ON BLOCKS WITH FROBENIUS INERTIAL QUOTIENT

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January 2001

1. Introduction

Let p be a prime number, k a field of characteristic p, G a finite group and b a block of kG, that is, a primitive idempotent of the center of kG. In [23] Okuyama and Tsushima showed that the center $Z(kGb)$ of the block algebra kGb is a symmetric algebra if and only if b has abelian defect groups and trivial inertial quotient, in other words if and only if b is a nilpotent block with abelian defect groups.

In this paper we study the connection between symmetry properties of the stable center and the $p-$local structure of group algebras and their blocks. The stable center of a finite dimensional k-algebra A is defined as follows. Denoting by A^0 the opposite algebra of A, the algebra A has a natural structure as an $A \otimes_k A^0-$module via left and right multiplication. It is easy to see that the map $(z \to (a \to az))$ gives an isomorphsim between the center $Z(A)$ of A and the ring $\mathrm{End}_{A\otimes_k A^o}(A)$ of $A \otimes_k A^o$-module endomorphisms of A. We denote by $Z^{pr}(A)$ the ideal in $Z(A)$ consisting of those elements of $Z(A)$ whose image under the above isomorphism is an endomorphism which factors through a projective $A \otimes_k A^o$- module. The stable center $\bar{Z}(A)$ is then defined to be the quotient $Z(A)/Z^{pr}(A)$.

It turns out that the property of $\bar{Z}(kGb)$ being symmetric puts a strong restriction on the $p-$local structure of b. We assume that k is large enough for all the algebras appearing in the statements of this section to be split. In other words we assume that the semi-simple quotients of these algebras are direct products of matrix algebras over k.

Theorem 1.1. *Let G be a finite group having a non-trivial Sylow-p-subgroup P and let b be the principal block of kG. Then $\bar{Z}(kGb)$ is a symmetric algebra if and only if P is abelian and $N_G(P)/C_G(P)$ acts freely on $P - \{1\}$.*

One implication in the preceding theorem can be formulated for arbitrary blocks of kG.

Theorem 1.2. *Let b be a block of kG and let (P, e) be a maximal b-Brauer pair. If P is non-trivial abelian and $N_G(P, e)/C_G(P)$ acts freely on $P - \{1\}$, then $\bar{Z}(kGb)$ is a symmetric algebra.*

The question of whether the converse of Theorem 1.2 holds for arbitrary blocks is open. We will give some necessary and sufficient conditions for a block to have symmetric stable center in Theorem 3.1 below. Combining 1.1 and 1.2 yields the following statement.

Corollary 1.3. *Let G be a finite group having a non-trivial Sylow p-subgroup P. Then $\bar{Z}(kG)$ is a symmetric algebra if and only if P is abelian and $N_G(P)/C_G(P)$ acts freely on $P - \{1\}$.*

Proof. If $\bar{Z}(kG)$ is symmetric, so is $\bar{Z}(kGb)$, where b is the principal block of kG. Thus P is abelian and $N_G(P)/C_G(P)$ acts freely on $P - \{1\}$ by 1.1. Conversely, if P is abelian and $N_G(P)/C_G(P)$ acts freely on $P - \{1\}$, then for any non-trivial subgroup Q, $N_G(Q)/C_G(Q)$ acts freely on $Q - \{1\}$. Thus, for any block b of kG with a non-trivial defect group, $\bar{Z}(kGb)$ is symmetric by 1.2. If b is a block of kG with the trivial defect group 1, then kGb is a separable algebra, and hence $Z(kGb) = Z^{pr}(kGb)$, or equivalently, $\bar{Z}(kGb) = \{0\}$. Consequently, $\bar{Z}(kG)$ is symmetric. □

The stable center of a symmetric algebra A is the degree zero component of the Tate analogue of the Hochschild cohomology (whose definition is given in §5 below). The Hochschild cohomology of Brauer tree algebras has been computed by Holm [16] and Erdmann-Holm [10]; in the particular case of blocks with cyclic defect groups, an alternative approach appears in the work of Siegel and Witherspoon [31]. We extend their methods to computing $\hat{HH}^*(kGb)$ in terms of the Tate cohomology of P for blocks fulfilling the hypotheses of 1.2.

Theorem 1.4. *Let G be a finite group, b a block of kG and (P, e) a maximal $b-$Brauer pair. Suppose that P is non-trivial abelian and that $E = N_G(P, e)/C_G(P)$ acts freely on $P - \{1\}$. There is an isomorphism of graded $k-$algebras*

$$\hat{HH}^*(kGb) \cong (kP \underset{k}{\otimes} \hat{H}^*(P, k))^E \ ,$$

where E acts diagonally on the tensor product via its natural action on P.

Remark 1.5 Theorem 1.4 holds more generally for a complete local Noetherian ring with k as residue field instead of k, as follows easily from the proof. Also, we should point out that $\bar{Z}(kGb)$ is symmetric if and only if its socle has dimension one. More generally, a finite-dimensional split local commutative $k-$algebra is symmetric if and only if its socle has dimension one.

This paper is divided into six sections. In section 2, we collect some general results on symmetric algebras. In particular, we give necessary and sufficient conditions for such an algebra to have a symmetric stable center. We then interpret these results in the specific context of block algebras in section 3. In section 4, we prove Theorems 1.1 and 1.2. In section 5, we describe the Tate analogue of the Hochschild cohomology of a symmetric algebra and prove 1.4. We will introduce relevant notation and terminology in each section as the need arises.

2. Some results on symmetric algebras

Let k be a field of prime characteristic p. For a finite-dimensional k-algebra A, we say that k is large enough for A if A is split. We denote by $l(A)$ the number of isomorphism classes of simple $A-$modules. Recall that a finite-dimensional k-algebra A is called *symmetric* if there exists a k-linear form $s : A \to k$ such that $s(ab) = s(ba)$ for every pair of elements a, b of A, and such that no non zero left or right ideal of A is contained in the kernel of s. Any such form s is called a *symmetrising form of A*. Note that the commutator subspace $[A, A]$ of a symmetric algebra A is contained in the kernel of any symmetrising form of A.

Any group algebra kG of a finite group G is symmetric, with the canonical symmetrising form mapping 1_G to 1_k and any non-trivial element of G to zero. Furthermore, if E is a $p'-$subgroup acting on G, then the algebra $(kG)^E$, of $E-$fixed points in kG is still symmetric, since the restriction to $(kG)^E$ of the canonical symmetrising form on kG remains a symmetrising form.

Given a symmetrising form s of A, for any k-subspace U of A, we denote by $U^\perp$ the k-subspace $\{a \in A \ : s(au) = 0 \text{ for all } u \in U\}$ (this is a slight abuse of notation, since $U^\perp$ depends in general on the choice of the symmetrising form). It is easy to check that $(U^\perp)^\perp = U$ and that $\dim_k(U) + \dim_k(U^\perp) = \dim_k(A)$. In the following proposition we gather a few well-known standard facts about symmetric algebras; we refer to Külshammer [19] for proofs as well as further properties of symmetric algebras.

Proposition 2.1. *Suppose that A is a symmetric k-algebra with symmetrising form $s : A \to k$. Then the following hold.*
(i) $Z(A)^\perp = [A, A]$.
(ii) $J(A)^\perp = \operatorname{soc}(A)$.
(iii) $Z(A) \cap \operatorname{soc}(A) \subseteq \operatorname{soc}(Z(A))$.
(iv) $Z^{pr}(A) \subseteq Z(A) \cap \operatorname{soc}(A)$.
Moreover, if A is split then
(v) $\dim_k(Z(A) \cap \operatorname{soc}(A)) = \dim_k(A/(J(A) + [A, A])) = l(A)$.

Lemma 2.2. *Let T be a finite-dimensional local commutative k-algebra and suppose that J is an ideal of T such that T/J is a symmetric algebra. Then, for every ideal I of T such that $J \subseteq I$, either $\operatorname{soc}(T) \subseteq I$ or $I = J$.*

Proof. Let T and J be as in the proposition. Since T/J is local and symmetric, $\operatorname{soc}(T/J)$ is simple. Thus, if I is an ideal of T properly containing J then $\operatorname{soc}(T/J) \subseteq I/J$. In particular, $(\operatorname{soc}(T) + J)/J \subseteq I/J$, hence $\operatorname{soc}(T) \subseteq I$. □

We will need the following proposition (the proof we present here, which shortens our original argument is due to the Referee).

Proposition 2.3. *Let A be a symmetric $k-$algebra. Assume that A has a simple module of dimension prime to p. Then $Z(A) \cap \operatorname{soc}(A) \nsubseteq Z(A) \cap [A, A]$. In particular, $\operatorname{soc}(Z(A)) \nsubseteq Z(A) \cap [A, A]$.*

Proof. If $Z(A) \cap \operatorname{soc}(A) \subseteq [A, A]$, taking perpendicular spaces yields $[A, A] + J(A) \supseteq Z(A)$. However, the elements in $[A, A] + J(A)$ have trace zero on every simple A-module while 1_A has non zero trace on any A-module of dimension prime to p. This proves the first statement of the proposition; the second follows from part (iii) of Proposition 2.1. □

Corollary 2.4. *Let A be a symmetric $k-$algebra. Assume that A has a simple module of dimension prime to p. Let J be an ideal of $Z(A)$ contained in $Z(A) \cap [A, A]$. Then $Z(A)/J$ is symmetric if and only if $J = Z(A) \cap [A, A]$.*

Proof. By 2.3, we have $\operatorname{soc}(Z(A)) \nsubseteq Z(A) \cap [A, A]$. Thus, if $Z(A)/J$ is symmetric, then $J = Z(A) \cap [A, A]$ by 2.2. Conversely, since $Z(A)^\perp = [A, A]$ by 2.1(i), any symmetrising form on A induces a symmetrising form on $Z(A)/(Z(A) \cap [A, A])$. □

Corollary 2.4 is going to be applied below in the case where $J = Z^{pr}(A)$. In an arbitrary symmetric algebra A, the projective ideal $Z^{pr}(A)$ need not be contained in

$[A,A]$, but the following easy (and well-known) observation will imply that whenever A is a block algebra or block source algebra with non trivial defect groups, then indeed $Z^{pr}(A) \subseteq [A,A]$.

Let H, L be subgroups of a finite group G and let A be a finite-dimensional $k-$algebra on which G acts. We denote as usual by A^H the subalgebra of elements $a \in A$ which are fixed under the action of H on A. If $L \subset H$, we denote by

$$\mathrm{Tr}_L^H : (kG)^L \to (kG)^H$$

the *relative trace map* (cf. [14]) which sends $a \in A^L$ to $\sum_{x \in [H/L]} {}^x a \in A^H$ and we set $A_L^H = \mathrm{Im}(\mathrm{Tr}_L^H)$; this is easily seen to be an ideal in A^H. If P is a $p-$subgroup of G, we set $A(P) = A^P / \sum_Q A_Q^P$, where Q runs over the set of proper subgroups of P, and we denote by $\mathrm{Br}_P^A : A^P \to A(P)$ the canonical surjective algebra homomorphism (called *Brauer homomorphism*; see [32, §11]). If $A = kG$, there is a canonical isomorphism $A(P) \cong kC_G(P)$. Recall from [26] (see also [32]), that an *interior $G-$algebra* is an algebra A endowed with a group homomorphism $G \to A^\times$. In particular, G acts on A by conjugation with the images in $A^\times$ of the elements of G.

Lemma 2.5. *Let G be a finite group and let A be an interior $G-$algebra of finite dimension over k. Then for any two subgroups, H and L of G such that $L \subseteq H$ and such that p divides the index of L in H, we have $A_L^H \subseteq [A,A]$. In particular, for every non-trivial p-subgroup P of G, we have $A_1^P \subseteq \ker(\mathrm{Br}_P^A) \subseteq [A,A]$.*

Proof. Let $a \in A^L$. We have $\mathrm{Tr}_L^H(a) - [H:L]a = \sum_{x \in H/L}[xa, x^{-1}] \in [A,A]$. Since p divides $[H:L]$, the first statement follows. The second part of the Lemma is an immediate consequence of the first since $\ker(\mathrm{Br}_P^A) = \sum_{R \subsetneq P} A_R^P$. □

3 A criterion for blocks with symmetric stable center

Let G be a finite group and let b be a block of kG; that is, b is a primitive idempotent in $Z(kG)$. For any $p-$subgroup P of G, we denote by $\mathrm{Br}_P : (kG)^P \to kC_G(P)$ the Brauer homomorphism obtained from composing Br_P^{kG} with the canonical isomorphism $(kG)(P) \cong kC_G(P)$. A $b-$Brauer pair is a pair (P,e) consisting of a $p-$subgroup P of G and a block e of $C_G(P)$ satisfying $\mathrm{Br}_P(b)e \neq 0$. Similarly, a $b-$Brauer element is a pair (u,f) consisting of a $p-$element in G and a block f of $C_G(u)$ satisfying $\mathrm{Br}_{<u>}(b)f \neq 0$. The set of $b-$Brauer pairs is partially ordered, and the group G acts transitively by conjugation on the set of maximal $b-$Brauer pairs (cf. [1]). We say that the $b-$Brauer element (u,f)belongs to the $b-$Brauer pair (P,e) if $(<u>, f) \leq (P,e)$. A $p-$subgroup P of G is a defect group of b if it is minimal such that $b \in (kG)_P^G$, or equivalently, if it is maximal such that $\mathrm{Br}_P(b) \neq 0$. In particular, the maximal $b-$Brauer pairs are precisely the $b-$Brauer pairs (P,e) in which P is a defect group of b. Since $\mathrm{Br}_P(b) \neq 0$ when P is a defect group of b, there is a primitive idempotent i in $(kGb)^P$ such that $\mathrm{Br}_P(i) \neq 0$. The interior $P-$algebra $ikGi$, with structural homomorphism mapping $u \in P$ to $ui \in (ikGi)^\times$, is a *source algebra of* b. This concept is due to L. Puig [26]. We refer to [32] for a detailed exposition of the material from block theory that we use here.

The above definitions make sense for arbitrary fields of characteristic p, but for the remainder of the paper, unless stated otherwise, we assume that k is large enough for all the block algebras appearing below.

The stable Grothendieck group of a block b of G is the quotient of the Grothendieck group of finitely generated $kGb-$modules by the subgroup generated by the images of the finitely generated projective $kGb-$modules. This is an abelian $p-$group whose order is the determinant of the Cartan matrix of kGb, and whose $p-$rank is the number of non-trivial elementary divisors (with multiplicities) of the Cartan matrix.

Theorem 3.1. *Let G be a finite group, b a block of kG and let (P,e) be a maximal $b-$Brauer pair. Suppose that P is non-trivial. The following are equivalent.*

(i) $\bar{Z}(kGb)$ is a symmetric algebra.

(ii) The stable Grothendieck group of kGb is cyclic, for every non-trivial $b-$Brauer element (u,f), the algebra $kC_G(u)f$ has a unique isomorphism class of simple modules, and there exists $x \in G$ such that ${}^x(u,f) \in (P,e)$ and ${}^xu \in Z(P)$.

(iii) We have $(kC_G(Q)f)_Q^{N_G(Q,f)} = \{0\}$ for any b-Brauer pair (Q,f) such that Q is conjugate to a non-trivial proper subgroup of P.

We break up the proof in a series of Lemmas. The first one is a collection of elementary (and well-known) observations which hold without the assumption on k being large enough; we include a proof for the convenience of the reader.

Lemma 3.2. *Let G be a finite group, let b be a block of kG and let P be a defect group of b.*

(i) For any subgroup Q of P, $(kGb)_Q^G \cap \ker(\mathrm{Br}_Q) = \sum_R (kGb)_R^G$, where R runs over the set of proper subgroups of Q.

(ii) $Z(kGb) \cap \ker(\mathrm{Br}_Q) \subseteq Z(kGb) \cap \ker(\mathrm{Br}_P)$ for any subgroup Q of P.

(iii) $Z(kGb) \cap (kGb)_Q^P \subseteq (kGb)_Q^G$ for any subgroup Q of P.

(iv) $Z(kGb) \cap (kGb)_1^P = (kGb)_1^G = Z^{pr}(kGb)$.

Proof. For any conjugacy class C of G, let $\underline{C}$ denote the sum in kG of all elements of C. Let $Cl(G)$ denote the set of conjugacy classes of G.

Let $a = \sum_{C \in Cl(G)} \alpha_C \underline{C}$ be an element of $Z(kG)$ and let Q be a p-subgroup of G. Identifying Br_Q with the canonical surjection of $(kG)^Q$ onto $kC_G(Q)$, we have $\mathrm{Br}_Q(a) = \sum_{C \in Cl(G)} \sum_{x \in C \cap C_G(Q)} \alpha_C x$. From this it follows that $a \in \ker(\mathrm{Br}_Q)$ if and only if $\alpha_C = 0$ for all $C \in Cl(G)$ such that Q is contained in a Sylow p-subgroup of $C_G(x)$ for some $x \in C$; (ii) is immediate from this observation. Also, $a \in (kG)_Q^G$ if and only if α_C is zero except when a Sylow p-subgroup of $C_G(x)$ is contained in Q for some $x \in C$. Thus $a \in (kG)_Q^G \cap \ker(\mathrm{Br}_Q)$ if and only if α_C is zero except when a Sylow p-subgroup of $C_G(x)$ is properly contained in Q for some $x \in C$. In other words, $(kG)_Q^G \cap \ker(\mathrm{Br}_Q) = \sum_R (kG)_R^G$, and this proves (i).

Now let Q be a subgroup of P and let $a = \sum_{C \in Cl(G)} \alpha_C \underline{C}$ be a non-zero element of $(kG)_P^G \cap (kG)_Q^P$. Choose C in $Cl(G)$ such that α_C is non-zero. Since, $a \in (kG)_P^G$, there is x in C such that a Sylow p-subgroup of $C_G(x)$ is contained in P. On the other hand, the space $(kG)_Q^P$ is spanned by elements of the form $\mathrm{Tr}_{C_P(y)}^P(y)$ such that $C_P(y) \subset Q$. Thus there is c in P such that $C_P({}^cx) \subset Q$. But $C_P({}^cx) = C_{{}^cP}({}^cx)$ is a Sylow p-subgroup of $C_G({}^cx)$. Thus Q contains a Sylow p-subgroup of $C_G({}^cx)$ and it follows that $\underline{C}$ and hence a is in $(kG)_Q^G$. Since $Z(kGb) \subset (kG)_P^G$, we have (iii). Statement (iv) follows from (iii) by putting $Q = \{1\}$. □

The first statement of the next Lemma is due to L. Puig [26, 3.5].

Lemma 3.3. *Let G be a finite group, b a block of G, P a defect group of b and let i be a primitive idempotent in $(kGb)^P$ such that $\mathrm{Br}_P(i) \neq 0$. Multiplication by i induces an isomorphism of the centers $Z(kGb) \cong Z(ikGi)$. This isomorphism maps $Z^{pr}(kGb)$ onto $Z^{pr}(ikGi)$ and $Z(kGb) \cap \ker(\mathrm{Br}_P)$ onto $Z(ikGi) \cap \ker(\mathrm{Br}_P)$.*

Proof. By [26, 3.5], multiplication by i induces a Morita equivalence between kGb and $ikGi$. Any Morita equivalence preserves centers and projective ideals. Since i commutes with P, clearly multiplication by i maps $Z(kGb) \cap \ker(\mathrm{Br}_P)$ into $Z(ikGi) \cap \ker(\mathrm{Br}_P)$. If $z \in Z(kGb)$ such that $\mathrm{Br}_P(iz) = 0$, then $\mathrm{Br}_P(\mathrm{Tr}_P^G(i)z) = \mathrm{Br}_P(\mathrm{Tr}_P^G(iz)) = \mathrm{Tr}_P^{N_G(P)}(\mathrm{Br}_P(iz)) = 0$, where we use the formula [32, 11.9]. By [25, Prop. 1] (see also [32, 9.3]), $\mathrm{Tr}_P^G(i)$ is invertible (this is where we use that k is large enough), and since Br_P is an algebra homomorphism, it follows that $\mathrm{Br}_P(z) = 0$. □

Lemma 3.4. *Let G be a finite group, let b be a block of kG, let P be a defect group of b and let i be a primitive idempotent in $(kGb)^P$ such that $\mathrm{Br}_P(i) \neq 0$. Then*

$$Z(ikGi) \cap \ker(\mathrm{Br}_P) = Z(ikGi) \cap [ikGi, ikGi] \ .$$

Proof. If P is trivial, both sides in 3.4 are zero. If $P \neq 1$, the left side is contained in the right side by 2.5. By 3.3, multiplication by i maps $Z(kGb) \cap \ker(\mathrm{Br}_P)$ onto $Z(ikGi) \cap \ker(\mathrm{Br}_P)$. Thus the quotient $Z(ikGi)/(Z(ikGi) \cap \ker(\mathrm{Br}_P))$ is isomorphic to $\mathrm{Br}_P(Z(kGb)) = \mathrm{Br}_P((kGb)_P^G) = (kC_G(P)\mathrm{Br}_P(b))_P^{N_G(P)}$ (cf. [32, 11.9]), and by Broué [4, Prop. III (1.1)], this is isomorphic to the symmetric algebra $(kZ(P))^E$. Since $ikGi$ has a simple module of dimension prime to p (cf. [27, 14.6]), the equality in 3.4 follows from 2.4. □

Lemma 3.5. *Let G be a finite group and let b be a block of kG having a non-trivial defect group P. Denote by r_b the $p-$rank of the stable Grothendieck group of b. Then*

$$l(b) = \dim_k(Z^{pr}(kGb)) + r_b \leq \dim_k(Z(kGb) \cap \ker(\mathrm{Br}_P)) + 1 \ .$$

In particular, the stable Grothendieck group of b is cyclic if and only if $\dim_k(Z^{pr}(kGb)) = l(b) - 1$.

Proof. The proof is based on Brauer's work [3] on lower defect groups and its further developments in [18], [4], [24], [7], as exposed in [22]. For any conjugacy class C of G, let $\delta(C)$ denote the set of G-conjugates of the Sylow p-subgroups of the centralizer in G of an element x of C, and let $\underline{C}$ denote the sum in kG of all elements of C. Let $Cl(G)$ denote the set of conjugacy classes of G and let $Bl(G)$ denote the set of blocks of kG. Then, by [22, Chapter 5, Theorem 11.3], we have a partition

$$Cl(G) = \cup_{e \in Bl(G)} \Omega_e, \quad (\Omega_e \cap \Omega_{e'} = \phi, \quad if \quad e \neq e')$$

such that $\{\underline{C}e|\ C \in \Omega_e\}$ is a k-basis of $Z(kGe)$ for every e in $Bl(G)$. Such a partition is called a block partition of $Cl(G)$.

Let $G_{p'}$ denote the set of p-regular elements of G, and $Cl(G_{p'})$, the set of conjugacy classes of p-regular elements of G. We have $Z^{pr}(kGb) = (kGb)_1^G$ (cf. 3.2(iv)). For any conjugacy class C in G and any $x \in C$, we have $\mathrm{Tr}_1^G(x) = |C_G(x)|\underline{C}$; this is non zero if and only if $|C_G(x)|$ is prime to p. Thus the set

$$\{\underline{C}b \mid C \in \Omega_b \cap Cl(G_{p'}); \delta(C) = 1\} \qquad (*)$$

is a k-basis for $Z^{pr}(kGb)$. On the other hand, for any $C \in \Omega_b$ such that $\delta(C)$ consists of conjugates of proper subgroups of P, we have $\underline{C}b \in \ker(\mathrm{Br}_P^{kGb}) \cap Z(kGb)$.

Now by [22, Chapter 5, Theorem 11.5],

$$|\Omega_b \cap Cl(G_{p'})| = l(b) \ , \tag{**}$$

there exists a unique C in $\Omega_b \cap Cl(G_{p'})$ such that $\delta(C)$ consists of the $G-$conjugates of P, and for all other C' in $\Omega_b \cap Cl(G_{p'})$, the elements of $\delta(C')$ are conjugate to proper subgroups of P. This implies the inequality $l(b) - 1 \leq \dim_k(Z(kGb) \cap \ker(\mathrm{Br}_P))$.

Finally, the orders of the groups in $\delta(C)$, as C varies over the set of $p'-$conjugacy classes in Ω_b, are exactly the elementary divisors of the Cartan matrix of kGb, and thus the $p-$rank r_b of the stable Grothendieck group of kGb is precisely the number of $p'-$conjugacy classes C in Ω_b such that $\delta(C) \neq \{1\}$. Combining (*) and (**) yields the first equality. In particular, the stable Grothendieck group of b is cyclic if and only if $\delta(C) = \{1\}$ for all but one $p'-$conjugacy class C in Ω_b, from which the last statement follows. □

Proposition 3.6. *Let G be a finite group, let b be a block of kG and let (P, e) be a maximal $b-$Brauer pair. Denote by $\mathcal{Z}$ a set of representatives of the $G-$conjugacy classes of $b-$Brauer elements (u, f) for which there is $x \in G$ such that ${}^x(u, f) \in (P, e)$ and ${}^x u \in Z(P)$; denote by $\mathcal{Z}'$ a set of representatives of the $G-$conjugacy classes of all other $b-$Brauer elements. Then*

$$\dim_k(Z(kGb) \cap \ker(\mathrm{Br}_P)) = \sum_{(u,f)\in\mathcal{Z}} (l(f) - 1) \ + \sum_{(u,f)\in\mathcal{Z}'} l(f) \ .$$

Proof. We use a counting argument similar to what appears in the proof of [30, Theorem 3]. Set $E = N_G(P, e)/PC_G(P)$. By considering again Broué's isomorphism $Z(kGb)/(\ker(\mathrm{Br}_P) \cap Z(kGb)) \simeq (kZ(P))^E$, we get that

$$\dim_k(Z(kGb)) = \dim_k(Z(kGb)) \cap \ker(\mathrm{Br}_P)) + \dim_k((kZ(P))^E) \ .$$

By [22, Ch. 5, Theorem 4.13], we have that

$$\dim_k(Z(kGb)) = \sum_{(u,f)\in\mathcal{Z}\cup\mathcal{Z}'} l(f) \ .$$

Now, let (u, f), (v, g) be two $b-$Brauer elements contained in (P, e) such that $u, v \in Z(P)$, and suppose that $(u, f) = {}^x(v, g)$ for some element $x \in G$. Then, in fact $(u, f) = {}^n(v, g)$ for some element $n \in N_G(P, e)$ (cf. [22, Ch. 5, Lemma 9.9]). Thus

$$\dim_k((kZ(P))^E) = |\mathcal{Z}| \ .$$

The Proposition follows from combining these three equalities. □

As a consequence of 3.6 we get a criterion for when the inequality in 3.5 is an equality.

Corollary 3.7. *Let G be a finite group, b a block of G and let (P, e) be a maximal $b-$Brauer pair. The following are equivalent:*

(i) $\dim_k(Z(kGb) \cap \ker(\mathrm{Br}_P)) = l(b) - 1$.

(ii) For every non-trivial $b-$Brauer element (u, f) we have $l(f) = 1$, and there is $x \in G$ such that ${}^x(u, f) \in (P, e)$ and ${}^xu \in Z(P)$.

Proof. We use the notation of 3.6. The set $\mathcal{Z}$ contains the trivial Brauer pair $(1, b)$, which contributes the value $l(b) - 1$ to the sum in 3.6. Thus (i) holds if and only if all other summands are zero. This is clearly equivalent to $l(f) = 1$ for every non-trivial Brauer element $(u, f) \in \mathcal{Z}$ and $\mathcal{Z}' = \emptyset$, thus to statement (ii). □

Lemma 3.8. *Let G be a finite group, b a block of kG, let (P, e) be a maximal $b-$Brauer pair and set $E = N_G(P, e)/PC_G(P)$. Suppose that P is non-trivial. Let i be a primitive idempotent in $(kGb)^P$ such that $\mathrm{Br}_P(i) \neq 0$. The following are equivalent.*

(i) $\bar{Z}(kGb)$ is a symmetric algebra.

(ii) $\bar{Z}(ikGi)$ is a symmetric algebra.

(iii) $\bar{Z}(kGb) \cong \bar{Z}(ikGi) \cong (kZ(P))^E$.

(iv) $Z^{pr}(ikGi) = Z(ikGi) \cap [ikGi, ikGi]$.

(v) $Z^{pr}(kGb) = Z(kGb) \cap \ker(\mathrm{Br}_P)$.

(vi) $Z^{pr}(kGb) = Z(kGb) \cap \ker(\mathrm{Br}_Q)$ for every non-trivial subgroup Q of P.

(vii) $Z^{pr}(kGb) = (kGb)^G_Q$ for any proper subgroup Q of P.

(viii) $\mathrm{Br}_Q((kGb)^G_Q) = 0$ for every non-trivial proper subgroup Q of P.

(ix) $\dim_k(Z^{pr}(kGb)) = l(b) - 1 = \dim_k(\ker(\mathrm{Br}_P) \cap Z(kGb))$.

Proof. Multiplication by i induces an isomorphism $Z(kGb) \cong Z(ikGi)$ mapping $Z^{pr}(kGb)$ onto $Z^{pr}(ikGi)$ by 3.3, whence (i) and (ii) are equivalent. The equivalence of (ii) and (iv) is just a particular case of 2.4, because the source algebra $ikGi$ has a simple module of dimension prime to p (cf. [27, 14.6]). The equivalence of (iv) and (v) follows from combining 3.3 and 3.4. As we have inclusions $Z^{pr}(kGb) \subseteq Z(kGb) \cap \ker(\mathrm{Br}_Q) \subseteq Z(kGb) \cap \ker(\mathrm{Br}_P)$ for any non-trivial subgroup Q of P by 3.2(i), we get the equivalence of (v) and (vi). The equivalence of (v) and (vii) follows from the inclusions $Z^{pr}(kGb) \subseteq (kGb)^G_Q$ for any subgroup Q of G and 3.2(iii). The equivalence of (vii) and (viii) follows from repeated use of 3.2 (i) and the fact that $Z^{pr}(kGb) \subseteq \ker(\mathrm{Br}_Q)$ for any non-trivial subgroup Q of P. The equivalence of (v) and (ix) is an immediate consequence of 3.5. Finally, the equivalence of (v) and (iii) follows again from Broué's isomorphism $Z(kGb)/(Z(kGb) \cap \ker(\mathrm{Br}_P)) \cong (kZ(P))^E$, which is a symmetric algebra. □

The above Lemmas contain all the required information to prove 3.1.

Proof of Theorem 3.1. Statement (i) is equivalent to the statement in 3.8 (ix). This, in turn, is equivalent to (ii) by 3.5 and 3.7. Statement (i) is also equivalent to the statement in 3.8 (viii) By the formula in [26, 11.9] statement (viii) of 3.8 is equivalent to the statement that $(kC_G(Q)\mathrm{Br}_Q(b))^{N_G(Q)}_Q = \{0\}$ for every non-trivial proper subgroup Q of P. Since different blocks of $kC_G(Q)$ appearing in a decomposition of $Br_Q(b)$ are orthogonal this in turn is equivalent to statement (iii) of Theorem 3.1. □

As a consequence of 3.1 (iii) we get the following necessary condition for a block to have symmetric stable center:

Corollary 3.9. *Let G be a finite group, let b be a block of kG and let P be a defect group of b. Suppose that $\bar{Z}(kGb)$ is symmetric. Let (Q, f) be a $b-$Brauer pair such* *that Q is conjugate to a non-trivial proper subgroup of P. Set $\bar{C} = C_G(Q)/Z(Q) \cong QC_G(Q)/Q$ and $\bar{N} = N_G(Q,f)/Q$. Denote by $\bar{f}$ the image of f in $k\bar{C}$. We have*

$$\mathrm{Tr}_{\bar{C}}^{\bar{N}}(Z^{pr}(k\bar{C}\bar{f})) = \{0\} \ .$$

Proof. We have $(kC_G(Q)f)_Q^{N_G(Q,f)} = \mathrm{Tr}_{QC_G(Q)}^{N_G(Q,f)}((kC_G(Q)f)_Q^{QC_G(Q)})$. The image of $(kC_G(Q)f)_Q^{QC_G(Q)}$ in $k\bar{C}\bar{f}$ is precisely $Z^{pr}(k\bar{C}\bar{f})$. Thus 3.9 follows from 3.1(iii). □

4. Proofs of the Theorems 1.1 and 1.2

Proof of Theorem 1.2. Let us use the notation of the Theorem. In addition, let (Q,f) be a $b-$Brauer pair such that Q is conjugate to a non-trivial proper subgroup of P. The hypothesis of the theorem implies that the block f of $kC_G(Q)$ is nilpotent with defect group P. In particular, the center $Z(kC_G(Q)f)$ of $kC_G(Q)f$ is isomorphic to kP ([8]). On the other hand, since P is abelian, and the inertial quotient of $kC_G(Q)f$ is 1, Broué's isomorphism applied to $Z(kC_G(Q)f)$ gives that $Z(kC_G(Q)f)/(Z(kC_G(Q)f)\cap\ker(\mathrm{Br}_P)) \cong kP$. Hence, $Z(kC_G(Q)f)\cap\ker(\mathrm{Br}_P) = \{0\}$. Since Q is a proper subgroup of P, $(kC_G(Q)f)_Q^{C_G(Q)} \subseteq Z(kC_G(Q)f)\cap\ker(\mathrm{Br}_P) = \{0\}$. In particular, $(kC_G(Q)f)_Q^{N_G(Q,f)} = \{0\}$. The result now follows from statement (iii) of 3.1. □

Remark 4.1. The present proof of Theorem 1.2 follows essentially a suggestion by the Referee; our original proof of 1.2 used a result of Puig [28], stating that under the hypotheses of 1.2, there is a stable equivalence of Morita type between the algebra kGb and the block algebra of the Brauer correspondent of b. Since the stable center is invariant under stable equivalences of Morita type by a result of Broué (cf. [5] or [6, Proposition 5.4]) it suffices therefore to show 1.2 under the assumption, that the defect group P is normal in G. In that case, by results of Külshammer or Puig, the block algebra is known to be Morita equivalent to a twisted group algebra of the form $k_*(P \rtimes \hat{E})$, where $\hat{E}$ is a central k^*-extension of the inertial quotient $E = N_G(P,e)/C_G(P)$. A straightforward computation shows that the stable center of this algebra is isomorphic to the symmetric algebra $(kP)^E$.

Proof of Theorem 1.1. If P is abelian and $N_G(P)/C_G(P)$ acts regularly on P, then $\bar{Z}(kGb)$ is symmetric by 1.2. Suppose conversely that $\bar{Z}(kGb)$ is symmetric. From Theorem 3.1 it follows that for every non-trivial element u of P, the principal block of $kC_G(u)$ has one simple module; hence $C_G(u)$ is a p-nilpotent group. Also, a conjugate of u lies in the center of P. Thus we are done by Proposition 4.2 below. □

Proposition 4.2. *Let G be a finite group and let P be a Sylow$-p-$subgroup of G. Suppose that every non-trivial p-element u of G is conjugate to an element in $Z(P)$ and that $C_G(u)$ is p-nilpotent. Then P is abelian and $N_G(P)/C_G(P)$ acts freely on $P - \{1\}$.*

Proof. We first consider the case that p is odd. We follow the first part of the proof of [2, 9.2]. Since the centralisers of non-identity p elements are p-nilpotent, $(Z_p \times Z_p) \rtimes Sl(2,p)$ is not involved in G. Let P be a Sylow p-subgroup of G. By a theorem of Glauberman in [11], we know that $N_G(Z(J(P)))$, where $J(P)$ denotes the Thompson subgroup of P, controls p-fusion in G. Let x be an element of P . If there exists $g \in G$ such that $y = {}^g x \in Z(P)$, then $y = {}^t x$ for some $t \in N_G(Z(J(P)))$; hence

$x = {}^{t^{-1}}y \in Z(J(P))$. If every p-element of G is conjugate to an element of $Z(P)$,then $P \subset Z(J(P))$, and P is abelian.

Now , we consider the case that $p = 2$. Here we follow a strategy suggested by R. Solomon. Let G be a minimum counter-example to the proposition. Then it is clear that $O_{2'}(G) = 1$.

We first show that G is a simple group. Let N be a maximal normal subgroup of G and suppose, if possible, that $1 \neq N \neq G$. Let P be a Sylow 2-subgroup of G and $Q = P \cap N$, a Sylow 2-subgroup of N. Since $O_{2'}(G) = 1$, we have $Q \neq 1$. On the other hand, N satisfies the hypothesis of the proposition, namely the centraliser of every non-trivial 2-element of N is 2-nilpotent and contains a Sylow 2-subgroup of N. Hence Q is abelian and in particular a proper subgroup of P.

By the Frattini argument, $G = N.N_G(Q)$, and hence $N_G(Q)/N_N(Q) \simeq G/N$ is a simple group. Hence, either $C_G(Q)N_N(Q) = N_G(Q)$ or $C_G(Q) \subset N_N(Q) \subset N$. Let $x \in P - Q$. By the hypothesis, some conjugate of x centralises P,and in particular, Q; hence $C_G(Q)$ is not contained in N. Thus, $C_G(Q)N_N(Q) = N_G(Q)$. Since, $O_{2'}(C_G(Q))$ is a normal subgroup of $N_G(Q)$, either $O_{2'}(C_G(Q))N_N(Q) = N_G(Q)$ or $O_{2'}(C_G(Q)) \subset N_N(Q)$. Suppose, if possible that $O_{2'}(C_G(Q))N_N(Q) = N_G(Q)$. Then the Sylow 2 subgroups of $N_N(Q)$ have the same order as the Sylow 2-subgroups of $N_G(Q)$, hence $Q = P$, and P is abelian, a contradiction. So we may assume that $O_{2'}(C_G(Q)) \subset N_N(Q) \subset N$. The fact that $C_G(Q)$ is 2-nilpotent, now implies that $N_G(Q)/N_N(Q)$ is a 2-group, and since $N_G(Q)/N_N(Q)$ is simple $N_G(Q)/N_N(Q)$ has order 2. Thus, N has index 2 in G. Let $x \in P - Q$, and suppose that $gxg^{-1} \in Z(P)$. Then, $P =< gxg^{-1}, Q >$ is abelian, a contradiction. Thus G is simple.

In [12], Gorenstein has classified all simple groups the centralisers of whose involutions are 2-nilpotent. Explicitly, G must be isomorphic to one of the groups $PSL(2, 2^n)$ (with $n \geq 4$), $PSL(2, q)$ (with $q > 3$, q odd), $Sz(2^n)$ (with $n \geq 3$), A_7 or $PSL(3, 4)$.

The Sylow 2-subgroups of $PSL(2, 2^n)$ are abelian, while for the other groups, it can be checked that either the Sylow 2-subgroups are abelian or the exponent of a Sylow 2-subgroup is strictly greater than the exponent of the center of a Sylow 2-subgroup. In the latter case, it is impossible for every element of a Sylow 2 group to be conjugate to an element of the center of the Sylow 2-subgroup.Hence, P is abelian.

Let $e \in E$ and $u \in P - \{1\}$ such that $e \in C_G(u)$. Since $P \subseteq C_G(u)$ and $C_G(u)$ is $p-$nilpotent, we have $e = 1$. Thus E acts regularly on P, which concludes the proof. □

Remark 4.3. Ron Solomon has pointed out the following consequence of an unpublished result of David Goldschmidt: Suppose that G is a finite group and that P is a Sylow 2-subgroup of G such that $Z(P)$ is not elementary abelian, and such that $C_G(x)$ is 2-nilpotent for every non-trivial element of x. Then $N_G(J(P))$ controls p-fusion in G, where $J(P)$ is the Thompson subgroup of P. From this result it is possible to deduce the proof of Proposition 4.2 in the case that $p = 2$, without invoking Gorensteins classification of groups whose involutions have 2-nilpotent centralisers. Indeed, with the hypothesis of Proposition 4.2, and in the case that $p = 2$, if $Z(P)$ is elementary abelian, then P has exponent 2 and hence is elementary abelian. If, on the other other hand, $Z(P)$ has exponent greater than 2 then by the result quoted above, $N_G(J(P))$, and hence $N_G(Z(J(P)))$ controls p-fusion in G. Since $Z(P) \subset Z(J(P))$, we may deduce immediately that P is abelian.

Remark 4.4. We do not know at this stage whether the converse of Theorem 1.2

holds in general; a first step in this direction would be to settle the case where P is normal in G. As mentioned before, in this situation there is a central $k^\times-$ extension $\hat{E}$ of $E = N_G(P,e)/PC_G(P)$ such that the block algebra kGb is Morita equivalent to the twisted semi-direct product $k_*(P \rtimes \hat{E})$ (see [26, 14.6]). Even in this case we are unable to give a complete answer, but we have the following partial result, which holds without any assumption on the size of k.

Proposition 4.5. *Let P be a non-trivial finite $p-$group, let E be a $p'-$subgroup of* $\mathrm{Aut}(P)$ *and let $\hat{E}$ be a central $k^\times-$extension of E such that $k_*(P\rtimes\hat{E})$ has a symmetric stable center. Then P is abelian, and if $\hat{E}$ is the split extension of E by $k^\times$ or if E is abelian, then E acts freely on $P - \{1\}$.*

Proof. The algebra $k_*(P \rtimes \hat{E})$ has a split local center and its semi-simple quotient is isomorphic to the separable algebra $k_*\hat{E}$, and therefore has a simple module of dimensiom prime to p. Thus, by 2.4,

$$Z^{pr}(k_*(P \rtimes \hat{E})) = Z(k_*(P \rtimes \hat{E})) \cap [k_*(P \rtimes \hat{E}), k_*(P \rtimes \hat{E})] \ .$$

Let $u \in P$ and $e \in E$. Denote by $\hat{e}$ any inverse image of e in $\hat{E}$. We have $\mathrm{Tr}_1^{P\rtimes E}(u) = 0$ and for a non-identity element e of E, $\mathrm{Tr}_1^{P\rtimes E}(u\hat{e})$ is a linear combination of elements of the form $v\hat{f}$, where $v \in P$ and $f \in E-\{1\}$. In particular, $kP\cap Z^{pr}(k_*(P\rtimes\hat{E})) = 0$.

We show now that P is abelian. If not, let $u \in P - Z(P)$, and set $Q = C_P(u)$. Then $1 \subset Q \subset P$. Consider the element $z = \mathrm{Tr}_Q^{P\rtimes E}(u)$. Since P is normal in the group $P \rtimes \hat{E}$, and since E is a $p'-$ group, it follows that z is a non zero element of $Z(k_*(P\rtimes\hat{E}))$. Also, $z \in [k_*(P\rtimes\hat{E}), k_*(P\rtimes\hat{E})]$ since Q is proper in P (cf. 2.5). Thus $z \in Z^{pr}(k_*(P \rtimes \hat{E}))$, which is impossible as $z \in kP$. This shows that P is abelian.

Note that for $x \in P$ and $\hat{f} \in \hat{E}$, $\mathrm{Tr}_1^P(x\hat{f}) = \sum_{y\in P} x[f,y]\hat{f}$ where $[f,y]$ denotes the commutator of f and y. Thus $\mathrm{Tr}_1^P(x\hat{f})$ and consequently $\mathrm{Tr}_1^{P\rtimes E}(x\hat{f})$ is zero unless $C_P(f) = 1$. In other words, the elements of $\mathrm{Tr}_1^{P\rtimes E}(k_*(P \rtimes \hat{E}))$ consist of linear combinations of elements of the type $x\hat{f}$, where $x \in P$ and $f \in E$ such that $C_P(f) = 1$.

Let $1 \neq e \in E$ and suppose that $Q = C_P(e)$ is non-trivial. Let $1 \neq v \in Q$ and let $\hat{e}$ be any lift of e in $\hat{E}$. Consider the element $a = \mathrm{Tr}_Q^{P\rtimes E}(v\hat{e})$. Since Q is a proper subgroup of E, by 2.5 a is in the commutator subspace of $k_*(P \rtimes \hat{E})$, and hence, by Lemma 2.4, $a \in \mathrm{Tr}_1^{P\rtimes E}(k_*(P \rtimes \hat{E}))$. So $a = 0$, and hence, $\mathrm{Tr}_Q^{P\rtimes C_E(e)}(v\hat{e}) = 0$.

Now, $\mathrm{Tr}_Q^P(v\hat{e}) = \sum_{x\in[P/Q]} [x,e]v\hat{e}$. The element $u = \sum_{x\in[P/Q]} [x,e]$ is $C_E(e)$-invariant. and thus we have

$$0 = \mathrm{Tr}_Q^{P\rtimes C_E(e)}(v\hat{e}) = \mathrm{Tr}_1^{C_E(e)}(uv\hat{e}) = u\mathrm{Tr}_1^{C_E(e)}(v\hat{e}).$$

Since Q is $C_E(e)$ invariant, and $P = Q \times [P, < e >]$, the above equations imply that $\mathrm{Tr}_1^{C_E(e)}(v\hat{e}) = 0$. Consequently, $\mathrm{Tr}_1^{C_E(e)\cap C_E(v)}(\hat{e}) = 0$ for every pair of commuting non-identity elements $e \in E$ and $v \in P$.

If $\hat{E}$ is the split extension of E by $k^\times$, that is, $k_*(P \rtimes \hat{E}) \cong k(P \rtimes E)$, the above condition implies immediately that E acts freely on $P - \{1\}$, since for all $e \in E$, and $v \in P$, $\mathrm{Tr}_1^{C_E(e)\cap C_E(v)}(e) = |C_E(e) \cap C_E(v)|e$ which is clearly non-zero.

Suppose finally that E is abelian and suppose, if possible, that E does not act freely on $P - \{1\}$. We are going to show that there exists a non-identity element v of P such that $C_E(v)$ is cyclic. Write

$$\Omega_1(P) = Q_1 \times \cdots \times Q_r$$

such that for $1 \leq i \leq r$, Q_i is E-invariant and E acts indecomposably on Q_i. Since E is abelian, $E/C_E(Q_i)$ acts freely on $Q_i - \{1\}$ for $1 \leq i \leq r$; in particular, by [9, Theorem 5.3.2], $E/C_E(Q_i)$ is cyclic.

Let s, $1 \leq s \leq r$, be the least integer such that $\cap_{i=1}^{i=s} C_E(Q_i) = 1$. Such an s exists since by [13, Theorem 5.2.4], $\cap_{i=1}^{i=r} C_E(Q_i) = 1$. If $s = 1$, then E is cyclic. If $s > 1$, then $\cap_{i=1}^{i=s-1} C_E(Q_i)$ is a non-trivial cyclic subgroup of E. This is because, by choice of s, the canonical surjection of E onto $E/C_E(Q_s)$ restricts to an injection on $\cap_{i=1}^{i=s-1} C_E(Q_i)$. For $1 \leq i \leq s-1$ choose a non-identity element v_i of Q_i, and let $v = v_1 \cdots v_{s-1}$. Since $C_E(v_i) = C_E(Q_i)$ and each Q_i is E-invariant, we have that $C_E(v) = \cap_{i=1}^{i=s-1} C_E(Q_i)$ is non-trivial and cyclic. Let e be a generator of $C_E(v)$. Then $C_E(e) \cap C_E(v) = C_E(v)$ and since $\hat{e}$ commutes to all of its powers, it is fixed under the action of $C_E(v)$; hence $\mathrm{Tr}_1^{C_E(v)}(\hat{e}) = |C_E(v)|\hat{e}$ is non zero. This contradiction completes the proof of the Proposition. □

5 On the Hochschild cohomology of blocks with abelian defect

Let A be a finite-dimensional symmetric $k-$algebra. We define the Tate analogue of the Hochschild cohomology of A by setting

$$\hat{HH}(A) = \overline{\mathrm{Hom}}_{A \underset{k}{\otimes} A^0}(\Omega^n_{A \underset{k}{\otimes} A^0}(A), A)$$

for any integer n. Here $\overline{\mathrm{Hom}}_{A \underset{k}{\otimes} A^0}$ denotes the homomorphism space in the stable category of $A \underset{k}{\otimes} A^0-$modules, and $\Omega_{A \underset{k}{\otimes} A^0}$ is the Heller operator, mapping an $A \underset{k}{\otimes} A^0$-module to the kernel of a projective cover of that module. This makes sense, because A and thus $A \underset{k}{\otimes} A^0$ are symmetric, and therefore the Heller operator $\Omega_{A \underset{k}{\otimes} A^0}$ induces an equivalence on the stable module category of $A \underset{k}{\otimes} A^0$.

It is well-known, that for positive n, we have canonical isomorphisms $\hat{HH}^n(A) \cong HH^n(A) = \mathrm{Ext}^n_{A \underset{k}{\otimes} A^0}(A, A)$. In degree zero, we have canonical isomorphisms $HH^0(A) \cong Z(A)$ and $\hat{HH}^0(A) \cong \bar{Z}(A)$. While the Hochschild cohomology is zero in negative degrees, its Tate analogue fulfills the analogue of the Tate duality; that is, we have isomorphisms $\hat{HH}^{-n}(A) \cong (\hat{HH}^{n-1}(A))^*$ for any integer n.

For the proof of 1.4 we need the following (both 5.1 and 5.2 below hold over an arbitrary field k of characteristic p).

Lemma 5.1. *Let P be a non-trivial finite $p-$group, and let α be an automorphism on P having no non-trivial fixed point. Then* $\hat{\mathrm{Ext}}^*_{k(P \times P^0)}(kP, (kP)_\alpha) = \{0\}$.

Proof. Set $\Delta P = \{(u, u^{-1})\}_{u \in P} \subset P \times P^0$. Then $kP \cong \mathrm{Ind}_{\Delta P}^{P \times P^0}(k)$ as $k(P \times P^0)-$ module. By the Eckmann-Shapiro lemma, we have therefore $\hat{\mathrm{Ext}}^*_{k(P \times P^0)}(kP, (kP)_\alpha) \cong \hat{\mathrm{Ext}}^*_{k\Delta P}(k, (kP)_\alpha) = \hat{H}^*(\Delta P, (kP)_\alpha)$. Since α has no non-trivial fixed point, if u runs over all elements of P then so does $u\alpha(u^{-1})$. Thus ΔP acts regularly on the basis P of $(kP)_\alpha$. This shows that $(kP)_\alpha$ is projective as $k\Delta P-$module, and hence its Tate cohomology vanishes, which implies the result. □

Proposition 5.2. *Let P be a non-trivial finite abelian $p-$group, let E be a $p'-$subgroup of* $\mathrm{Aut}(P)$ *and let $\hat{E}$ be a central $k^\times-$ extension of E. Assume that E acts regularly on P. Then*

$$\hat{HH}^*(k_*(P \rtimes \hat{E})) \cong (kP \underset{k}{\otimes} \hat{H}^*(P,k))^E \ .$$

Proof. Set $\Delta E = \{(e, e^{-1})\}_{e\in E}$. The algebra $k((P \times P^0) \rtimes \Delta E))$ can be identified to a subalgebra of $k_*(P \rtimes \hat{E}) \underset{k}{\otimes} (k_*(P \rtimes \hat{E}))^0$ by identifying $P \times P^0$ to its obvious image and by sending (e, e^{-1}) to $\hat{e} \otimes \hat{e}^{-1}$, where $e \in E$ and $\hat{e} \in \hat{E}$ is any element which lifts e. There is a canonical isomorphism of $k_*(P \rtimes \hat{E}) - k_*(P \rtimes \hat{E})-$bimodules

$$k_*(P \rtimes \hat{E}) \cong k_*(P \rtimes \hat{E}) \underset{k}{\otimes} (k_*(P \rtimes \hat{E}))^0 \underset{k((P\times P^0)\rtimes \Delta E)}{\otimes} kP \ ,$$

and thus, by Eckmann-Shapiro, again we get

$$\begin{aligned}\hat{HH}^*(k_*(P \rtimes \hat{E})) &\cong \hat{\mathrm{Ext}}^*_{k((P\times P^0)\rtimes\Delta E)}(kP, k_*(P \rtimes \hat{E})) \\ &\cong (\hat{\mathrm{Ext}}^*_{k(P\times P^0)}(kP, k_*(P \rtimes \hat{E})))^E,\end{aligned}$$

where the last isomorphism comes from the fact that E is a $p'-$group. Now as a $kP - kP-$bimodule, we have $k_*(P \rtimes \hat{E}) \cong k(P \rtimes E) = \underset{e\in E}{\oplus} kPe$. Since non trivial elements of E act without fixed points on $P - \{1\}$, it follows from 5.1 that

$$\hat{HH}^*(k_*(P \rtimes \hat{E})) \cong (\hat{\mathrm{Ext}}^*_{k(P\times P^0)}(kP, kP))^E = (\hat{HH}^*(kP))^E \ .$$

By a result of T. Holm [15], we have an isomorphism $HH^*(kP) \cong kP \underset{k}{\otimes} H^*(P, k)$. It is easy to see that this isomorphism extends to an isomorphism $\hat{HH}^*(kP) \cong kP \underset{k}{\otimes} \hat{H}^*(P, k)$, which is $E-$invariant (see e.g. [21] for an explicit way of describing this isomorphism). □

Proof of Theorem 1.4. By [28], there is a stable equivalence of Morita type between the block algebra kGb and a twisted group algebra $k_*(P \rtimes \hat{E})$ for some central $k^\times-$extension $\hat{E}$ of E. Then $\hat{HH}^*(kGb) \cong \hat{HH}^*(k_*(P \rtimes \hat{E}))$ by the arguments in [20, 2.13], and now 1.4 follows from 5.2. □

6 Further remarks

Block algebras are symmetric algebras, but not every symmetric algebra is Morita equivalent to a block algebra:

Proposition 6.1. *Let A be a split symmetric k-algebra which is Morita equivalent to a block algebra with non-trivial defect groups of some finite group. Then the ideal $Z^{pr}(A)$ is strictly smaller than $Z(A) \cap \mathrm{soc}(A)$.*

Proof. By 3.5 and 2.1(v) we have $\dim_k(Z^{pr}(A)) \leq l(A) - 1 < \dim_k(Z(A) \cap \mathrm{soc}(A)) = l(A)$. Alternatively, we may assume that A is a block source algebra, in which case we have $Z^{pr}(A) \subseteq [A, A]$ by 2.5, but $Z(A) \cap \mathrm{soc}(A) \not\subseteq [A, A]$ by 2.4. □

As pointed out by the Referee, Proposition 6.1 follows also from classical results: $\dim_k(Z^{pr}(kGb))$ is the multiplicity of 1 as elementary divisor of the cartan matrix C of A, while $\dim_k(Z(A) \cap \mathrm{soc}(A))$ is the size of C. Since C has an elementary divisor equal to order of a defect group of the considered block, these two numbers cannot be equal.

Corollary 6.2. *Let G be a finite group and let b be a block of kG having non-trivial defect groups. Assume that kGb has exactly one isomorphism class of simple modules. Then $Z^{pr}(kGb) = \{0\}$.*

The previous statement is not true for arbitrary symmetric algebras with one isomorphism class of simple modules: if A is a split local symmetric algebra of dimension prime to p, then $Z^{pr}(A) = Z(A) \cap \mathrm{soc}(A) \cong k$.

Theorem 3.1 can be viewed as a generalisation of the following Theorem, due to L. Puig and A. Watanabe:

Theorem 6.3. *([30, Theorem 3]) Let G be a finite group and let b be a block with an abelian defect group P. If $l(f) = 1$ for any $b-$Brauer element (u, f), then the block b is nilpotent.*

Proof. We may assume that P is non-trivial. If $l(f) = 1$ for any $b-$Brauer element then in particular $l(b) = 1$. Then the stable Grothendieck group of the block algebra kGb is cyclic and $Z^{pr}(kGb) = \{0\}$ by 3.5. Thus, by 3.1, $\bar{Z}(kGb) \cong Z(kGb)$ is symmetric, and now it follows from the results in [23], that b is a nilpotent block. □

By a Theorem of Puig [29, Theorem 8.2], if there is a stable equivalence of Morita type between two block algebras over a complete discrete valuation ring of characteristic zero, and if one of the two blocks is nilpotent, so is the other. Under the assumption that the nilpotent block has abelian defect groups, we show that this result holds more generally over the field k.

Theorem 6.4. *Let G, H be finite groups, let b be a block of kG, let c be a block of kH, and let P, Q be defect groups of b, c, respectively. Suppose that there is a stable equivalence of Morita type between the algebras kGb and kHc. If P is abelian and b is nilpotent, then $Q \cong P$ and c is nilpotent; in particular, kGb and kHc are Morita equivalent to kP.*

Proof. We may assume that P and Q are non-trivial. By [8], if P is abelian and b is nilpotent, then kGb is Morita equivalent to kP. In particular, $\bar{Z}(kGb) \cong Z(kGb) \cong kP$ is symmetric. Thus $\bar{Z}(kHc) \cong kP$ is symmetric. Therefore, by 3.8, $kP \cong (kZ(Q))^E$, where E is the inertial quotient of c. Since kGb and kHc are stably equivalent, their Cartan matrices have the same elementary divisors, and hence $|P| = |Q|$. Thus $\dim_k(kZ(Q)^E) = |Q|$, which forces $Q = Z(Q)$ and $E = 1$. Thus c is nilpotent with the abelian defect group Q such that $kP \cong kQ$. For abelian $p-$groups such an isomorphism implies an isomorphism of the groups $P \cong Q$ by [9]. □

Acknowledgements. The authors would like to thank Ron Solomon for his help regarding the proof of Proposition 4.3, Christin Borge for pointing out the reference [9], and the Referee for a number of useful comments, leading in particular to simplifications of the proofs of 1.2 and 2.3.

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