SIMPLE FUSION SYSTEMS AND THE SOLOMON 2-LOCAL GROUPS

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Abstract. We introduce a notion of simple fusion systems which imitates the corresponding notion for finite groups and show that the fusion system on the Sylow-2-subgroup of a 7-dimensional spinor group over a field of characteristic 3 considered by Ron Solomon [18] and by Ran Levi and Bob Oliver [11] is simple in this sense.

Introduction

The bigger picture which motivates the content of the present paper is the intuition, formulated by D. J. Benson in [3], that associated with each fusion system on a finite p-group in the sense of Puig [15] there should be a p-complete topological space which generalises the concept of a classifying space of a finite group. Broto, Levi and Oliver developed in [4] a theory describing how such a space should look, leading to the notion of a *p-local finite group*, and they gave in particular a cohomological criterion for the existence and uniqueness of p-local finite groups. Using this criterion, Levi and Oliver showed in [11] that there is up to homotopy equivalence a unique 2-local group associated with Solomon's fusion system and they showed further that this coincides indeed with the space constructed earlier by Benson in [3]. Put in these terms, Solomon's fusion system provides an example of a simple 2-local finite group which is not the 2-complete classifying space of any finite group by [18]. In fact, Solomon's fusion system cannot even be the fusion system of any 2-block of a finite group by [9]. Besides the obvious question - can one classify simple fusion systems? - one might wonder, whether the problem of the existence and uniqueness of a p-local finite group associated with any fusion system can be reduced to simple fusion systems.

Section 1 contains a brief account of Puig's abstract notion of a fusion system and we recall in Section 2 how fusion systems occur in block theory. The following two sections introduce our notions of normal and simple fusion systems. Sections 5, 6, 7 contain simplicity results for fusion systems on dihedral 2-groups, fusion systems related to orthogonal groups and the Solomon's fusion system, respectively. Throughout this paper, p denotes a prime.

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1 Background material on fusion systems

We recall basic material on Puig's axiomatisation of the local structure of blocks [15]. If P, Q, R are subgroups of a finite group G, we denote by $\mathrm{Hom}_P(Q,R)$ the set of group homomorphisms $\varphi : Q \to R$ for which there is $y \in P$ satisfying $\varphi(u) = yuy^{-1}$ for all $u \in Q$; we write $\mathrm{Aut}_P(Q) = \mathrm{Hom}_P(Q,Q)$. Thus $\mathrm{Aut}_P(Q)$ is canonically isomorphic to $N_P(Q)/C_P(Q)$; in particular $\mathrm{Aut}_Q(Q) \cong Q/Z(Q)$ is the group of inner automorphisms of Q.

Definition 1.1. A *category on a finite p-group P* is a category $\mathcal{F}$ whose objects are the subgroups of P and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ consist, for any two subgroups Q, R of P, of injective group homomorphisms with the following properties:

(i) if Q is contained in R then the inclusion $Q \subseteq R$ is a morphism in $\mathcal{F}$;

(ii) for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, the induced isomorphism $Q \cong \varphi(Q)$ and its inverse are morphisms in $\mathcal{F}$;

(iii) composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Definition 1.2. Let $\mathcal{F}$ be a category on a finite p-group P. A subgroup Q of P is called *fully $\mathcal{F}$-centralised* if $|C_P(R)| \leq |C_P(Q)|$ for any subgroup R of P such that $R \cong Q$ in $\mathcal{F}$, and Q is called *fully $\mathcal{F}$-normalised* if $|N_P(R)| \leq |N_P(Q)|$ for any subgroup R of P such that $R \cong Q$ in $\mathcal{F}$.

The following definition is due to Broto, Levi and Oliver [4].

Definition 1.3. Let $\mathcal{F}$ be a category on a finite p-group P, and let Q be a subgroup of P. For any morphism $\varphi : Q \to P$ in $\mathcal{F}$, we set $N_\varphi = \{y \in N_P(Q)|\ \text{there is}\ z \in N_P(\varphi(Q))\ \text{such that}\ \varphi({}^y u) = {}^z\varphi(u) \text{for all}\ u \in Q\}$.

In other words, N_φ is the inverse image in $N_P(Q)$ of the group $\mathrm{Aut}_P(Q) \cap (\varphi^{-1} \circ \mathrm{Aut}_P(\varphi(Q)) \circ \varphi)$. Note that in particular $QC_P(Q) \subseteq N_\varphi \subseteq N_P(Q)$. Broto, Levi and Oliver use the groups N_φ in [4] to give a definition of fusion systems (called saturated fusion systems in [4]) which is equivalent to Puig's original definition (called full Frobenius systems there), which in turn has been simplified by Stancu [20]; we present here Stancu's version:

Definition 1.4. A *fusion system on a finite p-group P* is a category $\mathcal{F}$ on P such that $\mathrm{Hom}_P(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R)$ for any two subgroups Q, R of P, and such that the following two properties hold:

(I-S) $\mathrm{Aut}_P(P)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(II-S) every morphism $\varphi : Q \to P$ in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalised extends to a morphism $\psi : N_\varphi \to P$ (that is, $\psi|_Q = \varphi$).

The "extension axiom" (II-S) relates the role of N_φ as object of $\mathcal{F}$ to its image N_φ/Q in $\mathrm{Aut}_{\mathcal{F}}(Q)$. We show in the following three Propositions that definition 1.4 is equivalent to the definition given in [4, 1.2] which uses the a priori stronger axioms

(I-BLO) if Q is a fully $\mathcal{F}$-normalised subgroup of P then Q is fully $\mathcal{F}$-centralised and $\mathrm{Aut}_P(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$;

(II-BLO) given any subgroup Q of P, every morphism $\varphi : Q \to P$ such that $\varphi(Q)$ is fully $\mathcal{F}$-centralised extends to a morphism $\psi : N_\varphi \to P$ in $\mathcal{F}$ (that is, $\psi|_Q = \varphi$).

The Propositions 1.5 and 1.6 show that the axioms in 1.4 imply the "Sylow axiom" (I-BLO).

Proposition 1.5. *([20]) Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a subgroup of P. If Q is fully $\mathcal{F}$-normalised then Q is fully $\mathcal{F}$-centralised.*

Proof. Let $\varphi : R \to Q$ be an isomorphism in $\mathcal{F}$. Assume that Q is fully $\mathcal{F}$-normalised and that R is fully $\mathcal{F}$-centralised. By (II-S) in 1.4 there is a morphism $\psi : RC_P(R) \to P$ in $\mathcal{F}$ such that $\psi|_R = \varphi$. Hence ψ maps $C_P(R)$ to $C_P(Q)$, which implies that $|C_P(R)| \le |C_P(Q)|$, hence equality since R is fully $\mathcal{F}$-centralised. Thus Q is fully $\mathcal{F}$-centralised. □

Proposition 1.6. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a subgroup of P. Then Q is fully $\mathcal{F}$-normalised if and only if Q is fully $\mathcal{F}$-centralised and* $\mathrm{Aut}_P(Q)$ *is a Sylow-p-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(Q)$.

Proof. Assume that Q is fully $\mathcal{F}$-normalised. Then Q is fully CF-centralised by 1.5. Choose Q to be of maximal order such that $\mathrm{Aut}_P(Q)$ is not a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Then Q is a proper subgroup of P by 1.4.(I-S). Choose a p-subgroup S of $\mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\mathrm{Aut}_P(Q)$ is a proper normal subgroup of S. Let $\varphi \in S - \mathrm{Aut}_P(Q)$. Since φ normalises $\mathrm{Aut}_P(Q)$, for every $y \in N_P(Q)$ there is $z \in N_P(Q)$ such that $\varphi({}^y u) = {}^z\varphi(u)$ for all $u \in Q$. In other words, $N_\varphi = N_P(Q)$. Since Q is fully $\mathcal{F}$-normalised, it follows from 1.4.(II-S) that there is an automorphism ψ of $N_P(Q)$ in $\mathcal{F}$ such that $\psi|_Q = \varphi$. Since φ has p-power order, by decomposing ψ into its p-part and its p'-part we may in fact assume that ψ has p-power order. Let $\tau : N_P(Q) \to P$ be a morphism in $\mathcal{F}$ such that $\tau(N_P(Q))$ is fully $\mathcal{F}$-normalised. Now $\tau\psi\tau^{-1}$ is a p-element in $\mathrm{Aut}_{\mathcal{F}}(\tau(N_P(Q)))$, thus conjugate to an element in $\mathrm{Aut}_P(\tau(N_P(Q)))$. Therefore we may choose τ in such a way that there is $y \in N_P(\tau(N_P(Q)))$ satisfying $\tau\psi\tau^{-1}(v) = {}^y v$ for any $v \in \tau(N_P(Q))$. Since $\psi|_Q = \varphi$, the automorphism $\tau\psi\tau^{-1}$ of $\tau(N_P(Q))$ stabilises $\tau(Q)$. Thus $y \in N_P(\tau(Q))$. Since Q is fully $\mathcal{F}$-normalised we have $N_P(\tau(Q)) \subseteq \tau(N_P(Q))$, hence $\psi(u) = {}^{\tau^{-1}(y)}u$ for all $u \in N_P(Q)$. But then in particular $\varphi \in \mathrm{Aut}_P(Q)$, contradicting our initial choice of φ. The converse is easy since $|N_P(Q)| = |\mathrm{Aut}_P(Q)| \cdot |C_P(Q)|$. □

The next Proposition shows that the axioms in 1.4 imply also the "extension axiom" (II-BLO).

Proposition 1.7. *([20]) Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Q be a subgroup of P and let $\varphi : Q \to P$ be a morphism in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$-centralised. Then there is a morphism $\psi : N_\varphi \to P$ in $\mathcal{F}$ such that $\psi|_Q = \varphi$.*

Proof. Let $\rho : \varphi(Q) \to P$ be a morphism in $\mathcal{F}$ such that $R = \rho(\varphi(Q))$ is fully $\mathcal{F}$-normalised. Then $\rho \circ \mathrm{Aut}_P(\varphi(Q)) \circ \rho^{-1}$ is a p-subgroup of $\mathrm{Aut}_\mathcal{F}(R)$. Moreover, by 1.6, the group $\mathrm{Aut}_P(R)$ is a Sylow-p-subgroup of $\mathrm{Aut}_\mathcal{F}(R)$. Thus there is $\alpha \in \mathrm{Aut}_\mathcal{F}(R)$ such that $\alpha \circ \rho \circ \mathrm{Aut}_P(\varphi(Q)) \circ \rho^{-1} \circ \alpha^{-1} \subseteq \mathrm{Aut}_P(R)$. This means that after replacing ρ by $\alpha \circ \rho$, we may assume that $N_\rho = N_P(\varphi(Q))$. In particular, ρ extends to a morphism $\sigma : N_P(\varphi(Q)) \to P$. But then $N_\varphi \subseteq N_{\rho\circ\varphi}$, hence $\rho \circ \varphi$ extends to a morphism $\tau : N_\varphi \to P$. Then $\tau(N_\varphi) \subseteq \sigma(N_P(\varphi(Q))$, and hence we get a morphism $\sigma^{-1}|_{\tau(N_\varphi)} \circ \tau : N_\varphi \to P$ which extends φ as required. □

Definition 1.8. Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a subgroup of P.

(i) Q is *$\mathcal{F}$-centric* if $C_P(R) = Z(R)$ for any subgroup R of P such that $R \cong Q$ in $\mathcal{F}$.

(ii) Q is *$\mathcal{F}$-radical* if $O_p(\mathrm{Aut}_\mathcal{F}(Q)/\mathrm{Aut}_Q(Q)) = 1$.

(iii) Q is *$\mathcal{F}$-essential* if Q is $\mathcal{F}$-centric, $Q \neq P$, and $\mathrm{Aut}_\mathcal{F}(Q)/\mathrm{Aut}_Q(Q)$ has a strongly p-embedded proper subgroup M (that is, M contains a Sylow-p-subgroup S of $Aut_\mathcal{F}(Q)/\mathrm{Aut}_Q(Q)$ such that ${}^\varphi S \cap S = \{1\}$ for every $\varphi \in \mathrm{Aut}_\mathcal{F}(Q)/\mathrm{Aut}_Q(Q) - M$).

(iv) Q is *weakly $\mathcal{F}$-closed* if for every morphism $\varphi : Q \to P$ in $\mathcal{F}$ we have $\varphi(Q) = Q$.

(v) Q is *strongly $\mathcal{F}$-closed*, if for any subgroup R of P and any morphism $\varphi : R \to P$ in $\mathcal{F}$ we have $\varphi(R \cap Q) \subseteq Q$.

If Q is $\mathcal{F}$-centric, then Q is fully $\mathcal{F}$-centralised, and if Q is $\mathcal{F}$-essential, then Q is $\mathcal{F}$-radical. If Q is strongly $\mathcal{F}$-closed then Q is weakly $\mathcal{F}$-closed. One easily checks that if Q is strongly $\mathcal{F}$-closed then for any subgroup R of P and any morphism $\varphi : R \to P$ in $\mathcal{F}$ we have in fact $\varphi(R \cap Q) = \varphi(R) \cap Q$. Indeed, the left side is contained in the right side by the above definition, and the other inclusion is obtained by applying this inclusion to $\varphi(R)$ and the morphism φ^{-1} viewed as morphism from $\varphi(R)$ to P.

Definition 1.9 Let $\mathcal{F}$ be a category on a finite p-group P, and let Q be a subgroup of P. We define the category $N_\mathcal{F}(Q)$ on $N_P(Q)$ by $\mathrm{Hom}_{N_\mathcal{F}(Q)}(R, R') = \{\varphi : R \to R' |\ \varphi$ extends to a morphism $\psi : QR \to QR'$ in $\mathcal{F}$ such that $\psi(Q) = Q\}$, for any two subgroups R, R' of $N_P(Q)$. Similarly, we define the category $C_\mathcal{F}(Q)$ on $C_P(Q)$ by $\mathrm{Hom}_{C_\mathcal{F}(Q)}(R, R') = \{\varphi : R \to R' |\ \varphi$ extends to a morphism $\psi : QR \to QR'$ in $\mathcal{F}$ such that $\psi|_Q = \mathrm{Id}_Q\}$.

We have clearly inclusions of categories $C_\mathcal{F}(Q) \subseteq N_\mathcal{F}(Q) \subseteq \mathcal{F}$. If $\mathcal{F} = N_\mathcal{F}(Q)$ for some subgroup Q of P, then clearly Q is strongly $\mathcal{F}$-closed. The converse of this statement is not true, in general. If $\mathcal{F}$ is a fusion system on P such that $\mathcal{F} = C_\mathcal{F}(Z)$ for some (necessarily central) subgroup Z of P then the category on P/Z induced by $\mathcal{F}$ is a fusion system on P/Z, denoted by $\mathcal{F}/Z$. In that case, if $\mathcal{F}'$ is a fusion system on P contained in $\mathcal{F}$ we have $\mathcal{F}' = \mathcal{F}$ if and only if $\mathcal{F}'/Z = \mathcal{F}/Z$; this follows from

Alperin's fusion theorem 1.11 below together with the fact that if Q is a subgroup of P then the canonical map $\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}/Z}(Q/Z)$ has a p-group as kernel because any p'-automorphism of Q/Z lifts to a p'-automorphism of Q.

Proposition 1.10. *([15]) Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a subgroup of P. If Q is fully $\mathcal{F}$-centralised, then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$; if Q is fully normalised, then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$.*

A proof of this Proposition can be found in [4, A6] (applied to the cases where the group K occurring in the statement of [4, A6] is either trivial or equal to $\mathrm{Aut}(Q)$). By the previous remarks, Proposition 1.10 implies that if Q is fully $\mathcal{F}$-centralised then $C_{\mathcal{F}}(Q)/Z(Q)$ is a fusion system on $C_P(Q)/Z(Q)$. The following result is Alperin's fusion theorem [1], refined by Goldschmidt [8], and extended to arbitrary fusion systems by Puig [15].

Theorem 1.11. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Every isomorphism in $\mathcal{F}$ can be written as a composite of finitely many isomorphisms $\varphi : Q \cong R$ in $\mathcal{F}$ such that either $\varphi = \alpha|_Q$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ or there is an $\mathcal{F}$-essential subgroup E of P containing both Q, R, and an automorphism $\beta \in \mathrm{Aut}_{\mathcal{F}}(E)$ such that $\varphi = \beta|_Q$.*

Lemma 1.12. *([15]) Let $\mathcal{F}$ be a fusion system on a finite p-group P. Let Q, R be $\mathcal{F}$-centric subgroups of P such that $Q \subseteq R$, and let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(R)$. We have $\varphi|_Q = \mathrm{Id}_Q$ if and only if $\varphi \in \mathrm{Aut}_{Z(Q)}(R)$.*

Proof. Assume that $\varphi|_Q = \mathrm{Id}_Q$. We proceed by induction over $[R : Q]$. Consider first the case where Q is normal in R. Let $u \in Q$ and $v \in R$. Then ${}^v u \in Q$, hence ${}^v u = \varphi({}^v u) = {}^{\varphi(v)} u$, and thus $v^{-1}\varphi(v) \in C_R(Q) = Z(Q)$, or equivalently, $\varphi(v) = vz$ for some $z \in Z(Q)$. If φ has order prime to p in $\mathrm{Aut}(R)$ this forces $\varphi = \mathrm{Id}_R$. Therefore we may assume that the order of φ is a power of p. Upon replacing R by a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate we may assume that $\varphi \in \mathrm{Aut}_P(R)$. Since φ restricts to Id_Q and since Q is $\mathcal{F}$-centric this implies that $\varphi \in \mathrm{Aut}_{Z(Q)}(R)$. This proves 1.12 if Q is normal in R. In general, if $\varphi|_Q = \mathrm{Id}_Q$ then $\varphi(N_R(Q)) = N_R(Q)$. Thus $\varphi|_{N_R(Q)} \in \mathrm{Aut}_{Z(Q)}(N_R(Q))$ by the previous paragraph. Hence there is $z \in Z(Q)$ such that $c_z \circ \varphi|_{N_R(Q)} = \mathrm{Id}_{N_R(Q)}$, where c_z is the automorphism of R given by conjugation with z. By induction we get $c_z \circ \varphi \in \mathrm{Aut}_{Z(N_R(Q))}(Q)$. As all involved groups are $\mathcal{F}$-centric we have $Z(N_R(Q)) \subseteq Z(Q)$, and thus $\varphi \in \mathrm{Aut}_{Z(Q)}(R)$ as claimed. The converse is trivial. □

Lemma 1.13. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Q, R be $\mathcal{F}$-centric subgroups of P such that $Q \subseteq R$, and let φ, $\varphi' \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ such that $\varphi|_Q = \varphi'|_Q$. Then $\varphi(R) = \varphi'(R)$.*

Proof. Let $v \in N_R(Q)$. For every $u \in Q$ we have $\varphi({}^v u) = \varphi'({}^v u)$, hence $\varphi(v)^{-1}\varphi'(v) \in C_P(\varphi(Q)) = Z(\varphi(Q))$. It follows that $\varphi(N_R(Q)) = \varphi'(N_R(Q))$. By 1.12, $\varphi|_{N_R(Q)}$ and

$\varphi'_{N_R(Q)}$ differ by conjugation with an element in $Z(Q)$, and we may therefore assume that their restrictions to $N_R(Q)$ actually coincide. The equality $\varphi(R) = \varphi'(R)$ follows by induction. □

Given a fusion system $\mathcal{F}$ on a finite p-group P, we denote by $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$-centric subgroups of P; we denote by $\bar{\mathcal{F}}$ the *orbit category of* $\mathcal{F}$, which has the same objects as $\mathcal{F}$ but whose sets of morphisms are the quotient sets $\mathrm{Hom}_{\bar{\mathcal{F}}}(Q, R) = \mathrm{Aut}_R(R)\backslash\mathrm{Hom}_{\mathcal{F}}(Q, R)$ of morphisms in $\mathcal{F}$ modulo inner automorphisms of the corresponding subgroups of P. We denote by $\bar{\mathcal{F}}^c$ the image in $\bar{\mathcal{F}}$ of $\mathcal{F}^c$. The category $\mathcal{F}$ has the property that every morphism is a monomorphism, and every endomorphism is an automorphism. The orbit category $\bar{\mathcal{F}}$ has still the property that every endomorphism is an automorphism, but not every morphism is a monomorphism, in general. As observed in [14] in the context of fusion systems of finite groups, the straightforward consequence of 1.12 is that in the opposite category $(\bar{\mathcal{F}}^c)^0$ every morphism is a monomorphism, or equivalently:

Proposition 1.14. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Every morphism in the category $\bar{\mathcal{F}}^c$ is an epimorphism.*

Proof. Let Q, R, S be $\mathcal{F}$-centric subgroups of P, let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ and let $\psi, \psi' \in \mathrm{Hom}_{\mathcal{F}}(R, S)$. Assume that the images of $\psi \circ \varphi$ and $\psi' \circ \varphi$ in $\mathrm{Hom}_{\bar{\mathcal{F}}^c}(Q, S)$ coincide. Up to replacing ψ' by some S-conjugate, we may assume that $\psi \circ \varphi = \psi' \circ \varphi$. Thus the restrictions to $\varphi(Q)$ of ψ, ψ' coincide. It follows from 1.13 that $\psi(R) = \psi'(R)$. Thus $\psi^{-1} \circ \psi'$ is an automorphism of R which restricts to the identity on $\varphi(Q)$, hence $\psi^{-1} \circ \psi' \in \mathrm{Aut}_{Z(\varphi(Q))}(R)$ by 1.12. Thus the images of ψ, ψ' in the orbit category are equal. □

2 Fusion systems of finite groups and p-blocks

For expository purpose, we describe in this section briefly the well-known examples which motivate Puig's definition of a fusion system.

Definition 2.1 Let G be a finite group, and let P be a Sylow-p-subgroup of G. We denote by $\mathcal{F}_P(G)$ the category on P whose morphisms are the group homomorphisms $\varphi : Q \to R$ for which there is an element $x \in G$ such that $\varphi(u) = xux^{-1}$ for all $u \in Q$.

Equivalently, $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \mathrm{Hom}_G(Q, R)$; in particular, $\mathrm{Aut}_{\mathcal{F}_P(G)}(Q) = \mathrm{Aut}_G(Q) \cong N_G(Q)/C_G(Q)$. We leave the elementary proof of the following well-known statement to the reader.

Theorem 2.2. *Let G be a finite group, and let P be a Sylow-p-subgroup of G.*

(i) The category $\mathcal{F}_P(G)$ is a fusion system on P.

(ii) A subgroup Q of P is fully $\mathcal{F}_P(G)$-centralised if and only if $C_P(Q)$ is a Sylow-p-subgroup of $C_G(Q)$.

(iii) A subgroup Q of P is fully $\mathcal{F}_P(G)$-normalised if and only if $N_P(Q)$ is a Sylow-p-subgroup of $N_G(Q)$.

Following Alperin-Broué [2], there is a fusion system on a defect group of a p-block of a finite group which generalises the definition of $\mathcal{F}_P(G)$ above in the sense, that it coincides with $\mathcal{F}_P(G)$ if the considered block is the principal p-block of G. In order to describe this briefly, let k be a field of characteristic p, let G be a finite group, and let b be a block of kG; that is, b is a primitive idempotent in $Z(kG)$. A *b-Brauer pair* is a pair (Q, f) consisting of a p-subgroup Q of G and a block f of $kC_G(Q)$ such that $\mathrm{Br}_Q(b)f = f$. Here $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$ is the *Brauer homomorphism* mapping any element of $C_G(Q)$ to itself and any non trivial Q-conjugacy class sum of elements in G to zero. By [2], the set of b-Brauer pairs admits a partial order " $\subseteq$ " which is compatible with the action of G by conjugation on this set, such that the maximal b-Brauer pairs form a single G-conjugacy class. Given a maximal b-Brauer pair (P, e), for every subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$, and the group P is called a *defect group of the block b*. The choice of a maximal b-Brauer pair gives rise to a category on P (we follow the notation of [10]):

Definition 2.3. Let G be a finite group, let b be a block of kG, and let (P, e) be a maximal b-Brauer pair. For any subgroup Q of P, denote by e_Q the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. We denote by $\mathcal{F}_{(P,e)}(G, b)$ the category on P whose morphisms are the group homomorphisms $\varphi : Q \to R$ for which there is an element $x \in G$ such that $\varphi(u) = xux^{-1}$ for all $u \in Q$ such that $xe_Qx^{-1} = e_{xQx^{-1}}$, or equivalently, such that ${}^x(Q, e_Q) \subseteq (R, e_R)$, where Q, R are subgroups of P.

If S is a Sylow-p-subgroup of G containing the defect group P of b, then clearly $\mathcal{F}_{(P,e)}(G, b)$ is a subcategory of $\mathcal{F}_S(G)$, but it is not in general a full subcategory, because the elements x in G used to define the morphisms in $\mathcal{F}_{(P,e)}(G, b)$ have to fulfill the additional compatibility property ${}^x(Q, e_Q) \subseteq (R, e_R)$. If b is the *principal block of kG* (that is, b is the unique block of kG not contained in the augmentation ideal of kG), then P is a Sylow-p-subgroup of G and e_Q is the principal block of $kC_G(Q)$ for any subgroup Q of P, and hence $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(G)$ in this case. The following statement, which generalises 2.2, is essentially a reformulation of results in [2]; we sketch a proof for the convenience of the reader:

Theorem 2.4. *Let G be a finite group, let b be a block of kG, and let (P, e) be a maximal b-Brauer pair. For every subgroup Q of P, denote by e_Q the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$.*

(i) The category $\mathcal{F}_{(P,e)}(G, b)$ is a fusion system on P.

(ii) A subgroup Q of P is fully $\mathcal{F}_{(P,e)}(G, b)$-centralised if and only if $C_P(Q)$ is a defect group of $kC_G(Q)e_Q$.

(iii) A subgroup Q of P is fully $\mathcal{F}_{(P,e)}(G, b)$-normalised if and only if $N_P(Q)$ is a defect group of $kN_G(Q, e_Q)e_Q$.

Note that the last statement makes sense, as e_Q remains a block for the group $N_G(Q, e_Q)$ by [2, (2.9)]. The automorphism group in $\mathcal{F}_{(P,e)}(G, b)$ of a subgroup Q of P is isomorphic to $N_G(Q, e_Q)/C_G(Q)$. Thanks to the preceding Theorem, we can apply Alperin's fusion thereom to the fusion system $\mathcal{F}_{(P,e)}(G, b)$, which implies in particular, that $\mathcal{F}_{(P,e)}(G, b)$ is completely determined by the automorphism groups $N_G(Q, e_Q)/C_G(Q)$ for the $\mathcal{F}_{(P,e)}(G, b)$-essential subgroups Q of P. Specialising Theorem 2.4 to the case where b is the principal block of kG yields Theorem 2.2.

Proof of Theorem 2.4. We prove first (ii) and (iii). By [12, 7.6], for every subgroup Q of P the group $C_P(Q)$ is contained in a defect group of e_Q as block of $kC_G(Q)$, and there is $x \in G$ such that ${}^x(Q, e_Q) \subseteq (P, e)$ and such that $C_P({}^xQ)$ is a defect group of ${}^xe_Q$ as block of $kC_G({}^xQ)$. From this follows (ii). By [2, (2.9)], e_Q remains a block of $kN_G(Q, e_Q)$. As before, $N_P(Q)$ is contained in a defect group of e_Q as block of $kN_G(Q, e_Q)$, and there is $x \in G$ such that ${}^x(Q, e_Q) \subseteq (P, e)$ and such that $N_P({}^xQ)$ is a defect group of ${}^xe_Q$ as block of $kN_G({}^x(Q, e_Q))$. This proves (iii).

In order to see (i), observe first that $\mathcal{F}_{(P,e)}(G, b)$ is clearly a category on P in the sense of 1.1. By Brauer's First Main Theorem [23, (40.14)], the group $N_G(P, e)/PC_G(P)$ is a p'-group (called inertial quotient of b), and hence the group $\mathrm{Aut}_{\mathcal{F}_{(P,e)}(G,b)}(P) \cong N_G(P, e)/C_G(P)$ has $\mathrm{Aut}_P(P)$ as Sylow-p-subgroup. In particular, the Sylow axiom (I-S) holds. It remains to verify that $\mathcal{F}_{(P,e)}(G, b)$ has also the property (II-S). Let Q, R be subgroups of P such that $N_P(R)$ is a defect group of e_R as block of $kN_G(R, e_R)$, and let $x \in G$ such that ${}^x(Q, e_Q) = (R, e_R)$. Denote by $\varphi : Q \to P$ the morphism in $\mathcal{F}_{(P,e)}(G, b)$ defined by $\varphi(u) = {}^xu$ for all $u \in Q$. Then $N_\varphi = \{y \in N_P(Q) \mid \text{there is } z \in N_P(R) \text{ such that } {}^{xy}u = {}^{zx}u \text{ for all } u \in Q\}$. Thus ${}^xN_\varphi \subseteq N_P(R)C_G(R)$. Since R is fully $\mathcal{F}_{(P,e)}(G, b)$-normalised, $N_P(R)$ is a defect group of e_R viewed as block of $kN_G(R, e_R)$ by (ii), and hence $N_P(R)$ is still a defect group of e_R viewed as block of $N_P(R)C_G(R)$. Therefore $(N_P(R), e_{N_P(R)})$ is a maximal $(N_P(R)C_G(R), e_R)$-Brauer pair (cf. [23, (40.15)]) and contains hence a $C_G(R)$-conjugate of every other $(N_P(R)C_G(R), e_R)$-Brauer pair (cf. [2, 3.10]). Thus there is $c \in C_G(R)$ such that ${}^{cx}(N_\varphi, e_{N_\varphi}) \subseteq (N_P(R), e_{N_P(R)})$. Hence $\psi : N_\varphi \to P$ defined by $\psi(n) = {}^{cx}n$ for all $n \in N_\varphi$ is a morphism in $\mathcal{F}_{(P,e)}(G, b)$ which extends φ. □

For future reference we include another obvious reformulation of some results in [2].

Proposition 2.5. *Let G be a finite group, let b be a block of kG, and let (P, e) be a maximal b-Brauer pair. For every subgroup Q of P, denote by e_Q the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. Set $\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$.*

(i) If Q is a fully $\mathcal{F}$-centralised subgroup of P then $(C_P(Q), e_{QC_P(Q)})$ is a maximal $(C_G(Q), e_Q)$-Brauer pair and we have $\mathcal{F}_{(C_P(Q), e_{QC_P(Q)})}(C_G(Q), e_Q) = C_\mathcal{F}(Q)$.

(ii) If Q is a fully $\mathcal{F}$-normalised subgroup of P then $(N_P(Q), e_{N_P(Q)})$ is a maximal $(N_G(Q, e_Q), e_Q)$-Brauer pair and we have $\mathcal{F}_{(N_P(Q), e_{N_P(Q)})}(N_G(Q, e_Q), e_Q) = N_\mathcal{F}(Q)$.

Proof. (i) Suppose that Q is fully $\mathcal{F}$-centralised. By 2.4.(ii), $C_P(Q)$ is a defect group of e_Q as block of $C_G(Q)$. We have $C_{C_G(Q)}(C_P(Q)) = C_G(QC_P(Q))$, hence $(C_P(Q), e_{QC_P(Q)})$ is a maximal $(C_G(Q), e_Q)$-Brauer pair. Similarly, for any subgroup R of $C_P(Q)$, the pair (R, e_{QR}) is a $(C_G(Q), e_Q)$-Brauer pair contained in $(C_P(Q), e_{QC_P(Q)})$. If R, S are subgroups of $C_P(Q)$ and $x \in C_G(Q)$ such that ${}^x(R, e_{QR}) \subseteq (S, e_{QS})$ then the group homomorphism from R to S induced by conjugation with x extends to a group homomorphism from QR to QS which is the identity on Q. Statement (i) follows.

(ii) Suppose that Q is fully $\mathcal{F}$-normalised. By 2.4.(iii), $N_P(Q)$ is a defect group of e_Q as block of $N_G(Q, e_Q)$. We have $C_{N_G(Q)}(C_P(Q)) = C_G(N_P(Q))$, hence $(N_P(Q), e_{N_P(Q)})$ is a maximal $(N_G(Q, e_Q), e_Q)$-Brauer pair. Similarly, for any subgroup R of $N_P(Q)$, the pair (R, e_{QR}) is a $(N_G(Q, e_Q), e_Q)$-Brauer pair contained in $(N_P(Q), e_{N_P(Q)})$. If R, S are subgroups of $N_P(Q)$ and $x \in N_G(Q, e_Q)$ such that ${}^x(R, e_{QR}) \subseteq (S, e_{QS})$ then the group homomorphism from R to S induced by conjugation with x extends to a group homomorphism from QR to QS which restricts to an automorphism of Q in $\mathrm{Aut}_{\mathcal{F}}(Q)$. The result follows. □

3 Normal fusion systems

Definition 3.1 Let $\mathcal{F}$ be a category on a finite p-group P, and let $\mathcal{F}'$ be a category on a subgroup P' of P. We say that $\mathcal{F}$ *normalises* $\mathcal{F}'$ if P' is strongly $\mathcal{F}$-closed and if for every isomorphism $\varphi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q \cap P'$ we have

$$\varphi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \varphi^{-1}|_{\varphi(R)} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R')) \ .$$

We say that $\mathcal{F}'$ *is normal in* $\mathcal{F}$ and write $\mathcal{F}' \trianglelefteq \mathcal{F}$ if $\mathcal{F}'$ is contained in $\mathcal{F}$ and $\mathcal{F}$ normalises $\mathcal{F}'$.

In other words, $\mathcal{F}$ normalises $\mathcal{F}'$ if for any isomorphism $\varphi : Q \to Q'$ in $\mathcal{F}$ and any morphism $\psi : R \to R'$ in $\mathcal{F}'$ such that $< R, R' >\subseteq Q$, we have $< \varphi(R), \varphi(R') >\subseteq P'$ and the induced morphism $\varphi \circ \psi \circ \varphi^{-1} : \varphi(R) \to \varphi(R')$ is a morphism in $\mathcal{F}'$. Note that this implies that we have in fact an equality

$$\varphi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \varphi^{-1}|_{\varphi(R)} = \mathrm{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R')) \ .$$

Indeed, the left side is contained in the right side by the definition, and the other inclusion follows from applying this inclusion to φ^{-1}, $\varphi(R)$, $\varphi(R')$ instead of φ, R, R', respectively. Applied to $R = R'$ and $S = \varphi(R)$ and making use of Alperin's fusion theorem this implies in particular that if R, S are subgroups of P' which are isomorphic in $\mathcal{F}$ then $\mathrm{Aut}_{\mathcal{F}'}(R) \cong \mathrm{Aut}_{\mathcal{F}'}(S)$.

The unique category on the trivial subgroup $\{1\}$ of P is a fusion system which is normal in any fusion system $\mathcal{F}$ on P. The obvious motivating example for the definition of normal fusion systems is this:

Proposition 3.2. *Let G be a finite group, let P be a Sylow-p-subgroup of G, and let N be a normal subgroup of G. We have $\mathcal{F}_{P\cap N}(N) \trianglelefteq \mathcal{F}_P(G)$.*

Proof. Trivial. □

Proposition 3.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Then $\mathcal{F}_P(P)$ is normal in $\mathcal{F}$ if and only if $\mathcal{F} = N_{\mathcal{F}}(P)$.*

Proof. Suppose that $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$. Then in particular for any morphism $\varphi : R \to P$ in $\mathcal{F}$ and any $u \in N_P(R)$ there is $v \in N_P(\varphi(R))$ such that $\varphi({}^u r) = {}^v\varphi(r)$ for all $r \in R$. Whenever $\varphi(R)$ is fully $\mathcal{F}$-centralised, φ extends to a morphism $\psi : N_P(R) \to P$ in $\mathcal{F}$. In particular, this holds if R, and hence $\varphi(R)$, are $\mathcal{F}$-centric. But then also $N_P(R)$ and $\psi(N_P(R))$ are $\mathcal{F}$-centric. Inductively, it follows that φ can be extended to an automorphism of P belonging to $\mathcal{F}$. Thus, by Alperin's fusion theorem, we get $\mathcal{F} = N_{\mathcal{F}}(P)$. The converse is easy. □

In fact, Proposition 3.3 remains true with P replaced by any subgroup of P (cf. [21, 6.2] or [13, Corollary 2]).

Proposition 3.4. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. If Q is a strongly $\mathcal{F}$-closed abelian subgroup of P then $\mathcal{F}_Q(Q)$ is normal in $\mathcal{F}$.*

Proof. Since Q is abelian, the only morphisms in $\mathcal{F}_Q(Q)$ are inclusions $R \subseteq R'$ of subgroups R, R' of Q. Since Q is strongly $\mathcal{F}$-closed, the result follows. □

Proposition 3.5. *Let $\mathcal{F}$, $\mathcal{F}'$ be fusion systems on a finite p-group P such that $\mathcal{F}'$ is normal in $\mathcal{F}$. Then for every subgroup Q of P the index $[\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_{\mathcal{F}'}(Q)]$ is prime to p.*

Proof. Let Q be a subgroup of P, and let $\varphi : Q \to R$ be an isomorphism in $\mathcal{F}$ such that the subgroup R of P is fully $\mathcal{F}$-normalised. Then $\mathrm{Aut}_P(R)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$ by 1.5, and $\mathrm{Aut}_P(Q) \subseteq \mathrm{Aut}_{\mathcal{F}'}(R)$. Since $\mathcal{F}'$ is normal in $\mathcal{F}$, it follows that the Sylow-p-subgroup $\varphi^{-1} \circ \mathrm{Aut}_P(R) \circ \varphi$ of $\mathrm{Aut}_{\mathcal{F}}(Q)$ is contained in $\mathrm{Aut}_{\mathcal{F}'}(Q)$. Thus the index of $\mathrm{Aut}_{\mathcal{F}'}(Q)$ in $\mathrm{Aut}_{\mathcal{F}}(Q)$ is prime to p. □

Remark 3.6. Proposition 3.5 is not true, in general, without the assumption that $\mathcal{F}'$ is normal in $\mathcal{F}$. Consider the case of a fusion system $\mathcal{F}$ on P such that there is a subgroup Q of P which is fully $\mathcal{F}$-centralised but not fully $\mathcal{F}$-normalised, and set $\mathcal{F}' = \mathcal{F}_P(P)$. Then $\mathrm{Aut}_P(Q) = \mathrm{Aut}_{\mathcal{F}'}(Q)$ is not a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. The following is an example for this situation.

Example 3.7. Let $G = S_8$ be the symmetric group on eight letters, set $E_1 = <(15)(26)(37)(48)>$, $E_2 = <(13)(24),(57)(68)>$, $E_4 = <(12),(34),(56),(78)>$. Then $P = (E_4 \rtimes E_2) \rtimes E_1$ is a Sylow-2-subgroup of G. Set $\mathcal{F} = \mathcal{F}_P(G)$. The subgroup E_4 of P is $\mathcal{F}$-centric, hence $Q = E_4 \rtimes <(13)(24)(57)(68)>$ and $R = E_4 \rtimes E_1$ are $\mathcal{F}$-centric as well. Conjugating Q by $(35)(46)$ yields R, hence $Q \cong R$ in $\mathcal{F}$. Clearly Q is normal in P; in particular, Q is fully $\mathcal{F}$-normalised. Conjugating $(15)(26)(37)(48) \in R$ by $(13)(24) \in E_2$ yields $(17)(28)(35)(46)$. This is not an element in R since 7 does not belong to the R-orbit of 1 (which is equal to $\{1, 2, 5, 6\}$). Thus R is not normal in P, and hence R is not fully $\mathcal{F}$-normalised.

4 Simple fusion systems

Definition 4.1 A fusion system $\mathcal{F}$ on a non trivial finite p-group P is called *simple* if $\mathcal{F}$ has no proper non trivial normal fusion subsystem.

In view of work of Broto, Levi, Oliver [4] - introducing p-local finite groups as a generalisation of classifying spaces associated with fusion systems - we extend this terminology in the obvious way: a p-local finite group is called *simple* if its underlying fusion system is simple. In order to avoid confusion we point out that this definition is different from previous similar definitions such as fusion-simple groups (in a group theoretic context) or the notion of simple fusion systems introduced in [15].

Certainly the fusion system $\mathcal{F}_P(G)$ of a finite simple group G (with Sylow-p-subgroup P) does not have to be simple, but conversely, if a simple fusion system $\mathcal{F}$ on a finite p-group P is equal to $\mathcal{F}_P(G)$ for some finite group G containing P as Sylow-p-subgroup, then G can be chosen to be simple:

Proposition 4.2. *Let $\mathcal{F}$ be a simple fusion system on some finite p-group P. Suppose that $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G having P as Sylow-p-subgroup. If $O_{p'}(G) = 1$ and if $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P, then G is simple. In particular, if G has minimal order such that P is a Sylow-p-subgroup of G and such that $\mathcal{F} = \mathcal{F}_P(G)$, then G is simple.*

Proof. Suppose that $O_{p'}(G) = 1$ and that $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P. Let N be a non-trivial normal subgroup of G. Then $N \cap P$ is a Sylow-p-subgroup of N, and $\mathcal{F}_{N\cap P}(N)$ is a normal fusion system in $\mathcal{F}_P(G)$. As $O_{p'}(G) = 1$, we have $N \cap P \neq 1$. As $\mathcal{F}_P(G)$ is simple, this forces $P \subseteq N$ and $\mathcal{F}_P(N) = \mathcal{F}_P(G)$, hence $N = G$ by the assumptions. Let now G be a finite group of minimal order such that P is a Sylow-p-subgroup of G and such that $\mathcal{F} = \mathcal{F}_P(G)$. Then $O_{p'}(G) = 1$, because the canonical map $G \to G/O_{p'}(G)$ induces an isomorphism of fusion systems. By the minimality of G, we have $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P. Thus the second statement follows from the first. □

Proposition 4.3. *Let P be a finite p-group. Then $\mathcal{F}_P(P)$ is simple if and only if P is cyclic of order p.*

Proof. By 3.4, for every subgroup Z of $Z(P)$ we have $\mathcal{F}_Z(Z) \trianglelefteq \mathcal{F}_P(P)$, from which the statement follows. □

Proposition 4.4. *Let P be a finite abelian p-group and let $\mathcal{F}$ be a fusion system on P. Then $\mathcal{F}$ is simple if and only if P has order p and $\mathcal{F} = \mathcal{F}_P(P)$.*

Proof. If $\mathcal{F}$ is simple, then $\mathcal{F} = \mathcal{F}_P(P)$ by 3.4, and hence $|P| = p$ by 4.3. The converse is clear. □

The following Proposition is due to the referee and has greatly simplified the original version of this paper.

Proposition 4.5. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p-group P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$ and such that $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}'}(P)$. Then $\mathcal{F}' = \mathcal{F}$.*

Proof. Suppose that $\mathcal{F}' \neq \mathcal{F}$. Let Q be a subgroup of maximal order such that $\mathrm{Aut}_{\mathcal{F}'}(Q) \neq \mathrm{Aut}_{\mathcal{F}}(Q)$. By the assumptions, Q is a proper subgroup of P. Since $\mathcal{F}'$ is normal in $\mathcal{F}$ we may assume that Q is fully $\mathcal{F}$-normalised. Then $\mathrm{Aut}_P(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Moreover, $\mathrm{Aut}_{\mathcal{F}'}(Q)$ is a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ containing $\mathrm{Aut}_P(Q)$, and hence, by the Frattini argument, we have $\mathrm{Aut}_{\mathcal{F}}(Q) = N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q))\mathrm{Aut}_{\mathcal{F}'}(Q)$. By the extension axiom (II-S) in 1.4 every automorphism of Q in $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q))$ extends to an automorphism of $N_P(Q)$ in $\mathcal{F}$, hence in $\mathcal{F}'$ by the maximality assumption on Q. This in turn implies that $N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q)) \subseteq \mathrm{Aut}_{\mathcal{F}'}(Q)$, leading to the contradiction $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}'}(Q)$. □

Corollary 4.6. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Assume that $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_P(P)$ and that P has no proper non trivial strongly $\mathcal{F}$-closed subgroup. Then $\mathcal{F}$ is simple.*

Proof. Let $\mathcal{F}'$ be a fusion system on a non trivial subgroup P' of P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. Then P' is strongly $\mathcal{F}$-closed, hence $P' = P$ by the assumptions. Since $\mathrm{Aut}_P(P) \subseteq \mathrm{Aut}_{\mathcal{F}'}(P) \subseteq \mathrm{Aut}_{\mathcal{F}}(P)$, the assumptions imply further that $\mathrm{Aut}_{\mathcal{F}'}(P) = \mathrm{Aut}_{\mathcal{F}}(P)$. Thus $\mathcal{F}' = \mathcal{F}$ by 4.5. □

Corollary 4.7. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Suppose that P is generated by the set of its subgroups of order p, that all subgroups of order p in P are $\mathcal{F}$-conjugate and that $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_P(P)$. Then $\mathcal{F}$ is simple.*

Proof. Let Q be a non-trivial strongly $\mathcal{F}$-closed subgroup of P. Since all subgroups of order p of P are $\mathcal{F}$-conjugate it follows that Q contains all subgroups of order p of P. But then $Q = P$ by the assumptions on P, and hence $\mathcal{F}$ is simple by 4.6. □

5 Dihedral 2-local groups

In order to illustrate the terminology from previous sections, we determine for any fusion system on a dihedral 2-group all normal subsystems. In this section we set $P =< x > \rtimes < t >$, such that $x^{2^n} = 1 = t^2$ for some integer $n \geq 2$ and $txt = x^{-1}$; that is, P is a dihedral 2-group of order $2^{n+1} \geq 8$.

Then P has three conjugacy classes of involutions, namely the classes of the elements $z = x^{2^{n-1}}$, t and xt. Besides the trivial fusion system $\mathcal{F}_P = \mathcal{F}_P(P)$, there are two other systems, up to isomorphism. We denote by $\mathcal{F}_P^I$ the fusion system on P generated by $\mathcal{F}_P$ and an automorphism of order 3 of the Klein four group $< z > \times < t >$. Thus z and t are $\mathcal{F}_P^I$-conjugate, while z and xt are not; hence there are now two $\mathcal{F}_P^I$-conjugacy classes of involutions in P. We denote by $\mathcal{F}_P^{II}$ the fusion system on P generated by $\mathcal{F}_P$ and an automorphism of order 3 on each of the Klein four groups $< z > \times < t >$ and $< z > \times < xt >$. Thus all involutions in P are $\mathcal{F}_P^{II}$-conjugate. Any fusion system on P is isomorphic to one of $\mathcal{F}_P$, $\mathcal{F}_P^I$, $\mathcal{F}_P^{II}$ and any of these systems appear as fusion systems $\mathcal{F}_P(G)$ of some finite group G having P as Sylow-2-subgroup (this follows easily from Erdmann's list of examples in [7]). Any 2-block of a finite group having P as defect group has 1 or 2 or 3 isomorphism classes of simple modules, and then its fusion system is isomorphic to $\mathcal{F}_P$ or $\mathcal{F}_P^I$ or $\mathcal{F}_P^{II}$, respectively. The fusion systems $\mathcal{F}_P$, $\mathcal{F}_P^I$, $\mathcal{F}_P^{II}$ correspond to the cases (bb), (ab), (aa), respectively, in [6].

For notational convenience, if Q is a Klein four group, we denote by $\mathcal{F}_Q^I$ and by $\mathcal{F}_Q^{II}$ the unique fusion system on Q generated by some automorphism of order 3 of Q.

Theorem 5.1. *Let $\mathcal{F}$ be a fusion system on the dihedral 2-group P of order at least 8. Then $\mathcal{F}$ is simple if and only if $\mathcal{F} = \mathcal{F}_P^{II}$.*

One implication in 5.1 is a consequence of the following.

Lemma 5.2. *Let Q be the subgroup of index 2 of P generated by x^2 and t. Then $\mathcal{F}_Q^{II} \trianglelefteq \mathcal{F}_P^I$; in particular, Q is strongly $\mathcal{F}_P^I$-closed and $\mathcal{F}_P^I$ is not simple.*

Proof. Observe first that $\mathcal{F}_Q^{II}$ is contained in $\mathcal{F}_P^I$, because the three classes of involutions in Q represented by z, t, x^2t are all conjugate in $\mathcal{F}_P^I$. Indeed, this is clear for z and t by the definition of $\mathcal{F}_P^I$, and moreover, $x^2t = xtx^{-1}$. As $\mathcal{F}_Q^{II}$ is the unique maximal fusion system on Q, it suffices to show that Q is strongly $\mathcal{F}_P^I$-closed, which is easy. □

Proof of Theorem 5.1. All fusion systems $\mathcal{F}$ on P have the property $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_P(P)$. Let Q be a strongly $\mathcal{F}_P^{II}$-closed subgroup of P. Then Q contains all involutions of P, as all involutions of P are $\mathcal{F}_P^{II}$-conjugate. Hence $Q = P$, and 4.6 implies that $\mathcal{F}_P^{II}$ is simple. Conversely, $\mathcal{F}_P$ is not simple by 4.3 and $\mathcal{F}_P^I$ is not simple by 5.2. □

Remark 5.3. Let q be an odd prime power. If $q \equiv \pm 1(\mathrm{mod}8)$, then the group $PSL(2,q)$ has a dihedral Sylow-2-subgroup P, and $\mathcal{F}_P(PSL(2,q)) = \mathcal{F}_P^{II}$. In particular, $\mathcal{F}_P(PSL(2,q))$ is simple in that case. If $q \equiv \pm 3(\mathrm{mod}8)$ then $PSL(2,q)$ has a Klein four group Q as Sylow-2-subgroup, and hence $\mathcal{F}_P(PSL(2,q))$ cannot be simple. As pointed out by the referee, in this case the inclusion $\mathcal{F}_Q^{II} \trianglelefteq \mathcal{F}_P^{I}$ is realised by the inclusion $PSL_2(q) \trianglelefteq PGL_2(q)$. This yields an alternative proof of 5.2.

6 The 2-fusion system of $\Omega_7(q)$, $q \equiv \pm 3(\mathrm{mod}\ 8)$, is simple

The group theoretic background material needed in this and the next Section can be found in [5], [16], [17], [18], [22], [24].

Theorem 6.1. *Let q be an odd prime power such that $q \equiv \pm 3(\mathrm{mod}\ 8)$ and let P be a Sylow-2-subgroup of $\Omega_7(q)$. We have $\mathrm{Aut}_{\Omega_7(q)}(P) = \mathrm{Aut}_P(P)$ and P has no non-trivial proper strongly $\mathcal{F}_{\Omega_7(q)}$-closed subgroup. In particular, the fusion system $\mathcal{F}_S(\Omega_7(q))$ is simple.*

Proof. Since $Q \equiv \pm 3(\mathrm{mod}\ 8)$ the Sylow-2-subgroup P of $\Omega_7(q)$ is isomorphic to a Sylow-2-subgroup of the alternating group A_{12}, whose structure is as follows (cf. [16, §2]): the Thompson subgroup $A = J(P)$ is elementary abelian of order 2^6 and we have $P = A \rtimes D$ for D a dihedral group of order 8. In particular, P is generated by its set of involutions. Moreover, $Z(P)$ is a Klein four group contained in A. The statement $\mathrm{Aut}_{\Omega_7(q)}(P) = \mathrm{Aut}_P(P)$ is a particular case of [16, 2.1]. Let Q be a non-trivial strongly $\mathcal{F}_P(\Omega_7(q))$-closed subgroup of P. Then in particular Q is normal in P, hence $Q \cap Z(P) \neq 1$, and so $Q \cap A \neq 1$. By the remark preceding [16, 6.3], the cases [16, 4.7.(iii)], [16, 4.8.(iii)] and [16, 6.2.(iii)] correspond to the fusion system of $\Omega_7(q)$. It follows from [16, 4.7.(iii)] that the group $\mathrm{Aut}_{\Omega_7(q)}(A) \cong A_7$ acts irreducibly on A, and hence $A \subseteq Q$. By [16, 6.2.(iii)] every involution of P is $\Omega_7(q)$-conjugate to an involution in A. Thus Q contains all involutions in P, and hence $Q = P$ as P is generated by its set of involutions. The simplicity of the fusion system $\mathcal{F}_{\Omega_7(q)}$ follows from 4.6. □

7 The Solomon 2-local finite group Sol(3) is simple

Let q be an odd prime power such that $q \equiv \pm 3(\mathrm{mod}\ 8)$ and let P be a Sylow-2-subgroup of the 7-dimensional spinor group $\mathrm{Spin}_7(q)$ over $\mathbb{F}_q$. Then $\mathrm{Spin}_7(q)$ has a central involution z such that $\mathrm{Spin}_7(q)/ < z > \cong \Omega_7(q)$, and hence $P/ < z >$ is isomorphic to a Sylow-2-subgroup of $\Omega_7(q)$. R. Solomon showed in [18] that if $q \equiv \pm 3(\mathrm{mod}8)$, no finite group having P as Sylow-2-subgroup can have a fusion system which properly contains $\mathcal{F}_P(\mathrm{Spin}_7(q))$, in which all involutions of P are conjugate and which has the property that $C_{\mathcal{F}}(z)/ < z > \cong \mathcal{F}_S(\Omega(7,q))$. Levi and Oliver proved in [11, 2.1], that there is actually for any odd prime power q a fusion system $\mathcal{F}_{\mathrm{Sol}(q)}$ on P with the above properties, and that this fusion system is the underlying fusion system of a unique 2-local finite group; we are going to call this the *Solomon* 2-*local finite group* Sol(q). Kessar showed in [9] that the fusion system Sol(3) cannot even occur as fusion system of a 2-block of a finite group with P as defect group.

Theorem 7.1. *The Solomon* 2-*local finite group* Sol(3) *is simple.*

Proof. Let $\mathcal{F}$ be the underlying fusion system of Sol(3) on a Sylow-2-subgroup P of $\mathrm{Spin}_7(3)$ as constructed in [11, §2]. The normaliser of P in $\mathrm{Spin}_7(3)$ is the inverse image of the normaliser of a Sylow-2-subgroup of $\Omega_7(3)$, and hence $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathrm{Spin}_7(3)}(P) = \mathrm{Aut}_P(P)$, where the first equality uses [11, 2.1].

Let Q be a non trivial strongly $\mathcal{F}$-closed subgroup of P. In particular, Q is strongly $\mathcal{F}_P(\mathrm{Spin}_7(3))$-closed. Since all involutions in P are $\mathcal{F}$-conjugate, they are all contained in Q. Thus Q strictly contains $< z >$. Its image $\bar{Q} = Q/ < z >$ in $\bar{P} = P/ < z >$ is strongly $\mathcal{F}_{\bar{P}}(\Omega_7(3))$-closed. By 6.1 this forces $\bar{Q} = \bar{P}$, hence $Q = P$. Thus $\mathcal{F}$ is simple by 4.6. □

8 Characterisations of fusion systems

Proposition 4.5 would be false without the assumption on $\mathcal{F}'$ being normal in $\mathcal{F}$. For the sake of completeness, we include some statements regarding the situation of not necessarily normal subsystems.

The first result shows that a fusion system $\mathcal{F}$ on a finite p-group P is determined by its fusion on elements of order p in P and their centralisers in $\mathcal{F}$. If Q is a subgroup of P, we denote by $C_{\mathcal{F}}(Q)/Z(Q)$ the category on $C_P(Q)/Z(Q)$ whose morphisms are induced by morphisms in $C_{\mathcal{F}}(Q)$ via the canonical map $C_P(Q) \to C_P(Q)/Z(Q)$. By the remarks following 1.8, if Q is fully $\mathcal{F}$-centralised, then $C_{\mathcal{F}}(Q)/Z(Q)$ is a fusion system on $C_P(Q)/Z(Q)$.

Proposition 8.1. *Let P be a finite p-group, and let $\mathcal{F}$, $\mathcal{F}'$ be fusion systems on P such that $\mathcal{F}' \subseteq \mathcal{F}$. The following are equivalent.*

(i) $\mathcal{F} = \mathcal{F}'$.

(ii) For any fully $\mathcal{F}'$-centralised subgroup Z of order p of P we have $\mathrm{Hom}_{\mathcal{F}}(Z, P) = \mathrm{Hom}_{\mathcal{F}'}(Z, P)$ *and* $C_{\mathcal{F}}(Z) = C_{\mathcal{F}'}(Z)$.

Proof. Suppose that (ii) holds. Let Q be a non trivial subgroup of P and let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Let Z be a subgroup of order p of $Z(Q)$. Let $\psi : Z \to P$ be a morphism in $\mathcal{F}'$ such that $\psi(Z)$ is fully $\mathcal{F}'$-centralised. Since $Q \subseteq C_P(Z)$, the morphism ψ extends to a morphism $\tau : Q \to P$ in $\mathcal{F}'$. In order to show that φ is a morphism in $\mathcal{F}'$, it suffices to show that $\tau \circ \varphi \circ \tau^{-1}|_{\tau(Q)} \in \mathrm{Aut}_{\mathcal{F}'}(\tau(Q))$. Thus, after replacing Q by $\tau(Q)$, we may assume that Z is fully $\mathcal{F}'$-centralised. By the assumptions, the morphism $\varphi^{-1}|_{\varphi(Z)} : \varphi(Z) \to Z$ belongs to $\mathcal{F}'$, and hence extends to a morphism $\kappa : Q \to P$ in $\mathcal{F}'$ (since $Q = \varphi(Q) \subseteq C_P(\varphi(Z))$). Then $\kappa \circ \varphi : Q \to P$ restricts to the identity on Z, hence $\kappa \circ \varphi$ is a morphism in $C_{\mathcal{F}}(Z) = C_{\mathcal{F}'}(Z)$. In particular, $\kappa \circ \varphi$ is a morphism in $\mathcal{F}'$. But then so is φ, because κ is in $\mathcal{F}'$. Alperin's fusion theorem implies now (i). The converse is trivial. □

Corollary 8.2. *Let $\mathcal{F}$, $\mathcal{F}'$ be fusion systems on a finite p-group P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. If $\mathrm{Hom}_{\mathcal{F}}(Z,P) = \mathrm{Hom}_{\mathcal{F}'}(Z,P)$ and $C_{\mathcal{F}}(Z)/Z$ is a simple fusion system on $C_P(Z)/Z$ for any fully $\mathcal{F}'$-centralised subgroup Z of order p of P, then $\mathcal{F}' = \mathcal{F}$.*

Proof. We have $C_{\mathcal{F}'}(Z) \trianglelefteq C_{\mathcal{F}}(Z)$ and hence $C_{\mathcal{F}'}(Z)/Z \trianglelefteq C_{\mathcal{F}}(Z)/Z$. Thus, if $C_{\mathcal{F}}(Z)/Z$ is simple for any fully $\mathcal{F}'$-centralised subgroup Z of order p of P, then $C_{\mathcal{F}'}(Z)/Z = C_{\mathcal{F}}(Z)/Z$. Since p'-automorphisms lift uniquely through central p-extensions this implies $C_{\mathcal{F}'}(Z) = C_{\mathcal{F}}(Z)$, hence $\mathcal{F}' = \mathcal{F}$ by 8.1. □

Lemma 8.3. *Let $\mathcal{F}$, $\mathcal{F}'$ be fusion systems on a finite p-group P such that $\mathcal{F}' \subseteq \mathcal{F}$. Let $Q \xrightarrow{\varphi} R \xrightarrow{\psi} S$ be a sequence of two composable morphisms in $\mathcal{F}$ such that Q, R, S are $\mathcal{F}$-centric. If any two of the three morphisms φ, ψ, $\psi \circ \varphi$ are in $\mathcal{F}'$, so is the third.*

Proof. If φ, ψ are in $\mathcal{F}'$, so is $\psi\circ\varphi$. If ψ, $\psi\circ\varphi$ are in $\mathcal{F}'$, then so is $\varphi = \psi^{-1}|_{\mathrm{Im}(\psi\circ\varphi)}\circ\psi\circ\varphi$. Assume now that φ and $\psi \circ \varphi$ are morphisms in $\mathcal{F}'$. Up to replacing Q by $\varphi(Q)$, we may assume that φ is the inclusion $Q \subseteq R$. Let $v \in N_R(Q)$. Then, for any $u \in Q$, we have $\psi({}^v u) = {}^{\psi(v)}u$. Thus the morphism $\psi|_Q$ extends to a morphism $\tau : N_R(Q) \to P$ in $\mathcal{F}'$. By 1.11, we have $\tau(N_R(Q)) = \psi(N_R(Q))$ and hence $\psi^{-1} \circ \tau \in \mathrm{Aut}_{Z(Q)}(N_R(Q))$ by 1.10. Thus $\psi|_{N_R(Q)}$ is a morphism in $\mathcal{F}'$. It follows inductively, that ψ is a morphism in $\mathcal{F}'$. □

Proposition 8.4. *Let $\mathcal{F}$, $\mathcal{F}'$ be fusion systems on a finite p-group P such that $\mathcal{F}' \subseteq \mathcal{F}$. The following are equivalent.*

(i) $\mathcal{F} = \mathcal{F}'$.

(ii) $\mathrm{Hom}_{\mathcal{F}}(Q,P) = \mathrm{Hom}_{\mathcal{F}'}(Q,P)$ *for every minimal $\mathcal{F}$-centric subgroup Q of P.*

Proof. Assume that (ii) holds. Let R be an $\mathcal{F}$-centric subgroup of P, and let Q be a minimal $\mathcal{F}$-centric subgroup of P contained in R. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R,P)$. Then $\varphi|_Q \in \mathrm{Hom}_{\mathcal{F}}(Q,P) = \mathrm{Hom}_{\mathcal{F}'}(Q,P)$. But then $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(R,P)$ by 8.3. Alperin's fusion theorem implies (i). The converse is trivial. □

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