BLOCKS OF MINIMAL DIMENSION

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Abstract. Any block with defect group P of a finite group G with Sylow-p-subgroup S has dimension at least $|S|^2/|P|$; we show that a block which attains this bound is nilpotent, answering a question of G. R. Robinson.
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Theorem. *Let p be a prime and let $\mathcal{O}$ be a complete local Noetherian commutative ring with algebraically closed residue field k of characteristic p. Let G be a finite group, let b be a block of $\mathcal{O}G$ with a defect group P and let S be a Sylow-p-subgroup of G. Then $\mathrm{rk}_{\mathcal{O}}(\mathcal{O}Gb) \geq |S|^2/|P|$, and if $\mathrm{rk}_{\mathcal{O}}(\mathcal{O}Gb) = |S|^2/|P|$ then b is a nilpotent block, the block algebra $\mathcal{O}Gb$ is isomorphic to the matrix algebra $M_{|S|/|P|}(\mathcal{O}P)$ and the algebra $\mathcal{O}P$ is a source algebra of b.*

Nilpotent blocks were introduced in [2] as a block theoretic analogue of p-nilpotent finite groups. The proof of the Theorem is based on Puig's results in [6] on the bimodule structure of a source algebra of $\mathcal{O}Gb$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule and the structure theory of nilpotent blocks in [7]. Examples of blocks of minimal $\mathcal{O}$-rank include all blocks of p-nilpotent finite groups G with abelian $O_{p'}(G)$ and, with $P = 1$, the block of the Steinberg module of a finite group of Lie type in defining characteristic.

We refer to Thévenaz [8] for background material on p-blocks of finite groups. In particular, with the notation of the Theorem, by a block of $\mathcal{O}G$ we mean a primitive idempotent b in $Z(\mathcal{O}G)$, and a defect group of b is a minimal subgroup P of G such that $\mathcal{O}Gb$ is isomorphic to a direct summand of $\mathcal{O}Gb \underset{\mathcal{O}P}{\otimes} \mathcal{O}Gb$ as $\mathcal{O}Gb$-$\mathcal{O}Gb$-binodule. This is equivalent to requiring that P is a maximal p-subgroup of G such that $\mathrm{Br}_P(b) \neq 0$, where

$$\mathrm{Br}_P : (\mathcal{O}G)^P \longrightarrow kC_G(P)$$

is the *Brauer homomorphism* sending a P-stable element $\sum_{x\in G} \lambda_x x$ of the group algebra $\mathcal{O}G$ to the element $\sum_{x\in C_G(P)} \bar{\lambda}_x x$ in the group algebra $kC_G(P)$, where here $\bar{\lambda}_x$ is the canonical image of the coefficient $\lambda_x \in \mathcal{O}$ in the residue field k. The map Br_P is well-known to be a surjective algebra homomorphism. In particular, $\mathrm{Br}_P(b)$ is an idempotent in $Z(kC_G(P))$, hence a sum of blocks of $kC_G(P)$. The blocks occurring in $\mathrm{Br}_P(b)$ are all conjugate under $N_G(P)$. More generally, a *b-Brauer pair* is a pair (Q, e) consisting of a p-subgroup Q of G and a block e of $kC_G(Q)$ such that $\mathrm{Br}_Q(b)e \neq 0$. Following [1], the set of b-Brauer pairs admits a canonical structure of partially ordered G-set with respect to the conjugation action of G. This partial order has the property that for any b-Brauer pair (Q, e) and any subgroup R of Q there is a unique block f of $kC_G(R)$ such that (R, f) is a b-Brauer pair and such that $(R, f) \subseteq (Q, e)$. The block b is called *nilpotent* if $N_G(Q, e)/C_G(Q)$ is a p-group for any b-Brauer pair (Q, e). As a consequence of a theorem of Frobenius, the group G is p-nilpotent if and only if the principal block of $\mathcal{O}G$ is nilpotent, which explains the terminology.

Proof of the Theorem. The statement on the minimal possible rank of $\mathcal{O}Gb$ is well-known, but we include a proof for the convenience of the reader. Choose a Sylow-p-subgroup S of G such that $P \subseteq S$. Since $\mathcal{O}Gb$ is a direct summand of $\mathcal{O}G$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodule, there is an $\mathcal{O}$-basis X of $\mathcal{O}Gb$ which is stable under left and right multiplication with elements in S. For any subgroup R of S, the set of "diagonal" fixpoints

$$X^R = \{x \in X \mid uxu^{-1} = x \text{ for all } u \in R\}$$

is mapped by Br_R to a k-basis in $\mathrm{Br}_R((\mathcal{O}Gb)^R) = kC_G(R)\mathrm{Br}_R(b)$. Since P is maximal such that $\mathrm{Br}_P(b) \neq 0$, the set X^P is in particular non empty. Also, $\mathcal{O}Gb$ has vertex ΔP and trivial source as $\mathcal{O}(G\times G)$-module, hence is a direct summand of $\mathrm{Ind}_{\Delta P}^{G\times G}(\mathcal{O})$, where $\Delta P = \{(u, u) \mid u \in P\}$. Mackey's formula implies that every indecomposable direct summand of $\mathcal{O}Gb$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodule is of the form $\mathrm{Ind}_Q^{S\times S}(\mathcal{O})$ for some subgroup Q of $S \times S$ of the form $S \times S \cap {}^{(x,y)}\Delta P$ with $x, y \in G$; in particular, Q has order at most $|P|$. In other words, the stabiliser of any element $x \in X$ in $S \times S$ has at most order $|P|$.

Let $x \in X^P$. The stabiliser of x in $S \times S$ contains ΔP but has at most order $|P|$, hence is equal to ΔP. Thus the $\mathcal{O}S$-$\mathcal{O}S$-bimodule $\mathcal{O}[SxS]$ generated by x is a direct summand of $\mathcal{O}Gb$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodule isomorphic to $\mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S$. In particular, $\mathrm{rk}_{\mathcal{O}}(\mathcal{O}Gb) \geq \mathrm{rk}_{\mathcal{O}}(\mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S) = |S|^2/|P|$.

In order to show that b is nilpotent we use a result of Puig [6, 3.1] in the form as described in [4, 7.8]. Let $i \in (\mathcal{O}Gb)^P$ be a primitive idempotent in the algebra of fixpoints in $\mathcal{O}Gb$ with respect to the conjugation action by P on $\mathcal{O}Gb$ such that $\mathrm{Br}_P(i) \neq 0$; that is, i is a source idempotent for b and the algebra $i\mathcal{O}Gi$ is a source algebra of b. Since i commutes with the action of P, the source algebra $i\mathcal{O}Gi$ is also a direct summand of $\mathcal{O}Gb \cong \mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S$ as $\mathcal{O}P$-$\mathcal{O}P$-bimodule. As a consequence of results in [1], the choice of the source idempotent i determines a fusion system $\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$ on P, where e is the unique block of $kC_G(P)$ such that $\mathrm{Br}_P(i)e = \mathrm{Br}_P(i)$; this makes sense as $\mathrm{Br}_P(i)$ is a primitive idempotent in $kC_G(P)$. More precisely, for any subgroup Q of P we have $\mathrm{Aut}_{\mathcal{F}}(Q) \cong N_G(Q, e_Q)/C_G(Q)$ where e_Q is the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. See e.g. [3] or [5], for more details on fusion systems of blocks. Now let Q be a subgroup of P and let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Denote by ${}_\varphi\mathcal{O}Q$ the

$\mathcal{O}Q$-$\mathcal{O}Q$-bimodule which is, as $\mathcal{O}$-module, equal to $\mathcal{O}Q$ but with $u \in Q$ acting on the left by multiplication with $\varphi(u)$ and on the right by multiplication with u. By [4, 7.8], the $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule ${}_\varphi\mathcal{O}Q$ is isomorphic to a direct summand of $i\mathcal{O}Gi$ as $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule. Thus ${}_\varphi\mathcal{O}Q$ is isomorphic to a direct summand of $\mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S$ as $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule. This forces φ to be induced by conjugation with an element in $N_S(Q)$. In particular, φ is a p-automorphism of Q. Thus $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a p-group for all subgroups Q of P, and hence b is a nilpotent block.

By the general structure theory of nilpotent blocks [7], the block algebra $\mathcal{O}Gb$ is isomorphic to a matrix algebra $M_n(\mathcal{O}P)$; in particular, the block b has a unique isomorphism class of simple modules. If V is a simple $\mathcal{O}Gb$-module then V has the defect group P as vertex and an endo-permutation kP-module W as source. This source is trivial if and only if the source algebra $i\mathcal{O}Gi$ is isomorphic to $\mathcal{O}P$. Dimension counting yields $\mathrm{rk}_{\mathcal{O}}(\mathcal{O}Gb) = n^2|P| = |S|^2/|P|$, hence $\dim_k(V) = n = [S:P]$. Now V is a direct summand of $\mathrm{Ind}_P^G(W)$, hence by Mackey's formula, $\mathrm{Res}_S^G(V)$ is a direct sum of direct summands of $\mathrm{Ind}_{S\cap {}^xP}^S({}^xW)$ with $x \in G$. Green's indecomposability theorem [8, (23.6)] forces $S \cap {}^xP = {}^xP$ and $\dim_k(W) = 1$, hence W is the trivial kP-module. $\square$

Remark. If $\mathcal{O}Gb$ has $\mathcal{O}$-rank $|S|^2/|P|$ then the first part in the proof of the Theorem says that $\mathcal{O}Gb \cong \mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S$ as $\mathcal{O}S$-$\mathcal{O}S$-bimodules for any defect group P of b contained in S. Thus, if $x \in G$ such that ${}^xP \subseteq S$ then $\mathcal{O}S \underset{\mathcal{O}P}{\otimes} \mathcal{O}S \cong \mathcal{O}S \underset{\mathcal{O}^xP}{\otimes} \mathcal{O}S$, which forces ${}^xP = {}^uP$ for some $u \in S$. It follows that the set $\mathrm{Hom}_G(P,S)$ of group homomorphisms from P to S induced by conjugation with elements in G is equal to $\mathrm{Hom}_S(P,S) \circ \mathrm{Aut}_G(P)$ or equivalently, $N_G(P,S) = SN_G(P)$, where $N_G(P,S) = \{x \in G \mid {}^xP \subseteq S\}$. In other words, the fact that P is a defect group of a block of minimal $\mathcal{O}$-rank has implications for the fusion system of the group itself.

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