Reverse sensitivity testing: What does it take to break the model?

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Abstract

Sensitivity analysis is an important component of model building, interpretation and validation. A model comprises a vector of random input factors, an aggregation function mapping input factors to a random output, and a (baseline) probability measure. A risk measure, such as Value-at-Risk and Expected Shortfall, maps the distribution of the output to the real line. As is common in risk management, the value of the risk measure applied to the output is a decision variable. Therefore, it is of interest to associate a critical increase in the risk measure to specific input factors. We propose a global and model-independent framework, termed 'reverse sensitivity testing', comprising three steps: (a) an output stress is specified, corresponding to an increase in the risk measure(s); (b) a (stressed) probability measure is derived, minimising the Kullback-Leibler divergence with respect to the baseline probability, under constraints generated by the output stress; (c) changes in the distributions of input factors are evaluated. We argue that a substantial change in the distribution of an input factor corresponds to high sensitivity to that input and introduce a novel sensitivity measure to formalise this insight. Implementation of reverse sensitivity testing in a Monte-Carlo setting can be performed on a single set of input/output scenarios, simulated under the baseline model. Thus the approach circumvents the need for additional computationally expensive evaluations of the aggregation function. We illustrate the proposed approach through a numerical example of a simple insurance portfolio and a model of a London Insurance Market portfolio used in industry.

Keywords Robustness and sensitivity analysis, risk management, Value-at-Risk, Expected Shortfall, stress testing.

1 Introduction

1.1 Problem framing and contribution

Risk managers often use complex quantitative models as decision support tools. Sensitivity analysis is concerned with characterising and providing insight regarding the relation between inputs and outputs. Sensitivity analysis can have different aims, including

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identifying the most or least influential inputs (*factor prioritisation* or *factor fixing* respectively), detecting the direction of input/output relationships, and inferring model structure; see Saltelli et al. (2008); Borgonovo and Plischke (2016) for comprehensive reviews.

For the specific aim of factor prioritisation, a *sensitivity measure* is typically used, assigning a sensitivity score to each input. When model inputs are subject to uncertainty, *global* sensitivity measures are used, considering the whole possible space of multivariate input scenarios. Such methods typically involve a comparison of the unconditional and conditional output distributions, when individual inputs are fixed, see Borgonovo et al. (2016) for a unifying framework. Prominent methods use a (Hoeffding) decomposition of the output variance (Saltelli et al., 2000; Saltelli, 2002; Saltelli et al., 2008), as well as moment independent approaches (Borgonovo, 2007; Borgonovo et al., 2011). Alternative methods consider partial derivatives of statistical functionals of the output distribution in the direction of parameters of interest, see Glasserman and Liu (2010) for expectation-type and Hong (2009); Tsanakas and Millossovich (2016) for percentile-based functionals.

In this paper we develop a sensitivity analysis framework appropriate for contexts where the following considerations, typical in several fields, including probabilistic safety assessment, reliability analysis and financial/insurance risk management (Saltelli and Tarantola, 2002; Aven and Nøkland, 2010; Gourieroux et al., 2000; Tsanakas and Millossovich, 2016), hold:

- Model inputs are uncertain, hence sensitivity and uncertainty analyses are interlinked and global sensitivity analysis methods are called for.

- A decision criterion is derived by applying a *risk measure* on the distribution of the output. Risk measures are functionals mapping random variables to the real line (Artzner et al., 1999; Szegö, 2005). Risk measures are used in a variety of operations research and risk analysis applications, with Value-at-Risk (VaR) and Expected Shortfall (ES – also known as CVaR) particularly popular choices; indicatively see Rockafellar and Uryasev (2002); Tapiero (2005); Gotoh and Takano (2007); Ahmed et al. (2007); Asimit et al. (2017).

- The value of the risk measure, applied on the output distribution, gives an indication of criticality for the system whose uncertainty is analysed. For example, in the context of financial risk management, high values of output risk measures may indicate that a portfolio is not admissible, e.g. due to regulatory constraints (Artzner et al., 1999). In the context of probabilistic safety assessment, legislation postulates acceptable probabilities of failure, e.g. of fatality numbers exceeding a threshold (Borgonovo and Cillo, 2017). Hence it is of interest to identify which inputs would be influential in a change of the model that leads to an unacceptable increase in the value of the output risk measure.

- The relationship between model inputs and outputs is complex and not necessarily given in analytical form; furthermore, evaluations of the model are computationally expensive. Therefore, it should be possible to estimate sensitivity measures from a single sample of input and output scenarios (Plischke et al., 2013).

We propose a sensitivity analysis framework, adapted to the above context, termed *reverse sensitivity testing*. We work in the standard setting of sensitivity analysis, where a number of random *input factors* are mapped to a random *output* via an *aggregation* function. The *baseline probability measure* summarises the distribution of inputs and output in current specification of the model. Reverse sensitivity testing comprises the following steps. First, an output stress is defined, corresponding to an increase in the value of the output risk measure. We focus on the widely used risk measures VaR and ES. The increase in the value of the risk measure is specified so as to produce a stress that is problematic to a decision maker. For example, in a capital management context, a stress on VaR may lead to a situation where insufficient assets are available to satisfy regulatory requirements.

Secondly, a *stressed probability measure* is derived. This is a probability (a) under which the risk measure applied to the model output is at its stressed level and (b) that minimises the Kullback-Leibler (KL) divergence subject to appropriate constraints on the output probability distribution. Thus the stressed probability leads to the most plausible alternative model, under which the output distribution is subjected to the required stress. We derive analytical solutions of the stressed probability measure under an increase of VaR and ES. The form of the solutions allows for numerically efficient implementation via a single set of Monte-Carlo simulations.

Finally, the distribution of individual input factors is examined under the baseline and stressed models. Substantial changes in the distribution of a particular input indicate a large sensitivity to that input. A new class of *reverse sensitivity measures* is introduced, quantifying these input changes. The sensitivity measures are then used to identify the most influential input factors; in a sense, those factors that may be responsible for 'breaking the model'.

1.2 Relation to the literature

The sensitivity measures we derive ultimately reflect the joint distribution of individual input factors and output; hence our proposed method remains formally within the unifying framework discussed by Borgonovo et al. (2016) and thus are (distantly) related to variance-based (Saltelli et al., 2008) and moment-independent (Borgonovo, 2007) sensitivity measures. Conceptually, the reverse direction (from output to input) of the proposed method, is related to *regionalised sensitivity analysis* methods (Spear et al., 1994; Osidele and Beck, 2004). However, there is a key difference between regionalised sensitivity analysis and our approach: in the former, states of the output are identified that are 'out of control', while in the latter what is 'out-of-control' are not individual states but specifications of the output distribution.

In the practice of financial risk management and regulation, *reverse stress testing*, starting with a stressed output state and studying the corresponding surface of scenarios that provide the adverse outcome, is frequently used (BCBS, 2013; EIOPA, 2009). For example, "reverse stresses that result in a depletion of capital..." (Lloyd's, 2016) are used in the validation of insurance risk models. The academic literature on reverse stress testing is relatively sparse, with a recent focus towards identifying most likely stress scenarios (McNeil and Smith, 2012; Breuer et al., 2012; Glasserman and Xu, 2014). Once again, our approach differs from reverse stress testing, in that we consider most influential factors in relation to changes in the output distribution and not a particular output state.

The KL-divergence has been widely used in financial risk management, in particular in the context of model uncertainty, where several plausible specifications of the probability measure may co-exist. For example, Breuer and Csiszár (2013); Glasserman and Xu (2014) consider the worst-case probability measure with respect to all probabilities lying within a KL-divergence radius of the baseline probability. In contrast, reflecting our focus on sensitivity rather than model uncertainty, we consider the probability measure with minimal KL-divergence that satisfies given constraints. Our approach is closely related to the work of Cambou and Filipović (2017) with probability set constraints and Weber (2007) with risk measure constraints. In this paper, we provide additional risk measure constraints not studied in those papers and generalise some results of Weber (2007) by dropping the assumption of bounded random variables.

1.3 Structure of the paper

In Section 2, some preliminaries on risk measures and the KL-divergence are given. In Section 3, the optimisation problem yielding stressed probability measures is stated and solved under constraints arising from different risk measures, with emphasis on VaR and ES. Explicit solutions allow easy implementation and inspection of the distributional changes arising. The solutions and their properties are illustrated through an example of a non-linear insurance portfolio evaluated using Monte-Carlo simulation.

Section 4 is devoted to a comparison of the stressed and the baseline probability measures through stochastic order relations. The output under the baseline probability is first-order stochastically dominated by the stressed probability. A similar dominance relation is given for input factors, under the assumption of a non-decreasing aggregation function and positive dependence between input factors. Moreover, stressed probability measures stemming from different stress severities lead to stochastically ordered input factors and output.

In Section 5 we propose two sensitivity measures specifically tailored to the proposed reverse sensitivity testing approach. A reverse sensitivity measure quantifies the extent that the distribution of an input factor is distorted by the transition to a stressed probability. A forward sensitivity measure is an associated metric that considers the change in output from stressing a particular input. These sensitivity measures can be viewed as dependence metrics between individual input factors and the output. We conclude with an application of the reverse sensitivity testing framework to a commercially used insurance portfolio risk model.

2 Preliminaries

We consider a measurable space $(\Omega, \mathcal{A})$ and denote by $\mathcal{P}$ the set of all probability measures on $(\Omega, \mathcal{A})$. For a random variable Z on $(\Omega, \mathcal{A})$ we write $F_Z^Q(\cdot) = Q(Z \leq \cdot)$ for its distribution under $Q \in \mathcal{P}$, and similarly, $E^Q(\cdot)$ for its expectation. Throughout, we use the Kullback-Leibler divergence (*KL-divergence*, Kullback and Leibler (1951)) as a measure of discrepancy between two probability measures. For $Q^1, Q^2 \in \mathcal{P}$, the KL-divergence, also known as relative entropy, of Q^1 with respect to Q^2 is defined by

$$D_{\mathrm{KL}}(Q^1 \| Q^2) = \begin{cases} \int \frac{dQ^1}{dQ^2} \log\left(\frac{dQ^1}{dQ^2}\right) \mathrm{d}Q^2 & \text{if } Q^1 \ll Q^2 \\ +\infty & \text{otherwise.} \end{cases}$$

The KL-divergence is non-negative, vanishes if and only if $Q^1 \equiv Q^2$, and is in general not symmetric (Kullback, 1997; Cover and Thomas, 2012). The KL-divergence is a special case of the class of *f-divergences*, first introduced by Ali and Silvey (1966), for the choice $f(x) = x\log(x)$, $x > 0$. For a given convex function f, the f-divergence of Q^1 with respect to Q^2, for any $Q^1, Q^2 \in \mathcal{P}$, is defined through $D_f(Q^1 \| Q^2) = \int f\left(\frac{dQ^1}{dQ^2}\right) \mathrm{d}Q^2$.

Risk measures are tools used in risk management, which associate to every random variable a real number. The application of risk measures leads to a classification of different levels of risk severities, see Artzner et al. (1999); Föllmer and Schied (2011) for a overview. Moments, such as the mean and standard deviation, can be seen as risk measures. In recent years, percentile-based risk measures (Acerbi, 2002) have become prominent, with the most commonly used risk measures being Value-at-Risk (VaR) and Expected Shortfall (ES). These risk measures are used extensively in financial regulation for the calculation of capital requirements, specifically VaR for European insurance companies, EIOPA (2009), and ES for banks, BCBS (2012, 2013).

The VaR at level $\alpha \in [0,1]$ of a random variable Z is defined as the left α-quantile of the distribution of Z, $\mathrm{VaR}^Q_\alpha(Z) = F_Z^{Q,-1}(\alpha) = \inf\{z \in \mathbb{R} \,|\, F^Q_Z(z) \geq \alpha\}$, where, as usual, $\inf \varnothing = +\infty$. In particular, the essential supremum of Z is $\operatorname{ess\,sup}^Q Z = F_Z^{Q,-1}(1)$. The ES (also CVaR) of Z at level $\alpha \in [0,1)$ is defined by

$$\mathrm{ES}^Q_\alpha(Z) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}^Q_u(Z)\mathrm{d}u = \frac{1}{1-\alpha} E^Q\left(\left(Z - \mathrm{VaR}^Q_\alpha(Z)\right)_+\right) + \mathrm{VaR}^Q_\alpha(Z),$$

where, in the second representation, $\mathrm{VaR}^Q_\alpha(Z)$ can be replaced by any α-quantile of F^Q_Z. Unlike VaR, the ES takes into account the whole tail of the distribution of Z, that is all realisations larger than $\mathrm{VaR}^Q_\alpha(Z)$. See Föllmer and Schied (2011) for a comparison of the two risk measures.

Shortfall risk measures, associated with utility-type arguments, are defined through $\rho^Q(Z) = \inf\{z \in \mathbb{R} \,|\, E^Q(\ell(Z-z)) \leq z_0\}$ for $Q \in \mathcal{P}$, where ℓ is a non-decreasing, non-constant and convex loss function while z_0 is a point in the interior of the range of ℓ (Föllmer and Schied, 2002). Examples of shortfall risk measures include entropic risk measures, Gerber (1974), and the class of generalised quantiles called expectiles (Newey and Powell, 1987; Bellini et al., 2014).

3 Deriving the stressed model

3.1 Problem statement

We consider the standard setting of (reverse) sensitivity analysis, involving a (typically complicate) function, mapping model inputs to an output that is used in a decision making process. Mathematically, we define the *input factors* as a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ on the measurable space $(\Omega, \mathcal{A})$. The (measurable) function $g\colon \mathbb{R}^n \to \mathbb{R}$, is called the *aggregation function*, which gives, when applied to input factors $\boldsymbol{X}$ the one-dimensional random *output* of interest $Y = g(\boldsymbol{X})$. The variability of the output Y to changes in input factors is of fundamental importance in sensitivity analysis (Saltelli et al., 2008; Borgonovo and Plischke, 2016). We adopt throughout the convention that large values of the output correspond to adverse states.

We call the triple $(\boldsymbol{X}, g, P)$, the *baseline model* with *baseline probability measure* $P \in \mathcal{P}$. The probability P is seen as encoding current beliefs regarding (or software implementation of) the distribution of $\boldsymbol{X}$. Under the baseline probability P we suppress the superscript and write, for example, $F_Z(\cdot) = F^P_Z(\cdot)$ and $E(\cdot) = E^P(\cdot)$, and analogously for risk measures, $\mathrm{VaR}_\alpha(\cdot) = \mathrm{VaR}^P_\alpha(\cdot)$ and $\mathrm{ES}_\alpha(\cdot) = \mathrm{ES}^P_\alpha(\cdot)$. We call any $Q \in \mathcal{P}$ an *alternative probability measure* and $(\boldsymbol{X}, g, Q)$ an *alternative model*. A Radon-Nikodym (RN) density is a non-negative random variable ζ on $(\Omega, \mathcal{A})$ such that $E(\zeta) = 1$. We denote by Q^ζ the probability measure which is absolutely continuous with respect to P with RN-density ζ, that is, $\zeta = \frac{\mathrm{d}Q^\zeta}{\mathrm{d}P}$.

The starting point of reverse sensitivity analysis is to define a *stress* on the distribution of the output that would be problematic to a decision maker, such as a risk manager or regulator. For example, one may require that the probability of a particular event, representing system failure, increases to an extent that the risk of failure is no longer acceptable. Specific stress definitions using different risk measures are discussed in Sections 3.2-3.5. Subsequently, we call $(\boldsymbol{X}, g, Q)$ a *stressed model* with *stressed probability measure* $Q \in \mathcal{P}$ if, under Q, the output Y fulfils a set of probabilistic constraints (the stress) and Q has minimal KL-divergence with respect to P. Thus, a stressed probability measure is defined as a solution to

$$\min_{Q \in \mathcal{P}} D_{\mathrm{KL}}(Q \| P), \quad \text{s.t. constraints on the distribution of } Y \text{ under } Q. \tag{1}$$

The optimisation problem (1) is robust in the sense that convergence in the KL-divergence implies weak convergence of the

probability measures, Gibbs and Su (2002). This means that an alternative probability which satisfies the constraints of (1) and is close in KL-divergence to the stressed probability, is also close to the stressed probability in the Lévy metric.

Optimisation problem (1) under linear (i.e. moment) constraints was first studied in the seminal paper by Csiszár (1975). In the context of financial risk management, in particular when risk measures are used, optimisation problem (1) involves non-linear constraints and Csiszár's theory cannot be applied. Relevant research includes Cambou and Filipović (2017) who consider the optimisation problem for general f-divergences and probability set constraints. Weber (2007) works with bounded random variables and considers risk measure constraints such as ES and shortfall risk measures, see Sections 3.3 and 3.4 for a more detailed comparison. The related problem of finding a worst-case distribution with respect to alternative probabilities lying within a KL-divergence distance of the baseline probability is addressed in Breuer and Csiszár (2013) and Glasserman and Xu (2014) and Blanchet et al. (2017). We refer to Ben-Tal et al. (2013) for robust linear optimisation with general f-divergence constraints.

3.2 Probability constraints

Before studying problem (1) with constraints involving the risk measures of Section 2, we consider stresses under which the probabilities of (adverse) outcomes of $Y = g(\boldsymbol{X})$ are altered. These outcomes are captured by disjoint sets $B_1, \ldots, B_I \subseteq \mathbb{R}$, each set B_i associated with an event $\{Y \in B_i\}$ where the system being studied is failing or 'out of control'. In a financial context, where Y is interpreted as a loss, one can identify B_i with a region of extreme losses.

The following result is an immediate consequence of Theorem 3.1 in Csiszár (1975); we also refer to Cambou and Filipović (2017).

Proposition 3.1. Let $B_1, \ldots, B_I \subseteq \mathbb{R}$ be disjoint Borel sets with $P(Y \in B_i) > 0$, $i = 1, \ldots, I$, and $\alpha_1, \ldots, \alpha_I > 0$ such that $\alpha_1 + \cdots + \alpha_I \le 1$. Then there exists a unique solution to

$$\min_{Q \in \mathcal{P}} D_{\mathrm{KL}}(Q \| P), \qquad \text{s.t. } Q(Y \in B_i) = \alpha_i,\ i = 1, \ldots, I, \tag{2}$$

with RN-density given by $\zeta = \sum_{i=0}^{I} \frac{\alpha_i}{P(Y \in B_i)} \mathbb{1}_{\{Y \in B_i\}}$, where we write $\alpha_0 = 1 - \sum_{i=1}^{I} \alpha_i$ and $B_0 = (\bigcup_{i=1}^{I} B_i)^c$.

The RN-density ζ in Proposition 3.1 is a piecewise constant function of Y. This implies that all outcomes of Y within a set B_i receive the same probability re-weighting by the change to the stressed probability. In particular, if $\alpha_i > P(Y \in B_i)$, under the alternative probability Q the probability of all outcomes in B_i increases.

3.3 VaR constraints

We now consider optimisation problem (1) under a constraint on the risk measure VaR, applied to the output Y. A VaR constraint is not equivalent to a probability constraint of optimisation problem (2), when F_Y is not strictly increasing.

Proposition 3.2. Let $0 < \alpha < 1$ and $q \in \mathbb{R}$ such that $\mathrm{VaR}_\alpha(Y) < q < \operatorname{ess\,sup} Y$ and consider the optimisation problem

$$\min_{Q \ll \mathcal{P}} D_{\mathrm{KL}}(Q \| P), \quad \text{s.t. } \mathrm{VaR}_\alpha^Q(Y) = q. \tag{3}$$

There exists a unique solution to (3) if and only if $P(q - \varepsilon < Y < q) > 0$ for all $\varepsilon > 0$. The RN-density of the solution is given by

$$\zeta = \frac{\alpha}{P(Y < q)} \mathbb{1}_{\{Y<q\}} + \frac{1-\alpha}{P(Y \geq q)} \mathbb{1}_{\{Y \geq q\}}.$$

The assumption $P(q - \varepsilon < Y < q) > 0$ for all $\varepsilon > 0$, implies that q cannot be chosen arbitrarily. In particular, problem (3) does not have a solution, if the distribution of Y is constant to the left of q (q excluded); this includes the (uncommon in practice) case where Y is a discrete random variable. This complication arises from using the constraint $\mathrm{VaR}_\alpha^Q(Y) = q$ rather than $Q(Y \leq q) = \alpha$. If q cannot be chosen to fulfil the assumptions in Proposition 3.2, the form of ζ in Proposition 3.2 remains meaningful: by Proposition 3.1, it is the solution to an optimisation problem where the constraint $\mathrm{VaR}_\alpha^Q(Y) = q$ is replaced by $Q(Y < q) = \alpha$.

The RN-density ζ of the solution to (3) is a non-decreasing function of Y since $\alpha \leq P(Y \leq \mathrm{VaR}_\alpha(Y)) \leq P(Y < q)$. Hence, under the stressed probability, adverse realisations of the output are given higher probabilities of occurrence. This is now demonstrated by an example taken from insurance risk modelling.

Remark. Propositions 3.1 and 3.2 hold true for any f-divergence with a strictly convex function f. In particular, the RN-densities ζ of the solutions are independent of the choice of f-divergence. We do not provide a proof for this statement, however the steps of the proofs of Propositions 3.1 and 3.2 can be closely retraced if one substitutes the KL-divergence with a general f-divergence.

Example. The following insurance portfolio, similar to Example 1 in Tsanakas and Millossovich (2016), will be used as an illustrative example throughout the paper. An insurance company faces a loss L resulting from two lines of business. The two lines produce losses X_1, X_2 respectively, which are subject to the same multiplicative inflation factor X_3, such that $L = X_3(X_1 + X_2)$. The insurance company has a reinsurance contract on the loss L with limit l and deductible d. The total portfolio loss for the insurance company is

$$Y = L - (1 - X_4) \min\{(L - d)_+, l\},$$

where X_4 captures the percentage lost due to a default of the reinsurance company.

In this example, the two lines of business X_1, X_2 are truncated Log-Normal and Gamma distributed, with respective means 150, 200 and standard deviations 35, 20. The truncation point for X_1 is chosen to be the 99.9% quantile. The multiplicative factor X_3 follows a truncated Log-Normal distribution with mean 1.05, standard deviation 0.02 and truncation point equal to the 99.9% quantile. The default loss X_4 is modelled through a Beta distribution with mean 0.1 and standard deviation 0.2 and is dependent on the aggregated loss L through a Gaussian copula with correlation 0.6. X_1, X_2, X_3 are independent and X_4 is independent of (X_1, X_2, X_3) given L. The deductible of the insurance contract is $d = 380$ and the limit $l = 30$.

Consider optimisation problem (3) with a 10% increase in $\mathrm{VaR}_{0.9}$, that is

$$\min_{Q \ll \mathcal{P}} D_{\mathrm{KL}}(Q\|P), \quad \text{s.t. } \mathrm{VaR}^Q_{0.9}(Y) = 1.1\,\mathrm{VaR}_{0.9}(Y). \tag{4}$$

The solution to the problem (4) is estimated from a Monte-Carlo sample containing $M = 100{,}000$ simulated scenarios from $(\boldsymbol{X}, Y)$. The explicit form of the RN-density in Proposition 3.2 (as well as the subsequent Propositions 3.3-3.4), allows easy implementation of the change of measure in a Monte-Carlo simulation context. Note that the RN-density is a function of Y, in the sense that $\zeta(\omega) = \eta(Y(\omega))$. Then, one can follow the process:

1. Sample M multivariate scenarios $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(M)}$ from $\boldsymbol{X}$. Calculate $y^{(k)} = g(\boldsymbol{x}^{(k)})$, $k = 1, \ldots, M$.
2. Set $\zeta^{(k)} = \eta(y^{(k)})$, $k = 1, \ldots, M$.
3. The distributions of the output and inputs under the stressed measure Q are estimated by:

$$F^Q_Y(y) = \frac{1}{M} \sum_{k=1}^{M} \zeta^{(k)} \mathbb{1}_{\{y^{(k)} \leq y\}}, \; y \in \mathbb{R},$$
$$F^Q_{X_i}(x) = \frac{1}{M} \sum_{k=1}^{M} \zeta^{(k)} \mathbb{1}_{\{x_i^{(k)} \leq x\}}, \; x \in \mathbb{R}, \; i = 1, \ldots, n.$$

Note that this calculation allows stressing the model without the need to re-simulate scenarios under Q, which can be of practical importance if evaluation of g is computationally expensive.

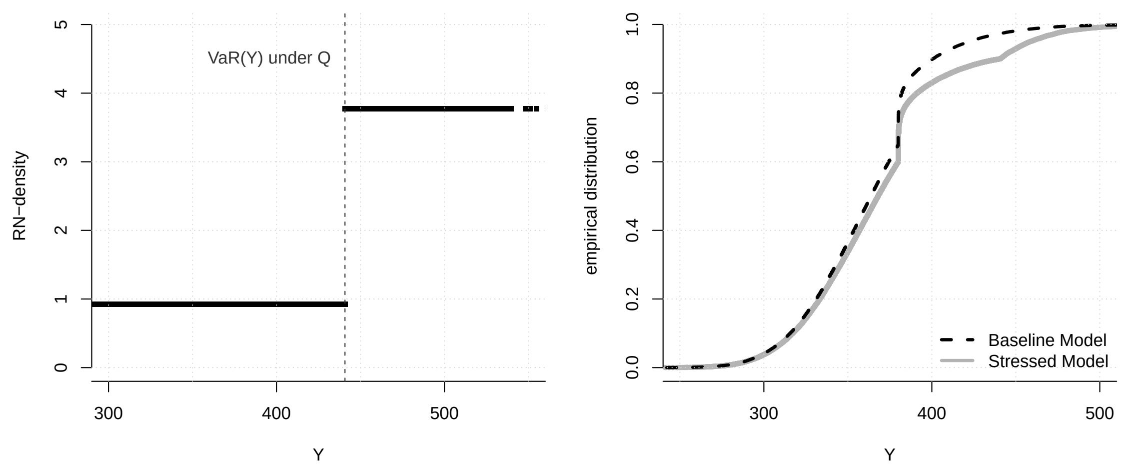

Figure 1: Left: simulated RN-density of the solution to (4). Right: simulated empirical distribution functions of the output under the baseline (dashed black) and the stressed (solid grey) model.

Simulated values of the RN-density ζ are plotted in the left of Figure 1, against samples from Y. It is seen that the RN-density is a non-decreasing function of Y and thus gives more weight to adverse outcomes of Y.

The empirical distribution functions of the total loss Y of the insurance company under the baseline probability (dashed black) and the stressed probability (solid grey) are displayed in the right of Figure 1. The output distribution under the stressed probability lies beneath, and therefore first-order stochastically dominates, the distribution of Y under the baseline probability. We refer to Section 4 for a more detailed discussion of stochastic comparisons of stressed and baseline probabilities.

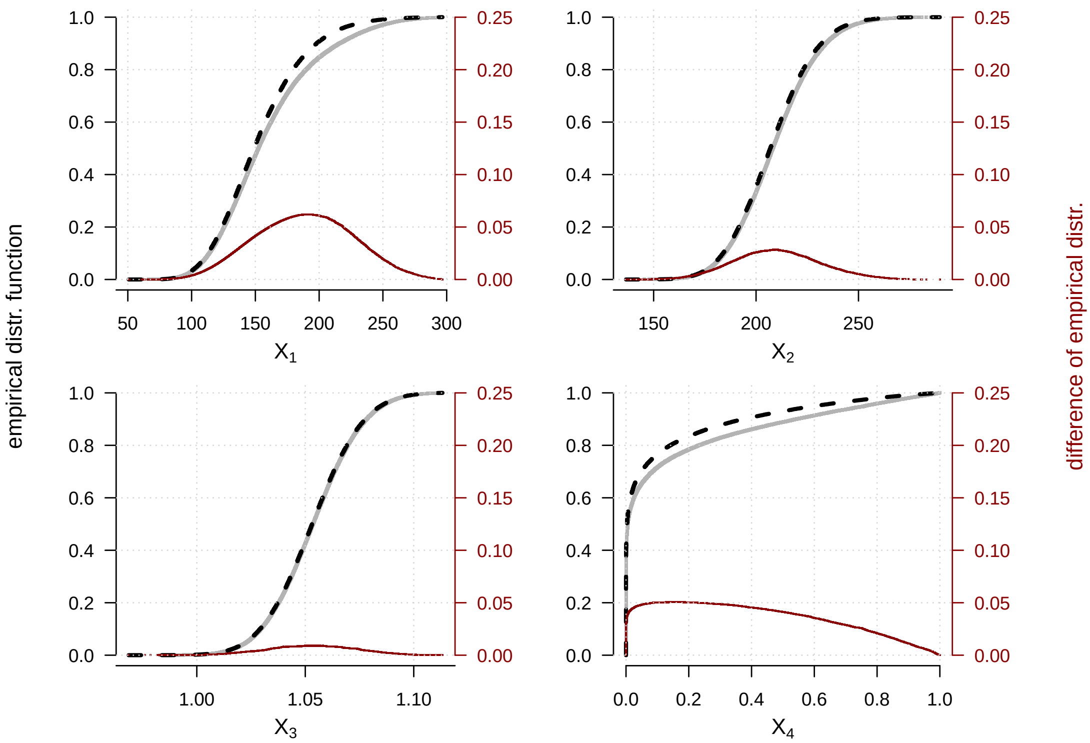

Figure 2: Empirical distribution functions of the input factors under the baseline (dashed black) and the stressed model (solid grey). The dark red line displays the difference of the distribution functions according to the axis on the right.

Figure 2 displays the change in distribution of the input factors when moving from the baseline model to the stressed model. It can be seen that all factors under the stressed probability first-order stochastically dominate the corresponding inputs under the baseline probability. However, not all input factors are impacted the same: the distributions of inputs X_1 and X_4 are stressed more compared to the baseline model. This indicates a higher sensitivity to X_1 and X_4, compared to X_2 and X_3. A specific sensitivity measure reflecting the above observations is introduced in Section 5.

Table 1 summarises basic characteristics of the change in the output and the input factors under the two models. Consistently with Figure 2, it is seen that X_1 and X_4 are the most affected input factors by the change of probability measure. For example, under the stressed probability, X_1, X_4 are subject to a relative increase of the standard deviation of 17%, 22%, respectively.

Table 1: Distributional characteristics of inputs and output under the baseline and stressed model.

Sensitivity	**Input factors**				**Output**
	X_1	X_2	X_3	X_4	Y
Mean under P	149.7	199.9	1.050	0.099	361.5
Mean under Q	156.0	201.4	1.050	0.135	369.3
$\big(E^Q(X_i) - E(X_i)\big)/E(X_i)$	4.21%	0.74%	0.04%	35.82%	2.16%
Standard deviation under P	34.5	20.1	0.020	0.199	36.1
Standard deviation under Q	40.5	20.8	0.020	0.243	44.6
$\big(\sigma^Q(X_i) - \sigma(X_i)\big)/\sigma(X_i)$	17.43%	3.51%	0.33%	22.07%	23.57%
Skewness under P	0.6	0.2	0.0	2.5	0.4
Skewness under Q	1.2	0.5	0.1	2.4	1.2
Excess kurtosis under P	0.5	0.1	-0.1	5.6	1.3
Excess kurtosis under Q	0.8	0.2	-0.1	3.9	1.4

3.4 VaR and ES constraints

This section addresses optimisation problem (1) with a constraint on both, VaR and ES. Adding to problem (3) a constraint on ES allows to stress the whole tail of the output distribution. Weber (2007) considers optimisation problem (1) with an ES constraint only. In that case there does not exist an analytic solution of the stressed probability and Weber (2007) offers a procedure for a numerical solution.

Proposition 3.3. Let $0 < \alpha < 1$ and $q, s \in \mathbb{R}$ such that $\mathrm{VaR}_\alpha(Y) < q < s < \operatorname{ess\,sup} Y$. Assume the cumulant generating function of $Y|Y > q$ under P exists in a neighbourhood of 0 and that $E(Y|Y > q) < s$. Consider the optimisation problem

$$\min_{Q\in\mathcal{P}} D_{\mathrm{KL}}(Q\|P), \qquad \text{s.t. } \mathrm{VaR}^Q_\alpha(Y) = q,\ \mathrm{ES}^Q_\alpha(Y) = s. \tag{5}$$

Define the sets $A_1 = \{Y \geq q\}$ and $A_2 = \{Y > q\}$ and, for $i = 1, 2$, denote by θ^*_i the unique positive solution of the equation

$$E\left((Y - s)e^{\theta(Y-q)} \,\middle|\, A_i\right) = 0. \tag{6}$$

There exists a unique solution to problem (5) under either

1. $P(q - \varepsilon < Y < q) > 0$ for all $\varepsilon > 0$ and $E\big(e^{\theta^*_1(Y-q)} \,\big|\, A_1\big) \leq \frac{P(A_1^c)/P(A_1)}{\alpha/(1-\alpha)}$.
2. $P(Y = q) > 0$ and $P(q - \varepsilon < Y < q) = 0$ for some $\varepsilon > 0$, and $E\big(e^{\theta^*_2(Y-q)} \,\big|\, A_2\big) \geq \frac{P(A_2^c)/P(A_2)}{\alpha/(1-\alpha)}$.

The corresponding RN-density of the solution is

$$\zeta_i = \frac{\alpha}{P(A_i^c)}\mathbb{1}_{A_i^c} + \frac{1-\alpha}{E\left(e^{\theta^*_i(Y-q)}\mathbb{1}_{A_i}\right)}e^{\theta^*_i(Y-q)}\mathbb{1}_{A_i},\ i = 1, 2.$$

Note that, compared to stressing solely the VaR, adding an ES constraint may provide a solution even for an output following a discrete distribution. The condition on the moment generating function in cases 1. and 2. restricts the choice of s and q, such that the stressed risk measure values cannot be chosen independently.

The RN-density of Proposition 3.3 under case 1., ζ_1, is a non-decreasing function of Y. Under Proposition 3.3 case 2., the RN-density ζ_2 is not monotone. However, both RN-densities are exponentially increasing for realisations of Y exceeding q. Thus, under the stressed model, adverse outcomes of Y, such as tail events, admit a higher likelihood compared to the baseline model.

Example (continued). We consider optimisation problem (5) with a 10% increase in $\mathrm{VaR}_{0.9}$ and a 13% increase in $\mathrm{ES}_{0.9}$. Figure 3 displays samples of the RN-density of the stressed probability measure, see Proposition 3.3 case 1. For high outcomes of the output Y, the RN-density ζ is exponentially increasing as a function of Y, hence inflates stressed tail probabilities. On the right hand side, the empirical distribution functions of the output under the baseline (dashed black) and the stressed model (solid grey) are shown.

Observe that the stressed distribution of the output appears similar to the stressed distribution of optimisation problem (4), see Figure 1. This is due to the fact that increasing $\mathrm{VaR}_{0.9}$ by 10% in optimisation problem (4), already leads to an increase of 8.5% in $\mathrm{ES}_{0.9}$ under the stressed model. However, comparing Tables 1 and 2 it is seen that the standard deviation, skewness and kurtosis of Y increase more when stressing VaR and ES, compared to stressing VaR alone.

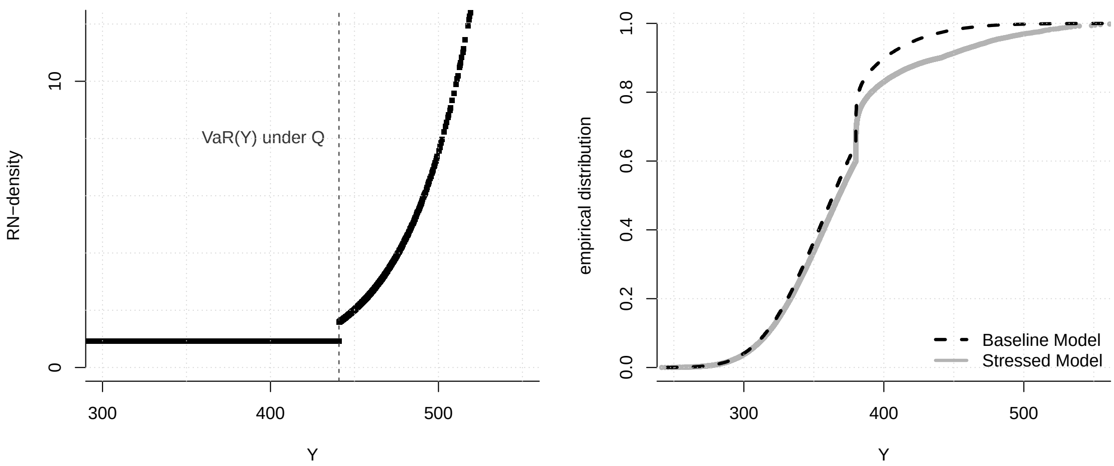

Figure 3: Left: simulated RN-density of the solution. Right: simulated empirical distribution functions of the output under the baseline (dashed black) and the stressed (solid grey) model.

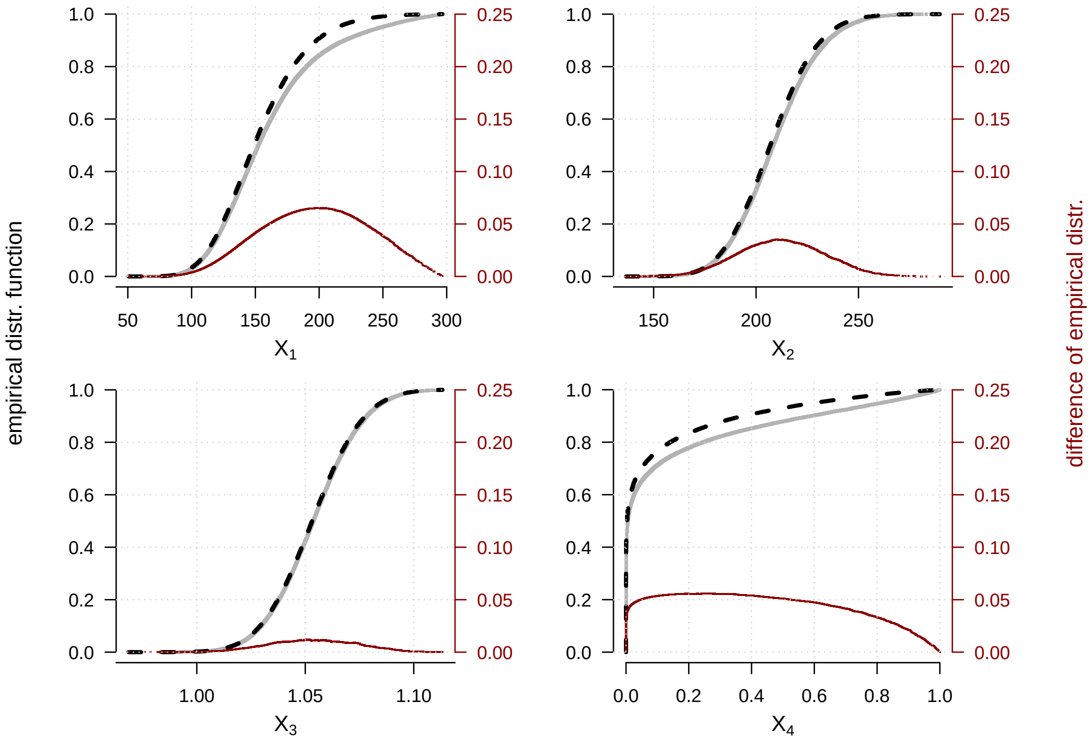

Figure 4: Empirical distribution functions of the input factors under the baseline (dashed black) and the stressed model (solid grey). The dark red line displays the difference of the distribution functions according to the axis on the right.

Table 2: Distributional characteristics under the baseline and the stressed model.

Sensitivity	**Input factors**				**Output**
	X_1	X_2	X_3	X_4	Y
Mean under P	149.7	199.9	1.050	0.099	361.5
Mean under Q	157.2	201.8	1.051	0.143	371.3
$\big(E^Q(X_i) - E(X_i)\big)/E(X_i)$	5.04 %	0.93%	0.06%	44.05%	2.73%
Standard deviation under P	34.5	20.1	0.020	0.199	36.1
Standard deviation under Q	43.1	21.1	0.020	0.257	49.8
$\big(\sigma^Q(X_i) - \sigma(X_i)\big)/\sigma(X_i)$	24.95%	5.24%	0.52%	29.29%	37.89%
Skewness under P	0.6	0.2	0.0	2.5	0.4
Skewness under Q	1.4	0.5	0.1	2.4	1.7
Excess kurtosis under P	0.5	0.1	-0.1	5.6	1.3
Excess kurtosis under Q	1.3	0.2	-0.1	3.7	2.8

Similar to optimisation problem (4), the output and the input factors under the baseline probability are first-order stochastically dominated by the stressed probability, as can be seen in Figure 3 and 4. We refer to Section 4 for a formal treatment of stochastic comparison of the stressed and baseline probabilities.

3.5 Shortfall risk measure constraints

Optimisation problem (1) with shortfall risk measure constraints is studied in Weber (2007) and is a direct application of Theorem 3.1 in Csiszár (1975). Nonetheless, we present the solution for completeness.

Proposition 3.4. Let ρ be a shortfall risk measure with loss function ℓ and y_0, and $q \in \mathbb{R}$ in the support of Y such that $E(\ell(Y-q)) < y_0$. If the moment generating function of $\ell(Y-q)$ exists in a neighbourhood of 0, then the optimisation problem

$$\min_{Q \in \mathcal{P}} D_{\mathrm{KL}}(Q\|P), \qquad \text{s.t. } \rho^Q(Y) = q, \tag{7}$$

has a unique solution whose density is given by $\zeta = \frac{1}{E\left(e^{\theta^*\ell(Y-q)}\right)} e^{\theta^*\ell(Y-q)}$, where θ^* is the unique positive solution of $E\left(\left(\ell(Y-q)-y_0\right)e^{\theta\ell(Y-q)}\right) = 0$.

4 Stochastic comparisons

The proposed reverse sensitivity testing framework is based on the change from a baseline probability measure P to a stressed probability Q. The optimisation problems of Section 3 ensure that under Q the value of particular risk measures applied on Y increases. But the broader changes in the distributions of input factors $\boldsymbol{X}$ and output Y arising from the change of measure are also of interest in a risk management context. For Q to be meaningfully called a 'stressed measure', we argue that three properties should be fulfilled. First, under Q the distribution of the output should dominate (in a suitable stochastic order relation) the output distribution under the baseline model. Second, under the assumptions of a non-decreasing aggregation function and positive dependence between input factors, the distribution of the input vector $\boldsymbol{X}$ under Q should stochastically dominate the distribution of $\boldsymbol{X}$ under P. Third, an increase in the extent to which risk measures are stressed should be reflected in the distributions of output and inputs under the corresponding stressed probabilities. In this section we aim to give precise conditions under which the above properties are fulfilled. Note that most of the discussion is not contingent on Q being a solution of one of the optimisation problems of Section 3.

We adopt the standard definitions of stochastic order relations. For distribution functions F, G we write $F \preceq_{st} G$ if G is larger than F in first-order stochastic dominance, that is $F(\boldsymbol{x}) \geq G(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$. For univariate F, G, we denote $F \preceq_{icx} G$ if G is larger than F in increasing convex (or stop-loss) order, that is $\int_u^1 F^{-1}(s)\mathrm{d}s \leq \int_u^1 G^{-1}(s)\mathrm{d}s$ for all $u \in (0,1)$. The following dependence concepts are of importance, see Denuit et al. (2006):

- An m-dimensional random vector $\boldsymbol{Z}$ is *stochastically increasing* (or *positively regression dependent*) in a random variable W, denoted by $\boldsymbol{Z} \uparrow_{si} W$, if $P(\boldsymbol{Z} > \boldsymbol{z} \,|\, W = w)$ is non-decreasing in w, for all $\boldsymbol{z} \in \mathbb{R}^m$.

- An m-dimensional random vector $\boldsymbol{Z}$ is *associated* if $\mathrm{Cov}(h_1(\boldsymbol{Z}), h_2(\boldsymbol{Z})) \geq 0$, for all non-decreasing functions $h_1, h_2 : \mathbb{R}^m \to \mathbb{R}$ for which the covariance exists.

- The random couple (W, Z) is *positively quadrant dependent* (PQD) if $P(W \leq w, Z \leq z) \geq P(W \leq w)P(Z \leq z)$ for all $w, z \in \mathbb{R}$.

- The random couple (W, Z) is *negatively quadrant dependent* (NQD) if $P(W \leq w, Z \leq z) \leq P(W \leq w)P(Z \leq z)$ for all $w, z \in \mathbb{R}$.

For a pair of random variables (W, Z) the above definitions are successively weaker: $Z \uparrow_{si} W$ implies that (Z, W) is associated, which implies PQD, see Esary et al. (1967). We write $\boldsymbol{Z}_{-k} = (Z_1, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_m)$, $1 \leq k \leq m$ for the $(m-1)$-dimensional sub-vector of $\boldsymbol{Z}$ deprived of its k-th component.

The next two propositions characterise the stochastic ordering of inputs and output under two different probabilities Q^1, Q^2, making alternative assumptions on distributions under P, on g and on the form of the corresponding two RN-densities.

Proposition 4.1. Let $Q^1, Q^2 \in \mathcal{P}$ be two probability measures with $\frac{\mathrm{d}Q^1}{\mathrm{d}P} = \eta_1(Y)$, $\frac{\mathrm{d}Q^2}{\mathrm{d}P} = \eta_2(Y)$, for some non-negative functions η_i, $i = 1, 2$. If the RN-densities cross once, such that for some $d \in \mathbb{R}$

$$\eta_2(y) \begin{cases} \leq \eta_1(y) & y < d \\ \geq \eta_1(y) & y \geq d, \end{cases} \tag{8}$$

then the following hold:

1. $F_Y^{Q^1} \preceq_{st} F_Y^{Q^2}$
2. For given $i \in \{1, \dots, I\}$, if $E\left((X_i - t)_+ | Y = y\right)$ is non-decreasing in y for all $t \in \mathbb{R}$, then $F_{X_i}^{Q^1} \preceq_{icx} F_{X_i}^{Q^2}$.
3. For given $i \in \{1, \dots, I\}$, if $X_i \uparrow_{si} Y$, then $F_{X_i}^{Q^1} \preceq_{st} F_{X_i}^{Q^2}$.

Proposition 4.2. Let $Q^1, Q^2 \in \mathcal{P}$ be two probability measures with $\frac{\mathrm{d}Q^1}{\mathrm{d}P} = \eta_1(Y)$, $\frac{\mathrm{d}Q^2}{\mathrm{d}P} = \eta_2(Y)$ for some non-negative functions η_i, $i = 1, 2$. Assume that $\eta_2 - \eta_1$ is non-decreasing. Then the following hold:

1. $F_Y^{Q^1} \preceq_{st} F_Y^{Q^2}$.
2. If the aggregation function g is non-decreasing in coordinate i and X_i is independent of $\boldsymbol{X}_{-i}$, then $F_{X_i}^{Q^1} \preceq_{st} F_{X_i}^{Q^2}$.
3. Assume that the aggregation function g is non-decreasing.
   (a) For given $i \in \{1, \dots, I\}$, if (X_i, Y) is PQD, then $F_{X_i}^{Q^1} \preceq_{st} F_{X_i}^{Q^2}$.
   (b) If $\boldsymbol{X}$ is associated, then $F_{\boldsymbol{X}}^{Q^1} \preceq_{st} F_{\boldsymbol{X}}^{Q^2}$.

Part 1. of both, Propositions 4.1 and 4.2, reflects the comparative impact of the stress on the output Y, while parts 2. and 3. characterise the impact of the stress on the inputs. An example where the assumption of Proposition 4.1, part 3., is satisfied is the following. Suppose the input vector $\boldsymbol{X}$ is multivariate normal and $Y = h(\sum_{i=1}^n w_i X_i)$ for an increasing function h and $w_i \in \mathbb{R}$ for all i. If $\mathrm{Cov}(X_i, h^{-1}(Y)) = \sum_{j=1}^n w_j \mathrm{Cov}(X_i, X_j) \geq 0$, then $X_i \uparrow_{si} Y$ holds. The assumption in Proposition 4.2 part 3.(a) holds for example if $\boldsymbol{X}_{-i} \uparrow_{st} X_i$ and g is non-decreasing.

Propositions 4.1 and 4.2 allow for a stochastic comparison of the output and the input factors under the stressed and the baseline model. In particular, Proposition 4.1 applies to the solutions of problems (3), (5) and (7) with $Q^2 = Q$ and $Q^1 = P$. Proposition 4.2 applies to optimisation problem (1), with $Q^2 = Q$ and $Q^1 = P$, if the RN-density of the solution is a non-decreasing function of Y. Recall that the RN-density of the solutions to (3), (5) case 1, and (7) are non-decreasing.

Proposition 4.1 also enables to contrast stressed probabilities corresponding to different stress levels. For example, when solving optimisation problem (3) with two different VaR constraints, the output under the stressed model corresponding to a higher VaR should stochastically dominate the output under the other stressed model. The next lemma associates Proposition 4.1 with solutions of the optimisation problems (3) and (5).

Lemma 4.3. The crossing condition of Proposition 4.1 is satisfied for:

1. Two solutions Q^1, Q^2 of optimisation problem (3) with constraints $\mathrm{VaR}_\alpha^{Q^1}(Y) = q_1$ respectively $\mathrm{VaR}_\alpha^{Q^2}(Y) = q_2$, and $q_1 < q_2$.
2. Two solutions Q^1, Q^2 of optimisation problem (5) with constraints $\mathrm{VaR}_\alpha^{Q^1}(Y) = \mathrm{VaR}_\alpha^{Q^2}(Y) = q$ and $\mathrm{ES}_\alpha^{Q^1}(Y) = s_1$, respectively $\mathrm{ES}_\alpha^{Q^2}(Y) = s_2$, and $s_1 < s_2$.

The second part of Lemma 4.3 holds true for both types of solutions of (5).

Example (continued). Applying Proposition 4.1 to the two optimisation problems in this example, we immediately verify that the output under the stressed probabilities first-order stochastically dominates the output under the baseline probability, see Figures 1 and 3. Moreover, the aggregation function g is non-decreasing and it can be verified that, for instance, (X_4, Y) is PQD. Hence, following Proposition 4.2 part 3.(a), the distribution of X_4 under the stressed probability first-order stochastically dominates that under the baseline probability. This can be seen in Figures 2 and 4.

An illustration of Lemma 4.3 is given in Figure 5. The left plot shows the RN-densities of solutions to (3) with two different stress levels. The black line corresponds to an increase of VaR of 10%, the same as in Figure 1, and the grey line to an increase of VaR of 15%. The plot to the right displays the RN-densities of solutions to (5) for an increase of 10% in VaR and 9% in ES (black) and an increase of 10% in VaR and 13% in ES (grey), see Figure 3. It is seen how in both cases, the two RN-densities satisfy the crossing condition of Proposition 4.1.

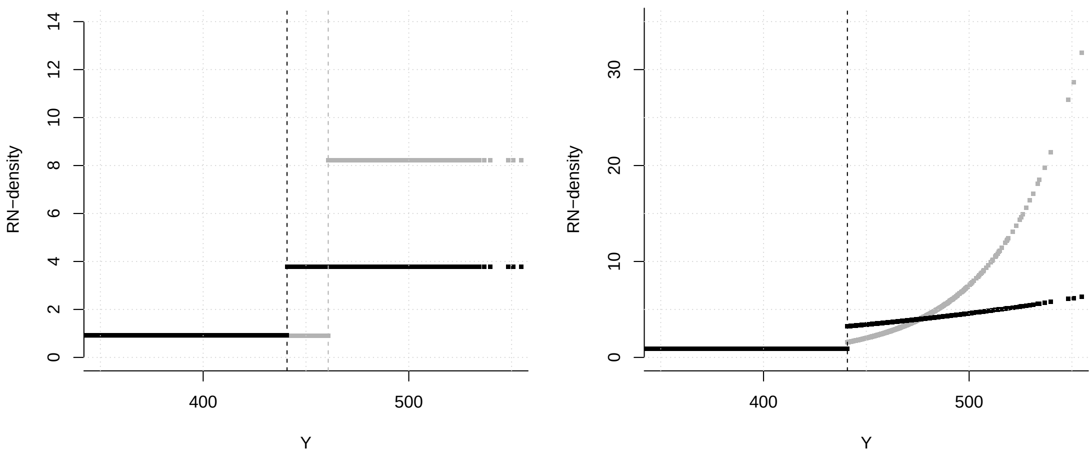

Figure 5: Left: simulated RN-densities of the solution to (4) with a 10% (black) and 15% (grey) increase in VaR. Right: simulated RN-densities of the solution to (5) case 1, with a 10% increase in VaR and 9% (black) and 13% (grey) increase in ES.

5 Sensitivity measures for importance ranking

5.1 Definition of sensitivity measures

Plots such as the ones shown in Figures 2 and 4 provide some insight into the sensitivity of the output risk measure to different input factors. In order to produce a ranking of inputs, it is necessary to introduce a formal sensitivity or importance measure; this is especially the case for models with large numbers of inputs for which succinct sensitivity summaries are needed. Here we develop a sensitivity measure that quantifies changes in input factors under the stressed model, compared to the baseline model.

Before proceeding to the definitions, some preliminaries are due. The random couple (V, W) is comonotonic if it can be written as $(V, W) \stackrel{d}{=} (F_V^{-1}(U), F_W^{-1}(U))$, for a uniformly distributed random variable U on $(0, 1)$. In contrast, (V, W) is counter-monotonic if $(V, W) \stackrel{d}{=} (F_V^{-1}(U), F_W^{-1}(1-U))$. Comonotonicity and counter-monotonicity correspond to extremal positive and negative dependence structures respectively, for a random couple with fixed marginals (Müller and Stoyan, 2002). For a random variable V, we denote by $V_{|W}, V_{|W^\dagger}$ the random variables satisfying $V_{|W} \stackrel{d}{=} V_{|W^\dagger} \stackrel{d}{=} V$, such that $(V_{|W}, W)$ is comonotonic and $(V_{|W^\dagger}, W)$ is counter-monotonic. Then for any $V' \stackrel{d}{=} V$ it holds that (Rüschendorf, 1983),

$$E(WV_{|W^\dagger}) \leq E(WV') \leq E(WV_{|W}).$$

The subsequent definition introduces a sensitivity measure that captures the extent to which a random variable is affected by a stress on the baseline model, that is, a change in probability measure.

Definition 5.1. Let Q^ξ be an alternative probability with RN-density $\xi = \frac{\mathrm{d}Q^\xi}{\mathrm{d}P}$. The sensitivity of a random variable Z to the change of measure is given by

$$\mathcal{S}(Z, \xi) = \begin{cases} \dfrac{E(Z\xi) - E(Z)}{\max_{\psi \stackrel{d}{=} \xi} E(Z\psi) - E(Z)} & E(Z\xi) \geq E(Z), \\ -\dfrac{E(Z\xi) - E(Z)}{\min_{\psi \stackrel{d}{=} \xi} E(Z\psi) - E(Z)} & \text{otherwise}, \end{cases}$$

where we use the convention $\pm\frac{\infty}{\infty} = \pm 1$ and $\frac{0}{0} = 0$.

In the definition of $\mathcal{S}(Z, \xi)$, the numerator $E(Z\xi) - E(Z)$ reflects the increase in the expectation of Z under the alternative model. The denominator normalises this difference, as it represents the maximal (or minimal) increase of the expectation of Z, under all alternative models with density ψ that are equal in distribution to ξ. This ensures normalisation of the sensitivity measure to $[-1, 1]$. If $\mathcal{S}(Z, \xi) = 1$ or $\mathcal{S}(Z, \xi) = -1$, the alternative model produces a maximal stress on the variable Z, representing a positive or negative impact of the changes in probability measure on Z respectively.

Note that $\arg\max_{\psi \stackrel{d}{=} \xi} E(Z\psi) = \xi_{|Z}$ and $\arg\min_{\psi \stackrel{d}{=} \xi} E(Z\psi) = \xi_{|Z^\dagger}$. This allows for a straightforward calculation of the sensitivity measure. If working within a Monte-Carlo simulation context, as is common in risk analysis, $\xi_{|Z}$, resp. $\xi_{|Z^\dagger}$, can be simply obtained by re-arranging samples of ξ to be sorted in the same, resp. opposite, way as samples from Z. This context gives a different perspective on the constraint $\psi \stackrel{d}{=} \xi$: if simulated elements of ξ represent a particular scheme for re-weighting simulated scenarios, ψ are vectors containing the same weights as ξ, but re-arranged to potentially prioritise different scenarios.

Next we define two sensitivity measures that are specific to the reverse sensitivity analysis framework of this paper.

Definition 5.2. Let Q^ζ be an alternative model with density $\zeta = \frac{\mathrm{d}Q^\zeta}{\mathrm{d}P} = \eta(Y)$, for a non-decreasing function η. For input X_i and output Y, we define the *reverse* and *forward sensitivity measures* Γ and Δ by:

$$\Gamma(X_i, Y, \zeta) = \mathcal{S}(X_i, \zeta),$$
$$\Delta(X_i, Y, \zeta) = \mathcal{S}(Y, \zeta_{|X_i}).$$

Here, $\zeta = \eta(Y)$ can be arrived at as the solution of optimisation problems (3), (5) or (7). $\Gamma(X_i, Y, \zeta)$ thus reflects the extent to which the reverse sensitivity test affects the expectation of the input factor X_i. Note that for $E(X_i\zeta) \geq E(X_i)$, we can write $\Gamma(X_i, Y, \zeta) = \frac{\mathrm{Cov}(X_i, \eta(Y))}{\max_{\psi \overset{d}{=} \eta(Y)} \mathrm{Cov}(X_i, \psi)}$, showing that the reverse sensitivity measure can also be understood as a dependence measure between X_i and Y. In this sense it is closely related to the dependence measure introduced by Kachapova and Kachapov (2012). Indeed, sensitivity measures considering the dependence between X_i and Y have a rich history in sensitivity analysis, for an overview see for example Borgonovo et al. (2016).

A possible criticism of the measure Γ and, by extension, the reverse sensitivity testing framework we propose, is as follows. Let $\Gamma(X_i, Y, \zeta)$ be high. This implies that stressing the model output Y leads to a substantial change in the distribution of the input factor X_i. However, this is not equivalent to a perturbation in the distribution of X_i leading to a sizeable stress in the distribution of the output Y. Such a discrepancy, though uncommon, is theoretically possible and has been termed *probabilistic dissonance* (Cooke and van Noortwijk, 1999).

This motivates the introduction of the forward sensitivity measure Δ, as a companion measure to Γ. The definition of the forward sensitivity measure Δ is analogous to that of Γ, but with a focus on the change in the expectation of Y when perturbing the distribution of the model input X_i. Recall that $\zeta_{|X_i} = \arg\max_{\psi \overset{d}{=} \zeta} E(\psi X_i)$. Therefore, $\zeta_{|X_i}$ is a RN-density with the same distribution as ζ that has the most adverse effect on the input factor X_i. Thus Δ captures the impact of a change in the input X_i on the output Y. Reporting Δ along with Γ can thus produce warning signs of probabilistic dissonance.

Properties of the sensitivity measures Γ and Δ, reflecting their nature as dependence measures, are summarised below.

Proposition 5.3. The sensitivity measures Γ and Δ are well-defined and have the following properties (suppressing the argument (X_i, Y, ζ)):

1. $-1 \leq \Gamma, \Delta \leq 1$.
2. $\Gamma = \Delta = 0$, if X_i, Y are independent.
3. $\Gamma = \Delta = 1$, if (X_i, Y) is comonotonic.
4. $\Gamma = \Delta = -1$, if (X_i, Y) is counter-monotonic.
5. $\Gamma = \Delta \geq 0$, if (X_i, Y) are PQD.
6. $\Gamma = \Delta \leq 0$, if (X_i, Y) are NQD.

The above defined sensitivity measures focus on the difference of expectations under an alternative and the baseline model. If the interest lies in other distributional properties, such as tails, Definition 5.2 can be extended to consider monotone transformations of input factors. Specifically, one can calculate $\Gamma(u(X_i), Y, \zeta)$, respectively $\Delta(u(X_i), Y, \zeta)$, for an appropriately chosen non-decreasing function u. As the couple $(u(X_i), X_i)$ is comonotonic, the interpretation of the sensitivity measures remains unchanged. One particular example is the choice

$$u_v(X_i) = (X_i - F_{X_i}^{-1}(v))_+ - (F_{X_i}^{-1}(1-v) - X_i)_+, \quad 0.5 \le v < 1. \tag{9}$$

For $v = 0.5$, the function $u_{0.5}$ is the identity and thus $\Gamma(u(X_i), Y, \zeta) = \Gamma(X_i, Y, \zeta)$, respectively $\Delta(u(X_i), Y, \zeta) = \Delta(X_i, Y, \zeta)$. When $v > 0.5$, the function u_v is zero whenever $X_i \in \left[F_{X_i}^{-1}(1-v), F_{X_i}^{-1}(v)\right]$ and linearly increasing otherwise. Thus, increasing v places higher emphasis on the tail behaviour of X_i. The random variable $u_v(Y)$ is defined and interpreted in a similar way.

We denote $\Gamma_v(X_i, Y, \zeta) = \Gamma(u_v(X_i), Y, \zeta)$ and $\Delta_v(X_i, Y, \zeta) = \Delta(X_i, u_v(Y), \zeta)$. It is easily seen that the properties of Proposition 5.3 still apply to Γ_v, Δ_v. In addition, it holds that

$$\Gamma_v(aX_i + b, Y, \zeta) = \text{sign}(a)\Gamma_v(X_i, Y, \zeta),$$

such that the reverse sensitivity measure is invariant under linear transformations of input factors.

Example (continued). Figure 6 displays the forward and reverse sensitivity measures Γ_v, Δ_v for $v \in [0.5, 0.999)$, for the stressed model arising from optimisation problem (5) with a 10% increase in VaR and a 13% increase in ES. Consistently with the example in Section 3.4, the highest sensitivity, for both reverse and forward measures, is displayed by X_1, followed by X_4, X_2 and X_3. Furthermore, the ranking is not affected by the level v and is thus not sensitive to emphasising the tails of the distributions. In the next section we present a situation where this no longer holds true.

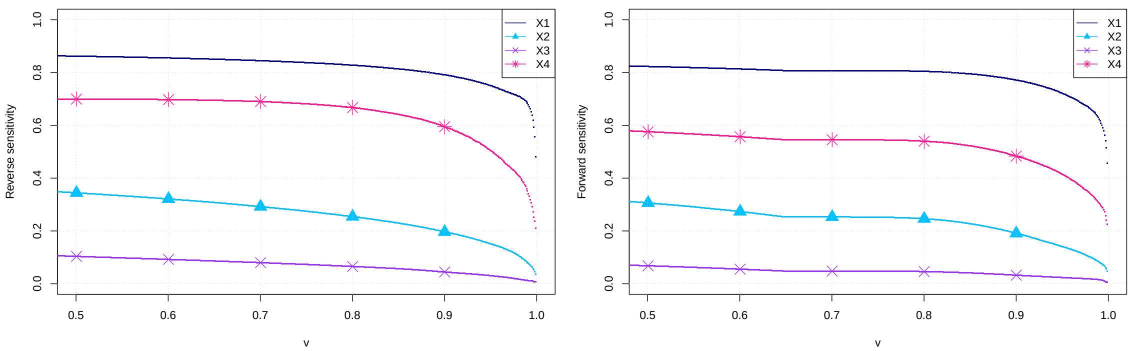

Figure 6: Reverse (left plot) and forward (right plot) sensitivity measures Γ_v, Δ_v with a 10% increase in VaR and 13% increase in ES.

5.2 Application to a model of London Insurance Market losses

In this section we demonstrate the use of the sensitivity measures Γ_v and Δ_v, in a more realistic insurance risk model with a higher number of inputs. This is a proprietary model of a London Insurance Market portfolio, currently in use by a participant in that market. For this model, we have been supplied by the model owner with a Monte-Carlo sample of size $n = 500{,}000$, containing simulated observations from input factors $\mathbf{X} = (X_1, \ldots, X_{72})$ and output Y. Each of the X_i's represents a normalised loss for a particular part of the portfolio and is measured on the same scale. The output Y stands for the portfolio loss.

The aggregation function g is linear, specifically

$$Y = g(\mathbf{X}) = \sum_{j=1}^{72} w_j X_j,$$

for a vector of weights $\mathbf{w} = (w_1, \ldots, w_{72})$. (The linearity of g is not used for sensitivity calculations, since the reverse sensitivity testing framework makes no assumptions on the form of g.) We do not have access to the joint probability distribution that was used to generate samples from the random vector $\mathbf{X}$; in fact the distribution of $\mathbf{X}$ is not given in closed form, as samples from $\mathbf{X}$ are themselves outputs of a different model, which remains a completely black box to us.

We consider optimisation problem (5) with risk measure constraints on VaR and ES given by $q = \mathrm{VaR}^Q_{0.95}(Y) = 1.08\mathrm{VaR}_{0.95}(Y)$ and $s = \mathrm{ES}^Q_{0.95}(Y) = 1.1\mathrm{ES}_{0.95}(Y)$. In Figure 7, the reverse and forward sensitivity measures Γ_v, Δ_v, for $v = 0.5$ and $v = 0.95$, are presented for all 72 inputs. The risk factors are ordered according to $\Gamma_{0.5}$ and the sizes of the markers reflect the weights w_i attached to the individual risk factors X_i.

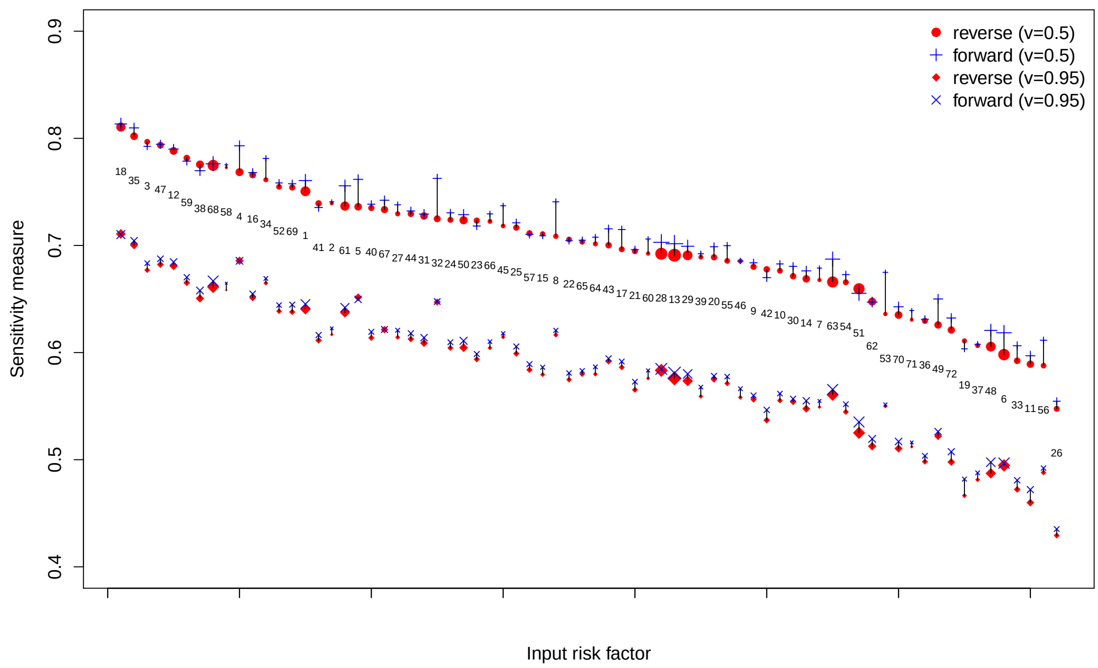

Figure 7: Reverse and forward sensitivity measures Γ_v, Δ_v for the London Insurance Market portfolio, for $v = 0.5$ and $v = 0.95$.

Observations on the plot of Figure 7:

- The ranking of risk factors according to $\Gamma_{0.5}$ and $\Gamma_{0.95}$ is not fully consistent; moving focus to the tails of input risk factors changes the order of the sensitivity measures.

Hence, under the stressed model, for some risk factors the expectation is affected more, while for others the impact is higher in the tail.

- For $v = 0.5$, the ranking produced by the reverse and forward sensitivity metrics is not equivalent. However, once the focus is moved towards the tails of risk factor distributions (e.g. $v = 0.95$), the discrepancy of the two sensitivity measures diminishes.

- There is no clear relation between the sizes of the markers and the ranking of risk factors. This means that the sensitivity measure $\Gamma_v(X_i, Y; \eta)$ does not solely reproduce the size of the weight w_i.

To elaborate on the last of those points, in Figure 8 (left), the reverse sensitivities $\Gamma_{0.95}(X_i, Y; \eta)$ are plotted against the weights w_i. There is a broadly increasing relation, which is not unreasonable. Given the linearity of the aggregation function, a higher weight w_i implies a higher *local sensitivity* $\frac{\partial g}{\partial x_i}$ (Borgonovo and Plischke, 2016). But the relation is by no means deterministic: weight is a weak predictor of the reverse sensitivity measure Γ_v.

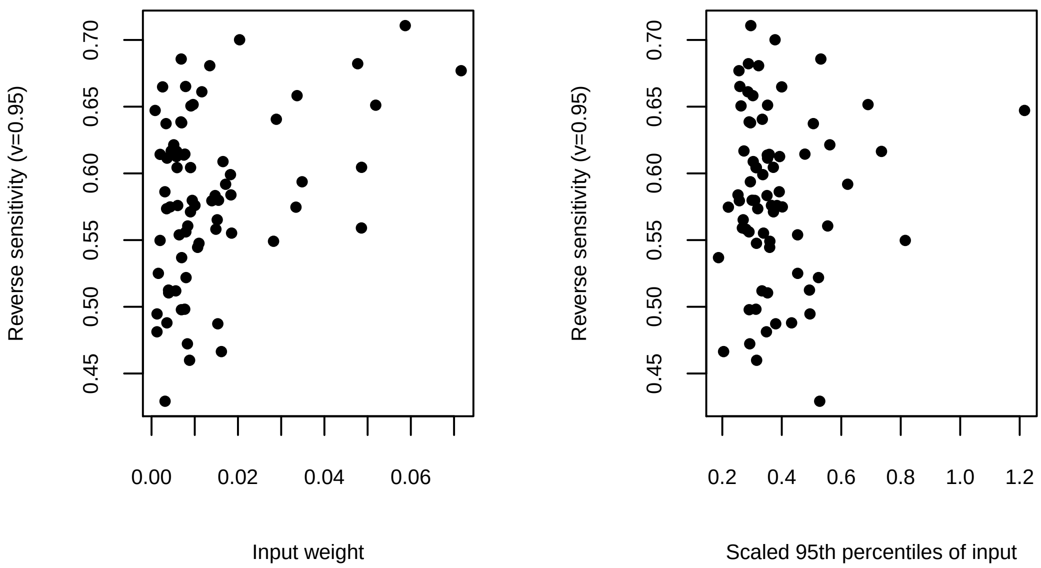

Figure 8: Reverse sensitivity measure $\Gamma_{0.95}(X_i, Y; \eta)$ for the London Insurance Market portfolio, against weights w_i (left) and scaled input percentiles $\frac{\mathrm{VaR}_{0.95}(X_i)}{E(X_i)} - 1$ (right).

Furthermore, the reverse sensitivity measure does not only reflect the shape of the input risk factor distributions. In Figure 8 (right), $\Gamma_{0.95}(X_i, Y; \eta)$ is displayed against the scaled percentiles $\frac{\mathrm{VaR}_{0.95}(X_i)}{E(X_i)} - 1$ which does not show a clear pattern. Hence the two plots in Figure 8 demonstrate that the proposed reverse sensitivity measure does not reproduce easily observed characteristics of the aggregation function g or of the distributions of the inputs X_i.

6 Conclusions

We proposed a reverse sensitivity testing framework that is appropriate for contexts where model inputs are uncertain and the relationship between model inputs and outputs is complex and not necessarily given in analytical form. At the core of the reverse sensitivity framework is a stress on the output distribution, corresponding to an increase in the value of a risk measure applied on the output and representing a plausible but adverse model change. This leads to stressed probabilities under which the output distribution is subjected to the required stress.

We provided analytical solutions of the stressed probability measure under an increase of the VaR and ES risk measures. These explicit solutions facilitate straightforward implementation in a Monte-Carlo simulation context and inspection of changes in the distributions of inputs. A new class of reverse sensitivity measures is introduced, quantifying the extent that the distribution of an input factor is distorted by the transition to a stressed probability. Analysis of stochastic order relations induced by the change of measure provides assurance that the proposed method has desirable properties.

The reverse sensitivity framework can be easily deployed by a risk analyst with access only to a set of input / output scenarios, simulated under the baseline model. Thus there is no need for a detailed consideration of the model structure or of simulating additional scenarios, involving computationally expensive model evaluations. Thus the proposed framework is immediately applicable to industry applications.

A Proofs

Proposition 3.1. A similar result can be found in Cambou and Filipović (2017), we also refer to Csiszár (1975) for the general form of the solution. It is immediately verified that ζ is a RN-density for which $Q^\zeta(Y \in B_i) = \alpha_i$, $i = 1, \dots, I$. Let ξ be any RN-density that satisfies $Q^\xi(Y \in B_i) = \alpha_i$, $i = 1, \dots, I$. Using Jensen inequality, the KL-divergence of Q^ξ with respect to P fulfils

$$\begin{aligned}
D_{\mathrm{KL}}(Q^\xi \| P) &= \sum_{i=0}^{I} E\left(\xi \log(\xi) \,|Y \in B_i\right) P(Y \in B_i) \\
&\geq \sum_{i=0}^{I} E\left(\xi \,|Y \in B_i\right) \log\left(E\left(\xi \,|Y \in B_i\right)\right) P(Y \in B_i) \\
&= \sum_{i=0}^{I} \alpha_i \log\left(\frac{\alpha_i}{P(Y \in B_i)}\right) \\
&= D_{\mathrm{KL}}(Q^\zeta \| P).
\end{aligned}$$

Therefore Q^ζ is a solution of (2). Uniqueness follows by strict convexity of the KL-divergence, see Csiszár (1975). □

Proposition 3.2. Assume that $P(q - \varepsilon < Y < q) > 0$ for all $\varepsilon > 0$. Then, it is immediate to verify that ζ is a RN-density such that $\mathrm{VaR}_\alpha^{Q^\zeta}(Y) = q$. Let $\xi = \frac{\mathrm{d}Q^\xi}{\mathrm{d}P}$ be a RN-density for which $\mathrm{VaR}_\alpha^{Q^\xi}(Y) = q$. By Jensen inequality, the KL-divergence of Q^ξ with respect to P is

$$\begin{aligned}
D_{\mathrm{KL}}(Q^\xi \| P) &= E\left(\xi \log(\xi) \,|\, Y < q\right) P(Y < q) + E\left(\xi \log(\xi) \,|\, Y \geq q\right) P(Y \geq q) \\
&\geq Q^\xi(Y < q) \log\left(\frac{Q^\xi(Y < q)}{P(Y < q)}\right) + Q^\xi(Y \geq q) \log\left(\frac{Q^\xi(Y \geq q)}{P(Y \geq q)}\right) \\
&= k\big(Q^\xi(Y < q), P(Y < q)\big),
\end{aligned}$$

where we define $k(x,y) = x\log(\frac{x}{y}) + (1-x)\log(\frac{1-x}{1-y})$, for $0 < x < 1$, $0 < y < 1$. Inspection shows that, for fixed $0 < y < 1$, $x \to k(x,y)$ is non-increasing on $(0, y]$. Moreover it holds

$$Q^{\xi}(Y < q) \leq \alpha \leq P(Y \leq \mathrm{VaR}_{\alpha}(Y)) \leq P(Y < q).$$

The KL-divergence of Q^{ξ} is thus larger than the KL-divergence of Q^{ζ},

$$\begin{aligned} D_{\mathrm{KL}}(Q^{\xi}\|P) &\geq k\big(Q^{\xi}(Y < q), P(Y < q)\big) \\ &\geq k\big(\alpha, P(Y < q)\big) \\ &= \alpha \log\left(\frac{\alpha}{P(Y < q)}\right) + (1-\alpha)\log\left(\frac{1-\alpha}{P(Y \geq q)}\right) \\ &= D_{\mathrm{KL}}(Q^{\zeta}\|P), \end{aligned}$$

and Q^{ζ} is a solution of (3). Uniqueness follows by strict convexity of the KL-divergence.

Assume now that there exists $\varepsilon > 0$ such that $P(q-\varepsilon < Y < q) = 0$. If $P(Y = q) = 0$, by the absolute continuity of the probability measures, the optimisation problem (3) does not admit a solution. Hence, we assume that $P(Y = q) > 0$. Let Q^{ξ} be a RN-density for which $\mathrm{VaR}_{\alpha}^{Q^{\xi}}(Y) = q$. Denote $r = Q^{\xi}(Y \leq q)$ and $p = P(Y \leq q)$. The KL-divergence of Q^{ξ} with respect to P is

$$\begin{aligned} D_{\mathrm{KL}}(Q^{\xi}\|P) &= E(\xi\log(\xi)\,|\,Y \leq q)p + E(\xi\log(\xi)\,|\,Y > q)(1-p) \\ &\geq r\log\left(\frac{r}{p}\right) + (1-r)\log\left(\frac{1-r}{1-p}\right) \\ &= D_{\mathrm{KL}}(Q^{\xi^r}\|P), \end{aligned}$$

where we define $\xi^u = \frac{\mathrm{d}Q^{\xi^u}}{\mathrm{d}P} = \frac{u}{p}\mathbb{1}_{\{Y\leq q\}} + \frac{1-u}{1-p}\mathbb{1}_{\{Y>q\}}$, $0 \leq u \leq 1$. The family of RN-densities ξ^u fulfil $\mathrm{VaR}_{\alpha}^{Q^{\xi^u}}(Y) = q$ if and only if $\alpha \leq u < \alpha\frac{p}{P(Y<q)}$. In particular this holds for the RN-density ξ^r. Hence the optimisation problem (3) is reduced to minimise $D_{\mathrm{KL}}(Q^{\xi^u}\|P)$ subject to $\alpha \leq u < \alpha\frac{p}{P(Y<q)}$. As a function of u the KL-divergence $D_{\mathrm{KL}}(Q^{\xi^u}\|P)$ is non-increasing on $(0, p]$, hence the optimisation problem does not admit a solution as $\alpha\frac{p}{P(Y<q)} < p$. □

Proposition 3.3. For $i = 1, 2$, equation (6) can be rewritten as

$$\frac{\partial}{\partial\theta}E(e^{\theta(Y-q)}|A_i) = \frac{E\big((Y-q)e^{\theta(Y-q)}|A_i\big)}{E(e^{\theta(Y-q)}|A_i)} = s - q.$$

The left hand side is increasing for positive θ, negative for $\theta = 0$ and diverges for $\theta \uparrow \theta_{max}$, where $\theta_{max} = \sup\{\theta > 0\,|\,E(e^{\theta Y}|A_i) < \infty\}$, by properties of the moment generating function. Thus, for $i = 1, 2$, there exists a unique positive solution θ_i^* of (6).

Case 1. The RN-density ζ_1 fulfils the constraints in (5) since $Q^{\zeta_1}(Y < q) = \alpha$, $Q^{\zeta_1}(Y \leq q) \geq \alpha$ and the ES constraint is equivalent to $(1-\alpha)(s-q) = E^{Q^{\zeta_1}}((Y-q)_+)$. Let $\xi = \frac{\mathrm{d}Q^{\xi}}{\mathrm{d}P}$ be a RN-density satisfying the constraints of problem (5) and denote $r = Q^{\xi}(A_1^c)$ and $p = P(A_1^c)$. Using Jensen's inequality, the KL-divergence of Q^{ξ} with respect to P fulfils

$$\begin{aligned} D_{\mathrm{KL}}(Q^{\xi}\|P) &= E\left(\xi\log(\xi)\mathbb{1}_{A_1^c}\right) + E\left(\xi\log(\xi)\mathbb{1}_{A_1}\right) + \theta_1^*(1-\alpha)(s-q) - E\left(\xi\log\left(e^{\theta_1^*(Y-q)}\right)\mathbb{1}_{A_1}\right) \\ &\geq r\log\left(\frac{r}{p}\right) + \theta_1^*(1-\alpha)(s-q) + E\left(\xi\log\left(\frac{\xi}{e^{\theta_1^*(Y-q)}}\right)\Big|A_1\right)(1-p). \end{aligned}$$

Recall that the perspective of a convex function f, defined by $h(x,y) = yf(x/y)$ is itself convex, see Boyd and Vandenberghe (2004). Applying then Jensen's inequality to $h(x,y) = y\log(\frac{y}{x})$, the third term becomes

$$
\begin{aligned}
E\left(\xi \log\left(\frac{\xi}{e^{\theta_1^*(Y-q)}}\right) \Big| A_1\right)(1-p) &\geq E(\xi \,|A_1)\log\left(\frac{E(\xi \,|\, A_1)}{E(e^{\theta_1^*(Y-q)} \,|\, A_1)}\right)(1-p) \\
&= (1-r)\log\left(\frac{(1-r)}{E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)}\right).
\end{aligned}
$$

Collecting all terms,

$$
\begin{aligned}
D_{\mathrm{KL}}(Q^{\xi}\|P) &\geq r\log\left(\frac{r}{p}\right) + \theta_1^*(1-\alpha)(s-q) + (1-r)\log\left(\frac{(1-r)}{E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)}\right) \\
&= k\big(r, p, E\big(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\big)\big),
\end{aligned}
$$

where we define $k(x,y,z) = x\log(\frac{x}{y}) + \theta_1^*(1-\alpha)(s-q) + (1-x)\log(\frac{1-x}{z})$, for $0 < x < 1$ and $y,z > 0$. For fixed $y,z > 0$, the function $x \to k(x,y,z)$ is decreasing on $\big(0, \frac{y}{y+z}\big]$. The condition on θ_1^* in 1. is equivalent to

$$
\alpha \leq \frac{p}{p + E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)}.
$$

Therefore, noting that $r \leq \alpha$, we obtain

$$
D_{\mathrm{KL}}(Q^{\xi}\|P) \geq k\big(r, p, E\big(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\big)\big) \geq k\big(\alpha, p, E\big(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\big)\big) = D_{\mathrm{KL}}(Q^{\zeta_1}\|P).
$$

The last equality follows since

$$
\begin{aligned}
D_{\mathrm{KL}}(Q^{\zeta_1}\|P) &= \alpha\log\left(\frac{\alpha}{p}\right) + \frac{1-\alpha}{E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)} E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\log\left(\frac{1-\alpha}{E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)}e^{\theta_1^*(Y-q)}\right)\right) \\
&= \alpha\log\left(\frac{\alpha}{p}\right) + (1-\alpha)\log\left(\frac{1-\alpha}{E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)}\right) \\
&\quad + \theta_1^*\frac{1-\alpha}{E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)}E\big(e^{\theta_1^*(Y-q)}(Y-q)_+\big) \\
&= \alpha\log\left(\frac{\alpha}{p}\right) + (1-\alpha)\log\left(\frac{1-\alpha}{E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)}\right) + \theta_1^*(1-\alpha)(s-q) \\
&= k\big(\alpha, p, E\big(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\big)\big).
\end{aligned}
$$

Therefore Q^{ζ_1} is a solution of (5). Uniqueness follows by strict convexity of the KL-divergence.

Case 2. The proof of case 2 is similar to that of case 1, replacing the set A_1 with A_2 and ζ_1 with ζ_2. The RN-density ζ_2 fulfils the constraints (5). Letting $\xi = \frac{\mathrm{d}Q^{\xi}}{\mathrm{d}P}$ be a RN-density satisfying the constraints of problem (5), then the KL-divergence of Q^{ξ} with respect to P can be bounded by

$$
D_{\mathrm{KL}}(Q^{\xi}\|P) \geq k\big(Q^{\xi}(A_2^c), P(A_2^c), E\big(e^{\theta_2^*(Y-q)}\mathbb{1}_{A_2}\big)\big),
$$

where the function $k(x, y, z)$ has been defined above. For fixed $y, z > 0$, the function $x \to k(x, y, z)$ is increasing on $\left[\frac{y}{y+z}, 1\right)$. Moreover, the condition on θ_2^* in 2. is equivalent to

$$\frac{P(A_2^c)}{P(A_2^c) + E\left(e^{\theta_2^*(Y-q)}\mathbb{1}_{A_2}\right)} \leq \alpha.$$

Since $\alpha \leq Q^\xi(A_2^c)$ we obtain

$$D_{\mathrm{KL}}(Q^\xi \| P) \geq k\big(\alpha, P(A_2^c), E\big(e^{\theta_2^*(Y-q)}\mathbb{1}_{A_2}\big)\big) = D_{\mathrm{KL}}(Q^{\zeta_2} \| P),$$

which is the KL-divergence of Q^{ζ_2}. □

Proposition 4.1. Let $y \leq d$, then $Q^2(Y \leq y) = E(\eta_2(Y)\mathbb{1}_{\{Y \leq y\}}) \leq E(\eta_1(Y)\mathbb{1}_{\{Y \leq y\}}) = Q^1(Y \leq y)$. For $y > d$, it holds $Q^2(Y \leq y) = 1 - Q^2(Y > y) = 1 - E(\eta_2(Y)\mathbb{1}_{\{Y > y\}}) \leq 1 - E(\eta_1(Y)\mathbb{1}_{\{Y > y\}}) = Q^1(Y \leq y)$. For the second part we have, for all $t \in \mathbb{R}$, using the tower property under P,

$$E^{Q^2}\left((X_i - t)_+\right) = E^{Q^2}\big(E\left((X_i - t)_+ | Y\right)\big) \geq E^{Q^1}\big(E\left((X_i - t)_+ | Y\right)\big) = E^{Q_1}\left((X_i - t)_+\right),$$

by first-order stochastic dominance of Y with respect to the measures Q^1, Q^2. The last claim follows using a similar argument. □

Proposition 4.2. The RN-densities have to cross once due to normalisation, therefore part 1. applies. In the rest of the proof, let $h = \eta_2 - \eta_1$.

To prove part 2., let g be non-decreasing in coordinate i and X_i independent of $\boldsymbol{X}_{-i}$. For any $t \in \mathbb{R}$, using the Fortuin-Kasteleyn-Ginibre inequality, we have

$$\begin{aligned} Q^2(X_i > t) - Q^1(X_i > t) &= E\left(h(Y)\mathbb{1}_{\{X_i > t\}}\right) = E\left(E\big(h(Y)\mathbb{1}_{\{X_i > t\}} \,|\, \boldsymbol{X}_{-i}\big)\right) \\ &\geq E\left(E\big(h(Y) \,\big|\, \boldsymbol{X}_{-i}\big)\right) P(X_i > t) = 0, \end{aligned}$$

proving first-order stochastic dominance.

To show part 3.(a), assume that g is non-decreasing and (X_i, Y) are PQD. Hence, for all $t \in \mathbb{R}$,

$$Q^2(X_i > t) - Q^1(X_i > t) = E\left(\mathbb{1}_{\{X_i > t\}} h(Y)\right) \geq 0,$$

where the last inequality follows from Lemma 3 in Lehmann (1966). Part 3.(b) follows by association of the vector $(h(Y), \boldsymbol{X})$, using a similar argument. □

Lemma 4.3. The first claim follows since $\alpha \leq P(Y < q_1) \leq P(Y < q_2)$. For part 2., consider first the case where $P(q - \varepsilon < Y < q) > 0$ for all $\varepsilon > 0$. Denote by θ_1^*, θ_2^* the solutions to (6) with q and s_1, respectively s_2. Hence, $\theta_1^* \leq \theta_2^*$, and there exists a $d > q$ such that for all $\omega \in \Omega$ with $Y(\omega) > d$ we have

$$e^{(\theta_2^* - \theta_1^*)(Y(\omega) - q)} \geq \frac{E\left(e^{\theta_2^*(Y-q)}\mathbb{1}_{A_1}\right)}{E\left(e^{\theta_1^*(Y-q)}\mathbb{1}_{A_1}\right)},$$

which implies $\eta_2 \geq \eta_1$ for all ω with $Y(\omega) > d$. Since on A_1^c, $\eta_1 = \eta_2$ P-a.s. the RN-densities admit a (unique) crossing point. The argument also holds if A_1 is replace with A_2. □

Proposition 5.3. We also refer to Theorem 6 in Kachapova and Kachapov (2012). The first two properties are immediate. For 3. if X_i and Y are comonotonic, ζ and $\zeta_{|X_i}$ are also comonotonic since ζ is a non-decreasing function of Y and $\zeta_{|X_i}$ a non-decreasing function of X_i. Part 4. follows by a similar argument. Properties 5. and 6. are consequences of the invariance of PQD (NQD) under non-decreasing (non-increasing) transformations, see Lemma 1 in Lehmann (1966). □

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