Hochschild and block cohomology varieties are isomorphic

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Abstract

We show that the varieties of the Hochschild cohomology of a block algebra and its block cohomology are isomorphic, implying positive answers to questions of Pakianathan and Witherspoon in [16] and [17]. We obtain as a consequence that the cohomology $H^*(G; k)$ of a finite group G with coefficients in a field k of characteristic p is a quotient of the Hochschild cohomology of the principal block of kG by a nilpotent ideal.

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1 Introduction

In [16], Pakianathan and Witherspoon raised the question whether the Hochschild cohomology $HH^*(B)$ of a block algebra B of a finite group G over an algebraically closed field k of prime characteristic p and the corresponding block cohomology $H^*(B)$ are isomorphic modulo nilpotent ideals. They showed this in a number of cases, including for blocks with cyclic defect groups and blocks of Frobenius groups; for blocks with an abelian defect group P and an abelian inertial quotient acting freely on $P - \{1\}$ this follows also from [9, 1.4]. Both $H^*(B)$ and $HH^*(B)$ are graded commutative Noetherian rings, and hence their maximal ideal spectra are affine varieties, denoted V_B and X_B, respectively. In [17], Pakianathan and Witherspoon described a stratification for the variety X_B, similar to Quillen's group cohomology stratification in [20], and deduced from this that the varieties X_B and V_B are at least homeomorphic for blocks of principal type. The purpose of this paper is to show that varieties X_B and V_B are isomorphic in general. As in [17], a key ingredient is a detection result for nilpotent elements in certain Ext-algebras, due to Siegel [17, Appendix, 5.1]. Other ingredients are the techniques developed in [17] applied to source algebras rather than block algebras, combined with the transfer technology from [10], in particular the canonical map [10, Theorem 5.6.(iii)] from block cohomology $H^*(B)$ to Hochschild cohomology $HH^*(B)$ of a block algebra B of a finite group.

Theorem 1.1. *Let B be a block algebra of a finite group G over an algebraically closed field k of prime characteristic p with defect group P. The canonical graded algebra homomorphism τ from block cohomology $H^*(B)$ to the Hochschild cohomology $HH^*(B)$ induces an isomorphism modulo nilpotent ideals, or equivalently, induces an isomorphism of varieties $X_B \cong V_B$.*

The interest of Theorem 1.1 lies in the fact that V_B is an invariant of the fusion system $\mathcal{F}$ of the block B while X_B is an invariant of the Morita equivalence class of B, and the relationship between the two is far from understood, to say the least. In one way or another, most prominent conjectures in block theory try to predict how invariants of the module category of B and invariants of the local structure of B determine each other. The isomorphism $X_B \cong V_B$ implies in particular that Quillen's stratification (extended to blocks in [13]) carries over to X_B, where it coincides for blocks of principal type with the stratification described in [17]. If B is the principal block of kG then τ is split injective, with retraction $HH^*(B) \to H^*(G,k)$ induced by the functor $- \otimes_B k$, and thus the above result has the following immediate consequence:

Corollary 1.2. *If B is the principal block of kG then the functor $- \otimes_B k$ induces a split surjective algebra homomorphism $HH^*(B) \to H^*(G,k)$ with nilpotent kernel.*

Since the principal block of kG is defined over the prime field $\mathbb{F}_p$, this statement remains true even if k is not algebraically closed. For non principal blocks it is not known whether τ is a split monomorphism, and hence we do not know whether block cohomology is a quotient of Hochschild cohomology. We list further easy consequences of Theorem 1.1 which are true for arbitrary blocks. The isomorphism in Theorem 1.1 restricts to the corresponding varieties of a finitely generated B-module M. The functor $- \otimes_B M$ induces a unitary algebra homomorphism $\mu_M : HH^*(B) \to \mathrm{Ext}^*_B(M,M)$; we denote by $X_B(V)$ the maximal ideal spectrum of $HH^*(B)/\ker(\mu_M)$, and by $V_B(M)$ the maximal ideal spectrum of $H^*(B)/\ker(\mu_M \circ \tau)$.

Corollary 1.3. *For any finitely generated B-module M, restriction along τ induces an isomorphism of varieties $X_B(M) \cong V_B(M)$.*

In other words, for block algebras the theory of varieties associated with Hochschild cohomology, as developed by Snashall and Solberg in [23], coincides with that using block cohomology in [11]. If the module M has a source of dimension prime to p then $V_B(M) = V_B$, and hence Corollary 1.3 implies the following.

Corollary 1.4. *For any finitely generated B-module M having a source of dimension prime to p, the canonical homomorphism $HH^*(B) \to \mathrm{Ext}^*_B(M,M)$ induced by the functor $- \otimes_B M$ has a nilpotent kernel.*

By a result of Snashall and Solberg [23, 4.4], for any finite-dimensional algebra A over some field having a separable semisimple quotient the canonical algebra homomorphism from $HH^*(A)$ to the graded center $Z^*(D^b(\mathrm{mod}(A)))$ of the derived bounded category of finitely generated A-modules has a nilpotent kernel; one can also use 1.4 to prove this in the special case of block algebras. The positive part of $HH^*(B)$ is invariant under stable equivalences of Morita type, and thus block varieties are invariant under stable equivalences of Morita type:

Corollary 1.5. *Let B, C be block algebras of finite groups G, H, respectively, and let M be a C-B-bimodule which is finitely generated projective as left and right module, and which induces a stable equivalence of Morita type between B and C. Then for any finitely generated B-module U we have an isomorphism of varieties $V_B(U) \cong V_C(M \otimes_B U)$. In particular, $V_B \cong V_C$.*

If the C-B-bimodule M inducing a stable equivalence of Morita type between the blocks B, C of G, H has a trivial source as $k(H \times G)$-module then M induces more precisely an isomorphism of the block cohomology rings $H^*(B) \cong H^*(C)$, but it is not known in general whether a stable equivalence of Morita type induces such an isomorphism; this is not even known for arbitrary Morita equivalences. The main step in the proof of Theorem 1.1 consists in showing first a slightly weaker statement, namely that τ induces an inseparable isogeny:

Theorem 1.6. *Let A be a source algebra of a block with defect group P and fusion system $\mathcal{F}$. Identify the block cohomology $H^*(B)$ with the subalgebra of $\mathcal{F}$-stable elements in $H^*(P;k)$ and* *denote by $\rho : H^*(B) \to \lim_{\mathcal{E}} (H^*(E;k))$ the map induced by the product of the restriction maps from P to $E \in \mathcal{E}$, where $\mathcal{E}$ is the full subcategory of $\mathcal{F}$ consisting of all elementary abelian subgroups of P. Denote by $\tau : H^*(B) \to HH^*(A)$ the canonical injective graded algebra homomorphism. Then there is a graded algebra homomorphism σ making the following diagram commutative*

$$\begin{tikzcd} HH^*(A) \arrow[rr, "\sigma"] & & \lim_{\mathcal{E}} (H^*(E;k)) \\ & H^*(B) \arrow[ul, "\tau"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

such that $\ker(\sigma)$ *is nilpotent; in particular, σ induces an inseparable isogeny.*

Section 2 contains some background material and technical statements on product structures in certain Ext-algebras of interior P-algebras, which is used in Section 3 in order to define the map σ in Theorem 1.6. In Section 4 this material is specialised to source algebras and then used in Section 5 to prove Theorem 1.6 and Corollary 1.2. The proof of Theorem 1.1 and the remaining corollaries are given in Section 6.

2 Products in Ext-algebras of interior P-algebras

Let G be a finite group and k a field. By the Eckmannn-Shapiro Lemma applied to the diagonal subgroup $\Delta G = \{(x,x)|\ x \in G\}$ of $G \times G$ there is a graded k-linear isomorphism $HH^*(kG) \cong H^*(\Delta G; kG)$. Siegel and Witherspoon showed in [22, 3.1] that the product structure of $HH^*(kG)$ can be identified in $H^*(\Delta G; kG)$ as being given by the cup product followed by multiplication in kG. In this section we show that this carries over to arbitrary interior algebras over some finite group - which we will denote by P, since the results of this section will be applied to source algebras of blocks with a defect group P. All tensor products without subscripts are tensor products over the base field k, which for now need not be algebraically closed or of positive characteristic. Let A be an interior P-algebra with structural homomorphism $\sigma : P \to A^\times$. If j is an idempotent in the fixpoint subalgebra $A^{\Delta P}$ with respect to the conjugation action then jAj is again an interior P-algebra, with structural homomorphism sending $u \in P$ to $\sigma(u)j = j\sigma(u) = j\sigma(u)j$. The algebra $A \otimes kP$ is an interior $k\Delta P$-algebra, with structural homomorphism sending (u,u) to $\sigma(u) \otimes u$, where $u \in P$. We view Aj as $A \otimes kP$-module with $a \otimes u$ acting on $b \in Aj$ by $ab\sigma(u^{-1})$, where $a \in A$ and $u \in P$. This restricts to a $k\Delta P$-module structure on jAj with (u,u) acting on $b \in jAj$ by $\sigma(u)b\sigma(u^{-1})$. A suitable version of Eckmann-Shapiro for restriction from $A \otimes kP$ to $k\Delta P$ via the bimodule $Aj \otimes kP$ yields a graded k-linear isomorphism $\mathrm{Ext}^*_{A \otimes kP}(Aj, Aj) \cong H^*(\Delta P; jAj)$ described in more detail in the proof of 2.1 below. The left side is an Ext-algebra, so has a product. The right side has a product, too, given by the cup product and multiplication in jAj as follows: denote by V a projective resolution of k as trivial $k\Delta P$-module, let $\mu \in H^m(\Delta P; jAj)$ and $\nu \in H^n(\Delta P; jAj)$ be represented by chain maps (denoted abusively by the same letters) from V to jAj viewed as complex of $k\Delta P$-modules concentrated in degree m and n, respectively. Then $\mu \otimes \nu$ is chain map from $V \otimes V$ to $jAj \otimes jAj$, the latter viewed as complex of $k\Delta P$-modules concentrated in degree $m+n$. Precomposing this with the homotopy equivalence $V \simeq V \otimes V$ lifting the isomorphism $k \cong k \otimes k$ and composing this with the map $jAj \otimes jAj \to jAj$ given by multiplication in A yields an element in $H^{m+n}(k\Delta P; jAj)$. We consider $H^*(k\Delta P; jAj)$ endowed with this product. Note that in order to define this product structure on $H^*(\Delta P; jAj)$ we only need the structure of A as ΔP-algebra - this definition does not make use of the interior P-algebra structure.

Proposition 2.1. *Let P be a finite group, A a finite-dimensional interior P-algebra over a field k and j an idempotent in $A^{\Delta P}$. There is a canonical isomorphism of graded k-algebras*

$$\mathrm{Ext}^*_{A\otimes kP}(Aj, Aj) \cong H^*(\Delta P; jAj)$$

Proof. Denote by $\sigma : P \to A^\times$ the structural homomorphism of A. We first show that we may assume that $j = 1$. Let W be a projective resolution of jAj as $jAj \otimes kP$-module. The restriction of W on the left is then a projective resolution of the projective jAj-module jAj, hence split. This implies that the complex $Aj \otimes_{jAj} W$ is a projective resolution of $Aj \otimes_{jAj} jAj \cong Aj$ as $A \otimes kP$-module; multiplying this resolution by j on the left yields again the resolution W. Thus adjunction via the A-jAj-bimodule Aj shows that multiplication by j yields a graded k-algebra isomorphism $\mathrm{Ext}^*_{A\otimes kP}(Aj, Aj) \cong \mathrm{Ext}^*_{jAj\otimes kP}(jAj, jAj)$. This shows that after replacing A by jAj we may assume $j = 1$. Note that there is a canonical isomorphism of $A \otimes kP$-modules

$$A \cong (A \otimes kP) \ \otimes_{k\Delta P} \ k$$

sending $a \in A$ to $(a \otimes 1_P) \otimes 1_k$, with inverse sending $(a \otimes u) \otimes \lambda$ to $\lambda a\sigma(u^{-1})$. Denote by V a projective resolution of the trivial $k\Delta P$-module k. Set $U = (A \otimes kP) \otimes_{k\Delta P} \ V$. Then, via the above isomorphism, U is a projective resolution of A as $A \otimes kP$-module. Adjunction via the $A\otimes kP$-$k\Delta P$-bimodule $A\otimes kP$ yields a graded k-linear isomorphism $\mathrm{Ext}^*_{A\otimes kP}(A, A) \cong H^*(\Delta P; A)$. In order to show that this is an algebra isomorphism we describe this map explicitly in terms of adjunction counits. Denote by ${}_{\Delta P}U$ the restriction of U to $k\Delta P$; this is a projective resolution of A as $k\Delta P$-module. The unitary map $\iota : k \to A$, viewed as homomorphism of $k\Delta P$-modules, is the adjunction counit of induction to $A \otimes kP$ followed by restriction back to $k\Delta P$ evaluated at k. This map lifts to a chain map of complexes of $k\Delta P$-modules $\mathrm{adj} : V \to {}_{\Delta P}U$, uniquely up to homotopy, and we may choose this chain map to send $y \in V$ to $(1_A \otimes 1_{kP}) \otimes y$, and again this is the appropriate adjunction counit evaluated at V. Multiplication in A is in particular a homomorphism of $k\Delta P$-modules from $A \otimes A$ to A, and hence lifts uniquely, up to homotopy, to a chain map $\pi : {}_{\Delta P}U \otimes {}_{\Delta P}U \to {}_{\Delta P}U$. The obvious commutative diagram of $k\Delta P$-modules with vertical maps given by multiplication in A

$$\begin{tikzcd} A \otimes k \arrow[r, "\mathrm{Id}\otimes\iota"] \arrow[d] & A \otimes A \arrow[d] \\ A \arrow[r, equal] & A \end{tikzcd}$$

lifts to a homotopy commutative diagram of chain complexes of $k\Delta P$-modules

(1)

$$\begin{tikzcd} {}_{\Delta P}U \otimes V \arrow[r, "\mathrm{Id}\otimes\mathrm{adj}"] \arrow[d, "\simeq"'] & {}_{\Delta P}U \otimes {}_{\Delta P}U \arrow[d, "\pi"] \\ {}_{\Delta P}U \arrow[r, equal] & {}_{\Delta P}U \end{tikzcd}$$

For any chain map of complexes of $A \otimes kP$-modules $\tau : U \to U'$, multiplication by A on the left of these two complexes yields a commutative diagram of chain complexes of $A \otimes kP$-modules

$$\begin{tikzcd} A \otimes U \arrow[r, "\mathrm{Id}\otimes\tau"] \arrow[d] & A \otimes U' \arrow[d] \\ U \arrow[r, "\tau"'] & U' \end{tikzcd}$$

where the vertical maps send $a \otimes u$ and $a \otimes u'$ to au and au', respectively, for $a \in A$, $u \in U$ and $u' \in U'$; the commutativity uses the fact that τ is a homomorphism of chain complexes of left A-modules. This diagram lifts, uniquely up to homotopy, to a homotopy commutative diagram of chain complexes of $A \otimes kP$-modules of the form

(2)

$$\begin{tikzcd} U \otimes U \arrow[r, "\mathrm{Id}\otimes\tau"] \arrow[d, "\pi"'] & U \otimes U' \arrow[d, "\pi'"] \\ U \arrow[r, "\tau"'] & U' \end{tikzcd}$$

Here the $A \otimes kP$-module structure in the first line is given by the left A-module structure on the left factor U in the tensor products and the right action of kP on the right factors. The k-linear graded isomorphism $\mathrm{Ext}^*_{A\otimes kP}(A, A) \cong H^*(\Delta P; A)$ is induced by the adjunction map $V \to {}_{\Delta P}U$. More precisely, this isomorphism sends an element in $\mathrm{Ext}^m_{A\otimes kP}(A, A)$ represented by a chain map

$$\zeta : U \longrightarrow U[m]$$

to the element in $H^m(\Delta P; A)$ represented by the chain map

$$\mu = \zeta \circ \mathrm{adj} : V \xrightarrow{\ \mathrm{adj}\ } {}_{\Delta P}U \xrightarrow{\ \zeta\ } {}_{\Delta P}U[m]$$

where we have used the same letter ζ for the restriction of ζ to a $k\Delta P$-homomorphism. In order to show that this graded isomorphism is a homomorphism of algebras we need to show that if

$$\tau : U \longrightarrow U[n]$$

is a chain map representing an element in $\mathrm{Ext}^n_{A\otimes kP}(A, A)$ and if

$$\nu = \tau \circ \mathrm{adj} : V \xrightarrow{\ \mathrm{adj}\ } {}_{\Delta P}U \xrightarrow{\ \tau\ } {}_{\Delta P}U[n]$$

then the chain map

$$U \xrightarrow{\ \zeta\ } U[m] \xrightarrow{\ \tau[m]\ } U[m+n]$$

representing the product $\zeta \cdot \tau$ in $\mathrm{Ext}^{m+n}_{A\otimes kP}(A, A)$ corresponds, via adjunction, to the chain map

$$V \cong V \otimes V \xrightarrow{\ \mu\otimes\nu\ } {}_{\Delta P}U[m] \otimes {}_{\Delta P}U[n] \xrightarrow{\ \pi[m+n]\ } {}_{\Delta P}U[m+n]$$

which represents the product of μ and ν in $H^{m+n}(\Delta P; A)$. That is, we have to show that the diagram of chain complexes of $k\Delta P$-modules

$$\begin{tikzcd} V \arrow[r, "\mathrm{adj}"] \arrow[d, equal] & {}_{\Delta P}U \arrow[r, "\zeta"] & {}_{\Delta P}U \arrow[r, "\tau[m]"] & {}_{\Delta P}U[m+n] \arrow[d, equal] \\ V \arrow[r, "\simeq"'] & V \otimes V \arrow[r, "\mu\otimes\nu"'] & {}_{\Delta P}U[m] \otimes {}_{\Delta P}U[n] \arrow[r, "\pi[m+n]"'] & {}_{\Delta P}U[m+n] \end{tikzcd}$$

is chain homotopy commutative. Since $\mu = \zeta \circ \mathrm{adj}$ this is equivalent to showing that the diagram

$$\begin{tikzcd} V \arrow[r, "\mu"] \arrow[d, "\simeq"'] & {}_{\Delta P}U[m] \arrow[r, "\tau[m]"] & {}_{\Delta P}U[m+n] \arrow[d, equal] \\ V \otimes V \arrow[r, "\mu\otimes\nu"'] & {}_{\Delta P}U[m] \otimes {}_{\Delta P}U[n] \arrow[r, "\pi[m+n]"'] & {}_{\Delta P}U[m+n] \end{tikzcd}$$

is chain homotopy commutative. By writing $\mu \otimes \nu = (\mathrm{Id} \otimes \nu) \circ (\mu \otimes \mathrm{Id})$ it follows that it suffices to show that the diagram

$$\begin{tikzcd} V \arrow[r, "\mu"] \arrow[d, "\simeq"'] & {}_{\Delta P}U[m] \arrow[rr, "\tau[m]"] \arrow[d, "\simeq"] & & {}_{\Delta P}U[m+n] \arrow[d, equal] \\ V \otimes V \arrow[r, "\mu\otimes\mathrm{Id}"'] & {}_{\Delta P}U[m] \otimes V \arrow[r, "\mathrm{Id}\otimes\nu"'] & {}_{\Delta P}U[m] \otimes {}_{\Delta P}U[n] \arrow[r, "\pi[m+n]"'] & {}_{\Delta P}U[m+n] \end{tikzcd}$$

is chain homotopy commutative. The left square in this diagram is homotopy commutative because it lifts the obvious commutative square

$$\begin{tikzcd} V \arrow[r, "\mu"] \arrow[d, "\cong"'] & {}_{\Delta P}U[m] \arrow[d, "\cong"] \\ V \otimes k \arrow[r, "\mu\otimes\mathrm{Id}"'] & {}_{\Delta P}U[m] \otimes k \end{tikzcd}$$

via the quasi-isomorphism $V \to k$. Therefore it remains to show that the diagram

$$\begin{tikzcd} {}_{\Delta P}U[m] \arrow[r, "\tau[m]"] & {}_{\Delta P}U[m+n] \\ {}_{\Delta P}U[m] \otimes V \arrow[r, "\mathrm{Id}\otimes\nu"'] \arrow[u, "\simeq"] & {}_{\Delta P}U[m] \otimes {}_{\Delta P}U[n] \arrow[u, "\pi[m+n]"'] \end{tikzcd}$$

is homotopy commutative. Using $\nu = \tau \circ \mathrm{adj}$ (and shifting by $-m$) it suffices to show that the diagram

$$\begin{tikzcd} {}_{\Delta P}U \arrow[r, equal] & {}_{\Delta P}U \arrow[r, "\tau"] & {}_{\Delta P}U[n] \\ {}_{\Delta P}U \otimes V \arrow[r, "\mathrm{Id}\otimes\mathrm{adj}"'] \arrow[u, "\simeq"] & {}_{\Delta P}U \otimes {}_{\Delta P}U \arrow[r, "\mathrm{Id}\otimes\tau"'] \arrow[u, "\pi"'] & {}_{\Delta P}U \otimes {}_{\Delta P}U[n] \arrow[u, "\pi[n]"'] \end{tikzcd}$$

is homotopy commutative. The left square in this diagram is homotopy commutative thanks to the diagram (1) above, and the right square is homotopy commutative thanks to (2). This completes the proof. $\square$

The following two results describe the obvious functoriality properties of the above isomorphism.

Proposition 2.2. *Let P be a finite group and $\sigma : A \to B$ a homomorphism of finite-dimensional ΔP-algebras over a field k. The map $\eta : H^*(\Delta P; A) \to H^*(\Delta P; B)$ induced by σ is a homomorphism of graded k-algebras.*

Proof. Let m, n be positive integers and V a projective resolution of k as $k\Delta P$-module. Let $\mu \in H^m(\Delta; A)$, $\nu \in H^n(\Delta P; A)$ represented by chain maps $\mu : V \to A[m]$, $\nu : V \to A[n]$. The element $\eta(\mu) \in H^m(\Delta; B)$ is represented by the chain map $\eta(\mu) = \sigma[m] \circ \mu : V \to B[m]$; similarly for $\eta(\nu)$. The result follows from the obvious homotopy commutative diagram

$$\begin{tikzcd}
V \arrow[r, "\simeq"] \arrow[d, equal] & V \otimes V \arrow[r, "\mu\otimes\nu"] & A[m] \otimes A[n] \arrow[r] \arrow[d, "\sigma[m]\otimes\sigma[n]"] & A[m+n] \arrow[d, "\sigma[m+n]"] \\
V \arrow[r, "\simeq"'] & V \otimes V \arrow[r, "\eta(\mu)\otimes\eta\nu"'] & B[m] \otimes B[n] \arrow[r] & B[m+n]
\end{tikzcd}$$

where the horizontal homotopy equivalences $V \simeq V \otimes V$ lift the isomorphism $k \cong k \otimes k$ and where the horizontal maps in the right square are induced by multiplication in A and B, respectively. The commutativity of the right square in this diagram uses the fact that σ is a homomorphism of ΔP-algebras. The first row of this diagram composed with $\sigma[m+n]$ represents η applied to the product of μ and ν, and the second row represents the product of $\eta(\mu)$ and $\eta(\nu)$, whence the result. $\square$

If $\sigma : A \to B$ above is actually a homomorphism of *interior* P-algebras then the algebra homomorphism induced by σ can be interpreted via the isomorphism of 2.1 as follows:

Proposition 2.3. *Let P be a finite group and $\sigma : A \to B$ a homomorphism of finite-dimensional interior P-algebras over a field k. Consider B as a B-A-bimodule with the right A-module structure induced via σ. The functor $B\otimes_A-$ induces a graded k-algebra homomorphism $\beta : \mathrm{Ext}^*_{A\otimes kP}(A, A) \to \mathrm{Ext}^*_{B\otimes kP}(B, B)$, and the graded k-linear map $\eta : H^*(\Delta P; A) \to H^*(\Delta P; B)$ induced by σ is an algebra homomorphism making the following diagram of graded k-algebras commutative:*

$$\begin{tikzcd}
\mathrm{Ext}^*_{A\otimes kP}(A, A) \arrow[r, "\cong"] \arrow[d, "\beta"'] & H^*(\Delta P; A) \arrow[d, "\eta"] \\
\mathrm{Ext}^*_{B\otimes kP}(B, B) \arrow[r, "\cong"'] & H^*(\Delta P; B)
\end{tikzcd}$$

where the horizontal isomorphisms are from 2.1.

Proof. We have an obvious isomorphism of B-A-bimodules $B \otimes_A A \cong B$ sending $b \otimes a$ to $b\sigma(a)$. Let U be a projective resolution of A as $A \otimes kP$-module and V a projective resolution of k as $k\Delta P$-module. As before denote by ${}_{\Delta P}U$ the restriction of U to a complex of ΔP-modules. The unitary map $k \to A$, viewed as homomorphism of $k\Delta P$-modules, lifts to a chain map $V \to {}_{\Delta P}U$, uniquely up to homotopy. Since A is projective as left A-module, the restriction to A on the left of U yields a split complex of projective A-modules with homology isomorphic to A in degree zero. Thus $B \otimes_A U$ is still a projective resolution of B as $B \otimes kP$-module, and hence the functor $B \otimes_A -$ induces indeed a homomorphism of graded k-algebras β as claimed. The unitary map $k \to B$, again viewed as homomorphism of $k\Delta P$-modules, lifts to a chain map $V \to {}_{\Delta P}(B \otimes_A U)$. The map σ, viewed as homomorphism of $k\Delta P$-modules, lifts to the chain map ${}_{\Delta P}U \to {}_{\Delta P}(B \otimes_A U)$ sending $u \in U$ to $1_B \otimes u$. Let n be a nonnegative integer and $\zeta \in \mathrm{Ext}^n_{A\otimes kP}(A, A)$ represented by a chain map $\zeta : U \to U[n]$. Then $\beta(\zeta)$ is represented by the chain map $\mathrm{Id} \otimes \zeta : B \otimes_A U \to B \otimes_A U[n]$. Since σ is a unitary homomorphism, the above chain maps make the following diagram of chain complexes of $k\Delta P$-modules chain homotopy commutative:

$$\begin{tikzcd}
V \arrow[r] \arrow[d, equal] & {}_{\Delta P}U \arrow[r, "\zeta"] \arrow[d] & {}_{\Delta P}U[n] \arrow[d] \\
V \arrow[r] & {}_{\Delta P}(B \otimes_A U) \arrow[r, "\mathrm{Id}\otimes\zeta"'] & {}_{\Delta P}(B \otimes_A U[n])
\end{tikzcd}$$

The first row in this diagram represents the image of ζ in $H^n(\Delta P; A)$, and the second row represents the image of $\beta(\zeta)$ in $H^n(\Delta P; B)$, which implies the result. $\square$

Given a graded k-algebra H^* we denote by $Z(H^*)$ the center of H^* in the graded sense; that is, the component of $Z(H^*)$ in degree n consists of all $\zeta \in H^n$ with the property that for all m and all $\xi \in H^m$ we have $\zeta\xi = (-1)^{nm}\xi\zeta$; this is also sometimes called *graded center* of H^* - but since we use this terminology already in the context of triangulated categories, we simply call $Z(H^*)$ the center of H^* as graded algebra. If $\mathrm{char}(k) = 2$ this is the ususal center of H^* as (ungraded) k-algebra; if $\mathrm{char}(k) > 2$ then the even part of $Z(H^*)$ is the center of the even part of H^*.

Lemma 2.4. *Let P be a finite group and A a finite-dimensional interior P-algebra over a field k. For any subgroup E of P, the unitary map $k \to A$ induces a graded k-algebra homomorphism $\iota^A_E : H^*(\Delta E; k) \to H^*(\Delta E; A)$, the induction functor $(A \otimes kE) \otimes_{k\Delta E} -$ induces a graded k-algebra homomorphism $\kappa^A_E : H^*(\Delta E; k) \to \mathrm{Ext}^*_{A\otimes kE}(A, A)$, the functor $A \otimes_{kE} -$ applied to kE-kE-bimodules induces a graded k-algebra homomorphism $HH^*(kE) \to \mathrm{Ext}^*_{A\otimes kE}(A, A)$ and diagonal induction from $E \cong \Delta E$ to $E\times E$ induces a graded k-algebra homomorphism $H^*(E; k) \to HH^*(kE)$ making the following diagram commutative*

$$\begin{tikzcd}
H^*(E; k) \arrow[r, "\cong"] \arrow[d, "\mathrm{Ind}^{E\times E}_{\Delta E}"'] & H^*(\Delta E; k) \arrow[r, equal] \arrow[d, "\kappa^A_E"] & H^*(\Delta E; k) \arrow[d, "\iota^A_E"] \\
HH^*(kE) \arrow[r, "A\otimes_{kE}-"'] & \mathrm{Ext}^*_{A\otimes kE}(A; A) \arrow[r, "\cong"'] & H^*(\Delta E; A)
\end{tikzcd}$$

where the lower right horizontal isomorphism is from 2.1. Moreover, the images of κ^A_E and ι^A_E are contained in the graded centers of $\mathrm{Ext}^*_{A\otimes kE}(A, A)$ and $H^*(\Delta E; A)$, respectively.

Proof. The unitary map $k \to A$ is a homomorphism of ΔE-algebras, hence induces an algebra homomorphisms $\iota^A_E : H^*(\Delta E; k) \to H^*(\Delta E; A)$ by 2.3. Let V be a projective resolution of the trivial $k\Delta E$-module k. Since $(A \otimes kE) \otimes_{k\Delta E} k \cong A$ as $A \otimes kE$-module, the complex $U = (A \otimes kE) \otimes_{k\Delta E} V$ is a projective resolution of A as $A \otimes kE$-module, and hence the functor $(A \otimes kE) \otimes_{k\Delta E} -$ induces an algebra homomorphism $\kappa_E^A : H^*(\Delta E; k) \to \mathrm{Ext}^*_{A \otimes kE}(A, A)$. Transitivity of induction functors induced by tensoring with bimodules implies the commutativity of the left square of the diagram. For the right square, let n be an integer and $\xi \in H^n(\Delta E; k)$ represented by a chain map $\xi : V \to k[n]$. Then $\kappa_E^A(\xi)$ is represented by the chain map $\mathrm{Id} \otimes \xi : U \to A \otimes k[n] = A[n]$. The isomorphism from 2.1 maps $\kappa_E^A(\xi)$ to the element in $H^n(\Delta E; A)$ represented by the chain map $V \to {}_{\Delta E}U$ obtained from precomposing $\mathrm{Id} \otimes \xi$ with the inclusion $V \to {}_{\Delta E}U$ sending $v \in V$ to $1 \otimes v$, which clearly is a chain map representing $\iota_E^A(\tau)$. This shows the commutativity of the right square in the diagram. The last statement follows from [17, Lemma 5.7]. □

We will need the following special case of a result of Snashall and Solberg:

Proposition 2.5 (cf. [23, Theorem 1.1])**.** *Let P be a finite group and A a finite-dimensional interior P-algebra over a field k such that A is projective as right kP-module, let E be a subgroup of P and j an idempotent in $A^{\Delta E}$. The image of the canonical graded k-algebra homomorphism $HH^*(A) \to \mathrm{Ext}^*_{A \otimes kE}(Aj, Aj)$ induced by tensoring with the A-kE-bimodule Aj is contained in the center $Z(\mathrm{Ext}^*_{A \otimes kE}(Aj, Aj))$ of the graded algebra $\mathrm{Ext}^*_{A \otimes kE}(Aj, Aj)$.*

3 On the Brauer construction in cohomology

Let k be an algebraically closed field of prime characteristic p. Let P be a finite group. For A a ΔP-algebra we denote by $A(\Delta P)$ the Brauer construction (cf. [24, §11]); that is, $A(\Delta P)$ is the quotient of the fixpoint subalgebra $A^{\Delta P}$ by the ideal of all elements which can be written as sums of proper trace $\sum_{x \in [P/Q]} xax^{-1}$ for some proper subgroup Q of P and some $a \in A^{\Delta Q}$, where $[P/Q]$ is a system of coset representatives of Q in P. Following the notation in [17, §2] and [22, §3] we denote by $\bar{H}^*(P; k)$ the quotient of $H^*(P; k)$ by the ideal $\sum_{Q<P} \mathrm{tr}_Q^P(H^*(Q; k))$, where Q runs over the proper subgroups of P and where $\mathrm{tr}_Q^P : H^*(Q; k) \to H^*(P; k)$ is the usual transfer map in group cohomology. If P is a not a p-group it is well-known that then $\bar{H}^*(P; k) = 0$, and if E is an elementary abelian p-group then $\bar{H}^*(E; k) \cong H^*(E; k)$ by [6, 6.3.4]. For M a kP-module we denote by $\bar{H}^*(P; M)$ the quotient of $H^*(P; M)$ by the subspace $\sum_{Q<P} \mathrm{tr}_Q^P(H^*(Q; \mathrm{Res}_Q^P(M)))$; in degree zero this is the usual Brauer construction applied to M. The canonical $H^*(P; k)$-module structure on $H^*(P; M)$ induces a $\bar{H}^*(P; k)$-module structure on $\bar{H}^*(P; M)$. The main purpose of this section is to prove that under suitable hypotheses, the canonical map $\iota_E^A : H^*(\Delta E; k) \to Z(H^*(\Delta E; A))$ from 2.4 is a split monomorphism.

Proposition 3.1. *Let P be a finite p-group and A a finite-dimensional ΔP-algebra over k. Suppose that A has a ΔP-stable k-basis X such that $X^{\Delta P} \neq \emptyset$, or equivalently, such that $A(\Delta P) \neq \{0\}$. Let E be a subgroup of P such that $Z(A(\Delta E))$ is local, or equivalently, such that the algebra $A(\Delta E)$ is indecomposable. There is a unique graded k-algebra homomorphism $\pi_E^A : Z(H^*(\Delta E; A)) \to \bar{H}^*(\Delta E; k)$ such that $\pi_E^A \circ \iota_E^A : H^*(\Delta E; k) \to \bar{H}^*(\Delta E; k)$ is equal to the canonical surjection, and such that π_E^A factors through the canonical map $Z(H^*(\Delta E; A)) \to Z(\bar{H}^*(\Delta E; A))$. In particular, if E is elementary abelian then $\pi_E^A \circ \iota_E^A = \mathrm{Id}_{H^*(\Delta E; k)}$.*

Before getting into the details of the proof - which we break up into two lemmas - we briefly describe where the hypotheses come into play. With the notation above, denote by $X^{\Delta E}$ the subset of all basis elements in X which are fixed by all elements in ΔE. As $k\Delta E$-module, A has a decomposition of the form $A = k[X^{\Delta E}] \oplus k[X - X^{\Delta E}]$. The Brauer homomorphism $\mathrm{Br}^A_{\Delta E}$: $A^{\Delta E} \to A(\Delta E)$ maps $X^{\Delta E}$ to a k-basis of $A(\Delta E)$, and every indecomposable direct summand of $k[X - X^{\Delta E}]$ is induced from a proper subgroup of ΔE. Thus, as graded k-modules, we have an isomorphism $\bar{H}^*(\Delta E; A(\Delta E)) \cong \bar{H}^*(\Delta E; A)$, where $A(\Delta E)$ is viewed as $k\Delta E$-module with the trivial action of ΔE. Both sides in this isomorphism have canonical algebra structures. The next result shows that there is a canonical choice for this isomorphism which is an isomorphism of graded k-algebras. We view $A^{\Delta E}$ and $A(\Delta E)$ as ΔE-algebras with the trivial action of ΔE.

Lemma 3.2. *Let P be a finite p-group and A a finite-dimensional ΔP-algebra over k. Suppose that A has a ΔP-stable k-basis X such that $X^{\Delta P} \neq \emptyset$, or equivalently, such that $A(\Delta P) \neq \{0\}$. Let E be a subgroup of P. The inclusion $A^{\Delta E} \to A$ induces a graded k-algebra homomorphism $\epsilon : H^*(\Delta E; A^{\Delta E}) \to H^*(\Delta E; A)$, the Brauer homomorphism $\mathrm{Br}^A_{\Delta E} : A^{\Delta E} \to A(\Delta E)$ induces a surjective graded k-algebra homomorphism $\beta : H^*(\Delta E; A^{\Delta E}) \to H^*(\Delta E; A(\Delta E))$ and there is a unique graded k-algebra isomorphism $\bar{\beta} : \bar{H}^*(\Delta E; A(\Delta E)) \cong \bar{H}^*(\Delta E; A)$ making the following diagram of graded k-algebra homomorphisms commutative:*

$$\begin{tikzcd}
 & H^*(\Delta E; A^{\Delta E}) \arrow[dl, "\beta"'] \arrow[dr, "\epsilon"] & \\
H^*(\Delta E; A(\Delta E)) \arrow[d] & & H^*(\Delta E; A) \arrow[d] \\
\bar{H}^*(\Delta E; A(\Delta E)) \arrow[rr, "\bar{\beta}", "\cong"'] & & \bar{H}^*(\Delta E; A) \\
\bar{H}^*(\Delta E; k) \arrow[u, "\bar{\iota}_E^{A(\Delta E)}"] \arrow[rr, equal] & & \bar{H}^*(\Delta E; k) \arrow[u, "\bar{\iota}_E^A"']
\end{tikzcd}$$

where the downward vertical arrows are the canonical surjections and the upward vertical arrows are induced by $\iota_E^{A(\Delta E)}$ and ι_E^A, respectively.

Proof. Let X be a ΔP-stable k-basis of A. The decomposition $A = k[X^{\Delta E}] \oplus k[X - X^{\Delta E}]$ as $k\Delta E$-modules yields, upon taking ΔE-fixpoints in both summands, a decomposition of k-vector spaces

$$A^{\Delta E} = k[X^{\Delta E}] \oplus (k[X - X^{\Delta E}])^{\Delta E}$$

Note that the second summand is equal to the ideal $\ker(Br^A_{\Delta E})$ and that the first summand is non zero, by the assumptions. Furthermore, since $\mathrm{Br}^A_{\Delta E}$ induces an isomorphism of $k\Delta E$-modules (with trivial action) $k[X^{\Delta E}] \cong A(\Delta E)$, the map β is surjective. Any element $\zeta \in H^*(\Delta E; A^{\Delta E})$ is the sum of its two components $\zeta_0 \in H^*(\Delta E; X^{\Delta E})$ and $\zeta_1 \in H^*(\Delta E; \ker(\mathrm{Br}^A_{\Delta E}))$. The component ζ_1 is contained in the kernel of both β and of ι composed with the surjection $H^*(\Delta E; A) \to \bar{H}^*(\Delta E; A)$. Similarly, the component ζ_0 is in the kernel of β composed with $H^*(\Delta E; A(\Delta E)) \to \bar{H}^*(\Delta E; A(\Delta E))$ if and only if ζ_0 is in the kernel of ϵ composed with $H^*(\Delta E; A) \to \bar{H}^*(\Delta E; A)$. This shows that there is an injective graded k-algebra homomorphism from $\bar{H}^*(\Delta E; A(\Delta E))$ to $\bar{H}^*(\Delta E; A)$ making the upper pentagon in the diagram commutative; since both sides have equal dimensions in each degree this is an isomorphism of graded k-algebras.

The lower square is commutative since the isomorphism $\bar{\beta}$ is induced by unitary algebra homomorphisms between the three algebras $A(\Delta E)$, $A^{\Delta E}$ and A. $\square$

This result plays back the calculation of $\bar{H}^*(\Delta E; A)$ to the same calculation with A replaced by $A(\Delta E)$, viewed as ΔE-algebra with trivial action. In that case one can be more precise; the following Lemma is a minor generalisation of [22, 3.2]:

Lemma 3.3. *Let P be a finite p-group and A a finite-dimensional k-algebra, considered as ΔP-algebra with ΔP acting trivially on A. For any subgroup E of P there are graded k-algebra isomorphisms*

$$A^{op} \otimes H^*(\Delta E; k) \cong H^*(\Delta E; A)$$

$$A^{op} \otimes \bar{H}^*(\Delta E; k) \cong \bar{H}^*(\Delta E; A)$$

induced by the maps sending $c \in A$ to right multiplication by c on A and by the graded algebra homomorphism $\iota_E^A : H^*(\Delta E; k) \to H^*(\Delta E; A)$.

Proof. Since ΔE acts trivially on A, standard homological algebra results imply that there is a k-linear isomorphism $H^*(\Delta E; k) \otimes A \cong H^*(\Delta E; A)$ sending $\zeta \otimes a$ to $\iota_E^A(\zeta)a$, where ι_E^A is as in 2.4 and where $a \in A$ is viewed as element in $H^0(\Delta E; A)$, whence the first isomorphism. Applying this isomorphism to subgroups of E and observing that these commute with transfer yields the second isomorphism $A^{op} \otimes \bar{H}^*(\Delta E; k) \cong \bar{H}^*(\Delta E; A)$, which concludes the proof. $\square$

Proof of Proposition 3.1. We construct π_E^A as follows. The canonical surjection $H^*(\Delta E; A) \to \bar{H}^*(\Delta E; A)$ induces a graded k-algebra homomorphism

$$Z(H^*(\Delta E; A)) \to Z(\bar{H}^*(\Delta E; A))$$

We then compose this map with the isomorphisms

$$Z(\bar{H}^*(\Delta E; A)) \cong Z(\bar{H}^*(\Delta E; A(\Delta E))) \cong Z(A(\Delta E)) \otimes \bar{H}^*(\Delta E; k)$$

induced by the isomorphisms in the two previous lemmas. Since $Z(A(\Delta E))$ is local, there is a unique unitary algebra homomorphism $Z(A(\Delta E)) \to k$. This map induces hence a graded algebra homomorphism

$$Z(A(\Delta E)) \otimes \bar{H}^*(\Delta E; A) \to \bar{H}^*(\Delta E; A)$$

The composition of the above maps is then denoted by π_E^A. It follows from the previous lemma that $\iota_E^A \circ \pi_E^A$ induces the canonical surjection $H^*(\Delta E; k) \to \bar{H}^*(\Delta E; k)$. The uniqueness of π_E^A is an easy consequence of the uniqueness of the unitary homomorphism from $Z(A(\Delta E))$ to k. $\square$

4 Fusion and cohomology

Fusion in a block algebra is induced by conjugation of Brauer pairs with elements of the underlying group. By a result of Puig [18], in a source algebra A of a block with defect group P, fusion can be identified as given by conjugation on images of subgroups of P in A with invertible elements of A, modulo a crucial technical detail: one has to consider for a subgroup Q of P not the image of Q in A but the images of Q in jAj, where j is a primitive local idempotent in $A^{\Delta Q}$. This technicality is the background for the following result. As in the previous section, k is an algebraically closed field of prime characteristic p. Given a finite p-group P and an interior P-algebra, we denote for any subgroup E of P and any idempotent $j \in A^{\Delta E}$ by

$$\delta_{(E,j)} : HH^*(A) \longrightarrow H^*(\Delta E; jAj)$$

the composition of the "restriction" map $HH^*(A) \to \mathrm{Ext}_{A\otimes kE}(Aj, Aj)$ induced by the the functor $-\otimes_A Aj$ followed by the canonical isomorphism $\mathrm{Ext}_{A\otimes kE}(Aj, Aj) \cong H^*(\Delta E; jAj)$ from Proposition 2.1. If $j = 1$ we write δ_E instead of $\delta_{(E,1)}$. By Lemma 2.5, the image of $\delta_{(E,j)}$ is contained in $Z(H^*(\Delta E; jAj))$, and we abusively denote the induced map from $HH^*(A)$ to $Z(H^*(\Delta E; jAj))$ again by $\delta_{(E,j)}$.

Proposition 4.1. *Let P be a finite p-group and A a finite-dimensional interior P-algebra over k such that A is projective as left and right kP-module. Let E, E' be subgroups of P and $j \in A^{\Delta E}$, $j' \in A^{\Delta E'}$ be idempotents. Suppose that there is a group isomorphism $\varphi : E \cong E'$ and an invertible element c in A such that for all $e \in E$ we have $cejc^{-1} = \varphi(e)j'$. Then the algebra isomorphism $\alpha : jAj \cong j'Aj'$ sending $a \in jAj$ to $cac^{-1} \in j'Aj'$ induces an isomorphism of graded k-algebras $\alpha^* : H^*(\Delta E'; j'Aj') \cong H^*(\Delta E; jAj)$ making the following diagram of graded k-algebras commutative:*

$$\begin{tikzcd}
HH^*(A) \arrow[r, "\delta_{(E',j')}"] \arrow[d, equal] & H^*(\Delta E'; j'Aj') \arrow[d, "\alpha^*"] & H^*(\Delta E'; k) \arrow[l, "\iota_{E'}^{j'Aj'}"'] \arrow[d, "\mathrm{res}_\varphi"] \\
HH^*(A) \arrow[r, "\delta_{(E,j)}"'] & H^*(\Delta E; jAj) & H^*(\Delta E; k) \arrow[l, "\iota_E^{jAj}"]
\end{tikzcd}$$

Proof. The commutativity of the right square is a trivial consequence of the fact that α is unitary. The hypotheses imply that the isomorphism α sends the image of E in jAj to the image of E' in $j'Aj'$. The commutativity of the left square in the above diagram expresses merely the fact that the elements in $HH^*(A)$ are "invariant under conjugation" by c because they are represented by chain maps of A-A-bimodules. Here are the rather tedious details. Let V, V' be projective resolutions of k as $k\Delta E$-module, $k\Delta E'$-module, respectively. Denote by $\Delta\varphi : \Delta E \cong \Delta E'$ the isomorphism induced by φ and by ${}_{\Delta\varphi}(V')$ the chain complex of ΔE-modules obtained from restricting V' along $\Delta\varphi$. Then ${}_{\Delta\varphi}(V')$ is also a projective resolution of the trivial $k\Delta E$-module, hence there is a unique (up to homotopy) homotopy equivalence of chain complexes of $k\Delta E$-modules $\nu : V \cong {}_{\Delta\varphi}(V')$. Since $jc^{-1} = c^{-1}j'$, right multiplication by c^{-1} induces an A-homomorphism $\gamma : Aj \cong Aj'$. Since $jec^{-1} = c^{-1}j'\varphi(e)$ for all $e \in E$, the map γ is more precisely an isomorphism of A-kE-bimodules $Aj \cong (Aj')_\varphi$, where as before the right kE-module structure of $(Aj')_\varphi$ is obtained from restricting the right kE'-module structure of Aj' via the group homomorphism $\varphi : E \to E'$. Let X be a projective resolution of A as A-A-bimodule. Then Xj, Xj' are projective resolutions of Aj, Aj' as an A-kE-bimodule, A-kE'-bimodule, respectively, because A is projective as a right kE-module and as a right kE'-module. Right multiplication by c^{-1} on Xj yields an isomorphism $\hat{\gamma} : Xj \cong (Xj')_\varphi$ of chain complexes of A-kE-bimodules which lifts the isomorphism γ. Let $n \geq 0$ be an integer and let $\zeta \in HH^n(A)$ be represented by a chain map of complexes of A-A-bimodules (denoted by the same letter) $\zeta : X \to A[n]$. Clearly ζ sends Xj to $Aj[n]$ and Xj' to $Aj'[n]$; thus the images of ζ in $\mathrm{Ext}^n_{A\otimes kE}(Aj, Aj)$ and $\mathrm{Ext}^n_{A\otimes kG'}(Aj', Aj')$ are obtained from restricting ζ to Xj and Xj', respectively. The maps γ, $\hat{\gamma}$ induce an isomorphism

$$\gamma^* : \mathrm{Ext}^*_{A\otimes kE'}(Aj', Aj') \cong \mathrm{Ext}^*_{A\otimes kE}(Aj, Aj)$$

given by $\gamma^*(\xi) = \gamma[n]^{-1} \circ \xi \circ \hat{\gamma}$, for any $\xi \in \mathrm{Ext}^n_{A\otimes kE'}(Aj', Aj')$, represented by a chain map $\xi : Xj' \to Aj'[n]$. Explicitly, for $x \in Xj$ we have $\gamma^*(\xi)(x) = \xi(xc^{-1})c$. In particular, if ξ is the image of an element $\zeta \in HH^n(A)$ then $\gamma^*(\xi)(x) = \zeta(xc^{-1})c = \zeta(x)$, which shows that the following diagram of graded k-algebras is commutative:

$$\begin{tikzcd}
HH^*(A) \arrow[r, "\delta_{(E',j')}"] \arrow[d, equal] & \mathrm{Ext}^*_{A\otimes kE'}(Aj', Aj') \arrow[d, "\gamma^*"] \\
HH^*(A) \arrow[r, "\delta_{(E,j)}"'] & \mathrm{Ext}^*_{A\otimes kE}(Aj, Aj)
\end{tikzcd}$$

Set $U = (Aj \otimes kE) \otimes_{k\Delta E} V$ and $U' = (Aj \otimes kE') \otimes_{k\Delta E'} V'$; these are projective resolutions of $(Aj \otimes kE) \otimes_{k\Delta E} k \cong Aj$ and of $(Aj' \otimes kE') \otimes_{k\Delta E'} k \cong Aj'$, hence uniquely homotopy equivalent to Xj and Xj' as complexes of $A \otimes kE$-modules and $A \otimes kE'$-modules, respectively. Through the above isomorphisms, the isomorphism $\gamma : Aj \cong Aj'$ gets identified with the isomorphism

$$(\gamma \otimes \varphi) \otimes \mathrm{Id} : (Aj \otimes kE) \otimes_{k\Delta E} k \ \cong\ (Aj' \otimes kE') \otimes_{k\Delta E'} k$$

This isomorphism lifts in the obvious way to a homotopy equivalence

$$\tilde{\gamma} = (\gamma \otimes \varphi) \otimes \nu : U \simeq (U')_\varphi$$

of complexes of A-kE-bimodules. Thus both $\hat{\gamma}$, $\tilde{\gamma}$ are lifts of γ, and hence make the diagram

$$\begin{tikzcd}
Xj \arrow[r, "\simeq"] \arrow[d, "\hat{\gamma}"'] & U \arrow[d, "\tilde{\gamma}"] \\
(Xj')_\varphi \arrow[r, "\simeq"'] & (U')_\varphi
\end{tikzcd}$$

homotopy commutative, where the horizontal homotopy equivalences lift the identity maps on Aj, Aj', respectively. Thus, using the projective resolutions U, U' instead of Xj, Xj', it follows that the isomorphism γ^* is given by

$$\gamma(\xi) = \gamma[n]^{-1} \circ \xi \circ \tilde{\gamma}$$

for $n \geq 0$ and $\xi \in \mathrm{Ext}^n_{A\otimes kE'}(Aj', Aj')$, represented by a chain map $\xi : U' \to Aj'[n]$. We have to compare the images of ξ and $\gamma^*(\xi)$ in $H^n(\Delta E'; j'Aj')$ and $H^n(\Delta E; jAj)$, respectively. Through the standard adjunction, the image of ξ in $H^n(\Delta E'; j'Aj')$ is represented by the map $\bar{\xi} : V' \to j'Aj'[n]$ given by $\bar{\xi}(v') = \xi((j' \otimes 1) \otimes v')$, where $v' \in V'$. Similarly, the image of $\gamma^*(\xi)$ is the chain map $\bar{\gamma}^*(\xi) : V \to jAj[n]$ given by $\bar{\gamma}(\xi)(v) = \gamma^*(\xi)((j \otimes 1) \otimes v) = (\gamma[n]^{-1} \circ \xi \circ \tilde{\gamma})((j \otimes 1) \otimes v) = (\gamma[n]^{-1} \circ \xi)((jc^{-1} \otimes 1) \otimes \nu(v)) = \xi(c^{-1}j' \otimes 1) \otimes \nu(v))c = c^{-1} \cdot (\xi((j' \otimes 1) \otimes \nu(v))) \cdot c$, where $v \in V$. In other words, we have a commutative diagram

$$\begin{tikzcd}
V \arrow[r, "\nu"] \arrow[d, "\bar{\gamma}^*(\xi)"'] & V' \arrow[d, "\bar{\xi}"] \\
jAj[n] \arrow[r, "\alpha[n]"'] & j'Aj'[n]
\end{tikzcd}$$

This shows that the canonical isomorphisms from Proposition 2.1 make the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Ext}^*_{A\otimes kE'}(Aj', Aj') \arrow[r, "\cong"] \arrow[d, "\gamma^*"'] & H^*(\Delta E'; j'Aj') \arrow[d, "\alpha^*"] \\
\mathrm{Ext}^*_{A\otimes kE}(Aj, Aj) \arrow[r, "\cong"'] & H^*(\Delta E; jAj)
\end{tikzcd}$$

Together with the first commutative diagram this completes the proof. □

In order to relate the maps $\delta_{(E,j)}$, $\delta_{(E,j')}$ for the same group E but different idempotents j, j' in $A^{\Delta E}$, we will need the following observation.

Proposition 4.2. *Let P be a finite p-group and A a finite-dimensional interior P-algebra over k. Let E be a subgroup of P and j an idempotent in $A^{\Delta E}$. The k-linear map $A \to jAj$ sending $a \in A$ to jaj is a split surjective homomorphism of $k\Delta E$-modules and induces a split surjective graded k-linear map $\pi : H^*(\Delta E; A) \to H^*(\Delta E; jAj)$. The restriction of π to $Z(H^*(\Delta E; A))$ induces a graded k-algebra homomorphism, still denoted $\pi : Z(H^*(\Delta E; A)) \to Z(H^*(\Delta E; jAj))$ making the following diagram of graded k-algebras commutative:*

$$\begin{tikzcd}
HH^*(A) \arrow[r, "\delta_E"] \arrow[d, equal] & Z(H^*(\Delta E; A)) \arrow[d, "\pi"] & H^*(\Delta E; k) \arrow[l, "\iota_E^A"'] \arrow[d, equal] \\
HH^*(A) \arrow[r, "\delta_{(E,j)}"'] & Z(H^*(\Delta E; jAj)) & H^*(\Delta E; k) \arrow[l, "\iota_E^{jAj}"]
\end{tikzcd}$$

Proof. The commutativity of the right square is an immediate consequence of the fact that π sends 1_A to j. For commutativity of the left square we will use the elementary ring theoretic fact that multiplication by an idempotent e in a ring R induces a ring homomorphism $Z(R) \to Z(R)e \subseteq Z(eRe)$, and hence $Z(R) \subseteq Z(eRe) \times Z((1-e)R(1-e))$. The k-linear map $A \to jAj$ sending $a \in A$ to jaj is a $k\Delta E$-homomorphism because j belongs to $A^{\Delta E}$, and it has the inclusion $jAj \to A$ as section, hence induces indeed a split surjective graded k-linear map $\pi : H^*(\Delta E; A) \to H^*(\Delta E; jAj)$. We have $H^0(\Delta E; A) = (A^{\Delta E})^{op}$, and hence j can be identified with an idempotent in $H^0(\Delta E; A)$. One easily checks that the map π is induced by left and right multiplication with this idempotent, hence induces an algebra homomorphism $Z(H^*(\Delta E; A)) \to Z(\Delta E; jAj))$ by the remarks at the beginning of the proof extended to centers of graded algebras in the obvious way; it is a trivial verification that π makes the diagram in the statement commutative. □

The last statement of this section is about the compatibility of the maps δ_E with restriction to subgroups:

Proposition 4.3. *Let P be a finite p-group and A a finite-dimensional interior P-algebra over k. Let E, F be subgroups of P such that $F \leq E$. Restriction from ΔE to ΔF induces a graded k-algebra homomorphism $\mathrm{res}^{\Delta E}_{\Delta F} : H^*(\Delta E; A) \to H^*(\Delta F; A)$ making the following diagram of graded* *k-algebras commutative:*

$$\begin{tikzcd}
HH^*(A) \arrow[r, "\delta_E"] \arrow[d, equal] & H^*(\Delta E; A) \arrow[d, "\mathrm{res}^{\Delta E}_{\Delta F}"] & H^*(\Delta E; k) \arrow[l, "\iota^A_E"'] \arrow[d, "\mathrm{res}^{\Delta E}_{\Delta F}"] \\
HH^*(A) \arrow[r, "\delta_F"'] & H^*(\Delta F; A) & H^*(\Delta F; k) \arrow[l, "\iota^A_F"]
\end{tikzcd}$$

Proof. Trivial verification, based on the fact that restriction is transitive and functorial. $\square$

5 Proof of Theorem 1.6

We start with the construction of the homomorphism σ in 1.6; the required block theoretic background material can be found in various sources such as [24], [8], or also [15]. For the rest of this section we use the following notation. Let G be a finite group and b a block of kG. The corresponding block algebra is denoted $B = kGb$. Let (P, e_P) be a maximal (G, b)-Brauer pair and i a primitive idempotent in $(kGb)^{\Delta P}$ such that $\mathrm{Br}_{\Delta P}(i)e_P = \mathrm{Br}_{\Delta P}(i) \neq 0$; that is, i is a source idempotent of b and the interior P-algebra $A = ikGi$ is a source algebra of b. The algebras A and B are Morita equivalent via the bimodules kGi and ikG. By [1, 3.4], for any subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $(Q, e_Q) \leq (P, e_P)$, and by [4, 1.8], the block e_Q is the unique block of $kC_G(Q)$ whose block algebra contains $\mathrm{Br}_{\Delta Q}(i)$, or equivalently, $\mathrm{Br}_{\Delta Q}(i)e_Q = \mathrm{Br}_{\Delta Q}(i)$. The fusion system on P determined by A has as morphisms all group homomorphisms $Q \to R$ induced by conjugation with an element $x \in G$ satisfying ${}^x(Q, e_Q) \leq (R, e_R)$, for any two subgroups Q, R of P. As a consequence of [4, Theorem 1.8], for any subgroup Q of P the algebra $A(\Delta Q)$ is indecomposable, and hence $Z(A(\Delta Q))$ is local. If Q is a fully $\mathcal{F}$-centralised subgroup of P one can be slightly more precise: in that case $C_P(Q)$ is a defect group of the block e_Q of $kC_G(Q)$, and the algebras $kC_G(Q)e_Q$ and $A(\Delta Q)$ are Morita equivalent (see for instance [15, 4.6]), which implies that $Z(A(\Delta Q)) \cong Z(kC_G(Q)e_Q)$. The source algebra A has a P-P-stable k-basis with at least one ΔP-fixpoint (because $A(\Delta P) \neq \{0\}$), and hence we can define for any elementary abelian subgroup E of P, a graded k-algebra homomorphism

$$\sigma_E : HH^*(A) \longrightarrow H^*(E; k)$$

as composition

$$HH^*(A) \xrightarrow{\delta^A_E} Z(H^*(\Delta E; A)) \xrightarrow{\pi^A_E} H^*(\Delta E; k) \xrightarrow{\cong} H^*(E; k)$$

where δ^A_E is as at the beginning of section 4, π^A_E is as in 3.1, and where the isomorphism $H^*(\Delta E; k) \cong H^*(E; k)$ is induced by the obvious identification $\Delta E \cong E$. We then define

$$\sigma : HH^*(A) \longrightarrow \prod_{E \in \mathcal{E}} H^*(E; k)$$

as product of the homomorphisms σ_E, the product taken over the full subcategory $\mathcal{E}$ of $\mathcal{F}$ consisting of all elementary abelian subgroups of P. By the definition of block cohomology we have $H^*(B) = \lim_{\mathcal{F}}(H^*(Q; k))$, which we identify to its image of $\mathcal{F}$-stable elements in $H^*(P; k)$. The algebra homomorphism $\rho : H^*(B) \to \lim_{\mathcal{E}}(H^*(E;k))$ in the statement of 1.6 is induced by the product of the maps $\rho_E : H^*(B) \to H^*(E;k)$, obtained from restricting res^P_E to $H^*(B)$, with E running over $\mathcal{E}$. By Quillen's stratification, extended to blocks in [13, 3.5, 4.2] and to arbitrary fusion systems in [3, 5.1, 5.2], the map ρ induces an inseparable isogeny. We identify $\lim_{\mathcal{E}}(H^*(E;k))$ to its canonical image in $\prod_{E\in\mathcal{E}} H^*(E;k)$. For any subgroup E of P, denote by

$$\rho_{\Delta E} : H^*(B) \longrightarrow H^*(\Delta E;k)$$

the composition of the inclusion $H^*(B) \subseteq H^*(P;k)$ followed by the canonical identification $H^*(P;k) \cong H^*(\Delta P;k)$, followed by the restriction $\mathrm{res}^{\Delta P}_{\Delta E} : H^*(\Delta P;k) \to H^*(\Delta E;k)$. Equivalently, $\rho_{\Delta E}$ is equal to the composition of ρ_E and the canonical identification $H^*(E;k) \cong H^*(\Delta E;k)$. We need to show that σ extends ρ, that $\ker(\sigma)$ is nilpotent and that $\mathrm{Im}(\sigma)$ is contained in $\lim_{\mathcal{E}}(H^*(E;k))$. We start with a Lemma describing the canonical map $\tau : H^*(B) \to HH^*(A)$.

Lemma 5.1. *Let E be a subgroup of P. The following diagram of graded k-algebras is commutative:*

$$\begin{tikzcd} H^*(B) \arrow[r, "\rho_{\Delta E}"] \arrow[d, "\tau"'] & H^*(\Delta E;k) \arrow[d, "\iota^A_E"] \\ HH^*(A) \arrow[r, "\delta^A_E"'] & Z(H^*(\Delta E;A)) \end{tikzcd}$$

Proof. We prove this first for $E = P$. The map τ is defined in terms of maps induced by diagonal induction from ΔP to $P\times P$ followed by the normalised transfer map $\mathrm{Tr}_{A_P} : HH^*(kP) \to HH^*(A)$ determined by A viewed as A-kP-bimodule (denoted by A_P). We briefly recall the definition and relevant properties of Tr_{A_P} from [10, §3]. We have algebra homomorphisms $HH^*(A) \to \mathrm{Ext}^*_{A\otimes kP}(A,A)$ and $HH^*(kP) \to \mathrm{Ext}^*_{A\otimes kP}(A,A)$ induced by the restriction and induction $A\otimes_{kP}-$, respectively. These homomorphisms are injective; indeed, this is an easy consequence of the well-known fact that A is isomorphic to a direct summand of $A\otimes_{kP} A$ as A-A-bimodule; see for instance [15, 4.2]. Following [10, 3.1.(iii)], an element $\xi \in HH^n(kP)$ is called A_P-stable if there is an element $\zeta \in HH^n(A)$ such that $\zeta\otimes \mathrm{Id}_{A_P} = \mathrm{Id}_{A_P}\otimes \xi$ in $\mathrm{Ext}^n_{A\otimes kP}(A,A)$. Moreover, in that case we have $\zeta = \mathrm{Tr}_{A_P}(\xi)$; one can take this as definition of Tr_{A_P} in this case. Thus the subalgebra $HH^*_{A_P}(kP)$ of A_P-stable elements in $HH^*(kP)$ yields a pullback diagram of the form

$$\begin{tikzcd} HH^*_{A_P}(kP) \arrow[r] \arrow[d, "\mathrm{Tr}_{A_P}"] & HH^*(kP) \arrow[d, "A\otimes_{kP}-"'] \\ HH^*(A) \arrow[r] & \mathrm{Ext}^*_{A\otimes kP}(A,A) \end{tikzcd}$$

The map $H^*(P;k) \to HH^*(kP)$ given by "diagonal" induction from $P \cong \Delta P$ to $P\times P$, composed with the map $HH^*(kP) \to \mathrm{Ext}_{A\otimes kP}(A,A)$ induced by $A\otimes_{kP} -$ is equal to ι^A_P, modulo the identification $H^*(P;k) \cong H^*(\Delta P,k)$. Moreover, by [10, 5.4], this map sends $H^*(B)$ to $HH^*_{A_P}(kP)$, and composing this with the normalised transfer map T_{A_P} is, by definition, the algebra homomorphism τ. As before, the diagonal induction functor from $P\cong \Delta P$ to $P\times P$ induces a graded homomorphism $H^*(P;k) \to HH^*(kP)$. Using the above maps, the commutative diagram from 2.4 and the isomorphism from 2.1 we get a commutative diagram

$$\begin{tikzcd}
H^*(B) \arrow[r, equal] \arrow[dddd, "\tau"'] & H^*(B) \arrow[r] \arrow[d] & H^*(P;k) \arrow[r, "\cong"] \arrow[d] & H^*(\Delta P;k) \arrow[dd, "\iota_P^A"] \\
 & HH^*_{A_P}(kP) \arrow[r] \arrow[d, "\mathrm{Tr}_{A_P}"'] & HH^*(kP) \arrow[d] & \\
 & HH^*(A) \arrow[r] \arrow[dll, equal] & \mathrm{Ext}^*_{A\otimes kP}(A,A) \arrow[r, "\cong"] & HH^*(\Delta P;A) \arrow[d, equal] \\
HH^*(A) \arrow[rrr, "\delta_P^A"'] & & & H^*(\Delta P;A)
\end{tikzcd}$$

Since the images of δ_P^A, ι_P^A are contained in $Z(H^*(\Delta P;k))$, the outer square of this diagram yields the commutative square of the statement in the case $E = P$. The general case follows from combining the outer commutative square of the previous diagram with the diagram from 4.3, together with the fact that the images of δ_E^A, ι_E^A are contained in $Z(H^*(\Delta E;A))$. $\square$

Proposition 5.2. *With the notation above, we have $\rho = \sigma \circ \tau$.*

Proof. Let E be a subgroup of P. Lemma 5.1 yields a commutative diagram

$$\begin{tikzcd}
H^*(B) \arrow[rr, "\rho_{\Delta E}"] \arrow[d, equal] & & H^*(\Delta E;k) \arrow[r, equal] \arrow[d, "\iota_E^A"] & H^*(\Delta E;k) \arrow[d, equal] \\
H^*(B) \arrow[r, "\tau"'] & HH^*(A) \arrow[r, "\delta_E^A"'] & Z(H^*(\Delta E;A)) \arrow[r, "\pi_E^A"'] & H^*(\Delta E;k)
\end{tikzcd}$$

Modulo identifying ΔE and E, the first row of the diagram above combined yields ρ_E, and the second row yields $\sigma_E \circ \tau$, whence the result. $\square$

Proposition 5.3. *With the notation above, the kernel of σ is nilpotent.*

Proof. Let $\zeta \in HH^*(A)$ be an element in the kernel of σ. Since $HH^0(A) \cong Z(A)$ is local and since $HH^*(A)$ is graded commutative, we may assume that ζ is homogeneous of positive degree. For any elementary abelian subgroup E of P the kernel of the map

$$Z(\bar{H}^*(\Delta E;A)) \cong Z(A(\Delta E)) \otimes H^*(\Delta E;k) \to H^*(\Delta E;k)$$

is nilpotent because $Z(A(\Delta E))$ is local. Thus, after replacing ζ by some power, we may assume that the image $\bar{\zeta}_E$ of ζ in $\bar{H}^*(\Delta E;A)$ is zero. Since the restriction map $HH^*(A) \to \mathrm{Ext}^*_{A\otimes kP}(A,A)$ is injective, in order to show that ζ is nilpotent it suffices, by Siegel's result in [17, 5.3], to show that the image ζ_E of ζ in $H^*(\Delta E;A)$ is nilpotent, for any elementary abelian subgroup E of P. We show this by induction over the order of E, following the arguments in the proof of [17, 2.5]. For $E = 1$ there is nothing to prove. Assume that E is nontrivial. After replacing ζ by some power, we may assume that $\zeta_F = 0$ for any proper subgroup F of E. For $a \geq 1$ denote by $S(E,a)$ the set of subgroups of index p^a in E. We show inductively that

$$(\zeta_E)^{p^{a-1}} \in \sum_{F \in S(E,a)} \mathrm{tr}_F^E(H^*(\Delta F; A))$$

for any positive integer a. For $a = 1$ this follows from the fact that $\bar{\zeta}_E = 0$. Suppose that $a \geq 1$ and that we can write

$$(\zeta_E)^{p^{a-1}} = \sum_{F \in S(E,a)} \mathrm{tr}_F^E(\xi_F)$$

for suitable $\xi_F \in H^*(\Delta F; A)$. Then the usual adjunctions yield

$$(\zeta_E)^2 = \sum_{F,F' \in S(E,a)} \mathrm{tr}_F^E(\xi_F)\mathrm{tr}_{F'}^E(\xi_{F'}) = \sum_{F,F' \in S(E,a)} \mathrm{tr}_F^E(\xi_F \cdot \mathrm{Res}_F^E(\mathrm{tr}_{F'}^E(\xi_{F'})))$$

For any $F \in S(E,a)$ we have

$$0 = \zeta_F = \mathrm{Res}_F^E(\zeta_E) = \sum_{F' \in S(E,a)} \mathrm{Res}_F^E(\mathrm{tr}_{F'}^E(\xi_{F'}))$$

For $F' \neq F$ the Mackey formula implies $\mathrm{Res}_F^E(\mathrm{tr}_{F'}^E(\xi_{F'})) \in \sum_{F''<F} \mathrm{tr}_{F''}^F(H^*(\Delta F''; A))$. Since $\zeta_F = 0$, this implies that also $\mathrm{Res}_F^E(\mathrm{tr}_F^E(\xi_F)) \in \sum_{F''<F} \mathrm{tr}_{F''}^F(H^*(\Delta F''; A))$. Thus $(\zeta_E)^2 \in \sum_{F \in S(E,a+1)} \mathrm{tr}_F^E(H^*(\Delta F; A))$. This induction comes to a halt when p^a is the order of E, showing that $(\zeta_E)^{p^a}$ is a trace from the trivial subgroup 1, hence zero. □

In order to show that the image of σ is contained in $\lim_{\mathcal{E}}(H^*(E;k))$ we use 4.2 to give an alternative description of the map $\sigma_E : HH^*(A) \to H^*(E;k)$, where E is an elementary abelian subgroup of P. If j is a primitive idempotent in $A^{\Delta E}$ satisfying $\mathrm{Br}_{\Delta E}(j) \neq 0$, we define a map

$$\sigma_{(E,j)} : HH^*(A) \longrightarrow H^*(E;k)$$

essentially as σ_E, but with A replaced by jAj. Explicitly, $\sigma_{(E,j)}$ is the composition of $\delta_{(E,j)}^A$, followed by π_E^{jAj}, and finally composed with the obvious isomorphism induced by the identification $\Delta E \cong E$:

$$HH^*(A) \xrightarrow{\delta_{(E,j)}^A} H^*(\Delta E; jAj) \xrightarrow{\pi_E^{jAj}} H^*(\Delta E; k) \xrightarrow{\cong} H^*(E;k)$$

Note that this makes sense: the map π_E^{jAj} is well-defined as $(jAj)(\Delta E)$ is local.

Lemma 5.4. *Let E be an elementary abelian subgroup of P and let j be a primitive idempotent in $A^{\Delta E}$ satisfying $\mathrm{Br}_{\Delta E}(j) \neq 0$. Then $\sigma_{(E,j)} = \sigma_E$; in particular, $\sigma_{(E,j)}$ does not depend on the local point of E on A to which j belongs. More precisely, we have a commutative diagram of graded algebras*

$$\begin{tikzcd}
HH^*(A) \arrow[r, "\delta_E^A"] \arrow[d, equal] & Z(H^*(\Delta E; A)) \arrow[r, "\pi_E^A"] \arrow[d] & H^*(\Delta E; k) \arrow[d, equal] \\
HH^*(A) \arrow[r, "\delta_E^{jAj}"'] & Z(H^*(\Delta E; jAj)) \arrow[r, "\pi_E^{jAj}"'] & H^*(\Delta E; k)
\end{tikzcd}$$

where the vertical map is induced by the map $A \to jAj$ given by left and right multiplication with j.

Proof. The commutativity of the left square follows from the left square in 4.2, and the commutativity of the right square follows from the right square in 4.2, together with the fact that π_E^A is a retraction for ι_E^A (and similarly for jAj instead of A). □

Proposition 5.5. *With the notation above, we have* $\mathrm{Im}(\sigma) \subseteq \lim_{\mathcal{E}}(H^*(E;k))$.

Proof. We have to show that if $\varphi : E \to E'$ is a morphism in $\mathcal{F}$ between two elementary abelian subgroups E, E' of P then $\sigma_E = \mathrm{res}_\varphi \circ \sigma_{E'}$. It suffices to do this in the cases where φ is an inclusion or an isomorphism. For inclusions this is a trivial consequence of 4.3, combined with the fact that π_E^A is a retraction for ι_E^A, by 3.1. We may assume that φ is an isomorphism. Puig's characterisation of fusion in source algebras in [18, 2.12, 3.1], in the form as presented in [12, 7.5] implies that there are primitive idempotents $j \in A^{\Delta E}$, $j' \in A^{\Delta E'}$ satisfying $\mathrm{Br}_{\Delta E}(j) \neq 0$, $\mathrm{Br}_{\Delta E'}(j') \neq 0$ and there is an element $c \in A^\times$ such that ${}^c(uj) = \varphi(u)j'$ for all $u \in E$. This is the situation dealt with in 4.1, implying that $\sigma_{(E,j)} = \mathrm{res}_\varphi \circ \sigma_{(E',j')}$, where we use again 3.1. But then the previous lemma implies that $\sigma_E = \mathrm{res}_\varphi \circ \sigma_{E'}$, whence the result. □

Proof of Theorem 1.6. The three propositions 5.2, 5.3, and 5.5, together with the fact that ρ induces an inseparable isogeny complete the proof of Theorem 1.6. □

The proof of Corollary 1.2 uses the elementary fact that an inseparable isogeny induced by a split monomorphism is an isomorphism of varieties:

Lemma 5.6. *Let R, S be rings and $\rho : R \to S$, $\sigma : S \to R$ be ring homomorphisms such that $\sigma \circ \rho = \mathrm{Id}_R$ and such that for any $s \in S$ there is a positive integer n satisfying $s^n \in \mathrm{Im}(\rho)$. Then every element in $\ker(\sigma)$ is nilpotent.*

Proof. Let $s \in \ker(\sigma)$. By the assumptions there is $r \in R$ and $n > 0$ such that $s^n = \rho(r)$. Applying σ yields $0 = \sigma(s)^n = \sigma(\rho(r)) = r$, hence $s^n = 0$. □

Proof of Corollary 1.2. If B is the principal block then τ is a split monomorphism with retraction induced by the functor $- \otimes_B k$. Thus Lemma 5.6 and Theorem 1.6 imply that this retraction has a kernel consisting of nilpotent elements. Since $HH^*(B)$ is Noetherian graded commutative it follows that the kernel of this retraction is a nilpotent ideal. □

Remark 5.7. As pointed out earlier, this raises the question whether $\tau : H^*(B) \to HH^*(A)$ is a split monomorphism as homomorphism of graded k-algebras for arbitrary blocks. If that were the case, the argument in the proof of 1.2 would immediately yield a proof of Theorem 1.1.

6 Proof of Theorem 1.1

We use the notation from the previous section; in particular, A is a source algebra of the block b of kG with defect group P. We denote by $[A, A]$ the k-subspace of A spanned by the set of additive commutators $[a, a'] = aa' - a'a$, where $a, a' \in A$. The following elementary observation is an immediate consequence of well-known facts (which play a role in the context of bilinear forms on group algebras in [5]; see e.g. [24, §46]) and provides a weak linear analogue of the augmentation in principal blocks; we include a proof for the convenience of the reader.

Lemma 6.1. *There is a k-linear map $\mu : A \to k$ such that $\mu(1_A) = 1_k$ and such that $J(A)+[A,A] \subseteq \ker(\mu)$. Any such map μ is then a homomorphism of $k\Delta P$-modules, and has, for any subgroup E of P, the following properties:*

(i) There is a unique linear map $\bar{\mu}_E : A(\Delta E) \to k$ satisfying $\mu|_{A^{\Delta E}} = \bar{\mu}_E \circ \mathrm{Br}_{\Delta E} : A^{\Delta E} \to k$.

(ii) We have $\bar{\mu}_E(1_{A(\Delta E)}) = 1_k$ and $J(A(\Delta E)) + [A(\Delta E), A(\Delta E)] \subseteq \ker(\bar{\mu}_E)$.

(iii) The restriction of $\bar{\mu}_E$ to $Z(A(\Delta E))$ is equal to the unique unitary algebra homomorphism $Z(A(\Delta E)) \to k$.

Proof. By [19, 14.6] or [24, (44.9)] the source algebra A has a simple module V of dimension prime to p. Define μ as composition of the canonical map $A \to \mathrm{End}_k(V)$ followed by the map $\frac{1}{\dim_k(V)}\mathrm{tr}_V : \mathrm{End}_k(V) \to k$, where tr_V is the trace map. Then clearly $\mu(1_A) = 1_k$. The kernel of the map $A \to \mathrm{End}_k(V)$ contains $J(A)$, hence so does the kernel of μ. Since tr_V is a symmetric linear map, $\ker(\mu)$ contains $[A, A]$. Denoting by $\sigma : P \to A^\times$ the structural homomorphism, for any $u \in P$ and any $a \in A$ we have ${}^u a - a = [\sigma(u)a, \sigma(u^{-1})] \in [A, A] \subseteq \ker(\mu)$, which implies that μ is a homomorphism of $k\Delta P$-modules. We observe next that the space $J(A) + [A, A]$ contains every nilpotent element in A. Indeed, the image of $[A, A]$ in the direct product of matrix algebras $A/J(A)$ contains the product of all trace zero matrices, hence contains all nilpotent elements in $A/J(A)$, from which this observation follows. Thus, in particular, for any subgroup E of P we have $J(A^{\Delta E}) \subseteq \ker(\mu) \cap A^{\Delta E}$. Also, by [9, 2.5], we have $\ker(\mathrm{Br}_{\Delta E}) \subseteq [A, A] \subseteq \ker(\mu)$. Thus there is a unique linear map $\bar{\mu}_E : A(\Delta E) \to k$ satisfying (i). Note that $\bar{\mu}_E(1_{A(\Delta E)}) = 1_k$. The Brauer homomorphism $\mathrm{Br}_{\Delta E}$ from $A^{\Delta E}$ to $A(\Delta E)$ is surjective, hence maps $J(A^{\Delta E})$ onto $J(A(\Delta E))$ and $[A^{\Delta E}, A^{\Delta E}]$ onto $[A(\Delta E), A(\Delta E)]$. Thus $\ker(\bar{\mu}_E)$ contains $J(A(\Delta E)) + [A(\Delta E), A(\Delta E)]$, which shows (ii). This also shows that $\ker(\bar{\mu}_E)$ contains every nilpotent element in $A(\Delta E)$ by the above arguments. Thus $\ker(\mu_E)$ contains $J(Z(A(\Delta E))$. Since $\bar{\mu}_E$ sends $1_{A(\Delta E)}$ to 1_k and since $Z(A(\Delta E))$ is local it follows that the restriction of μ_E to $Z(A(\Delta E))$ is equal to the unique unitary algebra homomorphism from $Z(A(\Delta E))$ to k, which completes the proof of (iii). □

Fix a k-linear map $\mu : A \to k$ satisfying $\mu(1_A) = 1_k$ and $J(A) + [A, A] \subseteq \ker(\mu)$. Since the map μ is a homomorphism of $k\Delta P$-modules, it induces for any subgroup E of P a graded k-linear map

$$\nu_E : H^*(\Delta E; A) \to H^*(\Delta E; k)$$

In particular, the map ν_P is a k-linear (but not multiplicative) retraction for the algebra homomorphism $\iota_P^A : H^*(\Delta P; k) \to H^*(\Delta P; A)$ induced by the unitary map $k \to A$ (cf. 2.4), and the purpose of what follows is to show that ν_P can be used to "lift" σ to a graded k-linear map $HH^*(A) \to H^*(B)$ which is a retraction of the map $\tau : H^*(B) \to HH^*(A)$. We first lift σ to a map $HH^*(A) \to H^*(P; k)$ and then apply the projection $H^*(P; k) \to H^*(B)$ determined by a P-P-biset satisfying the fusion stability conditions in [3, 5.5]. As before, for any elementary abelian subgroup E of P, we denote by $\iota_E^A : H^*(\Delta E; k) \to Z(H^*(\Delta E; A))$ the graded map induced by the unitary map $k \to A$, we denote by $\pi_E^A : Z(H^*(\Delta E; A)) \to H^*(\Delta E; k)$ the canonical retraction of ι_E^A from Proposition 3.1, and we denote by $\delta_E^A : HH^*(A) \to H^*(\Delta E; A)$ the composition of the map induced by restriction from $A \otimes A^{op}$ to $A \otimes kE$, followed by the canonical isomorphism in Proposition 2.1, applied in the case $j = 1_A$.

Lemma 6.2. *Let E be an elementary abelian subgroup of P. The restriction of ν_E to $Z(H^*(\Delta E; A))$ is equal to $\pi_E^A : Z(H^*(\Delta E; A)) \to H^*(\Delta E; k)$.*

Proof. Since $\mu(1_A) = 1_k$ we have $\nu_E \circ \iota_E^A = \mathrm{Id}_{H^*(\Delta E;k)}$. In conjunction with 3.2, the commutative diagram (as diagram of $k\Delta E$-modules)

$$\begin{tikzcd}
A \arrow[ddrr, "\mu"] & & \\
A^{\Delta E} \arrow[u] \arrow[d, "\mathrm{Br}^A_{\Delta E}"'] \arrow[drr] & & \\
A(\Delta E) \arrow[rr, "\bar{\mu}_E"'] & & k
\end{tikzcd}$$

yields, upon taking cohomology, a commutative diagram

$$\begin{tikzcd}
 & H^*(\Delta E; A) \arrow[ddl] \arrow[ddrr, "\nu_E"] & & \\
 & H^*(\Delta E; A^{\Delta E}) \arrow[u, "\epsilon"] \arrow[d, "\beta"] \arrow[drr] & & \\
\bar{H}^*(\Delta E; A) \arrow[r, "\cong", "\bar{\beta}"'] & H^*(\Delta E; A(\Delta E)) \arrow[rr, "\bar{\nu}_E"'] & & H^*(\Delta E; k)
\end{tikzcd}$$

where ϵ, β, $\bar{\beta}$ are as in 3.2. In particular, ν_E factors through the canonical map $Z(H^*(\Delta E; A)) \to Z(\bar{H}^*(\Delta E; A)) \cong Z(A(\Delta E))) \otimes H^*(\Delta E; k)$. Since $\bar{\nu}_E$ is induced by μ_E, it follows from 6.1 (iii) that the kernel of $\bar{\nu}_E$ contains $J(Z(A(\Delta E))) \otimes H^*(\Delta E; k)$. Thus the restriction of $\bar{\nu}_E$ to $Z(H^*(\Delta E; A(\Delta E)))$ extends the unique unitary algebra homomorphism $Z(A(\Delta E)) \to k$, hence is equal to the unique unitary algebra homomorphism from $Z(H^*(\Delta E; A(\Delta E))$ to $H^*(\Delta E; k)$ which is a retraction for $\iota_E^{A(\Delta E)}$. This shows that the restriction of μ_E to $Z(H^*(\Delta E; A))$ is an algebra homomorphism, hence equal to π_E^A by the characterisation of π_E^A in 3.1. $\square$

Lemma 6.3. *For any elementary abelian subgroup E of P, the following diagram is commutative:*

$$\begin{tikzcd}
HH^*(A) \arrow[r, "\delta_P^A"] \arrow[d, equal] & H^*(\Delta P; A) \arrow[r, "\nu_P"] \arrow[d, "\mathrm{res}^{\Delta P}_{\Delta E}"] & H^*(\Delta P; k) \arrow[d, "\mathrm{res}^{\Delta P}_{\Delta E}"] \\
HH^*(A) \arrow[r, "\delta_E^A"] \arrow[d, equal] & H^*(\Delta E; A) \arrow[r, "\nu_E"] & H^*(\Delta E; k) \arrow[d, "\cong"] \\
HH^*(A) \arrow[rr, "\sigma_E"'] & & H^*(E; k)
\end{tikzcd}$$

Proof. The commutativity of the two upper squares of the diagram is a consequence of the transitivity and functoriality of restriction. The lower rectangle commutes by the previous lemma. $\square$

This lemma implies that ν_P, precomposed with δ_P^A, lifts σ. We do not know whether $\mathrm{Im}(\nu_P)$ is contained in $H^*(B) = \lim_{\mathcal{F}}(H^*(Q; k)) \subseteq H^*(P; k)$, but we can force the issue by making use of a finite P-P-biset X satisfying the fusion stability properties as in [3, 5.5]. The associated transfer map

$$\mathrm{tr}_X : H^*(P;k) \to H^*(B)$$

is a graded k-linear projection of $H^*(P;k)$ onto $H^*(B)$; see for instance [14, 3.2], or [21].

Lemma 6.4. *Denote by* $\mathrm{res}^P_{\mathcal{E}} : H^*(P;k) \to \prod_{E\in\mathcal{E}} H^*(E;k)$ *the product of the restriction maps* res^P_E*, with* $E \in \mathcal{E}$*. Identify* $\lim_{\mathcal{E}}(H^*(E;k))$ *with its canonical image in* $\prod_{E\in\mathcal{E}} H^*(E;k)$*. Let* $\zeta \in H^*(P;k)$ *such that* $\mathrm{res}^P_{\mathcal{E}}(\zeta) \in \lim_{\mathcal{E}}(H^*(E;k))$*. Then* $\mathrm{res}^P_{\mathcal{E}}(\zeta) = \mathrm{res}^P_{\mathcal{E}}(\mathrm{tr}_X(\zeta))$*.*

Proof. Let E be an elementary abelian subgroup of P. The restriction of the biset X to an E-P-biset is a disjoint union of transitive E-P-bisets of the form

$$X = \cup_{(F,\varphi)}\ E \times_{(F,\varphi)} P$$

where (F, φ) runs over a family of pairs consisting of a subgroup F of E and a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(F, P)$. Here $E \times_{(F,\varphi)} P$ denotes the E-P-biset obtained by taking the quotient of $E \times P$ modulo the equivalence relation $(uv, w) \sim (u, \varphi(v)w)$, where $u \in E$, $v \in F$, $w \in P$. The hypothesis on ζ implies that for any such (F, φ) we have $\mathrm{res}_\varphi(\zeta) = \mathrm{res}^P_E(\zeta)$. Thus

$$\mathrm{res}^P_E(\mathrm{tr}_X(\zeta)) = \sum_{(F,\varphi)} \mathrm{tr}^P_F(\mathrm{res}_\varphi(\zeta)) = \sum_{(F,\varphi)} \mathrm{tr}^P_F(\mathrm{res}^P_F(\zeta)) = \sum_{(F,\varphi)} [E:F]\mathrm{res}^P_E(\zeta)$$

Since $\sum_{(F,\varphi)}[E : F] = |X|/|P| \equiv 1(\mathrm{mod}\ p)$, the result follows. $\square$

Proof of Theorem 1.1. Denote by $\lambda : HH^*(A) \to H^*(B)$ the composition of graded k-linear maps

$$\lambda :\ HH^*(A) \xrightarrow{\delta^A_P} H^*(\Delta P; A) \xrightarrow{\nu_P} H^*(\Delta P; k) \xrightarrow{\cong} H^*(P; k) \xrightarrow{\mathrm{tr}_X} H^*(B)$$

As before, we denote by τ the canonical algebra homomorphism from $H^*(B)$ to $HH^*(A)$. Lemma 5.1, applied to $E = P$, implies that there is a commutative diagram

$$\begin{tikzcd}
H^*(B) \arrow[r, "\tau"] \arrow[d, "\rho_{\Delta P}"'] & HH^*(A) \arrow[rr, "\lambda"] \arrow[d, "\delta^A_P"] & & H^*(B) \\
H^*(\Delta P;k) \arrow[r, "\iota^A_P"'] & Z(H^*(\Delta P;A)) \arrow[r, "\nu_P"'] & H^*(\Delta P;k) \arrow[r, "\cong"'] & H^*(P;k) \arrow[u, "\mathrm{tr}_X"']
\end{tikzcd}$$

Since $\nu_P \circ \iota^A_P = \mathrm{Id}_{H^*(\Delta P;k)}$ and since tr_X restricts to the identity on $H^*(B)$, this implies that

$$\lambda \circ \tau = \mathrm{Id}_{H^*(B)}$$

and hence $HH^*(A) = \mathrm{Im}(\tau) \oplus \ker(\lambda)$. Since, by 6.3, $\nu_P \circ \delta^A_P$ lifts σ, it follows from 6.4 that λ lifts σ; in other words, we have a commutative diagram

$$\begin{tikzcd}
HH^*(A) \arrow[r, "\lambda"] \arrow[d, equal] & H^*(B) \arrow[d, "\mathrm{res}^{\mathcal{F}}_{\mathcal{E}}"] \\
HH^*(A) \arrow[r, "\sigma"'] & \lim_{\mathcal{E}}(H^*(E;k))
\end{tikzcd}$$

where $\mathrm{res}^{\mathcal{F}}_{\mathcal{E}}$ is induced by the product of the restrictions from P to elementary abelian subgroups of P. Thus $\ker(\lambda) \subseteq \ker(\sigma)$, hence $HH^*(A) = \mathrm{Im}(\tau) + \ker(\sigma)$. By Theorem 1.6 (or Proposition 5.3), the ideal $\ker(\sigma)$ is nilpotent. This completes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.3. Since A and B are Morita equivalent, we may replace B by A. Let M be a finitely generated A-module, and denote by $\mu_M : HH^*(A) \to \mathrm{Ext}^*_A(M, M)$ the graded algebra homomorphism induced by $- \otimes_A M$. The variety $X_B(M)$ is, by definition, the maximal ideal spectrum of $HH^*(A)/\ker(\mu_M)$. By [11, 4.1], the variety V_B is the maximal ideal spectrum of $H^*(B)/\ker(\mu_M \circ \tau)$. Modulo nilpotent elements, the graded algebras $HH^*(A)/\ker(\mu_M)$ and $H^*(B)/\ker(\mu_M \circ \tau)$ are isomorphic, by Theorem 1.1, whence the result. $\square$

Proof of Corollary 1.4. This follows from Corollary 1.3 and [11, 2.3] or [7, Theorem 1.1]. $\square$

Proof of Corollary 1.5. It follows from [11, 5.1] that a stable equivalence of Morita type between block algebras B, C given by a C-B-bimodule M as in the statement of 1.5 induces an isomorphism of varieties $X_B(U) \cong X_C(M \otimes_B U)$, for any finitely generated B-module U. Combined with 1.3 this yields 1.5. $\square$

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