ON SATURATED FUSION SYSTEMS AND BRAUER INDECOMPOSABILITY OF SCOTT MODULES

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ABSTRACT. Let p be a prime number, G a finite group, P a p-subgroup of G and k an algebraically closed field of characteristic p. We study the relationship between the category $\mathcal{F}_P(G)$ and the behavior of p-permutation kG-modules with vertex P under the Brauer construction. We give a sufficient condition for $\mathcal{F}_P(G)$ to be a saturated fusion system. We prove that for Scott modules with abelian vertex, our condition is also necessary. In order to obtain our results, we give a criterion for the categories arising from the data of (b, G)-Brauer pairs in the sense of Alperin-Broué and Broué-Puig to be saturated fusion systems on the underlying p-group.

1. INTRODUCTION

Let p be a prime number and k an algebraically closed field of characteristic p. For a finite group G, a p-subgroup Q of G, and a finite dimensional kG-module M, the Brauer quotient $M(Q)$ of M with respect to Q, is naturally a $kN_G(Q)/Q$-module and hence by restriction is a $kQC_G(Q)/Q$ module (see [4], [5], [17, Section 11]). We will say that M is *Brauer indecomposable* if for any p-subgroup Q of G, $M(Q)$ is indecomposable (or zero) as a $kQC_G(Q)/Q$-module.

For subgroups Q, R of G, let $\mathrm{Hom}_G(Q, R)$ denote the set of all group homomorphisms from Q to R which are induced by conjugation by some element of G. For a p-subgroup P of G, let $\mathcal{F}_P(G)$ denote the category whose objects are the subgroups of P; whose morphism set from an object Q to an object R is the set $\mathrm{Hom}_G(Q, R)$, and where composition of morphisms is the usual composition of functions. We prove the following result (for background on fusion systems and saturated fusion systems, we refer the reader to the articles [3] and [13]; we note that we will follow the notational conventions in [3] rather than those of [13] in that all fusion systems will not be assumed to be saturated).

Theorem 1.1. *Let G be a finite group, P a p-subgroup of G and M an indecomposable p-permutation kG-module with vertex P. If M is Brauer indecomposable, then $\mathcal{F}_P(G)$ is a saturated fusion system.*

The question of Brauer indecomposability of p-permutation modules (or rather bimodules) plays a role in the "glueing processes" used for proving categorical equivalences between p-blocks of finite groups as predicted by Broué's abelian defect group conjecture (see [10], [11]). Since splendid equivalences between blocks preserve local structure, it is not unexpected that there is a connection between saturation and the Brauer indecomposability condition. Theorem 1.1 provides a neat formulation of the connection.

The converse of Theorem 1.1 does not hold in general (see remarks after the proof of Theorem 1.1). However, in the special case that M is a Scott module, there seems to be some control in the reverse direction. For the definition and properties of Scott modules

we refer the reader to [4]. For a finite group G and a p-subgroup P of G, we denote by $\mathrm{S}_P(G, k)$ the kG-Scott module with vertex P.

Theorem 1.2. *Let P be an abelian p-subgroup of a finite group G. If $\mathcal{F}_P(G)$ is a saturated fusion system then $\mathrm{S}_P(G, k)$ is Brauer indecomposable.*

As a corollary, we obtain the following.

Corollary 1.3. *Suppose that the finite group G has cyclic Sylow p-subgroups and let P be a p-subgroup of G. Then $\mathrm{S}_P(G, k)$ is Brauer indecomposable.*

Another consequence is the following result, of use for proving categorical equivalences between principal blocks of finite groups.

Corollary 1.4. *Let G_1 and G_2 be finite groups with common abelian Sylow p-subgroup P and let $\Delta(P)$ be the diagonal subgroup $\{(x, x) : x \in P\}$ of $G_1 \times G_2$. If $\mathcal{F}_P(G_1) = \mathcal{F}_P(G_2)$, then $\mathrm{S}_{\Delta(P)}(G_1 \times G_2, k)$ is Brauer indecomposable.*

We do not know whether Theorem 1.2 holds without the assumption that P is abelian. Using D. Craven's construction in [6] of the Scott modules for the symmetric groups S_n, $n \leq 6$, we prove the following.

Proposition 1.5. *Let $G = S_n$, $n \leq 6$ and P a p-subgroup of S_n. If $\mathcal{F}_P(G)$ is a saturated fusion system, then $\mathrm{S}_P(G, k)$ is Brauer indecomposable.*

Let A be a p-permutation G-algebra, finite dimensional over k, and b a primitive idempotent in the subalgebra of G-fixed points of A. To each triple (A, b, G), there is associated a G-poset of Brauer pairs. These were introduced in [2] for the case $A = kG$, considered as a G-algebra via the conjugation action of G on itself; the general case was treated in [5]. Roughly speaking, an (A, b, G)-Brauer pair is a pair of the form (P, e), where P is a p-subgroup of G and e is a block of the Brauer quotient $A(P)$ of A in a prescribed relationship with b. For a maximal object (P, e) of the poset of (A, b, G)-Brauer pairs, we let $\mathcal{F}_{(P,e)}(A, b, G)$ denote the category whose objects are the subgroups of P and whose morphisms are group homomorphisms induced by the action of G on the underlying poset (for exact definitions we refer the reader to section 2). In case $A = kG$, the results of [2] imply that $\mathcal{F}_{(P,e)}(A, b, G)$ is a saturated fusion system (see [12]). In the general case, it is a consequence of [5] that $\mathcal{F}_{(P,e)}(A, b, G)$ is a fusion system in the sense of [3, Definition 1.1] (see Proposition 2.4). However, it is not the case that $\mathcal{F}_{(P,e)}(A, b, G)$ is in general saturated (see remarks after the proof of Theorem 1.1 in section 4). Theorem 1.1 is a special case of the following result, due to the first author, which gives a sufficiency criterion for saturation. For an (A, b, G)-Brauer pair, (P, e), let $C_G(P, e)$ denote the subgroup of $C_G(P)$ which stabilizes the block e of $A(P)$ under the natural action of $C_G(P)$ on $A(P)$.

Theorem 1.6. *Let G be a finite group, A a p-permutation G-algebra, and b a primitive idempotent of A^G. Suppose that*

    (i) b is a central idempotent of A; and

    (ii) For each (A, b, G)-Brauer pair (Q, f) the idempotent f is primitive in $A(Q)^{C_G(Q, f)}$.

Then for any maximal (A, b, G)-Brauer pair (P, e), $\mathcal{F}_{(P,e)}(A, b, G)$ is a saturated fusion system on P.

We will say that a triple (A, b, G) satisfying conditions (i) and (ii) of Theorem 1.6 is *a saturated triple* or that (A, b, G) is of saturated type. In this case, if G and b are clear from the context, we may also simply say that A is of saturated type. If $A = kG$, then the primitive idempotents of A^G are precisely the blocks of kG, and it is easy to see that (A, b, G) is a saturated triple, hence Theorem 1.6 may be viewed as a generalization of the fact that block fusion systems are saturated. But the class of p-permutation G-algebras is very large. One motivation, besides the relevance to Brauer indecomposability, for introducing the notion of saturated type triples is that they provide a new source of saturated fusion systems and hence may contribute to our understanding of these categories.

The paper is divided into four sections. In section 2, we recall the results and definitions of [2] and [5]. Section 3 contains the proof of Theorem 1.6. Section 4 deals with p-permutation modules, and contains the proofs of Theorem 1.1, Theorem 1.2, Corollary 1.3, Corollary 1.4 and Proposition 1.5.

2. BACKGROUND AND QUOTED RESULTS

In this section, we set up notation and recall definitions and background results on Brauer pairs from the papers [2] and [5]. For notation and terminology regarding fusion systems and saturated fusion systems, we refer the reader to [13],[3].

Let G be a finite group, and let A be a p-permutation G-algebra, finite dimensional over k. Recall that A is p-permutation if for any p-subgroup Q of G there is a k-basis of A stabilized by Q.

2.1. Let P be a subgroup of G. We denote by A^P the subalgebra consisting of the fixed points of A under P; if Q is a subgroup of P, the map $\mathrm{Tr}_Q^P : A^Q \to A^P$ is the k-linear map defined by the formula $\mathrm{Tr}_Q^P(a) = \sum_{x \in P/Q} {}^x a$. The image of Tr_Q^P, denoted by A_Q^P is a two-sided ideal of A^P and we denote by $A_{<P}^P$ the sum $\sum_Q A_Q^P$, where Q ranges over the proper subgroups of P. We denote by $A(P)$ the quotient $A^P/A_{<P}^P$, and we denote by Br_P^A the canonical morphism from A^P onto $A(P)$. Recall from [5, Proposition 1.5] that $A(P)$ is a p-permutation $N_G(P)$ algebra. For $g \in G$, the map which sends an element $\mathrm{Br}_P^A(a)$, where $a \in A^P$ to the element ${}^g(\mathrm{Br}_P^A(a)) := \mathrm{Br}_{gP}^A({}^g a)$ is an algebra isomorphism from $A(P)$ to $A({}^g P)$.

If $Q \le P$ are p-groups, then there exists an algebra morphism, $\mathrm{Br}_{P,Q}^A : \mathrm{Br}_Q^A(A^P) \to A(P)$ such that $\mathrm{Br}_{P,Q}^A(\mathrm{Br}_Q^A(a)) = \mathrm{Br}_P^A(a)$ for $a \in A^P$. Clearly, ${}^g\mathrm{Br}_{P,Q}^A(x) = \mathrm{Br}_{gP, gQ}^A({}^g x)$ for any $g \in G$, $x \in \mathrm{Br}_Q^A(A^P)$.

If, in addition, Q is normal in P, then $\mathrm{Br}_Q^A(A^P) = A(Q)^P$ and $\mathrm{Ker}(\mathrm{Br}_{P,Q}^A) = \mathrm{Ker}(\mathrm{Br}_P^{A(Q)})$. Thus, $\mathrm{Br}_{P,Q}^A$ induces an isomorphism $b_{P,Q}^A : A(Q)(P) \to A(P)$. Note that $b_{P,Q}^A$ satisfies and is completely determined by the condition

$$b_{P,Q}^A(\mathrm{Br}_P^{A(Q)}(\mathrm{Br}_Q^A(x))) = \mathrm{Br}_{P,Q}^A(\mathrm{Br}_Q^A(x)))) = \mathrm{Br}_P^A(x) \ \text{ for all } \ x \in A^P.$$

Further, ${}^g b_{P,Q}^A(w) = b_{gP,Q}^A({}^g w)$ for all $g \in N_G(Q)$ and $w \in A(Q)(P)$.

2.2. Let b be a primitive idempotent of A^G. Recall from [5, Definition 1.6] that a (b, G)-Brauer pair is a pair (P, e) where P is a p-subgroup of G such that $\mathrm{Br}_P(b) \ne 0$ and e is a

block of $A(P)$ such that $\mathrm{Br}_P(b)e \neq 0$. Here we recall that a block of a finite-dimensional algebra is a primitive idempotent of the center of the algebra. As we will consider Brauer pairs for different algebras simultaneously, we will adopt the more cumbersome notation (A, b, G)-Brauer pair for (b, G)-Brauer pair.

Recall from [5, Definition 1.6] the notion of inclusion of (A, b, G)-Brauer pairs: If (Q, f) and (P, e) are (A, b, G)-Brauer pairs, then $(Q, f) \leq (P, e)$ if $Q \leq P$ and whenever i is a primitive idempotent of A^P such that $\mathrm{Br}_P^A(i)e \neq 0$, then $\mathrm{Br}_Q^A(i)f \neq 0$.

Let (P, e) be an (A, b, G)-Brauer pair and let $x \in G$. The conjugate of (P, e) by x is the (A, b, G)-Brauer pair ${}^x(P, e) := ({}^xP, {}^xe)$. Clearly, conjugation by x preserves inclusion.

Recall the following fundamental property of inclusion of Brauer pairs [2, Theorem 3.4], [5, Theorem 1.8].

Theorem 2.1. *Let (P, e) be an (A, b, G)-Brauer pair, and let $Q \leq P$.*
(i) There exists a unique block f of $A(Q)$ such that (Q, f) is an (A, b, G)-Brauer pair and $(Q, f) \leq (P, e)$.
(ii) If (Q, f) is an (A, b, G)-Brauer pair and P normalizes Q, then $(Q, f) \leq (P, e)$ if and only if P fixes f and $\mathrm{Br}_{P,Q}^A(f)e = e$.
(iii) The set of (A, b, G)-Brauer pairs is a G-poset under the action of G defined above.

Recall also [2, Theorem 3.10] and [5, Theorem 1.14]).

Theorem 2.2. *Let A be a p-permutation G-algebra and let b be a primitive idempotent of A^G.*
(i) The group G acts transitively on the set of maximal (A, b, G)-Brauer pairs.
(ii) Let (P, e) be an (A, b, G)-Brauer pair. The following are equivalent.
(a) (P, e) is a maximal (A, b, G)-Brauer pair.
(b) $\mathrm{Br}_P^A(b) \neq 0$ and P is maximal amongst p-subgroups Q of G with the property that $\mathrm{Br}_Q^A(b) \neq 0$.
(c) $b \in \mathrm{Tr}_P^G(A^P)$ and P is minimal amongst subgroups H of G such that $b \in \mathrm{Tr}_H^G(A^H)$.

The equivalence of ii(b) above with ii(a) is not explictly stated in [5, Theorem 1.14], but is an immediate consequence of (i). For clearly, if P satisfies ii(b), then (P, e)-is a maximal (A, b, G)-Brauer pair. Conversely, if (P, e) is a maximal (A, b, G)-Brauer pair and $P \leq R$ is such that $\mathrm{Br}_R^A(b) \neq 0$, then there exists some block t of $A(Q)$ such that (R, t) is an (A, b, G)-Brauer pair. Let (S, u) be a maximal (A, b, G)-Brauer pair with $(R, t) \leq (S, u)$. Then by (i), (P, e) and (S, u) are G-conjugate. In particular, $|P| = |S| \geq |R| \geq |P|$, hence $P = R$.

If Q, R are subgroups of G and $g \in G$ is such that ${}^gQ \leq R$, then $\mathrm{c}_g : Q \to R$ denotes the map which sends an element x of Q to the element ${}^gx := gxg^{-1}$ of R.

Definition 2.3. *Let (P, e_P) be a maximal (A, b, G)-Brauer pair. For each subgroup Q of P, let (Q, e_Q) be the unique (A, b, G)-Brauer pair such that $(Q, e_Q) \leq (P, e_P)$. The category $\mathcal{F}_{(P,e_P)}(A, b, G)$ is the category whose objects are the subgroups of P, whose morphisms are given by*

$$\mathrm{Hom}_{\mathcal{F}_{(P,e_P)}(A,b,G)}(Q, R) := \{\mathrm{c}_g : Q \to R | g \in G, {}^g(Q, e_Q) \leq (R, e_R)\}$$

for $Q, R \leq P$, and where composition of morphisms is the usual composition of functions.

For any $Q \leq R$, the inclusion map from Q to R is a morphism in $\mathcal{F}_{(P,e_P)}(A,b,G)$. In particular, the identity map $Q \to Q$ is a morphism in $\mathcal{F}_{(P,e_P)}(A,b,G)$ and if $R,S \leq P$ and $g,h \in G$ are such that $^g(Q,e_Q) \leq (R,e_R)$ and $^h(R,e_R) \leq (S,e_S)$, then

$$^{hg}(Q,e_Q) \leq\ ^h(R,e_R) \leq (S,e_S),$$

so $\mathcal{F}_{(P,e_P)}(A,b,G)$ is a category. By the uniqueness of inclusion of Brauer pairs for $Q,R \leq P$ and $g \in G$, $^g(Q,e_Q) \leq (R,e_R)$ if and only if $^gQ \leq R$ and $^ge_Q = e_{^gQ}$ and this in turn holds if and only if $^gQ \leq R$ and $^g(Q,e_Q) \leq (P,e_P)$. Thus if $x \in P$, then since e_P is fixed by P, $^xe_P = e_P$. Hence, for $Q \leq P$,

$$^x(Q,e_Q) \leq\ ^x(P,e_P) = (P,e_P).$$

So, whenever $^xQ \leq R$, then $\mathrm{c}_x : Q \to R$ is a morphism in $\mathcal{F}_{(P,e_P)}(A,b,G)$.

Also, note that if $Q,R \leq P$ and $g \in G$ are such that $^g(Q,e_Q) \leq (R,e_R)$, then $\mathrm{c}_g : Q \to R$ factors as $\mathrm{c}_g : Q \to\ ^gQ$ followed by the inclusion of ^gQ into R. Summarizing the above discussion gives the following proposition, the last statement of which is immediate from the fact that any two maximal (A,b,G)-Brauer pairs are G-conjugate.

Proposition 2.4. *Let A be a p-permutation G-algebra, b a primitive idempotent of A^G and (P,e_P) a maximal (A,b,G)-Brauer pair. Then $\mathcal{F} := \mathcal{F}_{(P,e_P)}(A,b,G)$ satisfies the following.*

(i) $\mathrm{Hom}_P(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R) \subseteq \mathrm{Inj}(Q,R)$ for all $Q,R \leq P$.

(ii) For any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, the induced isomorphism $Q \cong \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$ and its inverse are morphisms in $\mathcal{F}$. In particular, every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion in $\mathcal{F}$.

Thus, $\mathcal{F}$ is a fusion system in the sense of [3, Definiton 1.1]. If $(P',e_{P'})$ is another maximal (A,b,G)-Brauer pair, then $\mathcal{F}_{(P',e_{P'})}(A,b,G)$ is isomorphic to $\mathcal{F}_{(P,e_P)}(A,b,G)$.

3. PROOF OF THEOREM 1.6

Throughout this section, G will denote a finite group, A a p-permutation G-algebra, and b a primitive idempotent of A^G. Recall from the introduction that (A,b,G) is a saturated triple if conditions (i) and (ii) of Theorem 1.6 hold. Thus, we will prove that if (A,b,G) is a saturated triple, then $\mathcal{F}_{(P,e_P)}(A,b,G)$ is saturated for any maximal (A,b,G)-Brauer pair (P,e_P). We need some preliminary results.

Lemma 3.1. *Let H be a finite group and let B be an H-algebra. Let R be a subgroup of H and let C be a normal subgroup of H. Suppose that $1_B \in \mathrm{Tr}_R^H(B^R)$ and 1_B is primitive in B^C. Then, RC/C contains a Sylow p-subgroup of H/C.*

Proof. Let $b \in B^R$ be such that

$$1_B = \mathrm{Tr}_R^H(b) = \mathrm{Tr}_{RC}^H(\mathrm{Tr}_R^{RC}(b)),$$

and set $u := \mathrm{Tr}_R^{RC}(b)$. Then, $u \in B^{RC} \subseteq B^C$. By hypothesis, the identity $1_B = 1_{B^C}$ of B^C is the only idempotent of B^C. In other words, B^C is a local algebra which means that $J(B^C)$ has co-dimension 1 in B^C. Thus, we may write $u = \lambda 1_B + v$ for some $\lambda \in k$ and $v \in J(B^C)$. Thus,

$$1_B = \mathrm{Tr}_{RC}^H(\lambda 1_B + v) = [H : RC]\lambda 1_B + \mathrm{Tr}_{RC}^H(v).$$

Now, since C is normal in H, H acts on B^C and hence on $J(B^C)$. In particular, $\mathrm{Tr}_{RC}^H(v) \in J(B^C)$. But $1_B \notin J(B^C)$. Hence, it follows from the above displayed equation that $[H : RC]$ is not divisible by p, proving the lemma. $\square$

For (A, b, G)-Brauer pairs $(Q, f) \leq (P, e)$, set

$$N_G(P, e) := N_{(A, b, G)}((P, e)) := \{x \in G : {}^x(P, e) = (P, e)\},$$

and

$$C_G(P, e) := N_G(P, e) \cap C_G(P).$$

Lemma 3.2. *Let H be a finite group, B a p-permutation H-algebra and e a primitive idempotent of B^H. If $e \in Z(B)$, then for a p-subgroup Q of H and a block f of $B(Q)$, (Q, f) is an (B, e, H)-Brauer pair if and only if $\mathrm{Br}_Q^B(e)f = f$.*

Proof. Suppose that $e \in Z(B)$ and let Q be a p-subgroup of H. Since

$$Z(B) \cap B^H \subseteq Z(B) \cap B^Q \subseteq Z(B^Q),$$

e is a central idempotent of B^Q. Hence, either $\mathrm{Br}_Q^B(e) = 0$ or $\mathrm{Br}_Q^B(e)$ is a central idempotent of $B(Q)$ and for any block f of $B(R)$, either $\mathrm{Br}_Q^B(e)f = f$, or $\mathrm{Br}_Q^B(e)f = 0$. The result follows. $\square$

For the next result, we note the following. For an (A, b, G)-Brauer pair (Q, e), $A(Q)$ is a $N_G(Q)$-algebra and e is an idempotent of $A(Q)^{N_G(Q,e)}$. Thus, if e is primitive in $A(Q)^{C_G(Q,e)}$, then e is a primitive idempotent of $A(Q)^H$ for any H such that $C_G(Q, e_Q) \leq H \leq N_G(Q, e_Q)$ and it makes sense to speak of $(A(Q), e, H)$-Brauer pairs.

Lemma 3.3. *Suppose that (Q, e) is an (A, b, G)-Brauer pair such that e is primitive in $A(Q)^{C_G(Q,e)}$ and let H be a subgroup of G with $C_G(Q, e) \leq H \leq N_G(Q, e)$.*

(i) The H-poset of $(A(Q), e, H)$-Brauer pairs is the H-subposet of $(A(Q), e, N_G(Q, e))$-Brauer pairs consisting of those pairs whose first component is contained in H.

(ii) The map

$$(R, \alpha) \to (QR, \alpha)$$

is an H-poset homomorphism from the set of $(A(Q), e, H)$-Brauer pairs to the set of $(A(Q), e, QH)$-Brauer pairs and induces a bijection between the set of $(A(Q), e, H)$-Brauer pairs whose first component contains $Q \cap H$ and the set of $(A(Q), e, QH)$-Brauer pairs whose first component contains Q.

(iii) If $Q \leq H$, then (Q, e) is the unique $(A(Q), e, H)$-Brauer pair with first component Q and (Q, e) is contained every maximal $(A(Q), e, H)$-Brauer pair.

Proof. (i) This is immediate from the definitions.

(ii) Since Q acts trivially on $A(Q)$, for any p-subgroup R of H, $A(Q)^R = A(Q)^{QR}$ and $\mathrm{Br}_R^{A(Q)} = \mathrm{Br}_{QR}^{A(Q)}$. The first assertion is immediate from this observation. The second assertion follows from the first and the fact that $R \to QR$ is a bijection between subgroups of H containing $Q \cap H$ and subgroups of QH containing Q.

(iii) By hypothesis, $A(Q)^Q = A(Q)$. Hence, $A(Q)_{<Q}^Q = 0$ and $\mathrm{Br}_Q^{A(Q)}$ is the identity map on $A(Q)$. Thus, the set of $(A(Q), e, H)$-Brauer pairs with first component Q consists precisely of the pairs (Q, α), where α is a block of $A(Q)$ such that $e\alpha \neq 0$. Since e itself is a block of $A(Q)$ and any two distinct blocks of $A(Q)$ are orthogonal, it follows that

(Q, e) is an $(A(Q), e, H)$-Brauer pair and that it is the unique one with first component Q. Since $^h(Q, e) = (Q, e)$ for all $h \in H$ and by Theorem 2.2(a) H acts transitively on the set of maximal $(A(Q), e, H)$-Brauer pairs, (Q, e) is contained in every maximal $(A(Q), e, H)$-Brauer pair. $\qquad\square$

To prove that a fusion system of a finite group G on a Sylow p-subgroup S of the group is saturated one applies Sylow's theorem to the local subgroups $N_G(Q)$ and $N_S(Q)C_G(Q)$ of G, for Q a p-subgroup of G. The proof of Theorem 1.6 is based on the same idea with triples of the form $(A(Q), e, N_G(Q, e_Q))$, $(A(Q), e, N_P(Q)C_G(Q, e_Q))$ playing the role of local subgroups and Theorem 2.2 and Lemma 3.1 playing the role of Sylow's theorem. The next result allows us to pass back and forth between (A, b, G)-Brauer pairs and $(A(Q), e, H)$-Brauer pairs. Recall the isomorphisms $b_{R,Q}^A : A(Q)(R) \to A(R)$ for p-subgroups $Q \trianglelefteq R$ of G introduced at the end of Section 2.1.

Lemma 3.4. *Suppose that (Q, e) is an (A, b, G)-Brauer pair such that e is primitive in $(A(Q))^{C_G(Q,e)}$ and let H be a subgroup of G with $QC_G(Q, e) \leq H \leq N_G(Q, e)$.*
The map
$$(R, \alpha) \to (R, b_{R,Q}^A(\alpha))$$
is an H-poset isomorphism between the subset of $(A(Q), e, H)$-Brauer pairs consisting of those pairs whose first component contains Q, and the subset of (A, b, G)-Brauer pairs containing (Q, e) and whose first component is contained in H.

In particular, H acts transitively on the subset of (A, b, G)-Brauer pairs which are maximal with respect to containing (Q, e) and having first component contained in H.

Proof. Let $\mathcal{P}_1$ be the subset of $(A(Q), e, H)$-Brauer pairs consisting of those pairs whose first component contains Q, and let $\mathcal{P}_2$ be the subset of (A, b, G)-Brauer pairs containing (Q, e) and whose first component is contained in H. Since $H \leq N_G(Q, e) \leq N_G(Q)$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are H-posets. Now let $Q \leq R \leq H$, and let α be a block of $A(Q)(R)$. By Lemma 3.2, $e = \mathrm{Br}_Q^A(b)e$, hence
$$\mathrm{Br}_{R,Q}^A(e) = b_{R,Q}^A(\mathrm{Br}_R^{A(Q)}(e)) = b_{R,Q}^A(\mathrm{Br}_R^{A(Q)}(\mathrm{Br}_Q^A(b)e)) = \mathrm{Br}_R^A(b)\mathrm{Br}_{R,Q}^A(e).$$

Suppose first that (R, α) is an $(A(Q), e, H)$-Brauer pair. By Lemma 3.2, $\alpha = \mathrm{Br}_R^{A(Q)}(e)\alpha$. Applying $b_{R,Q}^A$ to both sides of this equation, and using the displayed equation above, we get that
$$b_{R,Q}^A(\alpha) = \mathrm{Br}_{R,Q}^A(e)b_{R,Q}^A(\alpha) = \mathrm{Br}_R^A(b)\mathrm{Br}_{R,Q}^A(e)b_{R,Q}^A(\alpha).$$
In particular, $\mathrm{Br}_R^A(b)b_{R,Q}^A(\alpha) \neq 0$, whence $(R, b_{R,Q}^A(\alpha))$ is an (A, b, G)-Brauer pair. By Theorem 2.1 and the first equality above, $(Q, e) \leq (R, b_{R,Q}^A(\alpha))$ as (A, b, G)-Brauer pairs.

Conversely, if $(Q, e) \leq (R, b_{R,Q}^A(\alpha))$, then again by Theorem 2.1, $b_{R,Q}^A(\alpha) = \mathrm{Br}_{R,Q}^A(e)b_{R,Q}^A(\alpha)$. Applying the inverse of $b_{R,Q}^A$ yields that $\alpha = \mathrm{Br}_R^{A(Q)}(e)\alpha$, hence that (R, α) is an $(A(Q), e, H)$-Brauer pair. This shows that $(R, \alpha) \to (R, b_{R,Q}^A(\alpha))$ is a bijection between $\mathcal{P}_1$ and $\mathcal{P}_2$.

We show that the bijection is inclusion preserving. Let (R, α) and (S, β) be $(A(Q), e, H)$-Brauer pairs with $Q \lhd R \leq S$. By Theorem 2.1, it suffices to consider the case that $R \trianglelefteq S$. Clearly, α is S-stable if and only if $b_{R,Q}^A(\alpha)$ is S-stable. Further, the restrictions of the maps $b_{S,Q}^A \circ \mathrm{Br}_{S,R}^{A(Q)} \circ \mathrm{Br}_R^{A(Q)} \circ \mathrm{Br}_Q^R$ and $\mathrm{Br}_{S,R}^A \circ b_{R,Q}^A \circ \mathrm{Br}_R^{A(Q)} \circ \mathrm{Br}_Q^A$ to A^S both equal Br_S^A. Since $\mathrm{Br}_R^{A(Q)} \circ \mathrm{Br}_Q^R(A^S) = A(Q)(R)^S$, it follows that $b_{S,Q}^A \circ \mathrm{Br}_{S,R}^{A(Q)}$ is equal to the

restriction of $\mathrm{Br}^A_{S,R} \circ b^A_{R,Q}$ to $A(Q)(R)^S$. In particular, $\mathrm{Br}^{A(Q)}_{S,R}(\alpha)\beta = \beta$ if and only if $\mathrm{Br}^A_{S,R}(b^A_{R,Q}(\alpha))b^A_{S,Q}(\beta) = b^A_{S,Q}(\beta)$. Thus, by Theorem 2.1 $(R,\alpha) \le (S,\beta)$ if and only if $(R, b^A_{R,Q}(\alpha)) \le (S, b^A_{S,Q}(\beta))$, and the bijection is inclusion preserving. Since Q is normal in H,

$$b^A_{^hR,Q}(^h\alpha) = b^A_{^hR,^hQ}(^h\alpha) = {}^hb^A_{R,Q}(\alpha)$$

for all $h \in H$, all p-subgroups R of G containing Q as a normal subgroup and all $\alpha \in A(Q)(R)$, and hence the above bijection is compatible with the H-action on $\mathcal{P}_1$ and $\mathcal{P}_2$. This proves that the given map is an isomorphism of H-posets. In particular, the map induces a bijection between the set of maximal elements of $\mathcal{P}_1$ and $\mathcal{P}_2$. But by Lemma 3.3 (c), the set of maximal elements in $\mathcal{P}_1$ is precisely the set of maximal $(A(Q), e, H)$-Brauer pairs. The final assertion follows from this and from the fact that H acts transitively on the set of maximal $(A(Q), e, H)$-pairs (see 2.2 (a)). $\square$

We will prove Theorem 1.6 by using the the saturation axioms given by Roberts and Schpectorov in [16] . For this we recall the following terminology: If $\mathcal{F}$ is a fusion system on a finite p-group P, then a subgroup Q of P is *fully automized* if $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ and Q is *receptive* if for any isomorphism $\varphi : R \to Q$ in $\mathcal{F}$, there exists a morphism $\hat{\varphi} : N_\varphi \to P$ in $\mathcal{F}$ such that $\mathrm{Res}|_R\hat{\varphi} = \varphi$, where N_φ is the subgroup of $N_P(R)$ consisting of those elements $z \in N_P(R)$ such that $\varphi \circ c_z = c_x \circ \varphi$ for some $x \in N_P(Q)$.

Lemma 3.5. *Suppose that (A, b, G) is a saturated triple and let (P, e_P) be a maximal (A, b, G)-Brauer pair. For each $Q \le P$ let e_Q be the unique block of $A(Q)$ such that $(Q, e_Q) \le (P, e_P)$ and let $\mathcal{F} = \mathcal{F}_{(P,e_P)}(A, b, G)$. If $Q \le P$ is such that $(N_P(Q), e_{N_P(Q)})$ is maximal amongst (A, b, G)-Brauer pairs (R, f) with $(Q, e_Q) \le (R, f)$ and $R \le N_G(Q, e_Q)$, then Q is fully $\mathcal{F}$-automized and $\mathcal{F}$-receptive.*

Proof. Suppose that $(N_P(Q), e_{N_P(Q)})$ is maximal amongst (A, b, G)-Brauer pairs (R, f) such that $(Q, e_Q) \le (R, f)$ and $R \le N_G(Q, e_Q)$. Let

$$\alpha = b^A_{N_P(Q),Q}(e_{N_P(Q)}).$$

By Lemma 3.4, $(N_P(Q), \alpha)$ is a maximal $(A(Q), e_Q, N_G(Q, e_Q))$-Brauer pair. Thus, by Theorem 2.2 (b), $e_Q \in \mathrm{Tr}^{N_G(Q,e_Q)}_{N_P(Q)}(A(Q)^{N_P(Q)})$. Since e_Q is central in $A(Q)$, idempotent and an element of $A^{N_G(Q,e_Q)}$, multiplying on both sides by e_Q gives that

$$e_Q \in \mathrm{Tr}^{N_G(Q,e_Q)}_{N_P(Q)}((e_Q A(Q) e_Q)^{N_P(Q)}).$$

Now, $C_G(Q, e_Q)$ is a normal subgroup of $N_G(Q, e_Q)$ and since (A, b, G) is a saturated triple e_Q is a primitive idempotent of $(A(Q))^{C_G(Q,e_Q)}$ and hence also of $(e_Q A(Q) e_Q)^{C_G(Q,e_Q)}$. Thus, by Lemma 3.1 applied with $B = e_Q A(Q) e_Q$, $H = N_G(Q, e_Q)$, $C = C_G(Q, e_Q)$ and $R = N_P(Q)$, we have that $N_P(Q)C_G(Q, e_Q)/C_G(Q, e_Q)$ is a Sylow p-subgroup of $N_G(Q, e_Q)/C_G(Q, e_Q)$. Since $N_P(Q)C_G(Q, e_Q)/C_G(Q, e_Q) \cong N_P(Q)/C_P(Q) \cong \mathrm{Aut}_P(Q)$ and $N_G(Q, e_Q)/C_G(Q, e_Q) \cong \mathrm{Aut}_{\mathcal{F}}(Q)$, it follows that Q is fully $\mathcal{F}$-automised.

It remains to show that Q is $\mathcal{F}$-receptive. For this, we first observe that the hypothesis on Q implies that $(N_P(Q), e_{N_P(Q)})$ is also maximal amongst (A, b, G)-Brauer pairs (R, f) such that $(Q, e_Q) \le (R, f)$ and $R \le N_P(Q)C_G(Q, e_Q)$. Hence, by Lemma 3.4, now applied with $H = N_P(Q)(C_G(Q, e_Q)$, $(N_P(Q), e_{N_P(Q)})$ contains an $N_P(Q)C_G(Q, e_Q)$ conjugate of

any (A, b, G)-Brauer pair which contains (Q, e_Q) and whose first component is contained in $N_P(Q)C_G(Q, e_Q)$. Now let $\varphi : R \to Q$ be an isomorphism in $\mathcal{F}$, and let $g \in G$ induce φ, that is, ${}^g(R, e_R) = (Q, e_Q)$ and $\varphi(x) = gxg^{-1}$ for all $x \in R$. Then, it is an easy check that $N_\varphi = N_P(R) \cap {}^{g^{-1}}N_P(Q)C_G(Q, e_Q)$. Set $N' = {}^gN_\varphi = {}^gN_P(R) \cap N_P(Q)C_G(Q, e_Q)$, $e'_{N'} = {}^ge_{N_\varphi}$ and consider the (A, b, G)-Brauer pair (N', e_N). Since $(R, e_R) \leq (N_\varphi, e_{N_\varphi})$, $(Q, e_Q) \leq {}^g(N_\varphi, e_{N_\varphi}) = (N', e'_{N'})$. Also, $N' \leq N_P(Q)C_G(Q, e_Q)$. Thus, as pointed out above ${}^h(N', e'_{N'}) \leq (N_P(Q), e_{N_P(Q)})$ for some $h \in N_P(Q)C_G(Q, e_Q)$. Multiplying by some element of $N_P(Q)$ if necessary, we may assume that $h \in C_G(Q, e_Q)$. Since ${}^{hg}(N_\varphi, e_{N_\varphi}) \leq (P, e_P)$ and hence $\bar{\varphi} := c_{hg} : N_\varphi \to P$ is a morphism in $\mathcal{F}$. and since $h \in C_G(Q, e_Q)$, $\bar{\varphi}$ extends φ. Thus Q is $\mathcal{F}$-receptive. $\qquad\square$

We now give the proof of Theorem 1.6.

Proof. Keep the notation of the theorem, set $\mathcal{F} = \mathcal{F}_{(P,e_P)}(A, b, G)$ and for each $Q \leq P$, let e_Q be the unique block of $A(Q)$ such that $(Q, e_Q) \leq (P, e_P)$. We have shown in Proposition 2.4 that $\mathcal{F}$ is a fusion system on P. Thus, by Lemma 3.5 and by the saturation axioms of [16] it suffices to show that each subgroup of P is $\mathcal{F}$-conjugate to a subgroup Q of P such that $(N_P(Q), e_{N_P(Q)})$ is maximal amongst (A, b, G)-Brauer pairs (R, f) with $(Q, e_Q) \leq (R, f)$ and $R \leq N_G(Q, e_Q)$. So, let $Q' \leq P$, and let (T, α) be a maximal $(A(Q'), e_{Q'}, N_G(Q', e_{Q'}))$- Brauer pair. By Lemma 3.3 (c), $Q' \leq T$. Let $f = b_{R,Q}^A{}^{-1}(\alpha)$. By Lemma 3.4, (T, f) is an (A, b, G)-Brauer pair with $(Q', e_{Q'}) \leq (T, f)$. Since (P, e_P) is a maximal (A, b, G)-Brauer pair, we have

$$ {}^g(Q', e_{Q'}) \leq {}^g(T, f) \leq (P, e_P) $$

for some $g \in G$. Set $Q = {}^gQ'$. By the above, $c_g : Q' \to Q$ is a morphism in $\mathcal{F}$, so Q is $\mathcal{F}$-conjugate to Q'. We will show that $(N_P(Q), e_{N_P(Q)})$ has the required maximality property. Note that by Lemma 3.4, (T, f) is maximal amongst (A, b, G)-Brauer pairs which contain $(Q', e_{Q'})$ and whose first component is contained in $N_G(Q', e_{Q'})$. Thus, by transport of structure ${}^g(T, f)$ is maximal amongst (A, b, G)-Brauer pairs which contain (Q, e_Q) and whose first component is contained in $N_G(Q, e_Q)$. Since ${}^g(T, f) \leq (P, e_P)$, ${}^gT \leq N_P(Q)$ and ${}^gf = e_{{}^gT}$. Consequently, ${}^g(T, f) \leq (N_P(Q), e_{N_P(Q)})$. Since $(N_P(Q), e_{N_P(Q)})$ contains (Q, e_Q) and $N_P(Q)$ is contained in $N_G(Q, e_Q)$, the maximality of ${}^g(T, f)$ forces ${}^g(T, f) = (N_P(Q), e_{N_P(Q)})$, and completes the proof of the theorem. $\qquad\square$

4. p-PERMUTATION MODULES AND SATURATION

Let G be a finite group, M an indecomposable p-permutation kG-module, and P a vertex of M and set $A = \mathrm{End}_k(M)$. Then A is a G-algebra via the map

$$ G \times A \to A, $$

sending the pair (g, ϕ) to the element ${}^g\phi$ of A defined by

$$ {}^g\phi(m) = g\phi(g^{-1}m), \quad m \in M. $$

Since M is a p-permutation module, M is a p-permutation G-algebra and since M is indecomposable, $1_A = id_M$ is primitive in $\mathrm{End}_k(M)^G$.

Proposition 4.1. *With the notation above, the $(A, 1_A, G)$-Brauer pairs are the pairs $(Q, 1_{A(Q)})$ such that $M(Q) \neq 0$ and $(P, 1_{A(P)})$ is a maximal $(\mathrm{End}_k(M), 1_{\mathrm{End}_k(M)}, G)$-Brauer pair. Further,*

(i) $\mathcal{F}_{(P, 1_{A(P)})}(A, 1_A, G) = \mathcal{F}_P(G)$.

(ii) The triple $(A, 1_A, G)$ is of saturated type if and only if M is Brauer indecomposable.

Proof. Let Q be a p-subgroup of G. There is a natural action of $A(Q)$ on $M(Q)$ which induces an isomorphism of $kN_G(Q)/Q$-algebras. between $A(Q)$ and $\mathrm{End}_k(M(Q))$ (see for instance [17, Proposition 27.6]). Since the identity element is the only central idempotent of a matrix algebra, it follows that the $(A, 1_A, G)$-Brauer pairs are the pairs $(Q, 1_{A(Q)})$ such that $M(Q) \neq 0$. The maximality of $(P, 1_{A(P)})$ is immediate from the fact that P is a vertex of P and that $M(Q) \neq 0$ if and only if Q is contained in a vertex of M (see [17, Corollary 27.6]). Clearly, ${}^g 1_{A(Q)} = 1_{A({}^g Q)}$, for any $g \in G$ and (i) is immediate from this. Under the natural identification of $A(Q)$ and $\mathrm{End}_k(M(Q))$ $1_{A(Q)} = id_{M(Q)}$. Hence $1_{A(Q)}$ is primitive in $(A(Q))^{C_G(Q)}$ if and only if $M(Q)$ is an indecomposable $kQC_G(Q)/Q$-module. The equivalence of (ii) is immediate from this and the fact that 1_A is a central idempotent of A and hence of A^G. $\quad\blacksquare$

Proof of Theorem 1.1. In light of Proposition 4.1, this is a special case of Theorem 1.6.

Remarks 1. Let P be a p-subgroup of G. Since there exist indecomposable p-permutation kG-modules with vertex P, the analysis before the statement of Theorem 1.1 shows that given any p-subgroup P of a finite group G, there exists a p-permutation G-algebra A, and a primitive idempotent b of A^G such that there is a maximal (A, b, G)-Brauer pair, say (P, e_P) with first component P and such that $\mathcal{F}_{(P, e_P)}(A, b, G) = \mathcal{F}_P(G)$. On the other hand, there exist pairs P, G where G is a finite group and P is a p-subgroup of G such that $\mathcal{F}_P(G)$ is not a saturated system-for instance if P is a non-Sylow p-subgroup of G such that $N_S(P)$ strictly contains $PC_S(P)$ for some Sylow p-subgroup S of G containing P. Thus, the fusion system $\mathcal{F}_{(P, e_P)}(A, b, G)$ is not always saturated.

2. Suppose that b is a (non-principal) block of kG such that a defect group P of kGb is a Sylow p-subgroups of G, but $\mathrm{Br}_P^{kG}(b)$ is a sum of more than one block of $kC_G(P)$. Let M be an indecomposable p-permutation module kG-module in the block b and with vertex P. Then, since $N_G(P)$ acts transitively on the set $\mathcal{E}$ of blocks e of $kC_G(P)$ such that $\mathrm{Br}_P^{kG}(b)e = e$ and $M(P) \neq 0$, $M(P)e \neq 0$ for any $e \in \mathcal{E}$, and in particular, $M(P)$ is not indecomposable as $kC_G(P)$-module. However, since P is a Sylow p-subgroup of G, $\mathcal{F}_P(G)$ is a saturated fusion system on P (see [3]). Thus, the converse of Theorem 1.1 does not hold in general. Since Theorem 1.1 is a special case of Theorem 1.6, it follows also that the converse of Theorem 1.6 does not hold. It might be that the methods of proof of Theorem 1.6 can be refined to yield a condition on (A, b, G) which in certain situations (as in the one just discussed) is weaker than the condition of (A, b, G) being a saturated triple, and which in all cases is necessary and sufficient for the saturation of the corresponding fusion systems.

We now prove Theorem 1.2. We need some lemmas. The following is well known.

Lemma 4.2. *Let H be a finite group and N a normal subgroup of H such that H/N is a p'-group. Then, the restriction of the projective cover of the trivial kH-module to kN is indecomposable.*

Proof. Under the hypothesis, $J(kH) = J(kN)kH$. Let V be a projective kH-module. Then,

$$\mathrm{Res}^H_N Rad(V) = \mathrm{Res}^H_N J(kH)V = \mathrm{Res}^H_N J(kN)kHV = \mathrm{Res}^H_N J(kN)V = Rad(\mathrm{Res}^H_N V).$$

Consequently,

$$\mathrm{Res}^H_N(V/Rad(V)) = \mathrm{Res}^H_N V/Rad(\mathrm{Res}^H_N V).$$

The result is immediate.

Remark. The above indecomposability result holds for the projective cover of any simple kH-module whose restriction to N remains simple.

Lemma 4.3. *Let G be a finite group, P a p-subgroup of G and $M := \mathrm{S}_P(G, k)$ the Scott module of kG relative to P.*
(i) $M(P)$ is indecomposable as $kPC_G(P)/P$-module if and only if $N_G(P)/PC_G(P)$ is a p'-group.
(ii) If $\mathcal{F}_P(G)$ is a saturated fusion system, then $M(P)$ is indecomposable as $kPC_G(P)/P$-module.

Proof. (i) $M(P)$ is the projective cover of the trivial $kN_G(P)/P$-module and in particular is indecomposable as $kN_G(P)/P$-module. The forward implication follows from Lemma 3.1, applied with $B = \mathrm{End}_k(M(P))$, $H = N_G(P)$, $R = P$ and $C = C_G(P)$. The backward implication is clear from Lemma 4.2.

(ii) Suppose that $\mathcal{F}_P(G)$ is a saturated fusion system. Then, $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. On the other hand, the image of $\mathrm{Aut}_P(P)$ under the natural isomorphism from $\mathrm{Aut}_{\mathcal{F}}(P)$ to $N_G(P)/C_G(P)$ is $PC_G(P)/C_G(P)$. Thus, $N_G(P)/PC_G(P)$ is a p'-group. The result is immediate from (i).

Lemma 4.4. *Let G be a finite group, P a p-subgroup of G, $M = \mathrm{S}_P(G, k)$ the Scott module of kG relative to P. Suppose that $\mathcal{F}_P(G)$ is a saturated fusion system and let $Q \leq Z(P)$. If $M(Q)$ is indecomposable as $kN_G(Q)/Q$-module, then $M(Q)$ is indecomposable as $kC_G(Q)/Q$-module.*

Proof. Suppose that $M(Q)$ is indecomposable as $N_G(Q)/Q$-module and set $L = N_G(Q)$ and $C = C_G(Q)$. Since $Q \leq Z(P)$ the extension axiom for saturated fusion systems implies that $L = C[N_G(P) \cap L]$. We consider $M(Q)$ as kL-module via inflation. Since $M(Q)$ has vertex P and $P \leq C$, there exists an indecomposable p-permutation kC-module V with vertex P such that $M(Q)$ is a direct summand of $\mathrm{Ind}^L_C V$. Let W be an indecomposable summand of $\mathrm{Res}^L_C \mathrm{Ind}^L_C V$. By the Mackey formula, $W \cong {}^x V$ for some $x \in L$. In particular, ${}^x P$ is a vertex of ${}^x V$. By the decomposition of L given above, $x = uv$ for some $u \in C_G(Q)$, $v \in N_G(P)$. Thus, ${}^x P = {}^u P$ is C-conjugate to P, and it follows that P is a vertex of W. In particular, $W(P) \neq 0$. Let

$$\mathrm{Res}^L_C M(Q) = W_1 \oplus \cdots \oplus W_s$$

be a decomposition of $M(P)$ as a direct sum of indecomposable kC-modules and suppose if possible that $s > 1$. By the above argument, $W_i(P) \neq 0$ for i, $1 \leq i \leq s$, hence

$$\mathrm{Res}^{N_G(P)}_{C \cap N_G(P)} M(P) \cong (\mathrm{Res}^L_C M(Q))(P) = W_1(P) \oplus \cdots \oplus W_s(P)$$

is not indecomposable. Since $C_G(P) \leq C \cap N_G(P)$, it follows that $\mathrm{Res}^{N_G(P)}_{C_G(P)} M(P)$ is not indecomposable. This contradicts Lemma 4.3.

Proof of Theorem 1.2. Let $M = \mathrm{S}_P(G, k)$. Suppose that $\mathcal{F} := \mathcal{F}_P(G)$ is saturated and let $Q \leq P$. We will show that $M(Q)$ is indecomposable as $kC_G(Q)$-module. We proceed by induction on the index of Q in P. If $Q = P$, then by Lemma 4.3, $M(Q)$ is indecomposable as $kPC_G(P)/P$-module. Suppose now that Q is proper in P and that $M(R)$ is indecomposable as $kRC_G(R)/R$-module for any p-subgroup R of P properly containing Q. Since $P \leq N_G(Q)$, $\mathrm{S}_P(N_G(Q), k)$ is a direct summand of $\mathrm{Res}^G_{N_G(Q)} M$ (see [14, Chapter 4, Theorem 8.6]). Write

$$\mathrm{Res}^G_{N_G(Q)} M = \mathrm{S}_P(N_G(Q), k) \oplus X.$$

We claim that $X(Q) = 0$. Indeed, suppose if possible that there exists a direct summand, say N of X such that $N(Q) \neq 0$ and let R be a vertex of N. Since Q is normal in $N_G(Q)$, we have that $Q \leq R$. The group Q is not a vertex of the indecomposable kG-module M. Hence by the Burry-Carlson-Puig theorem (see [14, Chapter 4, Theorem 4.6 (ii)]), $\mathrm{Res}^G_{N_G(Q)} M$ does not have any indecomposable summand with vertex Q. Thus Q is a proper subgroup of R. On the other hand, since M is a summand of $\mathrm{Ind}^G_P k$, and N is a summand of $\mathrm{Res}^G_{N_G(Q)} M$, by the Mackey formula, N is relatively ${}^x P \cap N_G(Q)$-projective for some $x \in G$. Thus,

$$Q < R < {}^x P \quad \text{and} \quad Q < P.$$

In particular, conjugation by x is an $\mathcal{F}$-isomorphism from ${}^{x^{-1}}Q$ to Q. Now P is abelian and $\mathcal{F}$ is saturated. So, by the extension axiom there exists a $g \in N_G(P)$ such that $gx^{-1} \in C_G(Q)$. Setting $h = gx^{-1}$, and conjugating all terms in the above by h, we get

$$Q = {}^h Q < {}^h R < {}^{hx} P = {}^g P = P.$$

Since $h \in N_G(Q)$, replacing R by ${}^h R$, we may assume that $R \leq P$. Since N is a summand of X and $N(R) \neq 0$, we have $X(R) \neq 0$. Since $\mathrm{S}_P(N_G(Q), k)$ has vertex P and $R \leq P$, we also have that $\mathrm{S}_P(N_G(Q), k)(R) \neq 0$. The equation

$$\mathrm{Res}^G_{N_G(Q)} M = \mathrm{S}_P(N_G(Q), k) \oplus X,$$

implies that $M(R)$ is not indecomposable as $k[N_G(Q) \cap N_G(R)]$-module. Since $RC_G(R) \leq N_G(Q) \cap N_G(R)$, it follows that $M(R)$ is not indecomposable as $kRC_G(R)$-module or equivalently as $kRC_G(R)/R$-module, a contradiction. This proves the claim. Thus,

$$M(Q) = \mathrm{S}_P(N_G(Q), k)(Q) \oplus X(Q) = \mathrm{S}_P(N_G(Q), k)$$

as $kN_G(Q)$ and hence as $kN_G(Q)/Q$-module. In particular, $M(Q)$ is indecomposable as $kN_G(Q)/Q$-module. By Lemma 4.4, $M(Q)$ is indecomposable as $kQC_G(Q)/Q$-module, completing the proof.

Proof of Corollary 1.3. If G has cyclic Sylow p-subgroups, then it is easy to see that $\mathcal{F}_P(G)$ is saturated for any p-subgroup P of G. The result is immediate from the Theorem 1.2.

Proof of Corollary 1.4. With the hypothesis of the statement, it is immediate that $\mathcal{F}_{\Delta(P)}(G_1 \times G_2) \cong \mathcal{F}_P(G_1)$. Thus, since P is Sylow in G_1, $\mathcal{F}_P(G_1)$ and hence $\mathcal{F}_{\Delta(P)}(G_1 \times G_2)$ is a saturated fusion system on P (see [3]). The result follows from Theorem 1.2.

Finally, we prove Proposition 1.5. For this we set up some more notation and recall a few facts about Scott modules. Let $(K, \mathcal{O}, k)$-be a p-modular system (we assume here that k is an algebraic closure of the field of p elements). Let $G = S_n$, and let P be a p-subgroup of G. Let $M = \mathrm{S}_P(G, k)$ be the kG-Scott module with vertex P and let $\tilde{M} = \mathrm{S}_P(G, \mathcal{O})$ be the $\mathcal{O}G$-Scott module with vertex P, so that $M = k \otimes_{\mathcal{O}} \tilde{M}$. Let $\chi : \tilde{M} \to K$ be the character of the $\mathcal{O}G$-module $\tilde{M}$. Since $\tilde{M}$ is a p-permutation $\mathcal{O}G$-module, for any p-element x of G, $\dim_k M(\langle x \rangle) = \chi(x)$. In particular, if Q is a p-subgroup of G, then $\dim_k M(Q) \leq \chi(x)$ for any element x of Q, with equality if $Q = \langle x \rangle$.

Proof of Proposition 1.5. Suppose that $n \leq 6$ and that $\mathcal{F}_P(G)$ is saturated. We will show that $M(Q)$ is indecomposable as $kC_G(Q)/Q$-module for every subgroup Q of P. By Theorem 1.2, we may assume that P is not abelian. If P is a Sylow p-subgroup of G, then $M = k$ [4, Theorem 2.5] and the result is immediate. So, we may assume that P is a non-abelian, non-Sylow p-subgroup of G. Consequently, $p = 2$, $n = 6$ and P is isomorphic to the dihedral group of order 8.

By the Sylow axiom for saturated fusion systems, $PC_G(P)$ is a Sylow 2-subgroup of $N_G(P)$. So, up to G-conjugacy P is one of $\langle (1,2,3,4), (1,3) \rangle$, $\langle (1,2,3,4)(5,6), (1,3) \rangle$ $\langle (1,2,3,4), (1,3)(5,6) \rangle$ or $\langle (1,2,3,4)(5,6), (1,3)(5,6) \rangle$.

We will show that in each case above, M is Brauer indecomposable. It can be checked directly that $\mathcal{F}_P(G)$ is saturated in each case above- the second case corresponds to the nilpotent fusion system, the remaining three correspond to the saturated fusion system on D_8 in which the automorphism of exactly one Klein-4 subgroup contains an element of order 3. However, we do not prove saturation as by Theorem 1.1 this will follow after the fact of Brauer indecomposability.

Before embarking on our case by case analysis, we recall the 2-decomposition matrix of S_6 [7, Page 414]:

		1	4_1	4_2	16
		(1)	$(5,1)$	$(4,2)$	$(3,2,1)$
1	(6)	1			
5	$(5,1)$	1	1		
9	$(4,2)$	1	1	1	
16	$(3,2,1)$				1
10	$(4,1^2)$	2	1	1	
5	(3^2)	1		1	
10	$(3,1^3)$	2	1	1	
5	(2^3)	1		1	
9	$(2^2,1^2)$	1	1	1	
5	$(2,1^4)$	1	1		
1	(1^6)	1			

Case: $P = \langle(1,2,3,4),(1,3)\rangle$. Then P is a Sylow p-subgroup of S_5, naturally considered as a subgroup of S_6 as a one-point stabilizer, whence $\tilde{M}$ is a direct summand of $\mathrm{Ind}_{S_5}^{S_6}(\mathcal{O})$ (see for instance [4, Theorem 2.5]). On the other hand by [6, Page 32], M has dimension 6. So, $\tilde{M} = \mathrm{Ind}_{S_5}^{S_6}(\mathcal{O})$. Now, if $u = (1,3)$, then $\chi(u) = 4$ and if $u = (1,2)(3,4)$ or $u = (1,2,3,4)$ then $\chi(u) = 2$. Hence, it follows that unless $Q \leq P$ is G-conjugate to $\langle(1,3)\rangle$, the dimension of $M(Q) \leq 2$ and if $Q = \langle(1,3)\rangle$, then $M(Q)$ has dimension 4. On the other hand, since $M(P)$ as $kN_G(P)/P$-module is the projective cover of the trivial module, $M(P)$ has dimension at least 2. So, if $Q \leq P$ is not G-conjugate to $\langle(1,3)\rangle$, then for any $R \leq P$ containing Q as a normal subgroup, $M(Q) \cong M(R)$ as $k(N_G(Q) \cap N_G(R))$-module, hence as $kC_G(R)$-modules. Arguing inductively, it follows that $M(Q) \cong M(P)$ as $kC_G(P)$-modules. By Lemma 4.3, $M(P)$ is indecomposable as $kPC_G(P)/P$-module, hence as $kC_G(P)$-module. Since $C_G(P) \leq QC_G(Q)$, it follows that $M(Q)$ is indecomposable as $kQC_G(Q)/Q$-module.

Now suppose that $Q = \langle(1,3)\rangle$. Then $M(Q)$ is a 4-dimensional p-permutation $kN_G(Q)$-module. Let V be an indecomposable $kN_G(Q)$-module summand of $M(Q)$ and let $Q \leq R \leq N_G(Q)$ be a vertex of $M(Q)$. Then

$$M(R) = M(Q)(R) \neq 0,$$

whence ${}^gQ \leq {}^gR \leq P$ or ${}^gR \leq N_P({}^gQ)$. Since no transposition in P is central in P, R has order at most 4 (and for some summand V exactly 4). Let S be a Sylow p-subgroup of $N_G(Q)$ containing R. Since V is a direct summand of $\mathrm{Ind}_R^{N_G(Q)}(k)$, the Mackey formula and the Green indecomposability theorem imply that any direct summand of $\mathrm{Res}_S^{N_G(Q)}V$ is isomorphic to $\mathrm{Ind}_{{}^xR \cap S}^S k$ for some $x \in N_G(Q)$. In particular, the dimension of V is divisible by the index of R in V. Since the Sylow p-subgroups of $C_G(Q) = N_G(Q)$ have order 16 and R has order 8, it follows that V has dimension divisible by 4. Thus, $V = M(Q)$. In particular, $M(Q)$ is indecomposable as $kN_G(Q)$, and $N_G(Q) = C_G(Q)$.

Case: $P = \langle(1,2,3,4),(1,3)(5,6)\rangle$. By [6] M has composition factors 1_G, $4_1 \oplus 4_2$, 1_G. An inspection of the decomposition matrix and the character table of S_6 gives that $\chi = \chi_{(6)} + \chi_{(4,2)}$. Further, the values of χ on non-trivial 2 elements of G are as follows:

$$\chi((1,3)) = 4, \ \chi((1,3)(2,4)) = 2, \ \chi((1,2)(3,4)(5,6)) = 4,$$

$$\chi((1,2,3,4)) = 0, \ \chi((1,2,3,4)(5,6)) = 2.$$

Since $C_G(P)/Z(P)$ contains an element of order 2, it follows as in the previous case that $M(Q)$ is indecomposable as $kC_G(Q)$-module for any p-subgroup Q of G such that $M(Q)$ has dimension 2. From the above character calculations, we may assume that the only non-trivial elements of Q are in the G-conjugacy class of $(1,3)$ and $(1,2)(3,4)(5,6)$ and in particular are non-central involutions in P. If Q contains two such involutions, then $Q = P$, so we may assume that either $Q = \langle (1,3) \rangle$ or $Q = \langle (1,2)(3,4)(5,6) \rangle$. But now the result follows as above since both of these involutions are central in some Sylow p-subgroup and in both cases $M(Q)$ has dimension 4.

Case: $P = \langle (1,2,3,4), (1,3)(5,6) \rangle$. The image of P under the exceptional non-inner automorphism of S_6 is S_6-conjugate to $\langle (1,2,3,4), (1,3) \rangle$. The result follows from Case 1 by transport of structure.

Case: $P = \langle (1,2,3,4)(5,6), (1,3)(5,6) \rangle$. By [6] M is two dimensional with composition factors 1_G, 1_G. Since $M(P)$ has dimension at least 2, $M(Q) = M(P) = M$ for all $Q \leq P$. By Lemma 4.3, $M = M(P)$ is indecomposable as $kPC_G(P)/P$-module. Hence, $M(Q) = M$ is indecomposable as $kQC_G(Q)/Q$-module for all $Q \leq P$ as required. This completes the proof of Proposition 1.5.

Concluding Remarks. Given a saturated fusion system, $\mathcal{F}$ on a finite p-group P, Park has shown that there exists a finite group G with $P \leq G$ and such that $\mathcal{F} = \mathcal{F}_P(G)$ (cf.[15]). We pose the following question:

Given a saturated fusion system $\mathcal{F}$ on a finite p-group P, does there exist a saturated triple (A, b, G) such that $\mathcal{F} = \mathcal{F}_{(P,e_P)}(A, b, G)$ for some maximal (A, b, G)-Brauer pair (P, e_P)?

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