On the Hilbert series of Hochschild cohomology of block algebras

Radha Kessar and Markus Linckelmann
City University London
Northampton Square
London EC1V 0HB
United Kingdom

Abstract

We show that the degrees and relations of the Hochschild cohomology of a p-block algebra of a finite group over an algebraically closed field of prime characteristic p are bounded in terms of the defect groups of the block and that for a fixed defect d, there are only finitely many Hilbert series of Hochschild cohomology algebras of blocks of defect d. The main ingredients are Symonds' proof of Benson's regularity conjecture and the fact that the Hochschild cohomology of a block is finitely generated as a module over block cohomology, which is an invariant of the fusion system of the block on a defect group.

Let p be a prime and k an algebraically closed field of characteristic p. Let G be a finite group and B a block algebra of kG; that is, B is an indecomposable direct factor of kG as a k-algebra. A *defect group of* B is a minimal subgroup P of G such that B is isomorphic to a direct summand of $B \otimes_{kP} B$ as a B-B-bimodule. The defect groups of B form a G-conjugacy class of p-subgroups of G, and the *defect of* B is the integer $d(B)$ such that $p^{d(B)}$ is the order of the defect groups of B. Donovan's conjecture predicts that there should be only finitely many Morita equivalence classes of block algebras of a fixed defect d. If true, this would imply that there are only finitely many isomorphism classes of Hochschild cohomology algebras of block algebras with a fixed defect d. We showed in [9] that for any fixed integer $n \geq 0$ the dimension of $HH^n(B)$ is bounded in terms of the defect groups of B. Our first result adds to this that there is a bound on the degrees of generators and relations of the Hochschild cohomology $HH^*(B)$ depending only on the defect of B.

Theorem 1. *There is a function $g : \mathbb{N}_0 \to \mathbb{N}_0$ such that for any finite group and block algebra B of kG of defect d the Hochschild cohomology $HH^*(B)$ of B is determined, as a graded commutative k-algebra, by homogeneous generators and relations in degrees less or equal to $g(d)$.*

Using the aforementioned bounds on the coefficients of the Hilbert series of $HH^*(B)$ and theorems of Hilbert and Serre, we show the following result:

Theorem 2. *Let d be a nonnegative integer. There are only finitely many Hilbert series of Hochschild cohomology algebras $HH^*(B)$, with B running over the block algebras with defect d of finite group algebras over k.*

For the sake of completeness we mention that this result has a 'converse':

Theorem 3. *Let $h \in \mathbb{Z}[[t]]$. There are at most finitely many positive integers d such that there exists a block B of defect d of a finite group algebra over k whose Hochschild cohomology has h as its Hilbert series.*

In other words, the Hilbert series of the Hochschild cohomology of a block and its defect 'determine each other up to finitely many possibilities'. Since the degree zero coefficient of the Hilbert series of $HH^*(B)$ is the dimension of $Z(B)$, Theorem 3 would follow from a positive solution to Brauer's problem 21, stipulating that the defect of a block B should be bounded in terms of a function depending only on the dimension of $Z(B)$. By [10, Theorem], a positive solution of Brauer's problem 21 would in turn be a consequence of an affirmative answer to the Alperin-McKay conjecture, stating that a block and its Brauer correspondent should have the same number of ordinary irreducible characters of height zero.

Whether there are actually only finitely many isomorphism classes of Hochschild cohomology algebras of blocks with a given defect d remains at this point an open problem. In view of Theorem 1, in order to show this, it would suffice to bound the size of the field generated by the coefficients in the relations between a suitably chosen set of generators. Evidence in that direction includes the work of Cliff, Plesken and Weiss [6], showing that $HH^0(B) \cong Z(B)$ has always a k-basis with multiplicative constants in the prime field $\mathbb{F}_p$. Since principal blocks are defined over $\mathbb{F}_p$, we can conclude that there are only finitely many isomorphism classes of Hochschild cohomology algebras of principal blocks with a given defect d; see Remark 7 below for more details.

The key ingredient for proving the above results is Symonds' proof in [13] of Benson's regularity conjecture. The extension of the Castelnuovo-Mumford regularity to graded-commutative rings with generators in arbitrary positive degrees is due to Benson [2, §4]. We follow the notational conventions in Symonds [13]. In particular, if p is odd and $T = \oplus_{n\geq 0} T^n$ is a finitely generated nonnegatively graded commutative k-algebra and M a finitely generated graded T-module, we denote by $\mathrm{reg}(T, M)$ the Castelnuovo-Mumford regularity of M as a graded T^{ev}-module, where $T^{ev} = \oplus_{n\geq 0} T^{2n}$ is the even part of T (which is then strictly commutative). We set $\mathrm{reg}(T) = \mathrm{reg}(T, T)$; that is, $\mathrm{reg}(T)$ is the Castelnuovo-Mumford regularity of T as a graded T^{ev}-module. We use the properties of $\mathrm{reg}(T, M)$ from [13, §1] and [14, §2]. Benson conjectured in [2, 1.1] that for any finite group G, $\mathrm{reg}(H^*(G; \mathbb{F}_p))$ should be zero, proved in [2, 4.2] the inequality $\mathrm{reg}(H^*(G; \mathbb{F}_p)) \geq 0$ and showed further in [4] that the conjecture holds in a number of cases. Symonds proved Benson's regularity conjecture in general in [13]. As an immediate consequence of the properties of $\mathrm{reg}(H^*(G; \mathbb{F}_p))$ listed in [13, §1], we have the following:

Lemma 4. *Let G be a finite group, H a subgroup of G and V a finitely generated kH-module. We have* $\mathrm{reg}(H^*(G; k), \mathrm{Ext}^*_{kG}(k; \mathrm{Ind}^G_H(V))) \cong \mathrm{reg}(H^*(H; k), \mathrm{Ext}^*_{kH}(k; V))$.

Proof. By Eckman-Shapiro we have a graded k-linear isomorphism $\mathrm{Ext}^*_{kG}(k; \mathrm{Ind}^G_H(V)) \cong \mathrm{Ext}^*_{kH}(k; V)$, which is easily seen to be an isomorphism as $H^*(G; k)$-modules, where on the right side the $H^*(G; k)$-module structure is obtained from restricting the $H^*(H; k)$-module structure via the graded algebra homomorphism $H^*(G; k) \to H^*(H; k)$ induced by Res^G_H. By a theorem of Evens and Venkov, $H^*(H; k)$ is finitely generated as a module over $H^*(G; k)$. It follows that $H^{ev}(H; k)$ is finitely generated as a module over $H^{ev}(G; k)$, and hence, by [13, 1.1], the regularity of $\mathrm{Ext}^*_{kH}(k; V)$ as a module over $H^*(H; k)$ coincides with that of $\mathrm{Ext}^*_{kH}(k; V)$ as a module over $H^*(G; k)$. □

Combining the above Lemma with Symonds' results in [13], [14] yields the following:

Proposition 5. *Let G be a finite group. We have $\mathrm{reg}(HH^*(kG)) = 0$, and for any block algebra B of kG we have $\mathrm{reg}(HH^*(B)) \leq 0$.*

Proof. Set $\Delta G = \{(x,x) \mid x \in G\} \subseteq G \times G$. The 'diagonal induction' functor $\mathrm{Ind}_{\Delta G}^{G\times G}$ induces a unitary graded algebra homomorphism $H^*(\Delta G; k) \to HH^*(kG)$. By Eckmann-Shapiro, we have an isomorphism of $H^*(\Delta G; k)$-modules $HH^*(kG) \cong H^*(\Delta G; kG)$. As a consequence of the theorem of Evens and Venkov, $HH^*(kG)$ is finitely generated as a $H^*(\Delta G; k)$-module, hence $HH^{ev}(kG)$ is finitely generated over $H^{ev}(\Delta G; k)$. Thus we have $\mathrm{reg}(HH^*(kG)) = \mathrm{reg}(H^*(\Delta G; k), HH^*(kG))$ by [13, 1.1]. Decomposing kG as a $k\Delta G$-module yields an isomorphism of $H^*(G; k)$-modules $H^*(\Delta G; kG) \cong \oplus_x H^*(\Delta G; \mathrm{Ind}_{\Delta C_G(x)}^{\Delta G}(k))$, where x runs over a set of representatives of the conjugacy classes of G. Using Lemma 4 it follows from [13, Corollary 0.2] that the regularity of each summand is zero. Using [13, Lemma 1.4. (4)] we get that $\mathrm{reg}(HH^*(kG)) = 0$. Since $HH^*(kG)$ is the direct product of the algebras $HH^*(B)$, with B runing over the blocks of kG, the second statement follows. □

Proof of Theorem 1. Let B be a block of kG with defect group P and fusion system $\mathcal{F}$. Denote by $H^*(B)$ the corresponding block cohomology; that is, $H^*(B)$ can be identified with the subalgebra $H^*_{\mathcal{F}}(P; k)$ of $H^*(P; k)$ of $\mathcal{F}$-stable elements. By [11, 5.6], [12, 4.3] there is an injective graded algebra homomorphism $\delta_B : H^*(B) \to HH^*(B)$ such that $HH^*(B)$ is finitely generated as a module over $H^*(B)$. Let R be a Noether normalisation of $H^{ev}(B)$. Then $HH^*(B)$ is finitely generated as a module over the polynomial algebra R via the homomorphism δ_B restricted to R. Since $\mathrm{reg}(HH^*(B)) \leq 0$ it follows from [13, 1.3] or [14, 2.1] that there is an integer $g(\mathcal{F})$ such that as a graded commutative k-algebra, $HH^*(B)$ is generated by elements and relations in degree at most $g(\mathcal{F})$, depending only on the degrees of the generators of R. Since for a fixed integer $d \geq 0$ there are at most finitely many isomorphism classes of finite p-groups of order p^d and for any finite p-group P there are only finitely many fusion systems on P it follows that there is an integer $g(d)$ as stated. □

Remark 6. We have made no effort to construct a best possible bound $g(d)$, but the proof of the main result indicates where to look for improvements: one needs to bound the degrees of the Noether normalisations of $H^*_{\mathcal{F}}(P; k)$ for any finite p-group P and any fusion system on P.

Remark 7. The principal block idempotent of a finite group algebra kG is contained in $\mathbb{F}_pG$, and Lemma 4 and Proposition 5 remain true with k replaced by an arbitrary field of characteristic p. A straightforward adaptation of the proof of Theorem 1 shows that for principal blocks, the conclusion of Theorem 1 holds with $\mathbb{F}_p$ instead of k. If B is the principal block of $\mathbb{F}_pG$ for some finite group G, then each $HH^i(B)$ is a finite set whose cardinality is bounded by the defect d of B, and hence there are only finitely many possible choices of generators and relations of $HH^*(B)$ in degree at most $g(d)$. Moreover, for every field k containing $\mathbb{F}_p$, the algebra $k \otimes_{\mathbb{F}_p} B$ is the principal block of kG, and we have $HH^*(k \otimes_{\mathbb{F}_p} B) \cong k \otimes_{\mathbb{F}_p} HH^*(B)$. It follows that there are only finitely many isomorphism classes of Hochschild cohomology algebras of principal blocks with defect d of finite group algebras over a fixed field of characteristic p.

Proof of Theorem 2. As before, let B be a block of a finite group algebra kG with defect group P and fusion system $\mathcal{F}$. Set $d = d(B)$; that is, $p^d = |P|$. Let $H^*(B)$ and R be as in the proof of Theorem 1. Then $HH^*(B)$ is finitely generated as an R-module. By the Hilbert-Serre theorem [1, Theorem 2.1.1], the Hilbert series $h_B(t) = \sum_{i\geq 0} \dim_k(HH^i(B))t^i$ of $HH^*(B)$ is a rational function of the form

$$h_B(t) = \frac{f(t)}{\prod_{i=1}^r (1 - t^{d_i})} \ ,$$

where $f \in \mathbb{Z}[t, t^{-1}]$ and where $d_i > 0$ is the degree of a homogeneous element ζ_i in R, such that the set $\{\zeta_i\}_{1 \le i \le r}$ generates R as a k-algebra. As noted earlier, for fixed d there are only finitely many isomorphism classes of block cohomology algebras $H^*(B)$, and hence the d_i and r are bounded in terms of d. Since $HH^*(B)$ vanishes in negative degrees we have $f \in \mathbb{Z}[t]$. It follows from Serre's formula [5, Theorem 4.4.3 (c)] that the degree of h_B as a rational function is bounded by the regularity of $HH^*(B)$, which is at most zero by Proposition 5 above, and hence $\deg(f) \le \sum_{i=1}^{r} d_i$. This implies that $\deg(f)$ is bounded in terms of a function depending only on d. By [9, Theorem 1], the coefficients $\dim_k(HH^n(B))$ of the series $h_B(t)$ are bounded in terms of functions depending only on n and d. Consider the equation

$$f(t) = h_B(t) \prod_{i=1}^{r} (1 - t^{d_i}) \ .$$

By the previous remarks, the absolute values of the coefficients of the right side are bounded in terms of d in each degree, thus the same is true for the coefficients of f. Since $\deg(f)$ is bounded in terms of d as well, this leaves only finitely many possibilities for f once the defect d is fixed. Thus there are only finitely many possibilities of Hilbert series $h_B(t)$ of Hochschild cohomology algebras of blocks B with a given defect d. □

Remark 8. There is a vast amount of literature on the interplay between the Castelnuovo-Mumford regularity and Hilbert series, predominantly for standard graded algebras. See [7] for an introduction to the subject. For the use of commutative algebra in finite group cohomology, see [3]. Both sources have long lists of further references.

Proof of Theorem 3. Write $h(t) = \sum_{n=0}^{\infty} a_n t^n$ with $a_n \in \mathbb{Z}$, $n \ge 0$. Suppose that B is a block of kG such that $\dim_k(HH^n(B)) = a_n$, for $n \ge 0$. By [12, Corollary 4.3], the Krull dimension of $HH^*(B)$ is equal to the rank of a defect group P of B. The Krull dimension of $HH^*(B)$ is also equal to its Krull dimension as a module over the even part $H^{ev}(B)$ of block cohomology, hence, by [1, Theorem 2.2.7], equal to the pole at $t = 1$ of h. In particular, h determines the rank of P. The following argument to bound the exponent is as in the proof of [10, Theorem]. By a result of Brauer, $\dim_k(Z(B))$ is equal to the sum $\sum_{(u,e)} \ell(C_G(u), e)$, where (u, e) runs over a set of representatives of the G-conjugacy classes of B-Brauer elements, and where $\ell(C_G(u), e)$ is the number of isomorphism classes of simple $kC_G(u)e$-modules. Therefore, $\dim_k(Z(B))$ bounds the number of summands in this sum, hence the number of different orders of elements in P; in particular, the exponent of P is bounded in terms of a function depending only on $\dim_k(Z(B)) = a_0$. Since the order of a finite p-group is bounded in terms of its rank and exponent (see Lemma 9 below) the result follows. □

For the convenience of the reader, we include the following elementary group theoretic fact used in the proof of Theorem 3:

Lemma 9. *Let r and n be positive integers. There are only finitely many isomorphism classes of finite p-groups of rank r and exponent p^n.*

Proof. Let P be a finite p-group of rank r and exponent p^n. Let A be a subgroup of P which is maximal with respect to being normal and abelian. Since the rank of A is at most r and the exponent of A is at most p^n, the order of A is at most p^{nr}. By [8, Lemma 5.3.12] we have $C_P(A) = A$. Thus P/A acts faithfully on A, hence is isomorphic to a subgroup of $\mathrm{Aut}(A)$. This shows that the orders of A and of P/A are bounded in terms of r, n, and hence so is the order of P. □

Acknowledgements. The authors would like to thank Dave Benson and Peter Symonds for helpful comments on the content of this note.

References

[1] D. J. Benson, *Polynomial invariants of Finite Groups*, London Math. Soc. **190**, Cambridge University Press (1993).

[2] D. J. Benson, *Dickson invariants, regularity and computation in group cohomology*, Illinois J. Math. **48** (2004), 171–197.

[3] D. J. Benson, *Commutative algebra in the cohomology of groups.* Trends in Commutative Algebra, MSRI Publication **51** (2004).

[4] D. J. Benson, *On the regularity conjecture for the cohomology of finite groups*, Proc. Edinburgh Math. Soc. **51** (2008), 273–284.

[5] W. Bruns and J. Herzog, *Cohen-Macaulay rings.* Cambridge studies adv. math. **39**, Cambridge University Press (1993).

[6] G. Cliff, W. Plesken, and A. Weiss, *Order-theoretic properties of the center of a block*, The Arcata Conference on Representation Theory of Finite Groups (1986).

[7] D. Eisenbud, *The Geometry of Syzygues.* Graduate Texts in Math. **229**, Springer Verlag, New York (2005).

[8] D. Gorenstein *Finite Groups*, Chelsea Publishing Company, New York (1980)

[9] R. Kessar and M. Linckelmann, *Bounds for Hochschild cohomology of block algebras*, J. Algebra 337 (2011), 318–322.

[10] B. Külshammer and G. R. Robinson, *Alperin-McKay implies Brauer's problem* 21, J. Algebra **180** (1996), 208–210.

[11] M. Linckelmann, *Transfer in Hochschild cohomology of blocks of finite groups*, Algebras Representation Theory **2** (1999), 107–135.

[12] M. Linckelmann, *Varieties in block theory*, J. Algebra **215** (1999), 460–480.

[13] P. Symonds, *On the Castelnuovo-Mumford regularity of the cohomology ring of a group*, J. Amer. Math. Soc. **23** (2010), 1159–1173.

[14] P. Symonds, *On the Castelnuovo-Mumford regularity of rings of polynomial invariants*, Ann. of Math. (2) **174** (2011), no. 1, 499–517.