ON GRADED CENTERS AND BLOCK COHOMOLOGY

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ABSTRACT. We extend the group theoretic notions of transfer and stable elements to graded centers of triangulated categories. When applied to the center $H^*(D^b(B))$ of the derived bounded category of a block algebra B we show that the block cohomology $H^*(B)$ is isomorphic to a quotient of a certain subalgebra of stable elements of $H^*(D^b(B))$ by some nilpotent ideal, and that a quotient of $H^*(D^b(B))$ by some nilpotent ideal is Noetherian over $H^*(B)$.

§1 INTRODUCTION

The graded center of a triangulated category $\mathcal{C}$ with shift functor Σ consists of all natural transformations $\mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ which commute modulo the sign $(-1)^n$ with Σ, where n runs over the integers. For B a source algebra or a block algebra of a p-block of a finite group, there are canonical graded maps from the block cohomology $H^*(B)$ and its Tate analogue $\hat{H}^*(B)$ to the centers $H^*(D^b(B))$ and $H^*(\overline{\mathrm{mod}}(B))$ of the bounded derived and stable module categories which factor through Hochschild cohomology $HH^*(B)$ and its Tate analogue $\hat{H}H^*(B)$, respectively. Analogously to questions raised in [13] it is natural to ask when the canonical maps $H^*(B) \to H^*(D^b(B))$ or $H^*(B) \to H^*(\overline{\mathrm{mod}}(B))$ are isomorphisms modulo nilpotent ideals. As in [13], the relevance of this type of question lies in the fact that $H^*(B)$ is an invariant of the local structure of the block while $H^*(D^b(B))$ and $H^*(\overline{\mathrm{mod}}(B))$ are invariants of the bounded derived and stable module category, respectively, and it is not known to what extent these categories determine the local structure. The following weaker version of this question has an affirmative answer: the image of the canonical map $H^*(B) \to H^*(D^b(B))$ lies in a certain subalgebra of stable elements, defined in §5, and the induced map is then indeed an isomorphism modulo nilpotent ideals; more precisely:

Theorem 1.1. *Let k be an algebraically closed field of prime characteristic p. Let G be a finite group and B be a source algebra of a block of kG with a defect group P. Denote by $\mathcal{G} : D^b(B) \to D^b(kP)$ the restriction functor. The canonical map $H^*(B) \to H^*(D^b(B))$ sends $H^*(B)$ to the subalgebra of $\mathcal{G}$-stable elements $H^*_{\mathcal{G}}(D^b(B))$ and there is a nilpotent ideal $\mathcal{N}$ in $H^*_{\mathcal{G}}(D^b(B))$ such that this map induces an isomorphism $H^*(B) \cong H^*_{\mathcal{G}}(D^b(B))/\mathcal{N}$.*

This is proved in section 9. For completeness, we include several straightforward observations, with proofs at the end of section 2:

Proposition 1.2. *Let k be an algebraically closed field of prime characteristic p, let G be a finite group and let B be a block algebra of kG. There is a nilpotent ideal $\mathcal{N}$ in $H^*(D^b(B))$ such that $H^*(D^b(B))/\mathcal{N}$ is Noetherian as $H^*(B)$-module. In particular, the graded algebra $H^*(D^b(B))/\mathcal{N}$ is finitely generated.*

Proposition 1.3. *Let k be a field of characteristic p and let P be a finite p-group. Let $\mathcal{N}$ be the kernel of the evaluation map $H^*(D^b(kP)) \to H^*(P, k)$. The ideal $\mathcal{N}$ is nilpotent and the canonical map $H^*(P, k) \to H^*(D^b(kP))$ induces an isomorphism $H^*(P, k) \to H^*(D^b(kP))/\mathcal{N}$.*

This implies in particular that if B is a nilpotent block then the canonical map $H^*(B) \to H^*(D^b(B))$ is an isomorphism modulo nilpotent ideals. Graded centers of stable module categories seem to be more elusive; neither is it clear whether the degree zero component $H^0(\overline{\mathrm{mod}}(B))$ is finite-dimensional, nor even what is the kernel of the canonical map $Z(B) \to H^0(\overline{\mathrm{mod}}(B))$. Rickard noted that almost vanishing morphisms in the sense of Happel [7] can be used to construct elements in $H^0(\mathcal{C})$ for $\mathcal{C}$ any triangulated category. Auslander-Reiten sequences for modules over a symmetric algebra A give rise to almost vanishing homomorphisms in $\overline{\mathrm{mod}}(A)$ of degree -1, and as a consequence we get that $H^{-1}(\overline{\mathrm{mod}}(A))$ is not finite-dimensional, in general:

Proposition 1.4. *Let A be a finite-dimensional symmetric algebra over an algebraically closed field k and let U be a finitely generated indecomposable non-projective A-module. Suppose that U is not perdiodic, or that U has even period, or that $\mathrm{char}(k) = 2$. There is an element $\zeta \in H^{-1}(\overline{\mathrm{mod}}(A))$ such that $\zeta(U) : U \to \Omega(U)$ represents an almost split sequence ending in U and such that $\zeta(V) = 0$ for any finitely generated indecomposable non-projective A-module V which is not isomorphic to $\Omega^n(U)$, for any integer n. In particular, if A has infinitely many Ω-orbits of isomorphism classes of finitely generated non projective modules whose period is either infinite or even then $H^{-1}(\overline{\mathrm{mod}}(A))$ is not finite-dimensional.*

Notation. Throughout this note, R is a commutative ring with unit element. Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be categories, let $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}, \mathcal{G}' : \mathcal{D} \to \mathcal{E}$ be covariant functors. For any natural transformation $\psi : \mathcal{G} \to \mathcal{G}'$ we denote by $\psi\mathcal{F} : \mathcal{G}\mathcal{F} \to \mathcal{G}'\mathcal{F}$ the natural transformation given by $(\psi\mathcal{F})(X) = \psi(\mathcal{F}(X)) : \mathcal{G}(\mathcal{F}(X)) \to \mathcal{G}'(\mathcal{F}(X))$ for any object X in $\mathcal{C}$. Similarly, for any natural transformation $\varphi : \mathcal{F} \to \mathcal{F}'$ we denote by $\mathcal{G}\varphi : \mathcal{G}\mathcal{F} \to \mathcal{G}\mathcal{F}'$ the natural transformation given by $(\mathcal{G}\varphi)(X) = \mathcal{G}(\varphi(X)) : \mathcal{G}(\mathcal{F}(X)) \to \mathcal{G}(\mathcal{F}'(X))$ for any object X in $\mathcal{C}$.

2 GRADED CENTERS OF GRADED CATEGORIES

Graded centers are being considered by a growing number of authors; see for instance [2], [5]. We review the terminology and basic properties.

Definition 2.1. Let $\mathcal{C}$ be an R-linear category and let $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ be an R-linear equivalence. We define a graded R−module $H^*(\mathcal{C})$ as follows. For any integer n we denote by $H^n(\mathcal{C})$ the R-module of natural transformations $\varphi : \mathrm{Id}_\mathcal{C} \to \Sigma^n$ satisfying $\Sigma\varphi = (-1)^n\varphi\Sigma$. We call $H^*(\mathcal{C})$ the *graded center of the graded category* $(\mathcal{C}, \Sigma)$.

Remark 2.2. In particular, $H^0(\mathcal{C})$ consists of all natural transformations $\varphi : \mathcal{C} \to \mathcal{C}$ satisfying $\Sigma\varphi = \varphi\Sigma$. This condition is void if $\Sigma = \mathrm{Id}_\mathcal{C}$. In that case, $H^0(\mathcal{C}) = Z(\mathcal{C})$ consists of all natural transformations $\varphi : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$, traditionally called the *center of the category* $\mathcal{C}$. It is easy to see that $Z(\mathcal{C})$ is commutative. If $\mathcal{C} = \mathrm{mod}(A)$, where A is any ring, we have a canonical isomorphism $Z(A) \cong Z(\mathcal{C})$ mapping $z \in Z(A)$ to the natural transformation consisting, for any A−module U, of the endomorphism given by multiplication with z on U.

Remark 2.3. The equality $\Sigma\varphi = (-1)^n\varphi\Sigma$ in 2.1 is an equality of natural transformations from Σ to Σ^{n+1}. The sign is motivated by the "turning triangles" axiom: if $(\mathcal{C}, \Sigma)$ is a triangulated category

and if $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} \Sigma(U)$ is an exact triangle in $\mathcal{C}$ then the triangle

$$\Sigma^n(U) \xrightarrow{(-1)^n \Sigma^n(f)} \Sigma^n(V) \xrightarrow{(-1)^n \Sigma^n(g)} \Sigma^n(W) \xrightarrow{(-1)^n \Sigma^n(h)} \Sigma^{n+1}(U)$$

is exact. Hence, if $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ is any natural transformation, we have a commutative diagram

$$
\begin{array}{ccccccc}
U & \xrightarrow{\ f\ } & V & \xrightarrow{\ g\ } & W & \xrightarrow{\ h\ } & \Sigma(U) \\
\varphi(U) \downarrow & & (-1)^n\varphi(V) \downarrow & & \varphi(W) \downarrow & & \downarrow (-1)^n\varphi(\Sigma(U)) \\
\Sigma^n(U) & \xrightarrow{(-1)^n\Sigma^n(f)} & \Sigma^n(V) & \xrightarrow{(-1)^n\Sigma^n(g)} & \Sigma^n(W) & \xrightarrow{(-1)^n\Sigma^n(h)} & \Sigma^{n+1}(U)
\end{array}
$$

Thus this diagram defines a morphism of triangles provided that $(-1)^n\varphi(\Sigma(U)) = \Sigma(\varphi(U))$.

The graded R-module $H^*(\mathcal{C})$ depends on Σ, while our notation does not take this into account; usually $\mathcal{C}$ will be a triangulated category and Σ will be its suspension functor which will be clear from the context. The graded R-module $H^*(\mathcal{C})$ becomes a graded R-algebra via the R-bilinear product mapping a pair $(\psi, \varphi) \in H^n(\mathcal{C}) \times H^m(\mathcal{C})$ to the composition $(\Sigma^m\psi) \circ \varphi \in H^{m+n}(\mathcal{C})$ for any two integers m, n. Strictly speaking, for this to make sense in negative degrees, we have to choose an inverse equivalence Σ^{-1} and natural isomorphisms $\Sigma \circ \Sigma^{-1} \cong \mathrm{Id}_{\mathcal{C}} \cong \Sigma^{-1} \circ \Sigma$, so that we can identify the functors $\Sigma^n \circ \Sigma^m \cong \Sigma^{n+m}$ for any two integers n, m, positive or negative. In practice, this may not be an issue because Σ is frequently not only an equivalence, but an *automorphism*; that is, $\Sigma \circ \Sigma^{-1} = \mathrm{Id}_{\mathcal{C}} = \Sigma^{-1} \circ \Sigma$. This is, for instance, the case if Σ is a shift functor in some category of complexes. The sign convention has the following immediate consequence:

Proposition 2.4. *Let $\mathcal{C}$ be an R-linear category and let $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ be an R-linear equivalence. The R-algebra $H^*(\mathcal{C})$ is graded commutative.*

Proof. Let m, n be two integers, let $\varphi \in H^m(\mathcal{C})$ and $\psi \in H^n(\mathcal{C})$. Since $\varphi\Sigma = (-1)^m\Sigma\varphi$ we have $\varphi\Sigma^n = (-1)^{mn}\Sigma^n\varphi$. Using that φ is a natural transformation from $\mathrm{Id}_{\mathcal{C}}$ to Σ^m we get a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Id}_{\mathcal{C}} & \xrightarrow{\ \varphi\ } & \Sigma^m \\
\psi \downarrow & & \downarrow \Sigma^m\psi \\
\Sigma^n & \xrightarrow[\varphi\Sigma^n]{} & \Sigma^{m+n}
\end{array}
$$

Together we get $\psi\varphi = (\Sigma^m\psi) \circ \varphi = (\varphi\Sigma^n) \circ \psi = (-1)^{mn}(\Sigma^n\varphi) \circ \psi = (-1)^{mn}\varphi\psi$. $\square$

Definition 2.5. Let $\mathcal{C}$ be an R-linear category and let $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ be an R-linear equivalence. For any two objects U, V in $\mathcal{C}$ we define a graded R-module $\mathrm{Ext}^*_{\mathcal{C}}(U, V)$ by setting $\mathrm{Ext}^n_{\mathcal{C}}(U, V) = \mathrm{Hom}_{\mathcal{C}}(U, \Sigma^n(V))$ for any integer n.

Remark 2.6. If U, V, W are three objects in an R-linear category $\mathcal{C}$ endowed with an R-linear equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$, there is a graded composition $\mathrm{Ext}^*_{\mathcal{C}}(V, W) \times \mathrm{Ext}^*_{\mathcal{C}}(U, V) \longrightarrow \mathrm{Ext}^*_{\mathcal{C}}(U, W)$ mapping a pair $(\psi, \varphi) \in \mathrm{Ext}^n_{\mathcal{C}}(V, W) \times \mathrm{Ext}^m_{\mathcal{C}}(U, V)$ to the composition $\Sigma^m(\psi) \circ \varphi \in \mathrm{Ext}^{m+n}_{\mathcal{C}}(U, W)$. This defines a graded R-algebra structure on $\mathrm{Ext}^*_{\mathcal{C}}(U, U)$, and $\mathrm{Ext}^*_{\mathcal{C}}(U, V)$ becomes an $\mathrm{Ext}^*_{\mathcal{C}}(V, V)$-$\mathrm{Ext}^*_{\mathcal{C}}(U, U)$-bimodule in this way. Moreover, for any object U in $\mathcal{C}$ there is a canonical graded

algebra homomorphism $H^*(\mathcal{C}) \longrightarrow \mathrm{Ext}^*_{\mathcal{C}}(U, U)$ mapping a natural transformation $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ to the morphism $\varphi(U) : U \to \Sigma^n(U)$, where n is any integer.

Example 2.7. Let A be an R-algebra. The derived bounded category of finitely generated A-modules $D^b(A)$ together with the shift functor is triangulated, and for any bounded complexes of A-modules U, V we have $\mathrm{Ext}^*_{D^b(A)}(U, V) = \mathrm{Ext}^*_A(U, V)$. If moreover A is finitely generated projective as R-module then $HH^*(A) = \mathrm{Ext}_{A \underset{R}{\otimes} A^{op}}(A, A)$; thus an element in $HH^n(A)$ can be represented by a morphism $\zeta : A \longrightarrow A[n]$ in the bounded derived category of A-A-bimodules $D^b(A \underset{R}{\otimes} A^{op})$. For any bounded complex U of A-modules, the functor $- \overset{\mathbb{L}}{\otimes} U$ applied to ζ yields a morphism $\varphi(U) : U \to U[n]$ in $D^b(A)$. This family of morphisms defines clearly a natural transformation from the identity functor on $D^b(A)$ to $[n]$. By chasing signs one verifies that this is in fact an element $\varphi \in H^*(D^b(A))$. Thus the map $\zeta \mapsto \varphi$ defines a homomorphism of graded algebras $HH^*(A) \longrightarrow H^*(D^b(A))$. This is not surjective, in general, not even in degree zero; see [9] for an example. For any bounded complex of A-modules U, evaluation at U defines a canonical graded algebra homomorphism (cf. 2.6) $H^*(D^b(A)) \to \mathrm{Ext}^*_A(U, U)$, and the composition of the two graded algebra homomorphisms $HH^*(A) \longrightarrow H^*(D^b(A)) \longrightarrow \mathrm{Ext}^*_A(U, U)$ is the canonical map induced by the functor $- \overset{\mathbb{L}}{\underset{A}{\otimes}} U$ from $D^b(A \underset{R}{\otimes} A^{op})$ to $D^b(A)$.

Example 2.8. Let A be an R-algebra which is finitely generated projective as R-module and suppose that A is relatively R-injective as left A-module; this is the case, for instance, if A is *symmetric* and hence in particular if $A = RG$ for some finite group G. Let $\overline{\mathrm{mod}}(A)$ be the R-stable category of the category $\mathrm{mod}(A)$ of finitely generated A-modules. That is, $\overline{\mathrm{mod}}(A)$ is the R-linear quotient category of $\mathrm{mod}(A)$ obtained from identifying to zero all relatively R-projective A-modules. Taking cokernels of relatively R-injective envelopes defines an equivalence Σ on $\overline{\mathrm{mod}}(A)$. Together with triangles induced by R-split exact sequences, $\overline{\mathrm{mod}}(A)$ endowed with Σ becomes a triangulated category (see e.g. [7]). Applied to $(\mathcal{C}, \Sigma) = (\overline{\mathrm{mod}}(A), \Sigma)$ and A-modules, we get Tate cohomology, $\mathrm{Ext}^*_{\mathcal{C}}(U, V) = \widehat{\mathrm{Ext}}^*_A(U, V)$. As in 2.7, for any A-module U we get graded algebra hommorphisms $\hat{H}H^*(A) \longrightarrow H^*(\overline{\mathrm{mod}}(A)) \longrightarrow \hat{\mathrm{Ext}}^*_A(U, U)$ whose composition is induced by the functor $- \underset{A}{\otimes} U$. This map is neither surjective nor injective, in general; see [8] for an example.

If $R = k$ is an algebraically closed field and $\mathcal{C}$ a triangulated category such that $H^*(\mathcal{C})$ is finitely generated as k-algebra, one can associate with any object U in $\mathcal{C}$ a variety $V_{\mathcal{C}}(U)$ in the spirit of Carlson's cohomology varieties for modules over group algebras, by taking for $V_{\mathcal{C}}(U)$ the maximal ideal spectrum of the quotient of $H^*(\mathcal{C})$ by the kernel of the algebra homomorphism $H^*(\mathcal{C}) \to \mathrm{Ext}^*_{\mathcal{C}}(U, U)$. In fact, for this to make sense it would suffice if the quotient of $H^*(\mathcal{C})$ by some nilpotent ideal is finitely generated as k-algebra. This motivates the next Proposition, which requires the following concept from Rouquier [14]. Let $(\mathcal{C}, \Sigma)$ be a triangulated category and let M be an object in $\mathcal{C}$. We denote by $\langle M \rangle_1$ the full additive subcategory of $\mathcal{C}$ consisting of all objects isomorphic to finite direct sums of summands of the objects $\Sigma^n(M)$, with $n \in \mathbb{Z}$. For $i \geq 2$ we define inductively $\langle X \rangle_i$ as the full additive subcategory of $\mathcal{C}$ consisting of all objects isomorphic to direct summands of objects Z for which there exists an exact triangle $X \to Y \to Z \to \Sigma(X)$ with X in $\langle M \rangle_{i-1}$ and Y in $\langle M \rangle_1$. Following [14, 3.6], the *dimension of* $\mathcal{C}$, denoted $\dim(\mathcal{C})$, is the smallest positive integer d for which there exists an object M in $\mathcal{C}$ such that $\langle M \rangle_{d+1} = \mathcal{C}$.

Proposition 2.9. *Let $(\mathcal{C}, \Sigma)$ be a triangulated category and let M be an object in $\mathcal{C}$ for which there exists a positive integer d satisfying $\langle M \rangle_{d+1} = \mathcal{C}$. Let $\mathcal{N}$ be the kernel of the canonical graded algebra homomorphism $H^*(\mathcal{C}) \to \mathrm{Ext}^*_{\mathcal{C}}(M, M)$. We have $\mathcal{N}^{2^d} = \{0\}$.*

Proof. Let m, n be integers and let $\varphi \in H^m(\mathcal{C})$ and $\psi \in H^n(\mathcal{C})$. If $\varphi \in \mathcal{N}$ then $\varphi(M) = 0$ and hence $\varphi(X) = 0$ for all X in $\langle M \rangle_1$. What we show is that if $\varphi(X) = 0 = \psi(X)$ for all objects in X in $\langle M \rangle_i$ then $(\psi\varphi)(Z) = 0$ for all objects in $\langle M \rangle_{i+1}$; the result follows then by induction. This is a standard argument: consider an exact commutative diagram in $\mathcal{C}$ of the form

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
{\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & {\scriptstyle \varphi(Z)}\nearrow & \downarrow & & \downarrow {\scriptstyle 0} \\
\Sigma^m(X) & \longrightarrow & \Sigma^m(Y) & \longrightarrow & \Sigma^m(Z) & \longrightarrow & \Sigma^{m+1}(X) \\
{\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & {\scriptstyle \psi(\Sigma^m(Z))}\downarrow & \swarrow & & \downarrow {\scriptstyle 0} \\
\Sigma^{m+n}(X) & \longrightarrow & \Sigma^{m+n}(Y) & \longrightarrow & \Sigma^{m+n}(Z) & \longrightarrow & \Sigma^{m+n+1}(X)
\end{array}
$$

with X in $\langle M \rangle_i$ and Y in $\langle M \rangle_1$. Since $\varphi(Z)$ composed with the morphism $\Sigma^m(Z) \to \Sigma^{m+1}(X)$ is zero, $\varphi(Z)$ factors through $\Sigma^m(Y)$. For similar reasons, $\psi(\Sigma^m(Z))$ factors through $\Sigma^{m+1}(X)$. But then $\psi(\Sigma^m(Z)) \circ \varphi(Z)$ factors through two consecutive morphisms in an exact triangle, hence is zero. $\square$

Proof of Proposition 1.2. Let B be a block algebra over an algebraically closed field k of prime characteristic p. We refer to [10, §5] for the definition of block cohomology. By [10, 5.6.(iii)] there is a canonical graded algebra homomorphism $H^*(B) \longrightarrow HH^*(B)$. Thus, for any B-module U, we have a sequence of canonical algebra homomorphisms

$$
H^*(B) \to HH^*(B) \to H^*(D^b(B)) \to \mathrm{Ext}^*_B(U, U)
$$

By 2.9 there exists a finitely generated B-module U such that $\mathcal{N} = \ker(H^*(D^b(B)) \to \mathrm{Ext}^*_A(U, U))$ is a nilpotent ideal (for instance $U = B/J(B)$ would have this property because then $\langle U \rangle_{d+1} = D^b(B)$ for some integer d bounded by the Loewy length of B by [14, 3.7]). Since $\mathrm{Ext}^*_B(U, U)$ is Noetherian over $H^*(G, k)$ (cf. [1, 4.2.4]) it follows from the commutative diagram in [11, 4.2.(iii)] that $\mathrm{Ext}^*_A(U, U)$ is Noetherian as module over $H^*(B)$. Thus its submodule isomorphic to $H^*(D^b(B))/\mathcal{N}$ is Noetherian over $H^*(B)$. $\square$

Proof of Proposition 1.3. Set $\Delta P = \{(u, u) \mid u \in P\} \subseteq P \times P$. The "diagonal" induction functor $\mathrm{Ind}_{\Delta P}^{P \times P}$ sends the trivial $k\Delta P$-module k to kP viewed as $k(P \times P)$-module via $(u, v) \cdot y = uyv^{-1}$ for $u, v, y \in P$. This functor induces hence a graded homomorphism $\delta_P : H^*(P, k) \to HH^*(kP)$ (this is well-known; see e.g. [10, §4] for some more details). When composed with the canonical homomorphisms $HH^*(kP) \to H^*(D^b(kP)) \to \mathrm{Ext}^*_{kP}(k, k) = H^*(P, k)$ this yields the identity on $H^*(P, k)$; in particular, the canonical map $H^*(D^b(kP)) \to H^*(P, k)$ is surjective. Since k is the unique simple kP-module, up to isomorphism, we have $D^b(kP) = \langle k \rangle_{d+1}$ for some positive integer d, and hence the kernel $\mathcal{N}$ of this map is nilpotent by 2.9. The result follows. $\square$

Proof of Proposition 1.4. Let $\zeta(U) \in \overline{\mathrm{Hom}}_A(U, \Omega(U))$ such that $\zeta(U)$ represents an Auslander-Reiten sequence ending in U. Then $\zeta(U)$ is almost vanishing in the sense of [Hap1].I.4.1; that is, for any morphism $\beta : W \to U$ in $\overline{\mathrm{mod}}(A)$ which is not a split epimorphism, we have $\zeta(U) \circ \beta = 0$, and, for any morphism $\gamma : \Omega(U) \to W$ in $\overline{\mathrm{mod}}(A)$ which is not a split monomorphism, we have $\gamma \circ \zeta(U) = 0$. For any integer n set $\zeta(\Omega^n(U)) = (-1)^n \Omega^n(\zeta(U))$; this is possible even if U has an odd period thanks to the assumption $\mathrm{char}(k) = 2$ in that case. For any indecomposable non projective A-module V not isomorphic to $\Omega^n(U)$ for any integer n set $\zeta(V) = 0$ in $\overline{\mathrm{Hom}}_A(V, \Omega(V))$.

Note that $\Omega^n(\zeta(U))$ represents an Auslander-Reiten sequence ending in $\Omega^n(U)$. In order to show that this family of morphisms defines an element in the graded center it suffices to show that for any two indecomposable non-projective A-modules X, Y and any $\alpha \in \overline{\mathrm{Hom}}_A(X, Y)$ the diagram in $\overline{\mathrm{mod}}(A)$

$$
\begin{array}{ccc}
X & \xrightarrow{\zeta(X)} & \Omega(X) \\
\alpha \downarrow & & \downarrow \Omega(\alpha) \\
Y & \xrightarrow{\zeta(Y)} & \Omega(Y)
\end{array}
$$

is commutative. It suffices in fact to do this for X and Y running over a set of representatives of the isomorphism classes of indecomposable non projective modules. If none of X, Y is isomorphic to $\Omega^n(U)$ for some integer n this holds trivially as then both $\zeta(X)$ and $\zeta(Y)$ are zero. Thus we may assume that one of X or Y is equal to $\Omega^n(U)$ for some integer n. By applying Ω^{-n} to such a diagram we may in fact assume that on of X or Y is equal to U. Suppose that both $X = Y = U$. Since k is algebraically closed, we can write $\alpha = \lambda \bar{\mathrm{Id}}_U + \rho$ for some $\lambda \in k$ and some $\rho \in J(\overline{\mathrm{End}}_A(U))$. If $\rho = 0$ then α is a multiple of the identity on U, hence $\Omega(\alpha)$ is the same multiple of the identity on $\Omega(U)$, and the commutativity of the above diagram is clear. If $\lambda = 0$ we have $\alpha = \rho$, hence neither α nor $\Omega(\alpha)$ is a split epimorphism or split monomorphism in $\overline{\mathrm{mod}}(A)$ and hence $\zeta(U) \circ \alpha = 0 = \Omega(\alpha) \circ \zeta(U)$ in that case. If $X = U$ but $Y \not\cong U$ then $\Omega(X) \not\cong \Omega(Y)$, hence $\Omega(\alpha) \circ \zeta(X) = 0$. We need to show that then also $\zeta(Y) \circ \alpha = 0$. If $Y \not\cong \Omega^n(U)$ for any integer n then $\zeta(Y) = 0$, so this is clear. Suppose that $Y \cong \Omega^n(U)$ for some integer n. Since $Y \not\cong X$, the morphism α is not a split epimorphism and therefore $\zeta(Y) \circ \alpha = 0$ because $\zeta(Y)$ represents an Auslander-Reiten sequence ending in Y. Similarly, if $Y = U$ but $X \not\cong U$, then α is not a split epimorphism, hence $\zeta(Y) \circ \alpha = 0$. To see that also $\Omega(\alpha) \circ \zeta(X) = 0$ we distinguish as before two cases: if $X \not\cong \Omega^n(U)$ for any integer n then $\zeta(X) = 0$, so the commutativity is clear. If $X \cong \Omega^n(U)$ for some integer n then $\zeta(X)$ represents an Auslander-Reiten sequence ending in Y and $\Omega(\alpha)$ is not a split monomorphism because $\Omega(X) \not\cong \Omega(Y)$. Hence $\Omega(\alpha) \circ \zeta(X) = 0$ as required. $\square$

3 Compatible adjunctions for functors between graded categories

Definition 3.1 Let $\mathcal{C}$, $\mathcal{D}$ be R−linear categories, endowed with R-linear equivalences $\Sigma : \mathcal{C} \to \mathcal{C}$ and $\Delta : \mathcal{D} \to \mathcal{D}$. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear functors such that $\mathcal{G}$ is left adjoint to $\mathcal{F}$, and let $f : \mathrm{Id}_{\mathcal{D}} \to \mathcal{F}\mathcal{G}$ be the unit and $g : \mathcal{G}\mathcal{F} \to \mathrm{Id}_{\mathcal{C}}$ the counit of an adjunction. Suppose that there are isomorphisms of functors $a : \Delta\mathcal{F} \cong \mathcal{F}\Sigma$ and $b : \mathcal{G}\Delta \cong \Sigma\mathcal{G}$.

We say that the left adjunction given by f and g is *compatible with Σ and Δ with respect to a and b* if the following diagrams of natural transformations are commutative:

$$
\begin{array}{ccc}
\Delta & \xrightarrow{\Delta f} & \Delta\mathcal{F}\mathcal{G} \\
f\Delta \downarrow & & \downarrow a\mathcal{G} \\
\mathcal{F}\mathcal{G}\Delta & \xrightarrow{\mathcal{F}b} & \mathcal{F}\Sigma\mathcal{G}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{G}\Delta\mathcal{F} & \xrightarrow{b\mathcal{F}} & \Sigma\mathcal{G}\mathcal{F} \\
\mathcal{G}a \downarrow & & \downarrow \Sigma g \\
\mathcal{G}\mathcal{F}\Sigma & \xrightarrow{g\Sigma} & \Sigma
\end{array}
$$

If $\mathcal{G}$ is also right adjoint to $\mathcal{F}$ and $f' : \mathrm{Id}_{\mathcal{C}} \to \mathcal{G}\mathcal{F}$ the unit, $g' : \mathcal{F}\mathcal{G} \to \mathrm{Id}_{\mathcal{D}}$ the counit of a right adjunction, we say that the adjunction of $\mathcal{F}$ and $\mathcal{G}$ given by the units and counints f, g, f', g' is *compatible with Σ and Δ with respect to a and b*, if the left adjunction of $\mathcal{G}$ to $\mathcal{F}$ is compatible with Σ and Δ with respect to a, b, and if the left adjunction of $\mathcal{F}$ to $\mathcal{G}$ is compatible with Σ and Δ with

respect to a^{-1}, b^{-1}; in other words, if in addition to the two previous diagrams the two following diagrams are commutative as well:

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\ \Sigma f'\ } & \Sigma\mathcal{G}\mathcal{F} \\
{\scriptstyle f'\Sigma}\downarrow & {\scriptstyle b^{-1}\mathcal{F}} & \downarrow \\
\mathcal{G}\mathcal{F}\Sigma & \xrightarrow[\ \mathcal{G}a^{-1}\ ]{} & \mathcal{G}\Delta\mathcal{F}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{F}\Sigma\mathcal{G} & \xrightarrow{\ a^{-1}\mathcal{G}\ } & \Delta\mathcal{F}\mathcal{G} \\
{\scriptstyle \mathcal{F}b^{-1}}\downarrow & {\scriptstyle \Delta g'} & \downarrow \\
\mathcal{F}\mathcal{G}\Delta & \xrightarrow[\ g'\Delta\ ]{} & \Delta
\end{array}
$$

Remark 3.2. Especially when dealing with derived categories, the natural transformations a and b will typically be equalities $\Delta\mathcal{F} = \mathcal{F}\Sigma$ and $\mathcal{G}\Delta = \Sigma\mathcal{G}$, in which case the four compatibility diagrams reduce to equalities of natural transformations $\Delta f = f\Delta$, $\ \Sigma g = g\Sigma$, $\ \Sigma f' = f'\Sigma$, $\ \Delta g' = g'\Delta$. Verdier [15] defined an exact functor of triangulated categories $(\mathcal{C}, \Sigma) \to (\mathcal{D}, \Delta)$ to be an additive functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ satisfying $\mathcal{F}\Sigma = \Delta\mathcal{F}$ and mapping exact triangles to exact triangles. Other authors allow more flexibility in that they require only a natural isomorphism $\mathcal{F}\Sigma \cong \Delta\mathcal{F}$ and 3.1 is intended to accomodate this extra degree of generality; the price to pay in this context is that the notion of an adjoint pair of functors between triangulated categories as in 3.1 becomes more formally involved.

The above definition is redundant in various ways - the following Lemmas describe this in detail. Lemma 3.3 says that the commutativity of the first two diagrams in 3.1 is equivalent:

Lemma 3.3. *With the notation of 3.1, if one of the following two diagrams is commutative, so is the other:*

$$
\begin{array}{ccc}
\Delta & \xrightarrow{\ \Delta f\ } & \Delta\mathcal{F}\mathcal{G} \\
{\scriptstyle f\Delta}\downarrow & {\scriptstyle a\mathcal{G}}\downarrow & \\
\mathcal{F}\mathcal{G}\Delta & \xrightarrow[\ \mathcal{F}b\ ]{} & \mathcal{F}\Sigma\mathcal{G}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{G}\Delta\mathcal{F} & \xrightarrow{\ b\mathcal{F}\ } & \Sigma\mathcal{G}\mathcal{F} \\
{\scriptstyle \mathcal{G}a}\downarrow & {\scriptstyle \Sigma g}\downarrow & \\
\mathcal{G}\mathcal{F}\Sigma & \xrightarrow[\ g\Sigma\ ]{} & \Sigma
\end{array}
$$

Proof. If the first diagram is commutative, there is a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{G}\Delta\mathcal{F} & \xrightarrow{\ \mathcal{G}\Delta f\mathcal{F}\ } & \mathcal{G}\Delta\mathcal{F}\mathcal{G}\mathcal{F} & \xrightarrow{\ \mathcal{G}\Delta\mathcal{F}g\ } & \mathcal{G}\Delta\mathcal{F} \\
{\scriptstyle \mathcal{G}f\mathcal{G}\Delta\mathcal{F}}\downarrow & & {\scriptstyle \mathcal{G}a\mathcal{G}\mathcal{F}}\downarrow & & \downarrow{\scriptstyle \mathcal{G}a} \\
\mathcal{G}\mathcal{F}\mathcal{G}\Delta\mathcal{F} & \xrightarrow{\ \mathcal{G}\mathcal{F}b\mathcal{F}\ } & \mathcal{G}\mathcal{F}\Sigma\mathcal{G}\mathcal{F} & \xrightarrow{\ \mathcal{G}\mathcal{F}\Sigma g\ } & \mathcal{G}\mathcal{F}\Sigma \\
{\scriptstyle g\mathcal{G}\Delta\mathcal{F}}\downarrow & & {\scriptstyle g\Sigma\mathcal{G}\mathcal{F}}\downarrow & & \downarrow{\scriptstyle g\Sigma} \\
\mathcal{G}\Delta\mathcal{F} & \xrightarrow[\ b\mathcal{F}\ ]{} & \Sigma\mathcal{G}\mathcal{F} & \xrightarrow[\ \Sigma g\ ]{} & \Sigma
\end{array}
$$

The upper left square in this diagram is obtained by composing the first diagram in 3.1 with $\mathcal{G}$ on the left and $\mathcal{F}$ on the right. The other squares are commutative because f and g are natural transformations. Moreover, the first row and the first column of this diagram are the identity transformation. Thus by considering the four outer corners of the above 3 by 3 diagram it follows that the second diagram in 3.3 is commutative. A dual argument shows that the commutativity of the second diagram in 3.3 implies the commutativity of the first one. $\square$

By the same argument, the commutativity of the third and fourth diagram in 3.1 is equivalent. When specialised to the case where a and b are equalities, this says that the equality $\Delta f = f\Delta$ holds if and only if $\Sigma g = g\Sigma$, and similarly $\Sigma f' = f'\Sigma$ if and only if $\Delta g' = g'\Delta$. If the first diagram in 3.1 commutes, then the natural isomorphisms a and b determine each other:

Lemma 3.4. *With the notation of 3.1, suppose the following diagram is commutative:*

$$\begin{array}{ccc} \Delta & \xrightarrow{\ \Delta f\ } & \Delta\mathcal{F}\mathcal{G} \\ {\scriptstyle f\Delta}\downarrow & & \downarrow{\scriptstyle a\mathcal{G}} \\ \mathcal{F}\mathcal{G}\Delta & \xrightarrow[\ \mathcal{F}b\ ]{} & \mathcal{F}\Sigma\mathcal{G} \end{array}$$

Then $a = (\mathcal{F}\Sigma g) \circ (\mathcal{F}b\mathcal{F}) \circ (f\Delta\mathcal{F})$ and $b = (g\Sigma\mathcal{G}) \circ (\mathcal{G}a\mathcal{G}) \circ (\mathcal{G}\Delta f)$.

Proof. In order to prove the formula for a we observe that there is a commutative diagram

$$\begin{array}{ccccc} \Delta\mathcal{F} & \xrightarrow{\ \Delta f\mathcal{F}\ } & \Delta\mathcal{F}\mathcal{G}\mathcal{F} & \xrightarrow{\ \Delta\mathcal{F}g\ } & \Delta\mathcal{F} \\ {\scriptstyle f\Delta\mathcal{F}}\downarrow & & \downarrow{\scriptstyle a\mathcal{G}\mathcal{F}} & & \downarrow{\scriptstyle a} \\ \mathcal{F}\mathcal{G}\Delta\mathcal{F} & \xrightarrow[\ \mathcal{F}b\mathcal{F}\ ]{} & \mathcal{F}\Sigma\mathcal{G}\mathcal{F} & \xrightarrow[\ \mathcal{F}\Sigma g\ ]{} & \mathcal{F}\Sigma \end{array}$$

Indeed, the left square is obtained by composing the first diagram in 3.1 (which is commutative by the assumptions) with $\mathcal{F}$ on the right. The second square uses that g is a natural transformation from $\mathcal{G}\mathcal{F}$ to $\mathrm{Id}_{\mathcal{C}}$. Moreover, the first row is the identity transformation on $\Delta\mathcal{F}$ by the general properties of adjunctions. The formula for a follows. The formula for b is obtained by composing the first diagram in 3.1 with $\mathcal{G}$ on the left and then using g again, getting a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{G}\Delta & \xrightarrow{\ \mathcal{G}\Delta f\ } & \mathcal{G}\Delta\mathcal{F}\mathcal{G} \\ {\scriptstyle \mathcal{G}f\Delta}\downarrow & & \downarrow{\scriptstyle \mathcal{G}a\mathcal{G}} \\ \mathcal{G}\mathcal{F}\mathcal{G}\Delta & \xrightarrow{\ \mathcal{G}\mathcal{F}b\ } & \mathcal{G}\mathcal{F}\Sigma\mathcal{G} \\ {\scriptstyle g\mathcal{G}\Delta}\downarrow & & \downarrow{\scriptstyle g\Sigma\mathcal{G}} \\ \mathcal{G}\Delta & \xrightarrow[\ b\ ]{} & \Sigma\mathcal{G} \end{array}$$

Observing again that the left column is the identity transformation on $\mathcal{G}\Delta$ yields the formula for b. $\square$

If all four diagrams in 3.1 are commutative, then the left and right adjunctions are related:

Lemma 3.5. *With the notation of 3.1, assume that all four diagrams in 3.1 are commutative. Then*
$$(\Delta g') \circ (\Delta f) = (g'\Delta) \circ (f\Delta)$$
$$(\Sigma g) \circ (\Sigma f') = (g\Sigma) \circ (f'\Sigma)$$

Proof. To see the first equality (of natural transformations on Δ) combine the first diagram in 3.1 with the fourth diagram in 3.1 rotated by 90 degree counterclockwise, and for the second equality (of natural transformations on Σ) combine the second diagram in 3.1 with the third diagram rotated by 90 degree clockwise. $\square$

Compatible adjunctions can be "added":

Proposition 3.6. *Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, respectively. Let $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}, \mathcal{G}' : \mathcal{D} \to \mathcal{C}$ be R-linear functors. Suppose that $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F}'$, $\mathcal{G}'$ are two pairs of adjoint functors which are both compatible with Σ, Δ with respect to natural isomorphisms of functors $a : \Delta \circ \mathcal{F} \cong \mathcal{F} \circ \Sigma$, $b : \mathcal{G} \circ \Delta \cong \Sigma \circ \mathcal{G}$ and $a' : \Delta \circ \mathcal{F}' \cong \mathcal{F}' \circ \Sigma$ $b' : \mathcal{G}' \circ \Delta \cong \Sigma \circ \mathcal{G}'$.*

Then the direct sum of of the adjunction isomorphisms of the adjoint pairs of functors $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F}'$, $\mathcal{G}'$ is an adjunction between the functors $\mathcal{F} \oplus \mathcal{F}'$ and $\mathcal{G} \oplus \mathcal{G}'$ which is compatible with Σ, Δ with respect to $a \oplus a'$, $b \oplus b'$.

Proof. Let U be an object in $\mathcal{C}$ and let V be an object in $\mathcal{D}$. The sum of the adjunction isomorphisms $\mathrm{Hom}_{\mathcal{C}}(U, \mathcal{G}(V)) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(U), V)$ and $\mathrm{Hom}_{\mathcal{C}}(U, \mathcal{G}'(V)) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}'(U), V)$ yields obviously a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(U, \mathcal{G}(V) \oplus \mathcal{G}'(V)) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(U) \oplus \mathcal{F}'(U), V)$$

showing that $\mathcal{F} \oplus \mathcal{F}'$ is left adjoint to $\mathcal{G} \oplus \mathcal{G}'$. Similarly one shows that $\mathcal{F} \oplus \mathcal{F}'$ is right adjoint to $\mathcal{G} \oplus \mathcal{G}'$. We determine the unit $\mathrm{Id}_{\mathcal{C}} \to (\mathcal{G} \oplus \mathcal{G}')(\mathcal{F} \oplus \mathcal{F}')$ of this adjunction in terms of the units $f : \mathrm{Id}_{\mathcal{C}} \to \mathcal{G}\mathcal{F}$ and $f' : \mathrm{Id}_{\mathcal{C}} \to \mathcal{G}'\mathcal{F}'$ as follows. The morphism $f(U) : U \to \mathcal{G}\mathcal{F}(U)$ is the image of $\mathrm{Id}_{\mathcal{F}(U)}$ under the appropariate adjunction isomorphism; similarly for $f'(U)$. Chasing $\mathrm{Id}_{\mathcal{F}(U) \oplus \mathcal{F}'(U)}$ through the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(U) \oplus \mathcal{F}'(U), \mathcal{F}(U) \oplus \mathcal{F}'(U)) \cong \mathrm{Hom}_{\mathcal{C}}(U, (\mathcal{G} \oplus \mathcal{G}')(\mathcal{F}(U) \oplus \mathcal{F}'(U))) =$$

$$= \mathrm{Hom}_{\mathcal{C}}(U, \mathcal{G}\mathcal{F}(U) \oplus \mathcal{G}\mathcal{F}'(U) \oplus \mathcal{G}'\mathcal{F}(U) \oplus \mathcal{G}'\mathcal{F}'(U))$$

shows that the counit $\mathrm{Id}_{\mathcal{C}} \longrightarrow (\mathcal{G} \oplus \mathcal{G}')(\mathcal{F} \oplus \mathcal{F}')$ is in fact simply obtained by taking the "sum" of f, f' evaluated at the object U, that is,

$$\begin{pmatrix} f(U) \\ f'(U) \end{pmatrix} : U \longrightarrow \mathcal{G}\mathcal{F}(U) \oplus \mathcal{G}'\mathcal{F}'(U)$$

followed by the canonical inclusion

$$\mathcal{G}\mathcal{F}(U) \oplus \mathcal{G}'\mathcal{F}'(U) \longrightarrow (\mathcal{G} \oplus \mathcal{G}')(\mathcal{F} \oplus \mathcal{F}')(U) = \mathcal{G}\mathcal{F}(U) \oplus \mathcal{G}\mathcal{F}'(U) \oplus \mathcal{G}'\mathcal{F}(U) \oplus \mathcal{G}'\mathcal{F}'(U))$$

A similar statement holds for the counits. Since units and counits for $\mathcal{F} \oplus \mathcal{F}'$, $\mathcal{G} \oplus \mathcal{G}'$ are essentially obtained by adding those for the two pairs of adjoint functors, the result follows. $\square$

Compatible adjunctions can be composed:

Proposition 3.7. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, Γ, respectively.*

Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be adjoint R-linear functors, compatible with Σ, Δ with respect to isomorphisms $a : \mathcal{F}\Sigma \cong \Delta\mathcal{F}$, $b : \mathcal{G}\Delta \cong \Sigma\mathcal{G}$.

Let $\mathcal{F}' : \mathcal{D} \to \mathcal{E}$, $\mathcal{G}' : \mathcal{E} \to \mathcal{D}$ be adjoint R-linear functors, compatible with Δ, Γ with respect to isomorphisms $a' : \mathcal{F}'\Delta \cong \Gamma\mathcal{F}'$, $b' : \mathcal{G}'\Gamma \cong \Delta\mathcal{G}'$.

The composition of adjunction isomorphisms of the adjoint pairs of functors $\mathcal{F}, \mathcal{G}$ and $\mathcal{F}', \mathcal{G}'$ is an adjunction between the compositions of functors $\mathcal{F}' \circ \mathcal{F}$, $\mathcal{G} \circ \mathcal{G}'$ which is compatible with Σ, Γ with respect to the isomorphisms $(a'\mathcal{F}) \circ (\mathcal{F}'a) : \mathcal{F}'\mathcal{F}\Sigma \cong \Gamma\mathcal{F}'\mathcal{F}$ and $(b\mathcal{G}') \circ (\mathcal{G}b') : \mathcal{G}\mathcal{G}'\Gamma \cong \Sigma\mathcal{G}\mathcal{G}'$.

Proof. Straightforward verification. $\square$

§4 Adjunction and transfer

In the context of group cohomology and Hochschild cohomology of symmetric algebras, transfer appears to be the translation of the adjunction principle to cohomology rings arising from triangulated categories. The following definition extends the pattern of various sources such as [3], [4], [10], [12].

Definition 4.1 Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear equivalences $\Sigma : \mathcal{C} \to \mathcal{C}$ and $\Delta : \mathcal{D} \to \mathcal{D}$. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear functors which are adjoint to each other, compatible with Σ and Δ with respect to natural transformations $a : \Delta \circ \mathcal{F} \cong \mathcal{F} \circ \Sigma$ and $b : \mathcal{G} \circ \Delta \cong \Sigma \circ \mathcal{G}$. Denote by $f : \mathrm{Id}_{\mathcal{D}} \to \mathcal{F}\mathcal{G}$, $g : \mathcal{G}\mathcal{F} \to \mathrm{Id}_{\mathcal{C}}$ and $f' : \mathrm{Id}_{\mathcal{C}} \to \mathcal{G}\mathcal{F}$, $g' : \mathcal{F}\mathcal{G} \to \mathrm{Id}_{\mathcal{D}}$, the units and counints of adjunctions which are compatible with Σ and Δ, with respect to a, b.

For any integer n and any natural transformation $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ we define the natural transformation $\mathrm{tr}_{\mathcal{F}}(\varphi) : \mathrm{Id}_{\mathcal{D}} \to \Delta^n$ to be the natural transformation equal to the composition

$$\mathrm{Id}_{\mathcal{D}} \xrightarrow{f} \mathcal{F}\mathcal{G} = \mathcal{F}\mathrm{Id}_{\mathcal{C}}\mathcal{G} \xrightarrow{\mathcal{F}\varphi\mathcal{G}} \mathcal{F}\Sigma^n\mathcal{G} \cong \Delta^n\mathcal{F}\mathcal{G} \xrightarrow{\Delta^n g'} \Delta^n \ ,$$

where the isomorphism $\mathcal{F}\Sigma^n\mathcal{G} \cong \Delta^n\mathcal{F}\mathcal{G}$ is given by applying repeatedly a. Similarly, for any two objects U, V in $\mathcal{D}$, we define the *relative transfer map* $\mathrm{tr}_{\mathcal{F}}(U,V) : \mathrm{Ext}_{\mathcal{C}}^*(\mathcal{G}(U), \mathcal{G}(V)) \longrightarrow \mathrm{Ext}_{\mathcal{D}}^*(U, V)$ to be the graded R-linear map sending a morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(U), \Sigma^n(\mathcal{G}(V)))$ to the morphism $\beta \in \mathrm{Hom}_{\mathcal{D}}(U, \Delta^n(V))$, which is equal to the composition

$$U \xrightarrow{f(U)} \mathcal{F}\mathcal{G}(U) \xrightarrow{\mathcal{F}(\alpha)} \mathcal{F}\Sigma^n\mathcal{G}(V) \cong \Delta^n\mathcal{F}\mathcal{G}(V) \xrightarrow{\Delta^n(g')} \Delta^n(V) \ ,$$

where the isomorphism $\mathcal{F}\Sigma^n\mathcal{G}(V) \cong \Delta^n\mathcal{F}\mathcal{G}(V)$ is again given by evaluating the natural transformation a.

Remarks 4.2.
(a) The definition of $\mathrm{tr}_{\mathcal{F}}$ makes use of the unit f and the counit g'; using the unit f' and counit g instead, one gets analogously $\mathrm{tr}_{\mathcal{G}}$. The transfer map $\mathrm{tr}_{\mathcal{F}}$ defined above depends on the choices of the adjunction isomorphisms as well as on the choices of the natural transformations a, b. Whenever we do not specify these choices in a statement, this means that we implicitly assert, that either the statement does not depend on this choice or the context determines such a choice canonically. It can in fact be useful to play two different choices off against each other. In practice, the natural isomorphisms a, b are frequently given canonically by the context; they may sometimes just be equalities of functors.

(b) Examples of pairs of biadjoint functors include induction/restriction between algebras of finite groups, Harish-Chandra induction/restriction, and more generally, functors between module categories of symmetric algebras induced by bimodules which are finitely generated projective on both sides; see §7 for details.

(c) With the notation of 4.1, the degree zero component $\mathrm{tr}_{\mathcal{F}}$ sends a natural transformation $\varphi : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ to the natural transformation

$$\mathrm{tr}_{\mathcal{F}}(\varphi) : \mathrm{Id}_{\mathcal{D}} \xrightarrow{f} \mathcal{F}\mathcal{G} = \mathcal{F}\mathrm{Id}_{\mathcal{C}}\mathcal{G} \xrightarrow{\mathcal{F}\varphi\mathcal{G}} \mathcal{F}\mathrm{Id}_{\mathcal{C}}\mathcal{G} = \mathcal{F}\mathcal{G} \xrightarrow{g'} \mathrm{Id}_{\mathcal{D}} \ .$$

Similarly, for any two objects U, V in $\mathcal{D}$, the degree zero component of the relative transfer map $\mathrm{tr}_{\mathcal{F}}(U,V)$ is the map $t_{\mathcal{F}}(U,V) : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(U), \mathcal{G}(V)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(U, V)$ sending a morphism $\alpha : \mathcal{G}(U) \to \mathcal{G}(V)$ to the morphism $\beta = g'(V) \circ \mathcal{F}(\alpha) \circ f(U) : U \to V$.

Theorem 4.3. *Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear equivalences $\Sigma : \mathcal{C} \to \mathcal{C}$ and $\Delta : \mathcal{D} \to \mathcal{D}$. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear functors which are adjoint to each other, compatible with Σ and Δ with respect to natural transformations $a : \Delta \circ \mathcal{F} \cong \mathcal{F} \circ \Sigma$ and $b : \mathcal{G} \circ \Delta \cong \Sigma \circ \mathcal{G}$. The map sending a natural transformation $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ to $\mathrm{tr}_{\mathcal{F}}(\varphi)$ for any integer n induces a graded R-linear map $\mathrm{tr}_{\mathcal{F}} : H^*(\mathcal{C}) \longrightarrow H^*(\mathcal{D})$ such that, for any object V in $\mathcal{D}$, the diagram*

$$
\begin{array}{ccc}
H^*(\mathcal{C}) & \longrightarrow & \mathrm{Ext}^*_{\mathcal{C}}(\mathcal{G}(V), \mathcal{G}(V)) \\
{\scriptstyle \mathrm{tr}_{\mathcal{F}}} \downarrow & & \downarrow {\scriptstyle \mathrm{tr}_{\mathcal{F}}(V,V)} \\
H^*(\mathcal{D}) & \longrightarrow & \mathrm{Ext}^*_{\mathcal{D}}(V, V)
\end{array}
$$

is commutative, where the horizontal maps are the canonical graded algebra homomorphisms. More explicitly, for any integer n, any natural transformation $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ and any object V in $\mathcal{D}$ we have $\mathrm{tr}_{\mathcal{F}}(\varphi)(V) = \mathrm{tr}_{\mathcal{F}}(V,V)(\varphi(\mathcal{G}(V)))$.

Proof. Let n be an integer and let $\varphi \in H^n(\mathcal{C})$; that is, $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ is a natural transformation satisfying $\varphi\Sigma = (-1)^n\Sigma\varphi$. We need to show that $\mathrm{tr}_{\mathcal{F}}(\varphi)\Delta = (-1)^n\Delta\mathrm{tr}_{\mathcal{F}}(\varphi)$. To see this, we consider the following diagram of functors and natural transformations:

The first row is $\Delta\mathrm{tr}_{\mathcal{F}}(\varphi)$ and the last row is $\mathrm{tr}_{\mathcal{F}}(\varphi)\Delta$. The left and right rectangle of the diagram commute by the compatibility assumption. The commutativity of the remaining squares is easily checked by using either the naturality, the compatibility condition or the assumption $\varphi\Sigma = (-1)^n\Sigma\varphi$. The unmarked isomorphisms are obtained from iterating a. We need to check the commutativity of the diagram in the statement of the theorem. The left side of that diagram is $\mathrm{tr}_{\mathcal{F}}(\varphi)$ evaluated at V, hence equal to the composition

$$
V \xrightarrow{f(V)} \mathcal{F}\mathcal{G}(V) \xrightarrow{\mathcal{F}\varphi\mathcal{G}(V)} \mathcal{F}\Sigma^n\mathcal{G}(V) \cong \Delta^n\mathcal{F}\mathcal{G}(V) \xrightarrow{\Delta^n g'(V)} \Delta^n(V)
$$

This is clearly equal to the map $\mathrm{tr}_{\mathcal{F}}(V,V)$ evaluated at $\varphi(\mathcal{G}(V))$. $\square$

Corollary 4.4. *With the notation of 4.3, setting $\pi_{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}})$, we have $\pi_{\mathcal{F}} \in H^0(\mathcal{D})$.*

Proof. The identity transformation $\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$ commutes obviously with Σ, hence belongs to $H^0(\mathcal{C})$ and thus its image $\pi_{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}})$ belongs to $H^0(\mathcal{D})$ by 4.3. $\square$

Definition 4.5. With the notation above, we call $\pi_{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}})$ the *relatively $\mathcal{F}$-projective element in $H^0(\mathcal{D})$.*

The transfer map $\mathrm{tr}_{\mathcal{F}}$ depends additively on $\mathcal{F}$:

Proposition 4.6. *Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, respectively. Let $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}, \mathcal{G}' : \mathcal{D} \to \mathcal{C}$ be R-linear functors. Suppose that $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F}'$, $\mathcal{G}'$ are two pairs of adjoint functors which are both compatible with Σ, Δ with respect to natural isomorphisms of functors $a : \Delta \circ \mathcal{F} \cong \mathcal{F} \circ \Sigma$, $b : \mathcal{G} \circ \Delta \cong \Sigma \circ \mathcal{G}$ and $a' : \Delta \circ \mathcal{F}' \cong \mathcal{F}' \circ \Sigma$ $b' : \mathcal{G}' \circ \Delta \cong \Sigma \circ \mathcal{G}'$.*

(i) We have $tr_{\mathcal{F} \oplus \mathcal{F}'} = tr_{\mathcal{F}} + tr_{\mathcal{F}'}$.

(ii) For any two objects V, W in $\mathcal{D}$, any $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V), \mathcal{G}(W))$ and any $\alpha' \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}'(V), \mathcal{G}'(W))$ we have $tr_{\mathcal{F} \oplus \mathcal{F}'}(V, W)(\alpha \oplus \alpha') = tr_{\mathcal{F}}(V, W)(\alpha) + tr_{\mathcal{F}'}(V, W)(\alpha')$.

Proof. Straightforward verification using 3.6. $\square$

Transfer maps compose as expected:

Proposition 4.7. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, Γ, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be adjoint R-linear functors, compatible with Σ, Δ with respect to isomorphisms $a : \mathcal{F}\Sigma \cong \Delta\mathcal{F}$, $b : \mathcal{G}\Delta \cong \Sigma\mathcal{G}$. Let $\mathcal{F}' : \mathcal{D} \to \mathcal{E}$, $\mathcal{G}' : \mathcal{E} \to \mathcal{D}$ be adjoint R-linear functors, compatible with Δ, Γ with respect to isomorphisms $a' : \mathcal{F}'\Delta \cong \Gamma\mathcal{F}'$, $b' : \mathcal{G}'\Gamma \cong \Delta\mathcal{G}'$.*

(i) We have $\mathrm{tr}_{\mathcal{F}' \circ \mathcal{F}} = \mathrm{tr}_{\mathcal{F}'} \circ \mathrm{tr}_{\mathcal{F}}$.

(ii) For any two objects V, W in $\mathcal{E}$ we have $\mathrm{tr}_{\mathcal{F}' \circ \mathcal{F}}(V, W) = \mathrm{tr}_{\mathcal{F}'}(V, W) \circ \mathrm{tr}_{\mathcal{F}}(\mathcal{G}(V), \mathcal{G}(W))$.

Proof. Straightforward verification, using 3.7. $\square$

The class of morphisms which are in the image of a transfer map form an "ideal":

Proposition 4.8. *Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear functors which are adjoint to each other, compatible with Σ, Δ with respect to natural isomorphisms of functors $a : \Delta \circ \mathcal{F} \cong \mathcal{F} \circ \Sigma$ and $b : \mathcal{G} \circ \Delta \cong \Sigma \circ \mathcal{G}$. Let U, V, W be objects in $\mathcal{D}$, and let $\alpha \in \mathrm{Ext}^*_{\mathcal{C}}(\mathcal{G}(V), \mathcal{G}(W))$.*

(i) For any $\beta \in \mathrm{Ext}^*_{\mathcal{D}}(U, V)$ we have $\mathrm{tr}_{\mathcal{F}}(V, W)(\alpha)\beta = \mathrm{tr}_{\mathcal{F}}(U, W)(\alpha\mathcal{G}(\beta))$.

(ii) For any $\gamma \in \mathrm{Ext}^*_{\mathcal{D}}(W, U)$ we have $\gamma\mathrm{tr}_{\mathcal{F}}(V, W)(\alpha) = \mathrm{tr}_{\mathcal{F}}(V, U)(\mathcal{G}(\gamma)\alpha)$.

Proof. We prove (i); the proof of (ii) is analogous. We may assume that α and β are homogeneous of degree m and n, respectively. That is,

$$\alpha : \mathcal{G}(V) \longrightarrow \Sigma^m \mathcal{G}(W)$$

is a morphism in $\mathcal{C}$ and

$$\beta : U \longrightarrow \Delta^n(V)$$

is a morphism in $\mathcal{D}$. The product $\alpha\mathcal{G}(\beta)$ is the composition of morphisms

$$\mathcal{G}(U) \xrightarrow{\mathcal{G}(\beta)} \mathcal{G}\Delta^n(V) \cong \Sigma^n \mathcal{G}(V) \xrightarrow{\Sigma^n(\alpha)} \Sigma^{n+m} \mathcal{G}(W)$$

With the notation of 4.3, applying $\mathrm{tr}_{\mathcal{F}}(U, W)$ to $\alpha\mathcal{G}(\beta)$ yields the composition of morphisms

$$U \xrightarrow{f(U)} \mathcal{F}\mathcal{G}(U) \xrightarrow{\mathcal{F}\mathcal{G}(\beta)} \mathcal{F}\mathcal{G}\Delta^n(V) \cong \mathcal{F}\Sigma^n\mathcal{G}(V) \xrightarrow{\mathcal{F}\Sigma^n(\alpha)}$$

$$\mathcal{F}\Sigma^{n+m}\mathcal{G}(W) \cong \Delta^{n+m}\mathcal{F}\mathcal{G}(W) \xrightarrow{\Delta^{n+m}g'(W)} \Delta^{n+m}(W)$$

We need to compare this to the product $\mathrm{tr}_{\mathcal{F}}(V,W)(\alpha)\beta$, which is equal to the composition

$$U \xrightarrow{\;\beta\;} \Delta^n(V) \xrightarrow{\;\Delta^n f(V)\;} \Delta^n \mathcal{FG}(V) \xrightarrow{\;\Delta^n \mathcal{F}(\alpha)\;}$$

$$\Delta^n \mathcal{F}\Sigma^m\mathcal{G}(W) \cong \Delta^{n+m}\mathcal{FG}(W) \xrightarrow{\;\Delta^{n+m}g'(W)\;} \Delta^{n+m}(W)$$

The right ends of these two morphisms coincide because $\mathcal{F}\Sigma^n(\alpha)$ and $\Delta^n\mathcal{F}(\alpha)$ are equal modulo the identifications induced by a and b. We need to show that the two left morphisms compose to the same morphism modulo identifications; that is, we need to show that the diagram

$$
\begin{array}{ccccc}
U & \xrightarrow{\;f(U)\;} & \mathcal{FG}(U) & \xrightarrow{\;\mathcal{FG}(\beta)\;} & \mathcal{FG}\Delta^n(V) \\
\Big\| & & & & \Big\downarrow{\scriptstyle\cong} \\
U & \xrightarrow{\;\beta\;} & \Delta^n(V) & \xrightarrow{\;\Delta^n f(V)\;} & \Delta^n\mathcal{FG}(V)
\end{array}
$$

commutes. Now we have a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\;f(U)\;} & \mathcal{FG}(U) \\
{\scriptstyle\beta}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathcal{FG}(\beta)} \\
\Delta^n(V) & \xrightarrow[\;f(\Delta^n(V))\;]{} & \mathcal{FG}\Delta^n(V)
\end{array}
$$

because f is a natural transformation, and we have a commutative diagram

$$
\begin{array}{ccc}
\Delta^n(V) & \xrightarrow[\;f(\Delta^n(V))\;]{} & \mathcal{FG}\Delta^n(V) \\
\Big\| & & \Big\downarrow{\scriptstyle\cong} \\
\Delta^n(V) & \xrightarrow[\;\Delta^n f(V)\;]{} & \Delta^n\mathcal{FG}(V)
\end{array}
$$

because of the compatibility conditions in the definition 4.1. Together they yield the required statement. $\square$

5 Stable elements

Definition 5.1. Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be an R-linear functor such that there is a natural isomorphism $a : \mathcal{F}\Sigma \cong \Delta\mathcal{F}$. Given an integer n, a natural transformation $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ is called $\mathcal{F}$-*stable*, if there is a natural transformation $\psi : \mathrm{Id}_{\mathcal{D}} \to \Delta^n$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\;\mathcal{F}\varphi\;} & \mathcal{F}\Sigma^n \\
\Big\| & & \Big\downarrow{\scriptstyle\cong} \\
\mathcal{F} & \xrightarrow[\;\psi\mathcal{F}\;]{} & \Delta^n\mathcal{F}
\end{array}
$$

commutes, where the isomorphism $\mathcal{F}\Sigma^n \cong \Delta^n\mathcal{F}$ is induced by a. An element $\varphi \in H^*(\mathcal{C})$ is called $\mathcal{F}$-*stable*, if there is $\psi \in H^*(\mathcal{D})$ such that the previous diagram is commutative for the components of φ and ψ in any degree n; in that case we write abusively $\mathcal{F}\varphi = \psi\mathcal{F}$. We denote by $H^*_{\mathcal{F}}(\mathcal{C})$ the set of $\mathcal{F}$-stable elements in $H^*(\mathcal{C})$.

The notion of $\mathcal{F}$-stability clearly depends on the choice of the natural isomorphism $a : \mathcal{F}\Sigma \cong \Delta\mathcal{F}$.

Proposition 5.2. *Let $\mathcal{C}$, $\mathcal{D}$ be $R-$linear categories endowed with $R-$linear self-equivalences Σ, Δ, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be an $R-$linear functor such that there is a natural isomorphism $a : \mathcal{F}\Sigma \cong \Delta\mathcal{F}$. Then $H_{\mathcal{F}}^*(\mathcal{C})$ is a graded subalgebra of $H^*(\mathcal{C})$.*

Proof. This follows from adding and composing commutative diagrams as in definition 5.1. $\square$

The image of a transfer map is a bimodule for the subalgebra of stable elements. This is a "general abstract nonsense version" of well-known Frobenius reciprocity type statements in cohomology such as [10, 3.4] in Hochschild cohomology, for instance.

Theorem 5.3. *Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear adjoint functors compatible with Σ, Δ. Let $\varphi \in H_{\mathcal{F}}^*(\mathcal{C})$ and $\psi \in H^*(\mathcal{D})$ such that $\mathcal{F}\varphi = \psi\mathcal{F}$.*
(i) We have $\varphi\mathcal{G} = \mathcal{G}\psi$; in particular, $\psi \in H_{\mathcal{G}}^*(\mathcal{D})$.
(ii) For any $\tau \in H^*(\mathcal{C})$ we have $\mathrm{tr}_{\mathcal{F}}(\varphi\tau) = \psi\mathrm{tr}_{\mathcal{F}}(\tau)$ and $\mathrm{tr}_{\mathcal{F}}(\tau\varphi) = \mathrm{tr}_{\mathcal{F}}(\tau)\psi$.
(iii) We have $\mathrm{tr}_{\mathcal{F}}(\varphi) = \pi_{\mathcal{F}}\psi$.
In particular, the image $\mathrm{Im}(\mathrm{tr}_{\mathcal{F}})$ is a sub-$H_{\mathcal{G}}^*(\mathcal{D})$-$H_{\mathcal{G}}^*(\mathcal{D})$-bimodule of $H^*(\mathcal{D})$ and $\mathrm{tr}_{\mathcal{F}}(H_{\mathcal{F}}^*(\mathcal{C})) = \pi_{\mathcal{F}}H_{\mathcal{G}}^*(\mathcal{D})$.

Proof. We may assume that φ, ψ are homogeneous of degree n and that τ is homogeneous of degree m. In order to prove (i) we will show that there is a commutative diagram of the following form:

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\varphi\mathcal{G}} & \Sigma^n\mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{G}\mathcal{F}\mathcal{G} & \xrightarrow{\mathcal{G}\mathcal{F}\varphi\mathcal{G}} & \mathcal{G}\mathcal{F}\Sigma^n\mathcal{G} \\
\| & & \downarrow \cong \\
\mathcal{G}\mathcal{F}\mathcal{G} & \xrightarrow{\mathcal{G}\psi\mathcal{F}\mathcal{G}} & \mathcal{G}\Delta^n\mathcal{F}\mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{G} & \xrightarrow{\mathcal{G}\psi} & \mathcal{G}\Delta^n
\end{array}
$$

Indeed, the upper square is obtained from the square of adjunction units

$$
\begin{array}{ccc}
\mathrm{Id}_{\mathcal{C}} & \xrightarrow{\varphi} & \Sigma^n \\
\downarrow & & \downarrow \\
\mathcal{G}\mathcal{F} & \xrightarrow{\mathcal{G}\mathcal{F}\varphi} & \mathcal{G}\mathcal{F}\Sigma^n
\end{array}
$$

composed with $\mathcal{G}$ on the right. The lower square is obtained from the square of adjunction counits

$$
\begin{array}{ccc}
\mathcal{F}\mathcal{G} & \xrightarrow{\psi\mathcal{F}\mathcal{G}} & \Delta^n\mathcal{F}\mathcal{G} \\
\downarrow & & \downarrow \\
\mathrm{Id}_{\mathcal{D}} & \xrightarrow{\psi} & \Delta^n
\end{array}
$$

composed with $\mathcal{G}$ on the left. For the square in the middle, we use the hypothesis on φ, ψ according to which we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\ \mathcal{F}\varphi\ } & \mathcal{F}\Sigma^n \\
\Big\| & & \Big\downarrow{\cong} \\
\mathcal{F} & \xrightarrow[\ \psi\mathcal{F}\ ]{} & \Delta^n\mathcal{F}
\end{array}
$$

which we then compose with $\mathcal{G}$ on both sides. Since the composition of adjunction units and counits $\mathcal{G} \to \mathcal{GFG} \to \mathcal{G}$ is the identity transformation, this proves (i). For the first equality in (ii) we will show that there is a commutative diagram of the form

$$
\begin{array}{ccccccccc}
\mathrm{Id}_{\mathcal{D}} & \longrightarrow & \mathcal{FG} & \xrightarrow{\mathcal{F}\varphi\tau\mathcal{G}} & \mathcal{F}\Sigma^{n+m}\mathcal{G} & \xrightarrow{\cong} & \Delta^{n+m}\mathcal{FG} & \longrightarrow & \Delta^{n+m} \\
\Big\| & & \Big\| & & \Big\downarrow{\cong} & & \Big\| & & \Big\| \\
\mathrm{Id}_{\mathcal{D}} & \longrightarrow & \mathcal{FG} & \xrightarrow{\psi\mathcal{F}\tau\mathcal{G}} & \Delta^n\mathcal{F}\Sigma^m\mathcal{G} & \xrightarrow{\cong} & \Delta^{n+m}\mathcal{FG} & \longrightarrow & \Delta^{n+m} \\
\psi\Big\downarrow & & \Big\downarrow{\psi\mathcal{FG}} & & \Big\| & & \Big\| & & \Big\| \\
\Delta^n & \longrightarrow & \Delta^n\mathcal{FG} & \xrightarrow{\Delta^n\mathcal{F}\tau\mathcal{G}} & \Delta^n\mathcal{F}\Sigma^m\mathcal{G} & \xrightarrow{\cong} & \Delta^{n+m}\mathcal{FG} & \longrightarrow & \Delta^{n+m}
\end{array}
$$

obtained as follows. The first row is $\mathrm{tr}_{\mathcal{F}}(\varphi\tau)$, where one notes that $\varphi\tau = \Sigma^m(\varphi) \circ \tau : \mathrm{Id}_{\mathcal{C}} \to \Sigma^{n+m}$. The second row is obtained from the first by making use of the hypothesis $\psi\mathcal{F} = \mathcal{F}\varphi$. The last row uses that ψ is a natural transformation. The last row is easily seen to be equal to $\Delta^n(\mathrm{tr}_{\mathcal{F}}(\tau))$ and hence upon composition with ψ (the left lower vertical arrow in the diagram) is equal to $\psi\mathrm{tr}_{\mathcal{F}}(\tau)$ as claimed. A similar argument proves the second equality in (ii). One can show this also using 4.3 and 4.8. Statement (iii) follows from (ii) applied to $\tau = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$. The last two statements follow from (ii) and (iii), respectively. $\square$

Corollary 5.4. *Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear adjoint functors compatible with Σ, Δ. We have $H^*_{\mathcal{F}}(\mathcal{C}) \subseteq H^*_{\mathcal{GF}}(\mathcal{C})$.*

Proof. Let $\varphi \in H^n(\mathcal{C})$ and $\psi \in H^n(\mathcal{D})$ such that $\mathcal{F}\varphi = \psi\mathcal{F}$, where n is an integer. By 5.3 we get $\mathcal{G}\psi = \varphi\mathcal{G}$, and hence $(\mathcal{GF})\varphi = \mathcal{G}\psi\mathcal{F} = \varphi(\mathcal{GF})$, which shows that φ is $\mathcal{GF}$-stable. $\square$

§6 NORMALISED TRANSFER

We extend some of the machinery in [10, §3] to the present context. The following is [10, 3.1.(ii)] in the case of Hochschild cohomology.

Definition 6.1 Let $\mathcal{C}$, $\mathcal{D}$ be $R-$linear categories endowed with $R-$linear equivalences $\Sigma : \mathcal{C} \to \mathcal{C}$ and $\Delta : \mathcal{D} \to \mathcal{D}$. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear functors which are adjoint to each other, compatible with Σ and Δ. Suppose that the relatively projective element $\pi_{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}})$ is invertible in $H^0(\mathcal{D})$. The *normalised transfer map* $\mathrm{Tr}_{\mathcal{F}} : H^*(\mathcal{C}) \longrightarrow H^*(\mathcal{D})$ is the graded R-linear map defined by $\mathrm{Tr}_{\mathcal{F}}(\varphi) = (\pi_{\mathcal{F}})^{-1}\mathrm{tr}_{\mathcal{F}}(\varphi)$ for all $\varphi \in H^*(\mathcal{C})$.

This makes sense as $H^*(\mathcal{D})$ is a module over $H^0(\mathcal{D})$. Unlike $\mathrm{tr}_{\mathcal{F}}$, the normalised transfer $\mathrm{Tr}_{\mathcal{F}}$ restricted to appropriate subalgebras of stable elements does no longer depend on the choices of a compatible adjunction (but still depends on the choice of the transformations a, b), provided that the relatively projective elements are invertible; this generalises [10, 3.6]:

Theorem 6.2. *Let $\mathcal{C}$, $\mathcal{D}$ be $R-$linear categories endowed with $R-$linear equivalences $\Sigma : \mathcal{C} \to \mathcal{C}$ and $\Delta : \mathcal{D} \to \mathcal{D}$. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear functors which are adjoint to each other, compatible with Σ and Δ with respect to isomorphisms $a : \mathcal{F}\Sigma \cong \Delta\mathcal{F}$ and $b : \mathcal{G}\Sigma \cong \Delta\mathcal{F}$. Suppose that the relatively projective element $\pi_{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}})$ is invertible in $H^0(\mathcal{D})$. The normalised transfer map $\mathrm{Tr}_{\mathcal{F}}$ induces a surjective homomorphism of graded R-algebras*

$$R_{\mathcal{F}} : H^*_{\mathcal{F}}(\mathcal{C}) \longrightarrow H^*_{\mathcal{G}}(\mathcal{D})$$

which is independent of the choice of a compatible adjunction with respect to a, b for which the relatively projective element $\pi_{\mathcal{F}}$ is invertible. In addition, if both $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{G}}$ are invertible in $H^0(\mathcal{D})$ and $H^0(\mathcal{C})$, respectively, then $R_{\mathcal{F}}$ and $R_{\mathcal{G}}$ are graded R-algebra isomorphisms

$$H^*_{\mathcal{F}}(\mathcal{C}) \cong H^*_{\mathcal{G}}(\mathcal{D})$$

which are inverse to each other.

Proof. Let $\varphi \in H^n(\mathcal{C})$ and $\psi \in H^n(\mathcal{D})$ such that $\mathcal{F}\varphi = \psi\mathcal{F}$, with the notation as in 5.1. By 5.3.(iii) we have

$$\mathrm{Tr}_{\mathcal{F}}(\varphi) = \psi \ .$$

This proves that $R_{\mathcal{F}}$ is a surjective algebra homomorphisms and that $R_{\mathcal{G}}$ is inverse to $R_{\mathcal{F}}$ if $\pi_{\mathcal{G}}$ is invertible as well. Since the equality $\mathcal{F}\varphi = \psi\mathcal{F}$ does not involve the adjunction isomorphism (but the transformation a) the independence statement in 6.2 follows as well. $\square$

The next Theorem generalises the cancellation property from [10, 3.8] to this context.

Theorem 6.3. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, Γ, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be adjoint functors compatible with Σ, Δ, and let $\mathcal{F}' : \mathcal{D} \to \mathcal{E}$, $\mathcal{G}' : \mathcal{E} \to \mathcal{D}$ be adjoint functors compatible with Δ, Γ. Suppose that the relatively projective element $\pi_{\mathcal{G}'}$ in $H^0(\mathcal{D})$ is invertible.*

Let n be a nonnegative integer, let $\zeta \in H^n(\mathcal{C})$ and let $\tau \in H^n(\mathcal{E})$ such that $\mathcal{F}'\mathcal{F}\zeta = \tau\mathcal{F}'\mathcal{F}$. Then

$$\mathcal{F}\zeta = \mathrm{Tr}_{\mathcal{G}'}(\tau)\mathcal{F}$$

In particular, we have $H^*_{\mathcal{F}'\mathcal{F}}(\mathcal{C}) \subseteq H^*_{\mathcal{F}}(\mathcal{C})$ and $\mathrm{Tr}_{\mathcal{G}'}(H^*_{\mathcal{G}\mathcal{G}'}(\mathcal{E})) \subseteq H^*_{\mathcal{G}}(\mathcal{D})$.

Proof. We will show that there is a commutative diagram of the following form:

The unmarked arrows are induced by the adjunction unit $\mathrm{Id}_{\mathcal{D}} \to \mathcal{G}'\mathcal{F}'$ and adjunction counit $\mathcal{G}'\mathcal{F}' \to \mathrm{Id}_{\mathcal{D}}$. The composition of these two is the relatively projective element $\pi_{\mathcal{G}'}$, which is invertible by the assumptions. This accounts for the commutativity of the right triangle in the diagram. The smaller square in the middle of the diagram is commutative, because this is the equality $\mathcal{F}'\mathcal{F}\zeta = \tau\mathcal{F}'\mathcal{F}$ composed with $\mathcal{G}'$ on the left. The lower part of the diagram is commutative, because this is the definition of $\mathrm{tr}_{\mathcal{G}'}(\tau)$ composed with $\mathcal{F}$ on the right. The pentagon in the upper right area of the diagram is commutative by the compatibility assumptions. Consider now the outer square of this diagram and invert the right vertical arrow (which is possible as $\pi_{\mathcal{G}'}$ is invertible). The resulting square expresses the equality $\mathcal{F}\zeta = \mathrm{Tr}_{\mathcal{G}'}(\tau)\mathcal{F}$ as stated. Thus in particular, $\zeta \in H_{\mathcal{F}}^n(\mathcal{C})$, which yields the inlcusion $H_{\mathcal{F}'\mathcal{F}}^*(\mathcal{C}) \subseteq H_{\mathcal{F}}^*(\mathcal{C})$, and by 5.3.(i) we also have $\mathrm{Tr}_{\mathcal{G}'}(\tau) \in H_{\mathcal{G}}^n(\mathcal{D})$, which proves the rest. $\square$

Corollary 6.4. *Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories endowed with R-linear self-equivalences Σ, Δ, respectively. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be adjoint functors compatible with Σ, Δ. Suppose that the relatively projective element $\pi_{\mathcal{F}}$ in $H^0(\mathcal{D})$ is invertible. Then $H_{\mathcal{F}}^*(\mathcal{C}) = H_{\mathcal{G}\mathcal{F}}^*(\mathcal{C})$.*

Proof. By 6.3 we have $H_{\mathcal{G}\mathcal{F}}^*(\mathcal{C}) \subseteq H_{\mathcal{F}}^*(\mathcal{C})$. The other inclusion follows from 5.4. $\square$

Remark 6.5. When applied to the degree zero component and the case where Σ, Δ are the identity functors on R-linear categories $\mathcal{C}$, $\mathcal{D}$, the notion of stability (with the notation of 5.1) reads as follows: an elements $\varphi \in Z(\mathcal{C})$ is called $\mathcal{F}$-stable, if there is $\psi \in Z(\mathcal{D})$ such that $\mathcal{F}\varphi = \psi\mathcal{F}$; that is, such that $\mathcal{F}(\varphi(U)) = \psi(\mathcal{F}(U))$ for any object U in $\mathcal{C}$. The set $Z_{\mathcal{F}}(\mathcal{C})$ of $\mathcal{F}$-stable elements in $Z(\mathcal{C})$ is a subalgebra of $Z(\mathcal{C})$, and Theorem 6.2 specialises to the following statements:

Corollary 6.6. *Let $\mathcal{C}$, $\mathcal{D}$ be R-linear categories, let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be R-linear functors which are adjoint to each other. Suppose there is an adjunction isomorphism such that that the relative projective elements $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{G}}$ are invertible in $Z(\mathcal{D})$ and $Z(\mathcal{C})$, respectively. Then, for any $\varphi \in Z(\mathcal{C})$ and any $\psi \in Z(\mathcal{D})$, we have $\mathcal{F}\varphi = \psi\mathcal{F}$ if and only if $\mathcal{G}\psi = \varphi\mathcal{G}$, and the correspondence mapping φ to ψ satisfying these equalities is an algebra isomorphism $Z_{\mathcal{F}}(\mathcal{C}) \cong Z_{\mathcal{G}}(\mathcal{D})$.*

§7 SYMMETRIC ALGEBRAS

7.1. Let k be a field. A k-algebra A is called *symmetric* if A is isomorphic to its k-dual $A^* = \mathrm{Hom}_k(A, k)$ as A-A-bimodule. In particular, A is self-injective; that is, A is both projective and injective as left and as right A-module. Any choice of a bimodule isomorphism $\Phi : A \cong A^*$ determines a *symmetrising form* $s : A \to k$ by $s = \Phi(1)$. Since A is generated as left and right A-module by 1, A^* is generated as left and right A-module by s; that is, for any $a \in A$ we have $\Phi(a) = a.s = s.a$, where $(a.s)(b) = s(ba)$ and $(s.a)(b) = s(ab)$ for any $b \in A$. Thus, in particular, $s(ab) = s(ba)$ for all $a, b \in A$, and Φ is determined by s. For any A-module U we have a natural isomorphism $\mathrm{Hom}_A(U, A) \cong \mathrm{Hom}(U, k) = U^*$ sending $\varphi \in \mathrm{Hom}_A(U, A)$ to $s \circ \varphi$. Indeed, this map is functorial in U and an isomorphism for $U = A$; as both sides are exact in U it follows that this is an isomorphism for all U. Given two symmetric k-algebras A, B and a B-A-bimodule M which is finitely generated projective as left B-module and as right A-module, the two functors $M \underset{A}{\otimes} - : \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(B)$ and $M^* \underset{B}{\otimes} - : \mathrm{Mod}(B) \longrightarrow \mathrm{Mod}(A)$ are left and right adjoint to each other (all this is well-known; see e.g. [10, Appendix] for a very brief review with some proofs). The algebra $A \underset{k}{\otimes} A^0$ is symmetric as well, and if we take $(\mathcal{C}, \Sigma) = (\overline{\mathrm{mod}}(A \underset{k}{\otimes} A^0), \Sigma_{A\otimes A^0})$, we get the Tate analogue of Hochschild cohomology $\mathrm{Ext}_{\mathcal{C}}^*(A, A) = \widehat{HH}^*(A)$. Applying the transfer formalism

to $D^b(A)$, $\overline{\text{mod}}(A)$, and related Ext-rings yields the well-known transfer maps in group cohomology, Tate cohomology and the standard reciprocity statements.

7.2. For $(\mathcal{C}, \Sigma) = (D^b(A \underset{k}{\otimes} A^0), [1])$ and $U = V = A$ the notation in 2.6 yields Hochschild cohomology; that is, $\text{Ext}^*_{\mathcal{C}}(A, A) \cong HH^*(A)$. Specialising the definition of transfer maps in 7.1 to this situation yields the transfer maps in Hochschild cohomology introduced in [10]. In order to describe this more precisely, let A, B two symmetric k-algebras. Let X be a bounded complex of B-A-bimodules which are finitely generated projective as left B-modules and as right A-modules. The k-dual $X^* = \text{Hom}_k(X, k)$ is then a bounded complex of A-B-bimodules which are finitely generated projective as left A-modules and as right B-modules; this uses the symmetry of A and B. The functors $\mathcal{F} = X \underset{A}{\otimes}$ $- : D^b(A) \longrightarrow D^b(B)$ and $\mathcal{G} = X^* \underset{B}{\otimes} - : D^b(B) \longrightarrow D^b(A)$ are left and right adjoint to each other; more precisely, any choice of bimodule isomorphisms $A \cong A^*$ and $B \cong B^*$ determines adjunction isomorphisms which commute with the shift functors. The transfer map $\text{tr}_X : HH^*(A) \to HH^*(B)$ in [10, 2.9] coincides with the transfer map $\text{tr}_{\mathcal{F}}(A, A) : \text{Ext}^*_{D^b(A \underset{k}{\otimes} A)}(A, A) \to \text{Ext}_{D^b(B \underset{k}{\otimes} B)}(B, B)$ with the notation from 7.1. It is an immediate consequence of the formal properties of the definition of transfer maps and stable elements that the canonical map $HH^*(A) \to H^*(D^b(A))$ restricts to an algebra homomorphism on stable elements $HH^*_{X^*}(A) \to H^*_{\mathcal{F}}(D^b(A))$ and we have a commutative diagram

7.3.

$$
\begin{array}{ccc}
HH^*(A) & \longrightarrow & H^*(D^b(A)) \\
{\scriptstyle \text{tr}_{X^*}} \downarrow & & \downarrow {\scriptstyle \text{tr}_{\mathcal{F}}} \\
HH^*(B) & \longrightarrow & H^*(D^b(B))
\end{array}
$$

The image of the relative projective element $\pi_{X^*} \in Z(A) = HH^0(A)$ in $H^0(D^b(A))$ is the relatively $\mathcal{G}$-projective element $\pi_{\mathcal{G}}$; in particular, if π_{X^*} is invertible, so is $\pi_{\mathcal{G}}$. Similarly, $\pi_{\mathcal{F}}$ is the image of π_X. Therfore, if both π_X and π_{X^*} are invertible, we get a commutative diagram of graded algebras

7.4.

$$
\begin{array}{ccc}
HH^*_{X^*}(A) & \longrightarrow & H^*_{\mathcal{F}}(D^b(A)) \\
{\scriptstyle R_X} \downarrow & & \downarrow {\scriptstyle R_{\mathcal{F}}} \\
HH^*_{X}(B) & \longrightarrow & H^*_{\mathcal{G}}(D^b(B))
\end{array}
$$

where the vertical arrows are isomorphisms by [10, 3.6] and 6.2, respectively.

7.5. With the notation in 7.2, let X' be a direct summand of the complex of B-A-bimodules X. The functors $\mathcal{F}' : X' \underset{A}{\otimes} - : D^b(A) \longrightarrow D^b(B)$ and $\mathcal{G}' : (X')^* \underset{B}{\otimes} - : D^b(B) \longrightarrow D^b(A)$ are again left and right adjoint to each other.

Lemma 7.6. *With the notation above, let n be an integer, let $\zeta \in H^n(D^b(A))$ be $\mathcal{F}$-stable and let $\tau \in H^n(D^b(B))$ such that $\mathcal{F}\zeta = \tau\mathcal{F}$. Then $\mathcal{F}'\zeta = \tau\mathcal{F}'$. In particular, we have $H^*_{\mathcal{F}}(D^b(A)) \subseteq H^*_{\mathcal{F}'}(D^b(A))$.*

Proof. The assumption $\mathcal{F}\zeta = \tau\mathcal{F}$ means that for any bounded complex of A-modules U the diagram

$$
\begin{array}{ccc}
X \underset{A}{\otimes} U & \xrightarrow{\mathrm{Id}_X \otimes \zeta(U)} & X \underset{A}{\otimes} U[n] \\
\| & & \| \\
X \underset{A}{\otimes} U & \xrightarrow[\tau(X \underset{A}{\otimes} U)]{} & X \underset{A}{\otimes} U[n]
\end{array}
$$

is commutative. Choose chain maps of B-A-modules $\iota : X' \to X$ and $\pi : X \to X'$ such that $\pi \circ \iota = \mathrm{Id}_{X'}$. Using that τ is a natural transformation we get a commutative diagram

$$
\begin{array}{ccc}
X' \underset{A}{\otimes} U & \xrightarrow{\mathrm{Id}_{X'} \otimes \zeta(U)} & X' \underset{A}{\otimes} U[n] \\
{\scriptstyle \iota \otimes \mathrm{Id}_U} \downarrow & & \downarrow {\scriptstyle \iota \otimes \mathrm{Id}_U[n]} \\
X \underset{A}{\otimes} U & \xrightarrow{\mathrm{Id}_X \otimes \zeta(U)} & X \underset{A}{\otimes} U[n] \\
\| & & \| \\
X \underset{A}{\otimes} U & \xrightarrow[\tau(X \underset{A}{\otimes} U)]{} & X \underset{A}{\otimes} U[n] \\
{\scriptstyle \pi \otimes \mathrm{Id}_U} \downarrow & & \downarrow {\scriptstyle \pi \otimes \mathrm{Id}_U[n]} \\
X' \underset{A}{\otimes} U & \xrightarrow[\tau(X' \underset{A}{\otimes} U)]{} & X' \underset{A}{\otimes} U[n]
\end{array}
$$

This shows that $\mathcal{F}'\zeta = \tau\mathcal{F}'$ and hence that ζ is also $\mathcal{F}'$-stable. $\square$

§8 Proof of Theorem 1.1

Let k be an algebraically closed field of prime characteristic p, let G be a finite group and let B be a source algebra of a block b of kG with P as defect group; that is, $B = ikGi$ for some primitive idempotent i in $(kGb)^P$ satisfying $\mathrm{Br}_P(i) \neq 0$, where $\mathrm{Br}_P : (kG)^P \to kC_G(P)$ is the Brauer homomorphism. The canonical symmetrising form on kG (sending 1_G to 1_k and $x \in G - \{1_G\}$ to 0) restricts to a symmetrising form $s : B \to k$. Set $X = B$, viewed as B-kP-bimodule. Then $X^* \cong B$, as kP-B-bimodule because B is symmetric. In other words, if we consider as in 7.2 the functors $\mathcal{F} = X \underset{kP}{\otimes} - : D^b(kP) \to D^b(B)$ and $\mathcal{G} = X^* \underset{B}{\otimes} - : D^b(B) \to D^b(kP)$ then $\mathcal{G}$ is the restriction functor. By [10, 5.6.(iii)], the canonical map from block cohomology $H^*(B)$ to Hochschild cohomology $HH^*(B)$ sends $H^*(B)$ to $HH_X^*(B)$. By 7.2, the canonical map from $HH^*(B)$ to the graded center $H^*(D^b(B))$ sends $HH_X^*(B)$ to $H_{\mathcal{G}}^*(D^b(B))$.

Lemma 8.1. *With the notation above, the relative projective elements $\pi_X \in Z(B)$, $\pi_{X^*} \in Z(kP)$, $\pi_{\mathcal{F}} \in H^0(D^b(B))$, $\pi_{\mathcal{G}} \in H^0(D^b(kP))$ are all invertible.*

Proof. For π_X and π_{X^*} this is proved in [10, 5.6.(i)]. Since $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{G}}$ are their images in $H^0(D^b(B))$ and $H^0(D^b(kP))$, respectively (cf. 7.2), the result follows. $\square$

Denote by $\gamma : H^*(B) \to HH_{X^*}^*(B)$ the canonical algebra homomorphism from [10, 5.6.(iii)]; this is obtained by composing the inclusion $H^*(B) \subseteq H^*(P, k)$ with the canonical "diganonal induc-tion" map $\delta_P : H^*(P, k) \to HH^*(kP)$ followed by the normalised transfer map $T_X : HH^*(kP) \to$

$HH^*(B)$. Denote further by $\epsilon : H^*(D^b(kP)) \to H^*(P,k)$ the canonical graded algebra homomorphism obtained from evaluating natural transformations at the trivial kP-module k, as in 2.8. By 1.2 the kernel of ϵ is a nilpotent ideal in $H^*(D^b(kP))$. Define $\eta : H^*_{\mathcal{F}}(D^b(B)) \to H^*(P,k)$ as the unique algebra homomorphism obtained from restricting ϵ to $H^*_{\mathcal{F}}(D^b(kP))$ and precomposing it with the normalised transfer $T_{\mathcal{G}} : H^*_{\mathcal{G}}(D^b(B)) \cong H^*_{\mathcal{F}}(D^b(kP))$ (this is an isomorphism by 8.1 and 6.2.). Using the last commutative diagram in 7.2 we get altogether a commutative diagram of graded algebra homomorphisms

8.2.

$$
\begin{array}{ccccccc}
H^*(B) & \xrightarrow{\ \gamma\ } & HH^*_X(B) & \longrightarrow & H^*_{\mathcal{G}}(D^b(B)) & \xrightarrow{\ \eta\ } & H^*(P,k) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} & & \Big\| \\
& & HH^*_{X^*}(kP) & \longrightarrow & H^*_{\mathcal{F}}(D^b(kP)) & & \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\| \\
H^*(P,k) & \longrightarrow & HH^*(kP) & \longrightarrow & H^*(D^b(kP)) & \xrightarrow[\ \epsilon\ ]{} & H^*(P,k)
\end{array}
$$

The unmarked vertical arrows in 8.2 are inclusions. Composing the three maps in the last row of this diagram yields the identity on $H^*(P,k)$. In order to show a similar statement for the first row of this diagram, we need to show the following statement:

Lemma 8.3. *With the notation above, the homomorphism η maps $H^*_{\mathcal{G}}(D^b(B))$ onto $H^*(B)$.*

Proof. The relative projective elements determined by $\mathcal{F}$, $\mathcal{G}$ are invertible by 8.1. Thus, by 6.2, we have $H^*_{\mathcal{G}}(D^b(B)) \cong H^*_{\mathcal{F}}(D^b(kP))$. By 6.4, we have $H^*_{\mathcal{F}}(D^b(kP)) = H^*_{\mathcal{GF}}(D^b(kP))$. Thus it suffices to show that the canonical map ϵ in the above diagram sends $H^*_{\mathcal{GF}}(D^b(kP))$ to $H^*(B)$. Let n be an integer and let $\zeta \in H^n(D^b(kP))$ such that there exists $\tau \in H^n(D^b(kP))$ satisfying $\mathcal{GF}\zeta = \tau\mathcal{GF}$. Since the functor $\mathcal{GF}$ is isomorphic to the functor $B \underset{kP}{\otimes} -$ on $D^b(kP)$ this is equivalent to the commutativity of the diagrams

$$
\begin{array}{ccc}
B \underset{kP}{\otimes} U & \xrightarrow{\ \mathrm{Id}\otimes\zeta(U)\ } & B \underset{kP}{\otimes} U[n] \\
\Big\| & & \Big\| \\
B \underset{kP}{\otimes} U & \xrightarrow[\ \tau(B\underset{kP}{\otimes}U)\ ]{} & B \underset{kP}{\otimes} U[n]
\end{array}
$$

for any bounded complex of kP-modules U, where B is regarded as kP-kP-bimodule. Any indecomposable direct summand of B is isomorphic to $kP \underset{kQ}{\otimes} {}_{\varphi}kP$ for some subgroup Q of P and some morphism $\varphi : Q \to P$ belonging to the fusion system of the source algebra B of B. By 7.6 the diagram

$$
\begin{array}{ccc}
kP \underset{kQ}{\otimes} {}_{\varphi}kP \underset{kP}{\otimes} U & \xrightarrow{\ \mathrm{Id}\otimes\zeta(U)\ } & kP \underset{kQ}{\otimes} {}_{\varphi}kP \underset{kP}{\otimes} U[n] \\
\Big\| & & \Big\| \\
kP \underset{kQ}{\otimes} {}_{\varphi}kP \underset{kP}{\otimes} U & \xrightarrow[\ \tau(kP\underset{kQ}{\otimes}{}_{\varphi}kP\underset{kP}{\otimes}U)\ ]{} & kP \underset{kQ}{\otimes} {}_{\varphi}kP \underset{kP}{\otimes} U[n]
\end{array}
$$

is then still commutative. By the standard adjunction we get a commutative diagram in $D^b(kQ)$ of the form

$$
\begin{array}{ccc}
_\varphi U & \xrightarrow{\;\zeta(U)\;} & _\varphi U[n] \\
\| & & \| \\
_\varphi U & \xrightarrow{\;\sigma(_\varphi U)\;} & _\varphi U[n]
\end{array}
$$

where we abusively denote again by $\zeta(U)$ its restriction via φ and where $\sigma(_\varphi U)$ is the map induced by τ. Now apply this to $U = k$, the trivial kP-module. Then $\mathrm{Res}^P_Q(U) = {}_\varphi U = k$, the trivial kQ-module. So in that case $(kP \underset{kQ}{\otimes} {}_\varphi kP) \underset{kP}{\otimes} k = (kP \underset{kQ}{\otimes} kP) \underset{kP}{\otimes} k = kP \underset{kQ}{\otimes} k$ hence $\tau((kP \underset{kQ}{\otimes} {}_\varphi kP) \underset{kP}{\otimes} k) = \tau((kP \underset{kQ}{\otimes} kP) \underset{kP}{\otimes} k) = \tau(kP \underset{kQ}{\otimes} k)$ and so $\sigma(_\varphi k)$ does not depend on the choice of the homomorphism φ from Q to P in the fusion system of iBi. But that means precisely that $\mathrm{Res}^P_Q(\zeta(k)) = \mathrm{Res}_\varphi(\zeta(k))$ for all morphisms φ in the fusion system of iBi, and thus $\zeta \in H^n(B)$. This shows that η maps $H^*_\mathcal{G}(D^b(B))$ to $H^*(B)$. This map is onto because the last row in the diagram 8.2 is the identity. $\square$

The proof of Theorem 1.1 concludes now as follows. Set $\mathcal{N} = \ker(\eta)$. Then $H^*(B) \cong H^*_\mathcal{G}(D^b(B))/\mathcal{N}$ by 8.3 and $\mathcal{N}$ is nilpotent as a consequence of the commutative diagram 8.2 in conjunction with the fact that $\ker(\epsilon)$ is nilpotent by Proposition 1.3.

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