VARIETIES IN BLOCK THEORY

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1 Introduction

The content of this paper is motivated by the question, whether the cohomological variety $V_G(U)$, as defined by J.F. Carlson in [9], of a bounded complex U of finitely generated $kG-$modules belonging to a block b of kG is an invariant of this block b, where G is a finite group and k an algebraically closed field of characteristic p.

We show that this is true if b is the principal block of kG, while this is not true in general - mainly because the definition of $V_G(U)$ involves the cohomology ring $H^*(G, k)$ of G, which is an invariant of the principal block of kG. However, it is "not far" from being true in general: we define in section 4 a variety $V_{G,b}(U)$ which comes along with a a finite surjective morphism

1.1.

$$V_{G,b}(U) \longrightarrow V_G(U)$$

and which is not only an invariant of U as complex of $kGb-$modules, but which is even invariant under splendid stable and derived equivalences (cf. [27], [18], [19]). The map in 1.1 is shown to be an isomorphism, if b is the principal block of kG.

The major ingredients for this are the following. In [20] we define for any block b of kG with defect pointed group P_γ a *cohomology ring* $H(G, b, P_\gamma)$ *of* b and show that there is an embedding into the Hochschild cohomology ring of the block algebra kGb,

1.2.

$$H^*(G, b, P_\gamma) \longrightarrow HH^*(kGb),$$

which should be thought of as a generalization of the well-known embedding $H^*(G, k) \longrightarrow HH^*(kG)$ induced by the "diagonal induction functor" $\mathrm{Ind}_{\Delta G}^{G\times G}$. Next, the functor $- \underset{kGb}{\otimes} U$ induces an algebra homomorphism $HH^*(kGb) \longrightarrow Ext^*_{kGb}(U, U)$; thus, by composing this with the homomorphism in 1.2 we obtain an algebra homomorphism

1.3.

$$H^*(G, b, P_\gamma) \longrightarrow Ext^*_{kGb}(U, U).$$

We denote by $I^*_{G,b}(U)$ the kernel of the homomorphism 1.3 and define $V_{G,b}(U)$ to be the maximal ideal spectrum of $H^*(G, b, P_\gamma)/I^*_{G,b}(U)$. The maximal ideal spectrum $V_G(b)$ of $H^*(G, b, P_\gamma)$ is called the *variety of the block* b. These definitions are, up to unique isomorphism, independent of the choice of P_γ, since all defect pointed groups of the block b are $G-$conjugate and $N_G(P_\gamma)$ acts trivially on $H^*(G, b, P_\gamma)$. We observe then that the restriction from G to the defect group P of b induces an algebra homomorphism

1.4.

$$H^*(G, k) \longrightarrow H^*(G, b, P_\gamma)$$

which is an isomorphism if b is the principal block of kG and which induces a finite surjective morphism as claimed in 1.1.

In section 5 we show that if H is another finite group and c a block of kH having also P as defect group, and if X is a splendid tilting complex of $kGb-kHc-$bimodules (or a splendid stable equivalence of Morita type), then the functor $X \underset{kHc}{\otimes} -$ induces an isomorphism of varieties

1.5.
$$V_{G,b}(X \underset{kHc}{\otimes} V) \cong V_{H,c}(V)$$

for any bounded complex V of finitely generated $kHc-$modules. The sections 2 and 3 contain the required material on transfer maps in the Hochschild cohomology of symmetric algebras and the theory of blocks of finite groups, respectively. Finally, section 6 is an attempt to generalize the notion of the *nucleus of* G introduced in [5] to arbitrary blocks of finite groups.

Notation. All algebras and rings are associative with unit element, all modules are finitely generated unitary, and, if not stated otherwise, left modules. If A, B are algebras over a commutative ring R, by an $A-B-$bimodule M we mean a bimodule whose left and right $R-$module structures coincide; that is, we may consider M as $A \underset{R}{\otimes} B^0-$module, where B^0 is the algebra obtained by endowing B with the opposite product. The $R-$dual $M^* = Hom_R(M, R)$ becomes then a $B-A-$bimodule through $(b.m^*.a)(m) = m^*(amb)$ for any $a \in A$, $b \in B$, $m \in M$ and $m^* \in M^*$. If M is an $A-A-$bimodule, we denote by M^A the subspace of $A-$invariant elements in M; that is, $M^A = \{m \in M | am = ma \quad \text{for any} \quad a \in A\}$. For a finite group G we consider any $RG - RG-$bimodule N as $R(G \times G)-$module with $(x, y) \in G \times G$ acting on $n \in N$ as xny^{-1} (and vice versa).

Remember that an $R-$algebra A is *symmetric* if A is finitely generated projective as $R-$module and $A \cong A^*$ as $A-A-$bimodules. The image of 1_A in A^* under such an isomorphism is called a *symmetrizing form on* A. The group algebra RG is symmetric through the isomorphism $RG \cong (RG)^*$ mapping $x \in G$ to the unique linear form on RG sending x^{-1} to 1_R and any other element of G to zero. The symmetrizing form corresponding to this isomorphism maps 1_G to 1_R and any nontrivial element of G to zero; we call this the *canonical symmetrizing form on* RG. See [20,section 6] for a short account on some formal properties of symmetric algebras that we need here.

Our notation and sign conventions when dealing with complexes are as in [20,1.2].

2 Transfer maps for symmetric algebras

We describe here the material on transfer maps in Hochschild cohomology of symmetric algebras, developed in [20], that we need in this paper. See [6], [17] for analogous concepts for Hochschild- and cyclic homology, respectively, and [14] for a transfer in cohomology of Hopf algebras.

In this section, R is a commutative ring with unit element, A, B, C are symmetric $R-$algebras (cf. [20, 6.3]), X, X' are bounded complexes of $A-B-$bimodules whose components are projective as left and right modules, $f : X \longrightarrow X'$ a chain homomorphism and Y is a bounded complex of $B-C-$bimodules whose components are projective as left and right modules. We denote by $\mathcal{P}_X \underset{\mu_X}{\longrightarrow} X$ a projective cover of X; that is, $\mathcal{P}_X$ is a right bounded complex of projective $A-B-$bimodules and μ_X is a quasi-isomorphism (that is, μ_X is a chain homomorphism inducing an isomorphism on homology).

2.1 The functors $X \underset{B}{\otimes} -$ and $X^* \underset{A}{\otimes} -$ between the categories $C(A)$ and $C(B)$ of complexes of $A-$modules and $B-$modules, respectively, are adjoint to each other (cf.

[20, section 6]). More precisely, any choice of symetrizing forms s on A and t on B (cf. [20, 6.3]) gives rise to natural isomorphisms of bifunctors

$$Hom_{C(A)}(X \underset{B}{\otimes} -, -) \cong Hom_{C(B)}(-, X^* \underset{A}{\otimes} -) \text{ and}$$
$$Hom_{C(B)}(X^* \underset{A}{\otimes} -, -) \cong Hom_{C(A)}(-, X \underset{B}{\otimes} -),$$

thus determine chain homomorphisms of complexes of bimodules

$$\epsilon_X : B \longrightarrow X^* \underset{A}{\otimes} X, \quad \eta_X : X \underset{B}{\otimes} X^* \longrightarrow A,$$
$$\epsilon_{X^*} : A \longrightarrow X \underset{B}{\otimes} X^*, \quad \eta_{X^*} : X^* \underset{A}{\otimes} X \longrightarrow B$$

representing the units and counits of this adjunction. Since for any projective resolution $\mathcal{P}_A$ of A as $A-A-$bimodule the total complex $X^* \underset{A}{\otimes} \mathcal{P}_A \underset{A}{\otimes} X$ is a projective resolution of $X^* \underset{A}{\otimes} X$, the above maps lift uniquely up to homotopy to chain homomorphisms, still denoted by the same letters,

$$\epsilon_X : \mathcal{P}_B \longrightarrow X^* \underset{A}{\otimes} \mathcal{P}_A \underset{A}{\otimes} X, \quad \eta_X : X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* \longrightarrow \mathcal{P}_A,$$

$$\epsilon_{X^*} : \mathcal{P}_A \longrightarrow X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^*, \quad \eta_{X^*} : X^* \underset{A}{\otimes} \mathcal{P}_A \underset{A}{\otimes} X \longrightarrow \mathcal{P}_B.$$

2.2 The transfer map associated with X (with respect to the choice of the symmetrizing forms s, t) is the graded $R-$linear map

$$t_X : HH^*(B) \longrightarrow HH^*(A)$$

mapping, for any nonnegative integer n, the homotopy class of a chain map $\zeta : \mathcal{P}_B \longrightarrow \mathcal{P}_B[n]$ to the homotopy class of the composition of chain maps

$$\mathcal{P}_A \xrightarrow{\epsilon_{X^*}} X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} X^* \xrightarrow{Id_X \otimes \zeta \otimes Id_{X^*}} X \underset{B}{\otimes} \mathcal{P}_B[n] \underset{B}{\otimes} X^* \xrightarrow{\eta_X[n]} \mathcal{P}_A[n]$$

The map t_X does depend on the choice of the symmetrizing forms s and t since the adjunction maps in 2.1 do depend on this choice, but one can exactly tell in which way t_X depends on this choice (see [20, 2.10]).

2.3 We recall from [20, 2.11 and 2.12] the following basic properties of transfer maps:

(i) $t_{X \oplus X'} = t_X + t_{X'}$.

(ii) $t_{X \underset{B}{\otimes} Y} = t_X \circ t_Y$.

(iii) $t_X = \sum_{n \in \mathbb{Z}} (-1)^n t_{X_n}$.

(iv) $t_{X[n]} = (-1)^n t_X$.

(v) $t_{C(f)} = t_{X'} - t_X$.

(vi) If X is acyclic then $t_X = 0$.

(vii) If f is a quasi-isomorphism then $t_X = t_{X'}$.

2.4 In order to study the behaviour of t_X with respect to the multiplicative structure in the Hochschild cohomology, we introduce the notion of stable elements. An elements $[\zeta] \in HH^*(A)$ is called $X-$*stable* ([20, 3.1]) if there is $[\tau] \in HH^*(B)$ such that for any nonegative integer n, the following diagram is homotopy commutative:

$$\begin{array}{ccc} \mathcal{P}_A \underset{A}{\otimes} X & \xrightarrow{\simeq} & X \underset{B}{\otimes} \mathcal{P}_B \\ {\scriptstyle \zeta_n \otimes Id_X}\downarrow & & \downarrow{\scriptstyle Id_X \otimes \tau_n} \\ \mathcal{P}_A[n] \underset{A}{\otimes} X & \xrightarrow[\simeq]{} & X \underset{B}{\otimes} \mathcal{P}_B[n] \end{array}$$

where ζ_n and τ_n represent the degree n component of $[\zeta]$ and $[\tau]$, respectively, and where the horizontal maps are the homotopy equivalences lifting the natural isomorphism of complexes $A \underset{A}{\otimes} X \cong X \cong X \underset{B}{\otimes} B$.

2.5 We denote by $\pi_X \in Z(A)$ the image of 1_A under the composition of $A-A-$bimodule homomorphisms

$$A \xrightarrow{\epsilon_{X^*}} X \underset{B}{\otimes} X^* \xrightarrow{\eta_X} A$$

and call π_X the *relatively* $X-$*projective element in* $Z(A)$; note that π_X depends again on the choice of the symmetrizing forms s and t (cf. [20, 3.2]).

If π_X is invertible in $Z(A)$, we denote by

$$T_X : HH^*(B) \longrightarrow HH^*(A)$$

the graded $R-$linear map defined by $T_X = (\pi_X)^{-1} t_X$ and call T_X the *normalized transfer map associated with* X.

2.6 The set $HH^*_X(A)$ of $X-$stable elements is a graded subalgebra in $HH^*(A)$, and if π_X is invertible, the normalized transfer T_X induces a surjective graded $R-$algebra homomorphism

$$HH^*_{X^*}(B) \longrightarrow HH^*_X(A).$$

Moreover, if both π_X and π_{X^*} are invertible, then T_X and T_{X^*} induce mutually inverse algebra isomorphisms between $HH^*_X(A)$ and $HH^*_{X^*}(B)$ (cf. [20, 3.6]).

2.7 Stable elements in Hochschild cohomology satisfy a "cancellation property" ([20, 3.8]): if π_Y is invertible in $Z(B)$ then $HH^*_{X \underset{B}{\otimes} Y}(A) \subset HH^*_X(A)$ and $T_Y(HH^*_{Y^* \underset{B}{\otimes} X^*}(C)) \subset HH^*_{X^*}(B)$. Also, for any direct summand X' of the complex X we have $HH^*_X(A) \subset HH^*_{X'}(A)$.

2.8 We have the following connection between transfer maps in Hochschild cohomology and ordinary cohomology of finite groups: if G is a finite group and $\Delta G = \{(x,x)\}_{x \in G} \subset G \times G$, the "diagonal induction functor" $\mathrm{Ind}_{\Delta G}^{G \times G}$ maps the trivial $R\Delta G-$module R to the $R(G \times G)-$module $\mathrm{Ind}_{\Delta G}^{G \times G}(R)$, which is, when viewed as $RG-RG-$bimodule, isomorphic to the regular bimodule RG. Thus $\mathrm{Ind}_{\Delta G}^{G \times G}$ maps a projective resolution of R as $R\Delta G-$module to a projective resolution of RG as $RG-RG-$bimodule and whence induces an injective graded $R-$algebra homomorphism (cf. [20,4.5])

$$\delta_G : H^*(G, R) \longrightarrow HH^*(RG).$$

Moreover, by [20, 4.8], if H is any subgroup of G, we have

$$Im(\delta_G) \subset HH^*_{(RG)_H}(RG),$$

where $(RG)_H$ is the regular $RG-RG-$bimodule RG restricted to RH on the right. We consider RG as symmetric $R-$algebra with respect to the canonical symmetrizing form $RG \longrightarrow R$ mapping 1_G to 1_R and any nontrivial element of G to zero. Then the following diagrams are commutative (cf. [20, 4.6, 4.7]):

$$\begin{array}{ccc} H^*(G,R) & \xrightarrow{res^G_H} & H^*(H,R) \\ \delta_G \downarrow & & \downarrow \delta_H \\ HH^*(RG) & \xrightarrow[t_{H(RG)}]{} & HH^*(RH) \end{array} \qquad \begin{array}{ccc} H^*(H,R) & \xrightarrow{t^G_H} & H^*(G,R) \\ \delta_H \downarrow & & \downarrow \delta_G \\ HH^*(RH) & \xrightarrow[t_{(RG)_H}]{} & HH^*(RG) \end{array}$$

where t^G_H denotes the usual transfer map on group cohomolgy.

2.9 For any bounded complex U of $A-$modules and any projective resolution $\mathcal{P}_A \underset{\mu_A}{\longrightarrow} A$ of A as $A-A-$bimodule, the total complex $\mathcal{P}_A \underset{A}{\otimes} U$, together with the chain map $\mu_A \otimes Id_U$, becomes a projective resolution of U. Thus the functor $- \underset{A}{\otimes} U$ induces an algebra homomorphism

$$\alpha_U : HH^*(A) \longrightarrow Ext^*_A(U,U)$$

mapping the homotopy class of a chain map $\zeta : \mathcal{P}_A \longrightarrow \mathcal{P}_A[n]$ to that of $\zeta \otimes Id_U$, where n is a nonnegative integer.

If G is a finite group and U a bounded complex of $RG-$modules, for any complex V of $RG-$modules there is a natural isomorphism of complexes of $RG-$modules

$$\mathrm{Ind}^{G\times G}_{\Delta G}(V) \underset{RG}{\otimes} U \cong V \underset{R}{\otimes} U$$

mapping $((x,y)\otimes v)\otimes u$ to $xv \otimes xy^{-1}u$, where $x,y \in G$, $u \in U$, $v \in V$, and where the complex of $R(G\times G)-$modules $\mathrm{Ind}^{G\times G}_{\Delta G}(V)$ is considered as complex of $RG-RG-$bimodules according to our conventions introduced in section 1; that is, $x \in G$ acts on the left and right of $m \in \mathrm{Ind}^{G\times G}_{\Delta G}(V)$ by $x.m = (x,1)m$ and $m.x = (1,x^{-1})m$, respectively. This isomorphism, applied to a projective resolution $\mathcal{P}_R$ of the trivial $RG-$module R instead of V, implies that the composition of $R-$algebra homomorphisms

$$H^*(G,R) \xrightarrow{\delta_G} HH^*(RG) \xrightarrow{\alpha_U} Ext^*_{RG}(U,U)$$

is equal to the algebra homomorphism

$$\gamma_U : H^*(G,R) \longrightarrow Ext^*_{RG}(U,U)$$

given by the functor $- \underset{R}{\otimes} U$.

3 Quoted results on blocks and their cohomology

We sketch here briefly some basic concepts and results from block theory. Most of the material we present here holds in more general situations, but we restrict this section to what we need in this paper. In particular, since we are mostly interested in providing techniques for dealing with varieties, our ground ring will be a field (of prime characteristic), and we leave it to the reader to check, that all statements, including the results in the sections 4 and 5 below, could be done more generally over a complete discrete valuation ring.

The Brauer homomorphism described in 3.1 below goes back to work of R. Brauer and has since then been generalized to $G-$algebras [8] and modules [13]. The systematical treatment of the $p-$local structure of blocks of finite groups in terms of Brauer pairs and pointed groups starts with work of Alperin-Broué [1], Broué-Puig [8] and Puig [21]. See Thévenaz' book [28] for a detailed exposition on block theory.

Let k be a field of prime characteristic p and G a finite group.

3.1 For any $p-$subgroup P of G the natural projection $kG \longrightarrow kC_G(P)$ mapping $x \in C_G(P)$ to x and $x \in G - C_G(P)$ to zero restricts to a surjective algebra homomorphism

$$Br_P^G : (kG)^P \longrightarrow kC_G(P),$$

called the *Brauer homomorphism of P in G*. Here $(kG)^P$ denotes the subalgebra of $P-$stable elements in kG with respect to the action of P by conjugation. If no confusion is possible we will write Br_P instead of Br_P^G. See [28, section 11] for more details and generalizations of this construction.

Recall from Puig [21] that a *point of P on kG* is a $((kG)^P)^\times-$conjugacy class γ of primitive idempotents in $(kG)^P$; we say that γ is a *local point of P on kG* if $Br_P(\gamma) \neq 0$. By standard theorems on lifting of idempotents, $Br_P(\gamma)$ is then a conjugacy class of primitive idempotents in $kC_G(P)$.

3.2 A *block of kG* is a primitive idempotent b in the center $Z(kG)$ of the group algebra kG. The algebra kGb is then called the *block algebra of the block b*. A *defect group of the block b* is a minimal subgroup P of G such that the map $kGb \underset{kP}{\otimes} kGb \longrightarrow kGb$ induced by multiplication in kGb splits as homomorphism of $kGb-kGb-$bimodules. Equivalently, P is a maximal $p-$subgroup of G such that $Br_P(b) \neq 0$. The defect groups of b form a $G-$conjugacy class of $p-$subgroups of G.

3.3 If P is a defect group of a block b of kG we have $Br_P(b) \neq 0$, and therefore there is a primitive idempotent $i \in (kGb)^P$ such that $Br_P(i) \neq 0$. The $((kG)^P)^\times-$conjugacy class γ of i in $(kG)^P$ is then a local point of P on kG contained in kGb. The pair P_γ is called a *defect pointed group of the block b*. Again, G acts transitively by conjugation on the set of defect pointed groups of b (cf. [21, 1.2]), thus in particular, $N_G(P)$ acts transitively on the set of local points of P on kG contained in kGb. The algebra $ikGi$, considered as *interior $P-$algebra* (cf. [21, 3.1]) via the group homomorphism $P \longrightarrow (ikGi)^\times$ mapping $u \in P$ to ui is called a *source algebra of the block b* (cf. [21, 3.2]).

By the preceding remarks, up to automorphisms of P induced by $N_G(P)$ all source algebras of b are isomorphic as interior $P-$algebras. The block algebra kGb and its source algebra $ikGi$ are Morita equivalent through the $kGb - ikGi$–bimodule kGi and the $ikGi - kGb$–bimodule ikG (cf. [21, 3.5]). Moreover, the p–local structure of the block b (in terms of Brauer pairs or local pointed groups) is in fact an invariant of the source algebra $ikGi$ (cf. [22]).

3.4 Let b be a block of G and P_γ a defect pointed group of b. Let $i \in \gamma$; that is, i is a primitive idempotent in $(kGb)^P$ such that $Br_P(i) \neq 0$. Then $Br_P(i)$ is a primitive idempotent in $kC_G(P)$ and thus there is a unique block e_P of $kC_G(P)$ such that $Br_P(i)e_P = Br_P(i)$. If Q is a subgroup of P, then in general $Br_Q(i)$ need no longer be primitive in $kC_G(Q)$, but we still have the following remarkable uniqueness property, due to Broué and Puig [8, 1.8]: for any subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $Br_Q(i)e_Q = Br_Q(i)$. Then $Z(Q)$ is contained in any defect group of e_Q, and we say that the Brauer pair (Q, e_Q) is *self-centralizing*, if $Z(Q)$ is a defect group of e_Q (cf. [28, section 41]).

For any two subgroups Q, R of P we denote by $\widetilde{Hom}(Q, R)$ the set of equivalence classes of group homomorphisms from Q to R modulo inner automorphisms of R and for any group homomrphism $\varphi : Q \longrightarrow R$ we denote by $\tilde{\varphi}$ its image in $\widetilde{Hom}(Q, R)$. We denote by $E_G((Q, e_Q), (R, e_R))$ the image in $\widetilde{Hom}(Q, R)$ of all group homomorphisms $\varphi : Q \longrightarrow R$ for which there is an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ and $xe_Qx^{-1} = e_{xQx^{-1}}$. In particular, $E_G((Q, e_Q), (Q, e_Q))$ is the image of $N_G(Q, e_Q)$ in the outer automorphism group of Q, whence isomorphic to $N_G(Q, e_Q)/QC_G(Q)$.

3.5 Let b be a block of kG and P_γ be a defect pointed group of b. Let $i \in \gamma$.

The *cohomology ring of the block* b (with respect to the defect pointed group P_γ) is the subring

$$H^*(G, b, P_\gamma)$$

of $H^*(P, k)$ of “stable elements with respect to the p–local structure of b ”; that is, $H^*(G, b, P_\gamma)$ consists of all $[\zeta] \in H^*(P, k)$ satisfying $res_\varphi([\zeta]) = res^P_Q([\zeta])$ for any subgroup Q in P and any group homomorphism $\varphi : Q \longrightarrow P$ such that $\tilde{\varphi} \in E_G((Q, e_Q), (P, e_P))$.

Any $\tilde{\varphi} \in E_G((Q, e_Q), (P, e_P))$ is, by Alperin's fusion lemma (see e.g. [28, (48.3)]), a composition of $\tilde{\psi} \in E_G((R, e_R), (R, e_R))$ with R running over the set of subgroups of P such that (R, e_R) is self-centralizing.

Thus $H^*(G, b, P_\gamma)$ is equal to the subring consisting of all $[\zeta] \in H^*(P, k)$ satisfying ${}^x res^P_R([\zeta]) = res^P_R([\zeta])$ for all $x \in N_G(R, e_R)$, where R runs over the set of subgroups of P such that (R, e_R) is self-centralizing.

This latter description, together with Puig's results in [22] on $ikGi$–fusion shows that $H^*(G, b, P_\gamma)$ is an invariant of the source algebra $ikGi$ of the block b. Also, the isomorphism class of the graded k–algebra $H^*(G, b, P_\gamma)$ does not depend on the choice of P_γ since all defect pointed groups of b are G–conjugate.

If b is the principal block of G then P is a Sylow-p–subgroup of G, for any subgroup Q of P the block e_Q is the principal block of $kC_G(Q)$ and $N_G(Q, e_Q) = N_G(Q)$; thus restriction from G to P induces an isomorphism $H^*(G, k) \cong H^*(G, b, P_\gamma)$ by the characterization of $H^*(G, k)$ in terms of stable elements in $H^*(P, k)$.

3.6 Blocks with “trivial p–local structure” are the so-called *nilpotent blocks*, introduced by Broué and Puig in [7]. With the notation of 3.5, the block b is *nilpotent* if P “controls fusion”; that is, if for any $\tilde{\varphi} \in E_G((Q, e_Q), (P, e_P))$ there is an element

$y \in P$ such that $\varphi(u) = yuy^{-1}$ for all $u \in Q$. If k is large enough, this is equivalent to requiring that $N_G(Q, e_Q)/QC_G(Q)$ is a $p-$group for any subgroup Q of P. For k large enough, the structure of nilpotent blocks has been determined by Puig in [23]: if b is nilpotent, there is an indecomposable $kP-$endopermutation module N (cf. [11], [12]) with vertex P such that $ikGi \cong S \underset{k}{\otimes} kP$ as interior $P-$algebra, where $S = End_k(N)$. In particular, kGb is Morita equivalent to kP and $H^*(G, b, P_\gamma) \cong H^*(P, k)$.

Even if b is arbitrary, nilpotent blocks come in systematically: for any subgroup Q of P such that (Q, e_Q) is self-centralizing, the block e_Q of $kC_G(Q)$ is nilpotent (this follows trivially from the fact that the defect group $Z(Q)$ of e_Q lies in the center of $kC_G(Q)$ and therefore there is no nontrivial fusion). In particular, e_P is always a nilpotent block of $kC_G(P)$.

Furthermore, it is shown in [7] that G is $p-$nilpotent if and only if the principal block of kG is nilpotent.

3.7 The "diagonal embedding" $\delta_G : H^*(G, k) \longrightarrow HH^*(kG)$ has an analogue for any block b of kG, if we assume additionally that k is large enough for the block algebra $kC_G(P)e_P$ to be *split*, which amounts to requiring that $kC_G(P)e_P$ is Morita equivalent to $kZ(P)$ as e_P is a nilpotent block.

Still with the notation of 3.5, we consider kGi as $kGb - kP-$bimodule and ikG as $kP - kGb-$bimodule. By [20, 5.6] the relative projective elements π_{kGi} and π_{ikG} are invertible in $Z(kGb)$ and $Z(kP)$, respectively (this is where we use that k is large enough). Moreover, by [20, 5.6(iii)], the composition of the inclusion $H^*(G, b, P_\gamma) \subset H^*(P, k)$, the diagonal embedding $\delta_P : H^*(P, k) \longrightarrow HH^*(kP)$ and the normalized transfer map $T_{kGi} : HH^*(kP) \longrightarrow HH^*(kGb)$ induce an injective algebra homomorphism

$$H^*(G, b, P_\gamma) \xrightarrow{T_{kGi} \circ \delta_P} HH^*(kGb)$$

whose image is contained in the subalgebra $HH^*_{kGi}(kGb)$ of $kGi-$stable elements in $HH^*(kGb)$. If b is the principal block of kG, this map is in fact equal to δ_G followed by the canonical projection from $HH^*(kG)$ onto $HH^*(kGb)$, as we will see in 4.2(ii) below.

4 Varieties for modules over a block algebra

We fix in this section an algebraically closed field k of prime characteristic p.

We remind the reader of the definition, due to J. F. Carlson [9], [10], of the cohomological variety $V_G(U)$ of a bounded complex U of $kG-$modules, where G is a finite group: the functor $- \underset{k}{\otimes} U$ induces an algebra homomorphism $\gamma_U : H^*(G, k) \longrightarrow Ext^*_{kG}(U, U)$. Denote by $I^*_G(U)$ the kernel of γ_U. The variety $V_G(U)$ is then defined to be the maximal ideal spectrum of the quotient $H^*(G, k)/I^*_G(U)$. Note that $\gamma_U = \alpha_U \circ \delta_P$ by 2.9.

The case where $U = k$ is the trivial $kG-$module had previously been considered by D. Quillen (see [24], [25]). See [3, Vol. II, section 5.1] for a more detailed historical overview on varieties in group representation theory and for an extensive bibliography on this subject.

Definition 4.1 Let G be a finite group, b a block of kG, P_γ a defect pointed group of b and let $i \in \gamma$. For any bounded complex U of $kGb-$modules denote by $I^*_{G,b}(U)$ the kernel in $H^*(G,b,P_\gamma)$ of the composition of $k-$algebra homomorphisms

$$H^*(G,b,P_\gamma) \xrightarrow{T_{kGi}\circ\delta_P} HH^*(kGb) \xrightarrow{\alpha_U} Ext^*_{kG}(U,U)$$

and let $V_{G,b}(U)$ be the maximal ideal spectrum of $H^*(G,b,P_\gamma)/I^*_{G,b}(U)$.

The isomorphism class of the variety $V_{G,b}(U)$ in 4.1 does not depend on the choice of P_γ. If b is the principal block, $V_{G,b}(U)$ is just the cohomological variety $V_G(U)$ (see 4.4 below).

The next theorem establishes a connection in general between $V_{G,b}(U)$ and $V_G(U)$.

Theorem 4.2. *Let G be a finite group, b a block of kG, P_γ a defect pointed group of b and let $i \in \gamma$.*

(i) The restriction res^G_P induces an algebra homomorphism

$$\rho_b : H^*(G,k) \longrightarrow H^*(G,b,P_\gamma)$$

such that $H^*(G,b,P_\gamma)$ becomes Noetherian as a module over $H^*(G,k)$.

(ii) The diagram of graded $k-$algebra homomorphisms

$$\begin{array}{ccc} H^*(G,k) & \xrightarrow{\delta_G} & HH^*(kG) \\ {\scriptstyle\rho_b}\downarrow & & \downarrow \\ H^*(G,b,P_\gamma) & \xrightarrow{T_{kGi}\circ\delta_P} & HH^*(kGb) \end{array}$$

is commutative, where the right vertical map is the canonical projection induced by multiplication with b.

(iii) For any bounded complex U of $kGb-$modules the diagram of graded $k-$algebra homomorphisms

$$\begin{array}{ccc} H^*(G,k) & \xrightarrow{\gamma_U} & Ext^*_{kG}(U,U) \\ {\scriptstyle\rho_b}\downarrow & & \downarrow\cong \\ H^*(G,b,P_\gamma) & \xrightarrow[\alpha_U\circ T_{kGi}\circ\delta_P]{} & Ext^*_{kGb}(U,U) \end{array}$$

is commutative.

Before we prove 4.2, let us note some consequences. By a result of Gerstenhaber [15], the Hochschild cohomology of an associative ring is graded commutative. Using the fact that $H^*(P,k)$ is Noetherian over $H^*(G,k)$ via restriction from G to P (cf. [3, Vol. II, 4.2.5]) and that by a result of T. Holm in [16] the Hochschild cohomology ring $HH^*(kG)$ is Noetherian as module over $H^*(G,k)$ via the algebra homomorphism δ_G, the diagram in 4.2(ii) implies the following:

Corollary 4.3. *(i) The algebra $HH^*(kGb)$ is Noetherian as module over $H^*(G, b, P_\gamma)$ through the homomorphism given by $T_{kGi} \circ \delta_P$.*

(ii) The Krull dimensions of $H^*(G, b, P_\gamma)$, $HH^*(kGb)$ and $H^*(P, k)$ coincide and are whence all equal to the rank of P.

The fact that $HH^*(kGb)$ and $H^*(P, k)$ have same Krull dimension has previously been observed by S. Siegel, who communicated a short direct proof of this statement to the author.

Using standard results from commutative algebra, in terms of maximal ideal spectra, statement 4.2(iii) translates to:

Corollary 4.4. *For any bounded complex U of $kGb-$modules we have $I_G^*(U) = \rho_b^{-1}(I_{G,b}^*(U))$; in particular, ρ_b induces a finite surjective map*

$$V_{G,b}(U) \longrightarrow V_G(U).$$

Moreover, if b is the principal block of kG, the above map is an isomorphism.

Proof of 4.2. Clearly the restriction res_P^G maps $H^*(G, k)$ to $H^*(G, b, P_\gamma)$, thus induces an algebra homomorphism $\rho_b : H^*(G, k) \longrightarrow H^*(G, b, P_\gamma)$. By [3, 4.2.5], $H^*(P, k)$ is Noetherian as module over $H^*(G, k)$ through restriction, thus $H^*(G, b, P_\gamma)$ is Noetherian as module over $H^*(G, k)$ through ρ_b. This proves (i).

Let S be a Sylow-$p-$subgroup of G containing P. Identify $H^*(G, k)$ to the subalgebra $res_S^G(H^*(G, k))$ of $H^*(S, k)$ of $G-$stable elements. Since restriction res_S^G followed by the transfer map t_S^G is multiplication by the index $[G : S]$ on $H^*(G, k)$, it follows from 2.8 (or [20,4.6]) that

$$[G : S]\delta_G = t_{(kG)_S} \circ \delta_S|_{H^*(G,k)}.$$

From [20,5.3] follows that

$$t_{(ikG)_S} \circ \delta_S|_{H^*(G,k)} = \frac{dim_k(ikG)}{|S|}\delta_P \circ \rho_b,$$

where $(ikG)_S$ is ikG viewed as $kP - kS-$bimodule. Moreover, by 2.3(ii) we have

$$t_{(ikG)_S} = t_{ikG} \circ t_{(kG)_S}.$$

By 2.7 and 2.8 we have

$$\delta_G(H^*(G, k)) \subset HH^*_{(kG)_P}(kG) \subset HH^*_{kGi}(kG)$$

and the projection $HH^*(kG) \longrightarrow HH^*(kGb)$ maps clearly $HH^*_{kGi}(kG)$ to $HH^*_{kGi}(kGb)$ (with the notational abuse of considering kGi as $kG - kP-$bimodule in the first place and then as $kGb - kP-$bimodule in the second).

Thus, by 2.6, the composition $T_{kGi} \circ T_{ikG}$ restricts to the identity on the image of $\delta_G(H^*(G, k))$ in $HH^*(kGb)$. It follows that

$$\delta_P \circ \rho_b = (\frac{dim_k(ikG)}{|S|})^{-1} t_{ikG} \circ t_{(kG)_S} \circ \delta_S|_{H^*(G,k)} =$$

$$(\frac{dim_k(ikG)}{|S|})^{-1}[G:S]t_{ikG}\circ\delta_G = T_{ikG}\circ\delta_G,$$

since $\pi_{ikG} = \frac{dim_k(ikG)}{|G|}$ by [20,5.6(i)]. Applying T_{kGi} to this equality yields the commutativity of the diagram in statement (ii).

The last statement follows then easily from the observation that the map $HH^*(kG) \longrightarrow Ext^*_{kG}(U,U) \cong Ext^*_{kGb}(U,U)$ induced by the functor $-\underset{kG}{\otimes} U$ factors through the natural projection $HH^*(kG) \longrightarrow HH^*(kGb)$. This concludes the proof of 4.2.

5 Invariance properties of varieties of modules

We show in this section roughly speaking, that the varieties $V_{G,b}(U)$ introduced in 4.1 are invariant under splendid stable and derived equivalences. This is based on the following general result:

Theorem 5.1. *Let A, B be symmetric algebras over a commutative ring R with symmetrizing forms $s \in A^*$, $t \in B^*$, and let X be a bounded complex of $A-B-$bimodules whose components are projective as left and right modules.*

If π_X is invertible in $Z(A)$, for any bounded complex U of $B-$modules there is a commutative diagram of graded $R-$algebra homomorphisms

$$\begin{array}{ccc} HH^*_{X^*}(B) & \xrightarrow{\alpha_U} & Ext^*_B(U,U) \\ {\scriptstyle T_X}\downarrow & & \downarrow{\scriptstyle \beta_{X,U}} \\ HH^*_X(A) & \xrightarrow[\alpha_{X\underset{B}{\otimes}U}]{} & Ext^*_A(X\underset{B}{\otimes}U, X\underset{B}{\otimes}U) \end{array}$$

where the horizontal maps are induced by the functors $-\underset{B}{\otimes}U$ and $-\underset{A}{\otimes}(X\underset{B}{\otimes}U)$, respectively, and where the right vertical map is induced by the functor $X\underset{B}{\otimes}-$.

Proof. Let n be a nonnegative integer, $\zeta : \mathcal{P}_A \longrightarrow \mathcal{P}_A[n]$ and $\tau : \mathcal{P}_B \longrightarrow \mathcal{P}_B[n]$ chain maps making the diagram

5.1.1

$$\begin{array}{ccc} \mathcal{P}_A\underset{A}{\otimes}X & \xrightarrow{\simeq} & X\underset{B}{\otimes}\mathcal{P}_B \\ {\scriptstyle \zeta\otimes Id_X}\downarrow & & \downarrow{\scriptstyle Id_X\otimes\tau} \\ \mathcal{P}_A[n]\underset{A}{\otimes}X & \xrightarrow[\simeq]{} & X\underset{B}{\otimes}\mathcal{P}_B[n] \end{array}$$

homotopy commutative, where the horizontal maps are homotopy equivalences lifting the natural isomorphism $A\underset{A}{\otimes}X \cong X \cong X\underset{B}{\otimes}B$ (and its shift by degree n).

Then we have $[\zeta] = T_X([\tau])$ by [20,3.4(ii)]. Thus $\alpha_{X\underset{B}{\otimes}U}\circ T_X$ maps $[\tau]$ to the element of $Ext^n_A(X\underset{B}{\otimes}U, X\underset{B}{\otimes}U)$ represented by the chain map

$$\mathcal{P}_A\underset{A}{\otimes}X\underset{B}{\otimes}U \xrightarrow{\zeta\otimes Id_X\otimes Id_U} \mathcal{P}_A[n]\underset{A}{\otimes}X\underset{B}{\otimes}U$$

and $\beta_{X,U} \circ \alpha_U$ maps $[\tau]$ to the element of $Ext^n_A(X \underset{B}{\otimes} U, X \underset{B}{\otimes} U)$ represented by the chain map

$$X \underset{B}{\otimes} \mathcal{P}_B \underset{B}{\otimes} U \xrightarrow{Id_X \otimes \tau \otimes Id_U} X \underset{B}{\otimes} \mathcal{P}_B[n] \underset{B}{\otimes} U.$$

Thus both coincide as can be seen by tensoring the diagram 5.1.1 by $- \underset{B}{\otimes} U$.

Before we state our invariance theorem, we need some properties of relatively projective elements. We show that the relative projective elements coming from derived or stable equivalences of Morita type between block algebras are invertible not only with respect to the canonical symmetrizing forms but in fact with respect to any symmetrizing form; we state this in the following slightly more general form:

Proposition 5.2. *Let A, B be indecomposable non simple symmetric algebras over a field k and let X be a bounded complex of $A-B-$bimodules whose components are projective as left and right modules. Suppose that*

$X \underset{B}{\otimes} X^* \simeq A \oplus U_A$, where U_A is a bounded complex of projective $A-A-$bimodules, and

$X^* \underset{A}{\otimes} X \simeq B \oplus U_B$, where U_B is a bounded complex of projective $B-B-$bimodules.

Then for any choice of symmetrizing forms on A and B, the corresponding relatively projective elements π_X and π_{X^*} are invertible in $Z(A)$ and $Z(B)$, respectively.

Proof. If π_X is not invertible, the composition of adjunction maps $A \longrightarrow X \underset{B}{\otimes} X^* \longrightarrow A$ is a nilpotent $A-A-$endomorphism of A, since A is indecomposable as $A-A-$bimodule. However, modulo the thick subcategory of $K^b(A)$ consisting of bounded complexes of projective $A-$modules (and similarly for B instead of A) the functor $X \underset{B}{\otimes} -$ induces an equivalence by the hypotheses on X, and so the image in the suitable quotient category of the composition of adjunction maps $A \longrightarrow X \underset{B}{\otimes} X^* \longrightarrow A$ has to be an isomorphism and thus cannot be nilpotent. This shows that π_X is invertible. The same argument shows that π_{X^*} is invertible.

Note that the above proposition covers the situation of derived and stable equivalences of Morita type: if U_A and U_B are zero, the complex X is a *two-sided tilting complex*, frequently called *Rickard complex* in the literature, and also *two-sided split endomorphism tilting complex* in [27]. If the complex X is concentrated in degree zero (that is, X is a bimodule), we are in the situation of what is called a *stable equivalence of Morita type*, a concept due to M. Broué.

We collect now some elementary properties of symmetric subalgebras of a symmetric algebra that we are going to apply in 5.4 to the particular case of the subalgebra kPi (which is clearly isomorphic to kP, whence symmetric) of the source algebra $ikGi$ of the block b, where the notation is as in 3.3 (or 5.4 below).

Lemma 5.3. *Let A be a symmetric algebra over a commutative ring R with symmetrizing form $s \in A^*$ and let B be a unitary symmetric subalgebra of A such that the restriction $s|_B \in B^*$ is a symmetrizing form of B. Then the following hold.*

(i) As $B-B-$bimodule, B is a direct summand of A, and B has a unique complement C in A contained in $ker(s)$.

(ii) The projection $\pi : A \longrightarrow B$ with kernel C maps any $a \in A$ to the unique element $b \in B$ satisfying $a.s|_B = b.s|_B$. Moreover, π maps $Z(A)$ to $Z(B)$.

(iii) If $C^B \subset J(A^B)$ then π maps $Z(A)^\times$ to $Z(B)^\times$.

Proof. Any $R-$linear form on A is equal to $a.s$ for a uniquely determined $a \in A$, where $a.s$ is defined by $(a.s)(a') = s(aa')$ for all $a' \in A$. Its restriction to B is thus equal to $b.s|_B$ for a uniquely determined $b \in B$, as $s|_B$ is a symmetrizing form for B. The map sending $a \in A$ to $b \in B$ defined this way is clearly a projection of A onto B as $B-B-$bimodule, and its kernel C is the unique complement of B in A contained in $ker(s)$. This map sends A^B to $B^B = Z(B)$, thus it sends in particular $Z(A)$ to $Z(B)$. This shows (i) and (ii). Let now $z \in Z(A)^\times$ and write $z = \pi(z) + c$ for some $c \in C$. Similarly, write $z^{-1} = \pi(z^{-1}) + c'$ for some $c' \in C$. Then in fact $c, c' \in C^B$. Write $cc' = \pi(cc') + d$ for some $d \in C$; again in fact $d \in C^B$. We have now $1_A = zz^{-1} = \pi(z)\pi(z^{-1}) + \pi(z)c' + c\pi(z^{-1}) + cc'$. As $\pi(1) = 1$ we obtain from the previous equation that $1 = \pi(z)\pi(z^{-1}) + \pi(cc')$. Therefore, if $C^B \subset J(A^B)$ then $\pi(cc') = cc' - d \in J(A^B)$, so in particular $\pi(z)$ is invertible.

Observe that the projection $\pi : A \longrightarrow B$ in 5.3(ii) is a $B-B-$homomorphism, thus the induced map $Z(A) \longrightarrow Z(B)$ is $R-$linear, but not multiplicative in general. In particular, the induced map $Z(A)^\times \longrightarrow Z(B)^\times$ in 5.3(iii) is not a group homomorphism in general.

Proposition 5.4. *Let k be an algebraically closed field of prime characteristic p, let G be a finite group, b a block of kG and P_γ a defect pointed group of b. Let $i \in \gamma$. Denote by $s \in (ikGi)^*$ the canonical symmetrizing form on $ikGi$.*

(i) We have $s(i) \in k^\times$ and $s(ui) = 0$ for any $u \in P - \{1\}$. In particular, the restriction of s to the subalgebra kPi of $ikGi$ is a symmetrizing form of kPi.

(ii) The subalgebra kPi of $ikGi$ has a unique complement C in $ikGi$ as $kP-kP-$bimodule such that $C \subset ker(s)$, and then $C^P \subset J((ikGi)^P)$.

(iii) The projection $\pi : ikGi \longrightarrow kPi$ with kernel C maps $a \in ikGi$ to $s(i)^{-1} \sum_{u \in P} s(u^{-1}a)ui$. Moreover, π maps $Z(ikGi)^\times$ to $Z(kP)^\times$.

(iv) For any choice of symmetrizing forms on $ikGi$ and on kP, the relatively $ikG-$projective element π_{ikG} is invertible in $Z(kP)$.

Proof. (i) The fact that $s(ui)$ is non zero if and only if $u = 1_P$ is proved in [20,5.5]. This means that through the obvious isomorphism $kP \cong kPi$, the restriction of s to kP is a non zero scalar multiple of the canonical symmetrizing form on kP, thus itself a symmetrizing form.

(ii) By (i) and 5.3(i), kPi has a unique complement in $ikGi$ as claimed. Now applying the Brauer construction [28, section 11] yields, by a result of Puig [24, 14.5], that

$(ikGi)(P) \cong kZ(P)$ (this isomorphism is also described in [28, (38.10)]), and clearly $(kPi)(P) \cong kZ(P)$. Thus $C(P) = \{0\}$, or, equivalently, $C^P \subset ker(Br_P)$, where Br_P denotes the Brauer homomorphism on $(ikGi)^P$ (cf. 3.1). Since i is primitive in $(ikGi)^P$, the latter is a local algebra. As $Br_P(i) \neq 0$ the ideal $ker(Br_P)$ of $(ikGi)^P$ is therefore contained in $J((ikGi)^P)$, which implies (ii).

(iii) Since $C \subset ker(s)$, the given formula in (iii) maps C to zero, and whence coincides with π on C. Let $x \in P$. Then, by (i), for any $u \in P$ we have $s(u^{-1}xi) = 0$ unless $u = x$. Thus the given formula is the identity on kPi, whence coincides again with π. The second statement in (iii) follows from (ii) and 5.3(iii).

(iv) With respect to the canonical symmetrizing forms on kP and $ikGi$, by [20,5.5], the adjunction map $kP \longrightarrow ikG \underset{kGb}{\otimes} kGi \cong ikGi$ sends $u \in P$ to ui, and the adjunction map $ikGi \cong ikG \underset{kGb}{\otimes} kGi \longrightarrow kP$ sends $a \in ikGi$ to $\sum_{u\in P} s(u^{-1}a)u$. Any other symmetrizing form on $ikGi$ is equal to $z.s$ for a uniquely determined $z \in Z(ikGi)^\times$. With respect to this new symmetrizing form on $ikGi$ (and still the canonical form on kP), the first adjunction map $kP \longrightarrow ikG \underset{kGb}{\otimes} kGi \cong ikGi$ maps $u \in P$ to zui, while the second adjunction map remains unchanged (this follows from the formulae given in [20, 2.4]). Thus, the relatively $ikG-$projective element π_{ikG} with respect to this choice of symmetrizing forms is equal to $\sum_{u\in P} s(u^{-1}zi)u$. It follows from (iii) that $\pi_{ikG}i = s(i)\pi(z)$. Since π maps $Z(ikGi)^\times$ to $Z(kPi)^\times$, indeed π_{ikG} is invertible. Modifying the symmetrizing form on kP has no influence on the property of π_{ikG} being invertible (this follows from [20, 3.2.2]), which completes the proof.

We state now the invariance theorem for the varieties $V_{G,b}(U)$ with respect to splendid derived and stable equivalences, again in a slightly more general form analogously to 5.2.

Theorem 5.5. *Let k be an algebraically closed field of prime characteristic p, let G, H be finite groups and b, c be blocks of kG, kH, respectively, having a common defect group P. Let γ, δ be local points of P on kGb, kHc, respectively, and choose $i \in \gamma$, $j \in \delta$. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $Br_Q^G(i)e_Q = Br_Q^G(i)$ and $Br_Q^H(j)f_Q = Br_Q^H(j)$.*

Assume that $E_G((Q, e_Q), (P, e_P)) = E_H((Q, f_Q), (P, f_P))$ for any subgroup Q of P.

Let X be a bounded complex of $kGb - kHc-$bimodules whose components are isomorphic to direct sums of direct summands of the bimodules $kGi \underset{kQ}{\otimes} jkH$, where Q runs over the set of subgroups of P.

Assume that $X \underset{kHc}{\otimes} X^* \simeq kGb \oplus U_b$, where U_b is a bounded complex of projective $kGb-kGb-$bimodules, and that $X^* \underset{kGb}{\otimes} X \simeq kHc \oplus U_c$, where U_c is a bounded complex of projective $kHc - kHc-$bimodules.

(i) For any choice of symmetrizing forms on kGb and kHc, the relatively projective elements π_X and π_{X^*} are invertible in $Z(kGb)$ and $Z(kHc)$, respectively. With respect to the canonical symmetrizing form on kP we have $\pi_{iXj} = \pi_{jX^*i} \in k^\times 1_{kP}$, where iXj and its dual jX^*i are considered as complexes of $kP - kP-$bimodules.

(ii) The map $T_{kGi} \circ \delta_P$ sends $H^*(G, b, P_\gamma)$ to $HH^*_X(kGb)$ and $T_{kHj} \circ \delta_P$ sends $H^*(H, c, P_\delta)$ to $HH^*_{X^*}(kHc)$, making the following diagram of graded $k-$algebras commutative:

$$\begin{array}{ccc} H^*(H,c,P_\delta) & \xrightarrow{T_{kHj}\circ\delta_P} & HH^*_{X^*}(kHc) \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle T_X} \\ H^*(G,b,P_\gamma) & \xrightarrow[T_{kGi}\circ\delta_P]{} & HH^*_X(kGb) \end{array}$$

(iii) For any bounded complex V of $kHc-$modules, the following diagram of graded $k-$algebras is commutative:

$$\begin{array}{ccc} H^*(H,c,P_\delta) & \xrightarrow{\alpha_V\circ T_{kHj}\circ\delta_P} & Ext^*_{kHc}(V,V) \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle \beta_{X,V}} \\ H^*(G,b,P_\gamma) & \xrightarrow[\alpha_{X\underset{kHc}{\otimes}V}\circ T_{kGi}\circ\delta_P]{} & Ext^*_{kGb}(X\underset{kHc}{\otimes}V, X\underset{kHc}{\otimes}V) \end{array}$$

In particular, we have $I^*_{G,b}(X\underset{kHc}{\otimes}V)=I^*_{H,c}(V)$, and whence,

$$V_{G,b}(X\underset{kHc}{\otimes}V)=V_{H,c}(V).$$

Proof. Statement (i) follows from 5.2 and [20,5.7(i)]. For the proof of (ii), we first show that $T_{kGi}\circ\delta_P$ maps $H^*(G,b,P_\gamma)$ to $HH^*_X(kGb)$. This is based on the cancellation properties for stable elements in 2.7. By [20,5.7(iii)] we have $\delta_P(H^*(G,b,P_\gamma))\subset HH^*_{iXj}(kP)$. As $iXj\cong iX\underset{kHc}{\otimes}kHj$ and π_{kHj} is invertible, the first of the cancellation properties in 2.7 implies that $HH^*_{iXj}(kP)\subset HH^*_{iX}(kP)$. As $iX\cong ikG\underset{kGb}{\otimes}X$ and π_{kGi} is invertible, the second of the cancellation properties in 2.7 shows that T_{kGi} maps $HH^*_{iX}(kP)$ to $HH^*_X(kGb)$. A similar argument shows that $T_{kHj}\circ\delta_P$ maps $H^*(H,c,P_\delta)$ to $HH^*_{X^*}(kHc)$.

Note that the diagram in (ii) does not depend on the choice of symmetrizing forms on kGb, kHc, as follows from [20,3.6(ii)].

Observe next that $H^*(G,b,P_\gamma)=H^*(H,c,P_\delta)$ as subalgebras of $H^*(P,k)$ by the assumptions. We consider kP endowed with the canonical symmetrizing form. By [20,3.2.3] we may choose a symmetrizing form on kHc such that $\pi_{kHj}=1_{kHc}$, or equivalently, such that $T_{kHj}=t_{kHj}$. Since π_X is invertible for any choice of a symmetrizing form on kHc, we may again by [20,3.2.3] choose a symmetrizing form on kGb such that $\pi_X=1_{kGb}$, or equivalently, such that $T_X=t_X$. Since we have not changed the symmetrizing form of kP, the relatively projective element π_{kGi} is still invertible in $Z(kGb)$. Moreover, by 5.4(iv), the relatively projective element π_{ikG} is invertible in $Z(kP)$. In order to show that the diagram in (ii) is commutative we have to show that the maps $T_X\circ T_{kHj}\circ\delta_P$ and $T_{kGi}\circ\delta_P$ coincide on $H^*(H,c,P_\delta)=H^*(G,b,P_\gamma)$. This is equivalent to showing that the map $T_{ikG}\circ T_X\circ T_{kHj}$ is the identity on $\delta_P(H^*(H,c,P_\delta))$. Now

$$T_{ikG}\circ T_X\circ T_{kHj}=(\pi_{ikG})^{-1}t_{ikG}\circ t_X\circ t_{kHj}=(\pi_{ikG})^{-1}t_{iXj}$$

and by [20,5.7(iv)], the map t_{iXj} acts as multiplication by π_{iXj} on $\delta_P(H^*(H,c,P_\delta))$. Thus it suffices to observe that $\pi_{ikG}=\pi_{iXj}$. Now with the notation from [20, 3.2], we have $\pi_{iXj}=t^0_{iXj}(1_{kP})=t^0_{ikG}\circ t^0_X\circ t^0_{kHj}(1_{kP})$, and by our choice of symmetrizing forms, we have $t^0_{kHj}(1_{kP}) = 1_{kHc}$ and $t^0_X(1_{kHc}) = 1_{kGb}$. Since $t^0_{ikG}(1_{kGb}) = \pi_{ikG}$, the proof of (ii) is complete.

The diagram in (iii) is just obtained by composing the diagram of (ii) together with the appropriate version of 5.1.

6 Some further remarks

Let k be an algebraically closed field of prime characteristic p and G a finite group.

Benson, Carlson and Robinson introduce in [5] the *nucleus of* kG as the subvariety Y_G of $V_G(k)$ which is the union of the images of the maps $(res^G_H)^*$ induced by the restriction maps $res^G_H : H^*(G,k) \longrightarrow H^*(H,k)$, with H running over the set of subgroups of G for which $C_G(H)$ is not $p-$nilpotent, with the convention $Y_G = \{0\}$ if G is $p-$nilpotent.

We define now a *nucleus* $Y_{G,b}$ *of a block* b *of* G and show that this coincides with Y_G if b is the principal block of G. Remind from 4.1 that $V_G(b)$ is the variety of the block b; that is, the maximal ideal spectrum of the cohomology ring of b.

Definition 6.1 Let b be a block of kG and P_γ a defect pointed group of b. Let $i \in \gamma$. For any subgroup Q of P denote by e_Q the unique block of $kC_G(Q)$ such that $Br_Q(i)e_Q = Br_Q(i)$. The *nucleus* $Y_{G,b}$ *of the block* b is the union of the images of all maps $V_Q(k) \longrightarrow V_G(b)$ induced by the restriction maps res^P_Q from $H^*(G,b,P_\gamma)$ to $H^*(Q,k)$, where Q runs over the set of subgroups of P such that the block e_Q of $kC_G(Q)$ is not nilpotent, with the convention $Y_{G,b} = \{0\}$ if the block b is nilpotent.

Proposition 6.2. *If b_0 is the principal block of kG then restriction from G to a Sylow-$p-$subgroup P of G induces isomorphisms*

$$V_G(b_0) \cong V_G(k) \quad \textit{and} \quad Y_{G,b_0} \cong Y_G \ .$$

Proof. If b_0 is the principal block of kG then the cohomology ring of b_0 is precisely the image of the restriction to P of $H^*(G,k)$, thus $V_G(b_0) \cong V_G(k)$. Moreover, for any subgroup Q of P, the block e_Q as defined in 6.1 is the principal block of $kC_G(Q)$. By [7], e_Q is nilpotent if and only if the group $kC_G(Q)$ is $p-$nilpotent. Thus res^G_P induces an injective map $Y_{G,b_0} \longrightarrow Y_G$. This map is also surjective; indeed, if H is a subgroup of G such that $C_G(H)$ is not $p-$nilpoptent, then for any Sylow-$p-$subgroup Q of H, the group $C_G(Q)$ is not $p-$nilpotent. The statement follows.

Remark 6.3 Let b be a block of kG, P_γ a defect pointed group of b, let $i \in \gamma$ and denote for any subgroup Q of P by e_Q the unique block of $kC_G(Q)$ such that $Br_Q(i)e_Q = Br_Q(i)$.

(i) We have $Y_{G,b} = \{0\}$ if and only if e_Q is nilpotent for every non trivial subgroup Q of P.

(ii) The nucleus $Y_{G,b}$ of b is an invariant of the $p-$local structure of b since both the cohomology ring of b and the property of e_Q to be nilpotent are so.

(iii) One might want to try to generalize the results in [4], [5] to arbitrary blocks. The principal obstacle at this stage is, that we do not yet have analogous concepts which would generalize the *representation theoretical nucleus* defined in [5, 10.1] or the concept of *trivial homology modules* [5, 2.1] to arbitrary blocks.

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