ON THE COXETER COMPLEX AND ALVIS-CURTIS DUALITY FOR PRINCIPAL ℓ-BLOCKS OF $GL_n(q)$

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Abstract. M. Cabanes and J. Rickard showed in [3] that the Alvis-Curtis character duality of a finite group of Lie type is induced in non defining characteristic ℓ by a derived equivalence given by tensoring with a bounded complex X, and they further conjecture that this derived equivalence should actually be a homotopy equivalence. Following a suggestion of R. Kessar, we show here for the special case of principal blocks of general linear groups with abelian Sylow-ℓ-subgroups that this is true, by an explicit verification relating the complex X to the Coxeter complex of the corresponding Weyl group.

Throughout this note, n is a positive integer, q a prime power and ℓ a prime divisor of $q-1$ such that $\ell > n$. We denote by $\mathcal{O}$ a complete discrete valuation ring having an algebraically closed residue field k of characteristic ℓ.

Set $G = GL_n(q)$ and let b be the principal block of $\mathcal{O}G$; that is, b is the unique primitive idempotent in $Z(\mathcal{O}G)$ which acts as the identity on the trivial $\mathcal{O}G$-module. We will say as usual that an irreducible character χ of G belongs to the principal block b if $\chi(b) = \chi(1)$. See [11] for more block theoretic background material. The ℓ-blocks of finite linear groups were first described by Fong and Srinivasan in [5]; see also [2]. For any ℓ'-subgroup H of G let e_H be the idempotent in $\mathcal{O}G$ defined by $e_H = \frac{1}{|H|}\sum_{x\in H} x$. Denote by T the maximal torus of diagonal matrices in G, by U the group of upper triangular matrices whose diagonal entries are 1 and set $B = UT$. Let W be the subgroup of permutation matricies of G; that is $W \cong S_n$. Denote by S the generating set of W corresponding to the set of permutations $(i-1, i)$, where $2 \leq i \leq n$. For any subset I of S denote by W_I the subgroup of W generated by I, by P_I the standard parabolic subgroup of G generated by B and W_I, by U_I the unipotent radical of P_I and by L_I a Levi complement of U_I in P_I, with the convention $W_\emptyset = 1$, $U_\emptyset = U$, $P_\emptyset = B$ and $L_\emptyset = T$.

Since $\ell > n$, the torus T contains a Sylow-ℓ-subgroup Q of G, and then T decomposes uniquely as direct product $T = Q \times T'$, where $T' = O_{\ell'}(T)$. The set of subsets of S is viewed as simplicial complex with respect to the order which is reverse to the inclusion of subsets. The complex X defined in [3] inducing the Alvis-Curtis duality is the complex

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of $\mathcal{O}G$-$\mathcal{O}G$-bimodules associated with the coefficient system sending $I \subseteq S$ to the $\mathcal{O}G$-$\mathcal{O}G$-bimodule $\mathcal{O}Ge_{U_I} \underset{\mathcal{O}P_I}{\otimes} e_{U_I}\mathcal{O}G$; the Coxeter complex (see e.g. [6, Section 1.15]) is the complex C of $\mathcal{O}W$-modules associated with the coefficient system sending I to the $\mathcal{O}W$-module $\mathcal{O}W/W_I$. See [1, Chapter 7] or [3] for more details on coefficient systems. We view X and C as cochain complexes with non zero components in the degrees 0 to $|S|$. The principal block b of $\mathcal{O}G$ has the Sylow-ℓ-subgroup Q as defect group. By a result of Puig [8, 3.4], the idempotent $i = e_{T'}e_U$ is a source idempotent of b (that is, i is primitive in $(\mathcal{O}Gb)^Q$ and $2_Q(i) \neq 0$) and there is an isomorphism of interior Q-algebras

$$\Phi : i\mathcal{O}Gi \cong \mathcal{O}(Q \rtimes W) ;$$

that is, Φ is an $\mathcal{O}$-algebra isomorphism mapping iui to u for every element $u \in Q$. Denote by $\Delta(Q \rtimes W)$ the diagonal subgroup of $(Q \rtimes W) \times (Q \rtimes W)$ and consider C as cochain complex of $\Delta(Q \rtimes W)$-modules in the obvious way (that is, with ΔQ acting trivially on the components of C).

Theorem 1. *With the notation above, there is an isomorphism of complexes of $i\mathcal{O}Gi$-$i\mathcal{O}Gi$-bimodules*

$$iXi \cong \mathrm{Res}_{\Phi}(\mathrm{Ind}_{\Delta(Q\rtimes W)}^{(Q\rtimes W)\times(Q\rtimes W)}(C)) .$$

In particular, iXi is homotopy equivalent to the bimodule ${}_{\sigma}(i\mathcal{O}Gi)$ viewed as complex concentrated in degree zero, where σ is the algebra automorphism of $i\mathcal{O}Gi$ induced via Φ by the sign representation of W.

Since $\mathcal{O}Gb$ and $i\mathcal{O}Gi$ are Morita equivalent via the bimodules $\mathcal{O}Gi$ and $i\mathcal{O}G$ (cf. [7, 3.5]) this implies immediately the following:

Corollary 2. *With the notation above, the functor $X \underset{\mathcal{O}G}{\otimes} -$ induces an equivalence on the homotopy category $K(\mathcal{O}Gb)$ of complexes of $\mathcal{O}Gb$-modules which extends, up to natural isomorphism, the Morita equivalence on* $\mathrm{Mod}(\mathcal{O}Gb)$ *induced by the sign representation of W.*

The proof of Theorem 1 is based on the following three Propositions, the first of which is a particular case of a result of Puig [8, 3.4]:

Proposition 3. *(Puig) With the notation above, for any subset I of S, the idempotent i is a source idempotent of the principal block of $\mathcal{O}P_I$ and there is an isomorphism of interior Q-algebras $\Phi_I : i\mathcal{O}P_Ii \cong \mathcal{O}(Q \rtimes W_I)$.*

Since $W = W_S$ we choose notation such that $\Phi = \Phi_S$. The various isomorphisms Φ_I are compatible in the following sense:

Proposition 4. *With the notation above, for any subset I of S there is an element $a_I \in 1 + J((\mathcal{O}(Q \rtimes W))^Q)$ such that $\Phi(i\mathcal{O}P_I i) = (\mathcal{O}(Q \rtimes W_I))^{a_I}$.*

Proof. For every $w \in W_I$, the elements w and $w' = \Phi(\Phi_I^{-1}(w))$ act in the same way on Q because Φ, Φ_I are homomorphisms of interior Q-algebras. Thus $w(w')^{-1} \in ((\mathcal{O}(Q \rtimes W))^Q)^\times$. Since the unit element of $\mathcal{O}(Q \rtimes W)$ is primitive in the algebra $(\mathcal{O}(Q \rtimes W))^Q$ it follows that the group $((\mathcal{O}(Q \rtimes W))^Q)^\times$ is isomorphic to $k^\times \times (1 + J((\mathcal{O}(Q \rtimes W))^Q))$. In other words, identifying $k^\times$ to its canonical preimage in $\mathcal{O}^\times$ (cf. [10, Ch. II, §4, Prop. 8]) there is a unique scalar $\zeta(w) \in k^\times$ such that $\zeta(w)^{-1}w(w')^{-1} \in 1 + J((\mathcal{O}(Q \rtimes W))^Q)$. One checks easily that the map sending w to $\zeta(w)$ is in fact a group homomorphism. Setting $a_I = \frac{1}{|W_I|} \sum_{w \in W_I} \zeta(w)^{-1}w(w')^{-1}$ it is clear that $a_I \in 1 + J((\mathcal{O}(Q \rtimes W))^Q)$ and an easy computation shows that $wa_I = \zeta(w)a_I w'$ for any $w \in W_I$, implying the result. □

Proposition 5. *Let I be a subset of S and denote by b_I the principal block of $\mathcal{O}P_I$. We have $e_{U_I} b = b_I$.*

Proof. Let c_I be the principal block of $\mathcal{O}L_I$. Since ℓ divides $q - 1$ it follows from [5] that the principal blocks b and c_I of $\mathcal{O}G$ and $\mathcal{O}L_I$ are the unique unipotent blocks of $\mathcal{O}G$ and $\mathcal{O}L_I$, respectively. Let χ be an irreducible character of G and let ψ be an irreducible character of L_I such that χ is a constituent of the Harish-Chandra induced character $R_{L_I}^G(\psi)$. Since Harish-Chandra induction is given by tensoring with the $\mathcal{O}G$-$\mathcal{O}L_I$-bimodule $\mathcal{O}Ge_{U_I}$, this is equivalent to $e_{U_I}e(\chi)e(\psi) \neq 0$ in KG, where K is the quotient field of $\mathcal{O}$ and where $e(\chi), e(\psi)$ are the primitive idempotents in $Z(KG), Z(KL_I)$ associated with χ, ψ, respectively. Since Harish-Chandra induction preserves Lusztig series, χ belongs to the principal block b of $\mathcal{O}G$ if and only if ψ belongs to the principal block c_I of $\mathcal{O}L_I$. This implies $e_{U_I}bc = 0$ for any non principal block c of $\mathcal{O}L_I$, and hence the equality $e_{U_I}b = e_{U_I}bc_I$. It implies also that $e_{U_I}b'c_I = 0$ for any non principal block b' of $\mathcal{O}G$, and hence the equality $e_{U_I}bc_I = e_{U_I}c_I$. Clearly $\mathcal{O}P_I e_{U_I} \cong \mathcal{O}L_I$, and since $U_I \subseteq O_{\ell'}(P_I)$ is in the kernel of the principal block b_I of $\mathcal{O}P_I$, we get the equality $b_I = e_{U_I}c_I$. The result follows. □

Proof of Theorem 1. Set $Y = \mathrm{Ind}_{\Delta(Q \rtimes W)}^{(Q \rtimes W) \times (Q \rtimes W)}(C)$, viewed as cochain complex of $\mathcal{O}(Q \rtimes W)$-$\mathcal{O}(Q \rtimes W)$-bimodules. For any integer r, the degree r term of Y is isomorphic to the direct sum of the bimodules

$$\mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_I)}{\otimes} \mathcal{O}(Q \rtimes W)$$

with I running over the set of subsets of S such that $|I| = r$. The differential of Y is an alternating sum of the canonical maps

$$a_{IJ} : \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_I)}{\otimes} \mathcal{O}(Q \rtimes W) \longrightarrow \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_J)}{\otimes} \mathcal{O}(Q \rtimes W)$$

for any $I \subseteq J \subseteq S$ such that $|I| + 1 = |J|$.

Let I be a subset of S. By Proposition 5, the $i\mathcal{O}Gi$-$i\mathcal{O}Gi$-bimodule $i\mathcal{O}Ge_{U_I} \underset{\mathcal{O}P_I}{\otimes} e_{U_I}\mathcal{O}Gi$ is isomorphic to $i\mathcal{O}Ge_{U_I} \underset{\mathcal{O}P_I b_I}{\otimes} e_{U_I}\mathcal{O}Gi$. Since i is still a source idempotent of b_I it follows from [7, 3.5] that this bimodule is isomorphic to $i\mathcal{O}Gi \underset{i\mathcal{O}P_I i}{\otimes} i\mathcal{O}Gi$.

Set $Y' = \mathrm{Res}_{\Phi^{-1}}(iXi)$. It follows from combining the Propositions 3 and 4 with the previous paragraph that the degree r term of Y' is isomorphic to the direct sum of bimodules

$$\mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_I)^{a_I}}{\otimes} \mathcal{O}(Q \rtimes W)$$

with I running again over the set of subsets of S such that $|I| = r$. Furthermore, if $I \subseteq J \subseteq S$ then Proposition 4 implies that $\mathcal{O}(Q \rtimes W_I)^{a_I} \subseteq \mathcal{O}(Q \rtimes W_J)^{a_J}$, and so the differential of Y' is again just an alternating sum of the canonical maps

$$a'_{IJ} : \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_I)^{a_I}}{\otimes} \mathcal{O}(Q \rtimes W) \longrightarrow \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_J)^{a_J}}{\otimes} \mathcal{O}(Q \rtimes W)$$

for any $I \subseteq J \subseteq S$ such that $|I| + 1 = |J|$.

In order to prove the first isomorphism in Theorem 1 we have to prove that $Y \cong Y'$ as complexes of $\mathcal{O}(Q \rtimes W)$-$\mathcal{O}(Q \rtimes W)$-bimodules. There is an isomorphism

$$\mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_I)}{\otimes} \mathcal{O}(Q \rtimes W) \cong \mathcal{O}(Q \rtimes W) \underset{\mathcal{O}(Q \rtimes W_I)^{a_I}}{\otimes} \mathcal{O}(Q \rtimes W)$$

mapping $x \otimes y$ to $xa_I \otimes a_I^{-1}y$ for any $I \subseteq S$ and any $x, y \in \mathcal{O}(Q \rtimes W)$. Thus the terms of Y and Y' are isomorphic. However, these isomorphisms need not commute to the differentials. In order to show that Y and Y' are actually isomorphic as complexes it suffices to show that they are both split and have cohomology concentrated in degree zero.

Since ℓ does not divide the order of W the complex C is split and its cohomology is concentrated in degree zero isomorphic to the sign representation of W (cf. [4, 66.28] or [9, §8]). As induction is exact it follows that Y is split with cohomology concentrated in degree zero isomorphic to ${}_\tau\mathcal{O}(Q \rtimes W)$, where τ is the automorphism of $\mathcal{O}(Q \rtimes W)$ mapping uw to $sgn(w)uw$ for any $u \in Q$ and any $w \in W$. It remains to show that Y' is split with cohomology concentrated in degree zero. To see that Y' is split we explicitly define a section s_{IJ} for the above map a'_{IJ} by setting

$$s_{IJ}(x \otimes y) = \frac{1}{[W_J : W_I]} \sum_v xva_J^{-1}a_I \otimes a_I^{-1}a_J v^{-1}y$$

for any $I \subseteq J \subseteq S$ such that $|I| + 1 = |J|$, any $x, y \in \mathcal{O}(Q \rtimes W)$, and where v runs over a system of representatives in W_J of the set of cosets W_J/W_I. A straightforward verification shows that s_{IJ} is well-defined and that $a'_{IJ} \circ s_{IJ}$ is the identity map.

Knowing that Y' is split, in order to see that the cohomology is concentrated in degree zero it suffices to show that the cohomology of the quotient complex $Y'/J(\mathcal{O}(Q \rtimes W))Y'$ is concentrated in degree zero. The point is here that the elements a_I are in $1 + J((\mathcal{O}(Q \rtimes W))^Q)$ which in turn is contained in $1 + J(\mathcal{O}(Q \rtimes W))$. Thus the quotients $Y/J(\mathcal{O}(Q \rtimes W))Y$ and $Y'/J(\mathcal{O}(Q \rtimes W))Y'$ are actually isomorphic as complexes; in particular, their cohomology is both concentrated in degree zero. Theorem 1 follows. □

Remark 6. We expect that it should be possible to extend Theorem 1 to the blocks considered by Puig in [8, 3.4], using similar techniques.

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References

1. D. J. Benson, *Representations and cohomology II*, Cambridge studies in advanced mathematics 31, Cambridge University Press (1991).
2. M. Broué, *Les ℓ-blocs des groupes $GL(n, q)$ et $U(n, q^2)$ et leurs structures locales*, Astérisque **133-134** (1986), 159–188.
3. M. Cabanes, J. Rickard, *Alvis-Curtis duality as an equivalence of derived categories*, in: Modular representation theory of finite groups (Charlottsville, VA, 1998), de Gruyter, Berlin (2001), 157–174.
4. C. W. Curtis, I. Reiner, *Methods of representation theory II*, Wiley-Interscience, New York (1987).
5. P. Fong, B. Srinivasan, *The blocks of finite general linear and unitary groups*, Invent. Math. **69** (1982), 109–153.
6. J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge (1990).
7. L. Puig, *Pointed groups and construction of characters*, Math. Z. **176** (1981), 265–292.
8. L. Puig, *Algèbres de source de certains blocs des groupes de Chevalley*, Astérisque **181-182** (1990), 221-236.
9. J. Rickard, *Splendid equivalences: derived categories and permutation modules*, Proc. London Math.Soc. **72** (1996), 331–358.
10. J.-P. Serre, *Corps locaux*, Hermann (Paris) (1968).
11. J. Thévenaz, *G-algebras and modular representation theory*, Oxford University Press, Oxford, New York, 1995.

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