LOCAL CONTROL IN FUSION SYSTEMS OF P-BLOCKS OF FINITE GROUPS

Radha Kessar, Markus Linckelmann, Geoffrey R. Robinson

August 2001

Abstract. If p is an odd prime, b a p-block of a finite group G such that $SL(2,p)$ is not involved in $N_G(Q,e)/C_G(Q)$ for any b-subpair (Q,e), then $N_G(Z(J(P)))$ controls b-fusion, where P is a defect group of b. This is a block theoretic analogue of Glauberman's ZJ-Theorem [6].

1 Introduction

Glauberman's ZJ-Theorem [6, Theorem B] states that if p is an odd prime and G is a finite group such that $Qd(p)$ is not involved in G, then $N_G(Z(J(P)))$ controls p-fusion in G, for P a Sylow p-subgroup of G. Here, $J(P)$ denotes the Thompson subgroup of P (that is, the subgroup generated by all abelian subgroups of P of maximal order) and $Qd(p)$ denotes the semi-direct product of $C_p \times C_p$ with $SL(2,p)$ (with the natural action). This has proved to be an extremely powerful tool in local group-theoretic analysis, as it gives a general condition which ensures that p-fusion is controlled by a single p-local subgroup.

In this paper, we establish block-theoretic analogues of this and other similar results. Along the way, we will obtain results which seem to be new even in the group-theoretic case. A key ingredient, allowing us to exploit the existing group-theoretic methods, is a result of Külshammer and Puig [11] on extensions of nilpotent blocks. We also show (both in a group-theoretic and in a block-theoretic context) that if a normal subgroup of a given group G has a single local subgroup which controls fusion, then G itself has a single local subgroup with the same property. We discuss some consequences of such control of fusion to other problems in block theory.

Throughout the paper, k will denote an algebraically closed field of prime characteristic p. A *block of a finite group* G is a primitive idempotent b in $Z(kG)$; following Alperin-Broué [1], a (G,b)-*subpair* is a pair (Q,e) consisting of a p-subgroup Q of G and a block e of $C_G(Q)$ such that $\mathrm{Br}_Q(b)e = e$, where $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$ is the *Brauer*

1991 *Mathematics Subject Classification.* 20C20.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

homomorphism [5]. The set of (G, b)-subpairs is a partially ordered set on which G acts by conjugation, and the maximal (G, b)-subpairs with respect to this partial order are all G-conjugate. If (P, e) is a maximal (G, b)-subpair, then P is called a *defect group of the block* b (this notion is due to Brauer [2]); moreover, for any subgroup Q of P there is a unique block e_Q of $C_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$ (cf. [1]). A detailed account of subpairs and their properties may be found in [14] (where subpairs are referred to as Brauer pairs). The local structure of b is the G-set of (G, b)-subpairs viewed as category; the following definition makes this precise.

Definition 1.1. Let G be a finite group, let b be a block of G and let (P, e) be a maximal (G, b)-subpair. For any subgroup Q of P denote by e_Q the unique block of $C_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. We denote by $\mathcal{F}_{(P,e)}(G, b)$ the category whose objects are the subgroups of P and whose sets of morphisms $\mathrm{Hom}_{\mathcal{F}_{(P,e)}(G,b)}(Q, R)$ are the sets of group homomorphisms $\varphi : Q \to R$ for which there exists an element $x \in G$ satisfying ${}^x(Q, e_Q) \subseteq (R, e_R)$ and $\varphi(u) = xux^{-1}$ for all $u \in Q$, where Q, R run over the set of subgroups of P.

Since all maximal (G, b)-subpairs are G-conjugate, the category $\mathcal{F}_{(P,e)}(G, b)$ does not depend on the choice of (P, e) up to isomorphism of categories. If b is the principal block of G then P is a Sylow-p-subgroup of G and e_Q is the principal block of $C_G(Q)$ for any subgroup Q of P; in this case we write $\mathcal{F}_P(G) = \mathcal{F}_{(P,e)}(G, b)$. Glauberman's ZJ-Theorem reads then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(Z(J(P))))$, provided that p is odd and $Qd(p)$ is not involved in G.

We need a block-theoretic replacement for the hypothesis on $Qd(p)$. Recall that if G is a finite group and b is a block of G, then a (G, b)-subpair (Q, f) is called *centric* if $Z(Q)$ is a defect group of f and (Q, f) is called *radical* if $O_p(N_G(Q, f)/QC_G(Q)) = 1$. The notion of centric subpairs - frequently called self-centralising pairs in the literature - goes back to Brauer [3].

Definition 1.2. Let G be a finite group. A block b of G is called *$SL(2, p)$-free* if $SL(2, p)$ is not isomorphic to a subquotient of any of the groups $N_G(Q, f)/C_G(Q)$, where (Q, f) is a centric and radical (G, b)-subpair.

The definition of an $SL(2, p)$-free block is really a local condition on the block, in that it can be formulated purely in terms of the category $\mathcal{F}_{(P,e)}(G, b)$, where (P, e) is a maximal subpair of a block b of G. Indeed, b is $SL(2, p)$-free if and only if $SL(2, p)$ is not involved in the automorphism group in $\mathcal{F}_{(P,e)}(G, b)$ of any subgroup Q of P such that (Q, e_Q) is centric and radical for the unique e_Q such that $(Q, e_Q) \leq (P, e)$. It may well happen that a non principal block b of G is $SL(2, p)$-free even though $SL(2, p)$ is involved in G. If, however, the principal block of G is $SL(2, p)$-free, then $Qd(p)$ is not involved in G (cf. Proposition 5.1 and [7, Lemma 10.6]). In this case, our hypothesis "$SL(2, p)$-free" is in fact slightly more restrictive, since (in the principal block case) it effectively excludes faithful action of $SL(2, p)$ on any p-subgroup of G, not just the natural action of $SL(2, p)$ on $C_p \times C_p$.

Examples of $SL(2,p)$-free blocks include all blocks with abelian defect groups and, for $p \geq 5$, all blocks of finite p-solvable groups, or more generally, all blocks for which the groups $N_G(Q,f)/C_G(Q)$ occurring in 1.2 are p-solvable.

Since Glauberman's control of fusion theorems also apply to some characteristic subgroups of p-groups other than the center of the Thompson subgroup, we make the following definitions, the first of which is given in [9, §5].

Definition 1.3 A *positive characteristic p-functor* is a map W sending any finite p-group P to a subgroup $W(P)$ of P, with the property that $W(P) \neq 1$ if $P \neq 1$ and that any isomorphism of finite p-groups $P \cong Q$ maps $W(P)$ onto $W(Q)$. A *Glauberman functor* is a positive characteristic p-functor W with the following additional property: whenever P is a Sylow-p-subgroup of a finite group L which satisfies $C_L(O_p(L)) = Z(O_p(L))$ and which does not have a subquotient isomorphic to $Qd(p)$, then $W(P)$ is normal in L.

Of course, by Glauberman's ZJ-Theorem the map sending a finite p-group P to $Z(J(P))$ is a Glauberman functor; in fact showing that this map is a Glauberman functor is the essential ingredient of the ZJ-Theorem. By [7, Theorem 14.8] any of the maps sending a finite p-group P to $K_\infty(P)$ or $K^\infty(P)$ are Glauberman functors, where $K_\infty(P)$, $K^\infty(P)$ are defined in [7, Section 12].

If W is a positive characteristic p-functor, then $W(P)$ is characteristic in P, for any finite p-group P; in particular, if P is a p-subgroup of a finite group G, then $N_G(W(P))$ contains $N_G(P)$. If H is any subgroup of G containing $N_G(P)$, there is a unique block c of H such that $\mathrm{Br}_P(b) = \mathrm{Br}_P(c)$, the *Brauer correspondent of* b (cf. [1] or [14]). Then P is again a defect group of c, and since $C_G(P) \subseteq H$, every maximal (G,b)-subpair (P,e) is also a maximal (H,c)-subpair.

We are now ready to state our results. In what follows, refer to 2.1 and 2.3 for the exact definition of control of fusion that we are using.

Theorem 1.4. *Let G be a finite group, let b be a block of G and let (P,e) be a maximal (G,b)-subpair. Let W be a Glauberman functor, set $N = N_G(W(P))$ and denote by c the unique block of N such that $\mathrm{Br}_P(b) = \mathrm{Br}_P(c)$. If p is odd and b is $SL(2,p)$-free, then $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(N,c)$. In other words, the group N controls fusion in $\mathcal{F}_{(P,e)}(G,b)$.*

The proof of 1.4 is given in section 6. If we specialise Theorem 1.4 to the case of principal blocks and $W(P) = Z(J(P))$, we obtain the conclusion of Glauberman's ZJ-Theorem (but, as mentioned above, our hypothesis "$SL(2,p)$-free" is slightly more restrictive).

Our next result shows that the property of being locally controlled by the normaliser of a single non-trivial subgroup of a defect group carries through normal extensions of blocks.

Theorem 1.5. *Let G be a finite group, H a normal subgroup of G, c a G-stable block of H and b a block of G such that $bc = b$. Let (P, e) be a maximal (G, b)-subpair. There is a P-stable maximal (H, c)-subpair (Q, f) such that $Q = P \cap H$ and $fe_Q \neq 0$, where (Q, e_Q) is the unique (G, b)-subpair contained in (P, e).*

Furthermore, if there is a normal subgroup V of Q such that $N_H(V)$ controls fusion in $\mathcal{F}_{(Q,f)}(H, c)$, then $N_G(W)$ controls fusion in $\mathcal{F}_{(P,e)}(G, b)$ where W is the subgroup of P generated by the set of $N_G(Q, f)$-conjugates of V.

An interesting consequence of Theorem 1.5 is that it allows us to prove that any block b of a finite group G lying over an $SL(2, p)$-free block of a normal subgroup N of G with non-trivial defect groups has again a local structure which is controlled by the normaliser of a single non-trivial p-subgroup of G, even though b itself need not be $SL(2, p)$-free:

Corollary 1.6. *Let G be a finite group, let b be a block of G and let (P, e) be a maximal (G, b)-subpair. If there is a normal subgroup H of G such that $H \cap P \neq 1$ and such that b covers an $SL(2, p)$-free block c of H, then there is a non-trivial normal subgroup W in P such that $N_G(W)$ controls fusion in $\mathcal{F}_{(P,e)}(G, b)$.*

In [7, Section 12] Glauberman showed that for $W = K_\infty$ or $W = K^\infty$, the subgroup $W(P)$ of P is self-centralising; that is, $C_P(W(P)) = Z(W(P))$. Thus, in the situation of Theorem 1.4, the (G, b)-subpair $(W(P), e_{W(P)})$ is centric; in other words, the normaliser in G of some centric (G, b)-subpair controls b-fusion. The next Theorem shows that there is a canonical choice for such a centric subpair. By results of Külshammer and Puig in [11, Theorem 1.8], associated with any centric (G, b)-subpair (Q, f) and any choice of a maximal $(N_G(Q, f), f)$-subpair (R, g), there is a canonical group extension

$$1 \longrightarrow Q \longrightarrow L \longrightarrow N_G(Q, f)/QC_G(Q) \longrightarrow 1$$

having the property that R is a Sylow-p-subgroup of L and $\mathcal{F}_{(R,g)}(N_G(Q, f), f) = \mathcal{F}_R(L)$ (we explain this in some more detail in 2.4 below); moreover, $O_{p'}(L) = 1$ and $C_L(Q) = Z(Q)$. Thus, if b is $SL(2, p)$-free, then $Qd(p)$ is not involved in L, and hence $W(R)$ is normal in L for any Glauberman functor W.

Theorem 1.7. *Let G be a finite group, let b be a block of G and let (P, e) be a maximal (G, b)-subpair. Assume that p is odd and that b is $SL(2, p)$-free. There is a unique minimal subgroup Q of P such that (Q, f) is centric and radical, where f is the unique block of $C_G(Q)$ such that $(Q, f) \subseteq (P, e)$. Moreover, Q is normal in P and we have $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(L)$, where L is the middle term of the Külshammer-Puig extension associated with (Q, f).*

Remark 1.8. Theorems 1.5 and 1.7 seem to add some new information even in the principal block case. Theorem 1.5 implies that if N is a normal subgroup of a finite group G such that $N_N(V)$ controls strong p-fusion in $P \cap N$ with respect to N for some normal subgroup V of $P \cap N$ then the subgroup, W, of P generated by all $N_G(P \cap N)$-conjugates of V has the property that $N_G(W)$ controls strong p-fusion in P with respect to G. Theorem 1.7 translates to the following statement: given a finite group G with a Sylow-p-subgroup P such that $SL(2,p)$ is not involved in $N_G(Q)/C_G(Q)$ for any p-subgroup Q of G, there is a unique minimal subgroup Q of P such that $Z(Q)$ is a Sylow-p-subgroup of $C_G(Q)$ and such that $O_p(N_G(Q)/QC_G(Q)) = 1$; moreover, $N_G(Q)$ controls strong p-fusion in P with respect to G.

A *classifying space of* b is a p-complete space $B(G,b)$ having the homotopy type of the p-completion of an $\mathcal{L}$-system associated with $\mathcal{F}_{(P,e)}(G,b)$ in the sense of Broto, Levi and Oliver [4]. Note that in the situation of Theorem 1.7, the local structure of b is the same as the local structure of the principal block of L. Thus, if we take for $B(G,b)$ the p-completion $BL^\wedge_p$ of the classifying space BL of L we obtain the following immediate consequence.

Corollary 1.9. *If p is odd, any $SL(2,p)$-free block has a classifying space, which is unique up to homotopy.*

Theorems 1.4 and 1.5 provide many examples of blocks whose fusion pattern is determined by the normaliser of a single non-trivial p-subgroup. The existence of such controlling subgroups has ramifications for the Dade Projective Conjectures (DPC).

Theorem 1.10. *Let G be a finite group, let b be a block of G and let (P,e) be a maximal (G,b)-subpair. Assume that there is a normal subgroup R in P such that $N_G(P,e) \subseteq N_G(R)$ and such that $N_G(R)$ controls fusion in $\mathcal{F}_{(P,e)}(G,b)$. Let c be the block of $N_G(R)$ which satisfies $\mathrm{Br}_P(c)e = e$; that is, c is the Brauer correspondent in $N_G(Q)$ of b.*

(i) If every section of G satisfies DPC, then there is a defect preserving bijection between the sets of irreducible characters of b and irreducible characters of c.

(ii) If every proper section of G satisfies DPC, then DPC holds for b if and only if there is a defect preserving bijection between the sets of irreducible characters of b and irreducible characters of c.

2 On local categories of blocks

We collect in this Section some standard terminology and properties of local categories of blocks.We fix a finite group G, a block b of G and a maximal (G,b)-subpair (P,e). For any subgroup Q of P, denote by e_Q the unique block of $C_G(Q)$ such that $(Q, e_Q) \subseteq (P,e)$ (in particular, $e = e_P$).

By the uniqueness of the inclusion of subpairs (cf. [1]) we have $\mathcal{F}_P(P) \subseteq \mathcal{F}_{(P,e)}(G,b)$. If we choose a Sylow-$p$-subgroup S of G containing P, we have also $\mathcal{F}_{(P,e)}(G,b) \subseteq \mathcal{F}_S(G)$.

Two subgroups Q, R of P are isomorphic as objects in $\mathcal{F}_{(P,e)}(G,b)$ if there is $x \in G$ such that ${}^x(Q,e_Q) = (R,e_R)$. Any subgroup Q of P is isomorphic in $\mathcal{F}_{(P,e}(G,b)$ to a subgroup R of P such that $N_P(R)$ is a defect group of e_R viewed as block of $N_G(R,e_R)$ (cf. [1] or [14]). We say that (Q,e_Q) is an *Alperin-Goldschmidt- pair* (for $\mathcal{F}_{(P,e)}(G,b)$), if (Q,e_Q) is centric, radical and $N_P(Q)$ is a defect group of $kN_G(Q,e_Q)e_Q$. If Q is normal in P, then P is a defect group of e_Q as block of $N_G(Q,e_Q)$, and hence (P,e_P) is also a maximal $(N_G(Q,e_Q),e_Q)$-subpair. It has been shown by Puig, that (Q,e_Q) is centric if and only if $C_P(R) = Z(R)$ for any subgroup R of P which is isomorphic to Q in $\mathcal{F}_{(P,e)}(G,b)$. Thus the property of being centric can be read off the category $\mathcal{F}_{(P,e)}(G,b)$. Furthermore, the automorphism group of Q in $\mathcal{F}_{(P,e)}(G,b)$ is canonically isomorphic to $N_G(Q,e_Q)/C_G(Q)$.

A *conjugation family for* $\mathcal{F}_{(P,e)}(G,b)$ is a set $\mathcal{C}$ of subgroups of P with the following property: every isomorphism in $\mathcal{F}_{(P,e)}(G,b)$ is the composition of isomorphisms of the form $\varphi : Q \to R$, where Q, R are subgroups of P, such that there exists a subgroups S in $\mathcal{C}$ containing both Q, R and an element $x \in N_G(S,e_S)$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in Q$.

It is well-known and easy to check that if $\mathcal{C}$ is a conjugation family for $\mathcal{F}_{(P,e)}(G,b)$, then any subset $\mathcal{C}'$ of $\mathcal{C}$ such that any object in $\mathcal{C}$ is isomorphic to an object of $\mathcal{C}'$ in $\mathcal{F}_{(P,e)}(G,b)$ is again a conjugation family.

By Alperin's fusion theorem (in its refined version by Goldschmidt and adapted to blocks, cf. [1, §4]), the set of subgroups Q of P for which (Q,e_Q) is an Alperin-Goldschmidt pair is a conjugation family for $\mathcal{F}_{(P,e)}(G,b)$, called the *Alperin-Goldschmidt conjugation family for* $\mathcal{F}_{(P,e)}(G,b)$.

Definition 2.1 A subgroup H of G *controls fusion in* $\mathcal{F}_{(P,e)}(G,b)$ if H contains P and if $\mathcal{F}_{(P,e)}(G,b) \subseteq \mathcal{F}_S(H)$ for some Sylow-p-subgroup S of H which contains P.

By Alperin's fusion theorem, a subgroup H of G containing P controls fusion in $\mathcal{F}_{(P,e)}(G,b)$ if and only if $N_G(Q,e_Q) = N_H(Q,e_Q)C_G(Q)$ for any subgroup Q of P.

Lemma 2.2. *Let W be a normal subgroup in P, and let H be a subgroup of G such that $P \subseteq H \subseteq N_G(W)$. Assume that H controls fusion in $\mathcal{F}_{(P,e)}(G,b)$. Then W is contained in any subgroup Q of P such that (Q,e_Q) is centric and radical.*

Proof. Let Q be a subgroup of P such that (Q,e_Q) is centric and radical. Since $N_G(Q,e_Q) = N_H(Q,e_Q)C_G(Q)$ and W is normal in H, the image of $N_W(Q)$ is normal in $N_G(Q,e_Q)/QC_G(Q)$, hence $N_W(Q) \subseteq QC_G(Q)$ as (Q,e_Q) is radical. Thus $N_W(Q) \subseteq Q$ because (Q,e_Q) is centric, and therefore $W \subseteq Q$. □

The first statement of the following Proposition is a variation of [10, Statement 1]. The second statement makes precise what it means, in certain circumstances, for a subgroup to control fusion.

Proposition 2.3. *Let Q be a subgroup of P, let H be a subgroup of $N_G(Q)$ containing $QC_G(Q)$, and let c be the unique block of H such that $\mathrm{Br}_Q(c)e_Q = e_Q$. Assume that c has a defect group R contained in P. Then (R, e_R) is a maximal (H, c)-subpair, and we have $\mathcal{F}_{(R,e_R)}(H, c) \subseteq \mathcal{F}_{(P,e)}(G, b)$; moreover, this inclusion is an equality if and only if H controls fusion in $\mathcal{F}_{(P,e)}(G, b)$.*

Proof. Since Q is normal in H, Q is contained in any defect group of H. If R is a defect group of c contained in P, then $C_G(R) \subseteq C_G(Q) \subseteq H$, and thus (R, e_R) is a - necessarily maximal - (H, c)-subpair. Let (S, f) be a centric radical (H, c)-subpair contained in (R, e_R). Again, since Q is normal in H, we have $Q \subseteq S$ by 2.2. Then $C_G(S) = C_H(S)$, and so $f = e_S$. Thus $N_H(S, f) = N_H(S, e_S) \subseteq N_G(S, e_S)$. The inclusion $\mathcal{F}_{(R,e_R)}(H, c) \subseteq \mathcal{F}_{(P,e)}(G, b)$ follows, using Alperin's fusion theorem.

Assume that H controls fusion in $\mathcal{F}_{(P,e)}(G, b)$. Then in particular $R = P$ is a defect group of c, as Q is normal in H and P is contained in H. Thus (P, e) is also a maximal (H, c)-subpair. Let now S be a subgroup of P such that (S, e_S) is a radical centric (G, b)-subpair. Thus $Q \subseteq S$ by 2.2. But then $C_G(S) \subseteq H$, and so (S, e_S) is also a centric (H, c)-subpair. Thus the inclusion $N_G(S, e_S) \subseteq N_H(S, e_S)C_G(S)$ translates to $\mathcal{F}_{(P,e)}(G, b) \subseteq \mathcal{F}_{(P,e)}(H, c)$, hence equality by the first statement. The rest is clear. □

Proposition 2.3 applies in the following two situations. If H contains $N_G(P)$ and if c is the unique block of H such that $\mathrm{Br}_P(c) = \mathrm{Br}_P(b)$, then (P, e) is also a maximal (H, c)-subpair. Thus if P has a subgroup Q such that $C_G(Q) \subseteq H \subseteq N_G(Q)$, we have $\mathcal{F}_{(P,e)}(H, c) \subseteq \mathcal{F}_{(P,e)}(G, b)$. The second situation, in which we are going to apply 2.3 arises if $H = N_G(Q, e_Q)$ for some subgroup Q of P and if $c = e_Q$ such that $N_P(Q)$ is a defect group of c (viewed as block of H).

The next Proposition is a particular case of Külshammer-Puig [11, Theorem 1.8], translated to our terminology (see also [10, Statement 8]).

Proposition 2.4. *Assume that $G = N_G(Q, e_Q)$ for some subgroup Q of P such that (Q, e_Q) is centric. Then $b = e_Q$, and there is a short exact sequence of finite groups*

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/QC_G(Q) \longrightarrow 1$$

such that P is a Sylow-p-subgroup of L and such that $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_P(L)$. Moreover, we have $O_{p'}(L) = 1$ and $C_L(Q) = Z(Q)$; in particular, L is p-constrained.

Proof. As Q is normal in G, the block idempotent b is contained in $kC_G(Q)$, and as G stabilises e_Q, we have $b = e_Q$ (this is a standard argument; see [1]). To establish the link with the terminology in [11, 1.8], note first that P is also a defect group of $\{b\}$ viewed as point of G on $kC_G(Q)$, because P is maximal with respect to the property $\mathrm{Br}_P(b) \neq 0$. The existence of a canonical exact sequence as stated such that P is a Sylow-p-subgroup of L is a particular case of [11, 1.8]. This extension has the property, that for any $y \in L$, the outer automorphisms of Q induced by conjugation with y and by conjugation with some element $x \in G$ such that $xQC_G(Q)$ is the image of y in $G/QC_G(Q)$ coincide.

In particular, if $y \in C_L(Q)$ then $x \in QC_G(Q)$, and hence $y \in Q$. This shows that $C_L(Q) = Z(Q)$, and since Q is normal in L, we have $O_{p'}(L) = O_{p'}(C_L(Q)) = 1$. The equality $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(L)$ is essentially a reformulation of [11, 1.8.2]; we reproduce the argument from [10, Statement 8]. Since Q is normal in L and in G, it suffices to show that the images in $\mathrm{Aut}(R)$ of $N_G(R, e_R)$ and $N_L(R)$ are equal, where R is a subgroup of P containing Q. As (Q, e_Q) is centric, so is (R, e_R). Similarly, as $C_L(Q) = Z(Q)$, we have $C_L(R) = Z(R)$. Setting $\bar{G} = G/QC_G(Q)$, with the notation of [11, 1.8] (which is defined in [11, 2.8]) we have $E_{G,\bar{G}}(R, e_R) = E_{L,\bar{G}}(R)$. By [11, (2.8.1)], the canonical maps $E_{G,\bar{G}}(R, e_R) \to E_G(R, e_R)$ and $E_{L,\bar{G}}(R) \to E_L(R)$ are surjective. Thus $E_G(R, e_R) = E_L(R)$. This implies the equality $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(L)$. □

We need the following generalisation of [10, Statement 9].

Proposition 2.5. *Let Q be a normal subgroup of P, set $H = N_G(Q)$ and denote by c the unique block of H such that $e_Q c = e_Q$. Suppose there is a finite group L having P as Sylow-p-subgroup such that $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(L)$. Then (P,e) is a maximal (H,c)-subpair, P is a Sylow-p-subgroup of $N_L(Q)$, and we have $\mathcal{F}_{(P,e)}(H,c) = \mathcal{F}_P(N_L(Q))$.*

Proof. Since Q is normal in P, the pair (P, e_P) is also a maximal (H,c)-subpair, and clearly P is a Sylow-p-subgroup of $N_L(Q)$. By 2.3, we have $\mathcal{F}_{(P,e)}(H,c) \subseteq \mathcal{F}_{(P,e)}(G,b)$. In order to show the equality $\mathcal{F}_{(P,e)}(H,c) = \mathcal{F}_P(N_L(Q))$, it suffices to show that $N_H(S,f)$ and $N_L(S) \cap N_L(Q)$ have the same images in $\mathrm{Aut}(S)$, where (S,f) is an (H,c)-Brauer pair contained in (P,e). Since Q is normal in H and $N_L(Q)$, we may assume that $Q \subseteq S$, by 2.2. Then $C_G(S) \subseteq H$ and $f = e_S$. The assumption $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(L)$ implies that given any $x \in N_G(S, e_S)$, there is $y \in N_L(S)$ such that ${}^x u = {}^y u$ for all $u \in S$. Since $Q \subseteq S$, clearly $x \in N_H(S, e_S)$ if and only if $y \in N_L(S) \cap N_L(Q)$. The equality $\mathcal{F}_{(P,e)}(H,c) = \mathcal{F}_P(N_L(Q))$ follows. □

The following Lemma appears in a slightly more general version in Puig [12].

Lemma 2.6. *Let G be a finite group, let b be a block of G and let (Q,e), (R,f) be centric (G,b)-subpairs such that $(Q,e) \subseteq (R,f)$. We have*

$$N_G(R,f) \cap C_G(Q) = Z(Q)C_G(R) \ .$$

Proof. Clearly the right side is contained in the left side. For the converse, assume first that Q is normal in R. Let $x \in N_G(R,f) \cap C_G(Q)$. It is easy to check that $[R,x] \subseteq C_R(Q) = Z(Q)$. Thus $[R,x,x] = 1$. If x is a p'-element, this forces $x \in C_G(R)$ by standard properties of coprime group actions (cf. [8]). Note that the image of a defect group of f as block of $N_G(R,f)$ is a Sylow-p-subgroup of $N_G(R,f)/C_G(R)$. Thus if x is a p-element, we may assume that x belongs to a defect group of f as block of $N_G(R,f)$, which implies $x \in Z(Q)$, as (Q,e) is centric. The general case follows by induction. □

Proposition 2.7. *Assume that there is a unique minimal subgroup R of P such that (R, e_R) is centric and radical. Then R is normal in P, the pair (P, e) is a maximal $(N_G(R, e_R), e_R)$-subpair, and we have $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N_G(R, e_R), e_R)$.*

Proof. The uniqueness of R implies that R is normal in P, and hence (P, e) is also a maximal $(N_G(R, e_R), e_R)$-subpair. Let S be a subgroup of P such that (S, e_S) is centric and radical. Then $R \subseteq S$ by the uniqueness of (R, e_R). If $x \in N_G(S, e_S)$, then ${}^x(R, e_R) \subseteq (S, e_S)$, and again, by the uniqueness of (R, e_R), we deduce that ${}^x(R, e_R) = (R, e_R)$. In other words, $N_G(S, e_S) \subseteq N_G(R, e_R)$, which implies $\mathcal{F}_{(P,e)}(G, b) \subseteq \mathcal{F}_{(P,e)}(N_G(R, e_R), e_R)$, hence the equality by 2.3. □

We provide a criterion for when the Alperin-Goldschmidt conjugation family has a unique minimal element.

Proposition 2.8. *Assume that $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N_G(Q, e_Q), e_Q)$ for some normal subgroup Q of P such that (Q, e_Q) is centric. Then there is a unique subgroup R of P containing Q such that $O_p(N_G(Q, e_Q)/QC_G(Q)) = RC_G(Q)/QC_G(Q)$. The group R is then the unique minimal subgroup of P such that (R, e_R) is centric and radical. In particular, R is normal in P and $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N_G(R, e_R), e_R)$.*

Proof. We may assume that $G = N_G(Q, e_Q)$ and hence that $b = e_Q$. The image of P in $G/QC_G(Q)$ is a Sylow-p- subgroup; since (Q, e_Q) is centric, this image is isomorphic to P/Q. Therefore, there is a unique subgroup R of P containing Q such that the image of R in $G/QC_G(Q)$ is $O_p(G/QC_G(Q))$. The uniqueness of R implies that R is normal in P. Note that b is still a block of $RC_G(Q)$, and then (R, e_R) is a maximal $(RC_G(Q), b)$-subpair. By our choice of R, the group $RC_G(Q)$ is normal in G, and since $RC_G(Q)$ acts transitively on the set of maximal $(RC_G(Q), b)$-subpairs, the Frattini argument shows that $G = N_G(R, e_R)C_G(Q)$.

Let S be a subgroup of P such that (S, e_S) is centric and radical. By Lemma 2.6, we have $N_G(S, e_S) \cap QC_G(Q) = QC_G(S)$. Thus the inclusion $N_G(S, e_S) \subset G$ induces an injective group homomorphism $N_G(S, e_S)/QC_G(S) \to G/QC_G(Q)$. The image of R in $G/QC_G(Q)$ is $O_p(G/QC_G(Q))$; thus the image of $N_R(S)$ in $N_G(S, e_S)/QC_G(S))$ is contained in $O_p(N_G(S, e_S)/QC_G(S))$, and hence the image of $N_R(S)$ in $N_G(S, e_S)/SC_G(S)$ is contained in $O_p(N_G(S, e_S)/SC_G(S)) = 1$. This forces $N_R(S) \subseteq SC_G(S)$. As (S, e_S) is centric, we get $N_R(S) \subseteq S$, hence $R \subseteq S$.

By Lemma 2.6 again, we have $N_G(R, e_R) \cap RC_G(Q) = RC_G(R)$. As $G = N_G(R, e_R)C_G(Q)$, it follows that $N_G(R, e_R)/RC_G(R) \cong G/RC_G(Q)$, and hence $O_p(N_G(R, e_R)/RC_G(R)) = 1$ by our choice of R. This shows that R is indeed the unique minimal subgroup of P such that (R, e_R) is centric and radical. The rest is clear by 2.7. □

3 Local control of characteristic p-functors

Let G be a finite group, let b be a block of G, let (P, e) be a maximal (G, b)-subpair, and for any subgroup Q of P, denote by e_Q the unique block of $C_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$.

Given a positive characteristic p-functor W and a subgroup Q of P, we set $W_1(Q) = Q$ and $P_1(Q) = N_P(Q)$. For any positive integer i, we define inductively $W_{i+1}(Q) = W(P_i(Q))$ and $P_{i+1}(Q) = N_P(W_{i+1}(Q))$. For all positive integers i we have $W_i(Q) \subseteq P_i(Q)$, and if $P_i(Q)$ is a proper subgroup of P, in fact $P_i(Q)$ is a proper subgroup of $P_{i+1}(Q)$. In particular, $P_i(Q) = P$ for all large enough i. We will say that Q is *well-placed in* P (with respect to W and $\mathcal{F}_{(P,e)}(G, b)$) if $P_i(Q)$ is a defect group of the block $e_{W_i(Q)}$ as block of $N_G(W_i(Q), e_{W_i(Q)})$ for all positive integer i. Clearly P is always well-placed in P.

The next Lemma states essentially that every subgroup of P is isomorphic to a well-placed subgroup with respect to $\mathcal{F}_{(P,e)}(G, b)$ and a positive characteristic p-functor.

Lemma 3.1. *Let W be a positive characteristic p-functor. For any subgroup Q of P, there is an element $x \in G$ such that ${}^x(Q, e_Q) \subset (P, e)$, ${}^xN_P(Q) \subseteq P$ and such that ${}^xQ$ is well-placed in P.*

Proof. Define sequences of subgroups and blocks as follows. Let $V_1 := Q$, $v_1 := e_Q$. Let (R_1, r_1) be a b-subpair which is maximal with respect to normalising (V_1, v_1) and such that $(N_P(Q), e_{N_P(Q)}) \leq (R_1, r_1)$. For $i \geq 1$ let $V_{i+1} = W(R_{i+1})$ and let (V_{i+1}, v_{i+1}) be the b-subpair contained in (R_{i+1}, r_{i+1}). Let (R_{i+1}, r_{i+1}) be a b-subpair which is maximal with respect to normalising (V_{i+1}, v_{i+1}) and such that $(R_i, r_i) \leq (R_{i+1}, r_{i+1})$. Note that if (S, f) is a maximal b-subpair containing (R_{i+1}, r_{i+1}), then $R_{i+1} = N_S(V_{i+1})$. On the other hand, $N_S(R_i) \subset N_S(V_{i+1})$. Thus, either $R_i = S$ or R_{i+1} properly contains R_i. In other words, there exists an integer t such that for all $i \geq t$, $(R_i, r_i) = (R_t, r_t)$ is a maximal b-Brauer pair, $(V_i, v_i) = (W(R_t), v_t)$. Let $x \in G$ be such that ${}^x(R_t, r_t) = (P, e)$. Then ${}^x(Q, e_Q) \leq (P, e)$, and since for every $i \geq 1$, ${}^xR_i \subset {}^gR_t = P$, it is clear that ${}^x(Q, e_Q)$ is well placed in (P, e). The second assertion is clear since $N_P(Q) \subset R_1 \subset {}^{x^{-1}}P$. □

The next results states roughly speaking, that "if a positive characteristic p-functor controls fusion locally, it controls fusion globally". This generalises a result by Alperin and Gorenstein (cf. [9, Ch. X, Theorem 9.3])

Proposition 3.2. *Let W be a positive characteristic p-functor. Assume that for any non-trivial subgroup Q of P and any maximal $(N_G(Q, e_Q), e_Q)$-subpair (R, f), the group $N_{N_G(Q,e_Q)}(W(R))$ controls fusion in $\mathcal{F}_{(R,f)}(N_G(Q, e_Q), e_Q)$. Then $N_G(W(P))$ controls fusion in $\mathcal{F}_{(P,e)}(G, b)$.*

Proof. Set $H = N_G(W(P))$. Suppose, if possible that the result is not true. Then by 3.1 above, there exists a non-trivial subgroup Q of P such that (Q, e_Q) is well placed in (P, e) such that $N_G(Q, e_Q)$ is not contained in $C_G(Q)N_H(Q, e_Q)$.

We introduce the following notation. For $i \geq 1$, let $W_i = W_i(Q)$, $P_i = P_i(Q)$, $e_i = e_{W_i}$, $N_i = N_G(W_i, e_i)$, $M_i = N_G(W_i)$ and $L_i = N_i \cap N_G(W_{i+1})$. Let f_i be the block of M_i satisfying $e_i f_i = e_i$. Let s_i be the block of L_i such that $\mathrm{Br}_{P_i}(s_i) = \mathrm{Br}_{P_i}(e_i)$.

Set $\mathcal{F}_i = \mathcal{F}_{(P_i, e_{P_i})}(N_i, e_i)$, set $\mathcal{G}_i = \mathcal{F}_{(P_i, e_{P_i})}(L_i, s_i)$, and set $\mathcal{H}_i = \mathcal{F}_{(P_i, e_{P_i})}(M_i, f_i)$.

It is clear from 2.3 that $\mathcal{G}_i \subset \mathcal{F}_i$. On the other hand, $P_i C_{M_{i+1}}(W_i) \subset L_i \subset N_{M_{i+1}}(W_i)$. Since $(W_{i+1}, e_{i+1}) \leq (P_i, e_{P_i})$, $\mathrm{Br}_{P_i}(f_i)e_{P_i} = e_{P_i}$ and hence by 2.3 it follows that $\mathcal{G}_i \subset \mathcal{H}_{i+1}$. Since, clearly $\mathcal{H}_{i+1} = \mathcal{F}_{i+1}$, we get that $\mathcal{G}_i \subset \mathcal{F}_{i+1}$.

By the hypothesis of proposition, we have that $\mathcal{G}_i = \mathcal{F}_i$, hence, we get that for all $i \geq 1$, $\mathcal{F}_1 \subset \mathcal{F}_i \subset \mathcal{F}_{i+1}$.

Let i be such that $P_i = P$, so that $\mathcal{F}_{i+1} = \mathcal{F}_{(P,e)}(H, c)$, where c is the Brauer correspondent of b. Let g be an element of $N_G(Q, e_Q)$. Then conjugation by g determines an element, say ϕ of of $\mathrm{End}_{\mathcal{F}_1}(Q)$. Then ϕ is induced by conjugation with an element $x \in H$, hence $g = zx$ for some $z \in C_G(Q)$. Thus, $N_G(Q, e_Q) \subset C_G(Q)(H \cap N_G(Q, e_Q))$, contradicting our choice of (Q, e_Q). □

4 On the local structure of central p-extensions

Let G be a finite group, let b be a block of G, and let (P, e) be a maximal (G, b)-subpair. We assume in this section that P contains a subgroup Z of $Z(G)$. We set $\bar{G} = G/Z$ and $\bar{P} = P/Z$; for any element or subset a of kG, we denote by $\bar{a}$ its canonical image in $k\bar{G}$. It is well-known that the image $\bar{b}$ of b in $k\bar{G}$ is a block of $\bar{G}$ having $\bar{P}$ as defect group. The following (equally well-known) Lemma relates the local structures of b and $\bar{b}$.

Lemma 4.1. *For every (G, b)-subpair (Q, f) there is a unique $(\bar{G}, \bar{b})$-subpair of the form $(\bar{Q}, g)$ such that $\bar{f}g = \bar{f}$, and then the canonical map $G \to \bar{G}$ induces a surjective group homomorphism $N_G(Q, f)/C_G(Q) \to N_{\bar{G}}(\bar{Q}, g)/C_{\bar{G}}(\bar{Q})$ whose kernel is an abelian p-group. In particular, if $O_p(N_G(Q, f)/QC_G(Q)) = 1$, this map induces an isomorphism $N_G(Q, f)/QC_G(Q) \cong N_{\bar{G}}(\bar{Q}, g)/\bar{Q}C_{\bar{G}}(\bar{Q})$.*

Proof. It is well-known (and easy to check) that the group $\overline{C_G(Q)}$ is a normal subgroup of $C_{\bar{G}}(\bar{Q})$ and that $C_{\bar{G}}(\bar{Q})/\overline{C_G(Q)}$ is an abelian p-group. Thus any block of $C_{\bar{G}}(\bar{Q})$ is contained in $k\overline{C_G(Q)}$. Hence the sum of the different $C_{\bar{G}}(\bar{Q})$- conjugates of $\bar{f}$ is the unique block g of $C_{\bar{G}}(\bar{Q})$ fulfilling $\bar{f}g = \bar{f}$, and we have $N_{\bar{G}}(\bar{Q}, g) = \overline{N_G(Q, f)}C_{\bar{G}}(\bar{Q})$. The Lemma follows. □

The above Lemma implies in particular, that the maximal (G, b)-subpair (P, e) determines a unique maximal $(\bar{G}, \bar{b})$-subpair $(\bar{P}, f)$ by the condition $\bar{e}f = \bar{e}$. With this choice of maximal subpairs, 4.1 translates to the following statement.

Proposition 4.2. *The canonical map $G \to \bar{G}$ induces a surjective functor $\mathcal{F}_{(P,e)}(G,b) \to \mathcal{F}_{(\bar{P},f)}(\bar{G},\bar{b})$. In particular, b is $SL(2,p)$-free, if and only if $\bar{b}$ is $SL(2,p)$-free.*

Proof. Clear by 4.1. □

Proposition 4.3. *Let H be a subgroup of G containing $N_G(P)$ and denote by c the unique block of H such that $\mathrm{Br}_P(c) = \mathrm{Br}_P(b)$. Assume that there is a subgroup Q of P containing Z such that Q is normal in H and such that $C_{\bar{G}}(\bar{Q}) \subseteq \bar{H}$. Then $\mathrm{Br}_{\bar{P}}(\bar{c}) = \mathrm{Br}_{\bar{P}}(\bar{b})$, and we have $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(H,c)$ if and only if $\mathcal{F}_{(\bar{P},f)}(\bar{G},\bar{b}) = \mathcal{F}_{(\bar{P},f)}(\bar{H},\bar{c})$.*

Proof. The equality $\mathrm{Br}_{\bar{P}}(\bar{c}) = \mathrm{Br}_{\bar{P}}(\bar{b})$ is clear by [10, Statement 5]. Suppose that $\mathcal{F}_{\bar{G},\bar{b}} = \mathcal{F}_{\bar{H},\bar{c}}$. Let (R,t) be a centric radical (G,b)-subpair. Let s be the unique block of $C_{\bar{G}}(\bar{R})$ such that $\bar{t}s = \bar{t}$. By Lemma 4.1, we have $N_G(R,t)/RC_G(R) \cong N_{\bar{G}}(\bar{R},s)/\bar{R}C_{\bar{G}}(\bar{R}) = N_{\bar{H}}(\bar{R},s)/\bar{R}C_{\bar{G}}(\bar{R}) \cap \bar{H}$. Now Q is normal in H, and thus the canonical image of $N_Q(R)$ is normal in $N_G(R,t)/RC_G(R)$. Therefore we have $N_Q(R) \subseteq RC_G(R)$. As the subpair (R,t) is centric, we have $N_Q(R) \subseteq R$, which forces $Q \subseteq R$. Thus $C_{\bar{G}}(\bar{R}) \subseteq \bar{H}$ by the assumptions, and so $(\bar{R},s)$ is also an $(\bar{H},\bar{c})$-subpair and (R,t) is an (H,c)-subpair. Therefore $N_H(R,t)/RC_G(R)$ is a subgroup of $N_G(R,t)/RC_G(R) \cong N_{\bar{G}}(\bar{R},s)/\bar{R}C_{\bar{G}}(\bar{R}) = N_{\bar{H}}(\bar{R},s)/\bar{R}C_{\bar{H}}(\bar{R})$. But then Lemma 4.1, applied to H and c instead of G and b, respectively, shows that $N_H(R,t) = N_G(R,t)$, which implies the equality $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(H,c)$. The converse is trivial. □

5 On $SL(2,p)$-free blocks

Proposition 5.1. *Let G be a finite group and let b be a block of G. Suppose that $SL(2,p)$ is involved in $N_G(Q,f)/C_G(Q)$ for some non-trivial (G,b)-subpair (Q,f). Then $SL(2,p)$ is involved in $N_G(Q,e)/C_G(Q)$ for some centric and radical (G,b)-subpair (Q,e).*

Proof. Fix a maximal (G,b)-subpair (P,e_P), and for any subgroup Q of P, denote by (Q,e_Q) the unique (G,b)-subpair contained in (P,e_P). Let Q be a subgroup of P with $|Q|$ maximum such that $SL(2,p)$ is involved in $N_G(Q,e_Q)/C_G(Q)$. Replacing (Q,e_Q) with a G-conjugate if necessary, we may assume that $N_P(Q)$ is a defect group of $kN_G(Q,e_Q)e_Q$, so that in particular, $R = QC_P(Q)$ is a defect group of $kQC_G(Q)e_Q$ and (R,e_R) is a maximal $(QC_G(Q),e_Q)$-pair. Since $QC_G(Q)$ is normal in $N_G(Q,e_Q)$, the Frattini argument gives $N_G(Q,e_Q) = C_G(Q)[N_G(R,e_R) \cap N_G(Q,e_Q)]$. But then, $N_G(Q,e_Q)/C_G(Q) \cong N_G(R,e_R) \cap N_G(Q,e_Q)/N_G(R,e_R) \cap C_G(Q)$. On the other hand, since $C_G(R) \subseteq C_G(Q)$, $N_G(R,e_R) \cap N_G(Q,e_Q)/N_G(R,e_R) \cap C_G(Q)$ is isomorphic to a subquotient of $N_G(R,e_R)/C_G(R)$. Hence $SL(2,p)$ is involved in $N_G(R,e_R)/C_G(R)$. The choice of Q now implies that $R = Q$ whence (Q,e_Q) is a centric (G,b)-pair.

Let M be the inverse image of $O_p(N_G(Q,e_Q)/QC_G(Q))$ in $N_G(Q,e_Q)$ and let $S = M \cap N_P(Q)$. Then S is a defect group of kMe_Q, (S,e_S) is a maximal (M,e_Q)-pair. Since $N_G(Q,e_Q)$ normalises M, the Frattini argument again gives that

$N_G(Q, e_Q) = M[N_G(S, e_S) \cap N_G(Q, e_Q)]$. But $M = (QC_G(Q))S$ whence $N_G(Q, e_Q) = C_G(Q)[N_G(S, e_S) \cap N_G(Q, e_Q)]$. Arguing as before, we conclude that $S = Q$ and hence that $M = QC_G(Q)$. This completes the proof. □

The main application of 5.1 is the following proposition which shows that the property of being $SL(2, p)$-free passes down to corresponding blocks of normalisers of subpairs.

Proposition 5.2. *Let G be a finite group and let b be an $SL(2, p)$-free block of G. For every (G, b)-subpair (Q, f) the block f of $N_G(Q, f)$ is $SL(2, p)$-free.*

Proof. Let (R, g) be a centric radical $(N_G(Q, f), f)$-subpair. Then $Q \subseteq R$ by 2.2, and hence $C_G(R) \subseteq N_Q(Q, f)$. Thus (R, g) is a (G, b)-Brauer pair, and hence $SL(2, p)$ is not a subquotient of $N_G(R, g)/C_G(R)$ by 5.1. But then $SL(2, p)$ is obviously not a subquotient of $N_{N_G(Q,f)}(R, g)/C_{N_G(Q,f)}(R)$. □

6 Proof of Theorem 1.4 and Theorem 1.7

Proof of Theorem 1.4. Let G be a finite group, let b be a block of G, let (P, e) be a maximal (G, b)-subpair, and for any subgroup Q of P, denote by e_Q the unique block of $C_G(Q)$ such that $(Q, e_Q) \subseteq (P, e)$. Let W be a Glauberman functor, set $N = N_G(W(P))$ and denote by c the unique block of N such that $\mathrm{Br}_P(c) = \mathrm{Br}_P(b)$. Assume that p is odd.

Suppose that Theorem 1.4 fails for the blocks b and c of G and N, respectively, and assume that $|G|$ has minimal order with this property. We are going to derive a contradiction, proceeding in several steps.

6.1. *We have $O_p(G) \neq 1$.*

Proof. If $O_p(G) = 1$, then for any nontrivial (G, b)-Brauer pair (Q, f), the group $N_G(Q, f)$ is a proper subgroup of G. Since f is $SL(2, p)$-free by 5.2, the induction hypothesis implies that Theorem 1.4 holds for the block f of $N_G(Q, f)$. But then 3.2 implies, that Theorem 1.4 holds for the block b of G, contradicting our choice of b. □

From now on, we set $Q = O_p(G)$. Since Q is normal in G, the block b lies in $kC_G(Q)$ (cf. [1, (2.9)(1)]). Thus $b = \mathrm{Tr}^G_{N_G(Q,e_Q)}(e_Q)$. But then $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N_G(Q, e_Q), e_Q)$. If $N_G(Q, e_Q)$ is a proper subgroup of G, the induction hypothesis implies that Theorem 1.4 holds for the block e_Q of $N_G(Q, e_Q)$, and hence for the block b, contradicting again our choice of b. This proves the following.

6.2. *We have $G = N_G(Q, e_Q)$ and $b = e_Q$.*

Then b is a block for any subgroup of G containing $C_G(Q)$. In particular, b is a block of $QC_G(Q)$. Set $R = QC_P(Q)$. Then (R, e_R) is a maximal $(QC_G(Q), b)$-subpair (cf. [1, (2.9)(6)]). Note that $C_G(R) \subseteq C_G(Q)$ and that R is normal in P. Since the maximal $(QC_G(Q), b)$-subpairs are $QC_G(Q)$-conjugate, a Frattini argument shows that

6.3. *we have* $G = N_G(R, e_R)C_G(Q)$.

If $R = Q$, then $(R, e_R) = (Q, b)$ is (G, b)-centric. The group L occurring in the Külshammer-Puig-extension [11, Theorem 1.8]

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/QC_G(Q) \longrightarrow 1$$

is p-constrained and does not have $SL(2, p)$ as subquotient (cf. 2.4). Thus $W(P)$ is normal in L. Since $\mathcal{F}_P(L) = \mathcal{F}_{(P,e)}(G, b)$, this implies $\mathcal{F}_{(P,e)}(G, b) = \mathcal{F}_{(P,e)}(N, c)$, contradicting our choice of b.

Thus Q is a proper subgroup of R. Since $Q = O_p(G)$, it follows that $N_G(R, e_R)$ is a proper subgroup of G. So Theorem 1.4 applies to the block e_R of $N_G(R, e_R)$, which has still P as defect group as R is normal in P.

We assume now that $PC_G(Q)$ is a proper subgroup of G, and derive a contradiction; that is, we are going to show that then (G, b) cannot be a counterexample to Theorem 1.4. We do this by showing that $N_G(S, e_S) = N_N(S, e_S)C_G(S)$ for any subgroup S of P containing Q such that (S, e_S) is (G, b)-centric. We argue by induction over the order of S. Up to replacing (S, e_S) by some G-conjugate, we may assume that $N_P(S)$ is a defect group of e_S as block of $N_G(S, e_S)$. The subgroup $N_G(S, e_S) \cap SC_G(Q)$ is normal in $N_G(S, e_S)$ and contains $C_G(S)$. Thus e_S is a block of $N_G(S, e_S) \cap SC_G(Q)$ having as defect group the group $T = N_P(S) \cap SC_G(Q) = N_P(S) \cap SC_P(Q) = N_{SR}(Q)$, as $R = QC_P(Q)$. Therefore, (T, e_T) is a maximal $(N_G(S, e_S) \cap SC_G(Q), e_S)$-subpair. The Frattini argument yields

$$N_G(S, e_S) = (N_G(S, e_S) \cap N_G(T, e_T)) \cdot (N_G(S, e_S) \cap SC_G(Q)) \ .$$

Now (S, e_S) is also a $(PC_G(Q), b)$-subpair contained in (P, e). As $PC_G(Q)$ is assumed to be a proper subgroup of G, it follows that $N_G(S, e_S) \cap SC_G(Q) = (N_N(S, e_S) \cap SC_G(Q))C_G(S)$. If S does not contain R, then S is properly contained in SR, hence properly contained in $T = N_{SR}(S)$. By induction, we get $N_G(T, e_T) = N_N(T, e_T)C_G(T)$. Together we get $N_G(S, e_S) \subseteq NC_G(S)$, hence $N_G(S, e_S) = N_N(S, e_S)C_G(S)$.

Thus, we may assume that $R \subseteq S$. Then $C_G(S) \subseteq C_G(R) \subseteq N_G(R, e_R) \cap PC_G(Q)$. Therefore, e_R is a block of the group $N_G(R, e_R) \cap PC_G(Q)$, having still (P, e) as maximal subpair. Let $x \in N_G(S, e_S)$. Since $G = N_G(R, e_R)C_G(Q)$, we can write $x = nc$ for some $n \in N_G(R, e_R)$ and some $c \in C_G(Q)$. Then ${}^c(S, e_S) = {}^{n^{-1}}(S, e_S)$. This implies that ${}^cS \subseteq N_G(R, e_R) \cap PC_G(Q)$ and that $(R, e_R) \subseteq {}^c(S, e_S)$. Thus ${}^c(S, e_S)$ is a $(N_G(R, e_R) \cap PC_G(Q), e_R)$-subpair. Therefore, there is $y \in N_G(R, e_R) \cap PC_G(Q)$ such that ${}^{yc}(S, e_S) \subseteq (P, e)$. We have $x = nc = (ny^{-1})(yc)$. The element yc belongs to the group $PC_G(Q)$, and conjugation by yc is a morphism in the category $\mathcal{F}_{(P,e)}(PC_G(Q), b)$ from S to ${}^{yc}S$. As $PC_G(Q)$ is assumed to be a proper subgroup of G, this implies that $yc \in (N \cap PC_G(Q))C_G(S)$. The element ny^{-1} belongs to the group $N_G(R, e_R)$, and conjugation by ny^{-1} is a morphism from ${}^{yc}S$ to ${}^xS = S$ in the category $\mathcal{F}_{(P,e)}(N_G(R, e_R), e_R)$. Since $N_G(R, e_R)$ is a proper subgroup of G, it follows that $ny^{-1} \in N_N(R, e_R)C_G({}^{yc}S)$. Together, we get $x = (ny^{-1})(yc) \in NC_G(S)$, hence $N_G(S, e_S) = N_N(S, e_S)C_G(S)$. This contradicts the fact that (G, b) is a counterexample to the Theorem. Therefore,

6.4. *we have $G = PC_G(Q)$.*

Set $Z = Z(P) \cap Q$; since Q is normal in G, the group Z is non-trivial. Set $\bar{G} = G/Z$, and denote by $\bar{b}$ the image of b in $k\bar{G}$. Thus $\bar{b}$ is a block of $k\bar{G}$ with defect group $\bar{P} = P/Z$. By 4.2, the block $\bar{b}$ is $SL(2,p)$-free. Denote by H the inverse image in G of $N_{\bar{G}}(W(\bar{P}))$. Then H is the normaliser in G of a subgroup of P which contains Q properly, and so H is a proper subgroup of G fulfilling the hypotheses of 4.3. Denote by d the unique block of H such that $\mathrm{Br}_P(d) = \mathrm{Br}_P(b)$, and denote by $\bar{d}$ the image of d in $k\bar{H}$. By the induction hypothesis, we have $\mathcal{F}_{(\bar{P},f)}(\bar{G},\bar{b}) = \mathcal{F}_{(\bar{P},f)}(\bar{H},\bar{d})$, where f is the unique block of $C_{\bar{G}}(\bar{P})$ such that $\bar{e}f = f$. But then 4.3 implies that we have $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(H,d)$. Since H is a proper subgroup of G, by induction again, we have $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_{(P,e)}(N,c)$ by 2.3. This contradicts our choice of b and completes the proof of Theorem 1.4. □

Proof of Theorem 1.7. Since b is $SL(2,p)$-free, we apply Theorem 1.4 with the Glauberman functor W mapping P to $Q = K_\infty(P)$. Then the subpair (Q, e_Q) is centric, and its normaliser controls fusion in $\mathcal{F}_{(P,e)}(G,b)$. The Theorem follows immediately from 2.8 and 2.7. □

7 Proof of Theorem 1.5 and 1.6

Proof of Theorem 1.5. In this section, we keep the notation of Theorem 1.5; that is, we let G be a finite group, and let H be a normal subgroup of G. Let c be a G-stable block of H and let b be a block of G, which covers c; that is, b satisfies $bc = b$. Let (P,e) be a maximal (G,b)-subpair and set $Q = P \cap H$. Then clearly Q is a defect group of the block c of N. Let (Q, e_Q) be the unique (G,b)-subpair contained in (P,e) and let f be a block of $C_H(Q)$ covered by the block e_Q of $C_G(Q)$, i.e. such that $e_Q f \neq 0$. Then (Q,f) is a maximal (H,c)-subpair. If $x \in N_G(Q,e_Q)$, then ${}^x f$ is a block of $kC_H(Q)$ which is covered by e_Q, hence $x = yz$ for some $y \in C_G(Q)$ and some $z \in [N_G(Q,e_Q) \cap N_G(Q,f)]$. In other words, we have

$$N_G(Q,e_Q) = C_G(Q)[N_G(Q,e_Q) \cap N_G(Q,f)].$$

The group in square brackets has a block which induces up to the block $kN_G(Q,e_Q)e_Q$ and thus contains a defect group of $kN_G(Q,e_Q)e_Q$. For some $y \in C_G(Q)$, we thus have

$${}^y P \leq N_G(Q,e_Q) \cap N_G(Q,f),$$

hence

$$P \leq N_G({}^{y^{-1}}(Q,e_Q)) \cap N_G({}^{y^{-1}}(Q,f)),$$

Since ${}^{y^{-1}}(Q) = Q$ and since $e_Q {}^{y^{-1}} f = {}^{y^{-1}}(e_Q f) \neq 0$, on replacing (Q,f) by ${}^{y^{-1}}(Q,f)$, we may assume that P stabilises f, and this proves the first statement of the Theorem.

For any subgroup R of P, we let e_R be the unique block of $C_G(R)$ such that $(R, e_R) \leq (P, e)$, and for a subgroup S of Q, we let f_S be the unique block of $C_H(S)$ such that $(S, f_S) \leq (Q, f)$. Note that whenever R is a subgroup of P, the pair $(R \cap H, f_{R\cap H})$ is stabilised by $N_P(R \cap H)$ because this last group stabilzes $R \cap H$ and (Q, f).

Let $\mathcal{F}$ denote the Brauer category $\mathcal{F}_{(P,e)}(G, b)$ and let $\mathcal{H}$ denote the Brauer category $\mathcal{F}_{(Q,f)}(H, c)$. Let $\mathcal{C}$ denote the Alperin-Goldschmidt conjugation family for $\mathcal{F}$.

Let $\mathcal{D}$ denote the set of objects R of $\mathcal{F}$ such that

(i) $N_P(R)$ is a defect group of $kN_G(R, e_R)e_R$.

(ii)$N_P(R \cap H)$ is a defect group of $kN_G(R \cap H, e_{R\cap H})e_{R\cap H}$.

(iii) $N_P(R \cap H)$ stabilises $f_{R\cap H}$.

7.1. *Every object in $\mathcal{F}$ is isomorphic to an object in $\mathcal{D}$ and $\mathcal{C} \cap \mathcal{D}$ is a conjugation family for $\mathcal{F}$.*

Proof. Consider $(R, e_R) \leq (P, e)$. Let (S, u) be a (G, b)-Brauer pair such that S is maximal with respect to normalising (R, e_R). Since $N_G(R, e_R) \leq N_G(R \cap H, e_{R\cap H})$, we may find a (G, b)- subpair (T, v) such that T is maximal with respect to normalising $(R \cap H, e_{R\cap H})$ and such that $S \leq T$. Note that T is a defect group of $kN_G(R \cap H, e_{R\cap H})e_{R\cap H}$ and S is a defect group of $N_G(R, e_R)e_R$.

For some element x of G, we have ${}^x(T, v) \leq (P, e)$. Thus we have ${}^x(R \cap H, e_{R\cap H}) \leq {}^x(R, e_R) \leq {}^x(S, u) \leq {}^x(T, v) \leq (P, e)$.

Clearly, ${}^xT$ is a defect group of $kN_G({}^x(R \cap H, e_{R\cap H}))^x e_{R\cap H}$, and ${}^xS$ is a defect group of $kN_G({}^x(R, e_R))^x e_R$. Also, ${}^xS = N_P({}^xR)$ and ${}^xT = N_P({}^x(R \cap H))$.

Hence, on replacing (R, e_R) by ${}^x(R, e_R)$, we may assume that (R, e_R) satisfies (i) and (ii) above. Statement (iii) is immediate from (i) and (ii), since P stabilses (Q, f). This proves the first part of the proposition. Since the set of objects R of $\mathcal{F}_{(P,e)}(G, b)$ for which (R, e_R) is a centric and radical (G, b)-subpair is invariant under $\mathcal{F}$ isomorphism, this proves also the second part of Statement 7.1. □

Let $\mathcal{E}$ be the Alperin-Goldschmidt conjugation family for $\mathcal{F}_{(Q,f)}(H, c)$.

7.2. *If $R \in \mathcal{C} \cap \mathcal{D}$, then $R \cap H \in \mathcal{E}$.*

Proof. Let $R \in \mathcal{C} \cap \mathcal{D}$ and let $\tilde{e}_{R\cap H}$ and $\tilde{f}_{R\cap H}$ respectively denote the blocks of $N_G(R \cap H)$ and $N_H(R \cap H)$ induced from $e_{R\cap H}$ and $f_{R\cap H}$. Since $N_P(R \cap H)$ is a defect group of $kN_G(R \cap H, e_{R\cap H})e_{R\cap H}$, $N_P(R \cap H)$ is a defect group of $kN_G(R \cap H)\tilde{e}_{R\cap H}$. Since the block $\tilde{e}_{R\cap H}$ of $kN_G(R \cap H)$ covers the block $\tilde{f}_{R\cap H}$ of $kN_H(R \cap H)$, $N_Q(R \cap H)$ is a defect group of $kN_H(R \cap H)\tilde{f}_{R\cap H}$; hence the defect groups of $kN_H(R \cap H, f_{R\cap H})f_{R\cap H}$ have order $|N_Q(R \cap H)|$. On the other hand, since $(R, e_R) \in \mathcal{D}$, $N_Q(R \cap H) \subseteq N_H(R \cap H, f_{R\cap H})$, thus $N_Q(R \cap H)$ is a defect group of $kN_H(R \cap H, f_{R\cap H})f_{R\cap H}$.

Next we show that $(R \cap H, f_{R\cap H})$ is (H, c)-centric. For this, by the above remarks, it suffices to show that $C_Q(R \cap H) = Z(R \cap H)$. Choose p-regular $y \in C_H(R \cap H) \cap N_G(R, e_R)$. Then $[R, y] \subseteq R \cap H$, so that $[R, y, y] = 1$, and hence $[R, y] = 1$ as y is p-regular. Hence $[C_H(R \cap H) \cap N_G(R, e_R)]/C_H(R)$ is a p-group. On the other hand, $C_H(R \cap H) \cap N_G(R, e_R)$ is clearly a normal subgroup of $N_G(R, e_R)$, and $O_p(N_G(R, e_R)/RC_G(R)) = 1$. Hence, $C_H(R \cap H) \cap N_G(R, e_R) \subseteq RC_G(R)$. Since

$C_P(R) = Z(R)$, we get $C_Q(R \cap H) \cap N_Q(R) \subseteq R$. Since R normalises $C_Q(R \cap H)$, this means that R is its own normaliser in the p-group $C_Q(R \cap H)R$ whence $C_Q(R \cap H) \subseteq R$.

It remains to show that $O_p(N_H(R \cap H, f_{R\cap H})/(R \cap H)C_H(R \cap H)) = 1$. So, let M be the full inverse image of $O_p(N_H(R \cap H, f_{R\cap H})/(R \cap H)C_H(R \cap H))$ in $N_H(R \cap H, f_{R\cap H})$. Since $N_P(R \cap H)C_H(R \cap H)/(R \cap H)C_H(R \cap H)$ is a Sylow-p subgroup of $N_H(R \cap H, f_{R\cap H})/(R \cap H)C_H(R \cap H))$, we have $M = (M \cap P)C_H(R \cap H)$. We will show that $M \cap P \subset R \cap H$.

We have $N_G(R \cap H, e_{R\cap H}) = C_G(R \cap H)[N_G(R \cap H, f_{R\cap H}) \cap N_G(R \cap H, e_{R\cap H})]$, and $C_H(R \cap H)$ is normal in $N_G(R \cap H, e_{R\cap H})$; hence $C_H(R \cap H)[M \cap N_G(R \cap H, e_{R\cap H})]$ is normal in $N_G(R \cap H, e_{R\cap H})$. Since $C_H(R \cap H) \subset [M \cap N_G(R \cap H, e_{R\cap H})]$, this means that $[M \cap N_G(R \cap H, e_{R\cap H})]$ is normal in $N_G(R \cap H, e_{R\cap H})$ and hence is normal in $N_G(R, e_R)$. By the definition of M, it follows that $M \cap N_G(R \cap H, e_{R\cap H})/C_H(R \cap H)$ is a p-group. On the other hand, we have shown before that $C_H(R \cap H) \cap N_G(R, e_R)/C_H(R)$ is a p-group. Hence, $M \cap N_G(R, e_R)/C_H(R)$ is a normal p subgroup of $N_G(R, e_R)/C_H(R)$, and is therefore isomorphic to a normal p-subgroup of $N_G(R, e_R)/C_G(R)$. But then by choice of (R, e_R) it follows that $M \cap N_G(R, e_R) \subset RC_G(R)$ whence $M \cap N_P(R) \subset RC_P(R) \cap H \subset R \cap H$. Since R normalises $M \cap P$, we see that $M \cap P \subset R \cap H$. This completes the proof. □

Now let V be a normal subgroup of Q and suppose that $N_H(V)$ controls fusion in $\mathcal{H}$ and let W be as in the statement of the Theorem.

7.3. *$N_H(W)$ controls fusion in $\mathcal{H}$. Further, if S is a subgroup of Q containing W then $N_G(S, e_S) \subset N_G(W)$.*

Proof. Let $(S, f_S) \leq (Q, f)$ and let $x \in N_G(Q, f)$. Since ${}^{x^{-1}}(S, f_S) \leq (Q, f)$, we have that $N_H({}^{x^{-1}}(S, f_S)) \subset C_H({}^{x^{-1}}S)N_H(V)$ whence $N_H(S, f_S) \subset C_H(S)N_H({}^{x}V)$.Thus $N_H({}^{x}V)$ controls fusion in $\mathcal{F}_{H,c}$ for all $x \in N_G(Q, f)$. It follows by Lemma 2.1 that if $S \in \mathcal{E}$, then $N_H(S, e_S) \subseteq N_H({}^{x}V)$ for all $x \in N_G(Q, f)$, so that in particular, $N_H(S, e_S) \subseteq N_H(W)$. Hence $N_H(W)$ controls fusion in $\mathcal{F}_{H,c}$.

Let S be a subgroup of Q containing W and let $x \in N_G(S, e_S)$. By the Frattini argument, we may write $x = yz$, where $y \in N_G(Q, f)$ and $z \in H$. Then ${}^{z}(S, f_S) = {}^{y^{-1}x}(S, f_S) \leq (Q, f)$. Since $N_H(W)$ controls fusion in $\mathcal{F}_{H,c}$, we may write $z = ct$, where $c \in C_H(S) \subset N_H(W)$ and $t \in N_H(W)$. Since by definition of W, $y \in N_G(W)$, we have $x = yct \in N_G(W)$. □

Let $R \in \mathcal{C} \cap \mathcal{D}$. Then by 7.2, $R \cap H \in \mathcal{E}$. In particular, by Lemma 2.1, we have that $W \subset R \cap H$ and it follows by 7.3 that $N_G(R \cap H, f_{R\cap H}) \subset N_G(W)$. Hence, $N_G(R, e_R) \subset N_G(R \cap H, e_{R\cap H}) \subset C_G(R \cap H)[N_G(R \cap H, e_{R\cap H}) \cap N_G(R \cap H, f_{R\cap H})] \subset N_G(W)$. Theorem 1.5 now follows from 7.2 and the fact that $P \subseteq N_G(W)$. □

Proof of 1.6. By a standard argument we may assume that G stabilises the block c. Then 1.6 is an immediate consequence of 2.3 and Theorems 1.4 and 1.5. □

Remark 7.4 The advantage of Theorem 1.6 is that if we wish to produce a single local subgroup controlling fusion in $\mathcal{F}_{(P,e)}(G, b)$, it is not really necessary to assume that b is $SL(2,p)$ free. This could be useful in some instances; for example, suppose that $G = XwrS_n$ for some large integer n and some non-Abelian finite simple group X, while H is the base-group of the wreath product. It is quite possible for automizers of "diagonal-type" (G,b)-subpairs to involve $SL(2,p)$ because of the action of the S_n, while automizers (in H) of (H,c) might not involve $SL(2,p)$.

8 Proof of 1.10

Proof of 1.10. It is clear that the pair (P,e) is a maximal $(N_G(R),c)$-subpair. For a subgroup Q of R, we let (Q,f_Q) be the unique $(N_G(R),c)$ subpair contained in (P,e).

In [13], it is shown that if we are considering a group G such that DPC holds in every section of G, then in calculating the various quantities $k_d(B,\lambda)$, it is only necessary to consider chains of (G,b)-pairs whose initial objects are pairs (Q,e_Q) contained in (P,e) which are (G,b)-centric and radical. By Lemma 2.2, we have that for any such subpair (Q,e_Q), $R \leq Q$, and thus $N_G(Q,e_Q) \subset C_G(Q)N_G(R) \subset N_G(R)$. The fact that $R \leq Q$ also implies that $f_Q = e_Q$. It follows that in the subpair version of (W)DPC, the contribution in kGb from chains beginning with (Q,e_Q) is the same as the contribution in $kN_G(R)c$ from chains beginning with (Q,e_Q). Similarly, it follows that if DPC holds in every proper section of G, then checking DPC for G reduces to checking that there is a defect-preserving bijection between irreducible characters of B lying over λ and irreducible characters in c lying over λ. $\square$

Acknowledgement. This work was completed while the third author was a Visiting Fellow at All Souls College, Oxford. The author is grateful to the College for its support and hospitality.

Note added in proof: Since this work was written, a related result of M. Lechuga (Theorem 7.11 in his thesis *Contribution à l'étude locale dans les groups finis*, Publ. Math. Univ. Paris 7, tome IV, 1994) has been brought to our attention. Lechuga's result concerns the particular Glauberman functor ZL (defined by L. Puig), is valid for $p \geq 5$, and makes use of J. G. Thompson's classification of quadratic pairs. While, as stated, it does not imply the involvement of $SL(2,p)$ in the relevant automizer, the $PSL(2,p^n)$ and $PSU(3,p^m)$ components he mentions arise because of quadratic action, so the presence of a genuine $SL(2,p)$ is implicit.

References

1. J. L. Alperin, M. Broué, *Local methods in block theory*, Ann. Math. **110** (1979), 143–157.
2. R. Brauer, *Zur Darstellungstheorie der Gruppen endlicher Ordnung*, Math. Z. **63** (1956), 406–444.
3. R. Brauer, *On the structure of blocks of characters of finite groups*, Lecture Notes in Mathematics **372** (1974), 103–130.
4. C. Broto, R. Levi, R. Oliver, *The homotopy theory of fusion systems*, preprint (2001).
5. M. Broué, *Brauer Coefficients of p-Subgroups Associated with a p-Block of a Finite Group*, J. Algebra **56** (1979), 356–383.
6. G. Glauberman, *A characteristic subgroup of a p-stable group*, Canadian J. Math. **20** (1968), 1101–1135.

7. G. Glauberman, *Global and local properties of finite groups*, Finite simple groups (eds. Powell-Higman), Academic Press, London, 1971, pp. 1–64.
8. D. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980.
9. B. Huppert, N. Blackburn, *Finite Groups III*, Springer-Verlag, Berlin Heidelberg New York, 1982.
10. R. Kessar, M. Linckelmann, *A block theoretic analogue of a theorem of Glauberman and Thompson*, Proc. Amer. Math. Soc. **131** (2003), 45–50.
11. B. Külshammer, L. Puig, *Extensions of nilpotent blocks*, Invent. Math. **102** (1990), 17–71.
12. L. Puig, *Unpublished notes.*
13. G. R. Robinson, *Cancellation theorems related to conjectures of Alperin and Dade*, J. Algebra **249** (2002), 196–219.
14. J. Thévenaz, *G-Algebras and Modular Representation Theory*, Oxford Science Publications, Clarendon Press, Oxford, 1995.

Radha Kessar
Math Tower
231 West 18th Avenue
Columbus, OH 43210
U.S.A.

Markus Linckelmann
Math Tower
231 West 18th Avenue
Columbus, OH 43210
U.S.A.

Geoffrey R. Robinson
School of Mathematics and Statistics
University of Birmingham
Birmingham B15 2TT
U.K.