non- dynamical elements. Because the formulation of the problem is close to that of pole assignment by structured static compensators the global linearization methodology was applied [LK95b]. Solvability conditions for the structured zero assignment as well as solutions were derived using a Quasi-Newton numerical approach, based on the degenerate compensator methodology, in the case of regular pencils with infinite zeros. As far as matrix pencils with no infinite zeros were concerned, conditions for complex zero assignability were derived, using the Dominant Morphism theorem [Bor91], [Hum75], associating them with invariants of the pencil.

Another approach for tuning the natural frequencies of an RLC network, within this structured framework, was introduced by [BHK12], where the effect of changes of a single dynamic or non-dynamic element was considered on the natural frequencies of the network. The authors were interested particularly in the movement of the natural frequencies rather than investigating the assignment of them, as in the previous work [LK09]. When the general case of RLC networks or a more simplified version, either the RL or the RC networks, where the impedance or the admittance models become matrix pencils is under consideration, the usefulness of the Determinantal Assignment approach in analyzing the spectrum is indisputable. It was shown that the study of the single variation problem was completely equivalent with the study of a Root Locus problem of a standard single-input and single-output (SISO) system. Given the network description and the transformation at stake, with the transformations representation being also a subject matter, the polynomials required for the Root Locus problem expression could be extracted and it was shown that the Root Locus problem may be of a fixed mode type. That is, the problem can be based on the polynomials of the numerator and denominator, for zeros and poles respectively, which were resulted when the transformation of interest was fixed. When selecting the transformation that may preserve or transforms the networks topology, the Root Locus problem gets fixed with the case where points in the Root Locus become fixed as well to be common. As this study was developed under the framework of exterior algebra, these points could be directly identified with its computation being degenerated into finding the Greatest Common Divisor. Due to the symmetry obtained in the admittance and impedance operators, along with the systems passivity, some interesting properties for the final Root Locus problem emerged. The

interlacing property of zeros and poles, an indicative example of spectrums properties, was derived for the whole family of such problems and for the case of the single parameter variation a movement towards a common direction referring to the locus could be shown to exist.

3.4 Research Agenda: Systems Theory and Redesign of Internal Implicit Models

Systems re-engineering implies changes in systems parameter, possibly changes of sub-systems and interconnection topology. Studying these problems requires a modeling framework that supports the study of evolution of system properties as a result of such transformations. Transfer function and state space models cannot support such studies. For the special case of passive RLC networks it has been shown [Kar11] that the impedance-admittance implicit (IAI) description models represented by the integral-differential operator $W(s)$ provide an appropriate description for the study of re-engineering problems. Network Theory [Dar99] has recently become a very active area of research focusing mostly on the classical problems of network synthesis [Smi02].

The study of properties of $W(s)$ provide a new direction for network research linked to the study of re-engineering of networks, which is different to the *RLC* realization of impedance-admittance scalar functions. Such a study involves the investigation of assignment of the natural frequencies of the network. This is a new area of research and provides a new direction to system theory, based on the properties of the integral-differential operator $W(s)$. This new area of research involves answering a number of questions which some of them we aim to address in this thesis. Central problems under study are:

- Study of properties of $W(s)$ operator as a rational matrix and in particular its McMillan degree.
- Investigation of the properties of the natural frequencies of the network and in particular their links to the network graph topology.

- The $W(s)$ introduces special graph topologies linked to the loop or nodal network representation and study of their properties is integral part of the structural properties of $W(s)$.
- The *linearization* of $W(s)$ by developing matrix pencil models that preserve the loop/nodal structure and provide a matrix pencil framework for studying assignment.
- The minimality issues of linearized representations of $W(s)$ are parts of the structural analysis and especially their links to the loop/nodal topologies.
- Examining the properties of the modified loop / nodal analysis models [WJ02, HRB75] to the $W(s)$ matrix pencil linearization is central in establishing the links between the different types of topologies and needs investigation.
- The $W(s)$ representation and its pencil linearization introduces an implicit system representation and issues of selecting inputs (orientation) involves evolution of structure that needs investigation.
- $W(s)$ provides the natural representation for the study of network re-engineering problems and providing a representation of such structural transformations in a form that is appropriate for frequency assignment is essential.
- Classifying structural re-engineering transformations into groups according to preservation of cardinality or McMillan degree provides a corresponding classification of assignment problems.
- The study of frequency assignment problems under the different network transformations is the main open issue.
- The new system representation introduced by $W(s)$ provides the means for studying a number of problems beyond re-engineering, such as the problem of network simplification that is linked to studies of structural evolution in networks [Kar08].

3.5 Conclusions

The emphasis in this chapter has been on reviewing the Network Re-engineering Problem and some fundamental background results on classical Network theory. Also this chapter

provided a summary of the results in the Determinantal Assignment Problem (DAP) and it provides a brief insight in the various techniques that have been utilized throughout the years. Results derived within the framework of Zero Assignment have been presented and emphasis has been given to those that concern the problem of tuning the natural frequencies of RLC networks. Finally, in the last section the research agenda defines the range of some of the new open issues motivating the research in this thesis.

Chapter 4

Implicit Network Descriptions and Their Properties

4.1 Introduction

The aim of this chapter is to familiarize the reader with fundamental notions from systems modelling, present the *Implicit Network Operator* $W(s)$ of an *RLC* network, its associated *Implicit Network Pencil* $P(s)$ and study their properties.

Particularly, in section 4.2 starting from Kirchhoff's second Law we demonstrate the derivation of the loop method or *impedance model* and we examine the *natural loop topology* that emerges from the basic topological structure of the network, i.e. the system graph. The *loop topology* naturally rises from the specifics of the loop analysis. Finally, in the last subsection of section 4.2 the development of independent set of loops in a system's graph is revised, which stems from the notion of fundamental circuits in graph theory, and leads to the derivation of independent set of loop equations.

In section 4.3 Kirchhoff's first Law is presented leading to the node method or *admittance model* formulation. Equivalently, the *natural vertex topology* is examined, which is linked with the nodal analysis. Next, the derivation of systems equations stemming from the two fundamental laws are introduced and various examples illustrating these methods are given.

All the previous analysis and the general modelling for passive *RLC* networks provides a description in terms of symmetric, integral-differential operators, which are presented in section 4.4 and from now on will be referred to as the *Implicit Network Description* or *Implicit Network Operator* $W(s)$ of the network.

In the next section, i.e. section 4.5, a preliminary result between these two *Implicit Network Descriptions* (Impedance and Admittance) is derived and this is demonstrated by means of an example.

In the following section (4.6) the relationship between the Implicit Network Description and the Network Pencil, which is a matrix pencil, is investigated and some fundamental properties of the two descriptions are established.

Section 4.7 is concerned about the regularity properties of the *Implicit Network Operator* $W(s)$. A alternative, generalized expression of the determinant of the Implicit description is given, along with a proof, which leads to a fundamental result relating the regularity of the network (or equivalently the regularity of this description) and the *connectivity* of the *RLC* network. Furthermore, equivalent regularity properties of the associated *Network Pencil* are examined and necessary and sufficient conditions are derived in terms of Toeplitz matrices.

All the above lead to the investigation of the natural frequencies of a *regular network* by examining the zero structure of the associated *Network Pencil* $P(s)$, which are developed in section 4.8.

4.2 Impedance Modeling, Loop Topology and Selection of Independent Loops

In this section Kirchhoff's second law is stated, which leads to the Loop method formulation. The loop or impedance model gives rise to the natural loop topology, which is presented next. In the final subsection of this section the derivation of independent set of loops is illustrated, which is based on the notion of fundamental circuits [SR61].

4.2.1 Impedance Modeling

Compatibility - The Path Law

The Path Law is a statement of the compatibility condition. It states that, for an oriented graph, the algebraic sum of the across - variables ¹ around any closed path is

¹variables that are defined by measuring a difference, or drop, across an element, that is between nodes on a graph (across one or more branches). These variables sum to zero around any closed loop

zero. The across variable differences are considered positive, if their orientation is in the direction of traverse around the closed path. Thus:

$$\sum_q v_{pq} = 0, q = 1, 2, \dots m, p = 1, 2, \dots \quad (4.1)$$

where the summation is considered around the closed path p and m is the number of elements contained in the closed path. The maximum number of different across variable terms v_{pq} is equal to the number of branches in the system graph. The question relating to the number of independent compatibility equations is investigated next and the main result is [Kar11]:

Lemma 4.1. *Given a system graph of two-terminal elements with n - vertices and b branches, only $b - (n - 1)$ of the path equations are linearly independent.* \square

If a graph has n vertices, then any tree should contain $(n - 1)$ branches. This is because the first branch included is incident on two vertices and its additional branch added includes one new vertex. If a graph has b branches, then there must be $[b - (n - 1)]$ co-trees (or co-spanning trees, or tree-links), since each tree must contain $(n - 1)$ branches.

Loop Method Formulation

In the loop method, the variables are selected such that the vertex law is automatically satisfied. Here, we consider only planar graphs. We then consider the variables associated with each of the meshes and we define as loop variables. This approach leads to that each branch through - variable will be the difference between the loop variables on each side of the branch. The path law is then written for each mesh and substitutions are made for the across variables in terms of the loop variables using the elemental equations. This way the overall system is reduced to a number of meshes, which are $(b - n + 1)$ [Kar11]. The process of working out the equations involves the selection of internal independent loops, the definition of loop currents and the transformation of current sources to equivalent voltage sources (Thevenins theorem). If we denote by (i_1, i_2, \dots, i_q) the set of the Laplace transforms of the loop currents and by (v_{s1}, \dots, v_{sq}) the set of Laplace transforms of equivalent voltage sources, then the loop or impedance

on the graph (they satisfy the compatibility conditions). Typical examples of across variables are: (i) velocity drop in mechanical systems, and (ii) voltage drop in electrical systems [Row08].

model is defined by:

$$\begin{bmatrix} z_{11} & -z_{12} & -z_{13} & \cdots & -z_{1q} \\ -z_{12} & z_{22} & -z_{23} & \cdots & -z_{2q} \\ -z_{13} & -z_{22} & z_{33} & \cdots & -z_{3q} \\ \vdots & \vdots & \vdots & & \vdots \\ -z_{1q} & -z_{2q} & -z_{3q} & \cdots & z_{qq} \end{bmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ \vdots \\ i_q \end{pmatrix} = \begin{pmatrix} v_{s1} \\ v_{s2} \\ v_{s3} \\ \vdots \\ v_{sq} \end{pmatrix} \quad (4.2)$$

where:

$z_{ii}(s)$: is the sum of impedances in loop i

$z_{ij}(s)$: is the sum of impedances common between loops i and j

and the sign in the off diagonal elements depends on the direction of the loop currents through the common branches in question, i.e (+) same direction, (-) opposite direction.

Equation (4.2) can be written in short as

$$Z(s)i(s) = v_s(s)$$

This is referred to as the *loop or impedance model* and the symmetric matrix $Z(s)$ is referred to as the *network impedance matrix*. The above are demonstrated in the following example:

Example 4.1. Consider the following network modelled using the loop analysis / method [Kar]: Applying the compatibility or path law to the corresponding loops of the

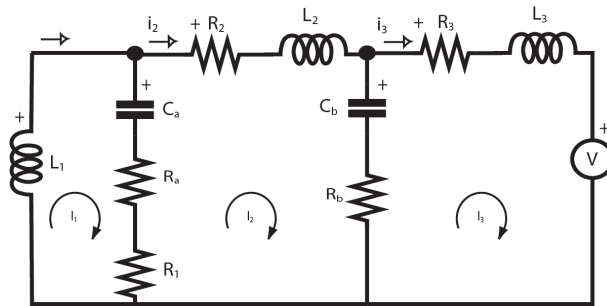


FIGURE 4.1: RLC network

network we have that:

- **Loop 1:** $v_{L_1} + v_{C_a} + v_{R_a} + v_{R_1} = 0$

- **Loop 2:** $-v_{R_1} - v_{R_a} - v_{C_a} + v_{R_2} + v_{L_2} + v_{C_b} + v_{R_b} = 0$
- **Loop 3:** $-v_{R_b} - v_{C_b} + v_{R_b} + v_{L_3} - v = 0$

Elemental relations, systems equations:

- **Loop 1:** $L_1 \frac{di_1}{dt} + \frac{1}{C_a} \int (i_1 - i_2) dt + R_a(i_1 - i_2) + R_1(i_1 - i_2) = 0$
- **Loop 2:** $-R_1(i_1 - i_2) - R_a(i_1 - i_2) - \frac{1}{C_a} \int (i_1 - i_2) dt + R_2 i_2 + L_2 \frac{di_2}{dt} + \frac{1}{C_b} \int (i_2 - i_3) dt + R_b(i_2 - i_3) = 0$
- **Loop 3:** $-R_b(i_2 - i_3) - \frac{1}{C_b} \int (i_2 - i_3) dt + R_3 i_3 + L_3 \frac{di_3}{dt} - v = 0$

or equivalently if we use the set of Laplace transforms:

- **Loop 1:** $\underbrace{\left[\frac{1}{C_a s} + (R_a + R_1) + L_1 s \right]}_{\triangleq z_{11}(s)} i_1 - \underbrace{\left[\frac{1}{C_a s} + (R_a + R_1) \right]}_{\triangleq z_{12}(s)} i_2 = 0$
 $- \underbrace{\left[\frac{1}{C_a s} + (R_a + R_1) \right]}_{\triangleq z_{21}(s)} i_1 + \underbrace{\left[\left(\frac{1}{C_a} + \frac{1}{C_b} \right) \frac{1}{s} + (R_a + R_1 + R_2 + R_b) + L_2 s \right]}_{\triangleq z_{22}(s)} i_2$
- **Loop 2:** $- \underbrace{\left[\frac{1}{C_a s} + R_b \right]}_{\triangleq z_{23}(s)} i_3 = 0$
- **Loop 3:** $- \underbrace{\left[\frac{1}{C_a s} + R_b \right]}_{\triangleq z_{32}(s)} i_2 + \underbrace{\left[\frac{1}{C_b s} + (R_b + R_3) + L_3 s \right]}_{\triangleq z_{33}(s)} i_3 = v$

Impedance Description of system equations & Impedance matrix:

$$\begin{cases} z_{11}(s)i_1 - z_{12}(s)i_2 = 0 \\ -z_{21}(s)i_1 + z_{22}(s)i_2 - z_{23}(s)i_3 = 0 \\ -z_{32}(s)i_2 + z_{33}(s)i_3 = v \end{cases} \quad (4.3)$$

or equivalently in matrix form:

$$\underbrace{\begin{bmatrix} z_{11} & -z_{12} & 0 \\ -z_{21} & z_{22} & -z_{23} \\ 0 & -z_{32} & z_{33} \end{bmatrix}}_{\triangleq Z(s)} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}}_{\triangleq i} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix}}_{\triangleq v} \quad (4.4)$$

□

4.2.2 Natural Loop Topology

In this subsection we are going to examine the *loop topology* [KLL14b] that emerges from the basic topological structure of the network i.e. the *system graph*. The *loop topology* [Kar11] is linked to the specifics of the Loop analysis considered in the previous subsection. The topological structure that stems from that depends on the nature of elements in the network, which are considered in Appendix A.

The loop topology is based on the following principle:

Every network of n vertices and b edges (branches) may be represented by $(b-n+1)$ loops leading to independent equations. All branches that are common between two loops may be represented by an *impedance function*. Specification of the values of through variables for the loops defines the values of all across variables in the network. The loop methodology implies the substitution of all through variable sources by equivalent across variable sources and this leads to the loop topology.

Definition 4.1. The *natural loop graph* of the network is a graph with no sources that defines completely the impedance matrix [Kar11]. □

It is crucial to state the following remark:

Remark 4.1. The natural loop graph is affected by the nature of the sources and the network graph is a *progenitor* of the natural loop graph. □

If the across variables sources are set to zero the graph that is obtained is a reduced graph referred to as *kernel loop graph*. The *kernel loop graph* contains sub-graphs defined by

the elements associated with the edges in series and these sub-graphs may be defined as follows:

Definition 4.2. [Kar11], (Appendix A) The A-loop sub-graph is generated by eliminating from the kernel loop graph all $T-$ and $D-$ type edges without opening up the loops. Equivalently, the T-loop sub-graph is formed by eliminating all $A-$ and $D-$ type edges and finally the D-loop sub-graph by eliminating all $A-$ and $T-$ type edges. If all $A-, T-, D-$ type elements are eliminated from the natural loop graph, then the remaining sub-graph represents the location of the across variable sources in the loops and it is defined as the S-loop sub-graph . \square

Remark 4.2. [Kar11], (Appendix A) The $A-, T-, D-, S-$ loop sub-graphs are by construction simple graphs with either loops or parallel edges. The corresponding adjacency matrices are all symmetric and Boolean. \square

If the natural loop graph of the network is denoted by G_l and the corresponding $A-, T-, D-, S-$ sub-graphs of G_l by $G_{l,a}, G_{l,t}, G_{l,d}, G_{l,s}$, then the natural loop graph G_l may be expressed as:

$$G_l = G_{l,a} \dot{\cup} G_{l,t} \dot{\cup} G_{l,d} \dot{\cup} G_{l,s} \quad (4.5)$$

Now, if $A_{l,a}; A_{l,t}; A_{l,d}; A_{l,s}$ represent the adjacency matrices of the sub-graphs $G_{l,a}, G_{l,t}, G_{l,d}, G_{l,s}$ respectively, then the quadruple $(A_{l,a}; A_{l,t}; A_{l,d}; A_{l,s})$ provides a representation of the loop topology of the network. Given that the selection of the independent loops is not necessarily unique, there is no unique loop topology.

4.2.3 Development of Independent set of Loops

The development of independent set of loops is based on the selection of a tree and then the use of the corresponding co-trees (or co-spanning trees) with the selected tree. In fact, if we insert any co-tree in a tree, this will cause the creation of a closed path. Each new closed path formed by separate addition to the co-trees (or co-spanning trees), one at a time, will be a new path since it will contain a branch, which was not in any previous subgraph. If the path law is applied to each of the paths formed this way, then each of the $[b - (n - 1)]$ equations will be independent of the others, since each equation will have a variable, which does not appear in any of the other equations. This proves

that there are only $(b - n + 1)$ linearly independent paths, or compatibility equations. It is worth noting, that the application of the path law to any path other than the ones formed by the addition of tree links will produce an equation, which can also be obtained by a linear combination of the previously obtained $[b - (n - 1)]$ equations. It is worth noting that the formulation of independent loops is based on the notion of fundamental circuits[SR61, Ruo13].

The development of independent set of loops is illustrated in the following figures. Let us begin from an arbitrary network presented in figure (4.2):

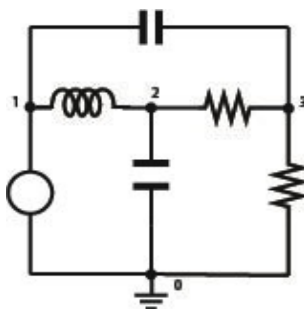


FIGURE 4.2: arbitrary network

The corresponding linear directed graph (or digraph) for this particular electrical network is demonstrated in figure (4.3). From this digraph the trees (or spanning trees)

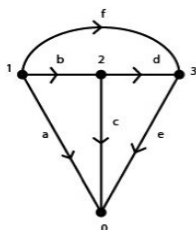


FIGURE 4.3: linear digraph of the network

along with the co-trees (or co-spanning trees) that can be formulated are demonstrated in figure (4.4). For each of the above figures a resulting set of independent loops (or

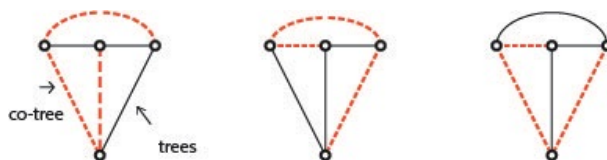


FIGURE 4.4: trees and co-trees of the linear graph

circuits) exists.

More precisely, for the first figure the resulting set of independent loops is presented in figure (4.5):

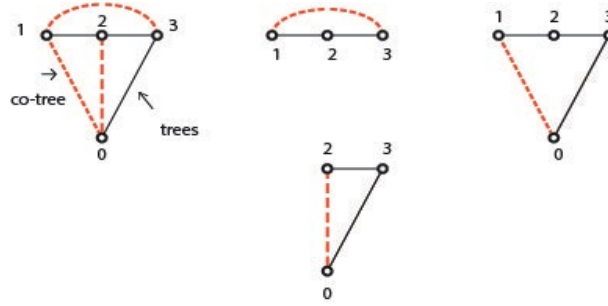


FIGURE 4.5: Figure 1 and the corresponding independent loops

Similarly, for figure 2 the corresponding loops are as in figure (4.6):

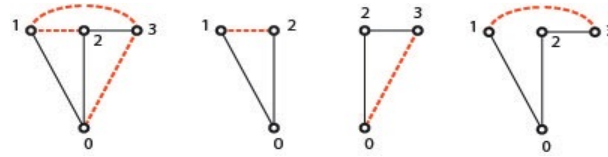


FIGURE 4.6: Figure 2 and the corresponding independent loops

Finally, for the figure (4.7) we have: Two important remarks can be stated from the

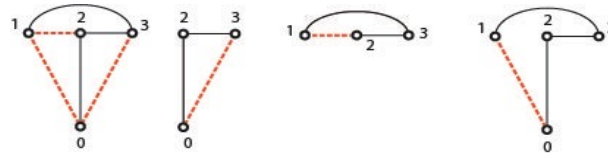


FIGURE 4.7: Figure 3 and the corresponding independent loops

previous example:

Remark 4.3. Any arbitrary choice of $(b - n + 1)$ closed paths of the original graph will not necessarily produce independent path equations [Kar11]. \square

Remark 4.4. The closed-paths formed by the addition of co-trees to a particular tree will produce one set of independent path equations. Such a set is not unique and depends on the selection of a particular tree. \square

4.3 Admittance Modeling and Vertex Topology

In this section Kirchhoff's first law is stated, which leads to the Node method formulation. The node or admittance model gives rise to the natural vertex topology, which is presented next. In the final subsection of this section the formulation of system equations is presented.

4.3.1 Admittance Modeling

Continuity - The Vertex Law

The law to be considered in this subsection is the Vertex Law, which expresses the continuity condition. It states that, for an oriented linear graph of a system, the algebraic sum of the through variables ² entering any vertex must be zero, i.e.

$$\sum_j i_{j_k} = 0, k = 1, 2, \dots, n, j = 1, 2, \dots, l \quad (4.6)$$

where k indicates one of the n vertices in the linear graph and l is the number of branches incident to the k -th vertex. Using the Vertex Law we can easily prove that a similar relation applies to any closed volume, which cuts through a system graph. A direct consequence of the above is:

Lemma 4.2. *Given a system graph of two-terminal elements with n - vertices, only $(n - 1)$ of the vertex equations are linearly independent.* \square

The above is readily established by drawing a volume of $(n - 1)$ internal vertices. If multi-terminal elements are included [Kar11] then the graph may have separate parts, which are not connected. If there are p separate parts, then the number of independent vertex equations becomes $(n - p)$.

²variables that are measured through an element, that is are considered as being transmitted through an element unchanged. These variables sum to zero at the nodes on a graph, and are said to satisfy the continuity condition. Typical examples of through variables are (i) current in electrical systems, and (ii) force in mechanical systems [Row08].

Node Method Formulation

In this method the across variables from each vertex to some reference vertex are chosen as the unknowns in terms of which the final set of equations is formulated; such variables are called node variables. These variables automatically satisfy the path laws, since the across variables between nodes is expressed simply as the difference between the appropriate variables. The vertex equation is then written at each node, and the through variables are then expressed directly in terms of the node variables as related by the elemental equations. The process eliminates all variables except the node variables and has a number of equations, which is in general $(n - 1)$ [Kar11]. The node method is the dual to the loop method and the basic steps involve the selection of internal nodes, definition of the corresponding node voltages and the transformation of the voltage sources to equivalent current sources (Nortons theorem). If we denote by (v_1, v_2, \dots, v_n) the set of the Laplace transforms of the node voltages and by (i_{s1}, \dots, i_{sn}) the set of Laplace transforms of equivalent current sources, then the node or admittance model is defined by:

$$\begin{bmatrix} y_{11} & -y_{12} & -y_{13} & \cdots & -y_{1n} \\ -y_{12} & y_{22} & -y_{23} & \cdots & -y_{2n} \\ -y_{13} & -y_{22} & y_{33} & \cdots & -y_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ -y_{1q} & -y_{2q} & -y_{3q} & \cdots & y_{nn} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} i_{s1} \\ i_{s2} \\ i_{s3} \\ \vdots \\ i_{sn} \end{pmatrix} \quad (4.7)$$

where:

$y_{ii}(s)$: is the sum of admittances in loop i

$y_{ij}(s)$: is the sum of admittances common between loops i and j

and can be written in short as

$$Y(s)v(s) = i_s(s)$$

This is referred to as the node or admittance model and the symmetric matrix $Y(s)$ is referred to as the network admittance matrix.

The above are demonstrated in the following example:

Example 4.2. Consider the following network modelled using the nodal analysis / method: Applying the continuity or vertex law to the corresponding nodes of the network

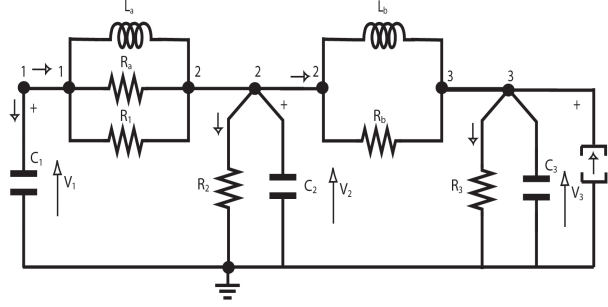


FIGURE 4.8: RLC network

we have that:

- **Node 1:** $i_{C_1} + i_{L_a} + i_{R_a} + i_{R_1} = 0$
- **Node 2:** $-i_{L_a} - i_{R_a} - i_{R_1} + i_{R_2} + i_{C_2} + i_{L_b} + i_{R_b} = 0$
- **Node 3:** $-i_{L_b} - i_{R_b} + i_{R_3} + i_{C_3} - i = 0$

Elemental relations, systems equations:

- **Node 1:** $C_1 \frac{dv_1}{dt} + \frac{1}{L_a} \int (v_1 - v_2) dt + \frac{1}{R_a} (v_1 - v_2) + \frac{1}{R_1} (v_1 - v_2) = 0$
- **Node 2:** $-\frac{1}{L_a} \int (v_1 - v_2) dt - \frac{1}{R_a} (v_1 - v_2) - \frac{1}{R_1} (v_1 - v_2) + \frac{1}{R_2} v_2 + C_2 \frac{dv_2}{dt} + \frac{1}{L_b} \int (v_2 - v_3) dt + \frac{1}{R_b} (v_2 - v_3) = 0$
- **Node 3:** $-\frac{1}{L_b} \int (v_2 - v_3) dt - \frac{1}{R_b} (v_2 - v_3) + \frac{1}{R_3} v_3 + C_3 \frac{dv_3}{dt} - i = 0$

or equivalently if we use the set of Laplace transforms:

- **Node 1:** $\underbrace{\left[\frac{1}{L_a s} + \left(\frac{1}{R_a} + \frac{1}{R_1} \right) + C_1 s \right]}_{\triangleq y_{11}(s)} v_1 - \underbrace{\left[\frac{1}{L_a s} + \left(\frac{1}{R_a} + \frac{1}{R_1} \right) \right]}_{\triangleq y_{12}(s)} v_2 = 0$
- **Node 2:** $-\underbrace{\left[\frac{1}{L_a s} + \left(\frac{1}{R_a} + \frac{1}{R_1} \right) \right]}_{\triangleq y_{21}(s)} v_1 + \underbrace{\left[\left(\frac{1}{L_a} + \frac{1}{L_b} \right) \frac{1}{s} + \left(\frac{1}{R_a} + \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_b} + C_2 s \right) \right]}_{\triangleq y_{22}(s)} v_2 - \underbrace{\left[\frac{1}{L_b s} + \frac{1}{R_b} \right]}_{\triangleq y_{23}(s)} v_3 = 0$
- **Node 3:** $-\underbrace{\left[\frac{1}{L_b s} + \frac{1}{R_b} \right]}_{\triangleq y_{32}(s)} v_2 + \underbrace{\left[\frac{1}{L_b s} + \left(\frac{1}{R_b} + \frac{1}{R_3} \right) + C_3 s \right]}_{\triangleq y_{33}(s)} v_3 = i$

Admittance Description of system equations & Admittance matrix:

$$\begin{cases} y_{11}(s)v_1 - y_{12}(s)v_2 = 0 \\ -y_{21}(s)v_1 + y_{22}(s)v_2 - y_{23}v_3 = 0 \\ -y_{32}(s)v_2 + y_{33}(s)v_3 = i \end{cases} \quad (4.8)$$

or equivalently in matrix form:

$$\underbrace{\begin{bmatrix} y_{11} & -y_{12} & 0 \\ -y_{21} & y_{22} & -y_{23} \\ 0 & -y_{32} & y_{33} \end{bmatrix}}_{\triangleq Y(s)} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\triangleq v} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ i \end{bmatrix}}_{\triangleq i} \quad (4.9)$$

□

4.3.2 Natural Vertex Topology

In this subsection the *vertex topology* is examined [KLL14b]. This topology emerges from the basic topological structure of the network i.e. the *system graph*. The *vertex topology* is linked to the specifics of the Nodal analysis, which is considered in the previous subsection. The topological structure that stems from that, depends on the nature of elements in the network, which are considered in Appendix A. Every network may be represented in terms of a set of vertices, or nodes and all branches between two vertices may be represented by an *admittance function*. Specification of the values of the across variables of the vertices defines the values of all through variables in the network. The vertex methodology implies the substitution of all across variable sources by equivalent through variable sources and define the resulting topology.

Remark 4.5. The nature of sources in the network plays a key role in deriving the natural vertex graph from the system graph. The network graph acts as a progenitor of the natural vertex graph [Kar11]. □

The nature of the elements in the branches of the *natural vertex graph* defines an element dependent topology, which is characterized by adjacency type matrices. If we set the external sources to zero, the reduced graph will be referred to as the *kernel vertex*

graph. The *kernel vertex graph* contains sub-graphs defined by the nature of the elements associated with the branches (edges) and these are defined as:

Definition 4.3. Similarly to the natural loop-topology, for a given *kernel vertex graph* we define A-vertex sub-graph by eliminating from the kernel vertex graph all $T-$ and $D-$ -type edges. Similarly, we define the T-vertex sub-graph by eliminating all $A-$ and $D-$ -type edges and the D-vertex sub-graph by eliminating all $A-$ and $T-$ -type edges. The sub-graph of the natural vertex graph obtained by eliminating all $T-$, $D-$, $A-$ -type elements represents the location of the through variable sources and will be called the source-vertex sub-graph, or simply S-vertex sub-graph [Kar11], (Appendix A). \square

Remark 4.6. The $A-$, $T-$, $D-$, $S-$ vertex sub-graphs are by construction simple graphs, that is they have loops, or parallel edges. The corresponding adjacency matrices are all symmetric Boolean matrices (Appendix A). \square

Equivalently to the natural loop topology, if we denote by G_v the *natural vertex graph* of a network and by $G_{v,a}$, $G_{v,t}$, $G_{v,d}$, $G_{v,s}$ the corresponding $A-$, $T-$, $D-$, $S-$ sub-graphs of G_v , then the latter define a decomposition of G_v , which may be denoted as:

$$G_v = G_{v,a} \dot{\cup} G_{v,t} \dot{\cup} G_{v,d} \dot{\cup} G_{v,s} \quad (4.10)$$

We can denote the adjacency matrices of the sub-graphs $G_{v,a}$, $G_{v,t}$, $G_{v,d}$, $G_{v,s}$ by $A_{v,a}$; $A_{v,t}$; $A_{v,d}$; $A_{v,s}$. In this case, the quadruple $(A_{v,a}; A_{v,t}; A_{v,d}; A_{v,s})$ provides a representation of the *vertex topology* of the network [Kar11].

Formulation of System Equations and Examples

The vertex and path laws along with the elemental equations allow the formulation of the system equations. From the discussion so far it follows [Kar11]:

Lemma 4.3. *A sufficient set of equations for determining the system equation for any output of any system (linear, or non-linear) is obtained by using a set of:*

- (i) *linearly independent vertex equations*

(ii) *linearly independent path equations*

(iii) *elemental equations, where s is the number of source branches.*

□

For a graph with a total of b branches of which s branches are sources, there are $2(b - s) + s = 2b - s$ unknowns, since each non-source has two unknowns (1 through and 1 across variable) and each source has one unknown (the complementary variable for that source branch). This set of equations is linearly independent and contains exactly $(2b - s)$ equations, since:

$$(n - 1)_{vertex} + (b - n + 1)_{path} + (b - s)_{elemental} = (2b - s)_{total} \quad (4.11)$$

For a linear system this forms the necessary and sufficient set of equations that can be solved. For a non-linear system, this set of equations is sufficient to determine the system performance, but it is not always possible to eliminate some variables.

All the results that were derived in sections 4.2 and 4.3 are demonstrated in the next example:

Example 4.3. Consider the mechanical translational system in figure 4.9, or equivalently in figure 4.10: with the associated linear graph demonstrated in figure (4.11):

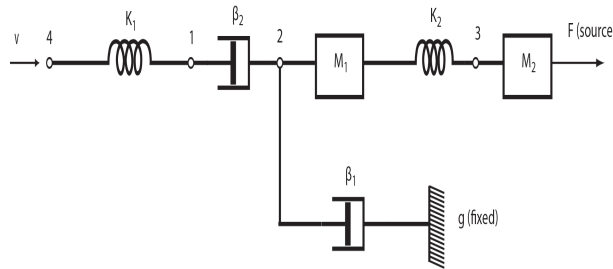


FIGURE 4.9: translational mechanical system

Loop Formulation:

When the mesh through variables are selected as above we have the following compatibility, or path equations:

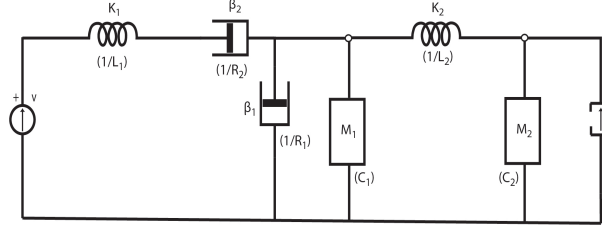


FIGURE 4.10: equivalent translational mechanical system

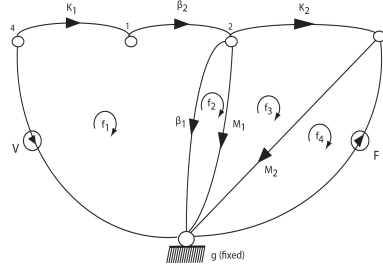


FIGURE 4.11: linear graph of mechanical system

- **Loop g-4-1-2-g:** $-v + \frac{s}{k_1} f_1 + \frac{1}{b_1} (f_1 - f_2) + \frac{1}{b_2} f_1 = 0$
- **Loop g-2-g:** $\frac{1}{b_1} (f_2 - f_1) + \frac{1}{m_1 s} (f_2 - f_3) = 0$
- **Loop g-2-3-g:** $\frac{1}{m_1 s} (f_3 - f_2) + \frac{s}{k_2} f_3 + \frac{1}{m_2 s} (f_3 - f_4) = 0$
- **Loop g-3-g:** $f_4 = -F$

and thus there is no need to sum across variables. The last condition is equivalent to expressing the through source F as an equivalent across source $\frac{-F}{m_2 s}$ with the m_2 element in series. The resulting equations are then:

$$\begin{bmatrix} \left(\frac{1}{b_1} + \frac{s}{k_1} + \frac{1}{b_2}\right) & -\left(\frac{1}{b_1}\right) & 0 \\ -\left(\frac{1}{b_1}\right) & \left(\frac{1}{b_1} + \frac{1}{m_1 s}\right) & -\left(\frac{1}{m_1 s}\right) \\ 0 & -\left(\frac{1}{m_1 s}\right) & \left(\frac{1}{m_1 s} + \frac{1}{m_2 s} + \frac{s}{k_2}\right) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} v \\ 0 \\ \frac{-F}{m_2 s} \end{bmatrix} \quad (4.12)$$

Using the analogy depicted in figure 4.10 and in Appendix A, where force \leftrightarrow current and velocity \leftrightarrow voltage, the mechanical system demonstrated in figures 4.9 and 4.10 has an equivalent electrical analogue:

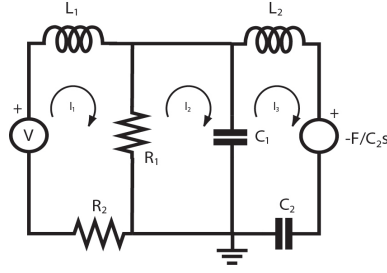


FIGURE 4.12: electrical analogue 1

$$\underbrace{\begin{bmatrix} R_1 + R_2 + L_1 s & -R_1 & 0 \\ -R_1 & R_1 + \frac{1}{C_1 s} & -\frac{1}{C_1 s} \\ 0 & -\frac{1}{C_1 s} & L_2 s + \frac{1}{C_1 s} + \frac{1}{C_2 s} \end{bmatrix}}_{\triangleq Z(s)} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{\triangleq \underline{i}} = \underbrace{\begin{bmatrix} V \\ 0 \\ \frac{-F}{C_2 s} \end{bmatrix}}_{\triangleq \underline{v}} \quad (4.13)$$

Node Formulation:

The *system graph* with node variables is shown in figure (4.11):

When the across variables for the nodes are selected as below the continuity equations may be expressed as:

- **Node 4:** Since node 4 is attached to the source V , u_4 is eliminated as an unknown, i.e. $v_4 = V$. This is equivalent to changing the across variable source to a through variable source.
- **Node 1:** Assuming through-variables as positive out of the node we have:

$$(v_1 - v_4) \frac{k_1}{s} + (v_1 - v_2) b_2 = 0$$
- **Node 2:** $(v_2 - v_1) b_2 + v_2 m_1 s + (v_2 - v_3) \frac{k_2}{s} + b_1 v_2 = 0$
- **Node 3:** $(v_3 - v_2) \frac{k_2}{s} + v_3 m_2 s - F = 0$

This, leads to the following model:

$$\begin{bmatrix} b_2 + \frac{k_1}{s} & -b_2 & 0 \\ -b_2 & (m_1 s + b_2 + b_1 + \frac{k_2}{s}) & -(\frac{k_2}{s}) \\ 0 & -(\frac{k_2}{s}) & (m_2 s + \frac{k_2}{s}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{k_1}{s} v \\ 0 \\ F \end{bmatrix} \quad (4.14)$$

Using the analogy depicted in figure 4.10 and in Appendix A, where force \leftrightarrow current and velocity \leftrightarrow voltage, the electrical analogue is presented below: with the equations

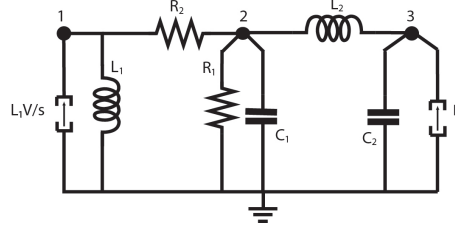


FIGURE 4.13: electrical analogue 2

in matrix form as follows:

$$\underbrace{\begin{bmatrix} \frac{1}{R_2} + \frac{1}{L_1 s} & -\frac{1}{R_2} & 0 \\ -\frac{1}{R_2} & \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{L_2 s} + C_1 s & -\frac{1}{L_2 s} \\ 0 & -\frac{1}{L_2 s} & \frac{1}{L_2 s} + C_2 s \end{bmatrix}}_{\triangleq Y(s)} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\triangleq v} = \underbrace{\begin{bmatrix} \frac{L_1 V}{s} \\ 0 \\ F \end{bmatrix}}_{\triangleq i} \quad (4.15)$$

It is clear that both impedance and admittance models are defined by integral differential operators, which are symmetric. In a system, which has considerably more nodes than loops, the loop method will be simpler to use and vice - versa. Specifically, using the number of equations for the nodes and loops we have:

Remark 4.7. Given that the number of independent node equations is $(n - 1)$ and the number of independent loop equations is $(b - n + 1)$, then:

1. If $b > 2(n - 1)$, we have fewer node than loop equations.
2. If $b < 2(n - 1)$, we have fewer loop than node equations.

□

Next, we will in examine the vertex and loop topologies of the associated linear graph of the mechanical system.

The Vertex Topology:

The vertex methodology, as discussed before, implies the substitution of all across variable sources by equivalent through variable sources. In this example, the equivalent graph with reduced number of independent vertices is:

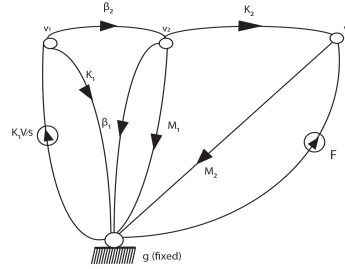


FIGURE 4.14: reduced vertex graph

The previous graph without the sources defines the admittance matrix and will be referred to as the *natural vertex graph* of the network.

The Loop Topology:

The loop topology is dual to the vertex topology. For the linear graph in figure (4.11) the *loop graph* is defined in the following picture. If the sources are omitted from the loop graph, the resulting graph characterizes the impedance matrix of the network and it is the *natural loop graph*. □

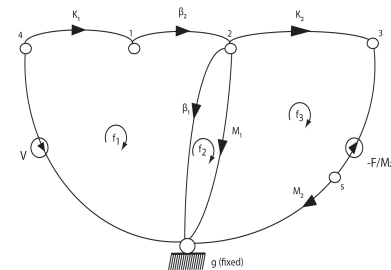


FIGURE 4.15: loop graph

4.4 The Internal Network Operator $W(s)$ and the Implicit Network Description

The general modeling for passive electrical networks provides a description of networks in terms of symmetric, integral, differential operators. The derivation of the *impedance* and *admittance* models shows that the corresponding matrices have the following general common structure:

$$W(s) = s\mathbf{B} + s^{-1}\mathbf{C} + \mathbf{D} \quad (4.16)$$

where in the case of *admittance* \mathbf{B} is the matrix of *A*-type elements, \mathbf{C} is the matrix of *T*-type elements and \mathbf{D} is the matrix of *D*-type elements (Appendix A) [Kar11, Liv12].

For the case of *impedance* the reverse holds true. Hence, \mathbf{B} is the matrix of *T*-type elements, \mathbf{C} is the matrix of *A*-type elements and \mathbf{D} is the matrix of *D*-type elements (Appendix A) [Kar11, Liv12].

Throughout this thesis the method adopted for modelling is the mesh / loop method. Hence, the equivalent expression for $W(s)$ in (4.16) will be:

$$W(s) = Z(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R} \quad (4.17)$$

where \mathbf{L} denotes the matrix of inductors, \mathbf{C} the matrix of capacitors, \mathbf{R} the matrix of resistors equivalently and $Z(s)$ denotes the impedance matrix of an *RLC* network.

The symmetric operator $W(s)$ is thus a common description of $Y(s)$ and $Z(s)$ matrices, i.e. it defines both impedance and admittance models / operators. The operator $W(s)$ describes the dynamics of the network and of special interest is the properties of its zeros.³ Furthermore, the structure of \mathbf{B} , \mathbf{C} and \mathbf{D} matrices characterizes the topology of *A*-, *T*- and *D*- type matrices associated with the network. Such matrices have a structure and properties, which underpin the development of a system theoretic framework based on network models. [Kar11]

³ $W^{-1}(s)$ is defined as the transfer function of an *RLC* network (see section 4.7), which defines the dynamics of the system. Hence, the poles of $W^{-1}(s)$ are the zeros of $W(s)$.

For the special cases where the network is characterized only by A - and D - type elements, or T - and D - type elements then $W(s)$ has the following special forms:

$$\begin{aligned}\widetilde{W}(s) &= s\mathbf{B} + \mathbf{D} \\ \widehat{W}(s) &= \widehat{s}\mathbf{C} + \mathbf{D}, \widehat{s} = s^{-1}\end{aligned}\tag{4.18}$$

which are *symmetric matrix pencils*. These pencils are derived from passive networks and thus inherit the passivity properties.

4.5 Relationship Between Impedance and Admittance Operators

We consider a network with m nodes and q loops and let us assume that $q \leq m$. We shall refer to m and q as the *nodal, loop* cardinality respectively. We assume that the corresponding Implicit Impedance and Admittance models are:

$$Y(s)\underline{v} = \underline{0} \quad \text{and} \quad Z(s)\underline{l} = \underline{0}\tag{4.19}$$

From the network topology the following Proposition is readily established:

Proposition 4.1. *If $q \leq m$, there exist a rational $m \times q$ matrix $\mathbf{T}(s)$ of the type $\mathbf{T}(s) = \mathbf{T}_0 + s\mathbf{T}_1 + s^{-1}\mathbf{T}_2$, where $\mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_2$ are $m \times q$ real matrices such that [KLL17]:*

$$\underline{v} = \mathbf{T}(s)\underline{l}\tag{4.20}$$

□

Proof. Assume that the number of nodes is larger or equal to the number of loops. Every nodal voltage can then be expressed as a function of the loop currents, corresponding admittances and possibly other loop currents. This readily establishes the relationship between loop currents and nodal voltages of the type indicated by 4.20 and this completes the proof. □

The above implies that there exists a relationship between the two *Implicit descriptions* $Y(s)$ and $Z(s)$, which needs further investigation.

The previous result may be illustrated via the following example.

Example 4.4. Consider the network illustrated in figure (4.16) with $m = 6$ nodes and $q = 3$ loops, i.e.

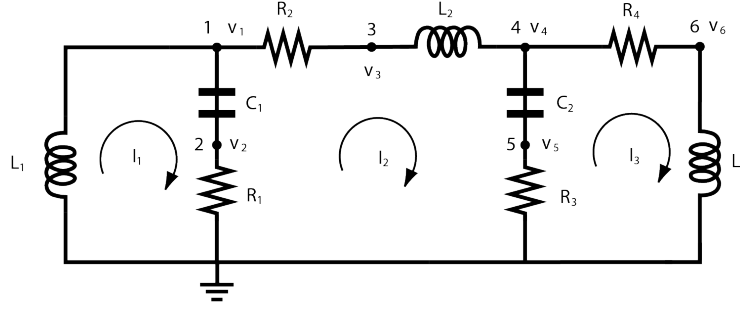


FIGURE 4.16: arbitrary network with $m = 6$ nodes and $q = 3$ loops

$$Z(s) = \begin{bmatrix} \frac{1}{s}C_1^{-1} + R_1 + sL_1 & -\frac{1}{s}C_1^{-1} - R_1 & 0 \\ -\frac{1}{s}C_1^{-1} - R_1 & \frac{1}{s}(C_1^{-1} + C_2^{-1}) + R_1 + R_2 + R_3 + sL_2 & -\frac{1}{s}C_2^{-1} - R_3 \\ 0 & -\frac{1}{s}C_1^{-1} - R_3 & -\frac{1}{s}C_2^{-1} + R_3 + R_4 + sL_3 \end{bmatrix} \quad (4.21)$$

$$Z(s)\underline{i}(s) = \underline{0} \quad (4.22)$$

We can now compute the Admittance model for the network having the 6 nodes.

$$Y(s) = \begin{bmatrix} \frac{1}{s}L_1^{-1} + sC_1 + R_2^{-1} & -sC_1 & -R_2^{-1} & 0 & 0 & 0 \\ -sC_1 & sC_1 + R_1^{-1} & 0 & 0 & 0 & 0 \\ -R_2^{-1} & 0 & \frac{1}{s}L_2^{-1} - R_2^{-1} & -\frac{1}{s}L_2^{-1} & 0 & 0 \\ 0 & 0 & \frac{1}{s}L_2^{-1} & \frac{1}{s}L_2^{-1} + sC_2 + R_4^{-1} & -sC_2 & -R_4^{-1} \\ 0 & 0 & 0 & -sC_2 & sC_2 + R_3^{-1} & 0 \\ 0 & 0 & 0 & R_4^{-1} & 0 & \frac{1}{s}L_3^{-1} - R_4^{-1} \end{bmatrix} \quad (4.23)$$

The first model is

$$Z(s)\underline{i}(s) = \underline{0} \quad \text{where} \quad \underline{i}(s) = \begin{bmatrix} i_1(s) \\ i_2(s) \\ i_3(s) \end{bmatrix} \quad (4.24)$$

$$Y(s)\underline{v}(s) = \underline{0} \quad \text{where} \quad \underline{v}(s) = \begin{bmatrix} v_1(s) \\ v_2(s) \\ v_3(s) \\ v_4(s) \\ v_5(s) \\ v_6(s) \end{bmatrix} \quad (4.25)$$

The next issue is to investigate the relationship between $Z(s)$ and $Y(s)$ and thus define the link between $\underline{v}(s)$ and $\underline{i}(s)$.

$$\left\{ \begin{array}{l} v_1 = sL_1 i_1 \\ v_2 = R_1(i_1 - i_2) \\ v_5 = R_3(i_2 - i_3) \\ v_6 = sL_3 i_3 \\ v_3 - v_1 = R_2 i_2 \quad \Leftrightarrow \quad v_3 = v_1 + R_2 i_2 = sL_1 i_1 + R_2 i_2 \\ v_4 - v_3 = sL_2 i_2 \quad \Leftrightarrow \quad v_4 = v_3 + sL_2 i_2 = sL_1 i_1 + R_2 i_2 + sL_2 i_2 \end{array} \right. \quad (4.26)$$

and thus

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} sL_1 & 0 & 0 \\ R_1 & -R_1 & 0 \\ sL_1 & R_2 & 0 \\ sL_1 & R_2 + sL_2 & 0 \\ 0 & R_3 & -R_3 \\ 0 & 0 & sL_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = Q(s) \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \quad (4.27)$$

$$\left\{ \begin{array}{l} v_1 = sL_1 i_1 \\ v_1 - v_2 = \frac{1}{s} C_1^{-1} (i_1 - i_2) \\ v_2 = R_1 (i_1 - i_2) \\ v_4 - v_3 = sL_2 i_2 \\ v_3 - v_1 = R_2 i_2 \\ v_5 - v_4 = \frac{1}{s} C_2^{-1} (i_2 - i_3) \\ v_5 = R_3 (i_2 - i_3) \\ v_6 - v_4 = R_4 i_3 \\ v_6 = sL_3 i_3 \end{array} \right. \quad (4.28)$$

□

4.6 The Network Pencil and its Relationship to the Internal Network Description

In this section, we are introducing the *Loop Network Pencil* $P(p)$ (or $P(s)$) and we examine its relationship with the internal network operator $W(s)$ (or $Z(s)$, because throughout the thesis we use impedance modelling) [KLL17]. Similar results may be derived in the case of the admittance operator $Y(s)$.

Consider a network with m nodes and q loops and let us assume that $m \geq q$. The corresponding *impedance model* is then given by the following expression, where p stands for the derivative operator:

$$W(p) \cdot \underline{i} = \underline{v}$$

where $W(p) = Z(p) = p\mathbf{L} + p^{-1}\mathbf{C} + \mathbf{R}$ is the impedance operator and $\mathbf{L}, \mathbf{R}, \mathbf{C}$ represent the matrices of inductors, resistors and capacitors respectively. Assuming that the network has no (input) voltage sources the previous equation can be written as:

$$W(p) \cdot \underline{i} = \underline{0} \quad (4.29)$$

We can define the new variables as: $p^{-1}\underline{i} = \widehat{\underline{i}}$ and $p\widehat{\underline{i}} = \underline{i}$ and thus the original implicit description 4.29 becomes:

$$p\mathbf{L}\underline{i} + \mathbf{C}\widehat{\underline{i}} + \mathbf{R}\underline{i} = \underline{0}$$

or

$$p \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \underline{i} \\ \widehat{\underline{i}} \end{bmatrix} + \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{i} \\ \widehat{\underline{i}} \end{bmatrix} = \underline{0} \quad (4.30)$$

Clearly, the vector $\underline{\xi}^t = \begin{bmatrix} \underline{i} & \widehat{\underline{i}} \end{bmatrix}^t$ is a state vector and the description defined by 4.30 is an implicit state space description, which is not necessarily minimal. This description preserves the loop structure of the network and it will be referred to as *loop implicit state space description* and the associated *matrix pencil*

$$P(s) = s \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} s\mathbf{L} + \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix} = s\mathbf{F} + \mathbf{G} \quad (4.31)$$

will be referred to as the *loop network pencil*. The relationship between $P(s)$ and $W(s)$ is established in the following proposition.

Proposition 4.2. *The following properties hold:*

- (i) *The determinants of $P(s)$ and $W(s)$, where $P(s)$ is the loop network pencil, $W(s)$ is the impedance description of the network and q is the number of loops, are related as:*

$$|W(s)| = |Z(s)| = s^{-q} |P(s)|$$

- (ii) *If $W(s) = Z(s) = s^{-1}\mathbf{Z}_a(s)$, then:*

$$|\mathbf{Z}_a(s)| = |\{s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}\}| = |P(s)|$$

□

Proof. (i) Using Schur's formula for $P(s)$ and expanding with respect to $s\mathbf{I}$ we have:

$$|P(s)| = |s\mathbf{I}| \cdot |s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R}| = s^q \cdot |W(s)|$$

□

This allows relating the zero structure of $W(s)$ with the zero structure of the associated pencil $P(s)$. In the following we examine the invariant structure properties of $P(s)$ which also characterize properties of $W(s)$. The linearized pencil is structured, but not symmetric in the general case. In section 4.8 we will further examine the zero structure properties of $P(s)$.

Remark 4.8. For the special cases where the network is characterized only by one type of dynamic elements, then the respective pencils are symmetric, preserve the network structure and inherit the passivity properties, i.e.

$$\begin{aligned} Y(s) &= s\mathbf{C} + \mathbf{R} \\ Z(\hat{s}) &= \hat{s}\mathbf{L} + \mathbf{R} \end{aligned}$$

where $\hat{s} = s^{-1}$.

□

Remark 4.9. The MFD factorization $Z(s) = [s\mathbf{I}_q]^{-1}\mathbf{Z}_a(s)$ is coprime at all finite s except possibly at $s = 0$. Thus the zeros of $Z(s)$ (or $W(s)$ equivalently) and $\mathbf{Z}_a(s)$ may differ only at $s = 0$.

□

4.7 Network Regularity and Invertibility of $W(s)$

In this section we investigate the regularity properties of $W(s)$ (or $Z(s)$) and we demonstrate the conditions under which is degenerate i.e. it loses rank over $\mathbb{R}(s)$. Furthermore, we present the equivalent regularity conditions for the associated pencil $P(s)$ in terms of the rank properties and structure of the corresponding *Toeplitz* matrices.

The implicit description of equation (4.29) may be expanded to an oriented (forced) description by selecting inputs $\underline{\tau}$ and outputs $\underline{\zeta}$ which transform the model to the form:

$$\begin{aligned} W(s)\underline{i} &= Q\underline{\tau}, \quad \underline{\zeta} = H\underline{i} \\ W(s) &= Z(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R} = s^{-1}\mathbf{Z}_a(s) \\ G(s) &= HW^{-1}(s)Q \end{aligned} \tag{4.32}$$

where $G(s)$ denotes the *explicit transfer function* of the oriented description (when inputs and outputs are introduced), whereas $W^{-1}(s)$ is the *implicit transfer function* of the non oriented description (4.29).

It is clear from the above that the ability to define transfer functions in a network depends on the invertibility of $W(s)$ (equivalently in the invertibility of $Z(s)$). A network will be called *regular* if $\det[W(s)] \neq 0$ over $\mathbb{R}(s)$. Note that $\mathbf{Z}_a(s) \in \mathbb{R}[s]^{q \times q}$ and can always be expressed as in equation (4.33) where $p_{ij} \in \mathbb{R}[s]$ are the polynomials resulting from the impedance functions between nodes i and j , all have positive coefficients $\hat{p}_{ii} = \sum_{j=1}^q p_{ij} + p_{ii}$. The above decomposition enables the computation of $\det[\mathbf{Z}_a(s)]$. In the following we will derive criteria for the characterization of this property. The computation of the expression for this determinant allows the characterization of the regularity property in graph terms. This computation requires some definitions and notation which are introduced first.

$$\begin{aligned} \mathbf{Z}_a(s) &= \begin{bmatrix} \hat{p}_{11} & -p_{12} & \cdots & -p_{1(m-1)} & -p_{1m} \\ -p_{12} & \hat{p}_{22} & \cdots & -p_{2(m-1)} & -p_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -p_{1(m-1)} & -p_{2(m-1)} & \cdots & \hat{p}_{(m-1)(m-1)} & -p_{(m-1)m} \\ -p_{1m} & -p_{2m} & \cdots & -p_{(m-1)m} & \hat{p}_{mm} \end{bmatrix} = \\ &= R(s) + T(s) = \text{diag}\{\dots p_{ii} \dots\} + T(s) \end{aligned} \quad (4.33)$$

We first present the definition, which is essential for the development of the proof for the regularity property of $W(s)$.

Definition 4.4. [KLL17] Let us denote by $\tilde{q} = \{1, 2, \dots, q\}$ and by

$\Omega_{k,q} = \{\omega_k = (i_1, i_2, \dots, i_k) \in Q_{k,q}, k \leq q\}$ [MM64], where $Q_{k,q}$ is the set of lexicographically ordered sequences of k integers from \tilde{q} and $\{p_{ij} \in \mathbb{R}[s], i, j = 1, 2, \dots\}$. We define:

(i) For any $\omega_k = (i_1, i_2, \dots, i_k) \in \Omega_{k,q}$, $r(\omega_k) = p_{i_1 i_1} p_{i_2 i_2} \cdots p_{i_k i_k}$ and

$$r(\omega_q) = p_{11} p_{22} \cdots p_{qq}.$$

(ii) If $\mathcal{A} = \{\rho = (j_1, j_2) \in Q_{2,q}\} = \left\{ \rho_1, \rho_2, \dots, \rho_\tau : \tau = \binom{q}{2} \right\}$, then $p(\rho) = p_{j_1} p_{j_2}$ for $\rho = (j_1, j_2) \in \mathcal{A}$.

(iii) Given any $\omega_k = (i_1, i_2, \dots, i_k) \in \Omega_{k,q}$, we denote by $\mathcal{A}(\omega_k)$ the subset of \mathcal{A} obtained by deleting the sequences $\rho = (j_1, j_2) \in Q_{2,q}$ based on the (i_1, i_2, \dots, i_k) set of indices. $\mathcal{A}(\omega_k)$ has $\vartheta = (q/2) - (k/2)$ elements.

(iv) Given $\mathcal{A}(\omega_k)$ we define

$$\mathcal{B}_k(\omega_k) = \left\{ \begin{array}{l} \sigma = (\rho_{l1}, \rho_{l2}, \dots, \rho_{l\nu}) \in Q_{\nu,\tau}, \\ \rho = (j_1, j_2) \in Q_{2,q} \end{array} \right\}$$

or simply $\mathcal{B}_k(\omega_k) = \{\sigma_1, \sigma_2, \dots, \sigma_\pi : \pi = (\tau/\nu)\}$, for $\nu \in \tilde{q}$. The elements of $\mathcal{A}(\omega_k)$, $\mathcal{B}_k(\omega_k)$ are lexicographically ordered.

(v) Given $\omega_k = (i_1, i_2, \dots, i_k) \in \Omega_{k,q}$ and the set $\mathcal{A}(\omega_k)$ we denote by $\mathcal{B}_k[\omega_k]$ the subset of $\mathcal{B}_k(\omega_k)$ that excludes all $\rho = (j_1, j_2) \in \mathcal{A}(\omega_k)$ sequences.

(vi) Let

$$\mathcal{B}_k[\omega_k] = \left\{ \begin{array}{l} \hat{\sigma} = (\hat{\rho}_{l1}, \hat{\rho}_{l2}, \dots, \hat{\rho}_{l\nu}) \in \mathcal{A}(\omega_k), \\ \hat{\rho}_{l\kappa}(j_1, j_2) \in Q_{2,q} \end{array} \right\} = \{\hat{\sigma}_1, \hat{\sigma}_2, \dots, \hat{\sigma}_{\pi'}\}$$

Every element $\hat{\sigma}$ of $\mathcal{B}_k[\omega_k]$ may be represented as

$$\hat{\sigma} = (\hat{\rho}_{l1}, \hat{\rho}_{l2}, \dots, \hat{\rho}_{l\nu}) = (j_{l11}, j_{l12}; j_{l21}, j_{l22}; \dots; j_{l\nu1}, j_{l\nu2})$$

The $\hat{\sigma}$ element will be called *proper*, if there are no more than $(k - l)$ repeated indices from the $\omega_k = (i_1, i_2, \dots, i_k) \in \Omega_{k,q}$ set; otherwise the element will be called *non-proper*. The subset of proper sequences of $\mathcal{B}_k[\omega_k]$ will be denoted by $\hat{\mathcal{B}}_k[\omega_k]$.

(vii) For any $\omega_k = (i_1, i_2, \dots, i_k) \in \Omega_{k,q}$ and a proper

$$\hat{\sigma} = (\hat{\rho}_{l1}, \hat{\rho}_{l2}, \dots, \hat{\rho}_{l\nu}) = (j_{l11}, j_{l12}; j_{l21}, j_{l22}; \dots; j_{l\nu1}, j_{l\nu2}) \in \hat{\mathcal{B}}_k[\omega_k]$$

we define as

$$r(\hat{\mathcal{B}}_k, \omega_k) = \sum_{\hat{\sigma} \in \hat{\mathcal{B}}_k[\omega_k]} p_{j_{l11}j_{l12}} p_{j_{l21}j_{l22}} \cdots p_{j_{l\nu1}j_{l\nu2}}$$

□

We demonstrate the above definition by an example:

Example 4.5. Let $\tilde{4} = \{1, 2, 3, 4\}$. Then for $\omega_4 = (1, 2, 3, 4)$ and $r(\omega_4) = p_{11}p_{22}p_{33}p_{44}$.

(i) If $\omega_3^a = (1, 2, 3)$, then $r(\omega_3^a) = p_{11}p_{22}p_{33}$ and $r(\widehat{\mathcal{B}}_3, \omega_3^a) = p_{14} + p_{24} + p_{34}$.

(ii) If $\omega_2^a = (1, 3)$, then $r(\omega_2^a) = p_{11}p_{33}$ and

$$r(\widehat{\mathcal{B}}_2, \omega_2^a) = p_{12}p_{14} + p_{14}p_{23} + p_{12}p_{24} + p_{14}p_{24} + p_{23}p_{24} + p_{12}p_{34} + p_{23}p_{34} + p_{24}p_{34} + p_{14}p_{34}.$$

(iii) If $\omega_1^a = (1)$, or $\omega_1^\beta = (2)$, or $\omega_1^\gamma = (3)$, or $\omega_1^\delta = (4)$, then $r(\omega_1^a) = p_{11}$, $r(\omega_1^\beta) = p_{22}$,

$r(\omega_1^\gamma) = p_{33}$, $r(\omega_1^\delta) = p_{44}$ and

$$\begin{aligned} r(\widehat{\mathcal{B}}_1, \omega_1^a) &= r(\widehat{\mathcal{B}}_1, \omega_1^\beta) = r(\widehat{\mathcal{B}}_1, \omega_1^\gamma) = r(\widehat{\mathcal{B}}_1, \omega_1^\delta) = p_{12}p_{13}p_{14} + p_{12}p_{13}p_{24} + p_{12}p_{13}p_{34} + \\ &+ p_{12}p_{14}p_{23} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{24} + p_{12}p_{23}p_{34} + p_{12}p_{24}p_{34} + p_{13}p_{14}p_{23} + p_{13}p_{14}p_{24} + \\ &+ p_{13}p_{23}p_{24} + p_{13}p_{23}p_{34} + p_{13}p_{24}p_{34} + p_{14}p_{23}p_{24} + p_{14}p_{23}p_{34} + p_{14}p_{24}p_{34} \end{aligned}$$

□

The computation of the determinant of the loop-impedance matrix, known as Kirchhoffs rule, have been discussed in an extensive depth by many authors [BSST09, Cau58, Fra25, Ku52, MS57, Oka55a, Oka55b, Rez58, Per53, Tal55, Wei58]. In this section we provide an alternative proof of this result, which is related to the connectivity of the network, as we will see later on. We may now state the following results:

Theorem 4.1. *We may express $\det \{\mathbf{Z}_{\mathbf{a}}(s)\}$ as a positive sum of polynomials with positive coefficients in terms of the elements of $\mathbf{Z}_{\mathbf{a}}(s)$, $p_{i,j(s)}$ and $\widehat{p}_{ii}(s)$ as:*

$$\det [\mathbf{Z}_{\mathbf{a}}(s)] = \sum_i p_{i1}p_{i2} \cdots p_{in}$$

□

Proof. The proof is made by induction.

Let $\mathbf{Z}_{\mathbf{a}}(s)$ be of the form (4.33), where $p_{ij} \in \mathbb{R}[s]$ are the polynomials resulting from the

impedance (or admittance) functions between nodes i and j , all have positive coefficients

$\hat{p}_{ii} = \sum_{j=1}^q p_{ij} + p_{ii}$. The structure of the matrix $T(s)$ in (4.33) is as follows:

- The elements in the main diagonal are all positive.
- The elements above and below the main diagonal are all negative.
- The sum of elements in each row of the matrix is equal to 0.
- Each of the p_{ij} 's represents the common impedances (or admittances) between loops (or nodes) i and j .
- Each of the elements in the main diagonal represents polynomials that contain the sum of the impedances (or admittances) that are common between loops (or nodes) i and j .
- Each element p_{ij} with the property $p_{ij} = p_{ji}$, represents the common impedance (or admittance) between loops (or nodes) i and j with a negative '-' sign.

We will prove by induction that $\det \{\mathbf{Z}_a(s)\}$ may be expressed as a positive sum of polynomials with positive coefficients. We show that this holds for $n = 3$, then we assume it applies for $n \leq k$ and then we demonstrate that is also verified for $n = k + 1$.

For $n = 3$ we can express $W(s)$ operator as:

$$W(s) = \frac{1}{s} \mathbf{Z}_a(s) = \frac{1}{s} \underbrace{\begin{bmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}}_{R(s)} + \frac{1}{s} \underbrace{\begin{bmatrix} p_{12} + p_{13} & -p_{12} & -p_{13} \\ -p_{12} & p_{12} + p_{13} & -p_{23} \\ -p_{13} & -p_{23} & p_{13} + p_{23} \end{bmatrix}}_{T(s)}$$

where the matrices $R(s)$ and $T(s)$ are as in (4.33) and have the properties and the structure that we defined previously. The computations of the determinant of $\mathbf{Z}_a(s)$ leads to the following result:

$$\det \{\mathbf{Z}_a(s)\} = p_{11}p_{22}p_{33} + p_{11}p_{22}(p_{13} + p_{23}) + p_{11}p_{33}(p_{12} + p_{23}) + p_{22}p_{33}(p_{12} + p_{13}) + (p_{11} + p_{22} + p_{33})(p_{12}p_{13} + p_{12}p_{23} + p_{13}p_{23})$$

which is a sum of polynomials with positive coefficients, as each of $p_{ij}, p_{ii} \geq 0$.

Induction Hypothesis: Let us assume that the hypothesis made previously holds for $n \leq k$. We use this assumption to prove that it holds for $n = k + 1$. $\mathbf{Z}_{\mathbf{a}_{k+1}}(s)$ can be expressed as follows:

$$\mathbf{Z}_{\mathbf{a}_{k+1}}(s) = \begin{bmatrix} p_{11} & 0 & \cdots & 0 \\ 0 & p_{22} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & p_{(k+1)(k+1)} \end{bmatrix} + T(s)$$

hence, the determinant of $\mathbf{Z}_{\mathbf{a}_{k+1}}(s)$ will have the following form:

$$\begin{aligned} \det(\mathbf{Z}_{\mathbf{a}_{k+1}}(s)) &= p_{11}T_1 + p_{22}T_2 + \dots + p_{(k+1)(k+1)}T_{(k+1)} + p_{11}p_{22}T_{12} + \\ &+ \dots + p_k p_{k+1}T_{k,(k+1)} + \dots + p_{11}p_{22} \cdots p_{(k+1)(k+1)} \end{aligned} \quad (4.34)$$

where each of T_i 's represent the determinants (minors) of the matrix $T(s)$ which result if we delete the i -th row and column from the initial matrix $T(s)$ with dimension $(k+1) \times (k+1)$. Equivalently, each of the T_{ij} 's represent the determinants (minors) of the matrix $T(s)$ of dimension $(k-1) \times (k-1)$ which result if we delete the i -th row and column and j -th row and column respectively and so on. Each of the T_i 's, T_{ij} 's and so on can be written as $R'(s) + T'(s)$, which have the same structure and properties with $R(s)$ and $T(s)$ as in (4.33). Thus, we can apply the induction hypothesis. Hence, in the case where $n = k + 1$ the resulting determinant will be as in (4.34). Thus, in the expression (4.34) each T_i , T_{ij} and so on, is a positive sum of polynomial products with positive coefficients and if we replace them, then the result will be as well a positive sum of polynomial products with positive coefficients.

We shall note that all the sub-matrices T that result from the deletion of rows and columns are of dimension $\leq k$ and they verify the properties and structure of the *induction hypothesis*. And this proves the theorem. \square

Lemma 4.4. *If we use the notation we established in definition (4.4), the resulting determinant can be written as [KLL17]:*

$$\det \{\mathbf{Z}_a(s)\} = p_{11}p_{22}\dots p_{qq} + \sum_{\omega \in \Omega(q-1,q)} r(\omega)r(\widehat{\mathcal{B}}_{q-1}, \omega) + \sum_{\omega \in \Omega(k,q)} r(\omega)r(\widehat{\mathcal{B}}_k, \omega) + \dots + (p_{11} + p_{22} + \dots + p_{qq})r(\widehat{\mathcal{B}}_1, \omega)$$

□

Lemma 4.5. *Let $j \in \widetilde{q}$ and for a given $\omega_k = (i_1, i_2, \dots, i_k) \in \Omega_{k,q}, j \notin \omega$. Then all $p_{ji}, i \neq j, i \in \widetilde{q}$ are terms in $r(\widehat{\mathcal{B}}_k, \omega)$.*

□

We will now state the main theorem for the regularity property of $W(s)$:

Theorem 4.2. *The network is regular, if $p_{ij} \neq 0$ in all loops/ nodes of the network.* □

Proof. To be identically equal to 0 the determinant in theorem (4.1) the following should hold:

$$\det [\mathbf{Z}_a(s)] = \sum_i p_{i1}p_{i2} \dots p_{in} = 0$$

Let us assume that

$$p_{i1}p_{i2} \dots p_{in} = A_{i,2n}s^{2n} + A_{i,2n-1}s^{2n-1} + \dots + A_{i,0}s^0$$

Equivalently,

$$\sum_i p_{i1}p_{i2} \dots p_{in} = \sum_i A_{i,2n}s^{2n} + \sum_i A_{i,2n-1}s^{2n-1} + \dots + \sum_i A_{i,0}s^0$$

For $\det \{\mathbf{Z}_a(s)\} = 0$ it follows that: $\sum_i A_{i,k} = 0$, where $k = 2n, 2n-1, \dots, 0 \Rightarrow A_{i,k} = 0 \forall i \Rightarrow p_{i1}p_{i2} \dots p_{in} = 0 \Rightarrow$ at least one of $p_{i1}, p_{i2}, \dots, p_{in} = 0$ and this proves the theorem. □

Example 4.6. Let $\widetilde{3} = \{1, 2, 3\}$. Then:

$$\det \{\mathbf{Z}_a(s)\} = p_{11}p_{22}p_{33} + p_{11}p_{22}(p_{13} + p_{23}) + p_{11}p_{33}(p_{12} + p_{23}) + p_{22}p_{33}(p_{12} + p_{13}) + (p_{11} + p_{22} + p_{33})(p_{12}p_{13} + p_{12}p_{23} + p_{13}p_{23})$$

which is a sum of polynomials with positive coefficients. Note that if $p_{11} = 0$, $p_{22} \neq 0$, $p_{33} \neq 0$, then $p_{22}p_{33}(p_{12} + p_{13}) = 0$ and thus $p_{12} = 0$, $p_{13} = 0$ and this demonstrates the result. \square

Next, based on theorem 4.2 for the regularity property, we state the following remark that gives an insight for the connectivity of an *RLC* network.

Remark 4.10. [KLL17] The network is regular if and only if the network is connected, that is there is no loop i (or respectively node) with all $p_{ij} = 0, j \in \tilde{q}$. \square

Note that network regularity is equivalent to that there is no j loop for which all $p_{ji} = 0, \forall i \in \tilde{q}$. Similar statement may be given for the admittance analysis. The conditions for regularity of $\mathbf{Z}_a(s)$, or $W(s)$ (equivalently $Z(s)$) may be expressed on the *loop network pencil* $P(s)$ and this leads to an algebraic characterization and some interesting properties of the associated impedance topology ([NMJ16]).

Next, we are going to examine the equivalent regularity properties for the loop network pencil $P(s)$.

Corollary 4.1. *The network is regular if and only if the loop network pencil $P(s)$ is regular. This implies that $P(s)$ has no column and no row minimal indices and that $\text{rank} [\mathbf{L}, \mathbf{R}, \mathbf{C}] = q$.* \square

Clearly, the singularity property of $W(s)$ is equivalent to the existence of $\underline{x}(s) \in \mathbb{R}^q[s]$, with $\deg \underline{x}(s) = k$ such that [KK88]:

$$\begin{aligned} W(s)\underline{x}(s) &= \underline{0} \\ \underline{x}(s) &= \underline{x}_0 + s\underline{x}_1 + \dots + s^k \underline{x}_k \end{aligned} \tag{4.35}$$

Given that $W(s) = Z(s) = s\mathbf{L} + \frac{1}{s}\mathbf{C} + \mathbf{R}$ the above two conditions lead to:

$$\{s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}\}(\underline{x}_0 + s\underline{x}_1 + \dots + s^k \underline{x}_k) = \underline{0}$$

or equivalently

$$\begin{aligned}
\mathbf{L}\underline{x}_k &= \underline{0} \\
\mathbf{R}\underline{x}_k + \mathbf{L}\underline{x}_{k-1} &= \underline{0} \\
\mathbf{C}\underline{x}_k + \mathbf{R}\underline{x}_{k-1} + \mathbf{L}\underline{x}_{k-2} &= \underline{0} \\
\mathbf{C}\underline{x}_{k-1} + \mathbf{R}\underline{x}_{k-2} + \mathbf{L}\underline{x}_{k-3} &= \underline{0} \\
&\vdots \\
\mathbf{C}\underline{x}_2 + \mathbf{R}\underline{x}_1 + \mathbf{L}\underline{x}_0 &= \underline{0} \\
\mathbf{C}\underline{x}_1 + \mathbf{R}\underline{x}_0 &= \underline{0} \\
\mathbf{C}\underline{x}_0 &= \underline{0}
\end{aligned} \tag{4.36}$$

The above conditions 4.36 may be expressed in a matrix form using *Toeplitz matrices* as [KK86]:

$$\underbrace{\begin{bmatrix} \mathbf{L} & \mathbf{0} & \mathbf{0} & \cdots & & \mathbf{0} \\ \mathbf{R} & \mathbf{L} & \mathbf{0} & \cdots & & \mathbf{0} \\ \mathbf{C} & \mathbf{R} & \mathbf{L} & & & \\ \mathbf{0} & \mathbf{C} & \mathbf{R} & & & \\ & & & \ddots & & \\ & & & & \ddots & \mathbf{L} & \mathbf{0} & \mathbf{0} \\ & & & & & \mathbf{R} & \mathbf{L} & \mathbf{0} \\ & & & & & \mathbf{C} & \mathbf{R} & \mathbf{L} \\ & & & & & \mathbf{0} & \mathbf{C} & \mathbf{R} \\ & & & & & \mathbf{0} & \mathbf{0} & \mathbf{C} \end{bmatrix}}_{\triangleq T_k} \begin{bmatrix} \underline{x}_k \\ \underline{x}_{k-1} \\ \underline{x}_{k-2} \\ \vdots \\ \underline{x}_2 \\ \underline{x}_1 \\ \underline{x}_0 \end{bmatrix} = \underline{0} \tag{4.37}$$

where the matrices

$$T_0 = \begin{bmatrix} \mathbf{L} \\ \mathbf{R} \\ \mathbf{C} \end{bmatrix}, T_1 = \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{R} & \mathbf{L} \\ \mathbf{C} & \mathbf{R} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}, \dots, T_k = \begin{bmatrix} \mathbf{L} & \mathbf{0} & \mathbf{0} & \dots & & \mathbf{0} \\ \mathbf{R} & \mathbf{L} & \mathbf{0} & \dots & & \mathbf{0} \\ \mathbf{C} & \mathbf{R} & \mathbf{L} & & & \\ \mathbf{0} & \mathbf{C} & \mathbf{R} & & & \\ & & & \ddots & & \\ & & & & \ddots & \mathbf{L} & \mathbf{0} & \mathbf{0} \\ & & & & & \mathbf{R} & \mathbf{L} & \mathbf{0} \\ & & & & & \mathbf{C} & \mathbf{R} & \mathbf{L} \\ & & & & & \mathbf{0} & \mathbf{C} & \mathbf{R} \\ & & & & & \mathbf{0} & \mathbf{0} & \mathbf{C} \end{bmatrix}$$

have dimensions respectively $q(k+3) \times q(k+1)$, where $(q \times q)$ is the dimension of $W(s)$. The set of T_k matrices will be referred to as the set of *Toeplitz network matrices*. The properties of such matrices characterize the regularity of the network as examined next. We first state some useful Lemmas.

Lemma 4.6. *For the set of matrices $\{T_k, k = 0, 1, \dots, \nu\}$ the following properties hold true:*

- (i) *If T_k is rank deficient, then all matrices $T_{k+\rho}$ are rank deficient $\forall \rho \geq 0$.*
- (ii) *If T_ν is full rank, then all matrices $\{T_i, i = 1, 2, \dots, \nu - 1\}$ are full rank.*

□

Proof. Part (i) readily follows from the Toeplitz structure of the matrices. Part (ii) follows from part (i) and by using contradiction arguments. □

Lemma 4.7. *Let $\underline{\xi}(s) = [\underline{x}(s), \underline{w}(s)]^t \in \mathbb{R}^{2q}[s]$ such that $P(s)(s)\underline{\xi}(s) = \underline{0}$. Then, $\{s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}\}\underline{w}(s) = \underline{0}$, $\underline{x}(s) = s\underline{w}(s)$ and $\deg\{\underline{x}(s)\} = \deg\{\underline{w}(s)\} + 1$. Furthermore, $P(s)$ is regular if and only if $W(s)$ is regular.* □

Proof. We first note that

$$\begin{bmatrix} \mathbf{I} & s\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -s\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Using the above we have

$$P(s) \begin{bmatrix} \mathbf{I} & s\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -s\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \underline{x}(s) \\ \underline{w}(s) \end{bmatrix} = \underline{0} \Leftrightarrow \begin{bmatrix} s\mathbf{L} + \mathbf{R} & s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x}(s) - s\underline{w}(s) \\ \underline{w}(s) \end{bmatrix} = \underline{0}$$

The equivalence of regularity between $P(s)$ and $W(s)$ follows from the fact that under unimodular equivalence we have:

$$P(s) \begin{bmatrix} \mathbf{I} & s\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} s\mathbf{L} + \mathbf{R} & s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

□

Lemma 4.8. *The pencil $P(s)$ has zero row minimal indices (rmi) if and only if:*

$$\text{rank} [\mathbf{L}, \mathbf{R}, \mathbf{C}] < q$$

where q represents the number of loops in an RLC network.

□

Proof. We note that the presence of zero - row minimal indices (rmi) implies the existence of a constant vector such that:

$$\begin{bmatrix} \underline{\beta}^t & \underline{\alpha}^t \end{bmatrix} \begin{bmatrix} s\mathbf{L} + \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix} = \underline{0} \Leftrightarrow \begin{bmatrix} \underline{\beta}^t & \underline{\alpha}^t \end{bmatrix} \begin{bmatrix} s\mathbf{L} + \mathbf{R} & s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} = \underline{0}$$

from which $\underline{\alpha}^t = \underline{0}$ and $\underline{\beta}^t (s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}) = \underline{0}$, or equivalently $\underline{\beta}^t (\mathbf{L}, \mathbf{R}, \mathbf{C}) = \underline{0}$.

□

Using the above results we may state the conditions for network regularity.

Theorem 4.3. *The regularity of $W(s)$ i.e. $\text{rank}_{\mathbb{R}(s)} [W(s)] = q$, is characterized by the following conditions:*

(i) Necessary conditions for $W(s)$ to be regular, is that matrices:

$$\{T_i, i = 1, 2, \dots, \nu, \nu \in \mathbb{Z}\}$$

have full rank.

(ii) Sufficient condition for $W(s)$ to be regular, is that $T_{2(q-1)}$ has full rank. Furthermore, if $\text{rank} [\mathbf{L}, \mathbf{R}, \mathbf{C}] = q$ then $T_{(2q-3)}$ has to be of full rank.

□

Proof. (i) Part (i) follows directly from the definition of degeneracy which is equivalent to the existence of a polynomial vector $\underline{x}(s)$ such that: $W(s) \cdot \underline{x}(s) = \underline{0}$ which in turn implies conditions 4.37. Clearly, regularity implies that there is no vector $\underline{x}(s)$ satisfying the above and this establishes part (i).

(ii) By lemma 4.7 the maximal column minimal index (cmi) of $P(s)$ denoted by ε yields as maximal cmi of $W(s)$ an index $\tilde{\varepsilon} = \varepsilon - 1$. From the pencil's dimensions $(2q \times 2q)$ and the Kronecker structure of the pencil [KV02a], it is clear that the maximal value for ε is $\varepsilon_{\max} = 2q - 1$ and thus, the maximal value of a cmi of $W(s)$ is: $\tilde{\varepsilon}_{\max} = \varepsilon_{\max} - 1 = 2q - 2 = 2(q - 1)$. Using lemma 4.6 it follows that a sufficient condition for $W(s)$ to be regular is that $T_{2(q-1)}$ is full rank.

Note that the presence of a cmi $\varepsilon_{\max} = 2q - 1$ for the pencil is obtained if $P(s)$ has at least a zero rmi. By lemma 4.8 it follows that if $\text{rank} [\mathbf{L}, \mathbf{R}, \mathbf{C}] = q$, there exists no rmi and thus $\varepsilon_{\max} < 2q - 1$, which in turn implies that $\tilde{\varepsilon}_{\max} < 2(q - 1)$. Thus, the condition that $T_{(2q-3)}$ has full rank is then sufficient condition for regularity.

□

Remark 4.11. Stronger sufficient conditions for regularity may be established by excluding the presence of certain values for rmi for $P(s)$, which may be expressed as rank

tests on a set of Toeplitz matrices of the type:

$$\tilde{T}_0 = [\mathbf{L}, \mathbf{R}, \mathbf{C}], \tilde{T}_1 = \begin{bmatrix} \mathbf{L} & \mathbf{R} & \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \mathbf{R} & \mathbf{C} \end{bmatrix}, \tilde{T}_2 = \begin{bmatrix} \mathbf{L} & \mathbf{R} & \mathbf{C} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \mathbf{R} & \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} & \mathbf{R} & \mathbf{C} \end{bmatrix} \quad (4.38)$$

□

Lower dimension tests for regularity are established by the next corollary.

Corollary 4.2. *If the Toeplitz matrix \tilde{T}_σ for some $\sigma = 0, 1, \dots, \sigma < 2q$, has full rank, then the sufficient condition for regularity is that $T_{2q-3-\sigma}$ has full rank.* □

Proof. The proof of the above follows similar lines to those of theorem 4.3. □

4.8 Natural Frequencies and the Network Pencil

In the previous section we introduced the *loop network pencil* $P(s)$ and we associated with the impedance operator $W(s)$ (equivalently $Z(s)$) of the network. In this section we will examine the zero structure properties of $P(s)$ taking into account that there exists a relationship between the two descriptions [KLL17].

The impedance operator $W(s)$ can be written as:

$$W(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R} = s^{-1}(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}) = s^{-1}\mathbf{Z}_a(s) \quad (4.39)$$

From the previous expression it follows that:

Proposition 4.3. *The following property holds true:*

$$|\mathbf{Z}_a(s)| = |(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})| = |P(s)|$$

□

Proof. It is clearly established from equation (4.39) and proposition 4.2. □

We can thus investigate the zero structure of the *RLC* network by examining the zero structure of the associated loop matrix pencil $P(s)$. From the structure of the pencil we have the following result:

Proposition 4.4. *Let us denote by $\rho_c = \text{rank}(\mathbf{C})$ and by $\rho_L = \text{rank}(\mathbf{L})$. Then the following properties hold true:*

- i. *The pencil $P(s)$ is regular.*
- ii. *The number of zero elementary divisors is $(q - \rho_C)$ and the number of infinite elementary divisors is $(q - \rho_L)$.*
- iii. *If r_f denotes the number of non-zero finite zeros of $P(s)$ or $\mathbf{Z}_a(s)$ then,*

$$r_f \leq \rho_L + \rho_C$$

with equality holding when all zero and infinite elementary divisors are linear, i.e. of multiplicity 1.

□

Proof. i. The pencil $P(s)$ defined in equation 4.31 is unimodular equivalent to:

$$P'(s) = \begin{bmatrix} s\mathbf{L} + \mathbf{R} & s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix}$$

However, $P'(s)$ has full rank since $(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}) = s \cdot W(s)$ has full rank, where $W(s)$ is the internal network operator.

- ii. Since $P(s)$ is regular, the number of infinite elementary divisors is defined by the rank deficiency of \mathbf{F} and the number of zero elementary divisors is defined by the rank deficiency of \mathbf{G} . Thus, the number of zero elementary divisors is: $(q - \rho_C)$, the nullity of \mathbf{G} and the number of infinite elementary divisors is: $(q - \rho_L)$, the nullity of \mathbf{F} .

iii. From proposition 4.3, it is clear that $\deg |\mathbf{Z}_a(s)| \leq 2q$. Assuming that the pencil has non-linear zero and infinite elementary divisors then:

$$r_f = 2q - (q - \rho_L) - (q - \rho_C) \leq \rho_L + \rho_C$$

with equality holding when all zero and infinite elementary divisors are linear.

□

Improved conditions for the degree of r_f may be obtained by working on the conditions defining the existence of nonlinear infinite and finite elementary divisors, which are considered next.

Definition 4.5. ([KK86]) Let $sF - G \in \mathbb{R}^{p \times p}[s]$ be a regular pencil. We define:

(i) The sequence of the ∞ -*Toeplitz* and 0-*Toeplitz* matrices respectively:

$$Q_1^\infty = [F], Q_2^\infty = \begin{bmatrix} F & 0 \\ -G & F \end{bmatrix}, \dots, \quad (4.40)$$

$$Q_k^\infty = \begin{bmatrix} F & 0 & 0 & \dots & 0 & 0 \\ -G & F & 0 & \dots & 0 & 0 \\ 0 & -G & F & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & F & 0 \\ 0 & 0 & 0 & \dots & -G & F \end{bmatrix}$$

$$\begin{aligned}
Q_1^0 &= [G], Q_2^0 = \begin{bmatrix} G & 0 \\ -F & G \end{bmatrix}, \dots, \\
Q_k^0 &= \begin{bmatrix} G & 0 & 0 & \cdots & 0 & 0 \\ -F & G & 0 & \cdots & 0 & 0 \\ 0 & -F & G & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & G & 0 \\ 0 & 0 & 0 & \cdots & -F & G \end{bmatrix}
\end{aligned} \tag{4.41}$$

and we shall denote by $\mathcal{L}^\infty = \{\eta_1^\infty, \eta_2^\infty, \dots, \eta_k^\infty, \dots\}$, $\mathcal{L}^0 = \{\eta_1^0, \eta_2^0, \dots, \eta_k^0, \dots\}$ the nullities of the corresponding matrices $\mathcal{Q}^\infty = \{Q_1^\infty, Q_2^\infty, \dots, Q_k^\infty, \dots\}$, $\mathcal{Q}^0 = \{Q_1^0, Q_2^0, \dots, Q_k^0, \dots\}$.

- (ii) We denote by $\mathcal{S}^\infty = \{q_1^\infty, q_2^\infty, \dots, q_\mu^\infty\}$, $\mathcal{S}^0 = \{q_1^0, q_2^0, \dots, q_\nu^0\}$ the set of integers defining the degrees of infinite and zero elementary divisors of the pencil, which is also referred as the *Segre Characteristic at infinity* and *Segre Characteristic at zero* respectively.

□

Using the previous definition we have the lemma:

Lemma 4.9. (*[KK86]*) *Let $sF - G \in \mathbb{R}^{p \times p}[s]$ be a regular pencil and let us denote by $\mathcal{S}^\infty = \{q_1^\infty, q_2^\infty, \dots, q_\mu^\infty\}$, $\mathcal{S}^0 = \{q_1^0, q_2^0, \dots, q_\nu^0\}$ the Segre Characteristic at infinity and Segre Characteristic at zero respectively of the pencil. Then,*

$$\begin{aligned}
\eta_k^\infty - \eta_{k-1}^\infty &\geq \eta_{k+1}^\infty - \eta_k^\infty \text{ or} \\
\eta_k^\infty &\geq (\eta_{k-1}^\infty + \eta_{k+1}^\infty)/2, k = 1, 2, \dots
\end{aligned} \tag{4.42}$$

$$\begin{aligned}
\eta_k^0 - \eta_{k-1}^0 &\geq \eta_{k+1}^0 - \eta_k^0 \text{ or} \\
\eta_k^0 &\geq (\eta_{k-1}^0 + \eta_{k+1}^0)/2, k = 1, 2, \dots
\end{aligned} \tag{4.43}$$

In particular:

- (i) Strict inequality holds if and only if $k \in \mathcal{S}^\infty$ for equation (4.42) and respectively $k \in \mathcal{S}^0$ for equation (4.43).
- (ii) Equality in equation (4.42) and in equation (4.43) holds if $k \notin \mathcal{S}^\infty$ and $k \notin \mathcal{S}^0$ respectively.

□

Based on lemma 4.9 we have the following corollary:

Corollary 4.3. *Let $sF - G \in \mathbb{R}^{p \times p}[s]$ be a regular pencil. Then,*

- (i) *If $\text{rank}(F) = \rho_\infty < p$ and $\eta_k^\infty = (\eta_{k-1}^\infty + \eta_{k+1}^\infty)/2, k = 1, 2, \dots, p$ then the pencil has only $p - \rho_\infty$ linear infinite elementary divisors.*
- (ii) *If $\text{rank}(G) = \rho_0 < p$ and $\eta_k^0 = (\eta_{k-1}^0 + \eta_{k+1}^0)/2, k = 1, 2, \dots, p$ then the pencil has only $p - \rho_0$ linear zero elementary divisors.*

□

The above results may now be used for the network pencil

$$\begin{aligned}
 P(s) &= s \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} = \\
 &= \begin{bmatrix} s\mathbf{L} + \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix} = sF + G \in \mathbb{R}^{2q \times 2q}[s]
 \end{aligned} \tag{4.44}$$

where q denotes the number of loops in an *RLC* network.

Proposition 4.5. *[KLL17] Consider a regular network and let $P(s) = sF + G \in \mathbb{R}^{2q \times 2q}[s]$ be the corresponding network pencil. Then, the matrices Q_k^∞, Q_k^0 defined by*

equations (4.40, 4.41) are equivalent over \mathbb{R} to the matrices:

$$\begin{aligned} \mathcal{P}_k^\infty &= \begin{bmatrix} \mathbf{L} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{R} & \mathbf{L} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{C} & \mathbf{R} & \mathbf{L} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{C} & \mathbf{R} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{C} & \mathbf{R} & \mathbf{L} \end{bmatrix}, k = 1, 2, \dots \\ \mathcal{P}_k^0 &= \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{R} & \mathbf{C} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{L} & \mathbf{R} & \mathbf{C} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{L} & \mathbf{R} & \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L} & \mathbf{R} & \mathbf{C} \end{bmatrix} \end{aligned} \quad (4.45)$$

□

Using the above results we may now state the criteria that characterizes the exact value of the degree of the zero polynomial.

Theorem 4.4. [KLL17] Consider a regular network defined by $P(s)$, or $W(s)$ and let us denote by $\rho_C = \text{rank}(C)$, $\eta_C = q - \text{rank}(C)$ and by $\rho_L = \text{rank}(L)$, $\eta_L = q - \text{rank}(L)$. Furthermore, let us denote by $\tilde{\mathcal{L}}^\infty = \{\eta_1^\infty, \eta_2^\infty, \dots, \eta_k^\infty, \dots\}$, $\eta_1^\infty = \eta_L$, $\tilde{\mathcal{L}}^0 = \{\eta_1^0, \eta_2^0, \dots, \eta_k^0, \dots\}$, $\eta_1^0 = \eta_C$ the nullities of the corresponding matrices:

$\mathcal{P}^\infty = \{\mathcal{P}_1^\infty, \mathcal{P}_2^\infty, \dots, \mathcal{P}_k^\infty, \dots\}$, $\mathcal{P}^0 = \{\mathcal{P}_1^0, \mathcal{P}_2^0, \dots, \mathcal{P}_k^0, \dots\}$. Then, the following properties hold true:

- (i) The number of zero elementary divisors is $q - \rho_C$ and the number of infinite elementary divisors is $q - \rho_L$.
- (ii) If r_f is the number of non-zero finite zeros of $P(s)$, or $\mathbf{Z}_a(s)$ then $r_f = \rho_C + \rho_L$, if and only if for all $k = 1, 2, \dots, m$ $\eta_k^\infty = k\eta_L$ and $\eta_k^0 = k\eta_C$.

□

Proof. The result readily follows for corollary 4.3 and proposition 4.4. In fact, corollary 4.3 implies that the sequences $\tilde{\mathcal{L}}^\infty = \{\eta_1^\infty, \eta_2^\infty, \dots, \eta_k^\infty, \dots\}$, $\eta_1^\infty = \eta_L$ and $\tilde{\mathcal{L}}^0 = \{\eta_1^0, \eta_2^0, \dots, \eta_k^0, \dots\}$, $\eta_1^0 = \eta_C$ are arithmetic progressions for all $k = 1, 2, \dots, 2q$. Note that the maximal possible degree of a zero, or infinite elementary divisor of $P(s)$ is $2q$, due to the dimensionality of $P(s)$. \square

4.9 Conclusions

The aim of this chapter was the investigation of fundamental system properties of an *RLC* network, in terms of the *Implicit Network Description* $W(s)$ [Kar11] and the associated *Network Pencil* $P(s)$. Initiating from the fundamental laws of Kirchhoff, we presented the derivation of the path and vertex equations and the formulation of the two types of modelling, i.e. the impedance and admittance, and their corresponding natural topologies. The *Implicit Network Operator* $W(s)$ was presented, which provides a unifying description of the network. Fundamental properties of this description were examined such as the notion of regularity of the network, that is invertibility of the $W(s)$ operator, which is strongly related with the notion of connectivity of the network. Moreover, the investigation of issues related to the linearisation of this Implicit Description gave rise to a matrix pencil representation of the network, i.e. the *Implicit Network Pencil* $P(s)$, which is not necessarily minimal but has the advantage that it preserves the natural loop or nodal topology as this is expressed by the corresponding triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$. Finally, issues of regularity and issues concerning the zero structure of the matrix pencil representation were examined using results derived for the characterization of infinite elementary divisors and cmi [KK86], utilizing Toeplitz matrices based on the triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$.

Chapter 5

Properties of Implicit Network Descriptions and The McMillan Degree

5.1 Introduction

In the previous chapter we examined some fundamental properties of the *Implicit Network operator* $W(s)$. In this chapter we address the problem of determining the *Implicit McMillan degree* δ_m of $W(s)^{-1}$, which defines the minimum number of dynamical elements required to describe the network fully, and relate it with the rank properties of the matrices of these elements. A result which is intuitively known but not rigorously proven in the circuit literature is that this degree has to be equal to the minimum number of independent dynamical elements in the network [LLK14]. In this chapter we investigate this result, proving that the maximum possible *Implicit McMillan degree* δ_m of such networks is given by $\text{rank}L + \text{rank}C$ and this value is attained when certain necessary and sufficient conditions are met.

Specifically, in section 5.2 the *Implicit McMillan degree* δ_m for a general *RLC* network is computed and a link is established between the McMillan degree and the *Implicit Network operator* $W(s)$. In section 5.3 necessary and sufficient conditions are derived for the *Implicit McMillan degree* δ_m to achieve its maximum value, which are expressed in various forms that are all testable. Explicitly, the first set of conditions are of *determinantal type* and relate the highest and lowest order coefficients of s in the expansion

of the determinant $\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})$ to the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$. The second set of conditions relates the property of these coefficients *to be nonzero* with some rank properties of matrices related to the three fundamental matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$. In section 5.4 the necessary and sufficient conditions derived before are implemented in terms of the graph incidence matrices of a network as an attempt for a graph systematic approach, which will provide the means for linking the McMillan degree with the topology of the *RLC* network. Furthermore, in section 5.5 an attempt is made to establish an expression for the maximum possible *Implicit McMillan degree* δ_m of an *RLC* network using the associated loop pencil $P(s)$ defined in Chapter 4. Finally, in section 5.6 all the results that are derived in this chapter are illustrated through two examples and the necessary and sufficient conditions are tested.

5.2 Implicit McMillan Degree and Its Calculation

In this section we establish a relationship between the $W(s)$ operator that describes a general *RLC* network and the *Implicit McMillan degree* of this network. Furthermore, we compute an upper bound for the Implicit McMillan degree that is strongly related to the ranks of the matrices of the dynamical elements (i.e. inductances and capacitors) [LLK14].

5.2.1 Problem Statement

The problem to be examined in this section is stated next:

For an *RLC* network that is described by the general operator:

$$W(s) = Z(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R}$$

find a relationship between the McMillan degree of the network and the rank of the matrices of the dynamical elements. The McMillan degree of the system may be computed in terms of the transfer function of the network, which is described by the $W(s)^{-1}$ operator. The main purpose of this chapter is to calculate this degree in terms of the elements of the network to derive testable conditions and to interpret the results.

5.2.2 The Implicit McMillan Degree and Its Calculation

The following theorem establishes the link between the McMillan degree ¹ of a general *RLC* network and its general operator $W(s)$. Furthermore, a formula for the computation of the *Implicit McMillan degree* is stated [LLK14].

Theorem 5.1. *Let $W^{-1}(s)$ be the transfer function of an *RLC* network ², where $W(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R}$ and $W(s)$ non-singular (a detailed proof can be found in section 4.7). Then the McMillan degree of $W(s)^{-1}$ is given by:*

$$\delta_m = n_{\max} - \min(n_{\min}, n)$$

where n_{\max} and n_{\min} are the maximum and minimum degrees of s in the expansion of the determinant:

$$\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})$$

and n denotes the cardinality of the network (number of independent loops / nodes). \square

Proof. The Smith-McMillan form [SS88, Kar09] of $W(s)^{-1}$ is described by the following equation:

$$W(s)^{-1} = V_1(s) \begin{bmatrix} \frac{\varepsilon_1(s)}{\psi_1(s)} & & \\ & \ddots & \\ & & \frac{\varepsilon_n(s)}{\psi_n(s)} \end{bmatrix} V_2(s) \quad (5.1)$$

where: $V_1(s), V_2(s)$ unimodular, $\varepsilon_i/\varepsilon_{i+1}, \psi_i/\psi_{i+1}$ and ε_i, ψ_i coprime polynomials. Computing the determinants at both sides of (5.1) we get:

$$\frac{s^n}{\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})} = \frac{\varepsilon_1(s) \cdots \varepsilon_n(s)}{\psi_1(s) \cdots \psi_n(s)} \quad (5.2)$$

The *McMillan degree* of $W(s)^{-1}$ is given by the degree of the polynomial:

$$p(s) = \psi_1(s) \cdots \psi_n(s)$$

¹The McMillan degree of a transfer-function matrix is the total number of poles in the diagonal elements of the matrix in its McMillan form. This number determines the order of any minimal state-space realization of the transfer-function matrix or the minimal order of coprime matrix-fraction models.

²defined in (4.32).

The polynomial $p(s)$ can be taken from the left hand part of 5.2 as the polynomial remaining from $\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})$ after the maximum possible cancellations of the powers of s in the corresponding left hand part ratio of 5.2. If we let:

$$\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}) = \alpha_{n_{\max}} s^{n_{\max}} + \alpha_{n_{\max}-1} s^{n_{\max}-1} + \dots + \alpha_{n_{\min}} s^{n_{\min}}$$

then the maximum possible term of s that can be canceled is $s^{\min(n_{\min}, n)}$, therefore:

$$p(s) = \psi_1(s) \cdot \psi_2(s) \dots \psi_n(s) = \alpha_{n_{\max}} s^{n_{\max} - \min(n_{\min}, n)} + \dots + \alpha_{n_{\min}} s^{n_{\min} - \min(n_{\min}, n)}$$

and hence the degree of $p(s)$ is $n_{\max} - \min(n_{\min}, n)$, which is the McMillan degree of $W(s)^{-1}$. \square

The next theorem establishes an upper bound for the degree of the determinant of the polynomial $\mathbf{Z}_a(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$ relatively to the ranks of the matrices of the dynamical elements, i.e. \mathbf{L}, \mathbf{C} [LLK14].

Theorem 5.2. *Let $\mathbf{Z}_a(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$ with $\text{rank}(\mathbf{L}) = p$, $\text{rank}(\mathbf{C}) = q$ and let the polynomial $\det[\mathbf{Z}_a(s)] = \alpha s^{n_2} + \dots + \beta s^{n_1}$ with the powers in descending order. Then: $n_2 - \min(n, n_1) \leq p + q$, when $n \geq n_1$ and $n_2 - \min(n, n_1) \leq p$, when $n < n_1$. Additionally, the maximum value for $n_2 - \min(n, n_1)$, which is $p + q$ is obtained when $n_2 = n + p$ and $n_1 = n - q$. \square*

Proof. Developing the determinant $\det[\mathbf{Z}_a(s)]$ we can get it as sums of determinants taking f_1 rows from $s^2\mathbf{L}$, f_2 rows from $s\mathbf{R}$ and the remaining rows from \mathbf{C} . In this case, the polynomial part of this term will be: $s^{2f_1+f_2}$. Furthermore, we have the following constraints for f_1, f_2 :

- (i) $f_1, f_2 \geq 0$
- (ii) $f_1 + f_2 \leq n$
- (iii) $f_1 \leq p$ (if we select more rows of L than its rank, the coefficient of $s^{2f_1+f_2}$ will be zero).

(iv) $n - f_1 - f_2 \leq q$ (for similar reasons as in (iii)).

Now as: $f_1 \leq p$, $f_1 + f_2 \leq n$ we get $2f_1 + f_2 \leq n + p$, with the equality achieved when both $f_1 = p$ and $f_1 + f_2 = n$ i.e. when: $f_1 = p$ and $f_2 = n - p$ (we can also see that all constraints are satisfied). Hence,

$$\max(2f_1 + f_2) = n + p \quad (5.3)$$

And this maximum value is achieved exactly when $f_1 = p$ and $f_2 = n - p$. Additionally, selecting f_3 rows from $s\mathbf{R}$ and f_4 rows from \mathbf{C} , the degree for n_1 is: $2(n - f_3 - f_4) + f_3$ and we have to minimize:

$$\min 2(n - f_3 - f_4) + f_3 \quad (5.4)$$

subject to the following constraints for f_3 and f_4 :

(i) $f_3, f_4 \geq 0$

(ii) $f_3 + f_4 \leq n$

(iii) $f_4 \leq q$.

The solution to this problem is: $f_3 + f_4 = n$, $f_4 = q$, thus $f_3 = n - q$ and the minimum degree is $(\min 2(n - f_3 - f_4) + f_3)$: $n - q$. Hence, for the McMillan degree $\delta_m = n_{\max} - \min(n_{\min}, n) = n_2 - \min(n_1, n)$ we distinguish the following two cases:

Case 1: When $n_1 \leq n$, then $\delta_m = n_2 - n_1$. To maximize δ_m we have to maximize n_2 and minimize n_1 . Thus, $\delta_{m_{\max}} = n + p - (n - q) = p + q$.

Case 2: When $n_1 > n$, then $\delta_m = n_2 - n$. To maximize δ_m we have to maximize n_2 , which is $n_2 = n + p$ and $\delta_{m_{\max}} = n + p - n = p$.

Hence, taking into account the two cases, the maximum possible McMillan degree is:

$$\delta_{m_{\max}} = p + q$$

when $n_2 = n + p$ and $n_1 = n - q$. □

5.3 Necessary and Sufficient Conditions For Determining The Implicit McMillan Degree

In this section we examine the necessary and sufficient conditions for determining the *Implicit McMillan degree* of an *RLC* network [LLK14].

The first theorem of this section provides a formula for the maximum and minimum coefficients of the determinant of the matrix representation of the circuit (i.e. $\mathbf{Z}_a(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$).

Theorem 5.3. *Let $\mathbf{Z}_a(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$ the matrix representation of a RLC circuit. Let k_{\max} , k_{\min} , n_{\max} , n_{\min} be the maximum and minimum coefficients and degrees of the determinant $\det[\mathbf{Z}_a(s)]$ respectively. Assume also that $\text{rank}(\mathbf{L}) = p$, $\text{rank}(\mathbf{C}) = q$ which implies that*

$$C_p(\mathbf{L}) = \alpha_1 \cdot \alpha_2^t, \alpha_1, \alpha_2 \in \mathbb{R}^{\binom{n}{p} \times 1}$$

and that

$$C_q(\mathbf{C}) = \beta_1 \cdot \beta_2^t, \beta_1, \beta_2 \in \mathbb{R}^{\binom{n}{q} \times 1}$$

Then the following hold true:

- (i) When $p < n$ then: $n_{\max} \leq n + p$ and n_{\max} takes the maximum possible value $n + p$ if and only if

$$k_{\max} = \text{tr}(C_p(\mathbf{L}) \cdot \text{Adj}_p(\mathbf{R})) = \alpha_2^t \cdot \text{Adj}_p(\mathbf{R}) \cdot \alpha_1 \neq 0.$$

In the case where $n = p$ then:

$$k_{\max} = \det(\mathbf{L}) \neq 0.$$

- (ii) When $q < n$ then: $n_{\min} \geq n - q$ and n_{\min} takes the minimum possible value $n - q$ if and only if

$$k_{\min} = \text{tr}(C_q(\mathbf{C}) \cdot \text{Adj}_q(\mathbf{R})) = \beta_2^t \cdot \text{Adj}_q(\mathbf{R}) \cdot \beta_1 \neq 0.$$

Particularly, when $n = q$ then:

$$k_{\min} = \det(\mathbf{C}) \neq 0.$$

□

Proof. Denote l_i, r_i, c_i the columns of the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$ respectively. The $\det[\mathbf{Z}_{\mathbf{a}}(\mathbf{s})]$ is the sum of the terms:

$$(-1)^\sigma \cdot \underbrace{l_{i_1} \wedge l_{i_2} \wedge \cdots \wedge l_{i_{f_1}}}_{f_1 \text{ from } \mathbf{L}} \wedge \underbrace{r_{j_1} \wedge r_{j_2} \wedge \cdots \wedge r_{j_{f_2}}}_{f_2 \text{ from } \mathbf{R}} \wedge \underbrace{c_{m_1} \wedge c_{m_2} \wedge \cdots \wedge c_{m_{n-f_1-f_2}}}_{n-f_1-f_2 \text{ from } \mathbf{C}} \cdot s^{2f_1+f_2} \quad (5.5)$$

(a) To find the maximum possible degree of the polynomial $\det[\mathbf{Z}_{\mathbf{a}}(\mathbf{s})]$ we have to solve the *integer-programming problem*:

$$\max \quad n = 2f_1 + f_2$$

s.t.

$$f_1, f_2 \geq 0, \quad f_1 + f_2 \leq n, \quad f_1 \leq p, \quad n - f_1 - f_2 \leq q$$

This has the obvious solution: $f_1 = p, f_2 = n - p$ and $n_{\max} = 2p + n - p = n + p$ i.e. take p columns from \mathbf{L} and $n - p$ columns from \mathbf{R} . In this case:

$$k_{\max} = \sum_{\omega \in Q_n^p} A_\omega$$

where A_ω are all $n \times n$ determinants of matrices formed by p rows from \mathbf{L} and $n - p$ complementary rows from \mathbf{R} . For a given selection of columns of \mathbf{L} : $\omega =$

$(i_1, i_2, \dots, i_p) \in Q_n^p$ the *Laplace Expansion Theorem* [Mey00] gives:

$$A_\omega = \begin{vmatrix} r_{j_1} \\ l_{i_1} \\ r_{j_2} \\ l_{i_2} \\ \vdots \\ l_{i_p} \\ r_{j_{n-p}} \end{vmatrix} = \sum_{\beta \in Q_n^p} C_p(\mathbf{L})_{\omega, \beta} \cdot \text{Adj}_p(\mathbf{R})_{\beta, \omega}$$

Therefore,

$$k_{\max} = \sum A_\omega = \sum_{\omega \in Q_n^p} \sum_{\beta \in Q_n^p} C_p(\mathbf{L})_{\omega, \beta} \cdot \text{Adj}_p(\mathbf{R})_{\beta, \omega} = \text{tr}(C_p(\mathbf{L}) \cdot \text{Adj}_p(\mathbf{R}))$$

Since, \mathbf{L} has rank p we have: $C_p(\mathbf{L}) = \alpha_1 \cdot \alpha_2^t$. Thus,

$$k_{\max} = \text{tr}(C_p(\mathbf{L}) \cdot \text{Adj}_p(\mathbf{R})) = \alpha_2^t \cdot \text{Adj}_p(\mathbf{R}) \cdot \alpha_1$$

(b) To find the minimum degree we have to solve the *integer-programming problem*:

$$\min 2(n - f_3 - f_4) + f_3$$

s.t.

$$f_3 + f_4 \leq n, \quad f_3, f_4 \geq 0, \quad f_4 \leq q$$

which has the obvious solution: $f_3 + f_4 = n$, $f_4 = q$ and thus, $f_3 = n - q$. In this case:

$$\min 2(n - f_3 - f_4) + f_3 = 2(n - n + q - q) + n - q = n - q$$

Then,

$$k_{\min} = \sum_{\omega \in Q_n^q} B_\omega$$

where B_ω are all the $n \times n$ determinants of matrices formed by q rows of \mathbf{C} and $n - q$ rows of \mathbf{R} . For $\omega = (i_1, i_2, \dots, i_q) \in Q_n^q$ using the *Laplace Expansion Theorem*

[Mey00] we have:

$$B_\omega = \begin{vmatrix} r_{j_1} \\ c_{i_1} \\ r_{j_2} \\ c_{i_2} \\ \vdots \\ c_{i_q} \\ r_{j_{n-q}} \end{vmatrix} = \sum_{\beta \in Q_n^q} C_q(\mathbf{C})_{\omega, \beta} \cdot \text{Adj}_q(\mathbf{R})_{\beta, \omega}$$

Therefore,

$$k_{\min} = \sum B_\omega = \sum_{\omega \in Q_n^q} \sum_{\beta \in Q_n^q} C_q(\mathbf{C})_{\omega, \beta} \cdot \text{Adj}_q(\mathbf{R})_{\beta, \omega} = \text{tr}(C_q(\mathbf{C}) \cdot \text{Adj}_q(\mathbf{R}))$$

Since, \mathbf{C} has rank q we have: $C_q(\mathbf{C}) = \beta_1 \cdot \beta_2^t$, proving that:

$$k_{\min} = \text{tr}(C_q(\mathbf{C}) \cdot \text{Adj}_q(\mathbf{R})) = \beta_2^t \cdot \text{Adj}_q(\mathbf{R}) \cdot \beta_1$$

□

The next proposition gives necessary conditions for the maximum and minimum coefficients k_{n+p} and k_{n-q} respectively to be non zero [LLK14].

Proposition 5.1. (1) A necessary condition for $k_{n+p} \neq 0$, is that the matrices $\begin{bmatrix} \mathbf{L} & \mathbf{R} \end{bmatrix}$,

$\begin{bmatrix} \mathbf{L} \\ \mathbf{R} \end{bmatrix}$ have **full rank**.

(2) A necessary condition for $k_{n-q} \neq 0$, is that the matrices $\begin{bmatrix} \mathbf{R} & \mathbf{C} \end{bmatrix}$, $\begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix}$ have **full rank**.

□

Proof. (1) As the coefficient of k_{n+p} is the sum of certain $n \times n$ minors of $\begin{bmatrix} \mathbf{L} & \mathbf{R} \end{bmatrix}$ or $\begin{bmatrix} \mathbf{L} \\ \mathbf{R} \end{bmatrix}$, if these matrices are not full rank all these minors have to be zero and therefore k_{n+p} must be zero.

(2) Similar to (1).

□

Proposition 5.2. Let $\mathbf{L} = \mathbf{L}' \cdot \mathbf{L}''$, $\mathbf{L}' \in \mathbb{R}^{n \times p}$, $\mathbf{L}'' \in \mathbb{R}^{p \times n}$ and $p < n$. Then:

$$C_p(\mathbf{L}'') \cdot \text{Adj}_p(\mathbf{R}) \cdot C_p(\mathbf{L}') = (-1)^p \cdot \begin{vmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{vmatrix}$$

□

Proof. Developing $A = \begin{vmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{vmatrix}$ with respect to the last p rows we get:

$$A = \sum_{\omega} (-1)^{n+1+n+2+\dots+n+p+j_1+j_2+\dots+j_p} \cdot |L''_{\omega}| \cdot |R_{\omega} \mathbf{L}'| \quad (5.6)$$

where $\omega = (j_1, j_2, \dots, j_p) \in Q_n^p$ and R_{ω} is the part of \mathbf{R} with j_1, j_2, \dots, j_p columns excluded, then expanding

$$\begin{vmatrix} R_{\omega} & \mathbf{L}' \end{vmatrix}$$

with respect to its last p columns (i.e. \mathbf{L}') we get:

$$|R_{\omega} \mathbf{L}'| = \sum_{\beta} (-1)^{n-p+1+n-p+2+\dots+n+f_1+f_2+\dots+f_p} \cdot |R_{\omega}| \cdot |L'_{\beta}| \quad (5.7)$$

where $\beta = (f_1, f_2, \dots, f_p) \in Q_n^p$ and R_{ω} is the part of \mathbf{R} with the ω rows and β columns excluded. Substituting (5.8) into (5.7) we get:

$$\begin{aligned} \begin{vmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{vmatrix} &= (-1)^{n+1+\dots+n_p+n-p+1+\dots+n} \cdot \sum_{\omega, \beta \in Q_n^p} (-1)^{j_1+j_2+\dots+j_p+f_1+f_2+\dots+f_p} \cdot |L_{\omega}| |R_{\omega, \beta}| |L'_{\beta}| = \\ &= (-1)^p \cdot C_p(\mathbf{L}') \cdot \text{Adj}_p(\mathbf{R}) \cdot C_p(\mathbf{L}'') \end{aligned}$$

□

Corollary 5.1. *Let $\mathbf{C} = \mathbf{C}' \cdot \mathbf{C}''$, $\mathbf{C}' \in \mathbb{R}^{n \times q}$, $\mathbf{C}'' \in \mathbb{R}^{q \times n}$ and $q < n$. Then:*

$$C_q(\mathbf{C}'') \cdot \text{Adj}_q(\mathbf{R}) \cdot C_q(\mathbf{C}') = (-1)^q \cdot \begin{vmatrix} \mathbf{R} & \mathbf{C}' \\ \mathbf{C}'' & \mathbf{0} \end{vmatrix}$$

□

The main theorem of this section, which is presented below, provides a description for the maximum coefficient of the determinant with respect to the rank properties of the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$ of an *RLC* network [LLK14].

Theorem 5.4. (i) *If $p < n$ then:*

$$k_{n+p} = C_p(\mathbf{L}'') \cdot \text{Adj}_p(\mathbf{R}) \cdot C_p(\mathbf{L}') \neq 0 \text{ (where } \text{Adj}_n(\mathbf{R}) = 1)$$

$$\text{if and only if } \text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) = n + \text{rank}(\mathbf{L})$$

$$(ii) \text{ If } p = n \text{ then: } \det(\mathbf{L}) \neq 0 \text{ if and only if } \text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) = n + \text{rank}(\mathbf{L})$$

□

Proof. Let $p = \text{rank}(\mathbf{L})$. Moreover,

$$\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) \leq \text{rank}(\mathbf{L}) + \text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \end{bmatrix} \right) = n + p$$

Therefore, for $\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) = n + p$ there must be $C_{n+p} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) \neq 0$. Taking into account the identity:

$$\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{L}' \end{bmatrix} \cdot \begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}'' \end{bmatrix}$$

by the Binet-Cauchy theorem [MM64] we have:

$$C_{n+p} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) = \det \left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix} \right) \cdot C_{n+p} \left(\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{L}' \end{bmatrix} \right) \cdot C_{n+p} \left(\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}'' \end{bmatrix} \right).$$

Hence,

$$C_{n+p} = \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) \neq 0 \text{ if and only if } \det \left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix} \right) \neq 0.$$

Since $k_{n+p} = (-1)^p \cdot \det \left(\begin{bmatrix} \mathbf{R} & \mathbf{L}' \\ \mathbf{L}'' & \mathbf{0} \end{bmatrix} \right)$ (proposition 5.2), we have that:
 $k_{n+p} \neq 0$ if and only if $\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) = n + p$. □

The following corollary states a similar result as theorem 5.4 for the minimum coefficient of the determinant with respect to the rank properties of the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$ of an *RLC* network [LLK14].

Corollary 5.2. (i) If $q < n$ then: $k_{n-q} \neq 0$ if and only if $\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \right) = n + \text{rank}(\mathbf{C})$.

(ii) If $q = n$ then: $\det(\mathbf{C}) \neq 0$ if and only if $\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \right) = n + \text{rank}(\mathbf{C})$

□

Corollary 5.3. Let δ_m be the McMillan degree of $W^{-1}(s) = (s\mathbf{L} + \mathbf{R} + 1/s\mathbf{C})^{-1}$. Then the following are equivalent:

(a) $\delta_m = \text{rank}(\mathbf{L}) + \text{rank}(\mathbf{C})$.

(b) $\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \right) = n + \text{rank}(\mathbf{L})$ and $\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \right) = n + \text{rank}(\mathbf{C})$.

□

Corollary 5.4. The necessary conditions for $\delta_m = \text{rank}(\mathbf{L}) + \text{rank}(\mathbf{C})$ are:

(a) $\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{L} \end{bmatrix} \right) = n$.

(b) $\text{rank} \left(\begin{bmatrix} \mathbf{R} & \mathbf{C} \end{bmatrix} \right) = n$.

(c) $\text{rank}(\mathbf{R}) \geq n - \min(\text{rank}(\mathbf{L}), \text{rank}(\mathbf{C}))$.

□

5.4 Graph Systematic Approach of Necessary and Sufficient Conditions

This section provides a graph systematic approach of the necessary and sufficient conditions that were developed in section 5.3. We emphasize mostly in implementing these conditions in terms of the *graph incidence matrices* of the $\mathbf{L}, \mathbf{R}, \mathbf{C}$ matrices of the network. Such an approach will provide a more clear result on the link of the *Implicit McMillan degree* δ_m and the topology of the *RLC* network. Firstly, we will introduce the notion of an *incidence matrix* of a graph or a network, which is crucial for the development of this graph approach.

Definition 5.1. An incidence matrix $G^T \in \mathbb{R}^{m \times n}$ is a matrix with $i, i = 1, \dots, m$ rows and $j, j = 1, \dots, n$ columns. Each row of the matrix corresponds to an element of the network, i.e. capacitor, inductance, resistor and each column corresponds to a loop or node of the given RLC network. Hence, an entry G_{ij} in the matrix is:

- a. 1 if element i is present in loop / node j and the current i_j flows through the element i in the clockwise direction.
- b. -1 if element i is present in loop / node j and the current i_j flows through the element i in the counter clockwise direction.
- c. 0 if element i is not present in loop j .

□

The following remark provides a description of the $\mathbf{L}, \mathbf{R}, \mathbf{C}$ matrices of an RLC network in terms of the associated incidence matrices defined in definition 5.1.

Remark 5.1. Each one of the elements $R_i, L_i, \frac{1}{C_i}$ can be decomposed into corresponding dyads as:

$$\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} R_i \begin{bmatrix} 0 & 1 & \cdots & 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} L_i \begin{bmatrix} 0 & 1 & \cdots & 0 & -1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} \frac{1}{C_i} \begin{bmatrix} 0 & 1 & \cdots & 0 & -1 & 0 \end{bmatrix}$$

with entries: 1 if element i is present in loop / node j and the current i_j flows through the element i in the clockwise direction, -1 if element i is present in loop / node j and the current i_j flows through the element i in the counter clockwise direction, or 0 if element i is not present in loop j . If all elements $R_i, L_i, \frac{1}{C_i}$ are gathered and the matrices $\mathbf{R}, \mathbf{L}, \mathbf{C}$ are formed accordingly then we have the following representation.

If G^T denotes the incidence matrix for the matrices $\mathbf{R}, \mathbf{L}, \mathbf{C}$ then these matrices can be represented by:

$$\begin{aligned}\mathbf{R} &= G_R \cdot D_R \cdot G_R^T \\ \mathbf{L} &= G_L \cdot D_L \cdot G_L^T \\ \mathbf{C} &= G_C \cdot D_C \cdot G_C^T\end{aligned}\tag{5.8}$$

where D_C, D_R, D_L represent the diagonal matrices with entries the capacitors, resistors and inductances respectively in a given network. \square

The following two theorems demonstrate equivalent expressions for the maximum and minimum coefficients k_{\max} and k_{\min} (as were developed in section 5.3) respectively not to be zero.

Theorem 5.5. *Let $\mathbf{L} = \mathbf{L}' \cdot \mathbf{L}''$, $\mathbf{L}' \in \mathbb{R}^{n \times p}$ and $\mathbf{L}'' \in \mathbb{R}^{p \times n}$. If we denote by $\mathbf{L}'' = D_L \cdot G_L^T$, $\mathbf{L}' = G_L$ then by theorems 5.1 and 5.3 and proposition 5.2 we have that:*

- *If G_L^T and G_R^T are **square incidence matrices** then an equivalent expression for $k_{\max} \neq 0$ is:*

$$C_p(G_L^T) \cdot \text{Adj}_p(G_R^T) \neq \underline{0}$$

- *If G_L^T and G_R^T are **not square incidence matrices** then an equivalent expression for $k_{\max} \neq 0$ is:*

$$C_p(G_L^T) \cdot J_{n,p} \cdot C_{n-p}(G_R) \neq \underline{0}$$

\square

Proof. We know from theorem 5.3 that $k_{\max} \neq 0$ if and only if

$$C_p(\mathbf{L}'') \cdot \text{Adj}_p(\mathbf{R}) \cdot C_p(\mathbf{L}') \neq \underline{0}\tag{5.9}$$

Lets denote by $\mathbf{L}'' = D_L \cdot G_L^T$ and by $\mathbf{L}' = G_L$ then using that $\mathbf{L} = \mathbf{L}' \cdot \mathbf{L}''$ and developing equation 5.9 we will have that:

$$\begin{aligned} & \det D_L \cdot C_p(G_L^T) \cdot \text{Adj}_p(\overbrace{G_R D_R G_R^T}^{n \times n}) \cdot C_p(G_L) = \\ &= \det D_L \cdot C_p(G_L^T) [J_{n,p} \cdot C_{n-p}(G_R D_R G_R^T) \cdot J_{n,p}^T] \cdot C_p(G_L) = \\ &= \det D_L \cdot C_p(G_L^T) [J_{n,p} \cdot C_{n-p}(G_R D_R G_R^T) \cdot J_{n,p}^T] \cdot C_p(G_L) \end{aligned} \quad (5.10)$$

Note: In equation (5.10) the adjoint $\text{Adj}_p(B)$ of an $n \times n$ matrix \mathbf{B} can be decomposed as:

$$\text{Adj}_p(\mathbf{B}) = (J_{n,p} \cdot C_{n-p}(\mathbf{B}) \cdot J_{n,p}^T)$$

Using for equation 5.10 the Binet-Cauchy theorem [MM64] we have that:

$$\begin{aligned} &= \det D_L \cdot C_p(G_L)^T [J_{n,p} \cdot C_{n-p}(G_R D_R G_R^T) \cdot J_{n,p}^T] \cdot C_p(G_L) \\ &= \det D_L \cdot C_p(G_L)^T \cdot J_{n,p} \cdot C_{n-p}(G_R) \cdot C_{n-p}(D_R) \cdot C_{n-p}(G_R)^T \cdot J_{n,p}^T \cdot C_p(G_L) \end{aligned}$$

Thus, for **non-square matrices** G_R^T, G_L^T the equivalent expression for $k_{\max} \neq 0$ is:

$$C_p(G_L)^T \cdot J_{n,p} \cdot C_{n-p}(G_R) \neq \underline{0}$$

□

Remark 5.2. For square matrices G_R^T, G_L^T the equivalent expression for $k_{\max} \neq 0$ is:

$$C_p(G_L)^T \cdot \text{Adj}_p(G_R^T) \neq \underline{0}$$

□

Theorem 5.6. Let $\mathbf{C} = \mathbf{C}' \cdot \mathbf{C}''$ $\mathbf{C}' \in \mathbb{R}^{n \times q}$, $\mathbf{C}'' \in \mathbb{R}^{q \times n}$. If we denote by $\mathbf{C}'' = D_C \cdot G_C^T$, $\mathbf{C}' = G_C$ then by theorems 5.1 and 5.3 and proposition 5.2 we have that:

- If G_C^T and G_R^T are **square incidence matrices** then an equivalent expression for $k_{\min} \neq 0$ is:

$$C_q(G_C^T) \cdot \text{Adj}_q(G_R^T) \neq \underline{0}$$

- If G_C^T and G_R^T are **not square incidence matrices** then an equivalent expression for $k_{\min} \neq 0$ is:

$$C_q(G_C^T) \cdot J_{n,q} \cdot C_{n-q}(G_R) \neq \underline{0}$$

□

Proof. We know from theorem 5.3 that $k_{\min} \neq 0$ if and only if

$$C_q(\mathbf{C}'') \cdot \text{Adj}_q(\mathbf{R}) \cdot C_q(\mathbf{C}') \neq \underline{0} \quad (5.11)$$

Lets denote by $\mathbf{C}'' = D_C \cdot G_C^T$ and by $\mathbf{C}' = G_C$ then using that $\mathbf{C} = \mathbf{C}' \cdot \mathbf{C}''$ and developing equation 5.11 we will have that:

$$\begin{aligned} \det D_C \cdot C_q(G_C^T) \cdot \text{Adj}_q(\overbrace{G_R D_R G_R^T}^{n \times n}) \cdot C_q(G_C) &= \\ = \det D_C \cdot C_q(G_C^T) [J_{n,q} \cdot C_{n-q}(G_R D_R G_R^T) \cdot J_{n,q}^T]^T \cdot C_q(G_C) &= \\ = \det D_C \cdot C_q(G_C^T) [J_{n,q} \cdot C_{n-q}(G_R D_R G_R^T) \cdot J_{n,q}^T] \cdot C_q(G_C) \end{aligned} \quad (5.12)$$

Using for equation 5.12 the Binet-Cauchy theorem [MM64] we have that:

$$\begin{aligned} &= \det D_C \cdot C_q(G_C)^T [J_{n,q} \cdot C_{n-q}(G_R D_R G_R^T) \cdot J_{n,q}^T] \cdot C_q(G_C) \\ &= \det D_C \cdot C_q(G_C)^T \cdot J_{n,q} \cdot C_{n-q}(G_R) \cdot C_{n-q}(D_R) \cdot C_{n-q}(G_R)^T \cdot J_{n-q}^T \cdot C_q(G_C) \end{aligned}$$

Thus, for **non-square matrices** G_R^T, G_C^T the equivalent expression for $k_{\min} \neq 0$ is:

$$C_q(G_C)^T \cdot J_{n,q} \cdot C_{n-q}(G_R) \neq \underline{0}$$

□

Remark 5.3. For **square matrices** G_R^T, G_C^T the equivalent expression for $k_{\min} \neq 0$ is:

$$C_q(G_C)^T \cdot \text{Adj}_q(G_R^T) \neq \underline{0}$$

□

The next theorem present under which conditions the maximum and minimum coefficients k_{\max} and k_{\min} are non zero.

Theorem 5.7. *For a given network represented by the matrices $\mathbf{R}, \mathbf{L}, \mathbf{C}$ and the associated incidence matrices G_L^T, G_R^T, G_C^T then:*

1. *The minimum coefficient of the McMillan degree is non-zero, i.e. $k_{\min} \neq 0$ if and only if*

$$C_q(G_C^T) \cdot \text{Adj}_q(G_R^T) \neq \underline{0}$$

where G_C^T and G_R^T are **square incidence matrices** or

$$C_q(G_C)^T \cdot J_{n,q} \cdot C_{n-q}(G_R) \neq \underline{0}$$

where G_R^T, G_C^T are **not square matrices**. Equivalently, at least one determinant formed by q rows of G_C^T and $(n - q)$ rows from G_R^T is non-zero.

2. *The maximum coefficient of the McMillan degree is non-zero, i.e. $k_{\max} \neq 0$ if and only if*

$$C_p(G_L^T) \cdot \text{Adj}_p(G_R^T) \neq \underline{0}$$

where G_R^T, G_L^T are **square incidence matrices** or

$$C_p(G_L)^T \cdot J_{n,p} \cdot C_{n-p}(G_R) \neq \underline{0}$$

where G_R^T, G_L^T **not square matrices**. Equivalently, at least one determinant formed by p rows of G_L^T and $(n - p)$ rows from G_R^T is non-zero.

□

Finally, the following corollary expresses the necessary conditions for the McMillan degree δ_m to achieve the upper bound. The necessary and sufficient conditions for this are presented in remark 5.4.

Corollary 5.5. *For a given network represented by the matrices $\mathbf{R}, \mathbf{L}, \mathbf{C}$ the **necessary conditions** for $\delta_m = \text{rank}(\mathbf{L}) + \text{rank}(\mathbf{C})$ are:*

$$\bullet \text{ rank } \begin{bmatrix} G_C^T \\ G_R^T \end{bmatrix} = n$$

$$\bullet \text{ rank } \begin{bmatrix} G_L^T \\ G_R^T \end{bmatrix} = n$$

□

Remark 5.4. The **necessary and sufficient** conditions for the McMillan degree of a network to be $\delta_m = \text{rank}(\mathbf{L}) + \text{rank}(\mathbf{C})$ are:

1. If there is a set of linearly independent lines formed by $(n - q)$ lines of the incidence matrix of \mathbf{R} and q lines of the incidence matrix of \mathbf{C} .
2. If there is a set of linearly independent lines formed by $(n - p)$ lines of the incidence matrix of \mathbf{R} and p lines of the incidence matrix of \mathbf{L} .

□

5.5 The Network Pencil $P(s)$ and Links to the McMillan Degree of the Network

In this section we try to establish an expression for the maximum possible McMillan degree δ_m of an *RLC* network using the associated loop pencil $P(s)$ defined in Chapter 4.

As mentioned in the previous sections the maximum possible McMillan degree of an *RLC* network is given by:

$$\delta_m = n_{\max} - \min(n_{\min}, n)$$

where n is the cardinality of the network and n_{\max} , n_{\min} are the maximum and minimum powers of s in the expansion of the determinant $\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C})$.

We can reformulate the above determinantal expression in terms of matrix pencils as:

$$\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}) = \det \begin{bmatrix} s\mathbf{L} + \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix} = \det \left(s \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \right) \quad (5.13)$$

To determine the maximum value of s in this determinantal expression, i.e. $s^{n_{\max}}$, which is s^{n+p} (theorem 5.3) we need to select all the last n rows from $s \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, p rows from $s \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix}$ and $n - p$ complementary rows from $\begin{bmatrix} \mathbf{R} & \mathbf{C} \end{bmatrix}$.

Hence,

$$A_{\omega} = \begin{vmatrix} l_1 & 0 \\ l_2 & 0 \\ \vdots & \vdots \\ l_p & 0 \\ r_{p+1} & c_{p+1} \\ \vdots & \vdots \\ r_n & c_n \\ \mathbf{0} & \mathbf{I} \end{vmatrix} = \begin{vmatrix} l_1 \\ l_2 \\ \vdots \\ r_{p+1} \\ \vdots \\ r_n \end{vmatrix} \quad (5.14)$$

and the coefficient of s^{n+p} , i.e k_{\max} is $k_{n+p} = \sum_{\omega} A_{\omega}$, where ω stands for different selections of l_1, l_2, \dots, l_p . To continue, we can use the same procedure as in section 5.3.

Equivalently, to determine the minimum power of s in the expansion of the determinant (5.13), we need to consider the following:

$$\det \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ -\mathbf{I} & s\mathbf{I} \end{bmatrix} = \det(s\mathbf{R} + \mathbf{C}) \quad (5.15)$$

Then, we will select q rows from \mathbf{C} and $n - q$ complementary rows from \mathbf{R} . Now, the minimum coefficient k_{\min} of s^{n-q} will be given by $k_{\min} = \sum_{\omega} B_{\omega}$, where ω stands for different selections of the rows of \mathbf{C} . To continue, we can use the same procedure as in section 5.3.

5.6 Examples

In this section we will demonstrate the use of previous theorems and test the necessary and sufficient conditions in the following examples [LLK14].

Example 5.1. First, let us investigate an RLC network with $n = 4$ loops, 2 inductors and 1 capacitor arranged as shown in Figure 5.1. The operator $\mathbf{Z}_a(s) = s \cdot W(s) = s^2 \mathbf{L} + s \mathbf{R} + \mathbf{C}$ is given by the following matrices:

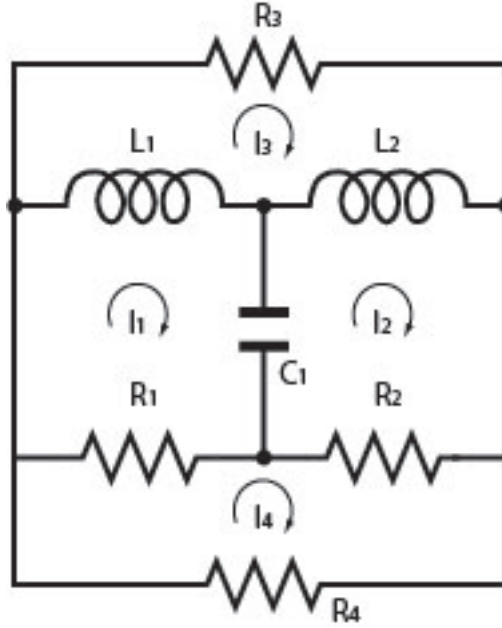


FIGURE 5.1: RLC autonomous network with $n = 4$, $p = 2$, $q = 1$

The autonomous network of the figure can be represented by the following symmetric matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$:

$$\mathbf{L} = \begin{bmatrix} L_1 & 0 & -L_1 & 0 \\ 0 & L_2 & -L_2 & 0 \\ -L_1 & -L_2 & L_1 + L_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.16)$$

$$\mathbf{R} = \begin{bmatrix} R_1 & 0 & 0 & -R_1 \\ 0 & R_2 & 0 & -R_2 \\ 0 & 0 & R_3 & 0 \\ -R_1 & -R_2 & 0 & R_1 + R_2 + R_4 \end{bmatrix} \quad (5.17)$$

$$\mathbf{C} = \begin{bmatrix} C_1^{-1} & -C_1^{-1} & 0 & 0 \\ -C_1^{-1} & C_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.18)$$

By inspection:

$$\text{rank}(\mathbf{L}) = p = 2$$

and

$$\text{rank}(\mathbf{C}) = q = 1$$

.

Using the formulas derived from Theorems 5.1, 5.2, 5.3 we may find the minimum and maximum coefficients of the determinant of the \mathbf{Z}_a operator. For these coefficients we need to compute:

(I) $C_p(\mathbf{L}) = C_2(\mathbf{L})$, because $p = 2$.

(II) $C_q(\mathbf{C}) = C_1(\mathbf{C}) = C$, because $q = 1$.

(III) $Adj_q(\mathbf{R}) = Adj_1(\mathbf{R})$ and $Adj_p(\mathbf{R}) = Adj_2(\mathbf{R})$.

Thus, we have:

$$\begin{aligned}
 C_2(\mathbf{L}) &= \begin{bmatrix} L_1 L_2 & -L_1 L_2 & 0 & L_1 L_2 & 0 & 0 \\ -L_1 L_2 & L_1 L_2 & 0 & -L_1 L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ L_1 L_2 & -L_1 L_2 & 0 & L_1 L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \\
 &= \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\alpha_1} \cdot \underbrace{\begin{bmatrix} L_1 L_2 & -L_1 L_2 & 0 & L_1 L_2 & 0 & 0 \end{bmatrix}}_{\alpha_2^t} \quad (5.19)
 \end{aligned}$$

$$\begin{aligned}
 C_1(\mathbf{C}) = \mathbf{C} &= \begin{bmatrix} C_1^{-1} & -C_1^{-1} & 0 & 0 \\ -C_1^{-1} & C_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{\beta_1} \cdot \underbrace{\begin{bmatrix} C_1^{-1} & -C_1^{-1} & 0 & 0 \end{bmatrix}}_{\beta_2^t} \quad (5.20)
 \end{aligned}$$

Finally, for the *compound adjoints* of R we have that:

$$\begin{aligned}
 Adj_1(\mathbf{R}) &= \begin{bmatrix} R_2 R_3 (R_1 + R_4) & R_1 R_2 R_3 & 0 & R_1 R_2 R_3 \\ R_1 R_2 R_3 & R_1 R_3 (R_2 + R_4) & 0 & R_1 R_2 R_3 \\ 0 & 0 & R_1 R_2 R_4 & 0 \\ R_1 R_2 R_3 & R_1 R_2 R_3 & 0 & R_1 R_2 R_3 \end{bmatrix} \\
 Adj_2(\mathbf{R}) &= \begin{bmatrix} R_3 (R_1 + R_2 + R_4) & 0 & R_2 R_3 & 0 & -R_1 R_3 & 0 \\ 0 & R_2 (R_1 + R_4) & 0 & R_1 R_2 & 0 & -R_1 R_2 \\ R_2 R_3 & 0 & R_2 R_3 & 0 & 0 & 0 \\ 0 & R_1 R_2 & 0 & R_1 (R_2 + R_4) & 0 & -R_1 R_2 \\ -R_1 R_3 & 0 & 0 & 0 & R_1 R_3 & 0 \\ 0 & -R_1 R_2 & 0 & -R_1 R_2 & 0 & R_1 R_2 \end{bmatrix}
 \end{aligned} \tag{5.21}$$

Hence, for the maximum and minimum coefficients using the following formulas:

$$k_{\max} = \alpha_2^t \cdot Adj_p(\mathbf{R}) \cdot \alpha_1$$

and

$$k_{\min} = \beta_2^t \cdot Adj_q(\mathbf{R}) \cdot \beta_1$$

we finally find that:

$$\begin{aligned}
 k_{\min} &= C_1^{-1} (R_1 + R_2) R_3 R_4 \\
 k_{\max} &= L_1 L_2 (R_3 R_4 + R_1 (R_3 + R_4) + R_2 (R_3 + R_4)) \\
 &= L_1 L_2 R_3 R_4 + L_1 L_2 R_1 R_3 + L_1 L_2 R_1 R_4 + L_1 L_2 R_2 R_3 + L_1 L_2 R_2 R_4
 \end{aligned}$$

and by subtracting their corresponding degrees n_{\max} , n_{\min} we get the McMillan degree:

$$\delta_m = 3.$$

Alternatively, we may use the composite matrices as denoted in Proposition 5.2 and Corollary 5.1:

$$(-1)^q \begin{vmatrix} R & C' \\ C'' & 0 \end{vmatrix} \quad (5.22)$$

$$(-1)^p \begin{vmatrix} R & L' \\ L'' & 0 \end{vmatrix} \quad (5.23)$$

Firstly, we need to decompose matrix \mathbf{C} from 5.18 to its corresponding dyads, $\mathbf{C} = \mathbf{C}' \cdot \mathbf{C}''$, as indicated below, where $\mathbf{C}' \in \mathbb{R}^{4 \times 1}$ and $\mathbf{C}'' \in \mathbb{R}^{1 \times 4}$. Then, \mathbf{C} can be written as:

$$\mathbf{C} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} C_1^{-1} & -C_1^{-1} & 0 & 0 \end{bmatrix}$$

Hence, the composite matrix which used to calculate the minimum coefficient of the $\det(\mathbf{Z}_a)$ operator, k_{min} , is expressed as:

$$(-1)^q \begin{vmatrix} R & C' \\ C'' & 0 \end{vmatrix} = (-1) \cdot \begin{vmatrix} R_1 & 0 & 0 & -R_1 & 1 \\ 0 & R_2 & 0 & -R_2 & -1 \\ 0 & 0 & R_3 & 0 & 0 \\ -R_1 & -R_2 & 0 & R_1 + R_2 + R_4 & 0 \\ C_1^{-1} & -C_1^{-1} & 0 & 0 & 0 \end{vmatrix}$$

Similarly, we need to decompose matrix \mathbf{L} from 5.16 to its corresponding dyads, $\mathbf{L} = \mathbf{L}' \cdot \mathbf{L}''$, where $\mathbf{L}' \in \mathbb{R}^{4 \times 2}$ and $\mathbf{L}'' \in \mathbb{R}^{2 \times 4}$. Then, \mathbf{L} can be written as:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_1 & 0 & -L_1 & 0 \\ 0 & L_2 & -L_2 & 0 \end{bmatrix}$$

and the composite matrix which used to calculate the highest coefficient k_{max} is expressed as:

$$(-1)^p \begin{vmatrix} R & L' \\ L'' & 0 \end{vmatrix} = (-1)^2 \cdot \begin{vmatrix} R_1 & 0 & 0 & -R_1 & 1 & 0 \\ 0 & R_2 & 0 & -R_2 & 0 & 1 \\ 0 & 0 & R_3 & 0 & -1 & -1 \\ -R_1 & -R_2 & 0 & R_1 + R_2 + R_4 & 0 & 0 \\ L_1 & 0 & -L_1 & 0 & 0 & 0 \\ 0 & L_2 & -L_2 & 0 & 0 & 0 \end{vmatrix}$$

Therefore, by computing the determinants of the composite matrices above we derive the minimum coefficient as:

$$k_{\min} = C_1^{-1}(R_1 + R_2)R_3R_4$$

and the maximum coefficient:

$$k_{\max} = L_1L_2R_3R_4 + L_1L_2R_1R_3 + L_1L_2R_1R_4 + L_1L_2R_2R_3 + L_1L_2R_2R_4$$

exactly same as before. Thus, it is verified that both computational methods produces the same results, i.e. McMillan degree $\delta_m = 3$.

Applying the *Graph Systematic Approach* discussed in section 5.5 and using the formulation derived in remark 5.1 we can express each one of the matrices $\mathbf{L}, \mathbf{R}, \mathbf{C}$ of the network as:

Matrix of capacitors \mathbf{C} :

$$\mathbf{C} = \begin{bmatrix} C_1^{-1} & -C_1^{-1} & 0 & 0 \\ -C_1^{-1} & C_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C_1^{-1} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{G_C} \cdot C_1^{-1} \cdot \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}}_{G_C^T}$$

Matrix of inductances \mathbf{L} :

$$\begin{aligned}
 \mathbf{L} &= \begin{bmatrix} L_1 & 0 & -L_1 & 0 \\ 0 & L_2 & -L_2 & 0 \\ -L_1 & -L_2 & L_1 + L_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = L_1 \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + L_2 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \\
 &= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \cdot L_1 \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \cdot L_2 \cdot \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} = \\
 &= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}}_{G_L} \cdot \overbrace{\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}}^{D_L} \cdot \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{G_L^T}
 \end{aligned}$$

Matrix of resistors \mathbf{R} :

$$\begin{aligned}
 \mathbf{R} &= \begin{bmatrix} R_1 & 0 & 0 & -R_1 \\ 0 & R_2 & 0 & -R_2 \\ 0 & 0 & R_3 & 0 \\ -R_1 & -R_2 & 0 & R_1 + R_2 + R_4 \end{bmatrix} = R_1 \cdot \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} + \\
 &+ R_2 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + R_3 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + R_4 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \\
 &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}}_{G_R} \cdot \overbrace{\begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix}}^{D_R} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{G_R^T}
 \end{aligned}$$

Next, we will test whether the necessary and sufficient conditions derived in corollary 5.5 and remark 5.4 for the McMillan degree of the network are met. Hence, the following composite matrices need to be formulated:

$$\begin{aligned} \text{a. } \begin{bmatrix} G_C^T \\ G_R^T \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & 0 & -1 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \\ \text{b. } \begin{bmatrix} G_L^T \\ G_R^T \end{bmatrix} &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ 1 & 0 & 0 & -1 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

From the first matrix we can choose $q = 1$ lines from G_C^T and $(n - q) = (4 - 1) = 3$ lines from G_R^T (these lines are demonstrated above in bold letters) that are linearly independent. Similarly, from the last composite matrix we can choose $p = 2$ lines from G_L^T and $(n - p) = (4 - 2) = 2$ lines from G_R^T (in bold) that are linearly independent with each other.

Thus, we conclude that the necessary and sufficient conditions for the McMillan degree are satisfied in this particular example.

□

Example 5.2. Now, let's examine a peculiar RLC network with $n = 2$ loops, 2 inductors, 1 capacitor and 1 resistance arranged as shown in Figure 5.6. The operator $\mathbf{Z}_a(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$ for the RLC network is:

$$\mathbf{Z}_a(s) = s^2 \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} + s \begin{bmatrix} R_1 & -R_1 \\ -R_1 & R_1 \end{bmatrix} + \begin{bmatrix} C^{-1} & -C^{-1} \\ -C^{-1} & C^{-1} \end{bmatrix}$$

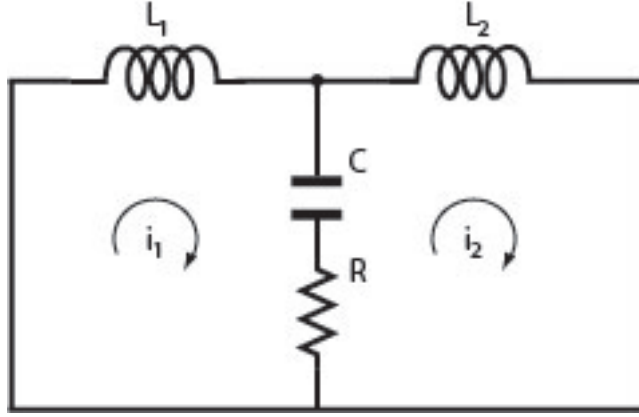


FIGURE 5.2: RLC autonomous network with $n = 2$, $p = 2$, $q = 1$

In this example, if we use the previous results, we expect the McMillan degree of the system to be equal with the number of dynamical elements (i.e. inductors and capacitors). So, $\delta_m = 3$. Then, we compute as previously the maximum and minimum coefficients and their corresponding degrees

$$k_{\max} = L_1 L_2 \cdot s^4$$

and $k_{\min} = C^{-1}(L_1 + L_2) \cdot s^2$. As we can see, $\delta_\mu = k_{\max} - k_{\min} = 4 - 2 = 2 \neq 3$ as we expected.

This is because the necessary and sufficient conditions are not valid in this case.

Applying the *Graph Systematic Approach* discussed in section 5.5 and using the formulation derived in remark 5.1 we can express each one of the matrices \mathbf{L} , \mathbf{R} , \mathbf{C} of the network as:

Matrix of capacitors \mathbf{C} :

$$\begin{bmatrix} C^{-1} & -C^{-1} \\ -C^{-1} & C^{-1} \end{bmatrix} = C^{-1} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{G_C} \cdot \overbrace{C^{-1}}^{D_C} \cdot \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{G_C^T}$$

Matrix of inductances L:

$$\begin{aligned}
\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} &= L_1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + L_2 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot L_1 \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} + \\
&+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot L_2 \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{G_L} \cdot \overbrace{\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}}^{D_L} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{G_L^T}
\end{aligned}$$

Matrix of resistors R:

$$\begin{bmatrix} R & -R \\ -R & R \end{bmatrix} = R \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{G_R} \cdot \overbrace{R}^{D_R} \cdot \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{G_R^T}$$

To determine whether the necessary and sufficient conditions derived in corollary 5.5 and remark 5.4 for the McMillan degree of the network are met, we need to formulate the following composite matrices:

$$\begin{aligned}
\text{a. } \begin{bmatrix} G_L^T \\ G_R^T \end{bmatrix} &= \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ 1 & -1 \end{bmatrix} \\
\text{b. } \begin{bmatrix} G_C^T \\ G_R^T \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned}$$

From the first matrix we can choose $p = 2$ lines from G_L^T and $(n - p) = (2 - 2) = 0$ lines from G_R^T (these lines are demonstrated above in bold letters) that are linearly independent. In contrast, from the last composite matrix we cannot choose $q = 1$ lines from G_C^T and $(n - q) = (2 - 1) = 1$ lines from G_R^T that are linearly independent with each other.

Thus, we conclude that the necessary and sufficient conditions for the McMillan degree are not met in this particular example.

□

5.7 Conclusions

The purpose of this chapter was to develop a framework with which *RLC* networks could be treated as control systems with a generalised transfer function $W^{-1}(s)$. For a general *RLC* network described by the *Implicit network operator* $W(s)$ we computed the McMillan degree δ_m , which expresses the maximum number of independent dynamical elements of the system (i.e. capacitors and inductances). We calculated an upper bound for this degree, which is $\delta_m = \text{rank}(\mathbf{L}) + \text{rank}(\mathbf{C})$ and this is achieved when certain regularity conditions for *RLC* networks are met [LLK14]. We established three different type of regularity conditions, i.e. determinantal, rank and graph theoretic. Furthermore, we reformulated this framework introducing matrix pencils theory and we tried to establish some of the results using the associated loop pencil of the network $P(s)$. Finally, we presented, as applications, a number of various examples where these regularity conditions were demonstrated.

Chapter 6

System Transformations Preserving or Altering Network Cardinality and Possibly the McMillan Degree

6.1 Introduction

A unifying description for the modelling of passive networks in terms of symmetric, integral, differential operators is the Implicit Network operator (or description) $W(s)$, defined in (4.16) [Kar11, KLL14b]. The aim of this chapter is to examine the effect of transformations [KLL14b, KLL14a] in *RLC* networks on the structure of the Implicit operator, or equivalently on the structure of the triple $\mathbf{B}, \mathbf{C}, \mathbf{D}$ matrices, which characterise the network. The cases to be examined are listed below:

1. Changing the values of the components of the system;
2. Altering the nature of components without changing the topology of the network;
3. Modifying the networks topology and possibly reducing the system by removing components / sub-systems;
4. Augmenting the system by adding components / sub-systems to the existing topology of the network.

These types of transformations [KLL14b] may or may not affect the network cardinality and the Implicit McMillan degree and are illustrated through various examples throughout the sections. Specifically, transformations preserving the network cardinality are defined and represented as additive transformations on the Implicit Network operator $W(s)$ in sections 6.2, 6.3, whereas transformations linked to the variation of network cardinality, that is augmentation or deletion of sub-networks, are represented as augmentation or reduction (in terms of dimension) of $W(s)$ in section 6.4. In section 6.5 the identification of fixed dynamics under such transformations in an RLC network, is considered, and the overall analysis leads to the derivation of the main result of this chapter.

Note: In the following examples we use loop - analysis (impedance model). Equivalent results may be obtained if nodal analysis (admittance method) is used.

6.2 RLC Network Transformations Preserving McMillan Degree and Network Cardinality

In this section, we are investigating the effect of transformations on the structure of $W(s)$ operator or more thoroughly, the structure of $(\mathbf{C}, \mathbf{B}, \mathbf{D})$ matrices, where these transformations do not affect the cardinality of the network or the McMillan degree. The case to be examined here [KLL14b] is *changing the values of the components of the system*. We will introduce the effect of these perturbations by means of an example using *impedance modeling*. The same results are obtained if we choose to use nodal analysis (admittance modeling), since the two methods are equivalent.

In more detail, given the transformed matrices $(\mathbf{C}', \mathbf{B}', \mathbf{D}')$, investigate the effect of perturbations on the structure of these matrices, where the variations can be expressed as follows:

$$\begin{aligned}\mathbf{C}' &= \mathbf{C} \pm \frac{1}{c}(x, b) \\ \mathbf{B}' &= \mathbf{B} \pm l(x, b) \\ \mathbf{D}' &= \mathbf{D} \pm r(x, b)\end{aligned}\tag{6.1}$$

where the matrices \mathbf{C} , \mathbf{B} and \mathbf{D} depend on the real parameter $x > 0$ and the position vector $b \in \mathbb{R}^k$. These single element variations have the basic form [BHK12]:

$$\overline{F}(x, b) = xbb^T \quad (6.2)$$

where $b = e_i$ for $i = j$ or $b = e_i - e_j$ for $i \neq j$. These may be illustrated via the following example.

Example 6.1. Lets assume that the initial network in figure 6.1.

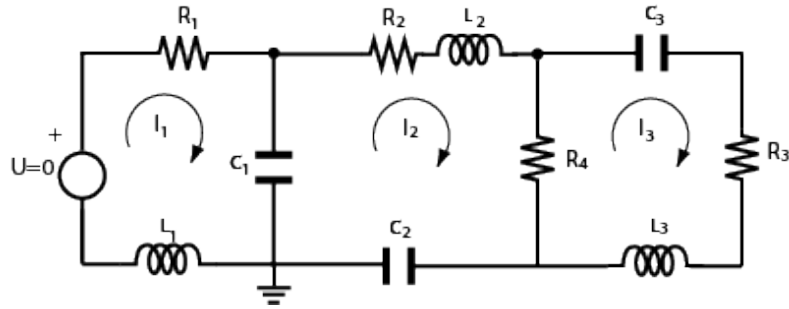


FIGURE 6.1: Initial RLC network

The *RLC* network in figure is described by the impedance operator $W(s)$, which here takes the form:

$$W(s) = s\mathbf{B} + s^{-1}\mathbf{C} + \mathbf{D}$$

We shall note here that $\mathbf{B} = \mathbf{L}$ represents the matrix of inductances, \mathbf{C} the matrix of capacitances and finally, $\mathbf{D} = \mathbf{R}$ the matrix of resistors, all of which are symmetric. The triple $(\mathbf{C}, \mathbf{B}, \mathbf{D})$ can be described as follows:

$$\begin{aligned} \mathbf{B} = \mathbf{L} &= \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} C_1^{-1} & -C_1^{-1} & 0 \\ -C_1^{-1} & C_1^{-1} + C_2^{-1} & 0 \\ 0 & 0 & C_3^{-1} \end{bmatrix} \\ \mathbf{D} = \mathbf{R} &= \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 \\ 0 & -R_4 & R_3 + R_4 \end{bmatrix} \end{aligned} \quad (6.3)$$

Suppose we alternate the values of the components by adding or subtracting a positive arbitrary value $\{x, y, z\}$ to R, L, C elements respectively. If the initial values of the components had the form:

$$R_i, i = 1, 2, \dots, k$$

$$L_j, j = 1, 2, \dots, l$$

$$\frac{1}{C_a}, a = 1, 2, \dots, m$$

where k, l, m is the number of components in the network then the final values of these components will have the following structure:

$$R_i' = R_i \pm x_i, i = 1, 2, \dots, k$$

$$L_j' = L_j \pm y_j, j = 1, 2, \dots, l$$

$$\frac{1}{C_a'} = \frac{1}{C_a} \pm z_a, a = 1, 2, \dots, m$$

and the resulting network is shown below: The network variables are the loop currents

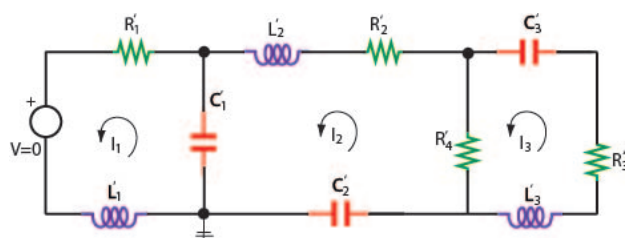


FIGURE 6.2: transformed RLC network

I_1, I_2, I_3 . The impedance model expresses the impedances in the three loops and thus

$W(s)$ operator can be written as:

$$\begin{aligned}
 W(s) &= s^{-1}\mathbf{C}' + \mathbf{D}' + s\mathbf{B}' = \\
 &\begin{bmatrix} \frac{1}{C'_1} & -\frac{1}{C'_1} & 0 \\ -\frac{1}{C'_1} & \frac{1}{C'_1} + \frac{1}{C'_2} & 0 \\ 0 & 0 & \frac{1}{C'_3} \end{bmatrix} s^{-1} + \begin{bmatrix} R'_1 & 0 & 0 \\ 0 & R'_2 + R'_4 & -R'_4 \\ 0 & -R'_4 & R'_3 + R'_4 \end{bmatrix} + \begin{bmatrix} L'_1 & 0 & 0 \\ 0 & L'_2 & 0 \\ 0 & 0 & L'_3 \end{bmatrix} s = \\
 &\begin{bmatrix} \frac{1}{C'_1} \pm z_1 & -\frac{1}{C'_1} \pm z_1 & 0 \\ -\frac{1}{C'_1} \pm z_1 & \frac{1}{C'_1} \pm z_1 + -\frac{1}{C'_2} \pm z_2 & 0 \\ 0 & 0 & \frac{1}{C'_3} \pm z_3 \end{bmatrix} s^{-1} + \begin{bmatrix} L_1 \pm y_1 & 0 & 0 \\ 0 & L_2 \pm y_2 & 0 \\ 0 & 0 & L_3 \pm y_3 \end{bmatrix} s + \\
 &+ \begin{bmatrix} R_1 \pm x_1 & 0 & 0 \\ 0 & (R_2 \pm x_2) + (R_4 \pm x_4) & -(R_4 \pm x_4) \\ 0 & -(R_4 \pm x_4) & (R_3 \pm x_3) + (R_4 \pm x_4) \end{bmatrix}
 \end{aligned} \tag{6.4}$$

Using the formulation (6.2) the above transformation can be expressed formally with modification to the corresponding matrices as shown below:

- **For the D-Type elements:**

$$\mathbf{D}' = \mathbf{D} \pm x_1 b_1 b_1^T \pm x_2 b_2 b_2^T \pm x_3 b_3 b_3^T \pm x_4 b_{23} b_{23}^T$$

where: $b_1 = e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $b_2 = e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, $b_3 = e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$
and $b_{23} = e_2 - e_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$. Thus, the

variations can be expressed as:

$$\begin{aligned}
 \mathbf{D}' &= \mathbf{D} \pm x_1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \pm x_2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \pm x_3 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \\
 &\pm x_4 \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}^T = \mathbf{D} \pm x_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \pm x_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \pm x_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &\pm x_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}
 \end{aligned}$$

• **For the A-Type elements:**

$$\mathbf{C}' = \mathbf{C} \pm z_1 b_{12} b_{12}^T \pm z_2 b_2 b_2^T \pm z_3 b_3 b_3^T$$

where: $b_{12} = e_1 - e_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T - \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$,

$b_2 = e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $b_3 = e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Thus, the variations are:

$$\begin{aligned}
 \mathbf{C}' &= \mathbf{C} \pm z_1 \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T \pm z_2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \\
 &\pm z_3 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T = \mathbf{C} \pm z_1 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \pm z_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &\pm z_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

- **For the T-Type elements:**

$$\mathbf{B}' = \mathbf{B} \pm y_1 b_1 b_1^T \pm y_2 b_2 b_2^T \pm y_3 b_3 b_3^T$$

where: $b_1 = e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $b_2 = e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $b_3 = e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Finally, we have that:

$$\begin{aligned} \mathbf{B}' &= \mathbf{B} \pm y_1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \pm y_2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \\ &\quad \pm y_3 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T = \mathbf{B} \pm y_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \pm y_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad \pm y_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

□

The above example leads to the statement of the following general observations:

Remark 6.1. [Liv12] The presence of an element of $A-$, $T-$, $D-$ type is expressed by an entry in the corresponding matrix \mathbf{C} , \mathbf{B} , \mathbf{T} respectively. In specific:

1. If an element is present in the i -th loop (node) then the alternation in the value of the component leads to the addition or subtraction (whether the value is increased or decreased) of the corresponding arbitrary value in its position in the respective matrix.
2. If an element is common in the i -th and j -th loop then the arbitrary value is added to (or subtracted from) its value in the i -th and j -th loop diagonal entries, as well as subtracted (added to) from the (i, j) and (j, i) position of the corresponding matrix.

□

Remark 6.2. What it is observed from the above is that, by altering the value of the components of the network (by addition or subtraction of a positive value) does not affect the cardinality of the network, i.e. the cardinality of the network is preserved. □

6.3 RLC Network Transformations Preserving Cardinality but Altering McMillan Degree

In this section, we are investigating the effect of transformations on the structure of $W(s)$ operator, i.e the structure of $(\mathbf{C}, \mathbf{B}, \mathbf{D})$ matrices, where these transformations do not affect the cardinality of the network but alter the McMillan degree [KLL14b]. The case to be examined here is *altering the nature of the components of the system without changing the cardinality of the system*. We will introduce the effect of these perturbations by means of an example using *impedance modeling*. The same results are obtained if we choose to use nodal analysis (admittance modeling), since the two methods are equivalent.

Example 6.2. Let us consider the electrical network illustrated in figure 6.1 and the associated impedance model stated in as in example 6.1. We assume that in the previous network we change the nature of the components as shown in figure 6.3:

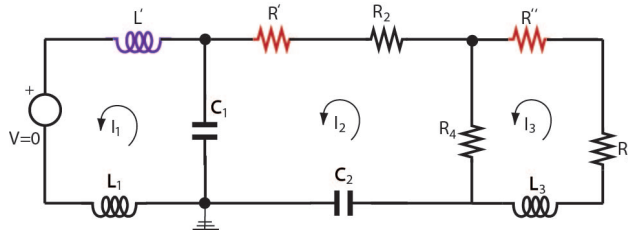


FIGURE 6.3: transformed RLC network

- Remove resistor R_1 from loop 1 and add inductance L' .
- Remove inductance L_2 from loop 2 and add resistor R' .
- Remove capacitor C_3 from loop 3 and add resistor R'' .

Using the formulation (6.2) the above transformation can be expressed formally with modification to the corresponding matrices as shown below:

- **For the T-Type elements:** The removal of inductance L_2 from loop 2 and the addition of inductance L' in loop 1 can be denoted by the following variations in the matrix \mathbf{B} as:

$$\mathbf{B}' = \mathbf{B} + L'b_1b_1^T - L_2b_2b_2^T$$

or equivalently:

$$\begin{aligned} \mathbf{B}' &= \mathbf{B} + L' \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} - L_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} \\ &+ L' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - L_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} L_1 + L' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_3 \end{bmatrix} \end{aligned}$$

- **For the A-Type elements:**

$$\mathbf{C}' = \mathbf{C} - \frac{1}{C_3}b_3b_3^T$$

where $b_3 = e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The above expresses the removal of capacitor C_3 from loop 3. Hence, we have:

$$\mathbf{C}' = \mathbf{C} - \frac{1}{C_3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \mathbf{C} - \frac{1}{C_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\mathbf{C}' = \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & 0 \\ 0 & 0 & \frac{1}{C_3} \end{bmatrix} - \frac{1}{C_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

• **For the D-Type elements:**

$$\mathbf{D}' = \mathbf{D} - R_1 b_1 b_1^T + R' b_2 b_2^T + R'' b_3 b_3^T$$

where: $b_1 = e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $b_2 = e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $b_3 = e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The above transformations express the removal of resistor R_1 from loop 1; the addition of resistor R' to loop 2 and the addition of resistor R'' to loop

3. In more detail:

$$\begin{aligned} \mathbf{D}' &= \mathbf{D} - R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + R' \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + R'' \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 \\ 0 & -R_4 & R_3 + R_4 \end{bmatrix} - R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + R' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + R'' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_2 + R_4 + R' & -R_4 \\ 0 & -R_4 & R_3 + R_4 + R'' \end{bmatrix} \end{aligned}$$

□

By establishing some general observations from example 6.2 the following remarks may be stated:

Remark 6.3. [Liv12] The presence of an element of $A-$, $T-$, $D-$ type is expressed by an entry in the corresponding matrix \mathbf{C} , \mathbf{B} , \mathbf{T} respectively. In specific:

1. If an element is removed from the i -th loop (node), then its value is replaced by 0 in the i -th position of the respective matrix.
2. If an element is removed from the i -th and j -th loop then its value is replaced by a 0 in the i -th and j -th loop diagonal entries, as well as subtracted from the (i, j) and (j, i) position of the corresponding matrix.

3. If the new element is present only in the i -th loop (node), then its value is added in the i -th position of the respective matrix. In opposite if the new element is common in the i -th and j -th loop (node), then its value is added in the i -th position of the respective matrix, as well as subtracted from the (i, j) and (j, i) positions of the corresponding matrix.

□

Remark 6.4. By modifying the nature of elements in the given impedance topology the cardinality of the network will not be altered, but the Implicit McMillan degree is possible to change. This depends upon the nature of elements in the resulting network each time.

□

6.4 RLC Network Transformations Altering Cardinality and the McMillan Degree

6.4.1 Modifying the Topology and Possibly Reducing the System by Removing Components - Subsystems

In this subsection, we examine the case where the given impedance topology (or nodal topology) and respective cardinality of the network are modified by removing single components of the system or sub-systems. These transformations are presented as operations on the $W(s)$ *Implicit* operator.

Example 6.3. Let us consider the electrical network illustrated in figure 6.1 and the associated impedance model stated in as in example 6.1. We assume that in the previous network we remove the following components R_1 , C_2 and L_3 as shown in figure 6.4: Using the formulation (6.2) the above transformation can be expressed formally with modification to the corresponding matrices as shown below:

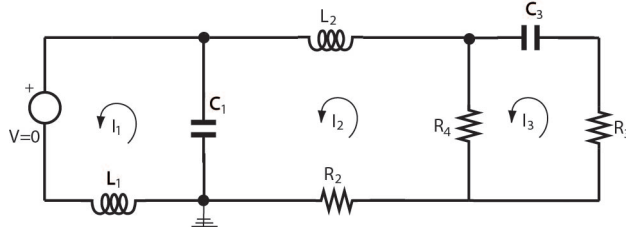


FIGURE 6.4: transformed RLC network

- **For the T-Type elements:** The removal of inductance L_3 from loop 3 is equivalent to the following variation:

$$\mathbf{B}' = \mathbf{B} - L_3 b_3 b_3^T$$

or more explicitly:

$$\mathbf{B}' = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} - L_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **For the A-Type elements:**

$$\mathbf{C}' = \mathbf{C} - \frac{1}{C_2} b_2 b_2^T$$

where $b_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. This variation expresses the removal of capacitor C_2 from loop 2. Hence,

$$\mathbf{C}' = \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & 0 \\ 0 & 0 & \frac{1}{C_3} \end{bmatrix} - \frac{1}{C_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} & 0 \\ 0 & 0 & \frac{1}{C_3} \end{bmatrix}$$

- **For the D-Type elements:**

$$\mathbf{D}' = \mathbf{D} - R_1 b_1 b_1^T$$

where $b_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. The previous denotes the removal of resistor R_1 from loop 1, or in terms of matrices:

$$\mathbf{D}' = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 \\ 0 & -R_4 & R_3 + R_4 \end{bmatrix} - R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 \\ 0 & -R_4 & R_3 + R_4 \end{bmatrix}$$

□

From example 6.3 the following remark can be stated:

Remark 6.5. The presence of an element of $A-$, $T-$, $D-$ type is expressed by an entry in the corresponding matrix \mathbf{C} , \mathbf{B} , \mathbf{T} respectively. Specifically, removing elements without changing the corresponding topology can be achieved by assuming reduction of the values of these elements until they become zero. The entries in the corresponding matrices are replaced by 0. The cardinality of the network is not affected but the McMillan degree possibly alters. □

Example 6.4. Let us consider the electrical network illustrated in figure 6.1 and the associated impedance model stated in as in example 6.1. We assume that in the previous network we remove loop 3. The corresponding network is shown in figure 6.5. The

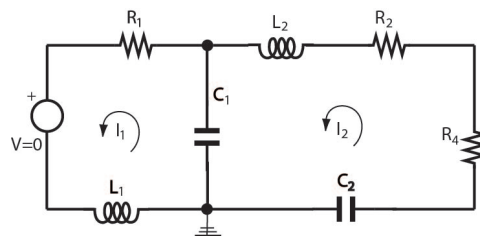


FIGURE 6.5: reduced RLC network

network variables are the loop currents I_1 , I_2 . By introducing the formulation used in (6.2) the above perturbations can be represented by altering the corresponding matrices $(\mathbf{C}, \mathbf{B}, \mathbf{D})$ as shown below:

- For the T-Type elements:

$$\mathbf{B}' = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$$

- For the A-Type elements:

$$\mathbf{C}' = \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} \end{bmatrix}$$

- For the D-Type elements:

$$\mathbf{R}' = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 + R_4 \end{bmatrix}$$

□

Remark 6.6. [Liv12] Removing a loop (node) from the initial *RLC* network has an effect on the structure of the corresponding matrices. A loop (node) removal leads to the reduction of the dimension by one of the corresponding matrices. This means that if k loops (nodes) are removed from the system then, if the initial dimension of the network was n then the final would be $n - k$. The same applies to the dimension of the corresponding matrices. □

The reduce impedance operator of the new network will be of the form:

$$W_{red} = \begin{bmatrix} sL_1 + 1/sC_1^{-1} + R_1 & -1/sC_1^{-1} \\ -1/sC_1^{-1} & sL_2 + 1/s(C_1^{-1} + C_2^{-1}) + R_2 + R_4 \end{bmatrix} \quad (6.5)$$

and its link to the impedance operator of the initial system as shown in figure (6.1) is depicted below:

$$W_{init} = \begin{bmatrix} W_{red} & 0 \\ & -R_4 \\ 0 & -R_4 & sL_3 + 1/sC_3^{-1} + R_3 + R_4 \end{bmatrix} \quad (6.6)$$

A more general result will be established later on, in section 6.5.

6.4.2 Augmenting the System by Adding Components - Subsystems to the Existing Topology

In this subsection, we investigate the variations that result as operations to the general operator $W(s)$ (or to the matrices $\mathbf{C}, \mathbf{B}, \mathbf{D}$ if the system is augmented either by adding separate components to the existing topology or by adding independent loops (nodes).

Example 6.5. Let us consider the electrical network illustrated in figure 6.1 and the associated impedance model stated in as in example 6.1. We assume that in the initial network we add the following components. The corresponding network is shown in figure 6.6.

- Add a resistor R_5 to loop 1.
- Add a common inductance L_4 between loops 1 and 2.
- Add a capacitor C_4 to loop 3.

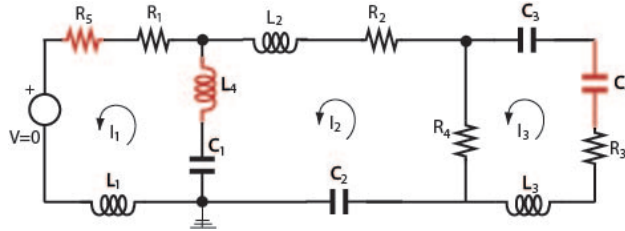


FIGURE 6.6: transformed RLC network

The network variables are the loop currents I_1 , I_2 and I_3 . By introducing the formulation used in (6.2) the above perturbations can be represented by altering the corresponding matrices $(\mathbf{C}, \mathbf{B}, \mathbf{D})$ as shown below:

- For the T-Type elements:

$$\mathbf{B}' = \mathbf{B} + L_4 b_{12} b_{12}^T$$

where $b_{12} = e_1 - e_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$. The addition of a common inductance L_4 between loops 1 and 2 can be represented as the following perturbation in the matrix of inductances \mathbf{B}' :

$$\begin{aligned} \mathbf{B}' &= \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} + L_4 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} L_1 + L_4 & -L_4 & 0 \\ 0 & L_2 + L_4 & 0 \\ 0 & 0 & L_3 \end{bmatrix} \end{aligned}$$

- **For the A-Type elements:**

$$\mathbf{C}' = \mathbf{C} + \frac{1}{C_4} b_3 b_3^T$$

where $b_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The above variation expresses the addition of capacitor C_4 to loop 3. Thus, for the matrix of capacitors \mathbf{C}' we will have:

$$\begin{aligned} \mathbf{C}' &= \begin{bmatrix} C_1^{-1} & -C_1^{-1} & 0 \\ -C_1^{-1} & C_1^{-1} + C_2^{-1} & 0 \\ 0 & 0 & C_3^{-1} \end{bmatrix} + \frac{1}{C_4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} C_1^{-1} & -C_1^{-1} & 0 \\ -C_1^{-1} & C_1^{-1} + C_2^{-1} & 0 \\ 0 & 0 & C_3^{-1} + C_4^{-1} \end{bmatrix} \end{aligned}$$

- **For the D-Type elements:**

$$\mathbf{D}' = \mathbf{D} + R_5 b_1 b_1^T$$

where $b_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. This variation denotes the addition of a resistance R_5 to loop 1. Thus, for the matrix of resistors \mathbf{R}' the following hold:

$$\begin{aligned} \mathbf{D}' &= \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 \\ 0 & -R_4 & R_3 + R_4 \end{bmatrix} + R_5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} R_1 + R_5 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 \\ 0 & -R_4 & R_3 + R_4 \end{bmatrix} \end{aligned}$$

□

Remark 6.7. [Liv12] The presence of an element of A -, T -, D - type is expressed by an entry in the corresponding matrix \mathbf{C} , \mathbf{B} , \mathbf{T} respectively. In specific:

1. If an element is present in the i -th loop (node), then its value is added in the i -th position of the respective matrix.
2. If an element is common to the i -th and j -th loop, then its value is added to the i -th and j -th loop diagonal entries, as well as subtracted from the (i, j) and (j, i) position of the corresponding matrix.

□

Here, we will investigate another type of transformation, i.e. the addition of another loop (or node) to the existing topology. This will affect the topology and the cardinality of the network and possibly the McMillan degree.

Example 6.6. Let us consider the electrical network illustrated in figure 6.1 and the associated impedance model stated in as in example 6.1. We assume that in the initial network we add another loop consisting of the elements R_5 , L_4 and C_4 , which affects only one loop. The corresponding network is shown in figure 6.7. The network variables are the loop currents I_1 , I_2 , I_3 and I_4 . By introducing the formulation used in (6.2) the

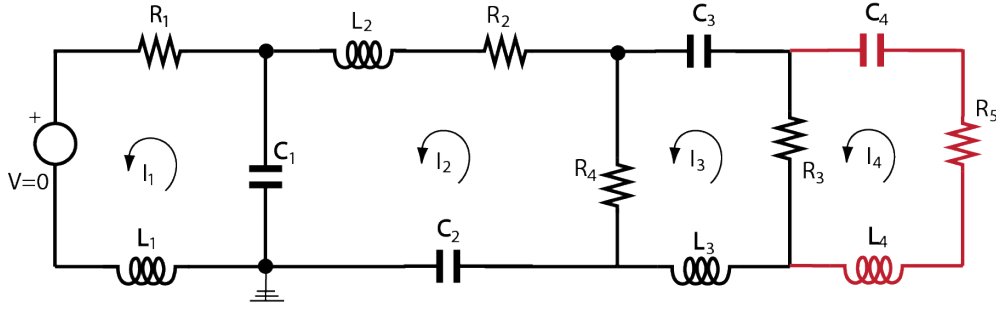


FIGURE 6.7: augmented RLC network by the addition of an extra loop

above perturbations can be represented by altering the corresponding matrices $(\mathbf{C}, \mathbf{B}, \mathbf{D})$ as shown below:

- **For the T-Type elements:**

$$\mathbf{B}' = \mathbf{B} + L_4 b_4 b_4^T$$

where the matrix \mathbf{B} is of dimension 4×4 (by the addition of the new loop, i.e. there is a change in the cardinality of the system) and $b_4 = e_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$. The addition of an extra loop can be represented as the following perturbation in the matrix of inductances \mathbf{B}' :

$$\begin{aligned} \mathbf{B}' &= \underbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}} + L_4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix} \end{aligned}$$

- **For the A-Type elements:** Similarly, for the A– type elements we have:

$$\mathbf{C}' = \mathbf{C} + \frac{1}{C_4} b_4 b_4^T$$

where the matrix \mathbf{C} is of dimension 4×4 (by the addition of the new loop in the system) and $b_4 = e_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$. The addition of an extra loop can be represented as the following perturbation in the matrix of capacitances \mathbf{C}' :

$$\begin{aligned} \mathbf{C}' &= \underbrace{\begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{C}} + \frac{1}{C_4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} & 0 \\ 0 & 0 & 0 & \frac{1}{C_4} \end{bmatrix} \end{aligned}$$

- **For the D-Type elements:** Finally, for the matrix of resistors \mathbf{D}' :

$$\mathbf{D}' = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 & 0 \\ 0 & -R_4 & R_3 + R_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - R_3 b_3 b_3^T + R_3 b_{34} b_{34}^T + R_5 b_4 b_4^T$$

where $b_{34} = e_3 - e_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T - \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T$,
 $b_4 = e_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ and the element R_3 is common now between loops 3

and 4. The addition of an extra loop can be represented as the following perturbation in the matrix of capacitances \mathbf{D}' :

$$\mathbf{D}' = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 & 0 \\ 0 & -R_4 & R_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + R_5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + R_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 & 0 \\ 0 & -R_4 & R_3 + R_4 & -R_3 \\ 0 & 0 & -R_3 & R_3 + R_5 \end{bmatrix}$$

□

Remark 6.8. [Liv12] The presence of an element of $A-$, $T-$, $D-$ type is expressed by an entry in the corresponding matrix \mathbf{C} , \mathbf{B} , \mathbf{T} respectively. In specific:

1. If an element is present in the i -th loop (node), then its value is added in the i -th position of the respective matrix.
2. If an element is common to the i -th and j -th loop then its value is added to the i -th and j -th loop diagonal entries, as well as subtracted from the (i, j) and (j, i) position of the corresponding matrix.
3. The addition of a loop (node) to the system has an effect in the structure of the operator $W(s)$. Especially if a loop (node) is added to the system then the corresponding matrices are augmented by one row and one column respectively. In general, if k loops (nodes) are added to the network, then the corresponding matrices of $A-$, $D-$, $T-$ type elements are augmented by k rows and columns.

□

At this point, we will demonstrate the addition of another loop (or node) consisting of the elements L_4, L_5 and C_4 . The added subsystem (loop) affects two other loops.

The topology, the cardinality of the network and possibly the McMillan degree will be affected.

Example 6.7. At this point, let's assume that in the initial RLC network (figure 6.1) we change the corresponding topology by adding another loop, consisting of the elements L_4 , L_5 and C_4 . The added subsystem (loop) affects two other loops. This is illustrated in the figure 6.8.

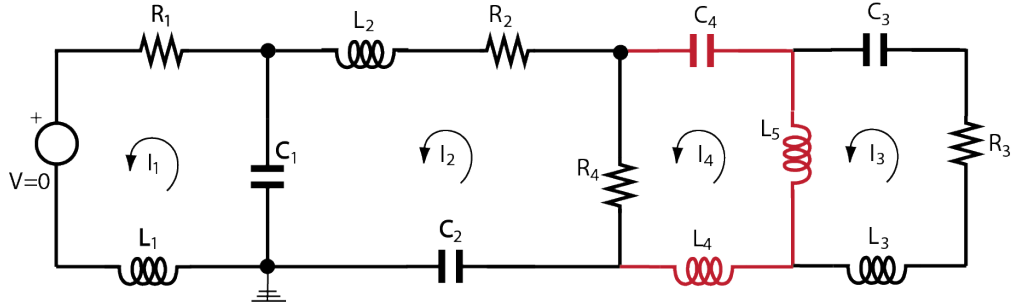


FIGURE 6.8: augmented RLC network by the addition of an extra loop, which affects two loops

The network variables are the loop currents I_1 , I_2 , I_3 and I_4 . We shall note here, as in the previous example, that the dimension of the \mathbf{B} , \mathbf{C} , \mathbf{D} matrices is altered due to the change of system's cardinality. By introducing the formulation used in (6.2) the above perturbations can be represented by altering the corresponding matrices (\mathbf{C} , \mathbf{B} , \mathbf{D}) as shown below:

- **For the T-Type elements:**

$$\mathbf{B}' = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - L_3 b_3 b_3^T + L_3 b_4 b_4^T + L_4 b_3 b_3^T + L_5 b_{34} b_{34}^T$$

where in this case, the matrices that represent the impedance operator are of dimension 4×4 due to the change of cardinality in the system (i.e the addition of the extra loop) and $b_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$, $b_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$, $b_{34} = e_3 - e_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T - \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T$. Hence, the above

perturbation can be written more analytically as follows:

$$\begin{aligned}
 B' &= \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - L_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + L_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + L_4 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \\
 &+ L_5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_4 + L_5 & -L_5 \\ 0 & 0 & -L_5 & L_3 + L_5 \end{bmatrix}
 \end{aligned}$$

- **For the A-Type elements:** Similarly,

$$\mathbf{C}' = \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{C_3} b_3 b_3^T + \frac{1}{C_3} b_4 b_4^T + \frac{1}{C_4} b_3 b_3^T$$

where the matrices of the system are augmented by one row and one column respectively and $b_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$ and $b_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$. Equivalently, the above expression is as follows:

$$\begin{aligned}
 \mathbf{C}' &= \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{C_3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{C_3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{C_4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & 0 & 0 \\ 0 & 0 & \frac{1}{C_4} & 0 \\ 0 & 0 & 0 & \frac{1}{C_3} \end{bmatrix}
 \end{aligned}$$

- **For the D-Type elements:** Finally, for the matrix of resistors \mathbf{D} the following hold:

$$\mathbf{D}' = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 & 0 \\ 0 & -R_4 & R_3 + R_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - R_3 b_3 b_3^T + R_3 b_4 b_4^T$$

where $b_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$ and $b_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$. More explicitly,

$$\begin{aligned} \mathbf{D}' &= \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 & 0 \\ 0 & -R_4 & R_3 + R_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - R_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + R_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 + R_4 & -R_4 & 0 \\ 0 & -R_4 & R_4 & 0 \\ 0 & 0 & 0 & R_3 \end{bmatrix} \end{aligned}$$

□

Remark 6.9. [Liv12] The presence of an element of A -, T -, D - type is expressed by an entry in the corresponding matrix \mathbf{C} , \mathbf{B} , \mathbf{T} respectively. In specific:

1. If an element is present in the i -th loop (node), then its value is added in the i -th position of the respective matrix.
2. If an element is common to the i -th and j -th loop, then its value is added to the i -th and j -th loop diagonal entries, as well as subtracted from the (i, j) and (j, i) position of the corresponding matrix.
3. The addition of a loop (node) to the system has an effect in the structure of the operator $W(s)$. Especially if a loop (node) is added to the system then the corresponding matrices are augmented by one row and one column respectively.

In general, if k loops (nodes) are added to the network, then the corresponding matrices of A -, D -, T - type elements are augmented by k rows and columns.

□

6.5 Fixed Dynamics of RLC Networks under Network Transformations

The problem we are investigating is finding *fixed dynamics* in network transformations. We aim to investigate the following problems:

Problem 6.1. How changes in the nature of a single element. i.e. changing of value, or nature within a given cardinality network leads to new dynamics, which have certain elements fixed. □

Problem 6.2. Investigating transformations where changes in network cardinality lead to new dynamics where part of which are fixed. □

Identifying the *fixed dynamics* and explaining how the rest of modified dynamics change are considered next. We can restrict our study to loop modelling and impedance functions, whereas the nodal modelling follows along similar lines. We will consider a generic example and try to develop some general rules through this.

Case I: Fixed cardinality transformations

We consider a network where in some loop we change the value, or nature of a given element. We take as a generic example the following network:

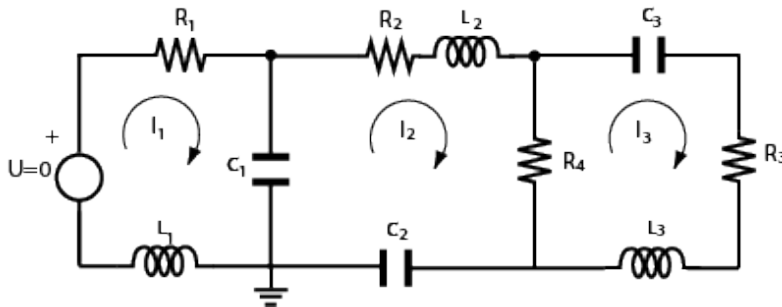


FIGURE 6.9: Network (1)

Example 6.8. The network is described by the *Implicit Network operator*

$$W_1(s) = \begin{bmatrix} sL_1 + 1/sC_1^{-1} + R_1 & -1/sC_1^{-1} & 0 \\ -1/sC_1^{-1} & sL_2 + 1/s(C_1^{-1} + C_2^{-1}) + R_2 + R_4 & -R_4 \\ 0 & -R_4 & sL_3 + 1/sC_3^{-1} + R_3 + R_4 \end{bmatrix} \quad (6.7)$$

Transformation 1.a: Single loop changes

Consider the modifications affecting only loop 2 with current i_2 in the following network: The resulting operator is given by:

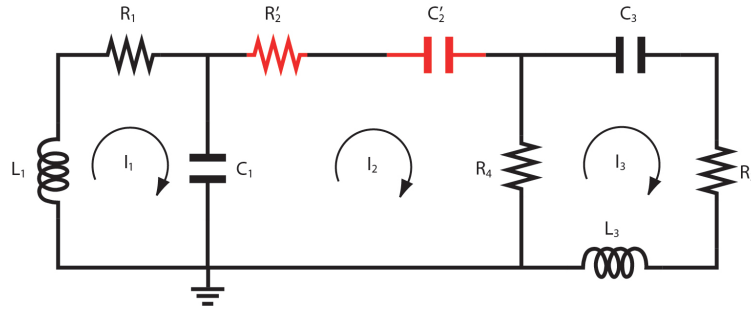


FIGURE 6.10: Network (1a)

$$W_{1a}(s) = \begin{bmatrix} sL_1 + 1/sC_1^{-1} + R_1 & -1/sC_1^{-1} & 0 \\ -1/sC_1^{-1} & 1/s(C_1^{-1} + C_2'^{-1}) + R_2' + R_4 & -R_4 \\ 0 & -R_4 & sL_3 + 1/sC_3^{-1} + R_3 + R_4 \end{bmatrix} \quad (6.8)$$

By changing rows 2 and 3 and then columns 2 and 3, the above matrix is equivalent to:

$$\begin{aligned} \widetilde{W}_{1a}(s) &= \begin{bmatrix} sL_1 + 1/sC_1^{-1} + R_1 & 0 & -1/sC_1^{-1} \\ 0 & sL_3 + 1/sC_3^{-1} + R_3 + R_4 & -R_4 \\ -1/sC_1^{-1} & -R_4 & 1/s(C_1^{-1} + C_2'^{-1}) + R_2' + R_4 \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{aligned} \quad (6.9)$$

where the sub-matrix in blue color indicates the fixed dynamics and it is clear that the dynamics of the first and third loop are not affected. Using *Schur formula*

[SP05] we have that:

$$\left| \widetilde{W}_{1a}(s) \right| = \begin{vmatrix} sL_1 + 1/sC_1^{-1} + R_1 & 0 \\ 0 & sL_3 + 1/sC_3^{-1} + R_3 + R_4 \end{vmatrix} \cdot \Delta \quad (6.10)$$

where Δ is defined from the *Schur formula* [SP05]: For $R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, if

$$|A| \neq 0 \Rightarrow |R| = |A| |D - CA^{-1}B|$$

and if

$$|D| \neq 0 \Rightarrow |R| = |D| |A - BD^{-1}C|$$

The modified dynamics are those expressed by Δ and are influenced by C_2' and R_2' .

Transformation 1.b: Two loop changes

Consider the modification affecting loops 2 and 3 demonstrated in the following figure: The corresponding operator is:

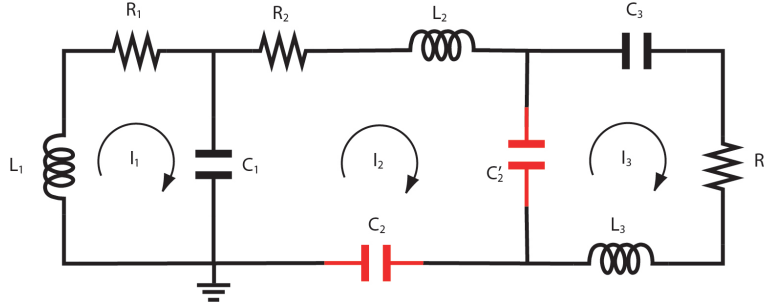


FIGURE 6.11: Network (1b)

$$\widetilde{W}_{1b}(s) = \begin{bmatrix} sL_1 + 1/sC_1^{-1} + R_1 & -1/sC_1^{-1} & 0 \\ -1/sC_1^{-1} & sL_2 + 1/s(C_2^{-1} + C_2'^{-1}) + R_2 & -1/sC_2'^{-1} \\ 0 & -1/sC_2'^{-1} & sL_3 + 1/s(C_3^{-1} + C_2'^{-1}) + R_3 \end{bmatrix} \quad (6.11)$$

where the blue color indicates the *fixed dynamics*. In this case the change affects loops 3 and 2 and thus the *fixed dynamics* are those of loop 1. \square

Case II: Varying Cardinality transformations

We now consider transformations which affect the network cardinality and we investigate again the problem of *fixed dynamics*.

Example 6.9. Consider the augmented network illustrated below with corre-

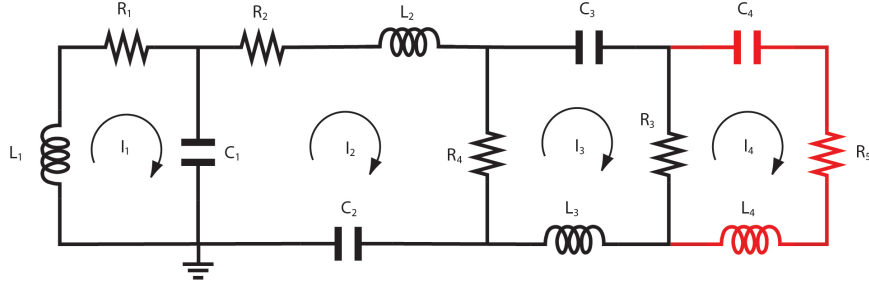


FIGURE 6.12: Augmented Network (2a)

sponding *Implicit operator*:

$$\widetilde{W}_{2a}(s) = \begin{bmatrix} sL_1 + \frac{1}{s}C_1^{-1} + R_1 & -\frac{1}{s}C_1^{-1} & 0 & 0 \\ -\frac{1}{s}C_1^{-1} & sL_2 + \frac{1}{s}(C_1^{-1} + C_2^{-1} + R_2 + R_4) & -R_4 & 0 \\ 0 & -R_4 & sL_3 + \frac{1}{s}C_3^{-1} + R_3 + R_4 & -R_3 \\ 0 & 0 & -R_3 & sL_4 + \frac{1}{s}C_4^{-1} + R_3 + R_5 \end{bmatrix} \quad (6.12)$$

What we observe is that the non affected loops define once again the *fixed dynamics* of the network, which are illustrated by blue color. \square

Mow let us consider the following example that illustrates a different augmentation:

Example 6.10. Consider the augmented network illustrated below: with corre-

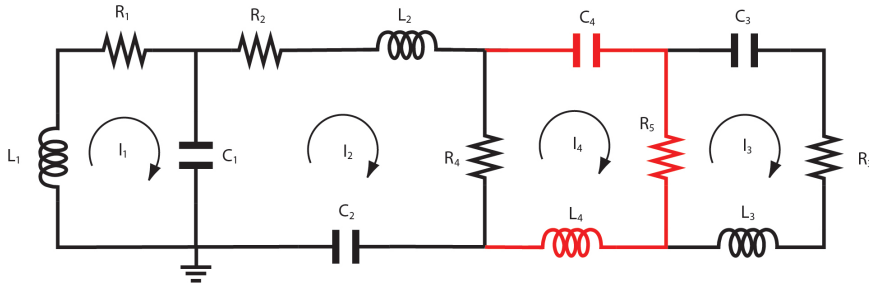


FIGURE 6.13: Augmented Network (2b)

sponding *Implicit operator*:

$$\widetilde{W}_{2b}(s) = \begin{bmatrix} sL_1 + \frac{1}{s}C_1^{-1} + R_1 & -\frac{1}{s}C_1^{-1} & 0 & 0 \\ -\frac{1}{s}C_1^{-1} & sL_2 + \frac{1}{s}(C_1^{-1} + C_2^{-1} + R_2 + R_4) & -R_4 & 0 \\ 0 & -R_4 & sL_4 + \frac{1}{s}C_4^{-1} + R_4 + R_5 & -R_5 \\ 0 & 0 & -R_5 & sL_3 + \frac{1}{s}C_3^{-1} + R_3 + R_5 \end{bmatrix} \quad (6.13)$$

The *fixed dynamics* are illustrated in blue color. □

Next, consider the following example:

Example 6.11. Consider the augmented network illustrated below: Obviously,

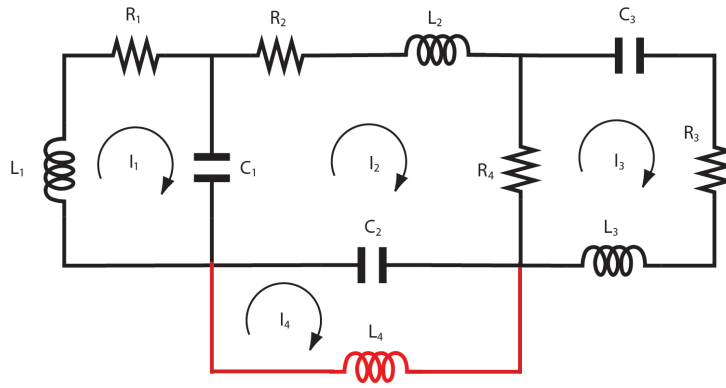


FIGURE 6.14: Augmented Network (2c)

the loops i_1, i_3 are not affected by the creation of this particular additional loop and thus the new network has dynamics of i_1, i_3 fixed.

The network depicted in figure (6.14) may be modified as presented next: In this

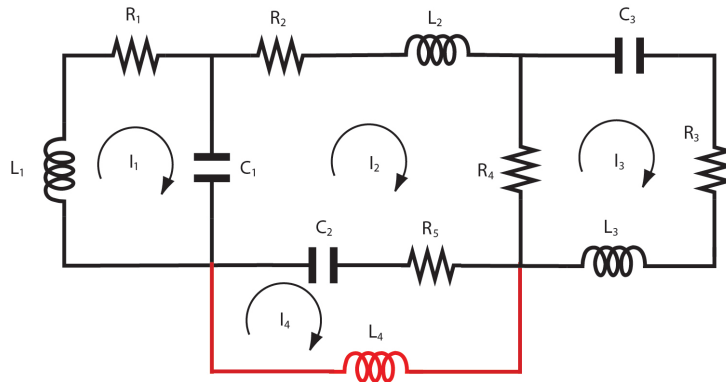


FIGURE 6.15: Augmented Network (2d)

network the loops i_1, i_3 are not affected, but loops i_2, i_4 are changing. We can

write the impedance matrix by considering an ordering of the loops as:

$$\underbrace{i_1, i_3}_{\text{not affected}} \quad \underbrace{i_2, i_4}_{\text{affected}}$$

and thus,

$$\widetilde{W}_{2d}(s) = \begin{bmatrix} sL_1 + 1/sC_1^{-1} + R_1 & 0 & -1/sC_1^{-1} & 0 \\ 0 & sL_3 + 1/sC_3^{-1} + R_3 + R_4 & -R_4 & 0 \\ -1/sC_1^{-1} & -R_4 & sL_2 + 1/sC_2^{-1} + R_2 + R_4 + R_5 & -1/sC_2^{-1} - R_5 \\ 0 & 0 & -1/sC_2^{-1} - R_5 & sL_4 + 1/sC_2^{-1} + R_5 \end{bmatrix} \begin{matrix} 1 \\ 3 \\ 2 \\ 4 \end{matrix} \quad (6.14)$$

The *fixed dynamics* are pictured in blue color. □

The final example is the following:

Example 6.12. Consider the augmented network shown in figure (6.16).

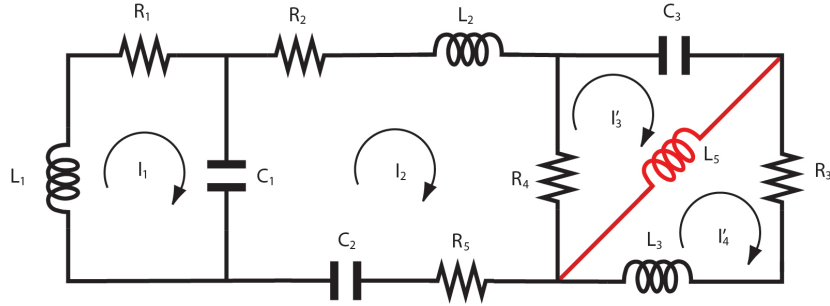


FIGURE 6.16: Augmented Network (2e)

In this example the original loop i_3 is destroyed and two new loops i'_3, i'_4 are created.

For the new network the dynamics of i_1, i_2 are invariant. In fact, arranging the impedance matrix as:

$$\underbrace{i_1, i_2}_{\text{not affected}} \quad \underbrace{i'_3, i'_4}_{\text{affected}}$$

and thus we have,

$$\widetilde{W}_{2e}(s) = \begin{bmatrix} sL_1 + 1/sC_1^{-1} + R_1 & -1/sC_1^{-1} & 0 & 0 \\ -1/sC_1^{-1} & sL_2 + 1/s(C_1^{-1} + C_2^{-1}) + R_2 + R_4 & -R_4 & 0 \\ 0 & -R_4 & sL_5 + 1/sC_3^{-1} + R_4 & -sL_5 \\ 0 & 0 & -sL_5 & s(L_4 + L_5) + R_3 \end{bmatrix} \begin{matrix} i_1 \\ i_2 \\ i'_3 \\ i'_4 \end{matrix} \quad (6.15)$$

The *fixed dynamics* are pictured in blue color. □

Note:

The results for the *fixed dynamics* under augmentation may be also applied in a reverse way for identifying the *fixed dynamics* under reduction of the network.

Remark 6.10. In the derivation of loop impedance, or nodal admittance models, the *ordering* of loop , or node numbering is *arbitrary*. Two different *orderings* lead to a symmetric row-column permutation of the corresponding impedance or admittance model. □

The above analysis motivates the development of results characterizing the existence of *fixed dynamics*, as well as those dynamics changing under some transformation on the network N , denoted as a transformation τ . Note that τ can be either a network transformation preserving the cardinality, or changing the network cardinality. We shall assume that N has μ independent loops with loop currents $\{i_1, i_2, \dots, i_\mu\}$. If τ is such a transformation then this leads to a modified network, denoted by N_τ , with loop currents $\{i_1, i_2, \dots, i_\nu\}$, where ν can be either $\nu = \mu$ or $\nu < \mu$ or $\nu > \mu$.

Remark 6.11. For the modified network N_τ there exists a set of indices $\sigma = (j_1, j_2, \dots, j_\rho)$ or $\sigma = \emptyset$ which describe loops of the original network N with a loop impedance not affected by the transformation τ . □

Next, we state the following definition:

Definition 6.1. A transformation τ for which $\sigma \neq \emptyset$ will be called proper transformation and if $\sigma = \emptyset$, then it will be called complete. □

Clearly, if τ is *complete* all dynamics of the evolved network N_τ are affected by the transformation. In the following we consider *proper* transformations.

For a *proper* transformation τ we may define an *ordering* of loops of N_τ as:

$$\omega = \{j_1, j_2, \dots, j_\rho; k_1, k_2, \dots, k_\nu\} = \{\sigma; \pi\} \quad (6.16)$$

The above *ordering* of loops, where $\sigma = (j_1, j_2, \dots, j_\rho)$ is the maximal set of non-affected loops of N_τ , which may be referred to as the invariant index of the transformation τ .

Using the ordering of loops as in (6.16) the corresponding impedance description of N_τ network (following remark 6.11) has the form:

$$W_{\tau,\omega} = \begin{bmatrix} W_\sigma(s) & X_{\sigma,\pi}(s) \\ X_{\sigma,\pi}^t(s) & W_\pi(s) \end{bmatrix} \quad (6.17)$$

where $W_\sigma(s)$ is the impedance matrix of the sub-network of N associated with the invariant index $\sigma = (j_1, j_2, \dots, j_\rho)$ of the transformation τ and $W_\pi(s)$ and $X_{\sigma,\pi}$ are the remaining parts of the representation, where $W_\pi(s)$ is the impedance of the sub-network of N_τ that is affected by the transformation τ . The above analysis leads to the following main result.

Theorem 6.1. *Let N be an RLC network and τ be a proper network transformation with an invariant index $\sigma = (j_1, j_2, \dots, j_\rho)$. If ω is the ordering of the loops of N_τ as*

$$\omega = \{\sigma; \pi\} = \{j_1, j_2, \dots, j_\rho; k_1, k_2, \dots, k_\nu\}$$

and $W_{\tau,\omega}(s)$ is the impedance representation according to this ordering, i.e.

$$W_{\tau,\omega} = \begin{bmatrix} W_\sigma(s) & X_{\sigma,\pi}(s) \\ X_{\sigma,\pi}^t(s) & W_\pi(s) \end{bmatrix}$$

then:

- i. The sub-network of N corresponding to the indices σ defines the fixed dynamics of N_τ under τ and $W_\sigma(s)$ is the corresponding fixed impedance matrix.*
- ii. The variable dynamics of N_τ are defined by the matrix:*

$$Z_\pi(s) = W_\pi(s) - X_{\sigma,\pi}^t(s) \cdot W_\sigma(s)^{-1} \cdot X_{\sigma,\pi}(s)$$

□

Proof. By selecting the ordering based on the invariant index σ as in (6.16) we have a representation of the impedance as in (6.17). Using the Schur formula [SP05] we have

that:

$$|W_{\tau,\omega}(s)| = |W_{\sigma}(s)| \cdot \left| W_{\pi}(s) - X_{\sigma,\pi}^t(s) \cdot W_{\sigma}(s)^{-1} \cdot X_{\sigma,\pi}(s) \right| = |W_{\sigma}(s)| \cdot |Z_{\pi}(s)|$$

and the result is established. \square

6.6 Conclusions

In this chapter, we examined *RLC* network transformations that preserve or alter network cardinality and possibly the Implicit McMillan degree of the network. These transformations were demonstrated through detailed examples leading to the derivation of a mathematical formulation. Four distinctive cases of transformations were investigated and results were established linked to the structure of the implicit operator $W(s)$, which reflects to the description of the triple $\mathbf{C}, \mathbf{B}, \mathbf{D}$. Specifically, transformations preserving the network cardinality were defined and represented as additive transformations on the Implicit Network operator $W(s)$, whereas transformations linked to the variation of network cardinality, that is augmentation or deletion of sub-networks were represented as augmentation or reduction (in terms of dimension) of the Implicit Network operator $W(s)$. The above analysis led in a natural way into the identification of fixed dynamics under such transformations in an *RLC* network, which was considered in section 6.5 and the main result was established.

Chapter 7

RLC Networks Redesign As Frequency Assignment Problems: Cardinality Preserving Transformations

7.1 Introduction

In this chapter we investigate the Arbitrary Frequency Assignment Problem of *RLC* Networks under a re-engineering context for the special case where the applied transformations preserve cardinality. Given an *RLC* network described by the *Implicit Network Operator* $W(s)$, we are interested in tuning the natural frequencies, which are strongly related to the topology of the network and the nature of elements.

Specifically, in section 7.2 the natural frequencies of a network are defined in terms of the zeros of $W(s)$ and the different types of frequency assignment under cardinality preserving transformations are presented. Next, in section 7.3, we formulate the special problem of zero assignment via diagonal perturbations, where non-dynamical elements are added to the network, in order to achieve desirable frequencies. We allow complex solutions to the problem and we are interested in the surjectivity property of the related Frequency Assignment Map. Specifically, for an *RLC* network with $n \geq p + q$ whenever this map is onto the problem can be solved generically. By using the Dominant Morphism theorem the problem is reduced to that of finding one point such that the differential of the frequency assignment map has full rank. We give a generic solution to the problem

by demonstrating an example where the differential has full rank and we ground the sufficient conditions for arbitrary frequency assignment.

Furthermore, in section 7.4 we aim to compute the number of solutions to the zero assignment problem in *RLC* networks, introduced in the previous section, in order to examine whether there exist real solutions to the problem. To achieve this, we compute the cohomology ring H^* of the compactified space. The new system of equations defining the problem is assigned into elements of this ring and via the cup product of H^* the number of solutions is determined the cases where $n = p + q$ and $n > p + q$.

Finally, in section 7.5 we examine the conditions under which the natural frequencies of an *RLC* network can be improved (from a stability perspective) using Zero Assignment under diagonal perturbations. Specifically, we establish the necessary conditions for which the zeros of $W(s)$ can be assigned into a specific area of the stability region.

7.2 Frequency Assignment by Cardinality Preserving Transformations

7.2.1 Introduction

Network re-engineering (or re-engineering of *RLC* networks) [KL06] as described in Chapter 1 of the thesis may be achieved by selecting different values both for dynamic and non-dynamic elements within a fixed or alternating interconnection topology, which may lead to evolution of the network (introducing or removing branches). The general re-engineering problem is more complex than the one considered in this thesis. The problem examined here is tuning the natural frequencies of an arbitrary *RLC* network, which can be formulated as a Frequency Assignment Problem (or zero assignment problem) for networks with specific structure, i.e. networks that may be represented by the *Implicit Network Operator* $W(s)$.

The natural frequencies of an *RLC* network depend on the topology of the network and the nature and values of the elements [LK09] and may be determined by the zeros of the *Implicit Network Operator* $W(s)$. Hence, by taking special interest in its zeros we can tune the natural frequencies of the network and achieve desirable properties. The

zeros of $W(s)$ may be computed as follows [LLK16]:

The general description of the *Implicit Network Operator* $W(s)$ can be re-written as:

$$W(s) = s\mathbf{L} + s^{-1}\mathbf{C} + \mathbf{R} = \frac{1}{s} (s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}) = \frac{s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}}{s} = \frac{N(s)}{D(s)} \quad (7.1)$$

The numerator of the above description $N(s)$ defines the *zeros* of the general operator $W(s)$.

Frequency Assignment by cardinality preserving transformations may be achieved in different ways. However, in this chapter we restrict ourselves in the case of Arbitrary Frequency Assignment via Diagonal Perturbations and specifically we consider the case where non-dynamical elements are added to the network [LLK16]. The problem can be extended to a more complex one, when the addition of dynamical elements (i.e. capacitors/ inductances or a combination of both) is necessary for frequency assignment purposes, but this case is not considered in this thesis. The approach adopted for tackling this problem belongs to the general class of Determinantal Assignment Problems (DAP), which is presented in (2.9). The problems discussed above can be formulated as:

Problem 7.1. Frequency Assignment by Non-Dynamical Perturbations: Tuning the resistors in an *RLC* network

Given an arbitrary passive *RLC* network described by the general operator $W(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$, where $\mathbf{R}, \mathbf{L}, \mathbf{C}$ are symmetric matrices that characterize the topology of the network, we need to determine a matrix of resistors \mathbf{R}' such that if we add it to the network, then:

$$\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} + s\mathbf{R}') = \tilde{p}(s) \quad (7.2)$$

where $\tilde{p}(s)$ is the desired polynomial to be assigned. □

Problem 7.2. Frequency Assignment by Dynamical Perturbations: Tuning the capacitors of an *RLC* network

Given an arbitrary passive *RLC* network described by the general operator $W(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$, where $\mathbf{R}, \mathbf{L}, \mathbf{C}$ are symmetric matrices that characterize the topology of the network, we need to determine a matrix of capacitors \mathbf{C}' such that if we add it to the network, then:

$$\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} + \mathbf{C}') = \tilde{p}(s) \quad (7.3)$$

where $\underline{p}(s)$ is the desired polynomial to be assigned. This case is not considered here. \square

Problem 7.3. Frequency Assignment by Dynamical Perturbations: Tuning the inductances of an RLC network

Given an arbitrary passive RLC network described by the general operator $W(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$, where $\mathbf{R}, \mathbf{L}, \mathbf{C}$ are symmetric matrices that characterize the topology of the network, we need to determine a matrix of inductances \mathbf{L}' such that if we add it to the network, then:

$$\det(s^2\mathbf{L} + s^2\mathbf{L}' + s\mathbf{R} + \mathbf{C}) = \tilde{p}(s) \quad (7.4)$$

where $\underline{p}(s)$ is the desired polynomial to be assigned. This case is not considered here. \square

In the following section we will examine a sub-problem of the problem 7.1, where the matrix of resistors that is added to the network has a specified structure, i.e. it is diagonal. This is considered in the following section.

7.3 Frequency Assignment in RLC Networks via Diagonal Perturbations

7.3.1 Problem Formulation

In this section the problem that we formulate is a special case of problem 7.1. The starting point of our work is the problem of arbitrary assignment of frequencies via static compensation presented in (2.8). We only consider the case where non-dynamical elements, i.e. resistors are added to the network, in order to achieve the desirable natural frequencies. The number of resistors that are generally added to the RLC network should always be equal or exceed the number of frequencies to be assigned, i.e. $n \geq p + q$, which consists a necessary and generically sufficient condition [LLK16].

The problem can be extended to a more complex one, where the addition of dynamical elements is necessary for frequency assignment purposes (this is necessary when the previous condition does not hold) as described in section 7.2. The problem examined here has common features with the arbitrary pole assignment problem via constant decentralized output feedback [LK95a], the zero assignment problem of matrix pencils

by additive structured transformations [LK09] and finally it is linked to work related with assigning frequencies via determinantal equations [Lev07]. Mathematically, the problem is equivalent to solving a system of algebraic equations or to finding intersection of varieties [Ful84]. Furthermore, it can be factored as a linear and a multi-linear problem, or an intersection of a linear variety with a nonlinear projective variety.

To tackle the complex solvability for the special case of zero assignment in *RLC* networks via Diagonal Perturbations, we apply the *Dominant Morphism theorem* [Bor91, Hum75, HM77], which was initially introduced in section 2.3. The problem is then reduced to that of determining one point such that the differential of the Frequency Assignment map of the problem has full rank. To find the point where the the differential has full rank it is sufficient that $n \geq p + q$.

Problem Formulation

As already stated, given an arbitrary passive *RLC* network that is described by the general operator $W(s) = s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}$, where $\mathbf{R}, \mathbf{L}, \mathbf{C}$ are symmetric matrices that characterize the topology of the network, we need to determine a matrix of resistors \mathbf{R}' such that if we add it to the network, then:

$$\det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} + s\mathbf{R}') = \underline{p}(s) \quad (7.5)$$

where $\underline{p}(s)$ is the desired polynomial to be assigned. If \mathbf{R}' is not diagonal then it is necessary to transform it into a diagonal matrix \mathbf{D} . To achieve this we can rewrite the above determinantal expression (7.5) as:

$$\begin{aligned} \det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} + s\mathbf{R}') &= \det(s^2\mathbf{L} + s\mathbf{R} + \mathbf{C} + s\mathbf{G}^T \cdot \mathbf{D} \cdot \mathbf{G}) \\ &= \det[\mathbf{G}^T \cdot (s^2\mathbf{G}^{T^{-1}}\mathbf{L}\mathbf{G}^{-1} + s\mathbf{G}^{T^{-1}}\mathbf{R}\mathbf{G}^{-1} \\ &\quad + \mathbf{G}^{T^{-1}}\mathbf{C}\mathbf{G}^{-1} + s\mathbf{D}) \cdot \mathbf{G}] \\ &= \det(\mathbf{G}^T) \cdot (\det s^2\mathbf{G}^{T^{-1}}\mathbf{L}\mathbf{G}^{-1} + s\mathbf{G}^{T^{-1}}\mathbf{R}\mathbf{G}^{-1} \\ &\quad + \mathbf{G}^{T^{-1}}\mathbf{C}\mathbf{G}^{-1} + s\mathbf{D}) \cdot \det(\mathbf{G}) \\ &= \lambda \cdot \det(s^2\mathbf{L}' + s\mathbf{R}'' + \mathbf{C}' + s\mathbf{D}) \end{aligned} \quad (7.6)$$

The matrices \mathbf{G}, \mathbf{G}^T in the previous expression are the graph incidence matrices defined in section 5.4 (see definition 5.1). Furthermore, if G^T denotes the *incidence matrix* for the matrices $\mathbf{R}, \mathbf{L}, \mathbf{C}$ then these matrices can be represented by remark 5.1:

$$\begin{aligned}\mathbf{R} &= G_R \cdot D_R \cdot G_R^T \\ \mathbf{L} &= G_L \cdot D_L \cdot G_L^T \\ \mathbf{C} &= G_C \cdot D_C \cdot G_C^T\end{aligned}$$

where D_C, D_R, D_L represent the diagonal matrices with entries the capacitors, resistors and inductances respectively in a given network. Hence, instead of solving the equation (7.5), it is equivalent to solve equation (7.6) as the determinant remains invariant.

Using the Binet-Cauchy Theorem [MM64] equation (7.6) can be factored as:

$$\begin{aligned}\det(s^2\mathbf{L}' + s\mathbf{R}'' + \mathbf{C}' + s\mathbf{D}) &= \\ &= \det \left(\begin{bmatrix} \mathbf{I} & \mathbf{D} \end{bmatrix} \cdot \begin{bmatrix} s^2\mathbf{L}' + s\mathbf{R}'' + \mathbf{C}' \\ s\mathbf{I} \end{bmatrix} \right) = \\ &= C_n \begin{bmatrix} \mathbf{I} & \mathbf{D} \end{bmatrix} \cdot C_n \begin{bmatrix} s^2\mathbf{L}' + s\mathbf{R}'' + \mathbf{C}' \\ s\mathbf{I} \end{bmatrix} = \underline{p}(s)\end{aligned}\tag{7.7}$$

It should be noted that $\text{rank}(\mathbf{D}) = n$, $\text{rank}(\mathbf{L}) = p$ and $\text{rank}(\mathbf{C}) = q$.

The matrix \mathbf{D} is assumed to be diagonal. If \mathbf{D} is non-diagonal then we transform it by multiplying it with an appropriate matrix \mathbf{G} , which is invertible (i.e. $\det(\mathbf{G}) \neq 0$, or \mathbf{G}^{-1} exists).

To attain complete frequency assignability the number of resistors that are added to the network should always be equal or exceed the network's implicit McMillan degree, i.e. $n \geq p + q$. Furthermore, the differential of the frequency assignment map plays a very vital part in the solvability of our problem as explained previously. Before we prove that the problem can be solved generically ¹, we will introduce the Frequency Assignment Map of the problem (FAP) and the differential related to this map, which are used to check whether the sufficient condition for complex solvability is satisfied.

¹In algebraic geometry, a property of an irreducible variety \mathcal{X} holds *generically* if it holds on a non-empty Zarisky open-set. In other words, it has to hold in the whole set apart from a set of measure zero.

7.3.2 Frequency Assignment Map

The Frequency Assignment Map of the Zero Assignment Problem in RLC networks via Diagonal Perturbations can be defined as follows:

Let P_t represent the Frequency Assignment Map:

$$P_t : \mathbb{C}^n \rightarrow \mathbb{C}^{p+q} \quad (7.8)$$

The Frequency Assignment Map (FAP) associated with the problem, is the map assigning $D = \text{diag}(d_1, d_2, \dots, d_n)$ to the coefficients of the powers of s (p_0, p_1, \dots, p_{p+q}) of the determinant (assuming that the polynomial is *monic*):

$$\det(s^2 \mathbf{L} + s(\mathbf{R} + \mathbf{D}) + \mathbf{C}) = (p_{p+q} \cdot s^{p+q} + \dots + p_1 \cdot s + p_0) \cdot s^{n-q} \quad (7.9)$$

where $n = \text{rank}(\mathbf{D})$, $p = \text{rank}(\mathbf{L})$ and $q = \text{rank}(\mathbf{C})$. \mathbf{D} is the diagonal matrix containing the non- dynamical elements (i.e. resistors) that are added to the network in order to obtain complete frequency assignability, \mathbf{L} is the matrix of inductors and \mathbf{C} is the matrix of capacitors.

7.3.3 Differential of Frequency Assignment Map

The *differential* of the *frequency assignment map* F associated with our problem, plays a very important role in the determination of the onto properties of the map and it has thus a crucial role in the solvability of the problem. The differential can be calculated in many ways; for a general square polynomial matrix $A(s)$ the following results hold [LK09].

Lemma 7.1. *If $A(s)$ is a polynomial matrix then,*

$$\det(A(s) + xB(s)) = \det(A(s)) + x \cdot \text{trace}[\text{adj}(A(s)) \cdot B(s)] + O(x^2).$$

□

Corollary 7.1. *If $\text{adj}(sA + B - \Lambda_0) = \underline{v}(s) \cdot \underline{g}^t(s)$ and $g_i(s), v_i(s)$ are the coordinates of these vectors, then the differential at the degenerate point DF_{Λ_0} can be represented by*

the coefficient matrix of the polynomial vector:

$$(g_1(s)v_1(s), \dots, g_n(s)v_n(s))$$

□

Using the above established results we will now introduce the *differential of an arbitrary RLC network* that has a description given by the general operator $W(s)$:

The differential of the Frequency Assignment Map, formulated above, at a point D_0 will be of the form [LLK16]:

$$\begin{aligned} DP_t|_{D_0}(B) &= \text{Coeef.Vector} [\text{trace} (\text{Adj} (s^2\mathbf{L} + s(\mathbf{R} + \mathbf{D}_0) + \mathbf{C}) \cdot B)] = \\ &= \text{Coeef.Vector} (p_1(s) \cdot \beta_1 + p_2(s) \cdot \beta_2 + \dots + p_n(s) \cdot \beta_n) \end{aligned} \quad (7.10)$$

where p_1, p_2, \dots, p_n are the diagonal entries of $(\text{Adj} (s^2\mathbf{L} + s(\mathbf{R} + \mathbf{D}_0) + \mathbf{C}))$ and B is diagonal.

7.3.4 Arbitrary Frequency Assignment via Diagonal Perturbations - Generic Results and Construction of Solutions

In this section we present the basic results of this chapter. Using the *Dominant Morphism Theorem* [Bor91, HM77, Hum75] we give a generic solution to the frequency assignment problem, by demonstrating an example and we establish sufficient conditions for arbitrary frequency assignment via diagonal perturbations [LLK16]. At this point, we shall note that the *Dominant Morphism Theorem* although proves the existence, is not appropriate for the construction of a solution. To construct the solutions we can utilize the usual methods based on the multi-linear/determinantal formulation and then solve the set of algebraic equations using Gröebner bases [BW93, Wai79] technique.

Generic Solution

Let us consider the set:

$$S_{p,q} = \{\mathbf{L}, \mathbf{R}, \mathbf{C} \in \mathbb{C}^{n \times n} : \text{rank}(\mathbf{L}) = p, \text{rank}(\mathbf{C}) = q, \mathbf{R}, \mathbf{L}, \mathbf{C} \text{ symmetric}\}$$

and also that $n \geq p + q$. For $t \in S_{p,q}$ consider the map: $P_t : \mathbb{C}^n \rightarrow \mathbb{C}^{p+q}$. This map, maps $D = \text{diag}(d_1, d_2, \dots, d_n)$ to the coefficients of the powers of s (p_0, p_1, \dots, p_{p+q}) of the determinant (assuming that the polynomial is *monic*):

$$\det(s^2 \mathbf{L} + s(\mathbf{R} + \mathbf{D}) + \mathbf{C}) = (p_{p+q} \cdot s^{p+q} + \dots + p_1 \cdot s + p_0) \cdot s^{n-q} \quad (7.11)$$

We will use the Dominant Morphism Theorem [Bor91, HM77, Hum75] stated in Chapter 2 to prove that the differential of the Frequency Assignment map defined in subsection 7.3.2 has full rank in an a point D_0 .

Now consider this point D_0 to be:

$$D_0 = \begin{bmatrix} 1 & 0 & \cdots & & & & & & 0 \\ 0 & 2 & & & & & & & \vdots \\ \vdots & & 3 & & & & & & \\ & & & \ddots & & & & & \\ & & & & p & & & & \\ & & & & & \frac{1}{p+1} & & & \\ & & & & & & \frac{1}{p+2} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \frac{1}{p+q} \\ & & & & & & & & & 1 \\ 0 & & & & & & & & & \ddots & 0 \\ 0 & \cdots & & & & & & & & 0 & 1 \end{bmatrix} \quad (7.12)$$

Then the differential $DP_t|_{D_0}$ is an $n \times (p + q)$ matrix depending polynomially at the parameters of $t \in S_{p,q}$. Therefore, the set:

$$S' = \{t \in S_{p,q} : \text{rank}(DP_t|_{D_0}) = p + q\} \quad (7.13)$$

is a *Zarisky open subset* of $S_{p,q}$. For the genericity property to hold, S' has to be nonvoid. To prove that S' is nonvoid it is sufficient to demonstrate an example, such that: $DP_t|_{D_0} = p + q$.

Note: Before we proceed to the example let us explain the *Zarisky topology*.

The *Zarisky topology* is useful for studying polynomial maps between algebraic varieties, as any polynomial map $\chi : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous. It is a topology defined on an algebraic variety \mathcal{X} such that all the closed sets are the subvarieties of \mathcal{X} . The open sets of this topology are $\mathcal{W} = \mathcal{X} \setminus \mathcal{X}'$, where \mathcal{X}' is a subvariety of \mathcal{X} . Hence, if an open set is nonvoid, it means it is the whole of \mathcal{X} apart from a set of measure zero. The above demonstrates that if a property holds $\forall w$ in a nonvoid Zarisky open set, then it holds for almost all $w \in \mathcal{W}$. Thus, such a topology is good for genericity arguments [Lev93].

Example 7.1. Here, we demonstrate an example that proves genericity, where the differential of the frequency assignment map P_t has full rank, i.e. $\text{rank}(DP_t|_{D_0}) = p + q$. Indeed consider the system (or *RLC* network) t_0 with matrices $\mathbf{L}_0, \mathbf{R}_0, \mathbf{C}_0$, where $n \geq p + q$, such that:

$$\begin{aligned}
 s^2 \mathbf{L}_0 + s \mathbf{R}_0 + \mathbf{C}_0 = & \left[\begin{array}{ccc} \underbrace{\begin{array}{c} s^2 \\ s^2 \\ \vdots \\ s^2 \end{array}}_p & & \\ & \underbrace{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}}_q & \\ & & \underbrace{\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array}}_{n-p-q} \end{array} \right] \\
 = & \left[\begin{array}{ccc} \underbrace{\begin{array}{c} s^2 \\ s^2 \\ \vdots \\ s^2 \end{array}}_p & & \\ & \underbrace{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}}_q & \\ & & \underbrace{\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array}}_{n-p-q} \end{array} \right] \quad (7.14)
 \end{aligned}$$

and the point D_0 defined in (7.12). Then,

$$s^2 \mathbf{L}_0 + s \mathbf{R}_0 + \mathbf{C}_0 + s \mathbf{D}_0 = \left[\begin{array}{cccc} \underbrace{\left. \begin{array}{ccc} s^2 + s & & \\ & s^2 + 2s & \\ & & \ddots \\ & & & s^2 + ps \end{array} \right\}^p}_{p} & & & \\ & \underbrace{\left. \begin{array}{ccc} \frac{1}{p+1}s + 1 & & \\ & \frac{1}{p+2}s + 1 & \\ & & \ddots \\ & & & \frac{1}{p+q}s + 1 \end{array} \right\}^q}_{q} & & \\ & & \underbrace{\left. \begin{array}{ccc} s & & \\ & s & \\ & & \ddots \\ & & & s \end{array} \right\}^{n-p}}_{\substack{n-p-q \\ (7.15)}} \end{array} \right]$$

and $\det((s^2 \mathbf{L}_0 + s(\mathbf{R}_0 + \mathbf{D}_0) + \mathbf{C}_0)) = \frac{p!}{(p+q)!} \cdot (s+1)(s+2) \cdots (s+p+q) \cdot s^{n-q} = f(s)$.

Then $DP_t|_{D_0}$ contains the matrix:

$$\begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \vdots \\ \underline{f}_{p+q} \end{bmatrix}$$

$f(s) = \frac{p!}{(p+q)!} (s+1)(s+2) \cdots (s+p+q)$ and \underline{f}_i is the coefficient matrix of the polynomial:

$$f_i(s) = \begin{cases} \frac{f(s)}{s+i}, 1 \leq i \leq p \\ i \cdot \frac{f(s)}{s+i}, p+1 \leq i \leq p+q \end{cases}$$

For this matrix $DP_t|_{D_0}$ to have rank $(p+q)$ is sufficient for the matrix:

$$F = \begin{bmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \vdots \\ \underline{f}_{p+q} \end{bmatrix}$$

to have rank $(p+q)$. Indeed if we call V the $(p+q) \times (p+q)$ Vandermode matrix:

$$V = \begin{bmatrix} (-1)^{p+q-1} & (-2)^{p+q-1} & (-3)^{p+q-1} & \dots & -(p+q)^{p+q-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^2 & 3^2 & \dots & (p+q)^2 \\ -1 & -2 & -3 & \dots & -(p+q) \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (7.16)$$

Then we have: $F \cdot V = \text{Diag}(\sigma_1, \sigma_2, \dots, \sigma_{p+q})$, where:

$$\sigma_i = \begin{cases} \frac{p!}{(p+q)!} \prod_{j=1, j \neq i}^{j=p+q} (i-j), & 1 \leq i \leq p \\ \frac{p!}{(p+q)!} \cdot i \prod_{j=1, j \neq i}^{j=p+q} (i-j), & p+1 \leq i \leq p+q \end{cases} \quad (7.17)$$

As, the matrices $V, \text{Diag}(\sigma_1, \sigma_2, \dots, \sigma_{p+q})$ are invertible so is F , which means that F has rank $(p+q)$ and therefore $DP_t|_{D_0}$ has rank $(p+q)$. \square

Main Theorem

Based on the above results, we present the main theorem of this chapter, which states that the frequency assignment map of our problem is (almost) onto, for an arbitrary network RLC which satisfies the condition $n \geq p+q$.

Theorem 7.1. *Given that $n \geq p+q$, for a general element $t \in S_{p,q}$ the zero assignment map:*

$$P_t : \mathbb{C}^n \rightarrow \mathbb{C}^{p+q}$$

is almost onto (i.e. the image of this map covers the whole \mathbb{C}^{p+q} , apart possibly from a set of measure zero).

□

Proof. Consider the subset S' of S_{p+q} defined below i.e.:

$$S' = \{t \in S_{p,q} : \text{rank}(DP|_{D_0}) = p + q\} \quad (7.18)$$

This is a Zarisky open subset of S_{p+q} and by the *Dominant Morphism Theorem* $\forall t \in S'$ the map: $P_t : \mathbb{C}^n \rightarrow \mathbb{C}^{p+q}$ is almost onto. Since the network t_0 defined as previously has the property: $DP|_{D_0} = p + q$ it implies that $t_0 \in S'$ and therefore S' is nonempty. Consequently, the subset of S_{p+q} such that P_t is not onto is a subset of $(S')^C$, which is contained in a proper sub-variety of S_{p+q} . This proves the theorem. □

Remark 7.1. [LLK16] The necessary and generically sufficient condition to obtain complete frequency assignability, i.e. $n \geq p + q$ arises from the fact that the zero assignment map:

$$P_t : \mathbb{C}^n \rightarrow \mathbb{C}^{p+q}$$

is almost onto when $n \geq p + q$. This can be established from the Dominant Morphism theorem and theorem 7.1. □

7.4 A Cohomology Approach to Frequency Assignment

7.4.1 Introduction

In the previous section we demonstrated that we can assign any frequency to an *RLC* network by addition of resistors, only as long as the number of resistors is equal or exceeds the number of zeros to be assigned, i.e. $n \geq p + q$. We examined the case where complex solutions were allowed. In this section, the number of solutions to the Zero Assignment Problem via diagonal perturbations is computed (for a known polynomial with desired frequencies) by using the cohomology ring H^* introduced in section 2.8.2. The number of solutions to the problem is determined for two cases, i.e. $n = p + q$ and $n > p + q$. The most important reason to calculate the number of solutions of the

Zero Assignment Problem is to determine whether we may have real solutions. Also, the complexity of the problem is related to the number of its solutions.

As explained explicitly in section 2.8.2, we may assign the system of equations defining our problem to a cycle in the cohomology ring H^* and the number of solutions may be determined via the cup product of H^* . H^* is an intersection ring where multiplication corresponds to intersection of varieties, addition to union of varieties and every subvariety coincides to an element in H^* (cycle). The equations defining the Zero Assignment problem are defined on the non compact space C^n , which can be compactified and the number of these solutions can be computed if we subtract the solutions at infinity from the total number of solutions.

7.4.2 Main Results

Let us now examine, in our case, the system of equations that describe the Zero Assignment Problem in *RLC* Networks, using the methodology explained in sections 2.8.1 and 2.8.2.

The Zero Assignment Problem in *RLC* Networks can be written in the form:

$$\det(s^2\mathbf{L} + s(\mathbf{R} + \mathbf{D}) + \mathbf{C}) = p(s) \quad (7.19)$$

where \mathbf{L} is the matrix of inductors, \mathbf{C} is the matrix of capacitors, \mathbf{R} is the matrix of resistors, all assumed to be known. \mathbf{D} defines the matrix of resistors to be assigned to the system to achieve desirable properties. Finally, $p(s)$ is a known polynomial with desired frequencies. The determinantal equation (7.19) can be decomposed as:

$$\det(s^2\mathbf{L} + s(\mathbf{R} + \mathbf{D}) + \mathbf{C}) = p(s) \Leftrightarrow \det \begin{bmatrix} s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}, & s\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} \\ \mathbf{D} \end{bmatrix} = p(s) \quad (7.20)$$

Using Binet-Cauchys theorem [MM64], equation (7.20) can be written as:

$$C_n \begin{bmatrix} s^2\mathbf{L} + s\mathbf{R} + \mathbf{C}, & s\mathbf{I} \end{bmatrix} \cdot C_n \begin{bmatrix} \mathbf{I} \\ \mathbf{D} \end{bmatrix} = p(s) \Leftrightarrow \\ \langle \underline{b}(s), \cdot [1, d_1, d_2, \dots, d_n, d_1d_2, d_1d_3, \dots, d_1d_n, d_1d_2d_3, \dots, \dots, d_1d_2d_3 \cdots d_n] \rangle = p(s) \quad (7.21)$$

If $\mathbf{D} = \text{diag}(d_1, d_2, \dots, d_n)$ is the diagonal matrix of resistors to be assigned and $\delta_\mu = (p + q)$ is the *implicit McMillan degree* of the network, this will result in $(p + q)$ equations with n unknowns of the form:

$$b_1(s) + b_2(s)d_1 + b_3(s)d_2 + \dots + b_k(s)d_1d_2 \cdots d_n = p(s) \quad (7.22)$$

In equation (7.22) we substitute s_1, s_2, \dots, s_{p+q} the roots of the polynomial $p(s)$. Hence, the new system of equations is:

$$\left\{ \begin{array}{l} b_{11} + b_{12}d_1 + b_{13}d_2 + \dots + b_{1k}d_1d_2 \cdots d_n = 0 \\ b_{21} + b_{22}d_1 + b_{23}d_2 + \dots + b_{2k}d_1d_2 \cdots d_n = 0 \\ \vdots \\ b_{(p+q)1} + b_{(p+q)2}d_1 + b_{(p+q)3}d_2 + \dots + b_{(p+q)k}d_1d_2 \cdots d_n = 0 \end{array} \right. \quad (7.23)$$

If we use compactification of \mathbb{C}^n , which is the affine space into $P^n(\mathbb{C})$, using the procedure in section 2.8.1 (i.e. homogenization with parameter λ) we result in excess intersection. Hence, it is evident to use another homogenization (i.e. $\lambda_1, \lambda_2, \dots, \lambda_n$). The new compact space will be: $\underbrace{P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times \dots \times P^1(\mathbb{C})}_{n \text{ times}}$.

To find the number of solutions we cannot use Bezouts theorem (see section 2.8.1), as it holds only for the projective space $P^n(\mathbb{C})$. In order to count the total number of solutions, we need to calculate the cohomology ring of:

$$H^*\left(\underbrace{P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times \dots \times P^1(\mathbb{C})}_{n \text{ times}}\right)$$

To accomplish that, we use Künneth decomposition, i.e.

$$\begin{aligned} H^*\left(\underbrace{P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times \dots \times P^1(\mathbb{C})}_{n \text{ times}}\right) &\simeq \\ \underbrace{H^*(P^1(\mathbb{C})) \otimes H^*(P^1(\mathbb{C})) \otimes \dots \otimes H^*(P^1(\mathbb{C}))}_{n \text{ times}} \end{aligned}$$

where $H^*(P^1(\mathbb{C})) = \frac{\mathbb{Z}[a]}{\{a^2=0\}}$.

Hence, we will have that:

$$\begin{aligned} H^*(P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times \dots \times P^1(\mathbb{C})) &= \frac{\mathbb{Z}[a_1]}{\{a_1^2=0\}} \otimes \frac{\mathbb{Z}[a_2]}{\{a_2^2=0\}} \otimes \dots \otimes \frac{\mathbb{Z}[a_n]}{\{a_n^2=0\}} = \\ &= \frac{\mathbb{Z}[a_1, a_2, \dots, a_n]}{\{a_i^2=0, i=1, 2, \dots, n\}} = \{b_0 + b_1 a_1 + \dots + b_{n+1} a_n + b_{n+2} a_1 a_2 + \dots + b_r a_1 a_2 \dots a_n, a_i^2 = 0, b_i \in \mathbb{Z}\} \end{aligned}$$

Using now the compactification procedure of \mathbb{C}^n i.e.:

$$(d_1, d_2, \dots, d_n) \rightarrow P^1(\mathbb{C})^n : [(\lambda_1, d_1), (\lambda_2, d_2), \dots, (\lambda_n, d_n)]$$

where $P^1(\mathbb{C})^n = \underbrace{P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times \dots \times P^1(\mathbb{C})}_{n \text{ times}}$, we substitute each $d_i \rightarrow \frac{d_i}{\lambda_i}, i = 1, 2, \dots, n$ using another homogenization technique. The new system of (7.23) becomes:

$$\begin{aligned} b_{11} \lambda_1 \lambda_2 \dots \lambda_n + b_{12} d_1 \lambda_2 \lambda_3 \dots \lambda_n + \dots + b_{1k} d_1 d_2 \dots d_n &= 0 \\ b_{21} \lambda_1 \lambda_2 \dots \lambda_n + b_{22} d_1 \lambda_2 \lambda_3 \dots \lambda_n + \dots + b_{2k} d_1 d_2 \dots d_n &= 0 \\ \vdots & \\ b_{(p+q),1} \lambda_1 \lambda_2 \dots \lambda_n + b_{(p+q),2} d_1 \lambda_2 \lambda_3 \dots \lambda_n + \dots + b_{(p+q),k} d_1 d_2 \dots d_n &= 0 \end{aligned} \tag{7.24}$$

At this point, let us introduce the notion of specialisation principle because we will use it. Specialisation principle states that if we change continuously (i.e. in a continuous way) the coefficients of the unknowns of the system of equations, we can simplify the system into new equations, which are assigned into elements $\{a_i\}$ of the cohomology ring. Subsequently, using the algebra of the cohomology ring we are able to count the number of solutions that it might have.

Note: This approach can be adopted only when the equations in (7.24) defining our problem are *independent*. From now we assume that the equations in (7.24) are *independent*.

Now, implementing the specialization principle (to find in which element of the cohomology ring H^* they are assigned) in one of the $(p+q)$ equations of (7.24) (we apply only to one because the rest are similar) we will have:

$$\begin{aligned} b_{11} \lambda_1 \dots \lambda_n + b_{12} d_1 \lambda_2 \lambda_3 \dots \lambda_n + \dots + b_{1k} d_1 d_2 \dots d_n &\xrightarrow{\text{specialisation principle}} 0 \\ (\mu_1 \lambda_1 + \nu_1 d_1) (\mu_2 \lambda_2 + \nu_2 d_2) \dots (\mu_n \lambda_n + \nu_n d_n) &= 0 \end{aligned}$$

Hence, the above will result in:

$$\begin{aligned}
\mu_1 \lambda_1 + \nu_1 d_1 = 0 & \xrightarrow{\text{corresponds}} a_1 \text{ in the cohomology ring} \\
\text{or} \\
\mu_2 \lambda_2 + \nu_2 d_2 = 0 & \xrightarrow{\text{corresponds}} a_2 \text{ in the cohomology ring} \\
\text{or} \\
\vdots \\
\text{or} \\
\mu_n \lambda_n + \nu_n d_n = 0 & \xrightarrow{\text{corresponds}} a_n \text{ in the cohomology ring}
\end{aligned} \tag{7.25}$$

Each one of these transformed equations (7.25) is assigned into: $(a_1 + a_2 + \dots + a_n)$, which is an element of the cohomology ring. The intersection of $(p + q)$ *independent* equations is assigned into the element: $(a_1 + a_2 + \dots + a_n)^{p+q}$.

At this point, we have to compute $(a_1 + a_2 + \dots + a_n)^{p+q}$ by expanding it. The expansion of this identity results in monomials of a_1, a_2, \dots, a_n , whose greatest exponent will be $(p + q)$. Due to the property of the cohomology ring, i.e. $a_i^2 = 0$ any monomial that is of degree equal or greater than 2 will disappear. Therefore, in the above expansion will be included factors of a_i either of exponent 0 or 1. The coefficient of these monomials will then be:

$$\frac{(p + q)!}{1! \cdot 1! \cdots 1!} = (p + q)!$$

Consequently, we will have that:

$$(a_1 + a_2 + \dots + a_n)^{p+q} = (p + q)! \sum a_{i_1} a_{i_2} \cdots a_{i_{p+q}} \tag{7.26}$$

At this point, we will distinguish two cases:

- **Case 1:** $n = (p + q)$

The previous identity, equation (7.26), will be equal to:

$$(a_1 + a_2 + \dots + a_n)^n = n! \cdot a_1 \cdot a_2 \cdots a_n \tag{7.27}$$

and we will have $n!$ points as solutions for our system of equations.

Before we proceed to the second case it is evident to discriminate some points that are crucial for our research:

1. In the case where the system consists of n equations with n unknowns (i.e. the case discussed above), then there will be $(n!)$ solutions and these will be translated into points in $H^*[(P^1(\mathbb{C})) \otimes (P^1(\mathbb{C})) \otimes \dots \otimes (P^1(\mathbb{C}))]$. These points can be counted and as we said are equal $(n!)$.
 2. In the case where $n > (p + q)$, which is case 2, the solution of the system do not result into specific points but results into varieties, whose class in the cohomology ring $H^* \left[\underbrace{(P^1(\mathbb{C})) \otimes (P^1(\mathbb{C})) \otimes \dots \otimes (P^1(\mathbb{C}))}_{n \text{ times}} \right]$ is presented in the next case.
- **Case 2:** $n > (p + q)$

At this case equation (7.26) will be equal to:

$$(a_1 + a_2 + \dots + a_n)^{p+q} = (p + q)! \sum a_{i_1} a_{i_2} \dots a_{i_{p+q}} \quad (7.28)$$

as proved before.

To determine specific solutions that topologically are translated into points we need to intersect the prior equations appropriately with $n - (p + q)$ equations (or surfaces), in order to result in a system of n equations with n unknowns. In other words, to do that we will have to multiply equation (7.28) with a variety of complementary dimension, i.e. $n - (p + q)$. That is:

$$(a_1 + a_2 + \dots + a_n)^{p+q} \cdot a = (p + q)! \left(\sum a_{i_1} a_{i_2} \dots a_{i_{p+q}} \right) \cdot a \quad (7.29)$$

where $a \triangleq$ the cohomology class corresponding to $n - (p + q)$ equations that we need to multiply our identity with, in order to derive solutions that match to points.

The expansion of this identity (7.29) results in monomials of a_1, a_2, \dots, a_n , whose greatest exponent will be $(p + q)$ multiplied by a , where a contains sums of common and uncommon monomials a_i with (7.28). Due to the property of the cohomology ring, i.e. $a_i^2 = 0$ any monomial that is of degree equal or greater than 2 will disappear. Therefore, in the above expansion will be included factors of a_i either

of exponent 0 or 1. Hence, an element of the following form will arise:

$$\lambda \cdot (p+q)! \cdot a_1 \cdot a_2 \cdots a_n \quad (7.30)$$

From this, we conclude that the number of solutions (or topologically the number of points) will be a multiple of $(p+q)!$.

Subsequently, we need to determine the number of solutions at infinity (if there are any). Let us give a short example for a 2-dimensional space and then we will

generalize the result for $H^* \left[\underbrace{(P^1(\mathbb{C})) \otimes (P^1(\mathbb{C})) \otimes \dots \otimes (P^1(\mathbb{C}))}_{n \text{ times}} \right]$.

In the case of the 2-dimensional space $H^* [P^1(\mathbb{C}) \otimes P^1(\mathbb{C})]$ we will have that:

$$\begin{cases} a_0 + a_1 d_1 + a_2 d_2 + a_3 d_1 d_2 = p_1 \\ b_0 + b_1 d_1 + b_2 d_2 + b_3 d_1 d_2 = p_2 \end{cases}$$

We substitute each $d_i \rightarrow \frac{d_i}{\lambda_i}$. Thus, using homogenization of the form:

$\{d_i\} \rightarrow \{d_i, \lambda_i\}$ the previous system will become:

$$\begin{aligned} & \left. \begin{aligned} a_0 + a_1 \frac{d_1}{\lambda_1} + a_2 \frac{d_2}{\lambda_2} + a_3 \frac{d_1 d_2}{\lambda_1 \lambda_2} &= p_1 \\ b_0 + b_1 \frac{d_1}{\lambda_1} + b_2 \frac{d_2}{\lambda_2} + b_3 \frac{d_1 d_2}{\lambda_1 \lambda_2} &= p_2 \end{aligned} \right\} \Rightarrow \\ & \Rightarrow \begin{cases} a_0 \lambda_1 \lambda_2 + a_1 d_1 d_2 + a_2 d_2 \lambda_1 + a_3 d_1 d_2 = p_1 \lambda_1 \lambda_2 \\ b_0 \lambda_1 \lambda_2 + b_1 d_1 d_2 + b_2 d_2 \lambda_1 + b_3 d_1 d_2 = p_2 \lambda_1 \lambda_2 \end{cases} \end{aligned} \quad (7.31)$$

Any solution at infinity will be determined when: $\lambda_1 \cdot \lambda_2 = 0$, which means either $\lambda_1 = 0$ or $\lambda_2 = 0$. Let us work out the case for $\lambda_1 = 0$. The case for $\lambda_2 = 0$ is similar. For $\lambda_1 = 0$ equation (7.31) becomes:

$$\left. \begin{aligned} a_1 d_1 \lambda_2 + a_3 d_1 d_2 &= 0 \\ b_1 d_1 \lambda_2 + b_3 d_1 d_2 &= 0 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} d_1 (a_1 \lambda_2 + a_3 d_2) &= 0 \\ d_1 (a_1 \lambda_2 + a_3 d_2) &= 0 \end{aligned} \right\} \xrightarrow{d_1 \neq 0} \begin{aligned} a_1 \lambda_2 + a_3 d_2 &= 0 \\ a_1 \lambda_2 + a_3 d_2 &= 0 \end{aligned} \quad (7.32)$$

$d_1 \neq 0$, because $(d_i, \lambda_i) = (0, 0)$ is rejected as solution.

If $\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} \neq 0$ we will have one solution i.e.: $(d_2, \lambda_2) = (0, 0)$, which is rejected as well. Therefore, we don't have any solutions at infinity with this homogenization and the case of excess intersection is excluded. All these hold only when the equations are *independent* of each other.

Next, we generalize this result for the case where we have n *independent* equations with n unknowns. To compute if there exist any solutions at infinity, we distinguish two cases:

– **Case 1:** $(p + q) = n$

The equations are of the form:

$$\left\{ \begin{array}{c} a_0 + a_1 d_1 + \dots + a_{n+1} d_n + \dots + a_k d_1 d_2 \cdots d_n = p_1 \\ \vdots \\ \underbrace{b_0 + b_1 d_1 + \dots + b_{n+1} d_n + \dots + b_k d_1 d_2 \cdots d_n = p_2}_{n \text{ equations}} \end{array} \right.$$

Homogenizing, i.e. $\{d_i \rightarrow \frac{d_i}{\lambda_i}\}$ will result in:

$$\left\{ \begin{array}{c} a_0 \lambda_1 \cdots \lambda_n + a_1 d_1 \lambda_2 \cdots \lambda_n + \dots + a_k d_1 d_2 \cdots d_n = p_1 \lambda_1 \cdots \lambda_n \\ \vdots \\ \underbrace{b_0 \lambda_1 \cdots \lambda_n + b_1 d_1 \lambda_2 \cdots \lambda_n + \dots + b_k d_1 d_2 \cdots d_n = p_2 \lambda_1 \cdots \lambda_n}_{n \text{ equations}} \end{array} \right. \quad (7.33)$$

The solutions (if any) at infinity are determined when: $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = 0$. Again we will investigate the case for $\lambda_1 = 0$, whereas the rest are done in a similar way.

For $\lambda_1 = 0$ equation (7.33) becomes:

$$d_1 (z_1 \lambda_2 \cdots \lambda_n + \dots + z_k d_2 d_3 \cdots d_n) = 0$$

We can divide with d_1 as $(d_1, \lambda_1) \neq (0, 0)$. Hence, we have that:

$$(z_1 \lambda_2 \cdots \lambda_n + \dots + z_k d_2 d_3 \cdots d_n) = 0 \quad (7.34)$$

Using the specialization principle equation (7.34) converts into:

$$\begin{array}{ccc}
 (k_1\lambda_2 + k_2d_2) & (k_3\lambda_3 + k_4d_3) & \cdots (k_p\lambda_n + k_{p+1}d_n) = 0 \Leftrightarrow \\
 \downarrow & \downarrow & \downarrow \\
 a_2 & a_3 & a_n \\
 (a_2 + a_3 + \dots + a_n)^n = \sum \frac{n!}{k_1!k_2!\dots k_n!} \cdot a_2^{k_2} a_3^{k_3} \cdots a_n^{k_n} = 0
 \end{array} \tag{7.35}$$

where k_i is the exponent of a_i and $k_2 + k_3 + \dots + k_n = n$. Because now, $\lambda_1 = 0$ (7.35) becomes:

$$a_1 \cdot (a_2 + a_3 + \dots + a_n)^n = a_1 \cdot \sum \frac{n!}{k_1!k_2!\dots k_n!} \cdot a_2^{k_2} a_3^{k_3} \cdots a_n^{k_n} = 0 \tag{7.36}$$

Equation (7.36) is equal to 0 because n has to be partitioned into $(n - 1)$ numbers. This holds if and only if at least one of $k_i = 2$. Now, because of the property of the cohomology ring i.e. $\{a_i^2 = 0\}$ this sum is equal to 0. This shows that the number of solutions at infinity is equal to 0.

The other case to investigate is:

– **Case 2:** $(p + q) < n$

From previous results we know that:

$$(a_1 + a_2 + \dots + a_n)^{p+q} \cdot a = \lambda \cdot (p + q)! \cdot a_1 a_2 \cdots a_n \tag{7.37}$$

where $a \stackrel{\Delta}{=} \text{the cohomology class corresponding to } n - (p + q) \text{ equations that we need to multiply our identity with, in order to derive solutions that match to points.}$

The procedure for defining (if any) the solutions at infinity is exactly the same as in the previous case.

Thus, we have to compute the following:

$$a_1 \cdot (a_2 + a_3 + \dots + a_n)^{p+q} \cdot a \tag{7.38}$$

where $a \stackrel{\Delta}{=} \text{the cohomology class corresponding to } n - (p + q) \text{ equations that have to be considered to derive solutions that match to points.}$

Equation (7.38) is equal to 0 because $n + 1$ factors have to be partitioned

using n symbols and this will result to factors whose exponent will be ≥ 2 and because of the property of the cohomology ring i.e. $\{a_i^2 = 0\}$, the number of solutions at infinity is equal to 0.

To summarize, in the case where $n = (p + q)$, or in other words, the case with n *independent* equations with n unknowns, the number of solutions (or topologically the number of points) of the Zero Assignment Problem is $n!$. In the other case, where $n > (p + q)$ the number of solutions to the problem is given by $\lambda \cdot (p + q)!$, which is a multiple of $(p + q)!$. Thus, in both cases we conclude that solutions exist, but we cannot determine whether these solutions are real.

7.5 Improving Natural Frequencies By Network Redesign: Frequency Assignment, Passivity and The Family Of Strongly Stable Polynomials

It is well known that passive *RLC* networks are stable under certain conditions. What we would like to investigate is whether under zero assignment via diagonal perturbations in an *RLC* network, the zeros of $W(s)$, or equivalently, the poles of $W(s)^{-1}$ can be assigned such that they belong in a certain area

$$A_\varphi = \{z_i = (-\varphi_i \pm i\vartheta_i) \in \mathbb{C} : \text{Re}(z_i) < -\varphi, \varphi > 0, i = 1, 2, \dots, n\}$$

of the stability region S , with $A_\varphi \subset S$.

7.5.1 Preliminary Analysis and Results

First, we will investigate the case of a polynomial of degree 2, thus $n = 2$. Let $t_2(s) = s^2 + a_1s + a_0$. Let $-\varphi_1, -\varphi_2$ be the roots of $t_2(s)$. We want to place the roots of $t_2(s)$ in A_φ , i.e. $-\varphi_1, -\varphi_2 \in A_\varphi$.

Let us investigate the following cases:

Case 1: 2 Real Roots, $-\varphi_1, -\varphi_2$ The polynomial $t_2(s)$ can be factored as follows:

$$\begin{aligned} t_2(s) &= s^2 + a_1s + a_0 = (s + \varphi_1)(s + \varphi_2) = \\ &= s^2 + (\varphi_1 + \varphi_2)s + (\varphi_1 \cdot \varphi_2) \end{aligned} \quad (7.39)$$

Hence, for $(-\varphi_1, -\varphi_2) \in A_\varphi$ we want:

$$(a) \ a_1 > 2\varphi$$

$$(b) \ a_0 > \varphi^2$$

Case 2: 2 Complex Conjugate Roots, $(-\varphi_1 + i\vartheta_1), (-\varphi_1 - i\vartheta_1)$

The polynomial $t_2(s)$ can be factored as follows:

$$\begin{aligned} s^2 + a_1s + a_0 &= (s + \varphi_1 + i\vartheta_1)(s + \varphi_1 - i\vartheta_1) = \\ &= s^2 + 2\varphi_1s + (\varphi_1^2 + \vartheta_1^2) \end{aligned} \quad (7.40)$$

where $\varphi_1 > \varphi$.

Hence, for $-\varphi_1 \pm i\vartheta_1 \in A_\varphi$ we want:

$$(a) \ a_1 > 2\varphi$$

$$(b) \ a_0 > \varphi^2$$

For $n = 3$, where $t_3(s) = s^3 + a_2s^2 + a_1s + a_0$ we will examine the following cases:

Case 1: 3 Real Roots, $-\varphi_1, -\varphi_2, -\varphi_3$. In this case $t_3(s)$ can be factored as follows:

$$\begin{aligned} t_3(s) &= s^3 + a_2s^2 + a_1s + a_0 = (s + \varphi_1)(s + \varphi_2)(s + \varphi_3) = \\ &= s^3 + (\varphi_1 + \varphi_2 + \varphi_3)s^2 + (\varphi_1 \cdot \varphi_2 + \varphi_1 \cdot \varphi_3 + \varphi_2 \cdot \varphi_3)s + \\ &\quad + (\varphi_1 \cdot \varphi_2 \cdot \varphi_3) \end{aligned} \quad (7.41)$$

Thus, the conditions under which $-\varphi_1, -\varphi_2, -\varphi_3 \in A_\varphi$ are:

$$(a) \ a_2 > 3\varphi$$

$$(b) \ a_1 > 3\varphi^2$$

$$(c) \ a_0 > \varphi^3$$

Case 2: 1 Real Root φ_1 and 2 Complex Conjugate Roots $(\varphi_2 + i\vartheta_2), (\varphi_2 - i\vartheta_2)$

In this case $t_3(s)$ can be factored as follows:

$$\begin{aligned} s^3 + a_2s^2 + a_1s + a_0 &= (s + \varphi_1)(s + \varphi_2 + i\vartheta_2)(s + \varphi_2 - i\vartheta_2) = \\ &= s^3 + (\varphi_1 + 2\varphi_2)s^2 + (2\varphi_1 \cdot \varphi_2 + \varphi_2^2 + \vartheta_2^2)s + \\ &\quad + \varphi_1 \cdot (\varphi_2^2 + \vartheta_2^2) \end{aligned} \quad (7.42)$$

From the above it is obvious that for $-\varphi_1 \in A_\varphi$ and $-\varphi_2 \pm i\vartheta_2 \in A_\varphi$:

$$(a) \ a_2 > 3\varphi$$

$$(b) \ a_1 > 3\varphi^2$$

$$(c) \ a_0 > \varphi^3$$

General Case for a polynomial with n roots

From the analysis so far, we can generalize these results for a polynomial t_n with n roots.

We demonstrate the *necessary* conditions under which $-\varphi_i \in A_\varphi$. These conditions relate the coefficients of the powers of s of the polynomial with the roots $-\varphi_i \pm i\vartheta_i$.

In general let $t_n(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_1s + a_0$ be a polynomial of n degree with n roots, real or complex conjugate, depending whether n is odd or even.

Theorem 7.2. *Let a polynomial $t_n(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_1s + a_0$ of n degree, with roots $(-\varphi_i \pm i\vartheta_i, i = 1, 2, \dots, n)$. The necessary conditions for $-\varphi_i \in A_\varphi$ are:*

$$\begin{aligned} a_{n-1} &> n \cdot \varphi \\ a_{n-2} &> \binom{n}{2} \cdot \varphi^2 \\ a_{n-3} &> \binom{n}{3} \cdot \varphi^3 \\ &\vdots \\ a_1 &> \binom{n}{n-1} \cdot \varphi^{n-1} \\ a_0 &> \varphi^n \end{aligned} \quad (7.43)$$

□

7.5.2 Inequalities and Zero Assignment Problem

In this section, we implement the previous analysis in an *RLC* network under zero assignment via diagonal perturbations. We establish the *necessary* conditions under which the zeros of $W(s)$ are placed in A_φ of the stability region S , with $A_\varphi \subset S$.

We need to investigate the solvability of the following determinantal equation with respect to a polynomial $p(s)$, using the analysis from the previous sections.

$$\det [s^2L + s(R + D) + C] = p(s) \Leftrightarrow \det \left[\begin{bmatrix} s^2L + s(R + D) + C, & sI \end{bmatrix} \cdot \begin{bmatrix} I \\ D \end{bmatrix} \right] = p(s) \quad (7.44)$$

Using Binet-Cauchy theorem [MM64] the previous equation can be written as:

$$C_n \begin{bmatrix} s^2L + s(R + D) + C, & sI \end{bmatrix} \cdot C_n \begin{bmatrix} I \\ D \end{bmatrix} = p(s) \Leftrightarrow \langle \underline{b}(s) \cdot (1, d_1, \dots, d_n, d_1d_2, d_1d_3, d_1d_n, d_1d_2d_3, \dots, d_1d_2d_3\dots d_n) \rangle = p(s) \quad (7.45)$$

If $D = \text{diag}(d_1, \dots, d_n)$ is the diagonal matrix of resistors to be assigned and $\delta_M = (p+q)$ is the implicit McMillan degree of the network, this will result in $(p+q)$ equations with n unknowns of the form:

$$b_1(s) + b_2(s)d_1 + b_3(s)d_2 + \dots + b_k(s)d_1d_2 \dots d_n = p(s) \quad (7.46)$$

Let,

$$\begin{aligned} b_1(s) &= b_{1,(p+q)} \cdot s^{(p+q)} + b_{1,(p+q-1)} \cdot s^{(p+q-1)} + \dots + b_{1,0} \cdot s^0 \\ b_2(s) &= b_{2,(p+q)} \cdot s^{(p+q)} + b_{2,(p+q-1)} \cdot s^{(p+q-1)} + \dots + b_{2,0} \cdot s^0 \\ &\vdots \\ b_k(s) &= b_{k,(p+q)} \cdot s^{(p+q)} + b_{k,(p+q-1)} \cdot s^{(p+q-1)} + \dots + b_{k,0} \cdot s^0 \end{aligned}$$

and

$$p(s) = p_{(p+q)} \cdot s^{(p+q)} + p_{(p+q-1)} \cdot s^{(p+q-1)} + \dots + p_0 \cdot s^0 \quad (7.47)$$

From equation (7.44) and (7.47) in order for the equality to hold true, we need:

$$\begin{aligned}
 [b_{1,(p+q)} + b_{2,(p+q)}d_1 + b_{3,(p+q)}d_2 + \dots + b_{k,(p+q)}d_1d_2\dots d_n] &= p_{(p+q)} \\
 [b_{1,(p+q-1)} + b_{2,(p+q-1)}d_1 + b_{3,(p+q-1)}d_2 + \dots + b_{k,(p+q-1)}d_1d_2\dots d_n] &= p_{(p+q-1)} \\
 &\vdots \\
 [b_{1,0} + b_{2,0}d_1 + b_{3,0}d_2 + \dots + b_{k,0}d_1d_2\dots d_n] &= p_0
 \end{aligned}$$

and

$$1 \cdot s^{(p+q)} + \frac{p_{(p+q-1)}}{p_{(p+q)}} \cdot s^{(p+q-1)} + \frac{p_{(p+q-2)}}{p_{(p+q)}} \cdot s^{(p+q-2)} + \dots + \frac{p_0}{p_{(p+q)}}$$

Thus, based on the inequalities presented in theorem 7.2 , the following should hold:

Let us assume that $p_{(p+q)} > 0$, then:

$$\begin{aligned}
 \frac{p_{(p+q-1)}}{p_{(p+q)}} > (p+q) \cdot \varphi &\Leftrightarrow p_{(p+q-1)} > p_{(p+q)} \cdot (p+q) \cdot \varphi \\
 \frac{p_{(p+q-2)}}{p_{(p+q)}} > \binom{p+q}{2} \cdot \varphi^2 &\Leftrightarrow p_{(p+q-2)} > p_{(p+q)} \cdot \binom{p+q}{2} \cdot \varphi^2 \\
 &\vdots \\
 \frac{p_0}{p_{(p+q)}} > \varphi^{(p+q)} &\Leftrightarrow p_0 > p_{(p+q)} \cdot \varphi^{(p+q)}
 \end{aligned} \tag{7.48}$$

Hence,

$$\begin{aligned}
 (b_{1,(p+q-1)} + b_{2,(p+q-1)}d_1 + b_{3,(p+q-1)}d_2 + \dots + b_{k,(p+q-1)}d_1d_2\dots d_n) &> p_{(p+q)} \cdot (p+q) \cdot \varphi \\
 &\vdots \\
 (b_{1,0} + b_{2,0}d_1 + b_{3,0}d_2 + \dots + b_{k,0}d_1d_2\dots d_n) &> p_{(p+q)} \cdot \varphi^{(p+q)}
 \end{aligned}$$

so, the following inequalities are derived:

$$\begin{aligned}
 (b_{1,(p+q)} + b_{2,(p+q)}d_1 + b_{3,(p+q)}d_2 + \dots + b_{k,(p+q)}d_1d_2\dots d_n) &> 0 \\
 (b_{1,(p+q-1)} + b_{2,(p+q-1)}d_1 + b_{3,(p+q-1)}d_2 + \dots + b_{k,(p+q-1)}d_1d_2\dots d_n) &> \\
 > (p+q) \cdot \varphi \cdot (b_{1,(p+q)} + b_{2,(p+q)}d_1 + b_{3,(p+q)}d_2 + \dots + b_{k,(p+q)}d_1d_2\dots d_n) & \\
 &\vdots \\
 (b_{1,0} + b_{2,0}d_1 + b_{3,0}d_2 + \dots + b_{k,0}d_1d_2\dots d_n) &> \\
 > \varphi^{(p+q)} \cdot (b_{1,(p+q)} + b_{2,(p+q)}d_1 + b_{3,(p+q)}d_2 + \dots + b_{k,(p+q)}d_1d_2\dots d_n) &
 \end{aligned} \tag{7.49}$$

Hence, the following theorem can be established:

Theorem 7.3. *Let an arbitrary RLC network under zero assignment via diagonal perturbations (where non dynamical elements are added to the network), described by the Implicit Network Operator $W(s)$. For a given $\varphi > 0$, the necessary conditions to assign the zeros of $W(s)$ in the region A_φ are given by the inequalities in (7.49). These inequalities have to be solved with respect to d_i , $i = 1, 2, \dots, n$. \square*

To define the space of solutions within which inequalities hold, is essential to use tools from Semi-Algebraic Geometry [Cos02], but this is out of the scope of this thesis. In the next section we illustrate the above results in an example.

7.5.3 Example

The results generated in the previous sections are presented below in an example.

Let an RLC network with 3 resistors, 1 inductor and 1 capacitor. In every loop we want to add a resistor, which will not be a common element for two loops. The network is illustrated below:

Thus, in this case we will have: $n = 3$ is the number of resistors to add, $p = \text{rank}(L) = 1$ is the rank of the matrix of inductors and $q = \text{rank}(C) = 1$ is the rank of the matrix of capacitors.

The network can be described by the implicit network operator $W(s)$ which is formulated below:

$$W(s) = \frac{1}{s} [s^2 \mathbf{L} + s(\mathbf{R} + \mathbf{D}) + \mathbf{C}]$$

where \mathbf{L} is the matrix of inductors of the form: $\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with $L = 1H$,

\mathbf{C} the matrix of capacitors: $\mathbf{C} = \begin{bmatrix} C^{-1} & 0 & -C^{-1} \\ 0 & 0 & 0 \\ -C^{-1} & 0 & C^{-1} \end{bmatrix}$ with $C = 1F$, \mathbf{R} the ma-

trix of resistors: $\mathbf{R} = \begin{bmatrix} R_1 & -R_1 & 0 \\ -R_1 & R_1 + R_2 & -R_2 \\ 0 & -R_2 & R_2 + R_3 \end{bmatrix}$ with $R_1 = R_2 = R_3 = 1\Omega$ and

finally \mathbf{D} represents the diagonal matrix of resistors we are adding to the network:

$$\mathbf{D} = \text{diag}(D_1, D_2, D_3).$$

Hence, the network operator will have the following form:

$$W(s) = \frac{1}{s} \left[s^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + s \begin{pmatrix} 1 + D_1 & -1 & 0 \\ -1 & 2 + D_2 & -1 \\ 0 & -1 & 2 + D_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \right]$$

The determinant of $W(s)$ can be written as:

$$W(s) = \frac{1}{s} (p_2 s^2 + p_1 s + p_0) \quad (7.50)$$

and based on the previous results we will have that:

$$\begin{aligned} p_2 &= 2 + 2d_1 + d_3 + d_1 d_3 \\ p_1 &= 4 + 4d_1 + 2d_2 + 2d_1 d_2 + 2d_3 + 2d_1 d_3 + d_2 d_3 + d_1 d_2 d_3 \\ p_0 &= 2 + 2d_1 + 3d_2 + d_1 d_2 + 2d_3 + d_2 d_3 \end{aligned}$$

with the following inequalities to hold true: $\frac{p_1}{p_2} > 2\varphi$, $\frac{p_0}{p_2} > \varphi^2$ and $d_1, d_2, d_3 > 0$.

Analyzing the first inequality will result in:

$$\begin{aligned} \frac{p_1}{p_2} > 2\varphi &\Leftrightarrow \\ \frac{4+4d_1+2d_2+2d_1d_2+2d_3+2d_1d_3+d_2d_3+d_1d_2d_3}{2+2d_1+d_3+d_1d_3} > 2\varphi &\Leftrightarrow \\ 4 + 4d_1 + 2d_2 + 2d_1d_2 + 2d_3 + 2d_1d_3 + d_2d_3 + d_1d_2d_3 > 2\varphi(2 + 2d_1 + d_3 + d_1d_3) & \\ &(7.51) \end{aligned}$$

and the second:

$$\begin{aligned} \frac{p_0}{p_2} > \varphi^2 &\Leftrightarrow \\ \frac{2+2d_1+3d_2+d_1d_2+2d_3+d_2d_3}{2+2d_1+d_3+d_1d_3} > \varphi^2 &\Leftrightarrow \\ 2 + 2d_1 + 3d_2 + d_1d_2 + 2d_3 + d_2d_3 > \varphi^2(2 + 2d_1 + d_3 + d_1d_3) & \\ &(7.52) \end{aligned}$$

For $\varphi = 1$ equations (7.51) and (7.52) become:

$$\begin{aligned} 4 + 4d_1 + 2d_2 + 2d_1d_2 + 2d_3 + 2d_1d_3 + d_2d_3 + d_1d_2d_3 &> 4 + 4d_1 + 2d_3 + 2d_1d_3 \Leftrightarrow \\ 2d_2 + 2d_1d_2 + d_2d_3 + d_1d_2d_3 &> 0 \end{aligned} \quad (7.53)$$

and

$$\begin{aligned} 2 + 2d_1 + 3d_2 + d_1d_2 + 2d_3 + d_2d_3 &> 2 + 2d_1 + d_3 + d_1d_3 \Leftrightarrow \\ 3d_2 + d_1d_2 + d_3 + d_2d_3 - d_1d_3 &> 0 \end{aligned} \quad (7.54)$$

To conclude the inequalities that are derived for this example are:

- (a) $2d_2 + 2d_1d_2 + d_2d_3 + d_1d_2d_3 > 0$
- (b) $3d_2 + d_1d_2 + d_3 + d_2d_3 - d_1d_3 > 0$
- (c) $2 + 2d_1 + d_3 + d_1d_3 > 0$
- (d) $d_1, d_2, d_3 > 0$

7.6 Conclusions

The problem of zero assignment via diagonal perturbations for an *RLC* network with general operator $W(s)$ has been considered in section 7.3. We examined the case where non-dynamical elements were added to the network to attain complete frequency assignability. The results established show that we can assign any frequency to a passive electrical network by adding resistors only as long as the number of resistors added is equal or exceeds the number of zeros that need to be assigned (or the McMillan degree of the network) when the sufficient condition is met. We proved that the sufficient condition, i.e. the differential of the algebraic map DF_x has full rank (equals to $p + q$) and that happens in general when $n \geq p + q$ and thus for every *RLC* network with that condition.

Furthermore, in section 7.4 after proving that the problem of zero assignment via diagonal perturbations can be solved generically for *RLC* networks that satisfy the condition $n \geq p + q$, we tried to determine the number of solutions to this problem for a known polynomial with desired frequencies. The polynomial equations describing our problem were defined on the affine space \mathbb{C}^n and by using homogenization we compactified \mathbb{C}^n into the projective space $P^n(\mathbb{C})$. By utilizing Bezout's theorem we resulted in excess intersection. By using another homogenization, the new compact space resulted to be the $P^1(\mathbb{C}^n)$. The total number of solutions of our problem was determined by calculating the cohomology ring of the new compactified space. We distinguished to different

cases. For the case where $n = p + q$ (when we have n independent equations with n unknowns) the number of solutions to the problem was $n!$. In the case where $n > p + q$, the solutions were equal to $\lambda \cdot (p + q)!$. We conclude that in both cases solutions exist to the problem but we cannot determine whether these solutions are real, as the number of solutions in both cases is even.

Finally, in section 7.5, the case of zero assignment in an *RLC* network via diagonal perturbations for natural frequency improvements was examined. We established the necessary conditions for the zeros of the Implicit Network Operator $W(s)$ to be assigned in an area A_φ of the stability region S , when resistors are added to the network. This conditions were given in terms of the inequalities in (7.49), which have to be solved with respect to d_i , $i = 1, 2, \dots, n$.

Chapter 8

Conclusions and Future Research

8.1 Conclusions

The thesis introduces the basic system properties of the *Implicit Network Description* provided by the integral-differential operator $W(s)$, representing the impedance or admittance models of *RLC* networks, without inputs or outputs and defines the network re-engineering transformations and their effect on structure assignment problems. This is an entirely new area of research emerging as a special case of the general problem of re-engineering systems. The main contribution is the specification of a new research area in network theory, which is different than the traditional problems of RLC realisations of the transfer functions. In particular, the achievements are in the following areas:

1. Derivation of the Implicit Network models in the form of integral-differential models and their corresponding equations. Such models are based on defining independent set of loops, correspondingly nodes, which in turn characterize the *loop* or *nodal cardinality*. The analysis uses the model of the smallest of the two cardinalities and the relationship between the admittance and impedance models is investigated. Such models introduce two additional network topologies defined by the loop or nodal structure in addition to the standard network graph topology. The generic Implicit model is described by the symmetric operator $W(s)$, which has the form of $W(s) = s\mathbf{L} + \frac{1}{s}\mathbf{C} + \mathbf{R}$, where the triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$ defines the topology and values of inductances, resistances and capacitances, respectively for the case of impedance modelling. It is this triple that completely characterizes the

topology and values of impedance modelling. Similar expression as a triple is used for the case of admittance modelling.

2. A number of fundamental system properties are examined in the thesis, such as:
 - (i) The notion of *regularity* of the network, that is invertibility of the $W(s)$ operator, which is strongly related with the notion of *connectivity* of the network. This property is equivalent to the existence of transfer functions for oriented models, i.e. models with inputs and outputs.
 - (ii) The *Implicit Network Description* gives rise to a matrix pencil representation of the network, which is not necessarily minimal but has the advantage that it preserves the natural loop or nodal topology as this is expressed by the corresponding triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$. Issues of regularity and issues concerning the zero structure of the matrix pencil representation were examined using results derived for the characterization of infinite elementary divisors and cmi [KK86], utilizing Toeplitz matrices based on the triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$.
 - (iii) The *Implicit McMillan degree* δ_m is defined and it is connected with the basic properties of the corresponding graph topology. This study also indicates the redundancy that may exist in the matrix pencil linearization of the network. Necessary and sufficient conditions were also derived linking the Implicit McMillan degree δ_m with the rank properties of the triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$.
3. The general problem of network re-engineering has been defined and the corresponding transformations have been expressed as additive perturbations or structural augmentation/ reduction of the corresponding $W(s)$ operator. All these transformations may be equivalently expressed on the triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$. Specifically:
 - (i) Changing the values or the nature of elements or changes in the network topology without affecting the corresponding network cardinality are expressed as additive transformations on $W(s)$ or on the corresponding triple $(\mathbf{L}, \mathbf{R}, \mathbf{C})$.
 - (ii) The problem of network augmentation or reduction is expressed as augmentation or reduction of $W(s)$ with preservation of the symmetry of the respective operator. Within this framework, the problem of identifying *fixed dynamics*, that is dynamics which remain invariant under the transformations is addressed and result leading to their identification is given. A by-product of this analysis is the definition of the problem of partial network re-engineering.

This is expressed by defining an operator that characterizes the emerged modified dynamics.

4. The problem of tuning the natural frequencies of an *RLC* network under re-engineering transformations is addressed. We focus ourselves to a special problem, which is tuning natural frequencies by altering the non-dynamical elements, i.e. resistances. Other types of transformations maybe expressed in a similar way.
 - (i) Such transformations maybe expressed as diagonal additive perturbations and properties of the resulting Frequency Assignment map P_t are investigated. It has been shown that a sufficient condition for the complex solvability of the problem is the map to be surjective [HM77] and we prove that this is true when the number of resistors to be assigned is equal or exceeds the Implicit McMillan degree δ_m of the network, i.e. $n \geq p + q$.
 - (ii) We have used tools from algebraic geometry [Mum76] and intersection theory [Ful84], i.e. the cohomology ring of a projective space as a computational tool, which leads to determining the number of solutions for the Zero Assignment problem in *RLC* networks via diagonal perturbations for a known polynomial with desired frequencies. The results show that for the the case where the number of resistors added to network is equal to the *Implicit McMillan degree* δ_m , i.e. $n = p + q$, the number of solutions is equal to $n!$, whereas for the case where the number of resistors added to the network exceeds the network's McMillan degree δ_m , i.e. $n > p + q$, the number of solutions is a multiple of $(p + q)!$. Hence, by following this approach we conclude that in both cases we have solutions, but we cannot conclude whether these are real solutions.
 - (iii) Given that the *RLC* network is passive, network transformations always lead to stable natural frequencies. Thus, the important problem is to assign the natural frequencies in a certain area of the stability region. We establish necessary conditions under which the natural frequencies of the *RLC* network are assigned in this area. These necessary conditions are given in terms of inequalities relating the resistors added to the network with the coefficients of the target polynomial.
 - (iv) The study of partial re-engineering provides the means for studying changes in the nature or value of any single dynamic or non-dynamic element by defining the corresponding operator as a function of a single variable. This

is equivalent to the results in [BHK12], where the problem is reduced to a standard Root Locus problem.

8.2 Future Research Work

1. **Further Network Research:** The current research has dealt with the re-engineering of *RLC* networks, and has opened up the road for further work within the current, as well as related problem areas. Issues related to this area deal with problems such as:

- (a) **Minimality of pencil realisation:** We need to define pencil realisations that preserve the loop or nodal topology of the network and have no redundancy.
- (b) **Relation between topologies:** The exact nature of the links between the natural graph topology of the network and the respective loop or nodal topologies as these are introduced by the corresponding models must be defined.
- (c) **Modified loop or nodal analysis and impedance / admittance modelling:** It has been shown recently [BKLew] that the modified nodal analysis has the potential to provide links to the natural graph topology, thus, establishing links of the latter to our modelling setup is required.
- (d) **Oriented Network Descriptions:** The current *Implicit Network Description* has no inputs or outputs. Defining sets of inputs and outputs for such models introduces orientation and expresses evolution leading to assignment of structural properties [KV02a] of the resulting system and transfer functions. Issues to be considered are:
 - i Defining the McMillan degree of the resulting oriented transfer function.
 - ii Computing the resulting finite and infinite zeros.
 - iii Evaluating controllability and observability properties and corresponding indices as well as Forney structural invariants [For75].

Note that each one of the above problems is associated with structure assignment problems where the selection of the set of inputs, outputs define the design parameters.

-
- (e) **Natural Frequency Assignment by Re-engineering:** This problem has been partially addressed and further topics requiring attention are:
- i Use of Global Linearisation methodology [LK95b] and the non-symmetric linearised pencil to investigate frequency assignment.
 - ii Extend results to dynamical elements, i.e. capacitances and/ or inductances re-engineering with changes or no changes to the corresponding cardinalities.
 - iii Develop properly the framework for assignment of the natural frequencies and in particular, the assignment of the frequency with the smallest value by properly setting up the problem within the framework of semi-algebraic geometry [Cos02].
 - iv Computational methods for Frequency Assignment: Use of Gröebner bases [BW93] for computing solutions to the non-linear set of equations of the Frequency Assignment Problem as well as further extending the derived sufficient conditions for solvability.
- (f) **System Simplification:** The system representation introduced by the *Implicit Network Description* provides a framework for discussing evolution of system properties under assumptions of simplification of the modelling. This involves the development of families of models when specific physical elements are ignored due to assumptions of negligible significance of elements, such as resistances, capacitances, inductances which in turn lead to nests of network models of variable complexity. Examining evolution of system properties within this nest of models remains an open question.
- (g) **Development of Dual models:** The development of analogue models between different physical domains remains a challenge. For systems beyond those of scalar impedance/ admittance descriptions. Extending such duality for matrix models in the case of cardinality greater than one is an open issue, in fact, this is a topic examining duality between models where loop cardinality becomes nodal and vice-versa.

2. **The general Re-engineering problem:** We have started with the need of developing a general methodology for re-engineering of general systems. Within this area, a major challenge remains the development of an appropriate *representation framework* that allows study of *evolution of system properties* as functions of

the introduced transformations. Such a framework is currently missing. However, some possible directions addressing partial problems are as follows:

- (a) **Re-engineering of input-output structures:** This topic is under investigation and relates the development of methods for selection of systems of sensors and actuators. Within this area, we have the problems of zero assignment by input-output squaring down [[LK08](#), [KG89](#), [KG84](#)]. This area of course assumes that the composition rule, i.e. interconnection topology that leads to the system formation, is fixed. For this case, we look for re-engineering of input-output structures.
- (b) **Interconnection topology Re-engineering:** The representation of composite systems introduced in [[Kar96](#)] introduces a description of the system as an action of the interconnection topology on the aggregate system. This has the potential to provide a framework for studying problems of re-engineering of topology, modifying sub-systems as well as redesigning the local sub-system level, local input-output structures. This involves a number of challenging problems and remains open.

Appendix A

Classification of Pure Elements

The two-terminal elements considered here are characterized by functional relationships between their through and across variables and they are represented in terms of branches of linear graphs. The nature of these functional relations introduces a coloring of these branches, which in turn provides a detailed structure of the resulting topological structure.

The ideal lumped elements are classified as energy-storage and dissipation elements. The mass, inertia and capacitance store energy by virtue of their across- variables (velocity, voltage) and they are referred to as *A*-type energy storage units [Kar11]. Springs and inductances store energy by virtue of their through- variables and are called *T*-type energy storage devices. The dampers and resistances dissipate energy and will be called *D*-type elements.

Table 1 summarizes the energy- storage and dissipation functions for the different types of ideal elements:

The behavior of all the ideal mechanical, electrical, fluid and thermal elements can be described by a single set of three elemental equations for *A*-type, *T*-type and *D*-type elements, written in terms of the generalized through and across variables f and V . The table above provides a summary of these relations, as well as expresses the related energy. The abstract elements used are:

C : stands for mass, inertia, electrical capacitance, fluid capacitance and thermal capacitance.

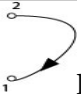
Classification of elements		
A-TYPE ELEMENTS	T-TYPE ELEMENTS	D-TYPE ELEMENTS
<ul style="list-style-type: none"> • Translational mass • Inertia • Electrical capacitance • Fluid capacitance • Thermal capacitance 	<ul style="list-style-type: none"> • Translational spring • Rotational spring • Inductance • Fluid inertance 	<ul style="list-style-type: none"> • Translational damper • Rotational damper • Electrical resistance • Fluid resistance • Thermal resistance
$f = C \frac{dV_{21}}{dt}$	$V_{21} = L \frac{df}{dt}$	$V_{21} = Rf$
$V_{21} = \frac{1}{C} \int_0^t f dt + (V_{21})_0$	$f = \frac{1}{L} \int_0^t V_{21} dt + f_0$	$f = \frac{1}{R} V_{21}$
$f = CpV_{21}$	$V_{21} = Lpf$	$V_{21} = Rf$
$V_{21} = \frac{1}{Cp} f$	$f = \frac{1}{Lp} V_{21}$	$f = \frac{1}{R} V_{21}$
$E_a = \frac{1}{2} C (V_{21})^2$	$E_t = \frac{1}{2} L f^2$	$P = \frac{V_{21}^2}{R}$
 linear digraph V_{21}, f		

TABLE 1: Classification of pure elements

L : stands for the reciprocal of spring constant, or inductance or fluid inertance.

R : stands for reciprocal damping, or electrical resistance, fluid resistance, thermal resistance.

The above classification is used in defining the topology of the network.

References

- [AV73] B.D.O Anderson and S. Vongpanitlerd. *Network analysis and synthesis*, volume 4. Prentice-Hall, Inc, Englewood Cliffs, N. J., 1973.
- [BB81] R. Brockett and C. Byrnes. Multivariable Nyquist criteria, root loci, and pole placement: A geometric viewpoint. *IEEE Transactions on Automatic Control*, 26(1):271–284, feb 1981.
- [BD49] R. Bott and R. J. Duffin. Impedance synthesis without use of transformers. *Journal of Applied Physics*, 20(8):816, 1949.
- [Bel68] Vitold Belevitch. *Classical network theory*. Holden-Day, San Fransisco, CA, 1968.
- [BHK12] Thomas Berger, George Halikias, and Nicos Karcianas. Effects of dynamic and non-dynamic element changes in RC and RL networks. *International Journal of Circuit Theory and Applications*, 43(1):36–59, 2012.
- [BJ66] F. H. Branin Jr. The Algebraic-Topological Basis for Network Analogies and the Vector Calculus. *Proceedings of the Symposium on Generalized Networks*, 16:453–491, 1966.
- [BKLew] T. Berger, N. Karcianas, and M. Livada. The mcmillan degree of implicit transfer functions of rlc networks. *Circuits, Systems & Signal Processing (CSSP)*, (Under review).
- [Bor91] Borel Armand. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer - Verlag, New York, NY, 1991.
- [Bru31] O. Brune. Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency. *J. Maths. Phys.*, 10:191–236, 1931.

- [Bry58] P. R. Bryant. A Topological Investigation of Network Determinants. *The Institution of Electrical Engineers*, (312):16–22, 1958.
- [Bry59] P. R. Bryant. The Order of Complexity of Electrical Networks. *Proceedings of the Institution of Electrical Engineers, Monographs*, 106(10):174, 1959.
- [BSST09] RL Brooks, CAB Smith, AH Stone, and WT Tutte. The dissection of rectangles into squares. In *Classic Papers in Combinatorics*, pages 88–116. Springer, 2009.
- [BW93] Thomas Becker and Volker H. Weispfenning. *Groebner Bases: A Computational Approach to Commutative Algebra*. Springer - Verlag, 1993.
- [Cau58] Wilhelm Cauer. *Synthesis of linear communication networks*, volume 1. McGraw-Hill, 1958.
- [CD82] Frank M. Callier and Charles A. Desoer. *Multivariable feedback systems*. Springer-Verlag, 1982.
- [Cos02] Michel Coste. An introduction to semialgebraic geometry. *RAAG network school*, 145, 2002.
- [CWZ16] Michael Z. Q. Chen, Kai Wang, and Yun Zou. A generalized theorem of reichert for biquadratic minimum functions. *International Journal of Circuit Theory and Applications*, 44(10):1840–1858, 2016. CTA-15-0163.R1.
- [Dar99] S. Darlington. A history of network synthesis and filter theory for circuits composed of resistors, inductors and capacitors. *IEEE Transactions on Circuits Syst. I*, 46(1):4–13, 1999.
- [DLM88] J. Descusse, J.F. Lafay, and M. Malabre. Solution to Morgan’s problem. *IEEE Transactions on Automatic Control*, 33(8):732–739, 1988.
- [For75] G. David Forney, Jr. Minimal Bases of Rational Vector Spaces, with Applications to Multivariable Linear Systems. *SIAM Journal on Control*, 13(3):493–520, may 1975.
- [Fra25] Philip Franklin. The electric currents in a network. *Studies in Applied Mathematics*, 4(1-4):97–102, 1925.
- [Ful84] William. Fulton. *Intersection Theory*. Springer-Verlag, second edition, 1984.

-
- [Gui77] Ernst A. (Ernst Adolph) Guillemin. *Synthesis of passive networks : theory and methods appropriate to the realization and approximation problems*. R.E. Krieger Pub. Co, 1977.
 - [HM77] Robert Hermann and Clyde Martin. Applications of algebraic geometry to systems theory—part i. *IEEE Transactions on Automatic Control*, 22(1):19–25, 1977.
 - [HRB75] Chung-Wen Ho, A Ruehli, and Pierce Brennan. The modified nodal approach to network analysis. *IEEE Transactions on circuits and systems*, 22(6):504–509, 1975.
 - [HS14] Timothy H Hughes and Malcolm C Smith. On the Minimality and Uniqueness of the Bott-Duffin Realization Procedure. *IEEE Transactions on Automatic Control*, 59(7):1858–1873, 2014.
 - [Hum75] James E. Humphreys. *Linear Algebraic Groups*. 1975.
 - [Kai80] Thomas Kailath. *Linear Systems*, volume 49. Prentice-Hall, Inc., 1980.
 - [Kal71] E. Kalman, R. Kronecker Invariants and Feedback. 1971.
 - [Kar] Nicos Karcantias. Systems Modelling (MSc Lecture notes).
 - [Kar87] Nicos Karcantias. An Introduction to Exterior Algebra, 1987.
 - [Kar94] Nicos Karcantias. The Selection of Input and Output Schemes for a System and the Model Projection Problems. *Kybernetika*, 30(6):585–596, 1994.
 - [Kar95] Nicos Karcantias. Integrated Process Design: A Generic Control Theory/Design Based Framework. *Comp. in Industry*, 26(3):291–301, aug 1995.
 - [Kar96] Nicos Karcantias. Control problems in global process instrumentation: A structural approach. *Computers & Chemical Engineering*, 20:S1101–S1106, jan 1996.
 - [Kar98] Nicos Karcantias. Systems and Control Issues in the Overall Production and Design of Industrial Processes. *ESPRIT Project, ICIMS NOE publication, State of the Art Survey, Systems and Control methodologies Committee*, 1998.

- [Kar08] N. Karcanias. Structure evolving systems and control in integrated design. *Annual Reviews in Control*, 32(2):161–182, 2008.
- [Kar09] Nicos Karcanias. Multivariable Poles and Zeros, 2009.
- [Kar11] Nicos Karcanias. Passive Network Redesign: System Structure Evolution. Technical report, City, University of London, London, 2011.
- [Kar13a] N Karcanias. Generalised Resultants, Dynamic Polynomial Combinants and the Minimal Design Problem. *International Journal of Control* N. *International Journal of Control*, 86(8611):1946–1964, 2013.
- [Kar13b] Nicos Karcanias. Systems of Systems: A Control Theoretic View. In *2013 IEEE International Conference on Systems, Man, and Cybernetics*, pages 1732–1737. IEEE, oct 2013.
- [KG84] N. Karcanias and C. Giannakopoulos. Grassmann invariants, almost zeros and the determinantal zero, pole assignment problems of linear multivariable systems. *International Journal of Control*, 40(4):673–698, 1984.
- [KG89] Nikos Karcanias and C. Giannakopoulos. Necessary and Sufficient Conditions for Zero Assignment by Constant Squaring Down. *Linear Algebra and its Applications*, 122/123/12:415–446, sep 1989.
- [Kir47] G. Kirchhoff. Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird. *Annalen der Physik und Chemie*, 148(12):497–508, 1847.
- [KK79] N. Karcanias and B. Kouvaritakis. The output zeroing problem and its relationship to the invariant zero structure : a matrix pencil approach. *International Journal of Control*, 30(3):395–415, sep 1979.
- [KK86] N. Karcanias and G. Kalogeropoulos. On the Segré, Weyr characteristics of right (left) regular matrix pencils. *International Journal of Control*, 44(4):991–1015, 1986.
- [KK88] N. Karcanias and G. Kalogeropoulos. Right, left characteristic sequences and column, row minimal indices of a singular pencil. *International Journal of Control*, 47(4):937–946, 1988.

-
- [KK89] N. Karcantias and G. Kalogeropoulos. Geometric theory and feedback invariants of generalized linear systems: A matrix pencil approach. *Circuits, Systems, and Signal Processing*, 8(3):375–397, sep 1989.
 - [KL06] Nicos Karcantias and John Leventides. Passive Electric Network Redesign and Matrix Pencils. Technical report, City, University of London, London, 2006.
 - [KL15] Nicos Karcantias and John Leventides. Solution of the determinantal assignment problem using the Grassmann matrices. *International Journal of Control*, 89(September 2015):1–16, feb 2015.
 - [KLG88] N. KARCANIAS, B. LAIOS, and C. GIANNAKOPOULOS. Decentralized determinantal assignment problem: fixed and almost fixed modes and zeros. *International Journal of Control*, 48(1):129–147, jul 1988.
 - [KLL14a] N. Karcantias, J. Leventides, and M. Livada. Matrix Pencil Representation of Structural Transformations of Passive Electrical Networks. In *International Symposium on Communications, Control, and Signal Processing*, Athens, Greece, 2014.
 - [KLL14b] N. Karcantias, J. Leventides, and M. Livada. Multi-parameter structural transformations of passive electrical networks and natural frequency assignment. In *2014 22nd Mediterranean Conference on Control and Automation, MED 2014*, pages 984–989, 2014.
 - [KLL17] N. Karcantias, M. Livada, and J. Leventides. System Properties of Implicit Passive Electrical Networks Descriptions. In *20th IFAC World Congress, IFAC 2017*, Toulouse, France, 2017.
 - [KM71] W.H. Kim and H.E. Meadows. *Modern Network Analysis*. John Wiley & Sons, New York, 1971.
 - [KM76] B. KOUVARITAKIS and A. G. J. MACFARLANE. Geometric approach to analysis and synthesis of system zeros Part 2. Non-square systems. *International Journal of Control*, 23(2):167–181, feb 1976.

- [KM80] N. Karcanas and P. MacBean. Structural invariants and canonical forms of linear multivariable systems. In *3rd IMA Conf. on Control Theory*, pages 257–282, London, 1980. Academic Press.
- [Kro33] Gabriel Kron. Tensor analysis of rotating machinery. *Electrical Engineering*, 52(1):51–51, 1933.
- [Kro34] Gabriel Kron. Non-riemannian dynamics of rotating electrical machinery. *Journal of Mathematics and Physics*, 13(1-4):103–194, 1934.
- [Ku52] YH Ku. Résumé of maxwell’s and kirchhoff’s rules for network analysis. *Journal of the Franklin Institute*, 253(3):211–224, 1952.
- [KV02a] N. Karcanas and K.G. Vafiadis. Model Orientation and Well Conditioning of System Models: System and Control Issues. In *15th Triennial World Congress*, volume 35, pages 315–320, Barcelona, Spain, 2002. Elsevier Publications.
- [KV02b] Nicos Karcanas and D. Vafiadis. Canonical Forms for State- Space Descriptions, 2002.
- [Lev93] John Leventides. *Algebriometric and Topological Methods in Control Theory*. PhD thesis, City, University of London, 1993.
- [Lev07] John Leventides. Assigning Frequencies via Determinantal Equations: New Counterexamples and Invariants. *IEEE TRANSACTIONS ON AUTOMATIC CONTROL*, 52(3):5800, mar 2007.
- [Liv12] Maria Livada. Structure Evolving Systems in the Context of Passive Electrical Networks: The Problem of Growth and Death. Master’s thesis, City, University of London, 2012.
- [LK] J. Leventides and N. Karcanas. Frequency assignment maps, relation to system invariants and bounded complexity solutions. In *Proceedings of 35th IEEE Conference on Decision and Control*, volume 4, pages 4126–4127. IEEE.
- [LK93] J. Leventides and N. Karcanas. The pole placement map, its properties, and relationships to system invariants. *IEEE Transactions on Automatic Control*, 38(8):1266–1270, 1993.

-
- [LK95a] J. Leventides and N. Karcantias. Sufficient conditions for arbitrary pole assignment by constant decentralized output feedback. *Mathematics of Control, Signals, and Systems*, 8(3):222–240, sep 1995.
 - [LK95b] John Leventides and Nicos Karcantias. Global asymptotic linearisation of the pole placement map: A closed form solution for the constant output feedback problem. *Automatica*, 31(9):1303–1309, sep 1995.
 - [LK08] J. Leventides and N. Karcantias. Structured squaring down and zero assignment. *International Journal of Control*, 81(2):294–306, feb 2008.
 - [LK09] John Leventides and Nicos Karcantias. Zero assignment of matrix pencils by additive structured transformations. *Linear Algebra and Its Applications*, 431(8):1380–1396, sep 2009.
 - [LLK14] J. Leventides, M. Livada, and N. Karcantias. McMillan Degree of Impedance, Admittance Functions of RLC Networks. In *21st International Symposium on Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, 2014.
 - [LLK16] J. Leventides, M. Livada, and N. Karcantias. Zero assignment problem in rlc networks. *IFAC-Papers On Line*, 49(9):92 – 98, 2016.
 - [LMZZ98] Jean Jacques Loiseau, Sabine Mondié, Ion Zaballa, and Petr Zagalak. Assigning the Kronecker invariants of a matrix pencil by row or column completions. *Linear Algebra and its Applications*, 278(1-3):327–336, jul 1998.
 - [LÖMK91] Jean J Loiseau, Kadrl Özcaldiran, Michel Malabre, and Nicos Karcantias. Feedback Canonical Forms of Singular Systems. 27(4), 1991.
 - [LPK14] John Leventides, George Petroulakis, and Nicos Karcantias. The approximate Determinantal Assignment Problem. *Linear Algebra and its Applications*, 461(April 2015):139–162, 2014.
 - [Max73] C. Maxwell, J. A *Treatise on Electricity and Magnetism*. 1873.
 - [Mey00] Carl D Meyer. *Matrix Analysis and Applied Linear Algebra*. SIAM, 2000.
 - [MH77] C. F. Martin and R. Hermann. Applications of algebraic geometry to systems theory, part ii: Feedback and pole placement for linear hamiltonian systems. *Proceedings of the IEEE*, 65(6):841–848, June 1977.

- [MH78] Clyde Martin and Robert Hermann. Applications of Algebraic Geometry to Systems Theory: The McMillan Degree and Kronecker Indices of Transfer Functions as Topological and Holomorphic System Invariants. *SIAM Journal on Control and Optimization*, 16(5):743–755, sep 1978.
- [MK76] A. G. J. MACFARLANE and N. KARCANIAS. Poles and zeros of linear multivariable systems : a survey of the algebraic, geometric and complex-variable theory. *International Journal of Control*, 24(1):33–74, jul 1976.
- [MM64] M. Marcus and H. Minc. *A survey of matrix theory and matrix inequalities*. Allyn and Bacon, Inc, Boston, 1964.
- [Mor73] A. S. Morse. Structural Invariants of Linear Multivariable Systems. *SIAM Journal on Control*, 11(3):446–465, aug 1973.
- [MS57] Wataru Mayeda and Sundaram Seshu. Topological Formulas for Network Functions. Technical report, University of Illinois, 1957.
- [Mum76] David Mumford. *Algebraic Geometry I - Complex Projective Varieties*. Springer - Verlag, 1976.
- [NMJ16] Karcantias Nicos, Livada Maria, and Leventides John. System Properties of Implicit Passive Electrical Networks Descriptions. Technical report, City, University of London, 2016.
- [Oka55a] S Okada. Algebraic and topological foundations of network synthesis. In *Proc. Symp. on Modern Network Synthesis*, pages 283–322, 1955.
- [Oka55b] S Okada. On node and mesh determinants. *Proceedings of the Institute of Radio Engineers*, 43(10):1527–1527, 1955.
- [Ott57] J Otterman. On the order of the differential equation describing an electrical network. *Proceedings of the Institute of Radio Engineers*, 45(7):1024–1025, 1957.
- [Per53] WS Percival. The solution of passive electrical networks by means of mathematical trees. *Proceedings of the IEE-Part III: Radio and Communication Engineering*, 100(65):143–150, 1953.

-
- [Poi00] H. Poincaré. Second Complement à l'Analyse Situs. *Proceedings of the London Mathematical Society*, s1-32(1):277–308, jan 1900.
 - [Pop72] V. M. Popov. Invariant Description of Linear, Time-Invariant Controllable Systems. *SIAM Journal on Control*, 10(2):252–264, may 1972.
 - [Rez55] FM Reza. Order of complexity and minimal structures in network analysis. In *Proceedings of the Symposium on Circuit Analysis*, pages 7–1, 1955.
 - [Rez58] FM Reza. Some topological considerations in network synthesis. *IRE Trans, on Circuit Theory, CT-5*, 1958.
 - [Rij91] John E. Rijnsdorp. *Integrated process control and automation*. Elsevier, 1991.
 - [Ros70] H. H. Rosenbrock. *State-space and multivariable theory*. Nelson, London, 1970.
 - [Rot55] P. Roth, J. An Application of Algebraic Topology to Numerical Analysis: On the Existence of a Solution to the Network Problem. *Proceedings of the National Academy of Sciences*, 41(6):518–521, 1955.
 - [Rot59] J Paul Roth. An Application of Algebraic Topology: Kron's Method of Tearing. *Quarterly of Applied Mathematics QUARTERLY OF APPLIED MATHEMATICS XVII APRIL*, 17(1):1–24, 1959.
 - [Row08] D. Rowell. Dynamics and Control II - Lecture 13, 2008.
 - [RR70] H.H. Rosenbrock and A. Rowe. Allocation of poles and zeros. *Proceedings of the Institution of Electrical Engineers*, 117(9):1879, 1970.
 - [Ruo13] Keijo Ruohonen. *Graph theory*. 2013.
 - [Smi02] Malcolm C Smith. Synthesis of Mechanical Networks : The Inerter. *IEEE TRANSACTIONS ON AUTOMATIC CONTROL*, 47(December):1657–1662, 2002.
 - [SP05] Fuzhen Zhang (eds.) Simo Puntanen, George P. H. Styan (auth.). *The Schur Complement and Its Applications*. Numerical Methods and Algorithms 4. Springer US, 1 edition, 2005.

- [SR61] Sundaram Seshu and Myril B. Reed. *Linear Graphs and Electrical Networks*. Addison Wesley, 1961.
- [SS88] C.B. Schrader and M.K. Sain. Research on system zeros: a survey. *Proceedings of the 27th IEEE Conference on Decision and Control*, 50(4):1407–1433, 1988.
- [SS90] SABERI and P. SANNUTI. Squaring down of non-strictly proper systems. *International Journal of Control*, 51(3):621–629, jan 1990.
- [Tal55] A Talbot. Some fundamental properties of networks without mutual inductance. *Proceedings of the IEE-Part C: Monographs*, 102(2):168–175, 1955.
- [Van60] M. E Van Valkenburg. *Introduction To Modern Network Synthesis*. John Wiley & Sons, New York, 1960.
- [Veb31] O. Veblen. *Analysis situs*. Colloquium Publications - American Mathematical Society. American Mathematical Society, 1931.
- [Vla83] Jiri Vlach. *Computer Methods for Circuit Analysis and Design*. Van Nostrand Reinhold Electrical/Computer Science and Engineering Series, New York, 1983.
- [Wai79] R. Wait. *The Numerical Solution of Algebraic Equations*. John Wiley & Sons, Chichester-New York-Brisbane-Toronto, 1979.
- [Wan94] Xiaochang Wang. Decentralized pole assignment and product grassmannians. *SIAM Journal on Control and Optimization*, 32(3):855–875, 1994.
- [Wei58] Louis Weinberg. Kirchhoff’s’ third and fourth laws’. *IRE Transactions on Circuit Theory*, 5(1):8–30, 1958.
- [WH78] J.C Willems and W.H. Hesselink. Generic Properties of the Pole Placement Problem. In *IFAC*, Helsinki, Finland, 1978.
- [WJ02] L. M. Wedepohl and L. Jackson. Modified nodal analysis: an essential addition to electrical circuit theory and analysis. *Engineering Science and Education Journal*, 11(3):84–92, Jun 2002.
- [Wol74] W. A. Wolovich. *Linear Multivariable Systems*, volume 11 of *Applied Mathematical Sciences*. Springer New York, New York, NY, 1974.

- [Won85] W. Murray. Wonham. *Linear multivariable control : a geometric approach*. Springer-Verlag, 1985.