The Distance to Strong Stability

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Abstract: The notion of "strong stability" has been introduced in a recent paper [KHP2]. This notion is relevant for state-space models described by physical variables and prohibits overshooting trajectories in the state-space transient response for arbitrary initial conditions. Thus, "strong stability" is a stronger notion compared to alternative definitions (e.g. stability in the sense of Lyapunov or asymptotic stability). This paper defines two distance measures to strong-stability under absolute (additive) and relative (multiplicative) matrix perturbations, formulated in terms of the spectral and the Frobenius norm. Both symmetric and non-symmetric perturbations are considered. Closed-form or algorithmic solutions to these distance problems are derived and interesting connections are established with various areas in matrix theory, such as the field of values of a matrix, the cone of positive semi-definite matrices and the Lyapunov cone of Hurwitz matrices. The results of the paper are illustrated by numerous computational examples.

Keywords: Matrix distance problems, Strong stability, Non-overshooting trajectory, Spectral norm, Frobenius norm, Field of Values, Convex Invertible Cone, Lyapunov Cone.

1. Introduction

A new notion of "strong stability" was defined in [KHP1], [KHP2] for the autonomous, linear, time-invariant (LTI) state-space system:

$$\mathcal{S}(A) : \dot{x}(t) = Ax(t), \quad x(t_0) = x_0 \tag{1}$$

This is a stronger notion compared to traditional definitions of stability, e.g., asymptotic or Lyapunov stability, related to the transient response of a system, e.g. its overshooting behaviour, initial exponential growth or transient energy [HP2]. It is also closely related to the theory of logarithmic norms which can be used to obtain exponential stability estimates in the solution of initial value problems and the numerical analysis of Ordinary Differential Equations [S].

The notion of strong stability is briefly reviewed in section 2, along with some other fundamental properties and definitions. Reference [KHP2] examined the dependence of strong stability on general coordinate transformations and established the existence of special coordinate frames for which we cannot have strong stability and the invariance of this property under orthogonal transformations. It was further shown that the violation of the strong stability property is intimately related to the eigen-frame skewnesss of the state-matrix of the system (A in (1)). Upper bounds on a measure of eigen-frame skewnesss were also established which guarantee the equivalence of the asymptotic and strong stability properties.

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Reference [HPK] considered the strong stabilization problem under state and output feedback. Simple necessary and sufficient conditions of strong stabilizability were established using a variety of techniques (polynomial, geometric, convex programming/LMI-based). Geometrically, strong stabilization was shown to be equivalent to the condition that the intersection between an affine hyperplane and a convex cone is non-empty, a condition which can be easily verified via Linear Matrix Inequalities [SW], [SIG]. Simpler equivalent conditions can also be established directly from the state-space realization of the system, along with a complete parametrization of all strongly-stabilizing state-feedback, output injection or output feedback matrices, respectively, depending on the nature of the problem. Note, that in the context of state or output-feedback control, a small measure of skewness in the eigenvectors of the (closed-loop) state-matrix (equivalently small deviation of the state matrix from normality) is a highly desirable property [KNvD] as it implies low eigenvalue sensitivity to model uncertainty [W]. As an alternative application of strong stabilisation, consider the linear system resulting from the linearisation of a nonlinear system around an equilibrium point regulated via state or output feedback. In this case, "large" state overshoots in the linear response imply that after the application of a disturbance, the state of the (nonlinear) system may drift far away from the equilibrium, in a region where the linearisation approximation is no longer valid, resulting in instability. This is less likely to happen if the (linearised) response decreases monotonically to zero from a perturbed initial condition.

In this paper the following problem is addressed: Suppose a square matrix A is asymptotically stable but is not strongly stable. Does it make sense to say in certain cases that A is "approximately strongly stable" and, if yes, can we make this notion precise? The main motivation for posing this question arises from the strong stabilization problem outlined in the previous paragraph. Although strong stability is a highly desirable closed-loop system property, it may be a very strong condition to impose in certain cases, e.g. it may require excessive actuator signal levels. In such cases relaxing the definition by introducing approximate notions may be appropriate.

In this paper the approximate notion of strong stability is made precise by defining the "distance" of an arbitrary matrix A from the set of all strongly stable matrices of the same dimension. Two methods are proposed for defining this metric. The first, involves the minimization of the norm of an additive perturbation Δ of A such that $A + \Delta$ is strongly stable. The second method considers multiplicative perturbations Δ (left or right) and minimizes the norm of Δ such that $A(I+\Delta)$ (equivalently $(I+\Delta)A$) is strongly stable. Additive and multiplicative perturbation models are two types of "unstructured" uncertainty used extensively in robust systems and control theory as they correspond to absolute and relative modelling errors, respectively [HP], [SIG]. Both general and symmetric perturbation matrices Δ are considered. In addition, two norms are considered in the formulation of the distance problem, the Frobenius and spectral norm (largest singular value). The solution to the problem is obtained in each case either in closed-form or algorithmically, and connections are established with various linear-algebraic notions, such as the field of values, the cone of positive semi-definite matrices and the Lyapunov cone of asymptotically stable matrices.

The structure of the paper is as follows: The mathematical notation used in the paper along with some background material is defined in section 2. A brief introduction to strong stability, along with some basic definitions and fundamental results related to this notion, is included in section 3. Sections 4 and

5 contain the main results of the paper, i.e. the formulation and solution of the two distance problems described above, discussion of issues related to existence and uniqueness of solutions and illustration of the optimization methods via computation examples. The results of the paper are summarized in section 6, while section 7 contains the list of references.

2. Notation and Preliminaries

The notation is mostly standard and is included here for ease of reference. $\mathcal{R}$ and $\mathcal{C}$ denote the fields of real and complex numbers, respectively. The set of positive and non-negative numbers is denoted by $\mathcal{R}_+$ and $\mathcal{R}_{+0}$, respectively. If k is an integer, then $\underline{k} = \{1, 2, \ldots, k\}$. If $f(x)$ is a real-valued function and $x \in \mathcal{X} \subseteq \mathcal{R}$, then $\inf^+\{f(x) : x \in \mathcal{X}\} := \max(\inf_{x\in\mathcal{X}} f(x), 0)$ and $\sup^+\{f(x) : x \in \mathcal{X}\} := \max(\sup_{x\in\mathcal{X}} f(x), 0)$ (and similarly for minimisation of maximisation of $f(x)$). The open (resp. closed) left half complex plane is denoted by $\mathcal{C}_-$ (resp. $\overline{\mathcal{C}}_-$). $\mathcal{R}^{n\times m}$ is the space of all $n \times m$ matrices over $\mathcal{R}$. For a set $\Omega \subseteq \mathcal{R}^{n\times m}$, $\bar{\Omega}$ denotes its closure in $\mathcal{R}^{n\times m}$ (with respect to a suitable norm $\|\cdot\|$) and $\partial\Omega = \bar{\Omega}\backslash\Omega$. The interior of a set Ω in denoted by $\text{int}(\Omega)$. The distance of $A \in \mathcal{R}^{n\times m}$ to Ω is defined as $\text{dist}(A, \Omega) = \inf_{X\in\Omega} \|A - X\|$. The cone generated by a set $\Omega \subseteq \mathcal{R}^{n\times n}$ is defined as $\text{cone}[\Omega] = \{x \in \mathcal{R}^{n\times n} : x = \lambda\omega, \omega \in \Omega, \lambda > 0\}$. A set $\Omega \subseteq \mathcal{R}^{n\times n}$ is called a convex invertible cone (cic) if it is a convex cone and $\omega \in \Omega \Rightarrow \omega^{-1} \in \Omega$.

The spectrum of a matrix $A \in \mathcal{R}^{n\times n}$ is the set of its eigenvalues $\lambda(A) = \{\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)\}$. The field of values of A is the set $F(A) = \{x^*Ax : x \in \mathcal{C}^n, x^*x = 1\}$ where $(\cdot)^*$ denotes the complex-conjugate transpose. The spectral radius of A is defined as $\rho(A) := \max\{|\lambda_1(A)|, |\lambda_2(A)|, \ldots, |\lambda_n(A)|\}$ and the numerical radius of A is $r(A) := \max\{|z| : z \in F(A)\}$. The set of all real $n \times n$ real symmetric matrices ($A = A'$) is denoted as $\mathcal{S}_n$ and the set of all $n \times n$ real skew-symmetric matrices ($A = -A'$) is denoted as $\mathcal{A}_n$. The inertia of $A \in \mathcal{S}_n$ is the triplet $\text{In}(A) = (\pi(A), \delta(A), \nu(A))$ of positive, zero, and negative eigenvalues of A, respectively. For $A \in \mathcal{S}_n$ we denote by $[A]_+$ ($[A]_-$) the matrix that results by setting all negative (resp. positive) eigenvalues in the spectral decomposition of A to zero. The set of all $n \times n$ positive-definite (positive semi-definite) matrices $A > 0$ ($A \neq 0$) is denoted by $\mathcal{S}_n^+$ ($\bar{\mathcal{S}}_n^+$) while $\mathcal{S}_n^-$ ($\bar{\mathcal{S}}_n^-$) denotes the set of all $n \times n$ negative-definite (negative semi-definite) symmetric matrices. It follows easily that the sets $\mathcal{S}_n^+$ (and $\mathcal{S}_n^-$) are convex invertible cones. If $A, B \in \mathcal{S}_n$, $A < B$ ($A \leq B$) means that $B - A > 0$ ($B - A \geq 0$).

$\|A\|$ (or $\bar{\sigma}(A)$) denotes the spectral norm of $A \in \mathcal{R}^{n\times n}$ and $\|A\|_F$ the Frobenius norm of A. In matrix-distance problems the convenience of using the Frobenius norm arises from the fact that it is induced by an inner product in $\mathcal{R}^{n\times n}$, $\langle A, B\rangle = \text{trace}\{B'A\}$, with $\|A\|_F^2 = \langle A, A\rangle$. Thus the space $(\mathcal{R}^{n\times n}, \mathcal{R})$ equipped with $\|\cdot\|_F$ is a Hilbert space (due to completeness) and can be written as the direct sum $\mathcal{R}^{n\times n} = \mathcal{S}_n \oplus \mathcal{A}_n$ of the spaces of all symmetric and skew-symmetric matrices, respectively; this is in fact an orthogonal decomposition with respect to the inner product $\langle\cdot,\cdot\rangle$ defined above.

The Kronecker product of two matrices $A \in \mathcal{R}^{m\times n}$ and $B \in \mathcal{R}^{p\times q}$ is denoted as $A \otimes B \in \mathcal{R}^{mp\times nq}$. Given $A \in \mathcal{R}^{n\times m}$, $\text{vec}(A) : \mathcal{R}^{n\times m} \to \mathcal{R}^{nm}$ denotes the usual vectorisation operation; this defines an isometric isomorphism between the spaces $\mathcal{R}^{n\times n}$ and $\mathcal{R}^{n^2}$, so that $\|A\|_F = \|\text{vec}(A)\|$ where $\|\cdot\|$ denotes the Euclidian norm. Note also that, $\text{vec}(\mathcal{S}_n) = \{\text{vec}(A) : A \in \mathcal{S}_n\} \subseteq \mathcal{R}^{n^2}$ is a linear subspace of $\mathcal{R}^{n^2}$ of dimension $r = n(n + 1)/2$. Let $\{w_1, w_2, \ldots, w_r\}$, be an orthonormal basis set for $\text{vec}(S_n)$ and define $W_{\mathcal{S}} = [w_1 \; w_2 \; \dots \; w_r]$. For each $A \in \mathcal{S}_n$ the column vector of co-ordinates of $\text{vec}(A)$ with respect to $\{w_1, w_2, \dots, w_r\}$ is denoted by $\overline{\text{vec}}_{\mathcal{S}}(A)$. Clearly we have that: $\text{vec}(A) = W_{\mathcal{S}}\overline{\text{vec}}_{\mathcal{S}}(A) \Rightarrow \overline{\text{vec}}_{\mathcal{S}}(A) = W'_{\mathcal{S}}\text{vec}(A)$. Also, $W'_{\mathcal{S}}W_{\mathcal{S}} = I_r$, $\mathcal{R}[W'_{\mathcal{S}}] = \mathcal{R}^r$ and $\mathcal{R}[W_{\mathcal{S}}] = \text{vec}(\mathcal{S}_n)$.

The characterization of positive semi-definite matrices in [All] is based on the fact that $A \in \bar{\mathcal{S}}_n^+$ can be written (e.g. via its spectral decomposition) as $A = \alpha B^2$ for some $B = B'$ and $\alpha \geq 0$. Let:

$$\mathcal{U}_{\mathcal{S}} := \{B \in \mathcal{R}^{n\times n} : B = B' \text{ and } \|B\|_F = 1\} \subseteq \mathcal{S}_n$$

Also define: $\Psi_{\mathcal{S}} := \{\text{vec}(B^2) : B \in \mathcal{U}_{\mathcal{S}}\} \subseteq \mathcal{R}^{n^2}$ and $\Omega_{\mathcal{S}} = \text{conv}[\Psi_{\mathcal{S}}]$. Then the following result is proved in [All]:

Lemma 2.1 *[All]:*

(i) $\text{vec}(\bar{\mathcal{S}}_n^+) = \text{cone}[\Omega_{\mathcal{S}}]$ *with* $\text{vec}(\mathcal{S}_n^+) = \text{int cone}[\Omega_{\mathcal{S}}]$.

(ii) $\overline{\text{vec}}_{\mathcal{S}}(\bar{\mathcal{S}}_n^+) = \text{cone}[W'_{\mathcal{S}}\Omega_{\mathcal{S}}]$ *with* $\overline{\text{vec}}_{\mathcal{S}}(\mathcal{S}_n^+) = \text{int cone}[W'_{\mathcal{S}}\Omega_{\mathcal{S}}]$.

(iii) $\Psi_{\mathcal{S}}$ *is a compact set,* $\Omega_{\mathcal{S}}$ *is a non empty convex compact set with* $\text{dist}(0, \Omega_{\mathcal{S}}) = 1/\sqrt{n}$ *and* $\text{cone}[\Omega_{\mathcal{S}}]$ *is a nonempty closed convex cone.* □

We will also make use of the following result:

Lemma 2.2 *[HJ]: Let* m, n *be given positive integers. There is a unique matrix* $P(m,n) \in \mathcal{R}^{m\times n}$ *such that* $\text{vec}(X') = P(m,n)\text{vec}(X)$ *for all* $X \in \mathcal{R}^{m\times n}$. $P(m,n)$ *depends only on the dimensions* m *and* n *and is given by*

$$P(m,n) = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij} \otimes E'_{ij} = [E_{ij}]_{i=1,\dots,m}^{j=1,\dots,n}$$

where each $E_{ij} \in \mathcal{R}^{m\times n}$ *has entry* 1 *in position* (i,j) *and all other entries are zero. Moreover* $P(m,n)$ *is a permutation matrix and* $P(m,n) = P'(n,m) = P(n,m)^{-1}$. □

We conclude the section by giving the following definitions: A matrix $A \in \mathcal{R}^{n\times n}$ is said to be strongly stable if $A + A' \in \mathcal{S}_n^-$. The set of all strongly-stable matrices of dimension $n \times n$ is denoted by $\mathcal{D}_n$ and is a convex invertible cone (cic) in $\mathcal{R}^{n\times n}$. Given $A \in \mathcal{R}^{n\times n}$ we define the Lyapunov cone of A as the set $\mathcal{P}_A = \{P \in \mathcal{S}_n^+ : AP + PA' \in \mathcal{S}_n^-\}$. Lyapunov's stability theorem for LTI systems states that A is Hurwitz (i.e. $\text{Re}\lambda_i(A) < 0$ for all $i \in \underline{n}$) if and only if $\mathcal{P}_A$ is a non-empty set [B], [HP], [BS], [H]. It is straightforward to verify that $\mathcal{P}_A$ is also a convex invertible cone (cic) in $\mathcal{R}^{n\times n}$; further note that $A \in \mathcal{D}_n$ if and only if $I_n \in \mathcal{P}_A$.

3. Strong Stability: Definitions and basic results

We begin by giving the two standard definitions of Lyapunov and asymptotic stability of linear time-invariant systems [B], [K]:

Definition 3.1*: For the linear system* $\mathcal{S}(A)$ *in (1) we define:*

1. *$\mathcal{S}(A)$ is Lyapunov stable if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|x(t_0)\| < \delta(\epsilon)$ implies that $\|x(t)\| < \epsilon$ for all $t \geq t_0$.*

2. *$\mathcal{S}(A)$ is asymptotically stable if it is Lyapunov stable and $\delta(\epsilon)$ in part (1) of the definition can be selected so that $\|x(t)\| \to 0$ as $t \to \infty$.* □

For linear time-invariant systems a necessary and sufficient condition for asymptotic stability of $\mathcal{S}(A)$ is that A is Hurwitz; a necessary and sufficient condition for Lyapunov stability is that the spectrum of A lies in the closed left-half plane ($\bar{\mathcal{C}}_- = \mathrm{Re}(s) \leq 0$) and, in addition, any eigenvalue on the imaginary axis has simple structure (i.e. equal algebraic and geometric multiplicity) [B], [K], [HP]. Note that asymptotic stability is here taken to mean that the origin is the unique equilibrium point.

In the paper we use a refined version of stability which characterizes systems with non-overshooting behaviour, in the sense that the Euclidian norm of their state trajectory is a monotonically decreasing/non-increasing function of time for arbitrary initial conditions in the state-space. We refine this notion by introducing the following definitions (see [KHP2] for details):

Definition 3.2***:*** *For the LTI system $S(A)$ we define:*

1. *The system $\mathcal{S}(A)$ is strongly Lyapunov stable if $\|x(t)\| \leq \|x(t_0)\|$, $\forall t > t_0$ and $\forall x(t_0) \in \mathcal{R}^n$.*

2. *The system $\mathcal{S}(A)$ is strongly asymptotically stable w.s. (in the wide sense), if $\|x(t)\| < \|x(t_0)\|$, $\forall t > t_0$ and $\forall x(t_0) \neq 0$.*

3. *The system $\mathcal{S}(A)$ is strongly asymptotically stable s.s. (in the strict sense, or simply strongly asymptotically stable) if $\frac{d\|x(t)\|}{dt} < 0$, $\forall t \geq t_0$ and $\forall x(t_0) \neq 0$.* □

The three definitions of strong stability introduced above make precise the notion of a non-overshooting state-space response. Thus, strong Lyapunov stability does not allow state trajectories to exit (at any time $t > t_0$) the (closed) hyper-sphere with centre the origin and radius the norm of the state vector at time t_0, $r_0 = \|x(t_0)\|$ (although motion on the boundary of the sphere $\|x(t)\| = r_0$ is allowed, e.g. an oscillator's trajectory). Strong asymptotic stability (strict sense) requires that all state trajectories enter each hyper-sphere $\|x(t)\| = r \leq r_0$ from a non-tangential direction, whereas for systems which are strongly asymptotically stable (wide-sense), tangential entry is allowed. It is clear that strong Lyapunov stability implies Lyapunov stability and strong asymptotic stability (in either sense) implies asymptotic stability. Moreover, strong asymptotic stability (s.s.) implies strong asymptotic stability (w.s.) which in turn implies strong Lyapunov stability. For further discussion and concrete examples of each type of strong stability see [KHP1] and [KHP2].

The characterization of the properties of LTI systems for which we may have, or can avoid, overshoots is a property depending entirely on the state matrix A. Necessary and sufficient conditions for each type of strong stability are stated below:

Theorem 3.1 *[KHP2]: For the system $\mathcal{S}(A)$, the following properties hold true:*

(i) *$\mathcal{S}(A)$ is strongly asymptotically stable (s.s.) if and only if $A + A' < 0$.*

(ii) $\mathcal{S}(A)$ *is strongly asymptotically stable (w.s.) if and only if one of the following two equivalent conditions hold:*

(a) $A + A' \leq 0$ *and* A *is Hurwitz.*

(b) $A + A' \leq 0$ *and the pair* $(A, A + A')$ *is observable.*

(iii) $\mathcal{S}(A)$ *is strongly Lyapunov stable, if and only if* $A + A' \leq 0$. □

In the remaining parts of the paper we consider only strong asymptotic stability in the strict sense (s.s.), which in the sequel is simply referred to as "strong stability". Note that since the present work addresses problems which involve the calculation of the distance of a matrix from the strong stability condition, the precise notion of strong stability which is used is not really important and affects only the classification of an optimal solution as "infimising" or "minimising".

4. Additive perturbations: Distance to the cone of strongly stable matrices

A direct approach for formalizing the notion of "approximate strong-stability" is to let A be perturbed to $A + \Delta$ and minimise the Frobenious norm of Δ such that $A + \Delta$ is strongly stable. Formally we define:

$$\gamma_0 = \inf\{\|\Delta\|_F : A + A' + \Delta + \Delta' < 0\} \tag{2}$$

and

$$\hat{\gamma}_0 = \inf\{\|\Delta\|_F : A + A' + \Delta + \Delta' < 0, \Delta = \Delta'\} \tag{3}$$

An analytic solution to both problems is provided by the following Theorem:

Theorem 4.1***:*** *Problems (2) and (3) above have the unique, identical infimizing solution* $\Delta^o = -\frac{1}{2}[A + A']_+$ *and*

$$\gamma_0 = \hat{\gamma}_0 = \frac{1}{2}\sqrt{\sum_{i=1}^{n} [\max\{\lambda_i(A + A'), 0\}]^2}$$

Δ^o *is a minimising solution if and only if* $A + A' < 0$ *in which case* $\gamma_0 = \hat{\gamma}_0 = 0$ *and* $\Delta^o = 0$.

Proof: Follows easily via a spectral factorisation argument. The symmetric nature of the optimal solution of (2) follows from the observation that the constraint in (2) depends only on the symmetric part of Δ, while any skew-symmetric part of Δ would increase the norm above γ_0, since $\|\Delta\|_F^2 = \|\Delta_s\|_F^2 + \|\Delta_u\|_F^2$. □

Remark 4.1***:*** Set $X = A + \Delta$. Then (2) and (3) can be formulated, respectively, as the two distance problems: $\gamma_0 = \text{dist}(A, \mathcal{D}_n) = \inf_{X \in \mathcal{D}_n} \|A - X\|_F$ and $\hat{\gamma}_0 = \text{dist}(A, \hat{\mathcal{D}}_n) = \inf_{X \in \hat{\mathcal{D}}_n} \|A - X\|_F$ where $\hat{\mathcal{D}}_n = \mathcal{D}_n \cap \{X : X - A \in \mathcal{S}_n\}$. These have have the unique, identical infimizing solution $X = \frac{1}{2}[A + A']_- + \frac{1}{2}(A - A')$. Thus, the optimal solution is obtained by decomposing A to its symmetric (A_s) and skew-symmetric (A_u) parts, and adding the negative part of A_s (obtained via spectral decomposition) to A_u. □

Example* 4.1*: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \Rightarrow A_s = \frac{1}{2}(A + A') = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

The eigenvalues of A_s are $\{\lambda_1 = \frac{5+\sqrt{17}}{2}, \lambda_2 = \frac{5-\sqrt{17}}{2}, \lambda_3 = -2\}$ and hence A is not strongly stable. The nearest strongly-stable matrix (in the Frobenius-norm sense) is:

$$X_{opt} = \frac{1}{2}(A + A')_- + \frac{1}{2}(A - A') = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note that the symmetric part of X_{opt} has eigenvalues $\{-2, 0, 0\}$ and that $\|A - X_{opt}\|_F = \sqrt{21} = \sqrt{\lambda_1^2 + \lambda_2^2}$ in agreement with Theorem 4.1. □

Next, we examine the distance problem with the Frobenius norm replaced by the spectral norm. Specifically, given $A \in \mathcal{R}^{n \times n}$ we aim to solve:

$$\gamma_1 = \inf\{\|\Delta\| : A + A' + \Delta + \Delta' < 0\} \tag{4}$$

where $\|\Delta\| = \bar{\sigma}(\Delta)$. We start by considering a relaxed version of the problem by assuming that $\Delta = \Delta'$, i.e.

$$\hat{\gamma}_1 = \inf\{\|\Delta\| : A + A' + \Delta + \Delta' < 0, \Delta = \Delta'\} \tag{5}$$

Lemma 4.2 below provides a solution to the distance problem (5). Lemma 4.3 gives a parametrization of all solutions to problem (5). We first need the following technical result:

Lemma* 4.1*: *Let $\Lambda_+ = diag(\Lambda_+) > 0$ and $X = X' \geq 0$ with $\Lambda_+ \in \mathcal{R}^{n \times n}$ and $X \in \mathcal{R}^{n \times n}$. Then*

(i) $\lambda_{max}(\Lambda_+ + X) = \|\Lambda_+ + X\| \geq \lambda_{max}(\Lambda_+)$.

(ii) Further if $\Lambda_+ = diag(\lambda_1 I_r, \hat{\Lambda}_+)$ with $\|\hat{\Lambda}_+\| < \lambda_1$, then (i) is an equality if and only if $X = diag(0, Y)$ where $Y \in \mathcal{R}^{(n-r) \times (n-r)}$ such that $\|\hat{\Lambda}_+ + Y\| \leq \lambda_1$.

Proof: Straightforward and therefore omitted. □

Lemma* 4.2*: *(i) If $A + A' \leq 0$, then $\hat{\gamma}_1 = 0$ and $\Delta^0 = 0$ is the unique optimal (infimising) solution of problem (5) (minimising if $A + A' < 0$). (ii) If at least one eigenvalue of $A + A'$ is positive, then the optimal distance in (5) is given by $\hat{\gamma}_1 = \frac{1}{2}\lambda_{max}(A + A')$. Further, one optimal solution in this case is $\Delta^o = -\frac{1}{2}[A + A']_+$.*

Proof: (i) Follows immediately since in this case $\Delta = 0$ is feasible or lies on the closure of the feasible set. (ii) Assume that the largest eigenvalue of $A + A'$ is positive and let $\frac{1}{2}(A + A')$ have a spectral decomposition $\frac{1}{2}(A + A') = U\Lambda U' = U_1\Lambda_+ U_1' + U_2\Lambda_{-0}U_2'$ with $\lambda_i(\Lambda_+) > 0$ and $\lambda_i(\Lambda_{-0}) \leq 0$. Set $\rho = \dim(\Lambda_+)$. Then, the distance problem is equivalent to:

$$\hat{\gamma}_1 = \inf\{\|\Delta\| : A + A' + \Delta + \Delta' < 0, \Delta = \Delta'\} = \inf\{\|\hat{\Delta}\| : \Lambda + \hat{\Delta} < 0, \hat{\Delta} = \hat{\Delta}'\} \tag{6}$$

where we have defined $\hat{\Delta} = U'\Delta U$, using the fact that the spectral norm is unitarily invariant and noting that the transformation $\Delta \to \hat{\Delta}$ is a bijection in $\mathcal{S}_n$. We claim that $\hat{\Delta}^o = -\text{diag}(\Lambda_+, 0)$ is an infimiser of the optimisation problem defined in equation (6), so that $U\hat{\Delta}^o U'$ is an infimiser of the original problem. Assume for contradiction that $\tilde{\Delta} = \Delta - \text{diag}(\Lambda_+, 0)$, $\Delta = (\Delta_{ij})_{i,j\in\{1,2\}}$ with $\Delta_{11} \in \mathcal{R}^{\rho\times\rho}$, is an infimising solution which satisfies $\|\tilde{\Delta}\| < \|\hat{\Delta}\| = \lambda_{max}(\Lambda_+)$. Since $\tilde{\Delta}$ lies inside the feasible set or on its closure, $\Lambda + \tilde{\Delta} \in \bar{\mathcal{S}}_n^-$ or, $\Delta + \text{diag}(0_{\rho\times\rho}, \Lambda_{-0}) \leq 0$ and in particular that $\Delta_{11} \leq 0$ and $\Delta_{22} \leq -\Lambda_{-0}$. Now $\|\tilde{\Delta}\| \geq \|\Lambda_+ - \Delta_{11}\| \geq \lambda_{max}(\Lambda_+)$ using Lemma 4.1 part (i) which is a contradiction and concludes the proof. □

The following Lemma gives a complete parametrisation of all solutions to the distance problem (5).

Lemma 4.3***:*** *Assume that $A + A'$ has at least one positive eigenvalue. Let $\frac{1}{2}(A + A')$ have a spectral factorization $\frac{1}{2}(A + A') = U\Lambda U' = U\,diag(\lambda_1 I_r, \hat{\Lambda}_+, \Lambda_{-0})U'$ with $\Lambda = diag(\Lambda)$, $UU' = U'U = I_n$, $0 < \lambda_i(\hat{\Lambda}_+) < \lambda_1$ for $i = 1, 2, \ldots, \pi_1 - r$ where $\pi_1 := \pi(A+A')$ and $\lambda_i(\Lambda_{-0}) \leq 0$ for $i = 1, 2, \ldots, \nu_1 + \delta_1$ where $\nu_1 := \nu(A + A')$ and $\delta_1 := \delta(A + A')$. Then:*

(i) The optimal distance in (5) is given by $\hat{\gamma}_1 = \frac{1}{2}\lambda_{max}(A + A') = \lambda_1$.

(ii) All optimal (infimising) solutions of (5) are given as $\Delta^o = U\,diag(-\lambda_1 I_r, \Delta)U'$ where $\Delta = \Delta'$, $diag(\hat{\Lambda}_+, \Lambda_{-0}) + \Delta \leq 0$ and $\|\Delta\| \leq \lambda_1$.

Proof: Using the spectral decomposition of $\frac{1}{2}(A + A')$ and following the steps of the first part of the proof of Lemma 4.2, shows that the optimisation problem is equivalent to: $\hat{\gamma}_1 = \inf\{\|\hat{\Delta}\| : \hat{\Delta} = \hat{\Delta}', \Lambda + \hat{\Delta} < 0\}$ where $\hat{\Delta} = U'\Delta U$. Let $\hat{\Delta} = \hat{\Delta}' = (\Delta_{ij})_{i,j\in\{1,2,3\}}$ be an arbitrary infimising solution with $\Delta_{11} \in \mathcal{R}^{r\times r}$, $\Delta_{22} \in \mathcal{R}^{(\pi_1-r)\times(\pi_1-r)}$ and $\Delta_{33} \in \mathcal{R}^{(\nu_1+\delta_1)\times(\nu_1+\delta_1)}$. Since $\hat{\Delta}$ is an infimising solution it must lie inside or on the closure of the feasible set, i.e. $\hat{\Delta} + \text{diag}(\lambda_1 I_r, \hat{\Lambda}_+, \Lambda_{-0}) \leq 0$ (which implies that $\lambda_i(\Delta_{11}) \leq -\lambda_1$ for all $i = 1, 2, \ldots, r$ and hence $\rho(\Delta_{11}) \geq \lambda_1$) and from Lemma 4.2 must have norm $\|\hat{\Delta}\| \leq \lambda_1$ (which implies that $\|\Delta_{11}\| \leq \lambda_1$). Since $\Delta_{11} = \Delta_{11}'$ it follows that $\lambda_i(\Delta_{11}) = -\lambda_1$ for all $i = 1, 2, \ldots, r$ and hence $\Delta_{11} = -\lambda_1 I_r$. Thus $\|\Delta_{11}\| = \|\hat{\Delta}\| = \lambda_1$ and hence $\Delta_{12} = 0$ and $\Delta_{13} = 0$ from which the parametrisation of part (ii) follows. □

The following Theorem gives a complete parametrisation to the optimal solutions of (4) and (5). Note that there is always an matrix which optimizes (5) in the set of optimal solutions of (4).

Theorem 4.2***:***

(i) If $A + A' \leq 0$ then $\gamma_1 = \hat{\gamma}_1 = 0$ and $\Delta = 0$ is the unique infimising solution for both problems (4) and (5) (minimising solution if $A + A' < 0$).

(ii) If $A + A'$ has at least one positive eigenvalue, then $\gamma_1 = \hat{\gamma}_1 = \frac{1}{2}\lambda_{max}(A + A')$ and the set of all optimal solutions of (5) described by Lemma 4.3 forms a subset of the set of all optimal solutions of problem (4). Further if Δ is an infimising solution of problem (4), then $\frac{1}{2}(\Delta + \Delta')$ is also an infimising solution of problem (4).

(iii) Let $\frac{1}{2}(A + A')$ have a spectral factorization: $\frac{1}{2}(A + A') = U\Lambda U' = U\,diag(\lambda_1 I_r, \hat{\Lambda}_+, \Lambda_{-0})U'$ with $\Lambda = diag(\Lambda)$, $UU' = U'U = I_n$, $0 < \lambda_i(\hat{\Lambda}_+) < \lambda_1$ for $i = 1, 2, \ldots, \pi_1 - r$, ($\pi_1 := \pi(A + A')$) and $\lambda_i(\Lambda_{-0}) \leq 0$ for $i = 1, 2, \ldots, \nu_1 + \delta_1$ where $\nu_1 := \nu(A + A')$ and $\delta_1 := \delta(A + A')$. Then all

optimal (infimising) solutions of problem (4) are given as $\Delta^o = U diag(-\lambda_1 I_r, D+E)U'$ *where* $D = D'$ *and* $E = -E'$ *satisfy* $diag(\hat{\Lambda}_+, \Lambda_{-0}) + D < 0$ *and* $\|D+E\| \leq \lambda_1$.

Proof: (i) If $A + A' \leq 0$, $\Delta = 0$ is feasible (or lies on the closure of the feasible set) and hence is the unique infimising solution for both problems; hence in this case $\gamma_1 = \hat{\gamma}_1 = 0$. (ii) Note that since the constraint set of problem (5) is a subset of the constraint set of problem (4), we have that $\gamma_1 \leq \hat{\gamma}_1$. Note also that if $\Delta = \Delta_s + \Delta_u$ with $\Delta_s \in \mathcal{S}_n$ and $\Delta_u \in \mathcal{A}_n$, we have

$$\begin{aligned}\|\Delta\| = \|\Delta_s + \Delta_u\| &= \max\{|x'\Delta y| : x \in \mathcal{R}^n, y \in \mathcal{R}^n, \|x\| = \|y\| = 1\} \\ &\geq \max\{|x'(\Delta_s + \Delta_u)x| : x \in \mathcal{R}^n, \|x\| = 1\} \\ &= \max\{|x'\Delta_s x| : x \in \mathcal{R}^n, \|x\| = 1\} \quad (\text{since } \ x'\Delta_u x = 0) \\ &= \rho(\Delta_s) = \|\Delta_s\| \quad (\text{since } \ \Delta_s = \Delta_s')\end{aligned}$$

Hence,

$$\begin{aligned}\gamma_1 &= \inf\{\|\Delta\| : A + A' + \Delta + \Delta' < 0\} \\ &= \inf\{\|\Delta_s + \Delta_u\| : A + A' + \Delta_s + \Delta_s' < 0, \Delta_s = \Delta_s', \Delta_u = -\Delta_u'\} \\ &\geq \inf\{\|\Delta_s\| : A + A' + \Delta_s + \Delta_s' < 0, \Delta_s = \Delta_s'\} \\ &= \hat{\gamma}_1\end{aligned}$$

We conclude that $\gamma_1 = \hat{\gamma}_1$. Further, since all optimal (infimising) solutions of problem (5) lie on the closure of the feasible set of problem (4), they are also infimising solutions of (4). Finally, let Δ be an infimising solution of problem (4). Decompose Δ as $\Delta = \Delta_s + \Delta_u$ with $\Delta_s \in \mathcal{S}_n$ and $\Delta_u \in \mathcal{A}_n$. Suppose for contradiction that Δ_s is not an infimising solution of problem (4). Then $\|\Delta\| = \gamma_1$ and, since $A + A' + 2\Delta_s \leq 0$, we must have that $\|\Delta_s\| > \gamma_1$ if Δ_s is not an infimiser. In this case, however $\gamma_1 < \|\Delta_s\| \leq \|\Delta\| = \gamma_1$, which is a contradiction. (iii) Using similar arguments with the first steps of the proof of Lemma 4.2 we conclude that

$$\gamma_1 = \inf\{\|\hat{\Delta}\| : \Lambda + \hat{\Delta}_s \leq 0, \hat{\Delta} = \hat{\Delta}_s + \hat{\Delta}_u, \hat{\Delta}_s = \hat{\Delta}_s', \hat{\Delta}_u = -\hat{\Delta}_u'\} \tag{7}$$

and all optimal Δ^o are given as $\Delta^o = U(\hat{\Delta}_s^o + \hat{\Delta}_u^o)U'$, where $\hat{\Delta}^o = \hat{\Delta}_s^o + \hat{\Delta}_u^o$ are the infimising solutions of (7). All symmetric infimisers $\Delta_s^o = U\hat{\Delta}_s^o U'$ are parametrised in Lemma 4.3 part (ii), and part (ii) of this Theorem shows that all infimisers are obtained by perturbing the symmetric minimisers Δ_s^o by a skew-symmetric part Δ_u^o so that the norm is unaffected, i.e. $\|\Delta_s^o + \Delta_u^o\| = \|\Delta_s^o\| = \gamma_1$. Hence all optimal Δ^o are of the form $U'\Delta^o U = \text{diag}(-\lambda_1 I_r, D) + E$ where $D \in \mathcal{S}_{n-r}$, $E \in \mathcal{A}_n$ such that $\text{diag}(\hat{\Lambda}_+, \Lambda_{-0}) + D < 0$ and $\|\text{diag}(-\lambda_1 I_r, D) + E\| = \lambda_1$. When $E = (E_{ij})_{i,j\in\{1,2\}}$ is partitioned conformally with $\text{diag}(-\lambda_1 I_r, D)$, the last equation implies that

$$\|E_{11} - \lambda_1 I_r\| \leq \lambda_1 \Rightarrow (E_{11} - \lambda_1 I_r)(E_{11}' - \lambda_1 I_r) \leq \lambda_1^2 I_r \Rightarrow E_{11}E_{11}' - \lambda_1(E_{11} + E_{11}') + \lambda_1^2 I_r \leq \lambda_1^2 I_r$$

and hence $E_{11} = 0$ since $E_{11} + E_{11}' = 0$. Similarly,

$$\left\| \begin{pmatrix} -\lambda_1 I_r & E_{12} \end{pmatrix} \right\| \leq \lambda_1 \Rightarrow \lambda_1^2 I_r + E_{12}E_{12}' \leq \lambda_1^2 I_r$$

and hence $E_{12} = 0$ from which the parametrisation of all optimal Δ^o follows. □

Remark 4.2 ***(Field of values):*** It is possible to give a geometric interpretation to the (spectral-norm) distance problem discussed in this section via the *field of values* of a matrix. Recall that for $A \in \mathcal{R}^{n\times n}$ the field of values of A is defined as the set $F(A) = \{x^*Ax : x \in \mathcal{C}^n, x^*x = 1\}$. $F(A)$ is a compact convex subset of the complex plane which contains the convex hull of the spectrum of A; in particular, if A is normal $F(A) = \text{co}(\lambda(A))$ [HJ]. Two useful properties of the field of values are: (i) the "shift property", i.e. $F(A + \alpha I_n) = \alpha + F(A)$, and (ii) the "projection property", i.e. $\text{Re}(F(A)) = F(A_s)$ where A_s denotes the symmetric part of A, $A_s = \frac{1}{2}(A + A') \in \mathcal{S}_n$ [HJ]. It can also be easily shown that $F(A) \subset \mathcal{C}_-$ if and only if $A \in \mathcal{D}_n$ (i.e. $A + A' < 0$), which defines a geometric necessary and sufficient condition for strong stability; hence:

$$\begin{aligned}\gamma_1 = \inf\{\|\Delta\| : A + \Delta \in \mathcal{D}_n\} &= \inf\{\|\Delta\| : \Delta \in \mathcal{R}^{n\times n},\ F(A+\Delta) \subset \mathcal{C}_-\} \\ \leq \min{}^+\{\alpha : F(A + \alpha I_n) \subset \overline{\mathcal{C}}_-\} &= \min{}^+\{\alpha : \alpha + F(A) \subset \overline{\mathcal{C}}_-\} := \tilde{\gamma}_1\end{aligned}$$

Assuming that $F(A)$ is not contained in $\overline{\mathcal{C}}_-$, $\tilde{\gamma}_1$ geometrically represents the minimum amount α that $F(A)$ must be shifted to the left (i.e. in the negative real-axis direction) so that it is contained entirely within $\overline{\mathcal{C}}_-$, i.e.

$$\tilde{\gamma}_1 = \max{}^+\{\text{Re}(z) : z \in F(A)\} \tag{8}$$

Since A is assumed real, $F(A)$ is symmetric (with respect to the real axis) i.e. $z \in F(A) \Leftrightarrow \bar{z} \in F(A)$. This follows from the equivalences $z = x^*Ax \Leftrightarrow \bar{z} = x^tA\bar{x}$ and $\|x\| = 1 \Leftrightarrow \|\bar{x}\| = 1$. The fact that $F(A)$ is symmetric (with respect to the real axis) and convex implies that maximum in (8) is attained on the real axis (since $z_0 \in F(A) \Rightarrow \bar{z}_0 \in F(A) \Rightarrow \text{Re}(z_0) = \frac{1}{2}(z_0 + \bar{z}_0) \in F(A)$). Now,

$$\lambda(A) \subseteq F(A) \Rightarrow \text{Re}(\lambda(A)) \subseteq \text{Re}(F(A)) = F(A_s)$$

using the "projection" property of $F(A)$. Note that $F(A_s)$ is the closed interval

$$F(A_s) = \left[\frac{1}{2}\lambda_n(A + A'),\ \ \frac{1}{2}\lambda_1(A + A')\right]$$

which lies on the real axis of the complex plane. Thus

$$\tilde{\gamma}_1 = \max{}^+\{x : x \in \text{Re}(F(A))\} = \max{}^+\{x : x \in F(A_s)\} = \max\left(\frac{1}{2}\lambda_1(A + A'), 0\right) \tag{9}$$

Note that this is actually equal to γ_1 (see Theorem 4.2 (ii)) and hence the inequality $\gamma_1 \leq \tilde{\gamma}_1$ is actually an equality. For connections between the field of values and stability questions in numerical analysis see [S2]. □

5. Multiplicative perturbations: Distance to Lyapunov cone

Recall that $\mathcal{D}_n$ denote the set of all strongly stable matrices in $\mathcal{R}^{n\times n}$ and assume that $A \in \mathcal{R}^{n\times n}$ is a Hurwitz (but not necessarily strongly stable) matrix. It is well known that in this case the Lyapunov inequality

$$AP + PA' < 0 \tag{10}$$

has a positive-definite solution $P = P' > 0$. Denote the set of all solutions to the Lyapunov inequality by $\mathcal{P}_A$. It can be easily shown that $\mathcal{P}_A$ is a convex invertible cone [CL], [L], [H]. Now, if A is not strongly stable, we have $I_n \notin \mathcal{P}_A$ and we can define:

$$\gamma_2 = \text{dist}(A, \mathcal{D}_n) = \inf_{P\in\mathcal{P}_A} \|I_n - P\|_F \tag{11}$$

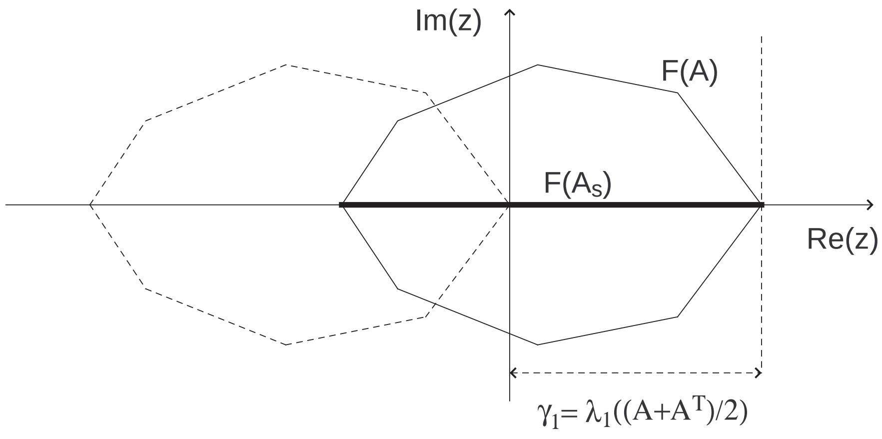

Figure 1: Field of values

Note that if $I_n \in \mathcal{P}_A$ we have $\mathrm{dist}(A, \mathcal{D}_n) = 0$.

Remark 5.1***:*** Let P_o be an infimising solution of (11) and set $\Delta_o = P_o - I_n$. Since $P_o \in \overline{\mathcal{P}}_A$ we have $AP_o + P_o A' \leq 0$. Since $\Delta_o = \Delta_o'$, this may be written as

$$A(I + \Delta_o) + (I + \Delta_o')A' = A(I + \Delta_o) + (I + \Delta_o)A' \leq 0$$

and hence the problem defined in (11) is equivalent to:

$$\gamma_2 = \inf\{\|\Delta\|_F : I_n + \Delta \in \mathcal{P}_A\} = \inf\{\|\Delta\|_F : \Delta = \Delta', A(I + \Delta) \in \mathcal{D}_n\}$$

This can be interpreted as the problem of finding the minimum-norm symmetric (right) multiplicative perturbation of A, such that the perturbed matrix is strongly stable. □

To compute this distance numerically, vectorize equation (10) to get:

$$-(I_n \otimes A + A \otimes I_n)\mathrm{vec}(\mathcal{P}_A) = \mathrm{vec}(\mathcal{S}_n^+)$$

Defining $\Phi_A = -(I_n \otimes A + A \otimes I_n)$, this can be written as:

$$\mathcal{P}_A = \mathrm{vec}^{-1}\left[\Phi_A^{-1}\mathrm{vec}(\mathcal{S}_n^+)\right]$$

Thus, $p \in \mathrm{vec}(\mathcal{P}_A)$ if and only if $p = \Phi_A^{-1} q$ for a vector $q \in \mathrm{vec}(\mathcal{S}_n^+)$ and hence:

$$\gamma_2 = \mathrm{dist}(A, \mathcal{D}_n) = \inf_{Q=Q'>0} \|I_n - \mathrm{vec}^{-1}\left[\Phi_A^{-1}\mathrm{vec}(Q)\right]\|_F = \inf_{Q=Q'>0} \|\mathrm{vec}(I_n) - \Phi_A^{-1}\mathrm{vec}(Q)\|$$

where $\|\cdot\|$ denotes the Euclidean norm. Using the relationship $\mathrm{vec}(Q) = W_{\mathcal{S}}\overline{\mathrm{vec}}_{\mathcal{S}}(Q)$, this can be written in the more "compact" form as:

$$\gamma_2 = \mathrm{dist}(A, \mathcal{D}_n) = \inf_{Q=Q'>0} \|\mathrm{vec}(I_n) - \Phi_A^{-1} W_{\mathcal{S}}\overline{\mathrm{vec}}_{\mathcal{S}}(Q)\|$$

The following Lemma shows that $\mathrm{dist}(A, \mathcal{D}_n)$ is well defined for Hurwitz matrices in the sense that Φ_A is invertible:

Lemma 5.1***:*** *If A is Hurwitz, Φ_A is invertible.*

Proof: The eigenvalues of Φ_A are given by the n^2 numbers $\{\lambda_i(A)+\lambda_j(A), i,j=1,2,\dots,n\}$, where $\{\lambda_i, i=1,2,\dots,n\}$ are the eigenvalues of A [HJ]. These have all negative real parts if the eigenvalues of A have all negative real parts. □

Remark 5.2***:*** (i) An alternative way of seeing that Φ_A is invertible when A is Hurwitz the Sylvester equation $AP-PB=0$; this has a nonzero solution P if and only if A and B have a common eigenvalue [HJ], which is impossible if $B=-A'$. (ii)Suppose that the Hurwitz matrix $A\in\mathcal{R}^{n\times n}$ has n linearly independent eigenvectors and hence is diagonalisable, i.e. $A=\sum_{i=1}^n \lambda_i w_i v_i'$ where $\{w_i\}$ and $\{v_i'\}$, $i=1,2,\dots,n$, denote the right and left eigenvectors of A, respectively. Now,

$$(Aw_i)\otimes w_j=(\lambda_i w_i)\otimes w_j \Rightarrow (A\otimes I_n)(w_i\otimes w_j)=\lambda_i(w_i\otimes w_j)$$

Similarly,

$$w_i\otimes(Aw_j)=w_i\otimes(\lambda_j w_j)\Rightarrow (I_n\otimes A)(w_i\otimes w_j)=\lambda_j(w_i\otimes w_j)$$

Adding the two equations above gives

$$(A\otimes I_n+I_n\otimes A)(w_i\otimes w_j)=(\lambda_i+\lambda_j)(w_i\otimes w_j)$$

and hence the n^2 vectors $\{w_i\otimes w_j\}$, $i=1,2,\dots,n$, $j=1,2,\dots,n$ are right eigenvectors of Φ_A. A similar argument shows that:

$$(v_i'\otimes v_j')(A\otimes I_n+I_n\otimes A)=(\lambda_i+\lambda_j)(v_i'\otimes v_j')$$

In fact, under the assumption that A is diagonalisable, Φ_A has only linear elementary divisors and the n^2 vectors $\{w_i\otimes w_j\}$ are linearly independent; thus, in this case, Φ_A has a spectral decomposition:

$$\Phi_A=\sum_{i=1}^n\sum_{j=1}^n(\lambda_i+\lambda_j)(w_i\otimes w_j)(v_i\otimes v_j)'$$

Since $\lambda_i+\lambda_j\neq 0$ for every pair (i,j) if A is stable,

$$\Phi_A^{-1}=\sum_{i=1}^n\sum_{j=1}^n\frac{1}{\lambda_i+\lambda_j}(w_i\otimes w_j)(v_i\otimes v_j)'$$

is an eigenvalue-eigenvector decomposition of Φ_A^{-1}. In the case when A has a Jordan form with m Jordan blocks of size p_i, $i=1,2,\dots,m$, Φ_A has Jordan blocks of size:

$$\{p_i+p_j-1,\ p_i+p_j-3,\dots,|p_i-p_j|+1\};\ \ i=1,2,\dots,m\ \text{ and }\ j=1,2,\dots,m$$

(see [G]). In this case an explicit expression for Φ_A^{-1} has a much more complex form. □

The following Theorem shows that the calculation of γ can be performed by calculating the distance of a vector to a convex cone. A concrete algorithm for this purpose is given later in this section.

Theorem 5.1***:*** *The distance problem defined in (11) is equivalent to*

$$\gamma_2=\inf_{k\in\mathcal{K}}\|f_1-k\| \tag{12}$$

where $[f_1\ f_2]' := U'vec(I_n)$ *with* $f_1 \in \mathcal{R}^r$ *and* $f_2 \in \mathcal{R}^{n^2-r}$*;* $\mathcal{K} := cone(LW'_{\mathcal{S}}\Omega_{\mathcal{S}})$ *in which* $L \in \mathcal{R}^{n^2\times r}$ *is defined by the factorization*

$$\Phi_A^{-1}W_{\mathcal{S}} = U\begin{pmatrix} L \\ 0 \end{pmatrix} \tag{13}$$

and $U \in \mathcal{R}^{n^2\times n^2}$ *orthogonal. Then: (i) L is non-singular; (ii)* $\mathcal{K}$ *is a convex cone, and (iii)* $f_2 = 0$ *. Further, if* $\hat{k}$ *denotes the (unique) infimiser of (12), then*

$$\hat{P}_o = vec^{-1}(\Phi^{-1}W_{\mathcal{S}}L^{-1}\hat{k}) \tag{14}$$

is the unique infimizer of (11) such that $A\hat{P}_o + \hat{P}_oA' \leq 0$ *and* $A\hat{P}_o + \hat{P}_oA'$ *is singular, unless* $\overline{vec}^{-1}(L^{-1}\hat{k}) \geq 0$*, in which case the infimum in (11) is uniquely attained by* $\hat{P}_o = I_n$ *and* $\gamma_2 = 0$*.*

Proof: Factorisation (13) can be easily performed (e.g. via QR or singular value decomposition). The form of the right factor and the fact that L is nonsingular follows immediately from the fact that $\text{rank}(W_{\mathcal{S}}) = r$. Thus,

$$\gamma_2 = \inf_{Q\in\mathcal{S}_n^+}\left\|\text{vec}(I_n) - U\begin{pmatrix} L \\ 0 \end{pmatrix}\overline{\text{vec}}_{\mathcal{S}}(Q)\right\| = \inf_{Q\in\mathcal{S}_n^+}\left\|\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \begin{pmatrix} L \\ 0 \end{pmatrix}\overline{\text{vec}}_{\mathcal{S}}(Q)\right\|$$

since the Euclidean norm is unitarily invariant. Thus,

$$\gamma_2 = \sqrt{\|f_2\|^2 + \inf_{Q\in\mathcal{S}_n^+}\|f_1 - L\overline{\text{vec}}_{\mathcal{S}}(Q)\|^2} = \sqrt{\|f_2\|^2 + \inf_{k\in\mathcal{K}}\|f_1 - k\|^2}$$

using Lemma 2.1(ii) and noting that $\mathcal{K} := \text{cone}(LW'_{\mathcal{S}}\Omega_{\mathcal{S}})$ is convex. To show that $f_2 = 0$, consider the linear map $\mathcal{S}_n \to \mathcal{S}_n : P \to AP + PA'$ when A is a fixed Hurwitz matrix. Since the Lyapunov equation $AP + PA' = Q$ has a unique (symmetric) solution P for every symmetric matrix Q [B], the map defined above is bijective and hence its inverse is well-defined. In vector form this inverse map can be represented as $\text{vec}(\mathcal{S}_n) \to \text{vec}(\mathcal{S}_n) : p = -\Phi_A^{-1}q$, where $p = \text{vec}(P)$ and $q = \text{vec}(Q)$, or equivalently as $\mathcal{R}^r \to \text{vec}(\mathcal{S}_n) : t \to p = -\Phi_A^{-1}W_{\mathcal{S}}t$. Thus, the columns of matrix $\Phi_A^{-1}W_{\mathcal{S}}$ form a basis of the (r-dimensional) subspace $\text{vec}(\mathcal{S}_n)$. Consider the indicated factorisation of $\Phi_A^{-1}W_{\mathcal{S}}$, and partition $U = (U_1\ U_2)$ where $U_1 \in \mathcal{R}^{n^2\times r}$ and $U_2 \in \mathcal{R}^{n^2\times(n^2-r)}$. It is clear that the columns of U_1 form an orthonormal basis of the subspace $\text{vec}(\mathcal{S}_n)$ while the columns of U_2 form an orthonormal basis of $(\text{vec}(\mathcal{S}_n))^\perp = \text{vec}(\mathcal{A}_n)$. Thus $U_2'f = 0$ for every vector $f \in \text{vec}(\mathcal{S}_n)$. In particular $f_2 = U_2'\text{vec}(I_n) = 0$ from which the result follows. □

Thus, the problem of computing γ_2 reduces to the calculation of the distance of a fixed vector from (the interior of) a convex cone, i.e. $\inf_{k\in\mathcal{K}}\|f_1 - k\|$, where $\mathcal{K} = \text{cone}(\Gamma)$, $\Gamma = LW'_{\mathcal{S}}\Omega_{\mathcal{S}}$. This can be solved numerically by an iterative algorithm given in [All] which is guaranteed to converge in a finite number of steps for any pre-specified tolerance ϵ.

Remark 5.3***:*** Distance problems to strong stability of a Hurwitz matrix A subject to symmetric left perturbations can formulated as:

$$\hat{\gamma}_2 = \inf_{\hat{P}\in\hat{\mathcal{P}}_A}\|I_n - \hat{P}\|_F \tag{15}$$

where $\hat{\mathcal{P}}_A$ denotes the dual Lyapunov cone $\hat{\mathcal{P}}_A = \{\hat{P} : \hat{P}A + A'\hat{P} < 0\}$ to $\mathcal{P}_A$ defined in (11). Again, let $\hat{P}_o$ be an infimising solution of (15) and set $\Delta_o = \hat{P}_o - I_n$. Since $\hat{P}_o \in \overline{\hat{\mathcal{P}}_A}$ we have $\hat{P}_oA + A'\hat{P}_o \leq 0$.

Since $\Delta_o = \Delta_o'$, this may be written as $(I+\Delta_o)A + A'(I+\Delta_o') \leq 0$ and we can formulate the problem defined in (15) as:

$$\hat{\gamma}_2 = \inf\{\|\Delta\|_F : I_n + \Delta \in \hat{\mathcal{P}}_A\} = \inf\{\|\Delta\|_F : \Delta = \Delta', (I+\Delta)A \in \mathcal{D}_n\}$$

This can be interpreted as the problem of finding the minimum-norm symmetric (left) multiplicative perturbation of A, such that the perturbed matrix is strongly stable. Note that $I_n \in \mathcal{P}_A$ if and only if $I_n \in \hat{\mathcal{P}}_A$ and that

$$\hat{\mathcal{P}}_A = \text{vec}^{-1}\left[\hat{\Phi}_A^{-1}\text{vec}(\mathcal{S}_n^+)\right]$$

where $\hat{\Phi}_A = -(I_n \otimes A' + A' \otimes I_n)$. Thus Theorem 5.1 (and the corresponding Algorithm) may be applied to calculate $\hat{\gamma}_3$ with only minor modifications (essentially replacing Φ_A by $\hat{\Phi}_A$). □

We can establish the following relations between the cones $\mathcal{P}_A$ and $\hat{\mathcal{P}}_A$:

Lemma 5.2***:*** *Let $A \in \mathcal{R}^{n\times n}$ be Hurwitz. Then: (i) $\mathcal{P}_A = \mathcal{P}_{A^{-1}}$; (ii) $\hat{\mathcal{P}}_A = \hat{\mathcal{P}}_{A^{-1}}$; (iii) $\mathcal{P}_{A'} = \hat{\mathcal{P}}_A$; (iv) $\mathcal{P}_A = \hat{\mathcal{P}}_{A'}$; (v) $\mathcal{P}_A = \hat{\mathcal{P}}_{(-1)^n adj(A)}$.*

Proof: (i) By definition, $\mathcal{P}_A = \{P : AP + PA' < 0\}$. Now if $P \in \mathcal{P}_A$, by Sylvester's law of inertia we have $A^{-1}(AP + PA')(A^{-1})' < 0$, or equivalently $P(A^{-1})' + A^{-1}P < 0$ and hence $P \in \hat{\mathcal{P}}_{A^{-1}}$, so that $\mathcal{P}_A \subseteq \hat{\mathcal{P}}_{A^{-1}}$. A dual argument shows that $\hat{\mathcal{P}}_{A^{-1}} \subseteq \mathcal{P}_A$ and hence $\hat{\mathcal{P}}_{A^{-1}} = \mathcal{P}_A$. (ii) Follows similarly to (i). Parts (iii) and (iv) are immediate from the definitions of the cones $\mathcal{P}_A$ and $\hat{\mathcal{P}}_A$. (v) Writing $A^{-1} = (\det(A))^{-1}(\text{adj}(A))'$ and noting that $\text{sign}(\det(A)) = \text{sign}(\prod_{i=1}^n \lambda_i(A)) = (-1)^n$, we conclude that $(-1)^n\text{adj}(A)$ is Hurwitz and hence the cone $\hat{\mathcal{P}}_{(-1)^n\text{adj}(A)}$ is non-empty. Hence, defining $\lambda = (-1)^n\det(A)$, we have that

$$\hat{\mathcal{P}}_{(-1)^n\text{adj}(A)} = \mathcal{P}_{(-1)^n\text{adj}(A)'} = \mathcal{P}_{(-1)^n\det(A)A^{-1}} = \mathcal{P}_{\lambda A^{-1}} = \mathcal{P}_{A^{-1}} = \mathcal{P}_A$$

using (i), (iii) and the fact that $\mathcal{P}_{\lambda A^{-1}} = \mathcal{P}_{A^{-1}}$ since $\mathcal{P}_{A^{-1}}$ is a cone and $\lambda > 0$. □

Corollary 5.1***:*** *Let $A \in \mathcal{R}^{2\times 2}$ be Hurwitz. Suppose that P^o and $\hat{P}^o$ be the (unique) infimisers of problems defined in equations (11) and (15) respectively. Then,*

$$P^o = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} \Leftrightarrow \hat{P}^o = \begin{pmatrix} p_3 & -p_2 \\ -p_2 & p_1 \end{pmatrix}$$

Further, $\gamma_2 = \hat{\gamma}_2$.

Proof: The two optimal solutions P^o and $\hat{P}^o$ are the projections of I_2 onto the closure of the cones $\mathcal{P}_A$ and $\hat{\mathcal{P}}_A$, respectively, denoted in the sequel as $P^o = \Pi_{\mathcal{P}_A}(I_2)$ and $\hat{P}^o = \Pi_{\hat{\mathcal{P}}_A}(I_2)$, respectively. Define

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and note that $J' = -J$ and $J'J = I_2$. Note also that for any $A \in \mathcal{R}^{2\times 2}$, $\text{adj}(A) = JAJ$. Using Lemma 5.2(v), $\hat{P}^o = \Pi_{\hat{\mathcal{P}}_A}(I_2) = \Pi_{\mathcal{P}_{\text{adj}(A)}}(I_2) = \Pi_{\mathcal{P}_{JAJ}}(I_2)$. Now,

$$\mathcal{P}_{JAJ} = \{P : (JAJ)P + P(JA'J) < 0\} = \{P : A(JPJ) + (JPJ)A' < 0\} = J\mathcal{P}_A J$$

Hence,

$$\hat{P}^o = \Pi_{J\mathcal{P}_A J}(I_2) = J\Pi_{\mathcal{P}_A}(I_2)J = JP^oJ$$

from which the result follows. Finally,

$$\hat{\gamma}_2 = \|I_2 - \hat{P}^o\|_F = \|I_2 - JP^oJ\|_F = \|J(I_2 - P^o)J\|_F = \|I_2 - P^o\|_F = \gamma_2$$

since the Frobenius norm is unitarily invariant. □

Next, we consider distance problems involving both symmetric and non-symmetric multiplicative perturbations of A expressed in terms of the spectral norm. Specifically we define the two distance problems

$$\hat{\gamma}_3 = \inf\{\|\Delta\| : A(I+\Delta) + (I+\Delta')A' < 0, \Delta = \Delta'\} \tag{16}$$

with symmetry constraints, and

$$\gamma_3 = \inf\{\|\Delta\| : A(I+\Delta) + (I+\Delta')A' < 0\} \tag{17}$$

without symmetry constraints. Note that the infimum in both problems can be easily computed via Linear Matrix Inequality (LMI) techniques [SW], [SIG]. The following Theorem parametrises all Δ for which $A(I_n + \Delta) \in \mathcal{D}_n$ (recall that $\mathcal{D}_n = \{A \in \mathcal{R}^{n\times n} : A + A' \in \mathcal{S}_n^-\}$).

Theorem* 5.2*: *(i) There exists $\Delta \in \mathcal{R}^{n\times n}$ such that $A(I_n + \Delta) \in \mathcal{D}_n$ if and only if A is non-singular. (ii) If A is non-singular, then all Δ such that $A(I_n + \Delta) \in \mathcal{D}_n$ are given as:*

$$\Delta = -\rho A' + \sqrt{\rho} L \Omega^{1/2}, \quad \rho > \rho_0 = \max\{0, \lambda_{max}(A^{-1} + (A^{-1})')\}$$

where $\Omega = \rho AA' - A - A'$ and $\|L\| < 1$. (iii) If $A + A' \leq 0$, we have $\gamma_3 = \hat{\gamma}_3 = 0$, the unique infimiser of problems (16) and (17) is $\Delta = 0$ (minimiser if $A + A' < 0\}$. If $A + A'$ has at least one positive eigenvalue, then we have:

$$\gamma_3 = \inf\{\|\sqrt{\rho} L \Omega_\rho^{1/2} - \rho A'\| : \rho > \rho_0, \|L\| < 1\}$$

and

$$\hat{\gamma}_3 = \inf\{\|\sqrt{\rho} L \Omega_\rho^{1/2} - \rho A'\| : \rho > \rho_0, \|L\| < 1, \sqrt{\rho} L \Omega_\rho^{1/2} - \rho A' = \sqrt{\rho}\Omega_\rho^{1/2} L' - \rho A\}$$

where γ_3 and $\hat{\gamma}_3$ are defined in (16) and (17), respectively.

Proof: (i) If A is non-singular then setting $\Delta = A^{-1}B - I_n$ where B is any strongly stable matrix shows that $A(I_n + \Delta)$ is strongly stable. Conversely, is A is singular then so is $A(I_n + \Delta)$ for every $\Delta \in \mathcal{R}^{n\times n}$. Hence $A(I_n + \Delta)$ cannot be Hurwitz and hence it cannot be strongly stable. (ii) Note that $A(I_n + \Delta)$ is strongly stable if and only if there exist $\rho > 0$ and $\Delta \in \mathcal{R}^{n\times n}$ such that

$$A + A' + A\Delta + \Delta' A' + \rho^{-1}\Delta'\Delta < 0$$

or equivalently

$$\rho^{-1}(\rho A + \Delta')(\rho A' + \Delta) < \rho AA' - A - A' = \Omega_\rho \Leftrightarrow \begin{pmatrix} -\rho I_n & \rho A' + \Delta \\ \rho A + \Delta' & -\Omega_\rho \end{pmatrix} < 0$$

This is further equivalent to:

$$\begin{pmatrix} \frac{1}{\sqrt{\rho}} I & \frac{1}{\sqrt{\rho}}(\rho A' + \Delta)\Omega_\rho^{-1} \\ 0 & \Omega_\rho^{-1/2} \end{pmatrix} \begin{pmatrix} -\rho I_n & \rho A' + \Delta \\ \rho A + \Delta' & -\Omega_\rho \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\rho}} I_n & 0 \\ \frac{1}{\sqrt{\rho}}\Omega_\rho^{-1}(\rho A + \Delta') & \Omega_\rho^{-1/2} \end{pmatrix} < 0$$

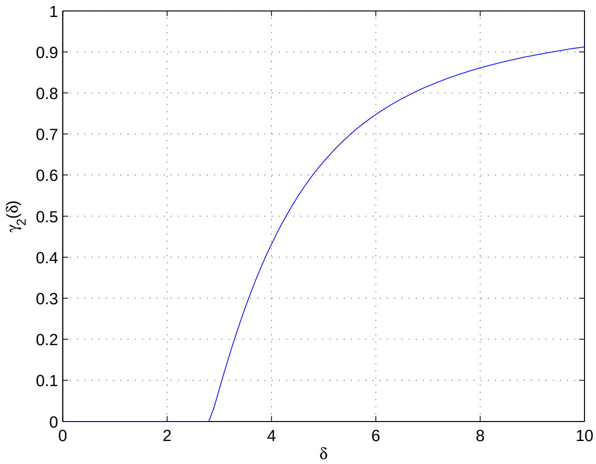

Figure 2: Cost function $\gamma_2(\delta)$

or

$$\text{diag}\left(-I_n + \frac{1}{\rho}(\rho A' + \Delta)\Omega_\rho^{-1}(\rho A + \Delta'), -I_n\right) < 0$$

which is equivalent to

$$(\rho A' + \Delta)\Omega_\rho^{-1}(\rho A + \Delta') < \rho I_n \Leftrightarrow (\rho A' + \Delta)\Omega_\rho^{-1/2} = \sqrt{\rho} L$$

for some contractive L ($\|L\| < 1$). Thus,

$$\Delta = -\rho A' + \sqrt{\rho} L \Omega_\rho^{1/2}, \quad \|L\| < 1 \tag{18}$$

as required. Conversely, it can easily be shown by reversing the steps of the above argument that if Δ has this form, then $A(I_n + \Delta) \in \mathcal{D}_n$ and therefore (18) defines all such Δ. Finally note that if A is non-singular,

$$\Omega_\rho = \rho AA' - A - A' > 0 \Leftrightarrow A[\rho I_n - (A^{-1})' - A^{-1}]A' > 0 \Leftrightarrow \lambda_{max}(A^{-1} + (A^{-1})') < \rho$$

and hence $\rho > \rho_0 = \max\{0, \lambda_{max}(A^{-1} + (A^{-1})')$. Note that since $\mathcal{D}_n$ is a convex invertible cone $A \in \mathcal{D}_n$ if and only if $A^{-1} \in \mathcal{D}_n$, and hence if A is nonsingular, $\rho_0 > 0$ if and only if $A \notin \bar{\mathcal{D}}_n$. Finally part (iii) follows directly from part (ii). □

Example 5.1***:*** Consider the matrix

$$A = \begin{pmatrix} -1 & \delta \\ 0 & -2 \end{pmatrix}$$

where δ is a real parameter which can be used to control the skewness of A. It can be easily verified that A is strongly stable if and only if $|\delta| < 2\sqrt{2}$. Consider the optimization problem $\gamma_2 = \min \|I_2 - P\|$ such that $AP + PA' \leq 0$. This was solved for 101 values of δ equally spaced in the interval $0 \leq \delta \leq 10$ using Algorithm 5.1. The plot of $\gamma_2(\delta)$ versus δ is shown in Figure 2. As expected the cost increases (above the critical value $\delta = 2\sqrt{2}$) as the skewness parameter δ increases. Next consider in detail the case $\delta = 4$. In this case the algorithm executed with a pre-set tolerance $\epsilon = 10^{-8}$ converges after 20 iterations, as shown in Table 2. Thus, $\gamma_2 = 0.4313669$ and the optimal P is

iteration index	current cost	lower bound
1	0.5745280951	0
2	0.4816416579	0.2962912575
3	0.4345016481	0.399206505
4	0.4332772565	0.4255259071
5	0.4326725181	0.4272459424
⋮	⋮	⋮
18	0.4313669448	0.4313668025
19	0.4313669398	0.4313668851
20	0.4313669372	0.4313669372

Table 1: Performance of Algorithm 5.1

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \begin{pmatrix} 1.18877 & -0.07046 \\ -0.07046 & 0.62515 \end{pmatrix} > 0$$

Matrix $AP + PA'$ has eigenvalues $\lambda = \{-5.44184, -1.59\ \ 10^{-8}\}$ and thus lies (almost) on the boundary of the feasible region. To check the solution, the optimisation is formulated as a non-linear programming problem with objective function:

$$f(p_1, p_2, p_3) = (p_1 - 1)^2 + 2p_2^2 + (p_3 - 1)^2$$

and inequality constraints:

$$\begin{aligned}
g_1(p_1, p_2, p_3) &= p_1 \geq 0 \\
g_2(p_1, p_2, p_3) &= p_3 \geq 0 \\
g_3(p_1, p_2, p_3) &= p_1 p_3 - p_2^2 \geq 0 \\
g_4(p_1, p_2, p_3) &= p_1 - 4p_2 \geq 0 \\
g_5(p_1, p_2, p_3) &= 8p_3(p_1 - 4p_2) - (3p_2 - 4p_3)^2 \geq 0
\end{aligned}$$

The Lagrangian of the problem has the form:

$$L(p, \mu) = f(p_1, p_2, p_3) - \sum_{i=1}^{5} \mu_i g_i(p_1, p_2, p_3)$$

At point $(p_1^*, p_2^*, p_3^*) = (1.18877, -0.07046, 0.62515)$ all five constraints are feasible but only the fifth constraint is active, i.e. $g_5(p_1^*, p_2^*, p_3^*) = 0$ (within error tolerance of 10^{-8}), so that $\mu_1^* = \mu_2^* = \mu_3^* = \mu_4^* = 0$. Solving simultaneously $L_{p_i}(p^*, \mu^*) = 0$, $i = 1, 2, 3$, shows that the three equations are consistent and

$$\mu_5^* = \frac{p_1^* - 1}{4p_3^*} = -\frac{2p_2^*}{4p_3^* + 9p_2^*} = \frac{p_3^* - 1}{4p_1^* - 4p_2^* - 16p_3^*} = 0.0755 > 0$$

and hence the Kuhn-Tucker conditions are satisfied. □

6. Conclusions

In this paper the notion of approximate strong stability has been formalised by formulating and solving distance problems from the convex invertible cone (cic) of all strongly stable matrices. Both the Frobenius and spectral norms were considered in the formulation of the distance metric, involving both additive and multiplicative perturbations. Closed-form or algorithmic solutions were derived, along with the parametrization of the optimal solution set, where this was possible. Interesting links were also developed with diverse concepts of matrix theory such as the field of values, the cone of positive semi-definite matrices and the Lyapunov cone of Hurwitz matrices. The results of the paper were illustrated via several numerical examples. Future work will apply the results of the paper to control synthesis problems involving relaxed notions of strong stability.

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