Applying AI for Modeling and Understanding Analogy-Based Classroom Teaching Tools & Techniques

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Abstract. This paper forms the final part of a short series of related articles[1, 2] dedicated to highlighting a fruitful type of application of cognitively-inspired analogy engines in an educational context. It complements the earlier work with an additional fully worked out example by providing a short analysis and a detailed formal model (based on the Heuristic-Driven Theory Projection computational analogy framework) of the Number Highrise, a tool for teaching multiplication-based relations in the range of natural numbers up to 100 to children in their first years of primary school.

1 Introduction

With this paper we want to complete a line of work dedicated to showing an additional way of how methods and technology developed in AI, and more precisely in the subfield of cognitive systems, can provide valuable support in the learning sciences and education studies on a conceptual level: Based on the theoretical underpinnings presented in [1], this paper complements the earlier [2] by presenting a second large-scale application case for the use of computational analogy engines in modeling, simulating, and analyzing analogy-related tools and scenarios from classroom teaching. We thus propose to apply intelligent systems not only at the stage of implementation and deployment but already earlier at the level of research and conceptualization of teaching methodology and tools.

As case study we first provide a description and short analysis of the analogy-based Number Highrise [3] used for teaching multiplication-based relations in the range of natural numbers up to 100 to children attending basic mathematics classes in primary school, before showing how a computational analogy-making framework as Heuristic-Driven Theory Projection (HDTP; [4]) can be used to provide a formal computational reconstruction of the highrise as an example of targeted analogy-use taken from a real-life teaching situation.

2 Case Study: The Number Highrise

In this section we develop and analyze the HDTP-based model for the Number Highrise. By constructing this kind of model we hope to identify underlying principles and factors which shed light on some of the basic mechanisms of human developmental learning and knowledge transfer.

Sorts:

$\mathit{natural\_number}, \mathit{operation}, \mathit{relation}.$

Entities:

$\mathit{zero}, \mathit{one}, \mathit{two}, \ldots, \mathit{one\_hundred}, N_1, N_2, N_3 : \mathit{natural\_number}. \quad \mathit{multiply} : \mathit{operation}. \quad <, >: \mathit{relation}.$

Functions:

$\mathit{apply} : \mathit{operation} \times \mathit{natural\_number} \times \mathit{natural\_number} \to \mathit{natural\_number}.$

Predicates:

$\mathit{succ} : \mathit{natural\_number} \times \mathit{natural\_number}. \quad \mathit{holds} : \mathit{relation} \times \mathit{natural\_number} \times \mathit{natural\_number}.$

Facts:

$(n_0)\ \mathit{succ}(\mathit{zero}, \mathit{one}). \quad (n_1)\ \mathit{succ}(\mathit{one}, \mathit{two}). \quad (n_2)\ \mathit{succ}(\mathit{two}, \mathit{three}). \quad \ldots \quad (n_{99})\ \mathit{succ}(\mathit{ninety\text{-}nine}, \mathit{one\_hundred}).$
$(n_{100})\ \mathit{apply}(\mathit{multiply}, \mathit{one}, \mathit{one}) = \mathit{one}. \quad (n_{101})\ \mathit{apply}(\mathit{multiply}, \mathit{one}, \mathit{two}) = \mathit{two}.$
$(n_{102})\ \mathit{apply}(\mathit{multiply}, \mathit{one}, \mathit{three}) = \mathit{three}. \quad \ldots \quad (n_{199})\ \mathit{apply}(\mathit{multiply}, \mathit{one}, \mathit{one\_hundred}) = \mathit{one\_hundred}.$
$(n_{200})\ \mathit{apply}(\mathit{multiply}, \mathit{two}, \mathit{one}) = \mathit{two}. \quad (n_{201})\ \mathit{apply}(\mathit{multiply}, \mathit{two}, \mathit{two}) = \mathit{four}. \quad \ldots$
$(n_{249})\ \mathit{apply}(\mathit{multiply}, \mathit{two}, \mathit{fifty}) = \mathit{one\_hundred}.$
$(n_{250})\ \mathit{apply}(\mathit{multiply}, \mathit{three}, \mathit{one}) = \mathit{three}. \quad \ldots \quad (n_{282})\ \mathit{apply}(\mathit{multiply}, \mathit{three}, \mathit{thirty\text{-}three}) = \mathit{ninety\text{-}nine}.$
$(n_{283})\ \mathit{apply}(\mathit{multiply}, \mathit{four}, \mathit{one}) = \mathit{four}. \quad \ldots \quad (n_{390})\ \mathit{apply}(\mathit{multiply}, \mathit{ten}, \mathit{ten}) = \mathit{one\_hundred}.$

Laws:

$(n_{391})\ \mathit{succ}(N_1, N_2) \to \mathit{holds}(<, N_1, N_2).$
$(n_{392})\ \mathit{holds}(<, N_1, N_2) \land \mathit{holds}(<, N_2, N_3) \to \mathit{holds}(<, N_1, N_3).$
$(n_{393})\ \mathit{holds}(<, N_1, N_2) \leftrightarrow \mathit{holds}(>, N_2, N_1).$

Table 1. An idealized form of the children's initial conception of the $[0;100]$ integer number space ($n_0 - n_{99}$) and the times tables up to 10 ($n_{100} - n_{390}$). *multiply* represents multiplication over $[0;100]$, laws $n_{391} - n_{393}$ define the successor and the lower and greater ordering relations.

2.1 Discovering the Number Space from 0 to 100

As described in [3], the Number Highrise is a mathematical toy world designed for exploring the space of the natural numbers up to 100 making explicit use of multiplication-based relations. It consists of 10 so called hallways each made up by 100 pearls. In each hallway the pearls are connected in chunks going in accordance with one of the basic times tables for numbers between 1 and 10 (the leftmost hallway corresponding to the one times table, the one to its right corresponding to the two times table, etc.).The resulting segments within each hallway are marked by separating slices of acrylic glass, the so called platforms (similar to half landings in a hallway). Not all of the hallways terminate in a platform, depending on the respective times table there might be some remaining pearls needed for reaching the 100.

The children interact with toy figures which can travel the different hallways up and down. Fitting with the difference in the number of pearls between two platforms, the toy figures can be equipped with different pairs of number shoes (having the respective segment size written on them). Wearing a specific pair of number shoes allows a toy figure to navigate from platform to platform within one specific hallway, namely the hallway with the corresponding distance between platforms. If two or several differently shod toy figures within their respective hallway can reach platforms on equal height with each other, they are called neighbors and can celebrate a party together. If this is the case, an elevator mechanism within the Number Highrise allows the toy figures to actually meet at the height of their platforms: The elevator is a horizontal batten movable vertically along the row of hallways. It can be stopped at each point along the height of the highrise and can be used for checking for numerical factor relations.

In class the Number Highrise can be used in several ways: From teaching the basic conception of the natural number space, through the discovery of concepts like the least common multiple and the greatest common factor, to finding and exploring the notion of prime numbers.

Sorts:

pearl, hallway, platform, number_shoe, toy_figure, batten, operation, relation, time, natural_number.

Entities:

$\mathit{zero}, \mathit{one}, \mathit{two}, \mathit{three}, \ldots, \mathit{one\_hundred}, P_1, P_2, P_3 : \mathit{pearl}.$ $\quad h_1, h_2, h_3, \ldots, h_{10}, H_1, H_2 : \mathit{hallway}.$
$\mathit{pl}_1, \mathit{pl}_2, \ldots, \mathit{pl}_n : \mathit{platform}.$ $\quad s_1, s_2, s_3, \ldots, s_{10}, S_1 : \mathit{number\_shoe}.$ $\quad \mathit{fi}_1, \mathit{fi}_2, \ldots, \mathit{fi}_m, F_1, F_2 : \mathit{toy\_figure}.$
$\mathit{elevator} : \mathit{batten}.$ $\quad \mathit{climb} : \mathit{operation}.$ $\quad \mathit{below}, \mathit{above} : \mathit{relation}.$ $\quad T_1, T_2 : \mathit{time}.$

Functions:

$\mathit{value} : \mathit{pearl} \to \mathit{natural\_number}.$ $\quad \mathit{location} : \mathit{hallway} \to \mathit{natural\_number}.$ $\quad \mathit{caption} : \mathit{number\_shoe} \to \mathit{natural\_number}.$
$\mathit{apply} : \mathit{operation} \times \mathit{natural\_number} \times \mathit{natural\_number} \to \mathit{natural\_number}.$

Predicates:

$\mathit{holds} : \mathit{relation} \times \mathit{pearl} \times \mathit{pearl}.$ $\quad \mathit{platf\_at} : \mathit{hallway} \times \mathit{pearl}.$ $\quad \mathit{stands\_on} : \mathit{toy\_figure} \times \mathit{hallway} \times \mathit{pearl} \times \mathit{time}.$
$\mathit{wears} : \mathit{toy\_figure} \times \mathit{number\_shoe}.$ $\quad \mathit{meets} : \mathit{batten} \times \mathit{hallway} \times \mathit{platform} \times time.$ $\quad \mathit{on\_top\_of} : \mathit{pearl} \times \mathit{pearl}.$
$\mathit{party\_location} : \mathit{hallway} \times \mathit{pearl} \times \mathit{time}.$ $\quad \mathit{neighbors} : \mathit{toy\_figure} \times \mathit{toy\_figure}.$

Facts:

$(h_0)\ \mathit{on\_top\_of}(\mathit{base}, \mathit{one}).$ $\quad (h_1)\ \mathit{on\_top\_of}(\mathit{one}, \mathit{two}).$ $\quad \ldots \quad$ $(h_{99})\ \mathit{on\_top\_of}(\mathit{ninety\text{-}nine}, \mathit{one\_hundred}).$
$(h_{100})\ \mathit{platf\_at}(h_1, \mathit{one}).$ $\quad (h_{101})\ \mathit{platf\_at}(h_1, \mathit{two}).$ $\quad \ldots \quad$ $(h_{199})\ \mathit{platf\_at}(h_1, \mathit{one\_hundred}).$
$(h_{200})\ \mathit{platf\_at}(h_2, \mathit{two}).$ $\quad (h_{201})\ \mathit{platf\_at}(h_2, \mathit{four}).$ $\quad \ldots \quad$ $(h_{249})\ \mathit{platf\_at}(h_2, \mathit{one\_hundred}).$
$(h_{250})\ \mathit{platf\_at}(h_3, \mathit{three}).$ $\quad \ldots \quad$ $(h_{390})\ \mathit{platf\_at}(h_{10}, \mathit{one\_hundred}).$
$(h_{391})\ \mathit{value}(\mathit{base}) = 0.$ $\quad (h_{392})\ \mathit{value}(\mathit{one}) = 1.$ $\quad (h_{393})\ \mathit{value}(\mathit{two}) = 2.$ $\quad \ldots \quad$ $(h_{491})\ \mathit{value}(\mathit{one\_hundred}) = 100.$
$(h_{492})\ \mathit{location}(h_1) = 1.$ $\quad \ldots \quad$ $(h_{501})\ \mathit{location}(p_{10}) = 10.$
$(h_{502})\ \mathit{caption}(s_1) = 1.$ $\quad \ldots \quad$ $(h_{511})\ \mathit{caption}(s_{10}) = 10.$

Laws:

$(h_{512})\ \mathit{on\_top\_of}(P_1, P_2) \to \mathit{holds}(\mathit{below}, P_1, P_2).$
$(h_{513})\ \mathit{holds}(\mathit{below}, P_1, P_2) \wedge \mathit{holds}(\mathit{below}, P_2, P_3) \to \mathit{holds}(\mathit{below}, P_1, P_3).$
$(h_{514})\ \mathit{holds}(\mathit{below}, P_1, P_2) \leftrightarrow \mathit{holds}(\mathit{above}, P_2, P_1).$
$(h_{515})\ T_1 < T_2 : \mathit{platf\_at}(H_1, P_1) \wedge \mathit{stands\_on}(F_1, H_1, P_1, T_1) \wedge \mathit{wears}(F_1, S_1) \wedge \mathit{location}(H_1) = \mathit{caption}(S_1) \wedge \mathit{value}(P_2) = \mathit{value}(P_1) + \mathit{caption}(S_1) \wedge \mathit{apply}(\mathit{climb}, 1, \mathit{caption}(S_1)) \to \mathit{stands\_on}(F_1, H_1, P_2, T_2).$
$(h_{516})\ T_1 < T_2, \forall n \in [0;100] : \mathit{platf\_at}(H_1, P_1) \wedge \mathit{stands\_on}(F_1, H_1, P_1, T_1) \wedge \mathit{wears}(F_1, S_1) \wedge \mathit{location}(H_1) = \mathit{caption}(S_1) \wedge \mathit{value}(P_2) = \mathit{value}(P_1) + \mathit{caption}(S_1) \wedge \mathit{apply}(\mathit{climb}, n, \mathit{caption}(S_1)) \to \mathit{stands\_on}(F_1, H_1, P_2, T_2) \wedge \mathit{apply}(\mathit{climb}, n-1, \mathit{caption}(S_1)).$
$(h_{517})\ \mathit{platf\_at}(H_1, P_1) \wedge \mathit{platf\_at}(H_2, P_2) \wedge \mathit{meets}(\mathit{elevator}, H_1, P_1, T_1) \wedge \mathit{meets}(\mathit{elevator}, H_2, P_2, T_1) \to \mathit{party\_location}(H_1, P_1, T_1) \wedge \mathit{party\_location}(H_2, P_2, T_1).$
$(h_{518})\ \mathit{stands\_on}(F_1, H_1, P_1, T_1) \wedge \mathit{stands\_on}(F_2, H_2, P_2, T_1) \wedge \mathit{meets}(\mathit{elevator}, F_1, P_1, T_1) \wedge \mathit{meets}(\mathit{elevator}, F_2, P_, T_1) \to \mathit{neighbors}(F_1, F_2).$
$(h_{519})\ \mathit{neighbors}(F_1, F_2) \leftrightarrow \mathit{neighbors}(F_2, F_1).$

Table 2. Partial formalization of the Number Highrise focusing on multiplication-based relations.

2.2 An HDTP-Based Model of the Number Highrise

We now reconstruct the Number Highrise as an analogy-based model for discovering and experiencing the space of natural numbers between 0 and 100, together with multiplication-based relations within this number space.

The analogy uses the Number Highrise as a base domain, transferring the structure and relational conception children acquire by playing with it into their previously acquired knowledge about natural numbers as target domain. Although already populated with previous knowledge, the latter domain is most likely initially relatively poor (especially with regard to the internal relational structure) as compared to the Number Highrise domain as only limited internal structure or relations have been acquired. The pre-existing relational concepts are most likely rather simple in that they are either based on equality between magnitues (i.e., "=") and ordering relations for simple magnitudes — as commonly represented by the "<" and ">" relations — or that they are based on the arithmetic operations of addition and subtraction. The notion of multiplication is known on a mostly abstract basis (as are the corresponding times tables), but has not yet been developed into a grounded, constructively applicable conceptualization. Also, the more complex multiplication-based concepts as, e.g., the least common multiple or prime numbers are not yet present.

An idealized version (i.e., a version featuring complete times tables, which in reality should be assumed to be rather incomplete or sparse) of the students' initial conceptu-

Sorts:

$\mathit{pearl/natural\_number}, \mathit{hallway}, \mathit{platform}, \mathit{number\_shoe}, \mathit{toy\_figure}, \mathit{batten}, \mathit{operation}, \mathit{relation}, \mathit{time}, \mathit{natural\_number}.$

Entities:

$\mathit{zero}, \mathit{one}, \ldots, \mathit{one\_hundred}, P_1, P_2, P_3 : \mathit{pearl/natural\_number}. \quad \mathit{Climb/Multiply} : \mathit{operation}.$
$\mathit{Below/{<}}, \mathit{Above/{>}} : \mathit{relation}.$
$(*)\ h_1, h_2, h_3, \ldots, h_{10}, H_1, H_2 : \mathit{hallway}. \quad (*)\ pl_1, pl_2, \ldots, pl_n : \mathit{platform}. \quad (*)\ s_1, s_2, s_3, \ldots, s_{10}, S_1 : \mathit{number\_shoe}.$
$(*)\ \mathit{fi}_1, \mathit{fi}_2, \ldots, \mathit{fi}_m, F_1, F_2 : \mathit{toy\_figure}. \quad (*)\ \mathit{elevator} : \mathit{batten}. \quad (*)\ T_1, T_2 : \mathit{time}.$

Functions:

$\mathit{apply} : \mathit{operation} \times \mathit{natural\_number} \times \mathit{natural\_number} \rightarrow \mathit{natural\_number}.$
$(*)\ \mathit{value} : \mathit{pearl/natural\_number} \rightarrow \mathit{natural\_number}. \quad (*)\ \mathit{location} : \mathit{hallway} \rightarrow \mathit{natural\_number}.$
$(*)\ \mathit{caption} : \mathit{number\_shoe} \rightarrow \mathit{natural\_number}.$

Predicates:

$\mathit{holds} : \mathit{relation} \times \mathit{pearl/natural\_number} \times \mathit{pearl/natural\_number}.$
$\mathit{GenSucc} : \mathit{pearl/natural\_number} \times \mathit{pearl/natural\_number}.$
$(*)\ \mathit{platf\_at} : \mathit{hallway} \times \mathit{pearl/natural\_number}. \quad (*)\ \mathit{stands\_on} : \mathit{toy\_figure} \times \mathit{hallway} \times \mathit{pearl/natural\_number} \times \mathit{time}.$
$(*)\ \mathit{wears} : \mathit{toy\_figure} \times \mathit{number\_shoe}. \quad (*)\ \mathit{meets} : \mathit{batten} \times \mathit{hallway} \times \mathit{platform} \times time.$
$(*)\ \mathit{party\_location} : \mathit{hallway} \times \mathit{pearl/number} \times \mathit{time}. \quad (*)\ \mathit{neighbors} : \mathit{toy\_figure} \times \mathit{toy\_figure}.$

Facts:

$(g_0)\ \mathit{GenSucc}(\mathit{zero}, \mathit{one}). \quad (g_1)\ \mathit{GenSucc}(\mathit{one}, \mathit{two}). \quad \ldots \quad (g_{99})\ \mathit{GenSucc}(\mathit{ninety\text{-}nine}, \mathit{one\_hundred}).$
$(g_{100}*)\ \mathit{platf\_at}(h_1, \mathit{one}). \quad (g_{101}*)\ \mathit{platf\_at}(h_1, \mathit{two}). \quad \ldots \quad (g_{199}*)\ \mathit{platf\_at}(h_1, \mathit{one\text{-}hundred}).$
$(g_{200}*)\ \mathit{platf\_at}(h_2, \mathit{two}). \quad (g_{201}*)\ \mathit{platf\_at}(h_2, \mathit{four}). \quad \ldots \quad (g_{249}*)\ \mathit{platf\_at}(h_2, \mathit{one\_hundred}).$
$(g_{250}*)\ \mathit{platf\_at}(h_3, \mathit{three}). \quad \ldots \quad (g_{390}*)\ \mathit{platf\_at}(h_{10}, \mathit{one\_hundred}).$
$(g_{391}*)\ \mathit{value}(\mathit{zero}) = 0. \quad (g_{392}*)\ \mathit{value}(\mathit{one}) = 1. \quad \ldots \quad (g_{491}*)\ \mathit{value}(\mathit{one\_hundred}) = 100.$
$(g_{492}*)\ \mathit{location}(h_1) = 1. \quad \ldots \quad (g_{501}*)\ \mathit{location}(p_{10}) = 10.$
$(g_{502}*)\ \mathit{caption}(s_1) = 1. \quad \ldots \quad (g_{511}*)\ \mathit{caption}(s_{10}) = 10.$

Laws:

$(g_{512})\ \mathit{GenSucc}(P_1, P_2) \rightarrow \mathit{holds}(\mathit{Below/{<}}, P_1, P_2).$
$(g_{513})\ \mathit{holds}(\mathit{Below/{<}}, P_1, P_2) \wedge \mathit{holds}(\mathit{Below/{<}}, P_2, P_3) \rightarrow \mathit{holds}(\mathit{Below/{<}}, P_1, P_3).$
$(g_{514})\ \mathit{holds}(\mathit{Below/{<}}, P_1, P_2) \leftrightarrow \mathit{holds}(\mathit{Above/{>}}, P_2, P_1).$
$(g_{515}*)\ T_1 < T_2 : \mathit{platf\_at}(H_1, P_1) \wedge \mathit{stands\_on}(F_1, H_1, P_1, T_1) \wedge \mathit{wears}(F_1, S_1) \wedge \mathit{location}(H_1) = \mathit{caption}(S_1) \wedge \mathit{value}(P_2) = \mathit{value}(P_1) + \mathit{caption}(S_1) \wedge \mathit{apply}(\mathit{Climb/Multiply}, 1, \mathit{caption}(S_1)) \rightarrow \mathit{stands\_on}(F_1, H_1, P_2, T_2).$
$(g_{516}*)\ T_1 < T_2, \forall n \in [0;100] : \mathit{platf\_at}(H_1, P_1) \wedge \mathit{stands\_on}(F_1, H_1, P_1, T_1) \wedge \mathit{wears}(F_1, S_1) \wedge \mathit{location}(H_1) = \mathit{caption}(S_1) \wedge \mathit{value}(P_2) = \mathit{value}(P_1) + \mathit{caption}(S_1) \wedge \mathit{apply}(\mathit{Climb/Multiply}, n, \mathit{caption}(S_1)) \rightarrow \mathit{stands\_on}(F_1, H_1, P_2, T_2) \wedge \mathit{apply}(\mathit{Climb/Multiply}, n-1, \mathit{caption}(S_1)).$
$(g_{517}*)\ \mathit{platf\_at}(H_1, P_1) \wedge \mathit{platf\_at}(H_2, P_2) \wedge \mathit{meets}(\mathit{elevator}, H_1, P_1, T_1) \wedge \mathit{meets}(\mathit{elevator}, H_2, P_2, T_1) \rightarrow \mathit{party\_location}(H_1, P_1, T_1) \wedge \mathit{party\_location}(H_2, P_2, T_1).$
$(g_{518}*)\ \mathit{stands\_on}(F_1, H_1, P_1, T_1) \wedge \mathit{stands\_on}(F_2, H_2, P_2, T_1) \wedge \mathit{meets}(\mathit{elevator}, F_1, P_1, T_1) \wedge \mathit{meets}(\mathit{elevator}, F_2, P_, T_1) \rightarrow \mathit{neighbors}(F_1, F_2).$
$(g_{519}*)\ \mathit{neighbors}(F_1, F_2) \leftrightarrow \mathit{neighbors}(F_2, F_1).$

Table 3. Generalized theory of the Number Highrise and the children's conception of the non-negative integers, expanded by the generalized forms of the candidate elements for analogical transfer from base to target domain (marked with $*$).

alization of the natural number space can formally be represented as shown in Table 1, whilst Table 2 gives a partial formal HDTP-style model of the Number Highrise (on the relational side focusing only on the multiplication-based aspects).

We shortly want to focus on some aspects of the respective formalizations. The formalization of the target domain of the later analogy, i.e., of an idealized version of the children's initial conception of the integer number space $[0;100]$, contains mostly facts the children have learned by heart, namely the order of the number terms between zero and one hundred, and the times tables up to 10 within this range. In reality it has to be assumed that the times tables are significantly more sparsely populated than in our formalization, corresponding to incomplete recall of the memorized full tables.

The base domain of the later analogy, i.e., the formalization of the Number Highrise, exhibits a rich structure concerning facts and laws alike. The facts represent the easily accessible structure of the highrise, namely the order of succession of the pearls, the distinction between the different hallways, and the placement of the platforms within the respective hallways. The laws cover the rules of the transformational and constructive process by which the children can interact with the highrise: For instance (h_{515}) and

Sorts:

natural_number, operation, relation, hallway, platform, number_shoe, toy_figure, batten, time.

Entities:

$\mathit{zero}, \mathit{one}, \ldots, \mathit{one\_hundred}, N_1, N_2, N_3 : \mathit{natural\_number}.$ $\quad$ $\mathit{multiply} : \mathit{operation}.$ $\quad$ $<, > : \mathit{relation}.$

$(*)\ h_1, h_2, h_3, \ldots, h_{10} : \mathit{hallway}.$ $\quad$ $(*)\ pl_1, pl_2, \ldots, pl_n : \mathit{platform}.$ $\quad$ $(*)\ s_1, s_2, s_3, \ldots, s_{10} : \mathit{number\_shoe}.$

$(*)\ fi_1, fi_2, \ldots, fi_m : \mathit{toy\_figure}.$ $\quad$ $(*)\ \mathit{elevator} : \mathit{batten}.$ $\quad$ $(*)\ T_1, T_2 : \mathit{time}.$

Functions:

$\mathit{apply} : \mathit{operation} \times \mathit{natural\_number} \times \mathit{natural\_number} \rightarrow \mathit{natural\_number}.$

$(*)\ \mathit{value} : \mathit{natural\_number} \rightarrow \mathit{natural\_number}.$ $\quad$ $(*)\ \mathit{location} : \mathit{hallway} \rightarrow \mathit{natural\_number}.$

$(*)\ \mathit{caption} : \mathit{number\_shoe} \rightarrow \mathit{natural\_number}.$

Predicates:

$\mathit{holds} : \mathit{relation} \times \mathit{natural\_number} \times \mathit{natural\_number}.$ $\quad$ $\mathit{succ} : \mathit{natural\_number} \times \mathit{natural\_number}.$

$(*)\ \mathit{platf\_at} : \mathit{hallway} \times \mathit{natural\_number}.$ $\quad$ $(*)\ \mathit{stands\_on} : \mathit{toy\_figure} \times \mathit{hallway} \times \mathit{natural\_number} \times \mathit{time}.$

$(*)\ \mathit{wears} : \mathit{toy\_figure} \times \mathit{number\_shoe}.$ $\quad$ $(*)\ \mathit{meets} : \mathit{batten} \times \mathit{hallway} \times \mathit{platform} \times \mathit{time}.$

$(*)\ \mathit{party\_location} : \mathit{hallway} \times \mathit{natural\_number} \times \mathit{time}.$ $\quad$ $(*)\ \mathit{neighbors} : \mathit{toy\_figure} \times \mathit{toy\_figure}.$

Facts:

$(e_0)\ \mathit{succ}(\mathit{zero}, \mathit{one}).$ $\quad$ $(e_1)\ \mathit{succ}(\mathit{one}, \mathit{two}).$ $\quad \ldots \quad$ $(e_{99})\ \mathit{succ}(\mathit{ninety\text{-}nine}, \mathit{one\_hundred}).$

$(e_{100})\ \mathit{apply}(\mathit{multiply}, \mathit{one}, \mathit{one}) = \mathit{one}.$ $\quad$ $(e_{101})\ \mathit{apply}(\mathit{multiply}, \mathit{one}, \mathit{two}) = \mathit{two}.$ $\quad \ldots$

$(e_{390})\ \mathit{apply}(\mathit{multiply}, \mathit{ten}, \mathit{ten}) = \mathit{one\_hundred}.$

$(e_{391}*)\ \mathit{platf\_at}(h_1, \mathit{one}).$ $\quad$ $(e_{392}*)\ \mathit{platf\_at}(h_1, \mathit{two}).$ $\quad \ldots \quad$ $(e_{490}*)\ \mathit{platf\_at}(h_1, \mathit{one\text{-}hundred}).$

$(e_{491}*)\ \mathit{platf\_at}(h_2, \mathit{two}).$ $\quad$ $(e_{492}*)\ \mathit{platf\_at}(h_2, \mathit{four}).$ $\quad \ldots \quad$ $(e_{540}*)\ \mathit{platf\_at}(h_2, \mathit{one\_hundred}).$

$(e_{541}*)\ \mathit{platf\_at}(h_3, \mathit{three}).$ $\quad \ldots \quad$ $(e_{681}*)\ \mathit{platf\_at}(h_{10}, \mathit{one\_hundred}).$

$(e_{682}*)\ \mathit{value}(\mathit{zero}) = 0.$ $\quad \ldots \quad$ $(e_{782}*)\ \mathit{value}(\mathit{one\_hundred}) = 100.$

$(e_{783}*)\ \mathit{location}(h_1) = 1.$ $\quad \ldots \quad$ $(e_{792}*)\ \mathit{location}(h_{10}) = 10.$

$(e_{793}*)\ \mathit{caption}(s_1) = 1.$ $\quad \ldots \quad$ $(e_{802}*)\ \mathit{caption}(s_{10}) = 10.$

Laws:

$(e_{803})\ \mathit{succ}(N_1, N_2) \rightarrow \mathit{holds}(<, N_1, N_2).$

$(e_{804})\ \mathit{holds}(<, N_1, N_2) \wedge \mathit{holds}(<, N_2, N_3) \rightarrow \mathit{holds}(<, N_1, N_3).$

$(e_{805})\ \mathit{holds}(<, N_1, N_2) \leftrightarrow \mathit{holds}(>, N_2, N_1).$

$(e_{806}*)\ T_1 < T_2 : \mathit{platf\_at}(H_1, N_1) \wedge \mathit{stands\_on}(F_1, H_1, N_1, T_1) \wedge \mathit{wears}(F_1, S_1) \wedge \mathit{location}(H_1) = \mathit{caption}(S_1) \wedge \mathit{value}(N_2) = \mathit{value}(N_1) + \mathit{caption}(S_1) \wedge \mathit{apply}(\mathit{multiply}, 1, \mathit{caption}(S_1)) \rightarrow \mathit{stands\_on}(F_1, H_1, N_2, T_2).$

$(e_{807}*)\ T_1 < T_2, \forall n \in [0;100] : \mathit{platf\_at}(H_1, N_1) \wedge \mathit{stands\_on}(F_1, H_1, N_1, T_1) \wedge \mathit{wears}(F_1, S_1) \wedge \mathit{location}(H_1) = \mathit{caption}(S_1) \wedge \mathit{value}(N_2) = \mathit{value}(N_1) + \mathit{caption}(S_1) \wedge \mathit{apply}(\mathit{multiply}, n, \mathit{caption}(S_1)) \rightarrow \mathit{stands\_on}(F_1, H_1, N_2, T_2) \wedge \mathit{apply}(\mathit{multiply}, n-1, \mathit{caption}(S_1)).$

$(e_{808}*)\ \mathit{platf\_at}(H_1, N_1) \wedge \mathit{platf\_at}(H_2, N_2) \wedge \mathit{meets}(\mathit{elevator}, H_1, N_1, T_1) \wedge \mathit{meets}(\mathit{elevator}, H_2, N_2, T_1) \rightarrow \mathit{party\_location}(H_1, N_1, T_1) \wedge \mathit{party\_location}(H_2, N_2, T_1).$

$(e_{809}*)\ \mathit{stands\_on}(F_1, H_1, N_1, T_1) \wedge \mathit{stands\_on}(F_2, H_2, N_2, T_1) \wedge \mathit{meets}(\mathit{elevator}, F_1, N_1, T_1) \wedge \mathit{meets}(\mathit{elevator}, F_2, N_2, T_1) \rightarrow \mathit{neighbors}(F_1, F_2).$

$(e_{810}*)\ \mathit{neighbors}(F_1, F_2) \leftrightarrow \mathit{neighbors}(F_2, F_1).$

Table 4. Analogically enriched conception of the non-negative integers.

(h_{516}) encompass the process of having the toy figure climb the highrise, and (p_{517}) to (p_{519}) add the notion of several figures meeting on the same level and celebrating a party together.

The HDTP mechanism can now be used for computing a common generalization of both domains, yielding a generalized theory like given in Table 3. The main domain elements defining the alignment of formulae are the matching between the entities of sort *pearl* and *natural_number*, between the *apply* functions, the matching of the *holds* predicates, the thus induced alignment of the respective *operation* and *relation* entities, as well as the alignment between the *successor* and *on_top_of* predicates (in turn induced by the alignment of the *operation* and *relation* entities and the resulting structural similarity between predicates).

In conclusion, the generalized theory forms the basis for transferring knowledge in an analogy-based way from the (originally richer) Number Highrise domain to the children's number domain, resulting in an expanded theory for the numbers as given in Table 4. The important aspect in this expanded version is the availability of the constructive relations and insights obtained in the interaction with the Number Highrise. These give, for instance, meaning to the multiplication relation via the assignment of the constructive process of climbing the highrise and the corresponding platforms for the intermediate steps and the result. Furthermore laws $(\mathrm{e}_{806}*)$ and $(\mathrm{e}_{807}*)$ allow for the independent computation of parts of the times tables that might not be obtainable from memory (i.e., that would not explicitly be present as a fact in a more realistic formalization of the number domain).

2.3 Example: Discovering the Least Common Multiple

We now can reconstruct, for instance, the children's discovery of the least common multiple of three numbers documented in [3]: At the cognitive level the analogy-based process of transfer of the gained insights from the Number Highrise into the children's conception of the $[0;100]$ number space can be exemplified in the given HDTP model. Taking the definition of the least common multiple *lmc* of two natural numbers x and y (i.e., $lmc|x$ and $lmc|y$ and $\nexists a \in \mathbb{N} : a < lmc, a|x, a|y$), it becomes obvious that the concept of a party location and the notion of neighborhood between toy figures (i.e., laws $(e_{808}*)$ to $(e_{810}*)$ in the analogically enriched formalization in Table 4, together with the game-immanent concept of having the toy figures start at level zero of the hallways and the task to find the first possible party location between all three neighbors, naturally induces the least common multiple concept in the children's number domain. Via analogical transfer from the Number Highrise, building upon the already pre-existing basic structure of natural numbers and times tables, the more complex idea of the least common multiple has been acquired by the children — experience-grounded concept learning has taken place in a rather abstract domain.

3 Conclusion

In general, this work is an example for the application of AI techniques and tools in gaining deeper understanding of the mechanisms underlying analogy-based teaching material. Modeling educational analogies, on the one hand, sheds new light on a particular analogy, in terms of which information is transferred, what the limitations of the analogy are, or whether it makes unhelpful mappings; and what potential extensions might be needed. On the other hand, it also contributes to a deeper general understanding of the basic principles and mechanisms underlying analogy-based learning (and subsequently teaching) in fairly high-level and abstract domains.

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