INDUCTION FOR INTERIOR ALGEBRAS

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Abstract. The aim of this note is to give, in Section 2, a short self-contained proof of transitivity properties of non-injective induction for interior G-algebras, due to Puig [2, 3]. It turns out to be convenient to do this in a slightly more general setup, which is described in Section 1. Section 3 contains some further formal properties of algebra induction.

1 Algebra induction - definition and examples

We define an induction for interior algebras which generalises Puig's concepts of induction, including its non-injective version, for interior G-algebras in [2], [3]. Throughout this note we fix a commutative ring R. By default, a module over an $R-$algebra A is a unitary left module. Any right $A-$module can be considered as a left module over the opposite algebra A^0. Given three $R-$algebras A, B, C, an $A-B-$bimodule M and an $A-C-$bimodule N, we denote by $\mathrm{Hom}_A(M,N)$ the $R-$module of all homomorphisms from M to N as left $A-$modules; this has a canonical structure of $C-B-$bimodule given by $(c.\varphi.b)(m) = \varphi(mb)c$ for any $b \in B$, $c \in C$ and $\varphi \in \mathrm{Hom}_A(M,N)$. Similarly, if N' is a $C-B-$bimodule, we denote by $\mathrm{Hom}_{B^0}(M,N')$ the $R-$module consisting of all homomorphisms from M to N' viewed as right $B-$modules; again, this has a canonical structure of $C-A-$bimodules given by $(c.\psi.a)(m) = c\psi(am)$ for any $a \in A$, $c \in C$ and $\psi \in \mathrm{Hom}_{B^0}(M,N')$.

Following the terminology of [1] (which extends that of [2, 3.1]), if A is an $R-$algebra, an *interior $A-$algebra* is an $R-$algebra B endowed with a unitary algebra homomorphism $\sigma : A \to B$. Note that in particular B becomes an $A-A-$bimodule through σ.

Definition 1.1 Let A, B be $R-$algebras, let C be an interior $B-$algebra, and let M be an $A-B-$bimodule. We set

$$\mathrm{Ind}_M(C) = \mathrm{End}_{C^0}(M \underset{B}{\otimes} C) \ ,$$

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considered as interior $A-$algebra with structural homomorphism $A \to \mathrm{Ind}_M(C)$ mapping $a \in A$ to the C^0-endomorphism of $M \underset{B}{\otimes} C$ given by left multiplication with a on $M \underset{B}{\otimes} C$.

Remark 1.2 The above definition makes sense for differential graded algebras. In order to limit technicalities, we leave it to the reader to verify, that the content of this note can easily be adapted to this more general context.

We briefly sketch, that this construction generalises the corresponding notions in [2, 3.3] and [3, 3.3].

Example 1.3 Let G be a finite group, H a subgroup of G and let C be an interior $H-$algebra over R; that is, C is an $R-$algebra endowed with a group homomorphism $H \to C^\times$ (cf. [2, 3.1]). This group homomorphism extends to an algebra homomorphism $RH \to C$, through which C becomes an interior $RH-$algebra in the sense defined above. Set $M = RG$, considered as $RG - RH-$bimodule. Then

$$\mathrm{Ind}_H^G(C) \cong \mathrm{Ind}_M(C) \ ,$$

where the left side is the induction defined in [2, 3.3]. To see this it suffices to apply the standard isomorphisms $\mathrm{Ind}_M(C) = \mathrm{End}_{C^0}(RG \underset{RH}{\otimes} C) \cong \mathrm{Hom}_{RH^0}(RG, RG \underset{RH}{\otimes} C) \cong RG \underset{RH}{\otimes} C \underset{RH}{\otimes} \mathrm{Hom}_{RH^0}(RG, RH) \cong RG \underset{RH}{\otimes} C \underset{RH}{\otimes} RG = \mathrm{Ind}_H^G(C)$. Explicitly, the map sending $x \otimes c \otimes y \in \mathrm{Ind}_H^G(C)$ to the C^0-endomorphism φ of $RG \underset{RH}{\otimes} C$ defined by $\varphi(z \otimes d) = x \otimes cyzd$ if $yz \in H$ and $\varphi(z \otimes d) = 0$ otherwise is an algebra isomorphism (where $x, y, z \in G$ and $c, d \in C$). See also [4, 10.7, 11.2].

Example 1.4 Let G, H be finite groups, let $\varphi : H \to G$ be a group homomorphism and let C be an interior $H-$algebra. Set $M = (RG)_\varphi$; that is, M is the $RG - RH-$bimodule which is equal to RG, endowed with the regular action of RG on the left and with the action of RH on the right given by restriction through φ. Then

$$\mathrm{Ind}_M(C) \cong \mathrm{Ind}_\varphi(C) \ ,$$

where the right side is the non-injective induction defined in [3, 3.3]. To see this, consider first the case where φ is surjective. Set $K = \ker(\varphi)$. Then $\mathrm{Ind}_M(C) = \mathrm{End}_{C^0}((RG)_\varphi \underset{RH}{\otimes} C) \cong \mathrm{End}_{C^0}(R \underset{RK}{\otimes} C) \cong \mathrm{Hom}_{RK^0}(R, R \underset{RK}{\otimes} C) \cong (R \underset{RK}{\otimes} C)^K$, which coincides with the expression in [3, 3.3.1]. The general case follows from combining this with 1.3 and the transitivity theorem 2.1 below.

Example 1.5 Let A, B be $R-$algebras, and let M be an $A-B-$bimodule such that the functor $M \underset{B}{\otimes} -$ induces an equivalence of categories $\mathrm{Mod}(B) \cong \mathrm{Mod}(A)$. Consider B as interior $B-$algebra in the trivial way; that is, via the identity map on B. Then Morita's theorem reads $\mathrm{Ind}_M(B) \cong A$.

2 Transitivity of algebra induction

The transitivity theorem below generalises and simplifies the corresponding statement in [3, §3]:

Theorem 2.1. *Let A, B, C be $R-$algebras, let D be an interior $C-$algebra, let M be an $A-B-$bimodule and let N be a $B-C-$bimodule. There is a canonical homomorphism of interior $A-$algebras*

$$\mathrm{Ind}_M(\mathrm{Ind}_N(D)) \longrightarrow \mathrm{Ind}_{M\underset{B}{\otimes}N}(D) \ ,$$

which is an isomorphism, if M is finitely generated projective as right $B-$module, or if N is finitely generated projective as right $C-$module, or if both $N\underset{C}{\otimes}D$ and $M\underset{B}{\otimes}N\underset{C}{\otimes}D$ are finitely generated projective as right $C-$modules.

The following technical lemma is needed in the proof of 2.1.

Lemma 2.2. *Let A, B, C be $R-$algebras, let M be an $A-B-$bimodule and let N, N' be $B-C-$bimodules. Then the $A-B-$homomorphism*

$$\Phi : \begin{cases} M\underset{B}{\otimes}\mathrm{Hom}_{C^0}(N,N') & \to \mathrm{Hom}_{C^0}(N, M\underset{B}{\otimes}N') \\ m\otimes\varphi & \mapsto (n\mapsto m\otimes\varphi(n)) \end{cases}$$

is natural in the three variables M, N, N'. Moreover, Φ is an isomorphism, if M is finitely generated projective as right $B-$module, or if N is finitely generated projective as right $C-$module, or if both N', $M\underset{B}{\otimes}N'$ are finitely generated projective as right $C-$modules.

Proof. The fact that Φ is a homomorphism of $A-B-$bimodules which is natural in M, N, N' is a straightforward verification. In order to show the last statement, it suffices to see that Φ is an $R-$linear isomorphism in any of the three cases stated above. If $M=B$ as right $B-$module, both sides are isomorphic to $\mathrm{Hom}_{C^0}(N,N')$. By naturality, it follows that Φ is an isomorphism whenever M is finitely generated projective as right $B-$module. Similarly, Φ is an isomorphism if $N=C$ as right $C-$module, and thus Φ is an isomorphism whenever N is finitely generated projective as right $C-$module. For the last case, if both N' and $M\underset{B}{\otimes}N'$ are finitely generated projective as right $C-$modules, we have canonical isomorphisms $M\underset{B}{\otimes}\mathrm{Hom}_{C^0}(N,N')\cong M\underset{B}{\otimes}N'\underset{C}{\otimes}\mathrm{Hom}_{C^0}(N,C)\cong\mathrm{Hom}_{C^0}(N,M\underset{B}{\otimes}N')$ by applying twice the right analogue of 4.3. A straightforward verification shows that the composition of the two isomorphisms coincides with Φ. $\square$

Proof of 2.1. Using the standard adjunction 4.2, we have a canonical $R-$isomorphism

$$\mathrm{Ind}_N(D) \cong \mathrm{Hom}_{C^0}(N, N \underset{C}{\otimes} D) \ .$$

Applying this again, we get a canonical $R-$isomorphism

$$\mathrm{Ind}_M(\mathrm{Ind}_N(D)) \cong \mathrm{Hom}_{B^0}(M, M \underset{B}{\otimes} \mathrm{Hom}_{C^0}(N, N \underset{C}{\otimes} D)) \ .$$

Applying this yet again to $M \underset{B}{\otimes} N$ yields an $R-$isomorphism

$$\mathrm{Ind}_{M \underset{B}{\otimes} N}(D) \cong \mathrm{Hom}_{C^0}(M \underset{B}{\otimes} N, M \underset{B}{\otimes} N \underset{C}{\otimes} D) \cong \mathrm{Hom}_{B^0}(M, \mathrm{Hom}_{C^0}(N, M \underset{B}{\otimes} N \underset{C}{\otimes} D)) \ .$$

Now the map Φ from 1.6 with $N \underset{C}{\otimes} D$ instead of N' induces the algebra homomorphism with the required properties. $\square$

Remark 2.3 The algebra homomorphism in 2.1 can explicitly be described as follows (using the explicit formulae from §4). Let $\varphi \in \mathrm{Ind}_M(\mathrm{Ind}_N(D))$ and denote by ψ the image in $\mathrm{Ind}_{M \underset{B}{\otimes} N}(D)$. Then, for any $m \in M$, any $n \in N$ and any $d \in D$ we have

$$\psi(m \otimes n \otimes d) = \sum_i m_i \otimes \varphi_i(n \otimes d) \ ,$$

where $m_i \in M$, $\varphi_i \in \mathrm{Ind}_N(D)$ such that $\varphi(m \otimes Id_{N \underset{C}{\otimes} D}) = \sum_i m_i \otimes \varphi_i$, and where i runs over a finite indexing set.

3 Functoriality of algebra induction

Proposition 3.1. *Let A, B be $R-$algebras, let M be an $A-B-$bimodule, and let $\gamma : C \to C'$ be a homomorphism of interior $B-$algebras. The functor $- \underset{C}{\otimes} C'$ induces a homomorphism of interior $A-$algebras*

$$\mathrm{Ind}_M(\gamma) : \mathrm{Ind}_M(C) \longrightarrow \mathrm{Ind}_M(C') \ .$$

Proof. Applying the functor $- \underset{C}{\otimes} C'$ to any C^0-endomorphism of $M \underset{B}{\otimes} C$ yields a $(C')^0-$endomorphism of $M \underset{B}{\otimes} C \underset{C}{\otimes} C' \cong M \underset{B}{\otimes} C'$, where the left $C-$module structure of C' is given by γ. $\square$

Let A, B be $R-$algebras, let M be an $A-B-$bimodule, and let C be an interior $B-$algebra. The algebra $\mathrm{Ind}_M(C)$ acts on $M \underset{B}{\otimes} C$ by $\alpha.n = \alpha(n)$ for any $\alpha \in \mathrm{Ind}_M(C)$

and any $n \in M \underset{B}{\otimes} C$. In this way, $M \underset{B}{\otimes} C$ becomes an $\mathrm{Ind}_M(C) - C-$ bimodule. This bimodule gives rise to induction and corestriction functors between the module categories of C and $\mathrm{Ind}_M(C)$, denoted by

$$\mathrm{Ind}_M = (M \underset{B}{\otimes} C) \underset{C}{\otimes} - : \mathrm{Mod}(C) \longrightarrow \mathrm{Mod}(\mathrm{Ind}_M(C)) ,$$

$$\mathrm{Res}_M = \mathrm{Hom}_{\mathrm{Ind}_M(C)}(M \underset{B}{\otimes} C, -) : \mathrm{Mod}(\mathrm{Ind}_M(C)) \longrightarrow \mathrm{Mod}(C) .$$

The following is just a particular case of the standard adjunction 4.2:

Proposition 3.2. *Let A, B be $R-$algebras, let M be an $A - B-$bimodule and let C be an interior $B-$algebra. The functor Ind_M is left adjoint to the functor Res_M.*

The following converse of 1.5 is again a particular case of Morita's theorem:

Proposition 3.3. *Let A, B be $R-$algebras, let M be an $A-B-$bimodule and let C be an interior $B-$algebra. If $M \underset{B}{\otimes} C$ is a progenerator as right $C-$module, then Ind_M and Res_M are mutually inverse equivalences between $\mathrm{Mod}(\mathrm{Ind}_M(C))$ and $\mathrm{Mod}(C)$. This case occurs in particular, if M is a progenerator as right $B-$module.*

An $R-$algebra A is called it symmetric, if A is finitely generated projective as $R-$module, and if A is isomorphic to its $R-$dual $A^* = \mathrm{Hom}_R(A, R)$ as $A-A-$bimodule.

Proposition 3.4. *Let A, B be $R-$algebras, let M be an $A - B-$bimodule and let C be an interior $B-$algebra. If C is symmetric, B is finitely generated projective as $R-$module and M is finitely generated projective as right $B-$module, then $\mathrm{Ind}_M(C)$ is symmetric.*

Proof. The hypotheses imply that $M \underset{B}{\otimes} C$ and hence $\mathrm{Ind}_M(C)$ is finitely generated projective as $R-$module. Using the isomorphism $\mathrm{Ind}_M(C) = \mathrm{End}_{C^0}(M \underset{B}{\otimes} C) \cong (M \underset{B}{\otimes} C) \underset{C}{\otimes} (M \underset{B}{\otimes} C)^*$ from 4.5 as well as 4.6 implies that $\mathrm{Ind}_C(M)$ is indeed symmetric. $\square$

4 Appendix

We collect in this section some of the (well-known) standard isomorphisms between modules and homomorphism spaces as we need them in this note. Let A, B, C be $R-$algebras, let M be an $A - B-$bimodule and let N be an $A - C-$bimodule. Recall from §1 that the space of homomorphisms $\mathrm{Hom}_A(M, N)$ from M to N as left $A-$modules becomes a $B - C-$bimodule via $(b.\varphi.c)(m) = \varphi(mb)c$; similarly, the space of homomorphisms $\mathrm{Hom}_{B^0}(M, N')$ of homomorphisms from M to N' as right $B-$modules becomes a $C - A-$bimodule via $(c.\psi.a)(m) = c\psi(am)$. There is an isomorphism of bifunctors

4.1.
$$\mathrm{Hom}_A(M \underset{B}{\otimes} -, -) \cong \mathrm{Hom}_B(-, \mathrm{Hom}_A(M, -))$$

on $\mathrm{Mod}(B) \times \mathrm{Mod}(A)$ given, for any $A-$module U and any $B-$module V, by the mutually inverse isomorphisms

4.2.
$$\begin{cases} \mathrm{Hom}_A(M \underset{B}{\otimes} V, U) & \cong & \mathrm{Hom}_B(V, \mathrm{Hom}_A(M, U)) \\ \varphi & \mapsto & (v \mapsto (m \mapsto \varphi(m \otimes v))) \\ (m \otimes v \mapsto \psi(v)(m)) & \leftarrow & \psi \end{cases} .$$

This isomorphism of bifunctors has an obvious analogue for right modules, namely the isomorphism $\mathrm{Hom}_{B^0}(- \underset{A}{\otimes} M, -) \cong \mathrm{Hom}_{A^0}(-, \mathrm{Hom}_{B^0}(M, -))$. There is a homomorphism of $B-C-$bimodules

4.3.
$$\begin{cases} \mathrm{Hom}_A(M, A) \underset{A}{\otimes} N & \rightarrow & \mathrm{Hom}_A(M, N) \\ \varphi \otimes n & \mapsto & (m \mapsto \varphi(m)n) \end{cases}$$

which is functorial in M and N. This is an isomorphism if one of M, N is finitely generated projective as $A-$module. Again, there is an obvious right analogue $N' \underset{B}{\otimes} \mathrm{Hom}_{B^0}(M, B) \rightarrow \mathrm{Hom}_{B^0}(M, N')$, which is an isomorphism if one of M, N' is finitely generated projective as right $B-$modules.

If A is symmetric and $s \in A^* = \mathrm{Hom}_R(A, R)$ is the image of 1_A under some $A-A-$bimodule isomorphism $A \cong A^*$, there is an isomorphism of $B-A-$bimodules

4.4.
$$\begin{cases} \mathrm{Hom}_A(M, A) & \cong & M^* \\ \varphi & \mapsto & s \circ \varphi \end{cases}$$

which is functorial in M. Similarly, if B is symmetric, there is an isomorphism of $B-A-$bimodules $\mathrm{Hom}_{B^0}(M, B) \cong M^*$ which is functorial in M. In other words, if A and B are symmetric, the three duality functors $\mathrm{Hom}_A(-, A)$, $\mathrm{Hom}_{B^0}(-, B)$ and $\mathrm{Hom}_R(-, R)$ are naturally isomorphic through isomorphisms depending on the choices of symmetrising forms on A and B.

Combining 4.3 and 4.4, if A is symmetric and if one of M, N is finitely generated projective as left $A-$module, there is a natural isomorphism of $B-C-$bimodules

4.5.
$$\mathrm{Hom}_A(M, N) \cong M^* \underset{A}{\otimes} N \ .$$

Similarly, if B is symmetric and one of M, N' is finitely generated projective as right $B-$modules, we have a natural isomorphism $\mathrm{Hom}_{B^0}(M, N') \cong N' \underset{B}{\otimes} M^*$ as $C-A-$bimodules. Finally, if A is symmetric and if M is finitely generated projective as $A-$module, there is a natural isomorphism of $C-B-$bimodules

4.6.
$$(M^* \underset{A}{\otimes} N)^* \cong N^* \underset{A}{\otimes} M \ .$$

Sketch of proof. The proof of 4.2 is a straightforward verification. The given map in 4.3 is easily seen to be functorial, and thus, in order to show that it is an isomorphism if one of M, N is finitely generated projective as left $A-$module, it suffices to verify this for $M = A$ or $N = A$, which is trivial. For the proof of 4.4 we use that $A \cong A^* = \mathrm{Hom}_R(A, R)$, together with the adjunction 4.2, from which we get $\mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_R(A, R)) \cong \mathrm{Hom}_A(A \underset{A}{\otimes} M, R) \cong M^*$. Statement 4.5 follows from combining 4.3 and 4.4; finally, statement 4.6 is obtained from 4.2 and 4.5 by $(M^* \underset{A}{\otimes} N)^* = \mathrm{Hom}_R(M^* \underset{A}{\otimes} N, R) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_R(M^*, R)) \cong \mathrm{Hom}_A(N, M) \cong N^* \underset{A}{\otimes} M$. $\square$

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