# On two theorems of Flavell

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**Abstract**

We extend two theorems due to P. Flavell [6] to arbitrary fusion systems.

The fusion system of a finite group $G$ at a prime $p$ encodes the structure of a Sylow-$p$-subgroup $S$ together with extra information on $G$-conjugacy within $S$ in category theoretic terms. From work of Alperin and Broué [1] it emerged that fusion systems of finite groups are particular cases of fusion systems of $p$-blocks of finite groups. In the early 1990's, Puig introduced the notion of an abstract fusion system on a finite $p$-group $S$ as a category whose objects are the subgroups of $S$ and whose morphism sets satisfy a list of properties modelled around what one observes in the case of fusion systems of finite groups and blocks. There are 'exotic' fusion systems which do not arise as fusion system of a block. Benson [2] suggested that nonetheless any fusion system should give rise to a $p$-complete topological space which should coincide with the $p$-completion of the classifying space $BG$ in case the fusion system does arise as fusion system of a finite group $G$. Broto, Levi and Oliver laid in [4] the homotopy theoretic foundations of such spaces - called $p$-local finite groups - and gave in particular a cohomological criterion for the existence and uniqueness of a $p$-local finite group associated with a given fusion system. While the existence and uniqueness of $p$-local finite groups for arbitrary fusion systems is still an open problem, there has been in recent years a steadily growing body of work by many authors trying to add to the understanding of fusion systems by extending classical concepts and results on the $p$-local structure of finite groups (some of which were relevant for the classification of finite simple groups) to all fusion systems. This is also the underlying philosophy of the present note. The following two theorems are Flavell's Theorem A and Theorem B in [6], extended to arbitrary fusion systems. Our general terminology on fusion systems follows [7]; in particular, by a fusion system we always mean a saturated fusion system.

**Theorem 1.** *Let $p$ be an odd prime, $S$ a finite $p$-group, $\mathcal{F}$ a fusion system on $S$ and $\alpha$ an automorphism of $S$ acting freely on $S-\{1\}$, which stabilises $\mathcal{F}$ and whose order is a prime number $r$ which does not divide the orders of the automorphism groups $\mathrm{Aut}_{\mathcal{F}}(R)$ for all $\mathcal{F}$-centric radical subgroups $R$ of $S$. Then $\mathcal{F} = N_{\mathcal{F}}(S)$.*

The hypothesis that $\alpha$ stablises $\mathcal{F}$ means that for any two subgroups $Q$, $R$ of $S$ and any morphism $\varphi : Q \to R$ in $\mathcal{F}$ the morphism $\alpha \circ \varphi \circ \alpha^{-1}|_{\alpha(Q)} : \alpha(Q) \to \alpha(R)$ is again a morphism in $\mathcal{F}$. Moreover, a subgroup $Q$ of $S$ is called *$\mathcal{F}$-centric* if $C_S(Q') = Z(Q')$ for all subgroups $Q'$ of $S$ such that $Q' \cong Q$ in $\mathcal{F}$; a subgroup $Q$ of $S$ is called *$\mathcal{F}$-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_Q(Q)$, the group of inner automorphisms of $Q$. By Alperin's fusion theorem, $\mathcal{F}$ is completely determined by the automorphism groups in $\mathcal{F}$ of $\mathcal{F}$-centric radical subgroups of $S$.

**Theorem 2.** *Let $p$ be an odd prime, $S$ a finite $p$-group and $\mathcal{F}$ a fusion system on $S$. Let $T$ be a subgroup of $Z(S)$ such that $N_{\mathcal{F}}(J(S)) \le N_{\mathcal{F}}(T)$. Then one of the following hold:*

*(a) $T$ is weakly $\mathcal{F}$-closed.*

*(b) There exists a non-trivial cyclic $p'$-subgroup $X$ of $\mathrm{Aut}_{\mathcal{F}}(T)$ such that $X$ acts transitively on $[T,X] - \{1\}$.*

Here $J(S)$ is the *Thompson subgroup of $S$*; that is, $J(S)$ is generated by all abelian subgroups of $S$ of maximal order, $[T,X]$ is the subgroup of $T$ generated by the set of elements of the form $\psi(t)t^{-1}$, where $\psi \in X$ and $t \in T$, and $T$ is *weakly $\mathcal{F}$-closed* if $\varphi(T) = T$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(T,S)$. A subgroup $Q$ of $S$ is called *fully $\mathcal{F}$-normalised* if $|N_S(Q)| \ge |N_S(Q')|$ for any subgroup $Q'$ of $S$ such that $Q' \cong Q$ in $\mathcal{F}$. In that case, by a result of Puig, there is a fusion system $N_{\mathcal{F}}(Q)$ on $N_S(Q)$, which is called the *normaliser of $Q$ in $\mathcal{F}$*. More precisely, for any two subgroups $U$, $V$ of $N_S(Q)$, the morphism set $\mathrm{Hom}_{N_{\mathcal{F}}(Q)}(U,V)$ consists of all group homomorphisms $\varphi : U \to V$ for which there exists a morphism $\psi : QU \to QV$ in $\mathcal{F}$ satisfying $\psi|_U = \varphi$ and $\psi(Q) = Q$. Moreover, $N_{\mathcal{F}}(Q)$ induces a fusion system $N_{\mathcal{F}}(Q)/Q$ on $N_P(Q)/Q$. Given two normal subgroups $Q$, $Q'$ of $S$ such that $\mathcal{F} = N_{\mathcal{F}}(Q) = N_{\mathcal{F}}(Q')$ one checks that then $\mathcal{F} = N_{\mathcal{F}}(QQ')$; we denote by $O_p(\mathcal{F})$ the largest normal subgroup $R$ of $S$ such that $\mathcal{F} = N_{\mathcal{F}}(R)$. Similarly, if $Q$ is a fully $\mathcal{F}$-centralised subgroup of $S$, there is a fusion system $C_{\mathcal{F}}(Q)$ on $C_S(Q)$ such that for any two subgroups $U$, $V$ of $C_S(Q)$, the morphism set $\mathrm{Hom}_{C_{\mathcal{F}}(Q)}(U,V)$ consists of all group homomorphisms $\varphi : U \to V$ for which there exists a morphism $\psi : QU \to QV$ in $\mathcal{F}$ satisfying $\psi|_U = \varphi$ and $\psi(Q) = \mathrm{Id}_Q$. If $Z$ is a subgroup of $Z(S)$ such that $\mathcal{F} = C_{\mathcal{F}}(Z)$ then $\mathcal{F}$ induces a fusion system, denoted by $\mathcal{F}/Z$, on $S/Z$.

The strategy to prove Theorem 2 is as in [7]: we show that a minimal counterexample is $p$-constrained, hence a fusion system of a finite group and thus [6, Theorem B] applies. Theorem 1 follows from Theorem 2 exactly along the lines of the proof of [6, Theorem A]. We need the following well-known statements:

**Lemma 3.** *Let $\mathcal{F}$ be a fusion system on $S$. Suppose that $T$ is a weakly $\mathcal{F}$-closed subgroup of $Z(S)$. Then $\mathcal{F} = N_{\mathcal{F}}(T)$.*

*Proof.* Let $R$ be an $\mathcal{F}$-centric subgroup of $S$ and let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(R)$. Since $R$ contains $Z(S)$, hence $T$, the assumptions on $T$ imply that $\varphi(T) = T$. Alperin's fusion theorem implies the lemma. □

**Lemma 4.** *Let $\mathcal{F}$ be a fusion system on $S$. For any subgroup $Q$ of $S$ there is a morphism $\tau : N_S(Q) \to S$ such that $\tau(Q)$ is fully $\mathcal{F}$-normalised.*

*Proof.* See for instance [7, Lemma 2.2]. □

**Lemma 5.** *Let $\mathcal{F}$ be a fusion system on $S$, let $Z$ be a subgroup of $Z(S)$ such that $\mathcal{F} = C_{\mathcal{F}}(Z)$. Set $\bar{S} = S/Z$ and $\bar{\mathcal{F}} = \mathcal{F}/Z$. We have $\mathcal{F} = N_{\mathcal{F}}(S)$ if and only if $\bar{\mathcal{F}} = N_{\bar{\mathcal{F}}}(\bar{S})$.*

*Proof.* This is a special case of more general results; see e.g. [7, Corollary 3.3]. □

**Lemma 6.** *Let $\mathcal{F}$ be a fusion system on $S$, let $Q$ be a subgroup of $S$ and $r$ a prime divisor of $|\mathrm{Aut}_{\mathcal{F}}(Q)|$. Then there is an $\mathcal{F}$-centric radical subgroup $R$ of $S$ such that $r$ divides $|\mathrm{Aut}_{\mathcal{F}}(R)|$.*

*Proof.* We may assume that $Q$ is fully $\mathcal{F}$-normalised. Then, by the Sylow axiom, the group $\mathrm{Aut}_S(Q)$ of automorphisms of $Q$ induced by conjugation with elements in $N_S(Q)$ is a Sylow-$p$-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$; hence $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_R(Q)$ for a unique subgroup $R$ of $N_S(Q)$ containing $QC_S(Q)$. The extension axiom implies that any automorphism in $\mathrm{Aut}_{\mathcal{F}}(Q)$ extends to an automorphism in $\mathrm{Aut}_{\mathcal{F}}(R)$. Thus $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a quotient of a subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$; in particular, $r$ divides $|\mathrm{Aut}_{\mathcal{F}}(R)|$. Since $Q < R$ unless $R$ is $\mathcal{F}$-centric radical, repeating this argument shows that $r$ divides $|\mathrm{Aut}_{\mathcal{F}}(R)|$ for some $\mathcal{F}$-centric radical subgroup $R$ of $S$. □

*Proof of Theorem 2.* We use the notation and hypotheses of 2. We argue by induction over the number of morphisms in $\mathcal{F}$. Let $\mathcal{F}$ be a minimal counterexample to 2. That is, $T$ is not weakly closed, and no non trivial cyclic $p'$-group $X$ of $\mathrm{Aut}_{\mathcal{F}}(T)$ acts transitively on the non identity elements in $[T, X]$. Note that the latter of these two conditions passes down to any subsystem of $\mathcal{F}$ on any subgroup of $S$ containing $T$.

The purpose of the first part of the proof is to show that $T \leq O_p(\mathcal{F})$; in particular, $O_p(\mathcal{F}) \neq 1$. By Alperin's fusion theorem, there exists a fully $\mathcal{F}$-normalised subgroup $Q$ of $S$ such that $T \leq Q$ and such that $T$ is not weakly $N_{\mathcal{F}}(Q)$-closed. We choose $Q$ with these properties such that $N_S(Q)$ has maximal order (but we do not require $Q$ to be centric). Note though that then $N_S(Q)$ is centric, hence $T \leq Z(S) \leq Z(N_S(Q))$; in particular, $T \leq J(N_S(Q))$.

Consider first the case where $N_S(Q) = S$. Then $J(N_S(Q)) = J(S)$. Since $N_{\mathcal{F}}(J(S)) \leq N_{\mathcal{F}}(T)$ by the hypotheses we get that $N_{N_{\mathcal{F}}(Q)}(J(S)) \leq N_{N_{\mathcal{F}}(Q)}(T)$. This shows that $N_{\mathcal{F}}(Q)$ is also a counterexample, hence $\mathcal{F} = N_{\mathcal{F}}(Q)$. This implies $T \leq Q \leq O_p(\mathcal{F}) \neq 1$ in the case $S = N_S(Q)$.

Consider next the case where $N_S(Q) < S$. By Lemma 4 applied to $J(N_S(Q))$ there is a morphism

$$\tau : N_S(J(N_S(Q))) \to S$$

such that $\tau(J(N_S(Q)))$ is fully $\mathcal{F}$-normalised. Note that

$$Q < N_S(Q) < N_S(N_S(Q)) \leq N_S(J(N_S(Q)))$$

Thus $\tau$ maps $N_S(Q)$ to $N_S(\tau(Q))$. Since $Q$ is fully $\mathcal{F}$-normalised, it follows that $\tau(N_S(Q)) = N_S(\tau(Q))$. Thus both $\tau(Q)$ and $J(N_S(\tau(Q))) = \tau(J(N_S(Q)))$ are fully $\mathcal{F}$-normalised. Next, observe that $\tau(T) = T$. Indeed, by Alperin's fusion theorem, $\tau$ is a composition of restrictions of automorphisms of fully $\mathcal{F}$-normalised subgroups of $S$ whose order is at least that of $N_S(J(N_S(Q)))$, which is bigger than the order of $N_S(Q)$. By the maximality assumptions on $Q$ we therefore have $\tau(T) = T$. Since $T$ is not weakly $N_{\mathcal{F}}(Q)$-closed, there is $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\varphi(T) \neq T$. Then, setting $\psi = \tau \circ \varphi \circ \tau^{-1}$ and using $\tau(T) = T$ we get that $\psi(T) \neq T$, and hence $T$ is not weakly $N_{\mathcal{F}}(\tau(Q))$-closed. Therefore, after possibly replacing $Q$ by $\tau(Q)$, we may assume that both $Q$ and $J(N_S(Q))$ are fully $\mathcal{F}$-normalised. The point of making this assumption is that then $N_{\mathcal{F}}(J(N_S(Q)))$ is a fusion system on $N_S(J(N_S(Q)))$. We claim that $N_{\mathcal{F}}(J(N_S(Q))) \leq N_{\mathcal{F}}(T)$. Since $Q$ was chosen with $N_S(Q)$ of maximal order subject to $T$ not being weakly $N_{\mathcal{F}}(Q)$-closed, $T$ is weakly $N_{\mathcal{F}}(J(N_S(Q)))$-closed. As $T \leq J(N_S(Q))$ this is equivalent to $N_{\mathcal{F}}(J(N_S(Q))) \leq N_{\mathcal{F}}(T)$. But then also $N_{N_{\mathcal{F}}(Q)}(J(N_S(Q))) \leq N_{N_{\mathcal{F}}(Q)}(T)$, hence $N_{\mathcal{F}}(Q)$ is a counterexample to Theorem 2 as well, a contradiction to the minimality of $\mathcal{F}$. This shows that $T \leq O_p(\mathcal{F}) \neq 1$.

The second part of the proof consists of showing that $O_p(\mathcal{F})$ is in fact $\mathcal{F}$-centric. Set $Q = O_p(\mathcal{F})$. Since $T$ is not weakly $\mathcal{F}$-closed there is $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\varphi(T) \neq T$. The morphism $\varphi$

extends to a morphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(QC_P(Q))$. Thus $T$ is not weakly $N_{\mathcal{F}}(QC_P(Q))$-closed, and hence $N_{\mathcal{F}}(QC_P(Q))$ is also a counterexample to Theorem 2. By the minimality of $\mathcal{F}$ it follows that $\mathcal{F} = N_{\mathcal{F}}(QC_P(Q))$, or equivalently, $QC_P(Q) \leq O_p(\mathcal{F}) = Q$, which implies that $Q$ is $\mathcal{F}$-centric.

It follows now from [3, Proposition C] that $\mathcal{F} = \mathcal{F}_S(L)$ for some finite group $L$. But then Flavell's Theorem B in [6] applies, showing that $\mathcal{F}$ cannot be a counterexample. This contradiction concludes the proof of Theorem 2. $\square$

*Proof of Theorem 1.* We show that Theorem 2 implies Theorem 1, closely following the proof of [6, Theorem A]. As before we argue by induction over the number of morphisms in $\mathcal{F}$. Let $\mathcal{F}$ be a minimal counterexample to 1; that is, with the notation and hypotheses of 1 we have $N_{\mathcal{F}}(S) < \mathcal{F}$.

We first show that $O_p(\mathcal{F}) = 1$. Suppose not; that is, $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some non trivial normal subgroup $Q$ of $S$. Since $\alpha$ stablises $\mathcal{F}$ it stablises $O_p(\mathcal{F})$. Choose a minimal $\alpha$-stable normal subgroup $V$ of $S$ such that $\mathcal{F} = N_{\mathcal{F}}(V)$. Since $Z(V)$ is then also $\alpha$-stable satisfying $\mathcal{F} = N_{\mathcal{F}}(Z(V))$, the minimality of $V$ implies that $V$ is abelian. The subgroup $\Omega_1(V)$ generated by all elements of order $p$ in $V$ is then again $\alpha$-stable and satisfies $\mathcal{F} = N_{\mathcal{F}}(\Omega_1(V))$. The minimality of $V$ implies that $V$ is elementary abelian and that $V$ is a simple module for the group $\mathrm{Aut}_{\mathcal{F}}(V) \rtimes \langle\alpha\rangle$ over $\mathbb{F}_p$. By Lemma 6, the prime $r$ does not divide the order of $\mathrm{Aut}_{\mathcal{F}}(V)$. Using that $\mathrm{Aut}_{\mathcal{F}}(V)$ and $\langle\alpha\rangle$ have coprime orders one checks easily that $V$ is in fact a faithful simple $\mathrm{Aut}_{\mathcal{F}}(V) \rtimes \langle\alpha\rangle$-module over $\mathbb{F}_p$. Since $\alpha$ acts freely on $V - \{1\}$, the hypotheses of [5, Theorem A] are satisfied, and hence $[\mathrm{Aut}_{\mathcal{F}}(V), \alpha]$ is a 2-group. Since $\alpha$ acts freely on $S - \{1\}$ we have $S = [S, \alpha]$. Thus the image of $S = [S, \alpha]$ in $\mathrm{Aut}_{\mathcal{F}}(V)$ is contained in the 2-group $[\mathrm{Aut}_{\mathcal{F}}(V), \alpha]$, and hence must be trivial as $p$ is odd. Therefore $S$ centralises $V$, or equivalently, $V \leq Z(S)$. It follows that $C_{\mathcal{F}}(V)$ is an $\alpha$-stable fusion system on $S$ satisfying the hypotheses of Theorem 1. If $C_{\mathcal{F}}(V) = \mathcal{F}$ then the fusion system $\bar{\mathcal{F}} = \mathcal{F}/V$ on $\bar{S} = S/V$ inherits the hypotheses in 1, and hence $\bar{\mathcal{F}} = N_{\bar{\mathcal{F}}}(\bar{S})$ by induction. Lemma 5 implies that $\mathcal{F} = N_{\mathcal{F}}(S)$, a contradiction. Thus $C_{\mathcal{F}}(V) < \mathcal{F}$, and hence $C_{\mathcal{F}}(V) = N_{C_{\mathcal{F}}(V)}(S)$ by induction. Since $\mathcal{F} = N_{\mathcal{F}}(V)$ and fusion in $V \leq Z(S)$ is controlled by $\mathrm{Aut}_{\mathcal{F}}(S)$ it follows again that $\mathcal{F} = N_{\mathcal{F}}(S)$, so the assumption $O_p(\mathcal{F}) \neq 1$ leads to a contradiction.

Set $T = \Omega_1(Z(S))$. Assume first that $T$ is weakly $\mathcal{F}$-closed. Then, by Lemma 3, we have $\mathcal{F} = N_{\mathcal{F}}(T)$, contradicting $O_p(\mathcal{F}) = 1$. Thus $T$ is not weakly $\mathcal{F}$-closed. Clearly $N_{\mathcal{F}}(S) \leq N_{\mathcal{F}}(J(S)) < \mathcal{F}$, and hence $N_{\mathcal{F}}(S) = N_{\mathcal{F}}(J(S))$ by induction. As $T = \Omega_1(Z(S)) \leq J(S)$ it follows that $N_{\mathcal{F}}(J(S)) \leq N_{\mathcal{F}}(T)$. Since $T$ is not weakly $\mathcal{F}$-closed, Theorem 2 implies that there is a non trivial cyclic $p'$-subgroup $X$ of $\mathrm{Aut}_{\mathcal{F}}(T)$ which acts transitively on the non identity elements in $[T, X]$. Now $T$ can be viewed as module of $\mathrm{Aut}_{\mathcal{F}}(T) \rtimes \langle\alpha\rangle$ over $\mathbb{F}_p$. Then [6, Theorem 3.2] implies that $\alpha$ has a non trivial fixpoint in $T$, a contradiction. This completes the proof of Theorem 1. $\square$

# References


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