ALPERIN'S WEIGHT CONJECTURE IN TERMS OF EQUIVARIANT BREDON COHOMOLOGY

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Abstract. Alperin's weight conjecture [1] admits a reformulation in terms of the cohomology of a functor on a category obtained from a subdivision construction applied to a centric linking system [7] of a fusion system of a block, which in turn can be interpreted as the equivariant Bredon cohomology of a certain functor on the G-poset of centric Brauer pairs.

The underlying general constructions of categories and functors needed for this reformulation are described in §1 and §2, respectively, and §3 provides a tool for computing the cohomology of the functors arising in §2. Taking as starting point the alternating sum formulation of Alperin's weight conjecture by Knörr-Robinson [10], the material of the previous sections is applied in §4 to interpret the terms in this alternating sum as dimensions of cohomology spaces of appropriate functors, using further work of Robinson [16, 17, 18].

Introduction

In its original version, Alperin's weight conjecture [1] is a numerical statement on the number $\ell(kG)$ of isomorphism classes of simple modules over the group algebra kG of a finite group G over an algebraically closed field k of prime characteristic p, in terms of invariants of normalisers of non-trivial p-subgroups of G. The subsequent reformulation of Knörr and Robinson [10] of this conjecture counts the number of ordinary irreducible characters of a p-block in terms of alternating sums indexed by G-conjugacy classes of chains of p-subgroups, suggesting there should be a complex behind these sums. Boltje [4] observed the existence of complexes whose Euler characteristic yields those alternating sums. The purpose of the present paper is to construct such a complex and relate it to the equivariant Bredon cohomology of G-posets of Brauer pairs associated with a block. It is, of course, at this stage pure speculation, whether any of this will be useful in view of actually proving Alperin's weight conjecture. The author's take is that if there is an intrinsic proof at all, such a proof would actually have to explain why the - on the surface miraculous - equalities predicted by this conjecture are true. If nothing more, bringing in topological concepts is at least an attempt towards a structural interpretation of Alperin's weight conjecture. It may be significant that the formulation we end up with - Theorem 4.3 below - involves the concept of a centric linking system,

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due to Broto, Levi, Oliver [7], which gives a structural perspective to attempts, going back to work of Benson [3], to attach a "classifying space" to each block of a finite group.

In the context of subgroup complexes, some of the ideas used in this paper appear already in work of Grodal [9], Symonds [21], Webb [23]. In particular, the covariant resolution in §3 below is a generalisation to EI-categories of the insight developed in [9] that Webb's sequences in [23], when they are not exact, compute certain higher limits, which in turn has been taken further by Symonds in [21] to give a non-blockwise version of Alperin's weight conjecture in terms of higher limits. In some sense, this paper is about the adjustments which need to be made in order to get such a formulation of the general blockwise version of Alperin's weight conjecture. The first adjustment is straightforward: subgroup complexes need to be replaced by complexes of Brauer pairs. Keeping in mind possible applications to p-local groups and their fusion systems, we try to set up the formalism more generally in terms of EI-categories. The second adjustment is that we need to be able to "glue together" a certain family of 2-cocycles of the automorphism groups of the underlying fusion system of a block - see [13, 4.2] or 4.2.2 and 4.4 below for more details on this issue.

This paper is subdivided into five Sections. The local structure of a block can be described in terms of its category of Brauer pairs [2], and the categories one obtains have all the property that any endomorphism of an object is an isomorphism. In any category with this property we can replace the objects by chains of non-isomomorphisms, generalising the concept of barycentric subdivision of a poset. The details of this construction are described in Section 1. Sending an object in a category to its automorphism group is in general not functorial. As it turns out, the subdivision categories constructed in Section 1 have the right formal properties for automorphisms groups to be functorial. Section 2 takes this observation as starting point for a technically slightly more involved construction sending objects to certain twisted automomorphism group algebras; this is necessary for the block theoretic applications to follow. Section 3 gives a simple description of a projective resolution of the constant functor on the poset of isomorphism classes in a subdivision category and relates this construction to Bredon cohomology of G-posets of subgroups of a finite group G. After introducing some block theoretic notation, the material developed in the previous Sections is applied in Section 4 to yield a formulation of Alperin's weight conjecture in terms of Bredon cohomology of G-posets of Brauer pairs in Theorem 4.3. In order to prevent getting drowned in technicalities, we describe briefly in Section 5, in which way Theorem 4.3 trivialises when applied to the two special cases of principal blocks and blocks with abelian defect groups.

1 Subdivisions of EI-categories

We describe in this section a construction which is a variation on the theme of subdivisions, designed to generalise the concept of a barycentric subdivision of a poset to EI-categories in a way that takes into account some of the basic properties of EI-categories. Recall that an *EI-category* is a small category with the property that every endomorphism of an object is an isomorphism. See [14] for more background material on EI-categories. If X, Y are non isomorphic objects in an EI-category $\mathcal{C}$ then at most one of the morphism sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(Y,X)$ is non empty. Indeed, if there is a morphism $\varphi : X \to Y$ and a morphism $\psi : Y \to X$ in $\mathcal{C}$, then $\psi \circ \varphi$ and $\varphi \circ \psi$ are automorphisms of X and Y, respectively. One easily deduces that both φ and ψ have to be isomorphisms. Thus the set of isomorphism classes of $\mathcal{C}$ has a partial order given by $[X] \le [Y]$ if $\mathrm{Hom}_{\mathcal{C}}(X,Y) \neq \emptyset$, where $[X]$, $[Y]$ are the isomorphism classes of the objects X, Y in $\mathcal{C}$, respectively. We denote by $[\mathcal{C}]$ the poset of isomorphism classes of $\mathcal{C}$, regarded itself as category in the usual way. There is an obvious functor $\mathcal{C} \longrightarrow [\mathcal{C}]$ mapping any object X in $\mathcal{C}$ to its isomorphism class $[X]$ and any morphism $\varphi : X \to Y$ in $\mathcal{C}$ to the unique morphism $[X] \le [Y]$ in $[\mathcal{C}]$.

In an EI-category $\mathcal{C}$, any morphism composed with any non-isomorphism yields again a non-isomorphism. The particular role of non-isomorphisms in EI-categories motivates the following definition.

Definition 1.1. The *subdivision of an EI-category* $\mathcal{C}$ is the category $S(\mathcal{C})$ defined as follows:

The objects of $S(\mathcal{C})$ are the chains

$$\mathbf{X} = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} X_m$$

of objects X_i, where $0 \le i \le m$, and of non isomorphisms φ_i, where $0 \le i \le m-1$, and where m is a non negative integer. Any such chain is called a *chain of length* m, where as usual chains of length 0 are just objects in $\mathcal{C}$.

A morphism in $S(\mathcal{C})$ from a chain

$$\mathbf{X} = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} X_m$$

to a chain

$$\mathbf{Y} = Y_0 \xrightarrow{\psi_0} Y_1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{n-1}} Y_n$$

is a family $\mu = (\mu_i)_{0 \le i \le m}$ where for each i there is $j(i) \in \{0, 1, .., n\}$ such that $\mu_i : X_i \to Y_{j(i)}$ is an isomorphism which makes the obvious diagrams commutative; that is,

$$\mu_{i+1} \circ \varphi_i \;=\; \psi_{j(i+1)-1} \circ \cdots \circ \psi_{j(i)+1} \circ \psi_{j(i)} \circ \mu_i$$

for any $i \in \{0, 1, .., m-1\}$.

The following Proposition collects some basic properties of subdivisions.

Proposition 1.2. *Let $\mathcal{C}$ be an EI-category. Then the subdivision $S(\mathcal{C})$ is an EI-category and every morphism in $S(\mathcal{C})$ is a monomorphism.*

Proof. The statements are trivial consequences of the fact that morphisms in $S(\mathcal{C})$ are families of isomorphisms. □

Clearly if $\mathcal{C}$ is itself a poset, this construction yields just the usual barycentric subdivision of $\mathcal{C}$ as poset. The subdivision construction is functorial with respect to covariant functors between EI-categories which preserve non-isomorphisms; more precisely, if $\mathcal{C}$, $\mathcal{C}'$ are EI-categories and if $\Phi : \mathcal{C} \longrightarrow \mathcal{C}'$ is a covariant functor which maps any non-isomorphism in $\mathcal{C}$ to a non-isomorphism in $\mathcal{C}'$, then Φ induces an obvious covariant functor $S(\mathcal{C}) \longrightarrow S(\mathcal{C}')$.

The next observation is one of the key reasons for working with subdivisions:

Proposition 1.3. *Let $\mathcal{C}$ be an EI-category. For any two objects $\mathbf{X}$, $\mathbf{Y}$ in $S(\mathcal{C})$ such that $\mathrm{Hom}_{S(\mathcal{C})}(\mathbf{X}, \mathbf{Y})$ is non empty, the group $\mathrm{Aut}_{S(\mathcal{C})}(\mathbf{X})$ acts regularly on the set $\mathrm{Hom}_{S(\mathcal{C})}(\mathbf{X}, \mathbf{Y})$.*

Proof. With the notation of 1.1, two morphisms μ, ν from $\mathbf{X}$ to $\mathbf{Y}$ in $S(\mathcal{C})$ are given by two families of isomorphisms $\mu_i : X_i \to Y_{j(i)}$ and $\nu_i : X_i \to Y_{j(i)}$, respectively, where $0 \leq i \leq m$. Thus the family of automorphisms $\rho_i = \nu_i^{-1}\mu_i$ of X_i defines the unique automorphism ρ of $\mathbf{X}$ such that $\mu = \nu \circ \rho$. □

By 1.2, if $\mathcal{C}$ is an EI-category, so is $S(\mathcal{C})$; in particular, we have a canonical functor $S(\mathcal{C}) \longrightarrow [S(\mathcal{C})]$ mapping a chain $\mathbf{X}$ to its isomorphism class $[\mathbf{X}]$. Furthermore, the subdivision $S(\mathcal{C})$ of $\mathcal{C}$ comes along with a canonical functor $S(\mathcal{C}) \longrightarrow \mathcal{C}$ which maps a chain

$$\mathbf{X} = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} X_m$$

to its maximal member X_m and maps the morphism μ from the last paragraph to the morphism $\varphi_{n-1} \circ \cdots \circ \varphi_{m+1} \circ \varphi_m \circ \mu_m : X_m \longrightarrow Y_n$. Finally, there is also a canonical functor $[S(\mathcal{C})] \longrightarrow [\mathcal{C}]$ mapping the isomorphism class of a chain

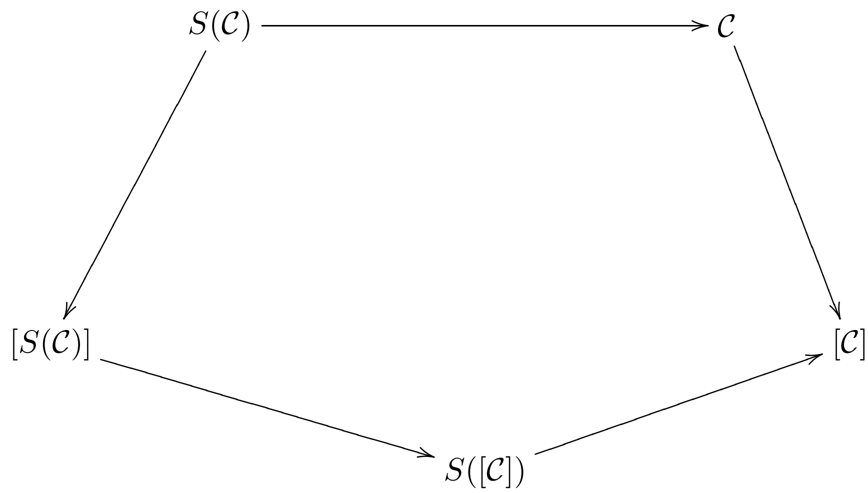
$$[X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} X_m]$$

to the chain of isomorphism classes

$$[X_0] \longrightarrow [X_1] \longrightarrow \cdots \longrightarrow [X_m]\ .$$

This need not be an equivalence of categories. Together with the canonical functors $\mathcal{C} \to [\mathcal{C}]$ and $S(\mathcal{C}) \to [S(\mathcal{C})]$ we get a commutative pentagon of canonical functors

1.4.



The subdivision construction of an EI-category is well-behaved with respect to extensions of categories in the following sense. Let $\mathcal{C}$ be an EI-category and let $\mathcal{Z} : \mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$ be a contravariant functor. Let $\mathcal{E}$ be an extension of $\mathcal{C}$ by $\mathcal{Z}$; that is, $\mathcal{E}$ is a category having the same objects as $\mathcal{C}$, for any two objects the abelian group $\mathcal{Z}(X)$ acts freely on $\mathrm{Hom}_{\mathcal{E}}(X,Y)$ whenever this set is non-empty, and there is a functor $\mathcal{E} \to \mathcal{C}$ which is the identity on the set of objects and which induces a bijection $\mathrm{Hom}_{\mathcal{E}}(X,Y)/\mathcal{Z}(X) \cong \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any two objects X, Y for which $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is non-empty. In other words, as a set we may identify $\mathrm{Hom}_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathcal{Z}(X)$, and the composition of two morphisms in $\mathcal{E}$,

$$X \xrightarrow{(\varphi,u)} Y \xrightarrow{(\psi,v)} Z$$

is then given by a formula of the form

$$(\psi, v) \circ (\varphi, u) = (\psi \circ \varphi, u + \mathcal{Z}(\varphi)(v) + \alpha(\psi, \varphi)) \ ,$$

where here $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $\psi \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, $u \in \mathcal{Z}(X)$, $v \in \mathcal{Z}(Y)$ and $\alpha(\psi,\varphi) \in \mathcal{Z}(X)$. The associativity of the composition of morphisms in $\mathcal{E}$ is equivalent to the equality

$$\alpha(\tau, \psi \circ \varphi) + \alpha(\psi, \varphi) = \alpha(\tau \circ \psi, \varphi) + \mathcal{Z}(\varphi)(\alpha(\tau, \psi)) \ ,$$

for any sequence of three composable morphisms $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{\tau} W$ in $\mathcal{C}$. The above equality means that α is a 2-cocycle in $Z^2(\mathcal{C}, \mathcal{Z})$ representing an element in $H^2(\mathcal{C}, \mathcal{Z})$, and, just as in the case of group extensions, the extension category $\mathcal{E}$ depends up to isomorphism of categories only on the image of α in $H^2(\mathcal{C}, \mathcal{Z})$. In other words, even though α itself depends on the choice of the identifications $\mathrm{Hom}_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathcal{Z}(X)$, its image in $H^2(\mathcal{C}, \mathcal{Z})$ does not. If α is constant 0, this construction is known as *Grothendieck construction*.

Note that $\mathcal{E}$ is again an EI-category. By composing $\mathcal{Z}$ with the canonical functor $S(\mathcal{C}) \longrightarrow \mathcal{C}$ we can consider $\mathcal{Z}$ as functor on $S(\mathcal{C})$, abusively still denoted by $\mathcal{Z}$. We can also consider any 2-cocycle α on $\mathcal{C}$ with values in $\mathcal{Z}$ as a 2-cocycle on $S(\mathcal{C})$ with values in $\mathcal{Z}$.

Proposition 1.5. *With the notation above, the canonical functor $S(\mathcal{E}) \longrightarrow S(\mathcal{C})$ induces an isomorphism of posets $[S(\mathcal{E})] \cong [S(\mathcal{C})]$.*

Proof. We have to check that two chains $\mathbf{X}$, $\mathbf{Y}$ in $S(\mathcal{E})$ whose images in $S(\mathcal{C})$ become isomorphic were isomorphic to begin with. It suffices to do this for chains of length one. Let $X \xrightarrow{\varphi} X'$ and $Y \xrightarrow{\psi} Y'$ be morphisms in $\mathcal{E}$, viewed as chains of lenght one in $S(\mathcal{E})$. Suppose there are isomorphisms $\bar{\mu} : X \cong Y$ and $\bar{\mu}' : X' \cong Y'$ in $\mathcal{C}$ such that $\bar{\mu}' \circ \bar{\varphi} = \bar{\psi} \circ \bar{\mu}$, where $\bar{\varphi}$, $\bar{\psi}$ are the images of φ, ψ in $\mathcal{C}$. Choose any lifts μ, μ' in $\mathcal{E}$ of the morphisms $\bar{\mu}$, $\bar{\mu}'$, respectively. Then $\mu' \circ \varphi$ and $\psi \circ \mu$ are two morphisms in $\mathcal{E}$ from X to Y' which lift the same morphism in $\mathcal{C}$, and hence, since $\mathrm{Hom}_{\mathcal{E}}(X, Y')/\mathcal{Z}(X) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y')$, there is $z \in \mathcal{Z}(X)$ such that $\mu' \circ \varphi = \psi \circ (\mu z)$. Thus, after replacing μ by μz we get that the pair (μ, μ') induces an isomorphism between the two considered chains of length one. □

The category $S(\mathcal{E})$ need not be an extension of $S(\mathcal{C})$ by $\mathcal{Z}$ because $S(\mathcal{E})$ may have "more" objects than $S(\mathcal{C})$. However, the previous Proposition tells us, that if we replace $S(\mathcal{E})$ by a suitable equivalent subcategory, we get such an extension:

Proposition 1.6. *With the notation above, the category $S(\mathcal{E})$ is equivalent to the extension of $S(\mathcal{C})$ by the functor $\mathcal{Z}$ corresponding to the 2-cocycle α, where both $\mathcal{Z}$ and α are viewed as functor and 2-cocycle on $S(\mathcal{C})$, respectively, via the canonical functor $S(\mathcal{C}) \to \mathcal{C}$.*

Proof. For any object in $S(\mathcal{C})$ choose a "lift" of that object in $S(\mathcal{E})$ and denote by $\mathcal{D}$ the full subcategory of $S(\mathcal{E})$ obtained in this way. By Proposition 1.5, the inclusion $\mathcal{D} \subseteq S(\mathcal{E})$ is an equivalence of categories. The canonical functor $S(\mathcal{E}) \to S(\mathcal{C})$ induced by $\mathcal{E} \to \mathcal{C}$ induces a functor $\mathcal{D} \to S(\mathcal{C})$. This functor induces in turn a bijection on the object sets, from which one easily sees that $\mathcal{D}$ is an extension of $S(\mathcal{C})$ by $\mathcal{Z}$ corresponding to α as claimed. □

Remark 1.7 If $\mathcal{C}$ is an EI-category with the additional property that every isomorphism in $\mathcal{C}$ is an automorphism, then one can show that the subdivision construction $S(\mathcal{C})$ is equivalent to the opposite of the category $s(\mathcal{C})$ defined in Słomińska [19, 1.1] in terms of a certain Grothendieck construction. In order to avoid confusion we also point out that the construction $S(\mathcal{C})$ is different from the division/subdivision constructions used in [8].

2 Automorphism functors

Let $\mathcal{C}$ be an EI-category. A morphism $X \to Y$ in $\mathcal{C}$ does not, in general, induce a map between the automorphism groups $\mathrm{Aut}_{\mathcal{C}}(X)$ and $\mathrm{Aut}_{\mathcal{C}}(Y)$. However, a morphism $\mathbf{X} \to \mathbf{Y}$ in $S(\mathcal{C})$ induces a group homomorphism $\mathrm{Aut}_{S(\mathcal{C})}(\mathbf{Y}) \to \mathrm{Aut}_{S(\mathcal{C})}(\mathbf{X})$; in other words, taking automorphism groups of objects in $S(\mathcal{C})$ becomes a contravariant functor. This is a consequence of 1.3, which motivates the following definition:

Definition 2.1. An EI-category $\mathcal{C}$ is called *regular* if for any two objects X, Y in $\mathcal{C}$ the group $\mathrm{Aut}_{\mathcal{C}}(X)$ acts regularly on the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ whenever this set is non-empty.

If $\mathcal{C}$ is an EI-category its subdivision $S(\mathcal{C})$ is a regular EI-category, by 1.2 and 1.3. Taking automorphism groups of objects in a regular EI-category is now contravariant functorial:

Proposition 2.2. *Let $\mathcal{C}$ be a regular EI-category. There is a contravariant functor mapping any object X in $\mathcal{C}$ to the automorphism group $\mathrm{Aut}_{\mathcal{C}}(X)$ and mapping any morphism $\varphi : X \to Y$ to the group homomorphism $\mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Aut}_{\mathcal{C}}(X)$ sending $\sigma \in \mathrm{Aut}_{\mathcal{C}}(Y)$ to the unique $\rho \in \mathrm{Aut}_{\mathcal{C}}(X)$ satisfying $\varphi \circ \rho = \sigma \circ \varphi$.*

Proof. If $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\sigma \in \mathrm{Aut}_{\mathcal{C}}(Y)$, then φ and $\sigma \circ \varphi$ are both morphisms from X to Y. Since $\mathrm{Aut}_{\mathcal{C}}(X)$ acts regularly on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, there is a unique $\rho \in \mathrm{Aut}_{\mathcal{C}}(X)$ such that $\varphi \circ \rho = \sigma \circ \varphi$. The rest is clear. □

Of course, in the situation of 2.2, we can linearise this to a contravariant functor $\mathcal{C} \to \mathrm{Mod}(k)$ mapping any object X in $\mathcal{C}$ to the group algebra $k\mathrm{Aut}_{\mathcal{C}}(X)$ of $\mathrm{Aut}_{\mathcal{C}}(X)$ over a commutative ring k. We will need a "twisted" version of this observation. Given a commutative ring k, we denote by $\underline{k}^{\times}$ the constant contravariant functor $\mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$ mapping any object X in $\mathcal{C}$ to the group $k^{\times}$ of invertible elements of k. A 2-cocycle $\alpha \in Z^2(\mathcal{C}, \underline{k}^{\times})$ is then a map sending any two composable morphisms φ, ψ in $\mathcal{C}$ to an element $\alpha(\psi,\varphi) \in k^{\times}$ such that for any three composable morphisms φ, ψ, τ in $\mathcal{C}$ we have the 2-cocycle identity, now written multiplicatively,

$$\alpha(\tau, \psi \circ \varphi)\alpha(\psi,\varphi) = \alpha(\tau \circ \psi, \varphi)\alpha(\tau,\psi) \ .$$

Such a 2-cocycle α restricts, for any object X in $\mathcal{C}$, to a 2-cocycle on the group $\mathrm{Aut}_{\mathcal{C}}(X)$ with values in $k^{\times}$. We denote by $k_{\alpha}\mathrm{Aut}_{\mathcal{C}}(X)$ the corresponding twisted group algebra; that is, as k-module, $k_{\alpha}\mathrm{Aut}_{\mathcal{C}}(X)$ is equal to $k\mathrm{Aut}_{\mathcal{C}}(X)$ but the multiplication in $k_{\alpha}\mathrm{Aut}_{\mathcal{C}}(X)$ is given by the formula $\psi\varphi = \alpha(\psi,\varphi)(\psi \circ \varphi)$ for any two φ, $\psi \in \mathrm{Aut}_{\mathcal{C}}(X)$. The 2-cocycle identity applied to $\mathrm{Id}_X, \mathrm{Id}_X, \varphi$ implies that $\alpha(\varphi, \mathrm{Id}_X) = \alpha(\mathrm{Id}_X, \mathrm{Id}_X)$ for any morphism $X \xrightarrow{\varphi} Y$ in $\mathcal{C}$. If we define the 1-cochain β by $\beta(\varphi) = \alpha(\varphi, \mathrm{Id}_X)$, the 2-cocycle α' defined by $\alpha'(\psi,\varphi) = \alpha(\psi,\varphi)\beta(\psi)^{-1}\beta(\varphi)^{-1}\beta(\psi \circ \varphi)$ represents the same cohomology class as α in $H^2(\mathcal{C}; \underline{k})$ and has the property that $\alpha'(\varphi, \mathrm{Id}_X) = 1$. Such a 2-cocycle α' is called *normalised.*

Proposition 2.3. *Let $\mathcal{C}$ be a regular EI-category. Let k be a commutative ring and let $\alpha \in Z^2(\mathcal{C}; \underline{k}^\times)$. There is a contravariant functor from $\mathcal{C}$ to the category of k-algebras mapping any object X in $\mathcal{C}$ to the twisted group algebra $k_\alpha \mathrm{Aut}_\mathcal{C}(X)$ and mapping any morphism $\varphi : X \to Y$ to the unique k-algebra homomorphism $k_\alpha\mathrm{Aut}_\mathcal{C}(Y) \to k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ sending $\sigma \in \mathrm{Aut}_\mathcal{C}(Y)$ to $\alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}\rho$, where ρ is the unique element of $\mathrm{Aut}_\mathcal{C}(X)$ satisfying $\varphi\circ\rho = \sigma\circ\varphi$. Up to isomorphism of functors, this functor depends only on the image of α in $H^2(\mathcal{C}; \underline{k}^\times)$.*

Proof. We have to check that the assignment $\sigma \mapsto \alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}\rho$ as in the statement is a k-algebra homomorphism $k_\alpha\mathrm{Aut}_\mathcal{C}(Y) \to k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ and that this assignment is functorial.

Let $X \xrightarrow{\varphi} Y$ be a morphism in $\mathcal{C}$. Let $\sigma, \sigma' \in \mathrm{Aut}_\mathcal{C}(Y)$ and $\rho, \rho' \in \mathrm{Aut}_\mathcal{C}(X)$ such that $\sigma\circ\varphi = \varphi\circ\rho$ and $\sigma'\circ\varphi = \varphi\circ\rho'$. The product of σ and σ' in $k_\alpha\mathrm{Aut}_\mathcal{C}(Y)$ is equal to $\alpha(\sigma,\sigma')\sigma\circ\sigma'$, and its image in $k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ under the above assignment is $\alpha(\sigma,\sigma')\alpha(\sigma\circ\sigma',\varphi)\alpha(\varphi,\rho\circ\rho')^{-1}\rho\circ\rho'$. The product of the images of σ and σ' in $k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ is $\alpha(\rho,\rho')\alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}\alpha(\sigma',\varphi)\alpha(\varphi,\rho')^{-1}\rho\circ\rho'$. In order to show that these two expressions coincide, we have to show the equality

2.3.1. $\alpha(\sigma,\sigma')\alpha(\sigma\circ\sigma',\varphi)\alpha(\varphi,\rho\circ\rho')^{-1} = \alpha(\rho,\rho')\alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}\alpha(\sigma',\varphi)\alpha(\varphi,\rho')^{-1}$.

Multiplying by $\alpha(\varphi,\rho\circ\rho')$ shows that this equation is equivalent to

2.3.2. $\alpha(\sigma,\sigma')\alpha(\sigma\circ\sigma',\varphi) = \alpha(\varphi,\rho\circ\rho')\alpha(\rho,\rho')\alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}\alpha(\sigma',\varphi)\alpha(\varphi,\rho')^{-1}$.

The 2-cocycle identity applied to the three composable morphisms φ,σ,σ' together with the equality $\sigma'\circ\varphi = \varphi\circ\rho'$ yields for the left side in 2.3.2 the equalities

2.3.3. $\alpha(\sigma\circ\sigma',\varphi)\alpha(\sigma,\sigma') = \alpha(\sigma,\sigma'\circ\varphi)\alpha(\sigma',\varphi) = \alpha(\sigma,\varphi\circ\rho')\alpha(\sigma',\varphi)$

and similarly we get

2.3.4. $\alpha(\varphi,\rho\circ\rho')\alpha(\rho,\rho') = \alpha(\varphi\circ\rho,\rho')\alpha(\varphi,\rho) = \alpha(\sigma\circ\varphi,\rho')\alpha(\varphi,\rho)$.

By applying 2.3.3 and 2.3.4 in 2.3.2 we get that 2.3.2 is equivalent to the equality

2.3.5. $\alpha(\sigma,\varphi\circ\rho') = \alpha(\sigma\circ\varphi,\rho')\alpha(\sigma,\varphi)\alpha(\varphi,\rho')^{-1}$,

and this is just the 2-cocycle identity applied to the three composable morphisms ρ',φ,σ. This shows that the assignment as defined in 2.3 yields algebra homomorphisms.

For the functoriality, we consider two composable morphisms $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$. Let $\rho \in \mathrm{Aut}_\mathcal{C}(X)$, $\sigma \in \mathrm{Aut}_\mathcal{C}(Y)$, $\tau \in \mathrm{Aut}_\mathcal{C}(Z)$ such that $\sigma\circ\varphi = \varphi\circ\rho$ and $\tau\circ\psi = \psi\circ\sigma$. The map $k_\alpha\mathrm{Aut}_\mathcal{C}(Z) \to k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ induced by $\psi\circ\varphi$ sends τ to $\alpha(\tau,\psi\circ\varphi)\alpha(\psi\circ\varphi,\rho)^{-1}\rho$, and the composition of the maps induced by φ and ψ sends τ to $\alpha(\tau,\psi)\alpha(\psi,\sigma)^{-1}\alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}\rho$. Thus we have to show the equality

2.3.6. $\alpha(\tau,\psi\circ\varphi)\alpha(\psi\circ\varphi,\rho)^{-1} = \alpha(\tau,\psi)\alpha(\psi,\sigma)^{-1}\alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}$.

Now $\alpha(\tau,\psi\circ\varphi)\alpha(\psi,\varphi) = \alpha(\tau\circ\psi,\varphi)\alpha(\tau,\psi) = \alpha(\psi\circ\sigma,\varphi)\alpha(\tau,\psi)$ and $\alpha(\psi\circ\varphi,\rho)\alpha(\psi,\varphi) = \alpha(\psi,\varphi\circ\rho)\alpha(\varphi,\rho) = \alpha(\psi,\sigma\circ\varphi)\alpha(\varphi,\rho)$. Thus the left side in 2.3.6 is equal to

2.3.7. $\alpha(\psi\circ\sigma,\varphi)\alpha(\psi,\sigma\circ\varphi)^{-1}\alpha(\tau,\psi)\alpha(\varphi,\rho)^{-1}$.

Finally, $\alpha(\psi\circ\sigma)\alpha(\psi,\sigma) = \alpha(\psi,\sigma\circ\varphi)\alpha(\sigma,\varphi)$ and hence $\alpha(\psi,\sigma\circ\varphi)\alpha(\psi,\sigma\circ\varphi)^{-1} = \alpha(\sigma,\varphi)\alpha(\psi,\sigma)^{-1}$. Applying this to 2.3.7 proves the equality 2.3.6.

It remains to check that this functor depends, up to isomorphism, only on the image of α in $H^2(\mathcal{C};\underline{k}^\times)$. If $\alpha' \in Z^2(\mathcal{C};\underline{k}^\times)$ represents the same cohomology class as α, there is a 1-cochain β (that is, a map sending any morphism in $\mathcal{C}$ to an element in $k^\times$) such that

$$\alpha'(\psi,\varphi) = \alpha(\psi,\varphi)\beta(\psi)\beta(\varphi)\beta(\psi\circ\varphi)^{-1} \ .$$

One checks that, for any object X in $\mathcal{C}$, the map sending $\rho\in \mathrm{Aut}_\mathcal{C}(X)$ to $\beta(\rho)\rho$ induces a k-algebra isomorphism $k_\alpha\mathrm{Aut}_\mathcal{C}(X)\cong k_{\alpha'}\mathrm{Aut}_\mathcal{C}(X)$ and that the algebra isomorphisms obtained in this way define a natural transformation. □

Given an algebra A over a commutative ring k we denote by $[A,A]$ the k-submodule of A generated by the set of additive comutators $ab-ba$, where $a,b\in A$. A k-linear map from A to k is central if and only if it has $[A,A]$ in its kernel, and hence the k-module of central k-linear maps from A to k can be identified canonically with $\mathrm{Hom}_k(A/[A,A],k)$. A k-algebra homomorphism $\alpha : A\to B$ does not, in general, induce an algebra homomorphism between the centers $Z(A)$, $Z(B)$ of A and B. It does though map $[A,A]$ to $[B,B]$, and therefore induces a k-linear map $\mathrm{Hom}_k(B/[B,B],k)\to \mathrm{Hom}_k(A/[A,A],k)$. The point of this observation is that if A is symmetric (cf. [22, §6]), then any choice of a symmetrising form $s : A\to k$ induces a k-linear isomorphism $Z(A)\cong \mathrm{Hom}_k(A/[A,A],k)$ sending $z\in Z(A)$ to the k-linear form $A/[A,A]\to k$ mapping $a+[A,A]$ to $s(za)$ for any $a\in A$. This applies in particular if A is a twisted group algebra, which is the situation underlying the next result.

Proposition 2.4. *Let $\mathcal{C}$ be a regular EI-category. Let k be a commutative ring and let $\alpha\in Z^2(\mathcal{C},\underline{k}^\times)$. The automorphism functor defined in 2.3 induces a covariant functor*

$$\mathcal{A} : \begin{cases} [\mathcal{C}] & \longrightarrow \mathrm{Mod}(k) \\ [X] & \mapsto \mathrm{Hom}_k(k_\alpha\mathrm{Aut}_\mathcal{C}(X)/[k_\alpha\mathrm{Aut}_\mathcal{C}(X),k_\alpha\mathrm{Aut}_\mathcal{C}(X)],k)\end{cases}$$

where X is a representative in $\mathcal{C}$ of the isomorphism class $[X]$. The functor $\mathcal{A}$ is up to isomorphism of functors independent of the choice of the representative X in each isomorphism class $[X]$ in $[\mathcal{C}]$.

Proof. By 2.3 we have a contravariant functor sending an object X in $\mathcal{C}$ to the twisted group algebra $k_\alpha\mathrm{Aut}_\mathcal{C}(X)$, and this functor maps a morphism $\varphi : X\to Y$ in $\mathcal{C}$ to a k-algebra homomorphism. Since k-algebra homomorphisms preserve commutators, this functor induces hence a contravariant functor mapping X to the k-module $k_\alpha\mathrm{Aut}_\mathcal{C}(X)/[k_\alpha\mathrm{Aut}_\mathcal{C}(X),k_\alpha\mathrm{Aut}_\mathcal{C}(X)]$. Given two different morphisms $\varphi,\psi : X\to Y$ in $\mathcal{C}$, there is a unique $\pi\in\mathrm{Aut}_\mathcal{C}(X)$ such that $\psi=\varphi\circ\pi$. The main point of the proof is to show that the two k-algebra homomorphisms $k_\alpha\mathrm{Aut}_\mathcal{C}(Y)\to k_\alpha\mathrm{Aut}_\mathcal{C}(X)$

induced by φ and ψ differ by an inner automorphism of $k_\alpha\mathrm{Aut}_\mathcal{C}(X)$, namely the one induced by conjugation with π. Before we get into the technicalities of this verification we show how this concludes the proof of 2.4: the two k-linear maps $k_\alpha\mathrm{Aut}_\mathcal{C}(Y)/[k_\alpha\mathrm{Aut}_\mathcal{C}(Y), k_\alpha\mathrm{Aut}_\mathcal{C}(Y)] \to k_\alpha\mathrm{Aut}_\mathcal{C}(X)/[k_\alpha\mathrm{Aut}_\mathcal{C}(X), k_\alpha\mathrm{Aut}_\mathcal{C}(X)]$ induced by φ and ψ are then actually equal. Therefore this functor factors through the canonical functor $\mathcal{C} \to [\mathcal{C}]$ and induces hence a contravariant functor from $[\mathcal{C}]$ to $\mathrm{Mod}(k)$ sending $[X]$ to $k_\alpha\mathrm{Aut}_\mathcal{C}(X)/[k_\alpha\mathrm{Aut}_\mathcal{C}(X), k_\alpha\mathrm{Aut}_\mathcal{C}(X)]$. By composing this with the contravariant k-duality functor $\mathrm{Hom}_k(-, k)$ we get the covariant functor $\mathcal{A}$ as claimed.

In order to show that the two algebra homomorphisms $k_\alpha\mathrm{Aut}_\mathcal{C}(Y) \to k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ induced by φ and ψ differ by conjugation with π, let $\sigma \in \mathrm{Aut}_\mathcal{C}(Y)$ and $\rho \in \mathrm{Aut}_\mathcal{C}(X)$ such that $\sigma\circ\varphi = \varphi\circ\rho$. Then $\sigma\circ\psi = \psi\circ\pi^{-1}\circ\rho\circ\pi$. Thus the two algebra homomorphisms induced by φ and ψ map σ to $\alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}\rho$ and $\alpha(\sigma,\psi)\alpha(\psi,\pi^{-1}\circ\rho\circ\pi)^{-1}\pi^{-1}\circ\rho\circ\pi$, respectively. We show that the second expression is obtained from the first by conjugation with π in $k_\alpha\mathrm{Aut}_\mathcal{C}(X)$. By 2.3 we may assume that α is normalised; thus $\alpha(\varphi, \mathrm{Id}_X) = 1$, the identity element of $k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ is Id_X and the inverse of π in $k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ is equal to $\alpha(\pi,\pi^{-1})^{-1}\pi^{-1}$. Thus conjugation of ρ by π in $k_\alpha\mathrm{Aut}_\mathcal{C}(X)$ yields the expression

$$\alpha(\pi,\pi^{-1})^{-1}\pi^{-1}\rho\pi = \alpha(\pi,\pi^{-1})^{-1}\alpha(\pi^{-1},\rho)\alpha(\pi^{-1}\circ\rho,\pi)\pi^{-1}\circ\rho\circ\pi\ .$$

Thus we have to show the equality

2.4.1. $\alpha(\sigma,\varphi)\alpha(\varphi,\rho)^{-1}\alpha(\pi,\pi^{-1})^{-1}\alpha(\pi^{-1},\rho)\alpha(\pi^{-1}\circ\rho,\pi) = \alpha(\sigma,\psi)\alpha(\psi,\pi^{-1}\circ\rho\circ\pi)^{-1}$.

Multiplying by $\alpha(\psi,\pi^{-1}\circ\rho\circ\pi)\alpha(\varphi,\rho)\alpha(\pi,\pi^{-1})$ implies that 2.4.1 is equivalent to the equality

2.4.2. $\alpha(\sigma,\varphi)\alpha(\pi^{-1},\rho)\alpha(\pi^{-1}\circ\rho,\rho)\alpha(\psi,\pi^{-1}\circ\rho\circ\pi) = \alpha(\sigma,\psi)\alpha(\varphi,\rho)\alpha(\pi,\pi^{-1})$.

The 2-cocycle identity applied to the three composable morphisms $\pi, \pi^{-1}\circ\rho, \psi$ together with the equality $\psi = \varphi\circ\pi$ yields $\alpha(\psi,\pi^{-1}\circ\rho\circ\pi)\alpha(\pi^{-1}\circ\rho,\pi) = \alpha(\psi\circ\pi^{-1}\circ\rho,\pi)\alpha(\psi,\pi^{-1}\circ\rho) = \alpha(\varphi\circ\rho,\pi)\alpha(\varphi\circ\pi,\pi^{-1}\circ\rho)$. Thus 2.4.2 is equivalent to

2.4.3. $\alpha(\sigma,\varphi)\alpha(\pi^{-1},\rho)\alpha(\varphi\circ\rho,\pi)\alpha(\varphi\circ\pi,\pi^{-1}\circ\rho) = \alpha(\sigma,\psi)\alpha(\varphi,\rho)\alpha(\pi,\pi^{-1})$.

The 2-cocycle identity applied to π, φ, σ yields $\alpha(\sigma,\varphi\circ\pi)\alpha(\varphi,\pi) = \alpha(\sigma\circ\varphi,\pi)\alpha(\sigma,\varphi)$; hence, after cancelling $\alpha(\sigma,\varphi\circ\pi) = \alpha(\sigma,\psi)$, the equality 2.4.3 is equivalent to

2.4.4. $\alpha(\varphi,\pi)\alpha(\pi^{-1},\rho)\alpha(\varphi\circ\pi,\pi^{-1}\circ\rho) = \alpha(\varphi,\rho)\alpha(\pi,\pi^{-1})$.

The 2-cocycle identity applied to $\rho, \pi^{-1}, \varphi\circ\pi$ yields $\alpha(\varphi\circ\pi,\pi^{-1}\circ\rho)\alpha(\pi^{-1},\rho) = \alpha(\varphi,\rho)\alpha(\varphi\circ\pi,\pi^{-1})$, and hence, after cancelling $\alpha(\varphi,\rho)$, the equality 2.4.4 is equivalent to

2.4.5. $\alpha(\varphi,\pi)\alpha(\varphi\circ\pi,\pi^{-1}) = \alpha(\pi,\pi^{-1})$,

and this is just the 2-cocycle identity applied to π^{-1}, π, ρ, where we use that $\alpha(\varphi, \mathrm{Id}_X) = 1$ since α was chosen normalised. This proves the equality 2.4.1, whence the result. □

Remark 2.5. In [14, 16.1] an EI-category $\mathcal{C}$ is called *free* if $\mathrm{Aut}_{\mathcal{C}}(Y)$ acts freely on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any two objects X, Y in $\mathcal{C}$ for which the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is non empty. Thus if $\mathcal{C}$ is regular in the sense of 2.1 then the opposite category $\mathcal{C}^{op}$ is free.

3 A covariant projective resolution for subdivisions

Given a small category $\mathcal{C}$ and a commutative ring k, the constant covariant functor $\underline{k}$ sending any object in $\mathcal{C}$ to k and any morphism in $\mathcal{C}$ to the identity on k has a well-known canonical projective resolution in the category of covariant functors from $\mathcal{C}$ to $\mathrm{Mod}(k)$. The degree n component of this resolution is a direct sum of projective functors of the form

$$\bigoplus_{X_0 \stackrel{\varphi_0}{\to} \cdots \stackrel{\varphi_{n-1}}{\to} X_n} k\mathrm{Hom}_{\mathcal{C}}(X_n, -) \ ,$$

where the sum is taken over the set of chains of n composable morphisms in $\mathcal{C}$, and where $k\mathrm{Hom}_{\mathcal{C}}(X_n,-)$ is the covariant functor sending an object Y in $\mathcal{C}$ to the free k-module $k\mathrm{Hom}_{\mathcal{C}}(X_n,Y)$ with basis $\mathrm{Hom}_{\mathcal{C}}(X_n,Y)$; by Yoneda's Lemma, this is indeed a projective functor. This construction applies, of course, also to the subdivision $S(\mathcal{C})$ as well as $[S(\mathcal{C})]$ of an EI-category $\mathcal{C}$, but since the objects of $S(\mathcal{C})$ are already chains themselves, this would yield rather clumsy direct sums, indexed by chains of chains. It turns out that there is a simpler projective resolution of the constant covariant functor $\underline{k}$ on $[S(\mathcal{C})]$ for any ordered category $\mathcal{C}$.

Proposition 3.1. *Let $\mathcal{C}$ be an EI-category, let k be a commutative ring, and denote by $\underline{k}$ the constant covariant functor on $[S(\mathcal{C})]$ mapping every object in $[S(\mathcal{C})]$ to k and every morphism in $[S(\mathcal{C})]$ to Id_k. There is a projective resolution of $\underline{k}$ in the category of covariant functors from $[S(\mathcal{C})]$ to $\mathrm{Mod}(k)$ whose degree n component is equal to*

$$\bigoplus_{[\mathbf{X}]} k\mathrm{Hom}_{[S(\mathcal{C})]}([\mathbf{X}], -) \ ,$$

where the direct sum is taken over all isomorphism classes of chains $\mathbf{X}$ in $S(\mathcal{C})$ of length n, for any $n \geq 0$. In particular, if $\mathcal{C}$ is finite, this projective resolution is bounded.

Proof. The functor $k\mathrm{Hom}_{S([\mathcal{C}])}([\mathbf{X}],-)$ sends the isomorphism class $[\mathbf{Y}]$ of a chain $\mathbf{Y}$ in $S(\mathcal{C})$ to k if $\mathbf{X}$ is isomorphic to a subchain of $\mathbf{Y}$, and to 0 otherwise. The differential is defined in the usual simplicial way: if $\mathbf{X}(i)$ is the chain of length $n-1$ obtained from deleting the i-th term in a chain $\mathbf{X}$ of length n, the unique morphism $[\mathbf{X}(i)] \leq [\mathbf{X}]$ induces a natural transformation

$$\rho_i : k\mathrm{Hom}_{[S(\mathcal{C})]}([\mathbf{X}], -) \longrightarrow k\mathrm{Hom}_{[S(\mathcal{C})]}([\mathbf{X}(i)], -)$$

and by taking the alternating sum $\sum\limits_{0 \leq i \leq n} (-1)^i \rho_i$ we get a map from the degree n component to the degree $n-1$-component. We are going to show that this is a differential and that the resulting complex is exact except in degree 0, where the homology is shown to be $\underline{k}$. To do this, we have to evaluate this sequence at an object $[\mathbf{Y}]$ in $[S(\mathcal{C})]$ and show that what we get is a complex with homology k in degree zero and homology 0 in all other degrees. Since $\mathrm{Hom}_{[S(\mathcal{C})]}([\mathbf{X}],[\mathbf{Y}]) = \emptyset$ unless $\mathbf{X}$ is isomophic to a subchain of $\mathbf{Y}$, the degree n component evaluated at $[\mathbf{Y}]$ is the free k-module

$$\bigoplus_{[\mathbf{X}]} k \ ,$$

where the sum is taken over the set of isomorphism classes $[\mathbf{X}]$ of chains $\mathbf{X}$ of length n for which there is a morphism $\mathbf{X} \to \mathbf{Y}$ in $S(\mathcal{C})$. Any such morphism identifies $\mathbf{X}$ to a subchain of $\mathbf{Y}$, and hence, if $\mathbf{Y} = Y_0 \to \cdots \to Y_m$ for some positive integer m, a subchain of lenght n of $\mathbf{Y}$ is uniquely determined by any subset of cardinal n of the totally ordered set $\{[Y_0], .., [Y_m]\}$. Thus this construction evaluated at $\mathbf{Y}$ yields the same as the projective resolution of $\underline{k}$ on the totally ordered set $\{[Y_0], .., [Y_m]\}$, so this is indeed a complex with homology concentrated in degree 0 isomorphic to k. □

The above projective resolution provides a way to compute the cohomology of covariant functors on $[S(\mathcal{C})]$:

Proposition 3.2. *Let $\mathcal{C}$ be an EI-category, let k be a commutative ring and let $\mathcal{A}$: $[S(\mathcal{C})] \longrightarrow \mathrm{Mod}(k)$ be a covariant functor. There is a cochain complex $C(\mathcal{A})$ whose component in degree $n \geq 0$ is equal to*

$$C(\mathcal{A})^n = \bigoplus_{[\mathbf{X}]} \mathcal{A}([\mathbf{X}]) \ ,$$

where the direct sum is taken over the set of isomorphism classes of chains $\mathbf{X}$ in $S(\mathcal{C})$ of length n, and whose cohomology is the cohomology of the functor $\mathcal{A}$; that is,

$$H^i(C(\mathcal{A})) \cong H^i([S(\mathcal{C})];\mathcal{A})$$

for any integer $i \geq 0$. In particular, if $\mathcal{C}$ is finite then $H^i([S(\mathcal{C})];\mathcal{A})$ is zero for all but finitely many integers i.

Proof. In order to compute the cohomology of $\mathcal{A}$ we need to apply the contravariant functor $\mathrm{Hom}(-,\mathcal{A})$ to a projective resolution of $\underline{k}$ on $[S(\mathcal{C})]$, where "Hom" means here taking natural transformations of functors. By Yoneda's Lemma, we have

$$\mathrm{Hom}(k\mathrm{Hom}_{[S(\mathcal{C})]}([\mathbf{X}],-),\mathcal{A}) \cong \mathcal{A}([\mathbf{X}])$$

and hence applying $\mathrm{Hom}(-,\mathcal{A})$ to the projective resolution in 3.1 yields a complex $C(\mathcal{A})$ with the properties as stated. □

Thus the complex $C(\mathcal{A})$ computes the cohomology of $\mathcal{A}$; if $\mathcal{A}$ is itself the constant covariant functor $\underline{k}$, the complex $C(\underline{k})$ computes the cohomology of the poset $[S(\mathcal{C})]$. If $\mathcal{C}$ is finite and k is a field then $C(\mathcal{A})$ is a bounded complex of vector spaces and hence $\sum_{i\geq 0} (-1)^i \dim_k(C(\mathcal{A})^i) = \sum_{i\geq 0} \dim(H^i(C(\mathcal{A})))$. In other words:

Corollary 3.3. *If $\mathcal{C}$ is finite and if k is a field then the Euler characteristic of the complex $C(\mathcal{A})$ is equal to*

$$\sum_{i\geq 0} (-1)^i \dim_k(H^i([S(\mathcal{C})];\mathcal{A})) \ .$$

The following two results establish a connection between the cohomology of covariant functors on $[S(\mathcal{C})]$ and Bredon cohomology in certain cases. Given a finite group G and subgroups Q, R of G we denote as usual by $\mathrm{Hom}_G(Q,R)$ the set of all group homomorphisms $\varphi : Q \to R$ for which there exists an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in Q$. If $\mathcal{P}$ is a G-poset of subgroups of G (that is, $\mathcal{P}$ is a set of subgroups of G closed under conjugation in G with partial order given by the inclusion of subgroups) we denote by $sd(\mathcal{P})$ the barycentric subdivision of $\mathcal{P}$; that is, $sd(\mathcal{P})$ is the G-poset of all totally ordered subsets $Q_0 < Q_1 < \cdots < Q_n$ in $\mathcal{P}$, ordered by inclusion.

Proposition 3.4. *Let G be a finite group and let $\mathcal{P}$ be a G-poset of subgroups of G. Denote by $\mathcal{C}$ the category having the same objects as $\mathcal{P}$ and morphism sets $\mathrm{Hom}_\mathcal{C}(Q,R) = \mathrm{Hom}_G(Q,R)$ for any two subgroups Q, R of G belonging to $\mathcal{P}$. The inclusion $sd(\mathcal{P}) \subseteq S(\mathcal{C})$ induces an isomorphism of posets $sd(\mathcal{P})/G \cong [S(\mathcal{C})]$.*

Proof. Any chain $Q_0 < Q_1 < \cdots < Q_n$ in $sd(\mathcal{P})$ is obviously an object in $S(\mathcal{C})$. Since $\mathcal{C}$ contains all homomorphisms induced by conjugation with elements in G it follows that G-conjugate chains in $sd(\mathcal{P})$ are isomorphic in $S(\mathcal{C})$; thus the inclusion $sd(\mathcal{P}) \subseteq S(\mathcal{C})$ induces a map $sd(\mathcal{P}) \to S(\mathcal{C})$. To see that this map is surjective, let $Q_0 \stackrel{\varphi_0}{\to} Q_1 \stackrel{\varphi_1}{\to} \cdots \stackrel{\varphi_{n-1}}{\to} Q_n$ be a chain of non isomorphisms belonging to $S(\mathcal{C})$. This chain is isomorphic, in $S(\mathcal{C})$, to the chain of subgroups $R_0 < R_1 < \cdots < R_n$, where $R_n = Q_n$ and $R_i = (\varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_i)(Q_i)$ for $0 \leq i < n$ via the isomorphism μ given by the family of group homomorphisms $\mu_n = \mathrm{Id}_{Q_n}$ and $\mu_i = \varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_i$ for $0 \leq i < n$. Thus this map is indeed surjective. For the injectivity, let $Q_0 < Q_1 < \cdots < Q_n$ and $R_0 < R_1 < \cdots < R_n$ be two chains in $sd(\mathcal{P})$ which are isomorphic in $S(\mathcal{C})$. This means that there is an isomorphism $\mu : Q_n \cong R_n$ such that $\mu(Q_i) = R_i$ for $0 \leq i \leq n$ and such that there is $x \in G$ satisfying $\mu(u) = xux^{-1}$ for all $u \in Q_n$. It follows that the two chains are G-conjugate, which implies the result. □

By a result of Słomińska [20] (see also Grodal [9, 7.1] for a proof), if $\mathcal{P}$ is a G-poset of subgroups of a finite group G, then the cohomology of covariant functors on $sd(\mathcal{P})/G$ coincides with equivariant Bredon cohomology. Thus, combined with the above Proposition, this reads:

Proposition 3.5. *Let G be a finite group and let $\mathcal{P}$ be a G-poset of subgroups of G. Denote by $\mathcal{C}$ the category having the same objects as $\mathcal{P}$ and morphism sets* $\mathrm{Hom}_\mathcal{C}(Q,R) =$

$\mathrm{Hom}_G(Q,R)$ *for any two subgroups* Q, R *of* G *belonging to* $\mathcal{P}$*. Let* k *be a commutative ring and let* $\mathcal{A} : [S(\mathcal{C})] \to \mathrm{Mod}(k)$ *be a covariant functor. We have*

$$H^*([S(\mathcal{C})];\mathcal{A}) \;\cong\; H^*_G(|\mathcal{P}|;\mathcal{A}) \ ,$$

where $H^*_G(|\mathcal{P}|;\mathcal{A})$ *is the* G*-equivariant Bredon cohomology of* $\mathcal{A}$ *viewed as coefficient system through the isomorphism* $[S(\mathcal{C})] \cong sd(\mathcal{P})/G$*.*

The above Propositions carry over in a straightforward way to G-posets of Brauer pairs, which is what we will use in the next Section to interpret Alperin's weight conjecture in terms of the Bredon cohomology of automorphism functors of the type as described in 2.4.

4 Alperin's conjecture in terms of functor cohomology

In this Section, p is a prime, $\mathcal{O}$ a complete discrete valuation ring having an algebraically closed residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p and quotient field K of characteristic zero.

Let G be a finite group, and let b be a block of $\mathcal{O}G$; that is, b is a primitive idempotent in $Z(\mathcal{O}G)$. In its original version, Alperin's weight conjecture [1] is a statement on the number $\ell(b)$ of isomorphism classes of simple $kG\bar{b}$-modules, where $\bar{b}$ is the canonical image of b in kG. Knörr and Robinson [10] reformulated this in terms of the number $\mathbf{k}(b)$ of ordinary irreducible characters associated with b: Alperin's weight conjecture is equivalent to an equality, involving an alternating sum, of the form

4.1.

$$\mathbf{k}(b) = \sum_{\sigma \in sd(\mathcal{P}^{\#})/G} (-1)^{|\sigma|}\ \mathbf{k}(b_\sigma) \ ,$$

where $|\sigma|$ denotes the length n of σ and where b_σ is the sum of all blocks of $kN_G(\sigma)$ which induce to b. Here $\mathcal{P}^{\#}$ is the G-poset of non-trivial p-subgroups, and hence $sd(\mathcal{P}^{\#})$ is the G-poset of all proper chains of non-trivial p-subgroups of G (excluding the empty chain which accounts for the slight difference in the formulation of Alperin's conjecture here and in [10]). The notation "$\sigma \in sd(\mathcal{P}^{\#})/G$" means that σ runs over a set of representatives of the G-conjugacy classes of chains in $sd(\mathcal{P}^{\#})$. It is shown in [10], that it is possible to replace $sd(\mathcal{P}^{\#})$ by certain subposets without changing the sum 4.1: one can take the set of all chains $\sigma = Q_0 < Q_1 < \cdots < Q_n$ of non-trivial p-subgroups of G where all Q_i are elementary abelian or where all Q_i are normal in Q_n or where all Q_i have the property $Q_i = O_p(N_G(Q_i))$. See [10] for more details and references.

Our aim is to show that after rewriting the sum 4.1 as a sum indexed by chains of Brauer pairs instead of p-subgroups, the numbers occurring in this alternating sum are the dimensions of a complex computing the cohomology of a functor on a suitable category. In order to introduce the relevant categories associated with the block b, we need some more notation.

A b-Brauer pair is a pair (Q, f) consisting of a p-subgroup Q of G and a block f of $kC_G(Q)$ such that $\mathrm{Br}_Q(b)f = f$, where $\mathrm{Br}_Q : (\mathcal{O}G)^Q \to kC_G(Q)$ is the *Brauer homomomorphism* induced by the $\mathcal{O}$-linear map sending $x \in G - C_G(Q)$ to zero and $x \in C_G(Q)$ to its canonical image in $kC_G(Q)$. Any b-Brauer pair (Q, f) has the property that $Z(Q)$ is contained in all defect groups of the block f, and we say that (Q, f) is *centric* if $Z(Q)$ is a defect group of f. Centric Brauer pairs have first been considered by Brauer, the general notion is due to Alperin and Broué [2]. It has been shown by Alperin and Broué in [2] that the set of b-Brauer pairs has a canonical structure of G-poset which has the following uniqueness property: if (Q, f) is a b-Brauer pair and if R is a subgroup of Q, there is a unique block g of $kC_G(R)$ such that (R, g) is a b-Brauer pair satisfying $(R, g) \leq (Q, f)$. Furthermore, any two maximal b-Brauer pairs are conjugate in G. See [2] or [22] for more details on the inclusion of Brauer pairs.

Let (P, e) be a maximal b-Brauer pair; that is, P is a maximal p-subgroups of G such that $\mathrm{Br}_P(b) \neq 0$ and e is a block of $kC_G(P)$ such that $\mathrm{Br}_P(e)b \neq e$. For any subgroup Q of P denote by e_Q the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. This information can be encoded in terms of a category, the *fusion system of* b; this is defined to be the category $\mathcal{F}$ having as objects the subgroups of P and having as morphism sets the sets of injective group homomorphisms $\varphi : Q \to R$ for which there exists $x \in G$ satisfying ${}^x(Q, e_Q) \subseteq (R, e_R)$ and $\varphi(u) = {}^xu = xux^{-1}$ for any $u \in Q$. The category $\mathcal{F}$ is clearly an EI-category. A subgroup Q of P is called $\mathcal{F}$*-centric* if the b-Brauer pair (Q, e_Q) is centric, or equivalently, if $Z(Q)$ is a defect group of e_Q. We denote by $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$ consisting of all $\mathcal{F}$-centric subgroups of P. A theorem of Külshammer and Puig in [12] associates with every $\mathcal{F}$-centric subgroup Q of P two pieces of information:

4.2.1. *there is a canonical class $\zeta(Q) \in H^2(\mathrm{Aut}_{\mathcal{F}}(Q), Z(Q))$, or equivalently, a canonical group extension*

$$1 \longrightarrow Z(Q) \longrightarrow L_Q \longrightarrow \mathrm{Aut}_{\mathcal{F}}(Q) \longrightarrow 1$$

with the property that if $N_P(Q)$ is a defect group of $kN_G(Q, e_Q)e_Q$ then $N_P(Q)$ is a Sylow-p-subgroup of L_Q and the above exact sequence restricts to the obvious exact sequence $1 \to Z(Q) \to N_P(Q) \to N_P(Q)/Z(Q) \to 1$;

4.2.2. *there is a canonical class $\alpha(Q) \in H^2(\mathrm{Aut}_{\bar{\mathcal{F}}}(Q), k^\times)$ such that the twisted group algebra $k_{\alpha(Q)}L_Q$ is Morita equivalent to $kN_G(Q, e_Q)e_Q$, where here $\alpha(Q)$ is viewed as an element of $H^2(L_Q; k^\times)$ via the canonical maps $L_Q \to \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$.*

By the work of Broto, Levi and Oliver [7] the existence of a classifying space of a block b is equivalent to the existence of a certain extension category $\mathcal{L}$ of $\mathcal{F}^c$ by the center functor $\mathcal{Z}$, called *centric linking system* in [7]. More precisely, the objects of $\mathcal{L}$ are again the $\mathcal{F}$-centric subgroups in P, for every $\mathcal{F}$-centric subgroup Q in P we have

$\mathrm{Aut}_{\mathcal{L}}(Q) = L_Q$ and there is a functor $\mathcal{L} \to \mathcal{F}^c$ which is the identity on objects, surjective on morphisms and which induces for each $\mathcal{F}$-centric Q the surjective map $L_Q \to \mathrm{Aut}_{\mathcal{F}}(Q)$ from the Külshammer-Puig exact sequence in 4.2.1. The classifying space of b is then obtained as the p-completion $|\mathcal{L}|_p^\wedge$ of the realisation of the nerve of $\mathcal{L}$. By the time of this writing, neither the existence nor the uniqueness of $\mathcal{L}$ seem to be known in general. If it exists, the extension category $\mathcal{L}$ determines a class $\zeta \in H^2(\mathcal{F}^c, \mathcal{Z})$ whose restriction to $\mathrm{Aut}_{\mathcal{F}}(Q)$ is the class $\zeta(Q)$ in 4.2.1 above, for every $\mathcal{F}$-centric subgroup subgroup Q of P. In other words, ζ "glues together" the classes $\zeta(Q)$.

In the same spirit, we conjectured in [13, 4.2] that there is a second cohomology class $\alpha \in H^2(\bar{\mathcal{F}}^c, \underline{k}^\times)$ whose restriction to $\mathrm{Aut}_{\bar{\mathcal{F}}}(Q)$ is the class $\alpha(Q)$ from 4.2.2 for any $\mathcal{F}$-centric subgroup Q of P. If α and $\mathcal{L}$ exist, we can view α as element of $H^2(S(\mathcal{L}); \underline{k}^\times)$ via the canonical functors $S(\mathcal{L}) \to S(\mathcal{F}^c) \to \mathcal{F}^c \to \bar{\mathcal{F}}^c$.

Under the assumption that both $\mathcal{L}$ and α exist, we can reformulate Alperin's weight conjecture in terms of the Euler characteristic of the twisted dualised automorphism functor as described in 2.4 on the subdivision category of the centric linking system $\mathcal{L}$.

Theorem 4.3. *Let G be a finite group, let b be a block of $\mathcal{O}G$ with positive defect, and suppose that α and a centric linking system $\mathcal{L}$ exist as above. Let $\mathcal{A}$ be the covariant functor defined on $[S(\mathcal{L})]$, mapping the isomorphism class $[\mathbf{X}]$ of a chain $\mathbf{X}$ in $S(\mathcal{L})$ to $\mathrm{Hom}_k(k_\alpha\mathrm{Aut}_{S(\mathcal{L})}(\mathbf{X})/[k_\alpha\mathrm{Aut}_{S(\mathcal{L})}(\mathbf{X}), k_\alpha\mathrm{Aut}_{S(\mathcal{L})}(\mathbf{X})], k)$, as defined in 2.4. Then Alperin's weight conjecture is logically equivalent to*

$$\mathbf{k}(b) = \sum_{i\geq 0} (-1)^i \dim_k(H^i([S(\mathcal{L})]; \mathcal{A})) \ .$$

Remark 4.4. The existence of $\mathcal{L}$ is not necessary for the formulation of Thereom 4.3 because $[S(\mathcal{L})] \cong [S(\mathcal{F}^c)]$ by 1.5, but we find the present formulation involving $\mathcal{L}$ structurally more appealing (and we expect anyway that every block has a centric linking system). The existence of α, though, is necessary and is not established in full generality by the time of this writing. "Logically equivalent" means that Alperin's weight conjecture holds for all p-blocks of finite groups if and only if the equality in 4.3 holds for all p-blocks of finite groups with positive defect, assuming the existence of α for all such blocks.

Before we prove 4.3 we need to rewrite the sum 4.1 as a sum taken over conjugacy classes of b-Brauer pairs rather than p-subgroups. Given a chain τ of p-subgroups of G we denote as before by b_τ the sum of all blocks of $kN_G(\tau)$ which induce to b. Given a chain of b-Brauer pairs $\sigma = (Q_0, d_0) < (Q_1, d_1) < \cdots < (Q_n, d_n)$, it follows from [10, 3.1] that $d_\sigma = d_n$ remains a block of $kN_G(\sigma, d_\sigma)$. The following version of Alperin's weight conjecture in terms of chains of Brauer pairs is well-known (see [18] for more general formulations related to Dade's conjectures); we include a proof for the convenience of the reader.

Proposition 4.5. *With the notation above, we have*

$$\sum_{\tau\in sd(\mathcal{P}^{\#})/G} (-1)^{|\tau|}\ \mathbf{k}(b_\tau) \ = \sum_{\sigma\in sd(\mathcal{B}^{\#})/G} (-1)^{|\sigma|}\ \mathbf{k}(d_\sigma)\ ,$$

where $\mathcal{P}^{\#}$ and $\mathcal{B}^{\#}$ are the G-posets of non-trivial p-subgroups of G and non-trivial b-Brauer pairs, respectively.

Proof. By [10, 3.1], given a chain $\tau = Q_0 < Q_1 < \cdots < Q_n$ of non trivial p-subgroups Q_i in G, every block of $kN_G(\tau)$ lies in $kC_G(Q_n)$, from which follows that $b_\tau = \mathrm{Br}_{Q_n}(b)$. Then either $b_\tau = 0$, or b_τ is a sum of $N_G(\tau)$-conjugacy classes of blocks of $kC_G(Q_n)$. In other words, if c is a block of $kN_G(\tau)$ such that $b_\tau c = c$, then $c = \mathrm{Tr}_{N_G(\tau,d)}^{N_G(\tau)}(d)$ for some block d of $kC_G(Q_n)$. It is well-known that d remains a block of $kN_G(\tau, d)$ and that the block algebras $kN_G(\tau, d)d$ and $kN_G(\tau)c$ are Morita equivalent through induction and restriction (truncated by the block idempotents). Thus in particular $\mathbf{k}(c) = \mathbf{k}(d)$. For any i such that $0 \le i \le n$ there is a unique block d_i of $kC_G(Q_i)$ such that $(Q_i, d_i) \subseteq (Q_n, d)$, by the uniqueness of the inclusion of Brauer pairs; in particular, $d_n = d$. Therefore, if we set $\sigma = (Q_0, d_0) < (Q_1, d_1) < \cdots < (Q_n, d_n)$ and $d_\sigma = d_n = d$ we get that $N_G(\sigma, d_\sigma) = N_G(\tau, d)$, from which the equality follows. □

Proposition 3.4 has various more or less straightforward generalisations to posets of Brauer pairs, centric Brauer pairs, pointed groups, local pointed groups or centric local pointed groups, of which we spell out only the version as needed for the proof of 4.3:

Proposition 4.6. *With the notation above, let $\mathcal{B}$ be the G-poset of b-Brauer pairs and let $\mathcal{B}^c$ be the G-subposet of centric Brauer pairs. The map sending a chain of b-Brauer pairs contained in (P, e) to its underlying chain of subgroups of P induces isomorphisms of posets $sd(\mathcal{B})/G \cong [S(\mathcal{F})]$ and $sd(\mathcal{B}^c)/G \cong [S(\mathcal{F}^c)]$.*

Proof. Let $\sigma = (Q_0, d_0) < (Q_1, d_1) < \cdots < (Q_n, d_n)$ be a chain of b-Brauer pairs. Up to replacing σ by a G-conjugate, we may assume that $(Q_n, d_\sigma) \subseteq (P, e)$. The underlying chain of p-subgroups of such a chain σ of Brauer pairs can be viewed as an object of the subdivision $S(\mathcal{F})$ of the fusion system $\mathcal{F}$ on P determined by the choice of the maximal b-Brauer pair (P, e). Conversely, every object in $S(\mathcal{F})$ is isomorphic to such a chain: any chain $Q_0 \xrightarrow{\varphi_0} Q_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} Q_n$ is ismorphic, in $S(\mathcal{F})$, to the chain $R_0 < R_1 < \cdots < R_n$, where we $R_n = Q_n$ and $R_i = (\varphi_{n-1} \circ \cdots \varphi_{i+1} \circ \varphi_i)(Q_i)$ for $0 \le i < n$. Thus there is a bijection between $sd(\mathcal{B})/G$ and $[S(\mathcal{F})]$ and hence a bijection between $sd(\mathcal{B}^c)/G$ and $[S(\mathcal{F}^c)]$ as required. □

Remark 4.7. The previous Proposition implies that analogously to 3.5 we have, for any covariant functor $\mathcal{A} : [S(\mathcal{L})] \to \mathrm{Mod}(k)$, an interpretation in terms of G-equivariant Bredon cohomology $H^*([S(\mathcal{L})]; \mathcal{A}) \cong H^*_G(|\mathcal{B}^c|; \mathcal{A})$, with $\mathcal{A}$ viewed as coefficient system via the isomorphisms of posets $sd(\mathcal{B}^c)/G \cong [S(\mathcal{L})]$ obtained from combining 4.6 and 1.5.

Proof of Theorem 4.3. As before, $\mathcal{B}^{\#}$ is the G-poset of non-trivial b-Brauer pairs and $\mathcal{B}^c$ is the subposet of centric b-Brauer pairs. Even though it is not *a priori* clear whether the sums $\sum_{\sigma \in sd(\mathcal{B}^{\#})/G} (-1)^{|\sigma|}\, \mathbf{k}(d_\sigma)$ and $\sum_{\sigma \in sd(\mathcal{B}^c)/G} (-1)^{|\sigma|}\, \mathbf{k}(d_\sigma)$ are equal, it has been shown by Robinson, that the equality $\mathbf{k}(b) = \sum_{\sigma \in sd(\mathcal{B}^{\#})/G} (-1)^{|\sigma|}\, \mathbf{k}(d_\sigma)$ holds for all p-blocks b with positive defect if and only if the equality $\mathbf{k}(b) = \sum_{\sigma \in sd(\mathcal{B}^c)/G} (-1)^{|\sigma|}\, \mathbf{k}(d_\sigma)$ holds for all b with positive defect.

Thus, in order to prove 4.3, it suffices to show that the right hand side in the last equality coincides with the right hand side of the equality showing up in the statement of 4.3. As a consequence of the result of Külshammer and Puig in [12] quoted in 4.2.2 above, one gets that given a chain σ of centric b-Brauer pairs contained in (P, e), the algebra $kN_G(\sigma)d_\sigma$ is Morita equivalent to the algebra $k_\alpha \mathrm{Aut}_{S(\mathcal{L})}(\mathbf{X})$, where as before d_σ is the block of the maximal member of σ and where $\mathbf{X}$ is the underlying chain of p-subgroups of σ, viewed as object of the category $S(\mathcal{L})$. It follows from 4.6 and 1.5 that $sd(\mathcal{B}^c)/G \cong [S(\mathcal{L})]$, and hence

$$\sum_{\sigma \in sd(\mathcal{B}^c)/G} (-1)^{|\sigma|}\, \mathbf{k}(d_\sigma) \;=\; \sum_{[\mathbf{X}] \in [S(\mathcal{L})]} (-1)^{|\mathbf{X}|}\, \mathbf{k}(k_\alpha \mathrm{Aut}_{S(\mathcal{L})}(\mathbf{X}))\ .$$

Since $\mathbf{k}(k_\alpha \mathrm{Aut}_{S(\mathcal{L})}(\mathbf{X})) = \dim_k(\mathcal{A}([\mathbf{X}]))$, this is now the Euler characteristic of the complex $C(\mathcal{A})$ defined in 3.2, hence, by 3.3, equal to the desired expression in the statement of 4.3. □

Remark 4.8. As a consequence of recent work of Robinson [18], we expect that Theorem 4.3 has a generalisation to Dade's projective conjecture with the fusion system $\mathcal{F}$ replaced by categories whose objects are pairs (Q, ζ) of p-subgroups Q of G and ordinary irreducible characters ζ of Q.

5 Special cases

We illustrate Theorem 4.3 by describing the particular cases of principal blocks and blocks with abelian defect groups. As in the previous Section we fix a prime p, an algebraically closed field k of characteristic p and a complete discrete valuation ring $\mathcal{O}$ with residue field k and quotient field K of characteristic zero. Let G be a finite group, let b be a block of $\mathcal{O}G$ and let (P, e) be a maximal b-Brauer pair; that is, P is a defect group of b and e is a block of $kC_G(P)$ such that $\mathrm{Br}_P(b)e = e$.

5.1. Principal blocks. Suppose that b is the principal block of $\mathcal{O}G$; that is, b is the unique primitive idempotent of $Z(\mathcal{O}G)$ not contained in the kernel of the augmentation homomorphism $\mathcal{O}G \to \mathcal{O}$. Then P is a Sylow-p-subgroup of G and e is the principal block of $kC_G(P)$. In fact, the b-Brauer pairs are in this case exactly the pairs (Q, f), where Q is a p-subgroup of G and f is the principal block of $kC_G(Q)$ (this is Brauer's Third Main Theorem). In particular, $N_G(Q,f) = N_G(Q)$, and hence the fusion system $\mathcal{F}$ on P of b is the same as that of G; that is, $\mathrm{Hom}_{\mathcal{F}}(Q,R) = \mathrm{Hom}_G(Q,R)$ for any two subgroups Q, R of P. The b-Brauer pair (Q,f) is centric if and only if Q is a centric p-subgroup of G; that is, if and only if $Z(Q)$ is a Sylow-p-subgroup of $C_G(Q)$. In this case we have $C_G(Q) = Z(Q) \times C_Q$, where $C_Q = O_{p'}(C_G(Q))$ is the maximal normal subgroup of order prime to p of $C_G(Q)$.

Following Broto, Levi, Oliver [6], the centric linking system $\mathcal{L}$ of the principal block can explicitly be described as follows: the objects of $\mathcal{L}$ are the centric subgroups of P, and for any two centric subgroups Q, R of P, we set $\mathrm{Hom}_{\mathcal{L}}(Q,R) = T_G(Q,R)/C_Q$, where $T_G(Q,R) = \{x \in G \mid xQx^{-1} \subseteq R\}$ and where the composition of morphisms in $\mathcal{L}$ is induced by multiplication of elements in G.

All 2-cocycles $\alpha(Q)$ appearing in 4.2.2 are trivial (where Q is any centric subgroup of P), and so we may take for α the trivial 2-cocycle in $Z^2(\bar{\mathcal{F}}^c; k^\times)$ mapping any pair of composable morphisms in $\bar{\mathcal{F}}^c$ to 1_k. For any chain $\mathbf{X} = (Q_0 < Q_1 < \cdots < Q_n)$ of centric subgroups of P, viewed as object in $S(\mathcal{L})$, we have $\mathrm{Aut}_{S(\mathcal{L})}(\mathbf{X}) = N_G(\mathbf{X})/C_{Q_n}$, where $N_G(\mathbf{X}) = \bigcap_{0 \le i \le n} N_G(Q_i)$, which describes the functor $\mathcal{A}$ in 4.3.

5.2. Abelian defect. Suppose that the defect group P of b is abelian. Then P is the only object of $\mathcal{F}^c$, hence of $\mathcal{L}$. Moreover, $\mathrm{Aut}_{\bar{\mathcal{F}}}(P) = \mathrm{Aut}_{\mathcal{F}}(P) = E$ is a p'-group (the intertial quotient of b) and hence the extension 4.2.1 for P is a split extension of the form

$$1 \longrightarrow P \longrightarrow P \rtimes E \longrightarrow E \longrightarrow 1 \ ,$$

The 2-cocycle $\alpha \in Z^2(\bar{\mathcal{F}}^c; k^\times)$ is completely determined by $\alpha(P) \in Z^2(E; k^\times)$ from 4.2.2 because P is the only object of $\mathcal{F}^c$. By results of Külshammer [11] and Puig [15], the twisted group algebra $k_\alpha(P \rtimes E)$ is a source algebra of $kN_G(P,e)e$. Theorem 4.3 predicts that $\mathbf{k}(b) = \dim_k(H^0([S(\mathcal{L})]; \mathcal{A}))$. Since $\mathcal{L}$ has P as unique object, this dimension is equal to $\dim_k(\mathcal{A}(P)) = \dim_k(\mathrm{Hom}_k(k_\alpha(P \rtimes E)/[k_\alpha(P \rtimes E), k_\alpha(P \rtimes E)], k)) = \dim_k(Z(k_\alpha(P \rtimes E))) = \dim_k(Z(kN_G(P,e)e)) = \mathbf{k}(c)$, where c is the unique block of $\mathcal{O}N_G(P)$ such that $\mathrm{Br}_P(c) = \mathrm{Br}_P(b)$. In other words, for a block b with abelian defect group P, Theorem 4.3 takes the familiar form of Alperin's weight conjecture predicting an equality of the numbers of irreducible characters of the block b and its Brauer correspondent c. The question as to whether the abelian defect case admits more subtle formulations in terms of contractible complexes has been investigated by Boltje [5].

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