STOCHASTIC MODELING AND MAINTENANCE OPTIMIZATION OF SYSTEMS SUBJECT TO DETERIORATION

By

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Table of Contents

List of Tables

List of Figures

Notation

General notation

Decision model

- ξ_r: partial repair threshold
- ξ_f: failure threshold
- ξ: degree of imperfect (partial) repair
- T_n: n^{th} inspection time
- τ_n: n^{th} inter-arrival inspection time
- τ: period of inspection
- β: repair alert parameter
- λ_0: baseline intensity function
- Λ_0: cumulative baseline intensity function
- λ: failure intensity function
- C: inspection cost of the system
- C_f: replacement cost of the system

- $C_r^v(v+\tau)$: minimal repair cost of the system subject to periodic inspection policy
- $C_r^v(v+\xi\tau)$: partial repair cost of the system subject to periodic inspection policy
- $C_r^v(v+\tau(x,v;\beta))$: minimal repair cost of the system subject to non-periodic inspection policy
- $C_r^v(v+\xi\tau(x,v;\beta))$: partial repair cost of the system subject to non-periodic inspection policy

Intensity control model

- $\hat{\mu}_{n+1}$: $(n+1)^{th}$ expected inspection time
- V_n: time between $(n-1)^{th}$ and n^{th} inspection time
- $q_{12}^u(t)$: transition rate from normal state ($X_t = 1$) to degraded state ($X_t = 2$) given control process $u_t = u$
- γ_i: inspection intensity of the system when the state of the damage process X_t is $i \in S = \{1, 2\}$
- K: inspection cost of the system when operating in state two ($X_t = 2$)
- $K - C$: inspection cost of the system when operating in state one ($X_t = 1$)
- $k_\epsilon(t, u)$: repair cost per unit time with repair degree u
- μ_i: revenue per unit time when the state of the damage process is $i \in S$
- ϕ_i: replacement cost when the state of the damage process is $i \in S$

Stochastic processes

- X_t: physical state (damage) process reflecting the effect of operating environment on the system

- V_t: virtual age process
- N_t: stochastic process counting the number of inspections up to time t
- $\mathcal{F}_t^N$: filtration generated by the counting process of inspections N_s up to time t
- X_n: physical state process just after n^{th} repair
- V_n: virtual age process just after n^{th} repair
- $\mathcal{A}_n$: filtration generated by the history of bivariate state process (X_k, V_k): $k = 1, 2, ..., n$
- u_t: control process at time t representing the decision at time t to perform a repair with repair degree u
- u_t^*: optimal control process at time t
- T^*: optimal production run length of the manufacturing system
- T_d: first passage time of damage process X_t from normal state to degraded state
- $\tau(x_n, v_n; \beta)$: inspection scheduling function $(x_n, v_n; \beta) \rightarrow \tau(x_n, v_n; \beta)$ adapted to partial information $\mathcal{A}_n$ measuring expected time between n^{th} and $(n+1)^{th}$ inspection given that the state of process just after n^{th} repair is $(X_n, V_n) = (x_n, v_n)$
- $F^u(t)$: sojourn time distribution in state one ($X_t = 1$) given control process $u_t = u$
- $F_t^{(X_t,V_t)}$: failure distribution function given bivariate state process (X_t, V_t)
- $\bar{F}_t^{(X_t,V_t)}$: survival function given bivariate state process (X_t, V_t)
- $\bar{R}_u^{(X_n,V_n)}$: failure state process just after n^{th} repair given bivariate state process (X_n, V_n)
- $R_u^{(X_n,V_n)}$: conditional survival function just after n^{th} repair given bivariate state process (X_n, V_n)

- $f_\tau(y|x)$: transition probability density function of $X_\tau = y$ given $X_0 = x$
- $f_\beta(y|x)$: transition probability density function of $X_{\tau(x,v;\beta)} = y$ given $(X_0, V_0) = (x, v)$
- $T^{(x,v)}_{\psi_{\xi_r^{(\tau,v)}}}$: hitting time of modified partial repair threshold $\psi_{\xi_r}^{(\tau_n,v_{n-1})}$ by the process X_n with periodic time to inspection τ given that the process initially started from state $(X_0, V_0) = (x, v)$
- $T^{(x,v)}_{\psi_{\xi_f^{(\tau,v)}}}$: first hitting time of modified failure threshold $\psi_{\xi_f}^{(\tau_n,v_{n-1})}$ by the process X_n with periodic time to inspection τ given that the process initially started from state $(X_0, V_0) = (x, v)$
- $T^{(x,v)}_{\psi_{\xi_r^{(\beta,v)}}}$: hitting time of modified partial repair threshold $\psi_{\xi_r}^{(\beta,v)}$ by the process X_t with expected time to next inspection $\tau(x, v; \beta)$ given that the process initially started from state $(X_0, V_0) = (x, v)$
- $T^{(x,v)}_{\psi_{\xi_f^{(\beta,v)}}}$: first hitting time of modified failure threshold $\psi_{\xi_f}^{(\beta,v)}$ by the process X_t with expected time to next inspection $\tau(x, v; \beta)$ given that the process initially started from state $(X_0, V_0) = (x, v)$
- $C_\tau^{(x,v)}$: cost of inspection and repair per cycle with period of inspection τ given that the process initially started from state $(X_0, V_0) = (x, v)$
- $\mu_{C_\tau^{(x,v)}}$: expected cost of inspection and repair per cycle with period of inspection τ given that the process initially started from state $(X_0, V_0) = (x, v)$
- $L_\tau^{(x,v)}$: length of a cycle with period of inspection τ given that the process initially started from state $(X_0, V_0) = (x, v)$
- $\mu_{L_\tau^{(x,v)}}$: expected length of a cycle with period of inspection τ given that the process initially started from state $(X_0, V_0) = (x, v)$
- $\mu_\tau^{(x,v)}$: expected cost of maintenance per unit time with period of inspection τ given that the process initially started from state $(X_0, V_0) = (x, v)$

- $C_\beta^{(x,v)}$: cost of inspection and repair per cycle with expected time to next inspection $\tau(x,v;\beta)$ given that the process initially started from state $(X_0,V_0)=(x,v)$

- $\mu_{C_\beta^{(x,v)}}$: expected cost of inspection and repair per cycle with expected time to next inspection $\tau(x,v;\beta)$ given that the process initially started from state $(X_0,V_0)=(x,v)$

- $L_\beta^{(x,v)}$: length of a cycle with expected time to next inspection $\tau(x,v;\beta)$ given that the process initially started from state $(X_0,V_0)=(x,v)$

- $\mu_{L_\beta^{(x,v)}}$: expected length of a cycle with expected time to next inspection $\tau(x,v;\beta)$ given that the process initially started from state $(X_0,V_0)=(x,v)$

- $\mu_\beta^{(x,v)}$: expected cost of maintenance per unit time with expected time to next $\tau(x,v;\beta)$ given that the process initially started from state $(X_0,V_0)=(x,v)$

- $\hat{\varphi}^u(n,t;i)$: probability measure of state $i\in S$ of damage process X at time $t\in[\hat{\mu}_n,\hat{\mu}_{n+1})$ given control process $u_t=u$

- $\hat{\gamma}(n,t;u)$: inspection intensity at time $t\in[\hat{\mu}_n,\hat{\mu}_{n+1})$ given control process $u_t=u$

- $\hat{\eta}_{n+1}$: inspection scheduling function adapted to partial information $\mathcal{F}^N$ measuring expected time between n^{th} and $(n+1)^{th}$ inspection

- $\hat{\lambda}(n,t;u)$: failure rate of the system at time $t\in[\hat{\mu}_n,\hat{\mu}_{n+1})$ given control process $u_t=u$

Abstract

During the past decades to prevent catastrophic failure of the system, avoiding potential costs arising from the system downtime and optimize maintenance costs, there has been an interest in maintenance optimization problem for repairable systems subject to deterioration.

Here to tackle the maintenance optimization problem two maintenance models for deteriorating repairable systems are proposed: optimal preventive maintenance scheduling model (decision model) and optimal maintenance-repair and inspection-scheduling model (intensity control model). In chapter 4 under both periodic and non-periodic inspection policy a novel approach to the determination of optimal repair and replacement decision rule subject to system parameters is presented. A renewal argument is used to derive expressions for the long-run average cost per unit time under theses two kinds of inspection policy. The second part of the research (see chapter 5) considers maintenance scheduling problem of manufacturing systems whose production process (resulting output) is subject to system state. The latter means, resulting outputs (revenue) from system depends on the deterioration level of the manufacturing system: the good state of the system results in more efficiency of the system and more resulting output (revenue); the bad state of the system leads to system malfunction and less revenue. To optimize revenue from the manufacturing system, using optimal intensity control model, an optimum repair and inspection policy to balance the the amount of maintenance requires to increase system efficiency against the loss of revenue arising from the system malfunction is presented. Our approach rests on assumption that the transition rate from good (normal) state to bad (degraded) state is linear/non-linear.

Deriving expression for long-run average cost per unit of time under both periodic and non-periodic inspection policy, applying the repair alert and virtual age process model, is the main advantage of the presented decision model to other maintenance models. In addition, using intensity control model, optimizing revenue from manufacturing systems subject to deterioration is a novel approach to maintenance scheduling of manufacturing systems whose production process is subject to the system state and represent an extension of the known maintenance models in which the maintenance process is restricted to inspections.

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Chapter 1

Problem Statement

In this chapter two maintenance optimization problems which are of great practical importance are considered: optimal preventive maintenance scheduling problem for a stochastically deteriorating system (Decision model) and optimal maintenance scheduling problem for a manufacturing system subject to deterioration (intensity control model). In both decision and intensity control model to optimize maintenance process we provide a solution to the optimal control of amount of maintenance in such a way that the maximum revenue from manufacturing system and minimum maintenance cost are derived respectively. In the first model (see chapter 5) which is a novel approach to the maintenance modeling, using intensity control model, the optimum revenue from manufacturing system is achieved by an optimal control (repair degree) process keeping a correct balance between revenue from system and amount of maintenance. In the second model (see chapter 4), using traditional method (embedded renewal process), we find an optimal solution the repair decision thresholds and system parameters. The optimal decision thresholds and system parameters as optimal (repair degree) control process provide a right balance between amount of maintenance and maintenance cost.

1.1 Maintenance scheduling problem of systems subject to non-self announced failure

Failure of many systems such as protective devices or stored items that are subject to random failure is not self-announced and can be found only by inspections which incur costs in terms of wage and material. It stands to clear the greater the probability of detection of failure, the higher the inspection costs. So, to minimize the cost keeping a correct balance between frequency of inspections and inspection costs is essential. The literature on the optimal inspection problem for systems subject to non-self announced failure is vast. Barlow et al (7) given some major assumptions which include the system failure is non-self announced and inspections do not impact on the failure characteristics, shows that optimal inspection times are the solution of system of equations. He proposes an algorithm to numerically solve the system of equations remarked by Barlow, Hunter and Proschan algorithm (BHP algorithm for short). The extreme sensitivity of the algorithm to initial value t_1 (first inspection time) is termed as the major problem of the Barlow et al model. Keller (37), and Kaio and Osaki (34) model the optimum inspection problem with respect to the inspection density that implies the number of checks per unit of time. Both Keller's model and the method of Kaio and Osaki to evaluate the inspection time sequence use the assumption that the time between the failure and its detection is half of the inspection interval. Lack of required accuracy resulting from assumption above is stated as a problem in both models. Munford and Shahani (48) define an inspection sequence characterized by the conditional failure probability of the system. The sequence of inspection times is optimized by minimizing the expected total cost. Chelbi and Ait-Kad (11) introduce an improved inspection model that the conditional failure probability should be an increasing function of the inspection number. Both conditional failure probability based models are known as one-parameter optimization models. As mentioned before, at survey times inspections due to either certain recovery actions (e.g. some adjustment, or partial repairs) or the system complexity do not impact on the failure characteristics of the system. That means, the system state at inspection times leaves unchanged. Such inspection problem which is most common in application is addressed by Jiang and Jardine (32). They present an optimum inspection schedule subject to an increasing failure rate. It is assumed that failures occur at random times and can be detected only through an inspection. Major problems of inspection models are that those either are restricted to some assumption, computationally complicated or in view point of optimal inspection times do not provide good accuracy. In summery the algorithms introduced above do not create an accurate solution for the inspection time sequence to the optimal inspection problem. Irrespective of the BHP model which is extremely sensitive to time t_1, it seems this problem arises from some subjective parameters.

The basic system of equations which is a basis to determine the optimal inspection sequence was first developed by Barlow et al (7). Barlow under a set of assumptions presents an optimal inspection model for systems which are subject to non-self announced failure. That means, failures event is random and can be detected only through an inspection. Based on Barlow maintenance model, inspections do not affect on the failure characteristics, inspection ceases upon detection of failure and at inspection times no repair takes place.

The system of equations associated with Barlow cost model that is

$$C = \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} \left[c_1 j + c_2(t_j - t)\right] dF(t) \tag{1.1.1}$$

is given by

$$\frac{\partial C}{\partial t_j} = 0 \qquad (1.1.2)$$

or, equivalently,

$$\delta_j = \frac{F(t_j) - F(t_{j-1})}{f(t_j)} - \frac{c_1}{c_2} \quad \forall j = 1, 2, ... \qquad (1.1.3)$$

where $F(t)$ is failure distribution function, $\{t_j = 1, 2, ...\}(t_0 = 0)$ and $\delta_j = t_{j+1} - t_j$ denote the inspection time sequence and inspection interval sequence respectively and c_1, c_2 refer to the cost per inspection and the cost per unit time of a system being unavailable due to an undetected failure, respectively.

Barlow to find a solution for the system of equations (1.1.3), which is a sequence of optimal inspection time, propose an algorithm widely called Barlow, Hunter and Proschan algorithm(BHP algorithm). The BHP algorithm is computationally cumbersome and has restrictive assumptions, being extremely sensitive to the value of t_1, the δ_j-sequence generated from equation (1.1.3) is the major problem of BHP algorithm. In preference to the Barlow model and a few nearly optimal algorithm, some often methods which fall into the categories outlined below are presented:

- *Improvement of the original algorithm* (see Nakagawa and Yasui (50));
- *Approximate methods associated with the concept of an inspection density function* (see Keller, Kaio and Osaki (34));
- *One-parameter optimization models.* This category contains an assumption on the failure model. For example, Munford and Shahani (48) assume that the conditional failure probability is constant. Chelbi, Ait-Kadi (11) suppose that the conditional failure probability is an increasing function of the inspection number.

1.1.1 Improved method of the original algorithm

The method presented by Nakagawa and Yasui (50) is associated with the parameter $\varepsilon \in (0, c_1/c_2)$ and a sufficiently large t_n such that

$$\delta_n \approx \delta_{n-1} - \varepsilon$$

or, equivalently,

$$\delta_n = \frac{F(t_n) - F(t_{n-1})}{f(t_n)} - \frac{c_1}{c_2} \approx \delta_{n-1} - \varepsilon \tag{1.1.4}$$

So, with respect to known values that are c_1, c_2, t_n and ε, the equation (1.1.4) gives the $(n-1)^{th}$ inspection time t_{n-1} and the rest of inspection times t_j, $j < n-1$, can be recursively evaluated from the equation (1.1.3). According to the Nakagawa and Yasui model (50), the process carries on until j^{th}(e.g. j=n-K) step where $F(t_{n-K-1}) < 0$ or $t_{n-K} > 2t_{n-K-1}$. Two problems associated with this algorithm are the resulting sequence of inspection times is sensitive to the parameter ε and the value t_n. Besides, if the value of t_1 resulting from above algorithm is applied in equation (1.1.3), the BHP algorithm produces a totally different solution. This deficiency arises from the fact that their algorithm introduces an inappropriate condition $F(t_{n-K-1}) < 0$ to replace the condition $t_0 = F(t_0) = 0$.

1.1.2 Approximate method associated with an inspection density function

Keller (34) with assumption that the time between the failure and its detection is half of the inspection interval, that is,

$$t_{j+1} - t = \delta_j/2$$

presents the inspection density concept $n(t)$ which denotes the number of checks per unit of time. Where t is the failure instant if the failure occurs within (t_j, t_{j+1}).
According to the Keller model the inspection time sequence in terms of $n(t)$ is obtained as

$$t_0 = 0, \quad t_j = t_{j-1} + 1/n(t_{j-1}), j = 1, 2, ... \tag{1.1.5}$$

where $n(t) = \sqrt{\frac{r(t)}{2c_1/c_2}}$ and $r(t)$ denotes the failure rate function.
Kaio and Osaki (34) show that the inspection time sequence resulting from the equation (1.1.5) is not accurate. This problem comes from dependency of t_j to the inspection intensity $n(t)$ evaluated at the left ending point of the inspection interval (t_{j-1}, t_j). To resolve the problem following equation to evaluate the inspection time sequence is suggested:

$$\int_{t_{j-1}}^{t_j} n(t)dt = 1, j = 1, 2, ... \tag{1.1.6}$$

The problem of above method is lack of improvement in accuracy of evaluation of inspection sequence. Moreover, it is computationally more complicated than BHP algorithm in the case that the failure distribution is not Weibull.

1.1.3 One-parameter optimization method

The conditional measure $p_j = 1 - R(t_j)/R(t_{j-1})$ which is the probability that the system fails over time interval (t_{j-1}, t_j) given that it survives beyond time t_{j-1} is considered as a basis to evaluate the inspection time sequence. By assuming above conditional probability is constant p, Munford and Shahani (48) define a sequence of inspection times.
Chelbi and Ait-Kadi (11) tackle the case that p_j should an increasing function of the

number of inspections (or, failure rate). More precisely,

$$p_j = p_1^{1/j}, j = 1, 2, ... \qquad (1.1.7)$$

In both methods which define an one-parameter optimization model the parameters p and p_j are determined such that the minimum expected total cost given by (1.1.1) is achieved.

1.2 Maintenance scheduling problem of systems subject to self announced failure

In literature the inspection strategy for system subject to self announced failure is introduced as a key tool to reduce level of the failure rate of the system in the form of detection and correction of minor defects before major breakdown occurs. However, every inspection also incurs costs. Hence the problem is to determine an optimal inspection maintenance strategy, resulting in the best level of system failure rate, which minimizes overall costs. The literature on the inspection-based maintenance schedule problem is vast. The issue of inspection for a piece of system with constant failure rate has been the theme of many research papers published in the literature. Relating system failure rate to inspection frequency, Jardine (29) assume that the breakdown rate varies inversely with the number of inspections and varies directly with arrival rate of breakdowns per unit of time when no inspection is made. Another model to consider the effect of inspection on failure rate was presented by Rao and Varaprasad (58) who assume that the failure rate after inspection is to be a function of inspection frequency and its effectiveness. More recently, Locket (45) with the same approach as Jardine (29) has assumed a simple inverse relationship between failure rate and inspection frequency.

Locket (45) to relate machine type to system failure rate defines a constant "k" which can be found from experience. As noted in all of the above cited models, the system failure rate over its entire life has a largely constant. But, in the deteriorating state it is expected the system has a steadily increasing failure rate. Hence, an optimal model for frequency of inspection needs to be time dependent. This ensures that the frequency of inspection can respond to variation in the failure rate over time. To meet this deficiency the time dependent inspection factor (I_F) is introduced as a key tool to reflect variations on the failure rate (Jardine (29); Dhillon (23); Kececioglu (36); Locket (45); Wild (76)). It is assumed that the failure rate is inversely related to the time dependent inspection factor. This realistic assumption ensures that as inspection is performed and the frequency of inspection increases, the failure rate decreases. Several variations including the inverse law (Jardine (29); Dhillon (23); Kececioglu (36); Locket (45); Wild (76)), negative exponential, increasing influence of hazard rate and decreasing influence of hazard rate (Mathew (47)) in defining I_F have been examined.

Nowadays, one of challenges facing industries with heavy utilization of systems (e.g. manufacturing systems) which are subject to deterioration is to set an inspection policy as a key tool to control the deterioration level of the system and balance it with economics (revenue). The problem is, on one hand insufficient inspection causes some malfunction of the system and may result in complete breakdown of the system. On the other hand frequent inspections of components to rectify faults leads to more inspection costs and loss of production (revenue) arising from the downtime. So, to maximize the revenue from the system which is in continuous operation, an inspection strategy to give a correct balance between frequency of inspections of system and the resulting output (revenue) is required. The literature on the maintenance maintenance scheduling problem of manufacturing systems whose resulting output (production process) is subject to deterioration is vast. The earlier works on preventive maintenance models with production processes with more than one operating state and a failure state are given by Derman ((21),(22)), who studies a process that deteriorates moving through a finite number of states according to a Markov chain. Derman assumes that the state of the process is known with accuracy at all discrete points in time and show that the optimal replacement policy is a control limit policy; that is, the equipment should be replaced as soon as it is observed to operate in a state worse than some critical state. The same result is also obtained by Kolesar (39) for a similar model but with a more general cost function. The process operating states can alternatively be expressed by the magnitude of the cumulative damage or wear of equipment. The process is assumed to be subject to exterior shocks that damages or causes wear to the equipment, thereby increasing its probability of failure. Shock models have been introduced by Taylor (72) and they have been studied by Feldman (25), Bergman (8) and Valez-Flores and Feldman (73). Another approach to model the process deterioration mechanism has been suggested by Kao (35). Kao uses a discrete time finite state semi-Markov process to formulate the problem. This approach can account both for changes in the process state and for the ageing process of equipment. Thus, Kao examines state-dependent policies as well as state-age dependent policies and proves that under reasonable conditions the optimal policy is of the control limit type. More recent literature for semi-Markovian deteriorating process can be found in So (69), Lam and Yeh (41) and Yeh (77). Sheu and Chen (68) presents an integrated model for the joint determination of both economic production quality (EPQ) and level of preventive maintenance (PM) for an imperfect production process. Sheu uses an increasing hazard rate to describe the deterioration of the production process. Applying the imperfect maintenance concept, Sheu models the effect of PM activities on the deterioration pattern of the process. The model is based on assumption that after PM, the aging of the system is reduced proportional to the PM level and the state of the production process is improved by minimal repair or stopped followed by the restoration work depending on it is in a type I out-of-control-state or in a type II out-of-control state. Using examples of Weibull shock models, Sheu shows that performing PM will yield reduction in the expected total cost. Wang and Pham (75) present a general repair model to maintenance optimization of production systems subject to deterioration. The effect of deterioration is reflected in higher production costs and lower product quality. To keep production costs down while maintaining good quality, Wang and Pham (75) suggest an optimal periodic maintenance model in which both preventive and corrective maintenance are imperfect.

Apart from the maintenance literature, deterioration of the process condition is also a standard feature of the statistical process control model in the quality field. The process is assumed to operate in the "good" quality state (in-control) until it shifts to an inferior quality state (out-of- control) as a consequence of the occurrence of some assignable causes. The time until transition to an out-of-control state (quality shift) is usually assumed to be exponentially distributed (Poisson process) but other distributions have been considered as well (Banerjee and Rahim (6)). In a model of a process with non-exponential transition times Rahim and Banerjee (64) introduce the use of the preventive maintenance actions to protect the equipment against quality shift. In particular, Tagaras (71) uses a Markovian approach similar to that of Derman ((21),(22)) to describe the evolution of production processes characterized by several quality states (a single in-control state and multiple out-of-control states) and a single failure state. He simultaneously considers quality control and quality control parameters that minimize the expected total costs.

In preference to above maintenance models, here we present a novel approach to maintenance scheduling problem of a manufacturing system subject self-announced failure. The model is superior to existing maintenance scheduling models based on inspection strategy. The approach rests on assumptions that the resulting output (revenue) from system is subject to the system state influenced by repair action and deterioration process. Insufficient maintenance leads to an increase in the number of detective items, low profit and low maintenance cost; excessive maintenance results reduces the proportion of defective items, high profit and high maintenance cost. In chapter 5 to tackle above maintenance problem, using optimal intensity control (10), an optimal solution to the determination of the amount of maintenance i.e., frequency of inspection and repair degree of the system is obtained. In contrast to former maintenance models, the intensity control based model presented in chapter 5 not only does not suffer from some subjective concepts, but also gives an insight into various measures such as prediction of system failure, conditional mean time to failure of the system and optimal inspection intensity and also optimal production run length of the system. Besides, it has the potential to be extended to tackle technical maintenance problems which are common in application (see chapter 6).

The second part of the thesis (see chapter 4) is devoted to preventive maintenance scheduling (Decision Modeling) of technical systems subject to deterioration. The decision modeling refers to determination of an optimal schedule of maintenance actions (optimal preventive maintenance (PM) policy) subject to cost aimed at the prevention of breakdown and catastrophic failure of the operating system. Precisely speaking, the objective is to use the processes in decision models which optimize the sequence of actions taken when the processes enter critical region. Typically the process crosses a critical boundary which represents the limit of acceptable performance and the decision maker must than take action to restore the situation.

1.3 Preventive maintenance scheduling problem

Decision modeling for deteriorating systems which is of great practical importance in industry have been widely studied in literatures. Jardine (31) under periodic inspection policy introduces a condition based maintenance (CBM) model subject to cost. The monitoring information to detect the condition of item is incorporated into proportional hazard model (PHM) (13) through underlying Markov stochastic process $Z(t)$. Decision making is given both the age and condition of components at inspection times. The basic decision rule which is used immediately after inspection instant is to replace the system if the item fails or deterioration level of the item (risk) described by the (PHM) $\big(h(t, Z(t))(t \geq 0)$ exceeds a threshold level (preventive replacement); otherwise operation can continue (see Figure 1.1) (30). He showed that the optimal replacement

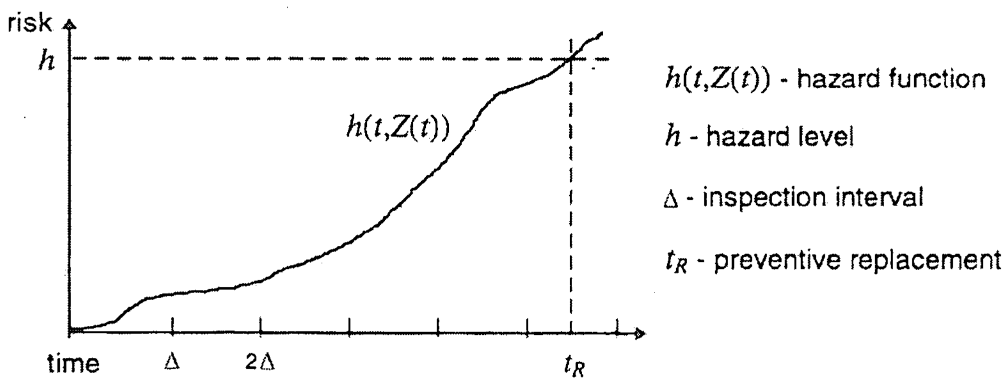

Figure 1.1: An evolution of the preventive replacement

policy is a control limit policy with respect to the hazard rate process $\big(h(t, Z(t)\big)$, i.e. a replacement is performed either at failure or when failure rate of the system reaches or exceeds an optimal failure threshold g^*:

$$T^* = \min\{T, \inf\{t \geq 0 : h(t, Z(t)) \geq g^*\}\}.$$

Where T and T^* denote the failure time and optimal replacement time of the system. Newby and Dagg (56) study an optimal inspection policy for a deteriorating system whose evolution is described by a stochastic process. The inspections and repair actions to avoid catastrophic failure of the system are with respect to warning limits (threshold) which classify the state space into some non-overlapping regions. At inspection times the decision maker subject to the system state has disposition to leave the system to continue to operate or replace it by new one (perfect repair). The cost is considered as a criterion to determine an optimal inspection policy. The model represented by (56) is appropriate for crack growth models (see (53), (54), (70), (12)) which is a basis for some physical phenomenons such as fatigue crack growth problems, offshore structure, and coastal flood barriers subject to erosion (57). Roughly speaking this suitability comes from the fact that the system is regarded as failed if the system state crosses the failure threshold (see Figures 1.2, 1.3).

Newby and Barker (55) present an approach to maintenance optimization subject to complex systems with gradual degradation. The evolution of the system state is characterized by the underlying Bessel process (40). The maintenance process is with respect to two critical thresholds ξ and $\mathcal{F} > \xi$. The threshold ξ denotes the repair actions determined by the probability that the process leaves $[0, \xi)$ and $\mathcal{F}$ defines the failure of the system followed by replacement. The inspection and maintenance policy is optimally determined subject to cost value by crossing the aggregate performance measure i.e. Bessel process of a critical threshold.

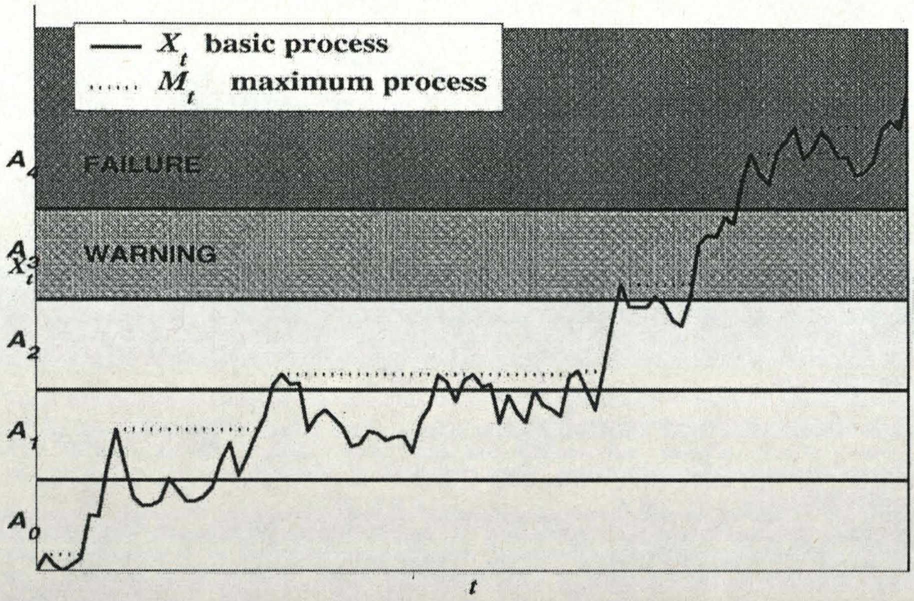

Figure 1.2: An evolution of the system state given Wiener process (65) and maximum process (56)

In chapter 4 we introduce a new approach to decision modeling for a stochastically deteriorating system whose state is characterized by bivariate state process (X, V) denoting damage and system's virtual age process respectively. The state of the system is revealed at inspection times. Repair and maintenance actions are carried out subject to the observed system state and decision thresholds-repair/replacement rule- ξ_r and ξ_f. The problem is to minimize the long-run average cost subject to the system parameters given periodic/non-periodic inspection policy. Using repair alert model (44), a novel approach to formulate time to next (non-periodic) inspection characterized by state process (X, V) is presented. Because new decision model demonstrated under both periodic and non-periodic inspection policy allows replacement if the system state crosses ξ_f, the replacement cycles constitute a renewal process. This property which makes our model

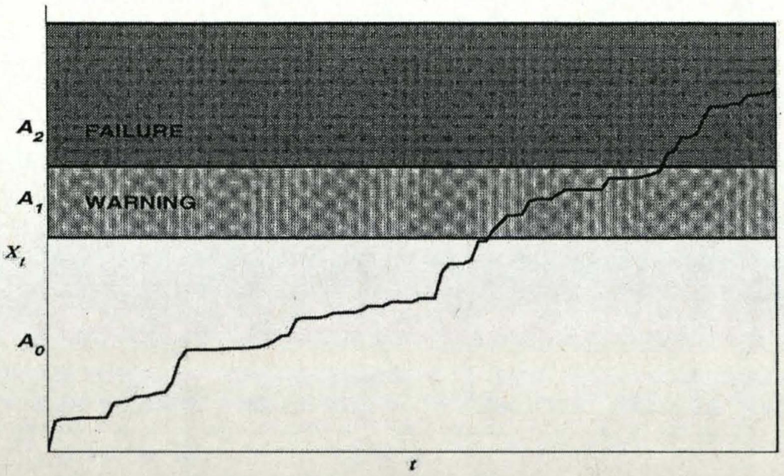

Figure 1.3: An evolution of the system state given Gamma process (74)

distinguished to the models (55) and (56) is used to derive expressions for the long-run average cost based on the decision rules ξ_r, ξ_f and the period of inspection/repair alert parameter.

Chapter 2

Some Fundamental Concepts

2.1 Introduction

To set up the repair and maintenance models which are preventive maintenance scheduling model for deteriorating systems (see Chapter 4) and maintenance scheduling model for a manufacturing system subject to deterioration (see Chapter 5) providing some mathematical tools is required. Following section is devoted to presenting an informal definition of some stochastic notions including intensity process, filtration, and martingale which are the solid basis in stochastic processes theory. The next section is oriented to give a formal definition of history of the process remarked by filtration $\mathcal{F}$, stochastic process and $\mathcal{F}$-adapted process. Section 4 and 5 give a detailed discussion of univariate, multivariate point process and measurability with respect to filtration $\mathcal{F}$. To provide fundamental requirements of the research, section 6 and 7 are assigned to give some mathematical techniques which are stopping time and martingale theory. Also, in subsequent section a formal definition of stochastic intensity with respect to filtration $\mathcal{F}$ is given. Finally, to model the failure of the system influenced by some variables such as environmental factors and system's age and repair and maintenance process ,which state history of the process, some intensity process models are represented.

2.2 An introduction to the basic concepts

Let X_1, X_2,...,X_n be n ($n \geq 1$) (uncensored) continuously distributed survival times from a survival function S with hazard rate function α; thus, $\alpha = f/(1-F)$ where $F = 1 - S$ is the distribution function and f the density of the X_i for $i = 1, 2, ..., n$. The hazard rate α completely determines the distribution through the relation

$$S_t = p(X_i > t) = \exp\left(-\int_0^t \alpha_s ds\right), \tag{2.2.1}$$

One can interpret α by the heuristic

$$p(X_i \in [t, t+dt] | X_i \geq t) = \alpha_t dt. \tag{2.2.2}$$

Typically, in survival analysis problem, complete observation of $X_1, X_2, ..., X_n$ is not possible. Rather, one only observes $(\tilde{X}_i, D_i)$, where D_i is a "censoring indicator," a zero-one valued random variable describing whether X_i or only a lower bound to X_i is observed; namely,

$$\begin{cases} X_i = \tilde{X}_i & \text{if } D_i = 1, \\ X_i > \tilde{X}_i & \text{if } D_i = 0. \end{cases} \tag{2.2.3}$$

We shall consider $\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n$ as random times; at these times, the value of the corresponding D_i becomes available, and we know whether the corresponding event is a failure or a censoring. Thus, all n survival periods start together at time $t = 0$.

As an example (see Andersen (3)), Figures 2.1 and 2.2 depict the observations of 10 randomly selected patients from the right censoring data on survival with malignant melanoma: First, in the original calendar time scale, and second, in the survival time

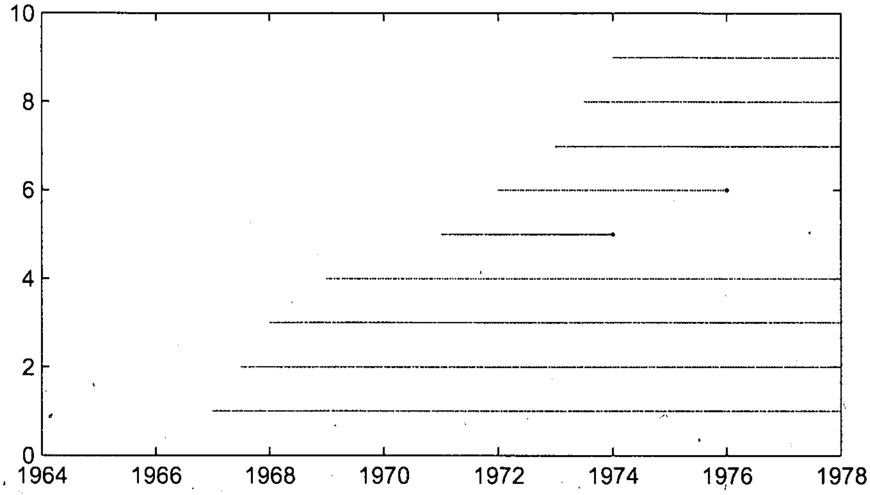

Figure 2.1: Ten observations from the malignant melanoma study, calendar time (years)

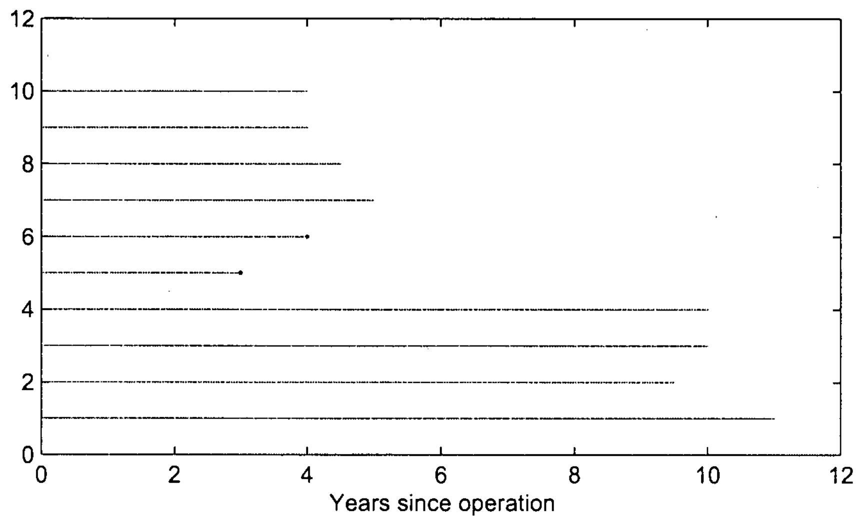

Figure 2.2: Ten observations from the malignant melanoma study, years since operation (survival time).

scale t years since operation. This latter time scale is the one we concentrate in our illustration of stochastic process concepts. A line with the filled circle corresponds to $D_i = 1$ (a failure), a simple line to $D_i = 0$ (a censoring).

Further analysis is difficult without an assumption of right censoring. we will make the most general assumption which still allows progress: the assumption of independent censoring, which means that at any time t (in the survival time scale) the survival experience in the future is not statistically altered (from what it would have been without censoring) by censoring and survival experience in the past. To formulate this notion, we must be able to talk mathematically about past and future. This will be done through the concept of a filtration or history $(\mathcal{F}_t)_{t\geq 0}$; $\mathcal{F}_t$ representing the available data at time t. We write $\mathcal{F}_{t}$- corresponding for the available data just before time t. A specification of $(\mathcal{F}_t)_{t\geq 0}$ can only be done relative some observer, and different observers may collect more or less information. But for all observers, as time proceeds, more information become available.

The notion of a filtration is defined as an increasing family of $\sigma-$algebras defined on the sample space. In our simple example, we will simply take $\mathcal{F}_t$ to mean the values of $\tilde{X}_i$ and D_i for all i such that $\tilde{X}_i \leq t$, otherwise just the information that $\tilde{X}_i > t$. For $\mathcal{F}_{t}-$ the obvious changes must be made: $\leq$ becomes $<$ and the $>$ becomes $\geq$.

The independent censoring assumption can now be written (still very informally) as

$$p(\tilde{X}_i \in [t, t+dt), D_i = 1 | \mathcal{F}_{t}\text{-}) = \begin{cases} \alpha_t dt & \text{if } \tilde{X}_i \geq t \\ 0 & \text{if } \tilde{X}_i < t \end{cases} \tag{2.2.4}$$

Compare this to (2.2.2). Replacing the probability on the left-hand side by the expectation of an indicator random variable, and summing over i values, we get

$$E\left(\#\left\{i:\tilde{X}_i\in[t,t+dt),D_i=1\right\}|\mathcal{F}_{t^-}\right)=\#\left\{i:\tilde{X}_i\geq t\right\}\alpha_t dt$$
$$=Y_t\alpha_t dt \tag{2.2.5}$$
$$=\lambda_t dt,$$

which we have defined the processes Y and λ by

$$Y_t=\#\left\{i:\tilde{X}_i\geq t\right\}$$

the number at risk just before time t for failing in the time interval $[t,t+dt)$, or the size of the risk set, and

$$\lambda_t=Y_t.\alpha_t$$

where $\#$ is a counting notation.

Now formula (2.2.5) can be interpreted as a martingale property involving a certain counting process (3); in this case, the process $N=(N_t)_{t\geq 0}$ counting the observed failures

$$N_t=\#\left\{i:\tilde{X}_i\leq t,D_i=1\right\}$$

and its intensity process λ. Let us write dN_t or $N(dt)$ for the increment $N_{t+dt^-}-N_{t^-}$ of N over the small time interval $[t,t+dt)$. Therefore, we can rewrite (2.2.5) as

$$E(dN_t|\mathcal{F}_{t^-})=\lambda_t dt \tag{2.2.6}$$

Note that the intensity process is random, through dependence on the conditioning random variable $\mathcal{F}_{t^-}$.

To explain the meaning of the martingale property, first define the integrated or cumulative intensity process Λ by

$$\Lambda_t=\int_0^t\lambda_s ds,\qquad t\geq 0,$$

and the compensated counting process or counting process martingale M by

$$M_t = N_t - \Lambda_t$$

Or, equivalently,

$$dN_t = d\Lambda_t + dM_t = \lambda_t dt + dM_t = Y_t \alpha_t dt + dM_t. \tag{2.2.7}$$

Consider the conditional expectation, given the strict past $\mathcal{F}_{t-}$, of the increment (or difference) of the process M over the small time interval $[t, t+dt)$; by (2.2.7), we find

$$\begin{aligned} E(dM_t|\mathcal{F}_{t^-}) &= E(dN_t - d\Lambda_t|\mathcal{F}_{t^-}) = E(dN_t - \lambda_t dt|\mathcal{F}_{t^-}) \\ &= E(dN_t|\mathcal{F}_{t^-}) - \lambda_t dt = 0, \end{aligned} \tag{2.2.8}$$

where the last step is precisely the equality (2.2.6), noting that λ_t is measurable with respect to the filtration $\mathcal{F}_{t-}$. Now, relation (2.2.8) says that Λ is the compensator of N, or that $M = N - \Lambda$ is a martingale for all t.

In wide generality, we have that any counting process N, that is, a process taking the values $0, 1, 2, ...$ in turn and registering by a jump from the value $(k-1)$ to k the time of the k^{th} occurrence of a certain type of event, has an intensity process λ defined by $\lambda_t dt = E(dN_t|\mathcal{F}_{t^-})$. The intensity process is characterized by the fact that $M = N - \Lambda$, where Λ is the corresponding cumulative intensity process, is a martingale remarked by a fair game (see section 2.7). The martingale property says that the conditional expectation of increments of M over small time intervals, given the past at the beginning of the interval, is zero. This is (heuristically at least) equivalent to the more familiar definition of a martingale

$$E(M_t|\mathcal{F}_s) = M_s \tag{2.2.9}$$

for all $s < t$, which, in fact, just requires the same property for all intervals $(s, t]$: for adding up the increments of M over small subintervals $[u, u + du)$ portioning $[s + ds, t +$

$dt) = (s, t]$, we find

$$\begin{aligned} E(M_t|F_s) - M_s &= E(M_t - M_s|\mathcal{F}_s) \\ &= E\left(\int_{s<u\leq t} dM_u|\mathcal{F}_s\right) \\ &= \int_{s<u\leq t} E\left(E(dM_u|\mathcal{F}_{u^-}|\mathcal{F}_s)\right) \\ &= 0. \end{aligned} \tag{2.2.10}$$

Version (2.2.9) of the martingale property is much easier to make the basis of a mathematical theory.

2.3 Filtration

We are going to model the occurrence in time of random events;in fact, discrete events occurring in continuous time. So we fix a continuous time interval

$$\eta = [0, \tau) \quad \text{or} \quad [0, \tau]$$

For a given terminal time τ, $0 < \tau \leq \infty$. Note that the terminal time point τ may or may not be included; this varies from application to application. We write $\bar{\eta} = [0, \tau]$, the time interval augmented with its endpoint if it was not first present.

Definition 2.3.1. (Filtration) (3) Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration

$$(\mathcal{F}_t : t \in \eta)$$

Also called a history, is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$. In the standard theory we use, it is often assumed also to be complete in the strong sense that, for every t, the σ-algebra $\mathcal{F}_t$ contains all P- null sets of $\mathcal{F}$. However, the assumption can be safely omitted, subject only to a very minor reformulation of the results of the standard theory.

When the complete set of assumptions hold, we say that $(\mathcal{F}_t)$ satisfies the usual conditions:

$$\begin{aligned} \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad & \text{for all } s < t \quad \text{(increasing)} \\ \mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t \quad & \text{for all } s \quad \text{(right continuous)} \\ A \subset B \in \mathcal{F}, P(B) = 0 \Rightarrow A \in \mathcal{F}_0 \quad & \text{(complete)} \end{aligned} \tag{2.3.1}$$

The σ-algebra $\mathcal{F}_t$ is interpreted as follows: It contains all events (up to null sets) whose occurrence or not is fixed by time t. There is also a pre-t σ-algebra containing all $\mathcal{F}_s$, $s < t$; it contains events fixed strictly before t.

It is most common the filtration to be described as the history generated by a stochastic process X. This means that $\mathcal{F}_t$ is the σ-algebra generated by X_s, $s \leq t$.

Definition 2.3.2. (Stochastic process) (5) A collection of random variables $X = X_t, t \geq 0$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called a stochastic process.

For any fixed $\omega \in \Omega$ the real function $X(., \omega)$ is called the path of a stochastic process X.

Definition 2.3.3. ($\mathcal{F}$-adapted) (5) A stochastic process is adapted to the filtration $\mathcal{F}$ if for any fixed $t \geq 0$ the random variable X_t is $\mathcal{F}_t$ measurable, i.e. for any Borel set B of $\mathbb{R}$ the event $\{X_t \in B\} \in \mathcal{F}_t$.

2.4 Point Processes

Definition 2.4.1. (Univariate counting Processes) (10) A realization of a point process over $[0, \infty)$ can be described by a sequence T_n in $[0, \infty)$ such that

$$T_0 = 0,$$

$$T_n < \infty \Rightarrow T_n < T_{n+1}.$$

This realization is, by definition, nonexplosive iff

$$T_\infty = \lim_{n\to\infty} T_n = \infty.$$

To each realization T_n corresponds a counting function N_t defined by

$$N_t = \begin{cases} n & \text{if } t \in [T_n, T_{n+1}), n \geq 0; \\ +\infty & \text{if } t \geq T_\infty. \end{cases} \tag{2.4.1}$$

N_t is therefore a right-continuous step function such that $N_0 = 0$, and its jumps are upward jumps of magnitude 1 (see Figure 2.3). In addition, if $E[N_t]$ is finite for all t then the point process is said to be integrable.

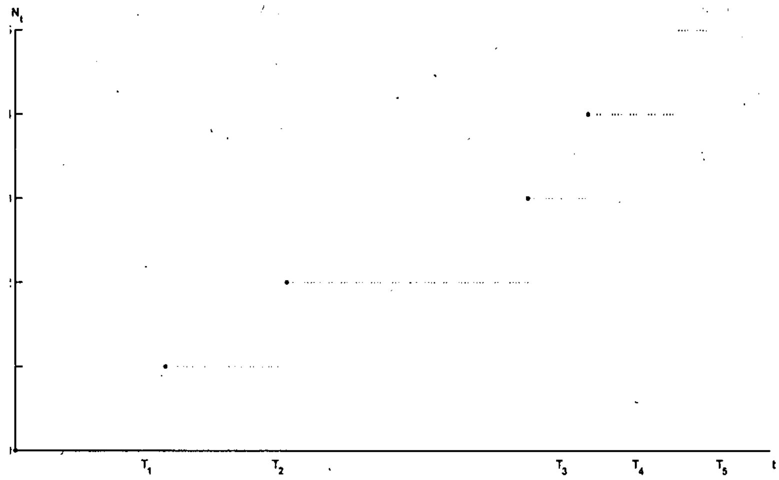

Figure 2.3: An evolution of the counting process N_t

Definition 2.4.2. (Multivariate Counting processes) (10) Let T_n be a point process defined on $(\Omega, \mathcal{F}, P)$, and let $(Z_n, n \geq 1)$ be a sequence of $1, 2, 3, ..., k$-valued random variables, also defined on $(\Omega, \mathcal{F}, P)$. Define for all i, $1 \leq i \leq k$, and all $t \geq 0$:

$$N_t(i) = \sum_{n\geq 1} 1(T_n \leq t) I(Z_n = i). \tag{2.4.2}$$

Both the k-vector process $N_t = (N_t(1), ...N_t(k))$ and the double sequence $(T_n, Z_n, n \geq 1)$ are called k-variate counting processes.

As noted the $N_t(i)'s$ have no common jumps. In general we say that two point processes $N_t(1)$ and $N_t(2)$ defined on $(\Omega, \mathcal{F}, P)$ have no common jumps if $\Delta N_t(1)\Delta N_t(2) = 0, t \geq 0$,$P$-a.s.

Following examples show application of both univariate and multivariate point process to the renewal process and Markov renewal process modeling.

Example 2.4.1. *(Renewal Process) Let $\{T_n\}$, $n = 0, 1, 2, ...$ denote a sequence of non-negative random variables defined on $(\Omega, \mathcal{F})$ and $T_0 = 0$. We introduce $X_i = T_i - T_{i-1}$, $i = 1, 2, ...$ as i^{th} independent inter-arrival times distributed identically with finite mean value $E(X) < \infty$. It means after each renewal the process restarts. It is easy to see that the counting process N_t:*

$$N_t = \sum_{n=1}^{\infty} 1(T_n \leq t)$$

as a univariate point process or renewal process counts the number of renewals in $[0, t]$.

Example 2.4.2. *(Markov Renewal Process) Let J_0 denote the initial state of a repair process; and for $n \geq 1$, let J_n be a Markov chain with transition probabilities p_{ij} denoting the repair state of the process following n^{th} repair and maintenance action (transition). So, the process $J_n, n = 1, 2, ...$ that can be called as the External Process is a Markov chain controlled by the transition probabilities p_{ij}.*

Now, let $N_i(t)$ refer to the number of repair times at which post repair states that are at disposition of a controller are i $(i = 1, 2, ..., k)$ value over time interval $(0, t]$. Now, if

$$N(t) = \sum_{i=0}^{k} N_i(t)$$

Then clearly, the process $Z(t)$ known Semi-Markov Process and given by

$$Z(t) = J_{N(t)}$$

can be treated as the repair state of the process at time t and the set

$$N(t) = (N_1(t), N_2(t), ..., N_k(t))$$

named a Markov Renewal Process can be treated as a k-variate counting process.

The Figure 2.4 shows an evolution of the multivariate counting process given $k = 3$.

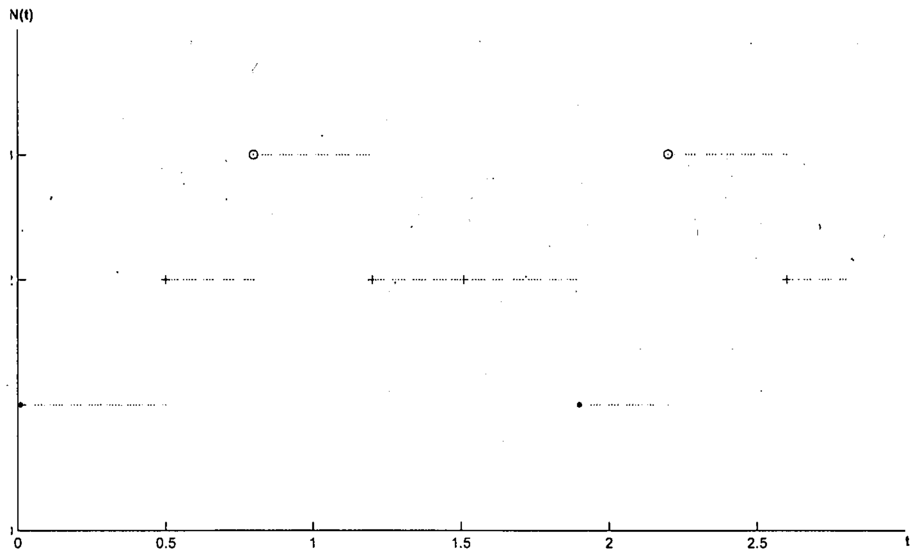

Figure 2.4: A realization of a 3-variate counting process

2.5 Measurability with respect to Filtration $\mathcal{F}$

Definition 2.5.1. ($\mathcal{F}$-Progressive) (4) A stochastic process X is $\mathcal{F}$-progressive or progressively measurable, if for every t the mapping $(s, \omega) \to X_s(\omega)$ on $[0, t] \times \Omega$ is measurable (see definition 2.3.3) with respect to the product σ-algebra $B([0, t]) \otimes \mathcal{F}_t$, where

$B([0,t])$ is the Borel σ-algebra on [0,t].

Obviously, every left-or right continuous adapted process is progressively measurable. A more measurability restriction over the stochastic processes leads to the subsequent definition.

Definition 2.5.2. ($\mathcal{F}$-Predictable) (4) Let $\mathcal{F}$ be a filtration on the basic probability space and let $P(\mathcal{F})$ be the σ-algebra on $(0,\infty)\times\Omega$ generated by the system of sets

$$(s,t]\times A, 0\leq s<t, \quad A\in\mathcal{F}_s, \quad t>0.$$

$P(\mathcal{F})$ is called the $\mathcal{F}$-predictable σ-algebra on $(0,\infty)\times\Omega$. A stochastic process $X=(X_t)$ is called $\mathcal{F}$-predictable, if X_0 is $\mathcal{F}_0$-measurable and the mapping $(t,\omega)\rightarrow X_t(\omega)$ on $(0,\infty)\times\Omega$ into R is measurable with respect to $P(\mathcal{F})$.

To get an impression on predictability of a stochastic process with respect to filtration, let

$$\mathcal{F}_{t-}=\bigvee_{s<t}\mathcal{F}_s=\sigma\left\{A_s, A_s\in\mathcal{F}_s, 0\leq s<t\right\}$$

Then, it is said to be stochastic process X_t an $\mathcal{F}$-predictable process if it is measurable from information available just before time t i.e. $\mathcal{F}_{t-}$. In other words,

$$E(X_t|\mathcal{F}_{t-})=X_t.$$

Some further important terms on predictability of stochastic process are as follows:

- if the stochastic process X_t is predictable then it is measurable with respect to the σ-algebra on $[0,\infty)\times\Omega$ generated by adapted left-continuous processes
- if X is a predictable process then the random variable X_t is $\mathcal{F}_{t-}$ measurable.

- The value of a predictable process is known at the moment t if the history in the time interval $[0, t)$ is known.

- Every left-continuous process adapted to $\mathcal{F}$ is $\mathcal{F}$-predictable.

2.6 Stopping Time

Before giving a formal definition of Stopping time, let have following example concerning the risk theory. Suppose the $\mathcal{F}_t$ adapted net profit rate $R = (R_t)$, $t \in \mathbb{R}_+$, which is non-increasing in time, denotes the difference between the flow of the income resulting from a deteriorating system and the total maintenance costs up to time t. The question is when to stop processing the system (optimal operating time) so that a correct balance between rewards from system and the increasing maintenance costs due to repair and maintenance actions which results in the maximum revenue is derived. A reasonable candidate for optimal operating time of the system is

$$\tau = \inf \{t : R_t \leq 0\}$$

which is the first time the risk process R_t falls below zero. The time τ which in risk theory is known the time to ruin and characterized by the information level $\mathcal{F}_t$ is informally called $\mathbb{F}$-stopping time. Obviously, this time point at which the stochastic process $R = (R_t)$ hits the certain level (0) is random, because its occurrence depends on evolution of the process. Formally, such random times which are based on the information level not anticipating the future are defined as follows.

Definition 2.6.1. (Stopping Time) (4) Suppose $\mathbb{F} = (\mathcal{F}_t)$, $t \in \mathbb{R}_+$, is a filtration on the measurable space $(\Omega, \mathcal{F})$. A random variable $\tau : \Omega \rightarrow [0, \infty]$ is said to be a stopping time if for every $t \in \mathbb{R}_+$,

$$\{\tau \leq t\} = \{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$$

2.7 Martingale Theory

In addition to the predictable processes explained above, the other kind of stochastic processes known as Martingales (4) or a pure noise part of a stochastic process plays a fundamental and complementary role in the general theory of stochastic processes. In subsequent section under Doob Meyer Decomposition Theorem (see Bagdonavicious (5)) we see how martingales are constructed through subtracting an increasing process Λ_t from a stochastic process X_t known as the sub-martingale. Roughly speaking, if X_t is a stochastic process then it can be decomposed as the sum of a drift or regression part Λ_t and an additive fluctuation described by a martingale M_t:

$$X_t = \Lambda_t + M_t$$

Also, in a slightly weakened version of Doob Meyer decomposition called Smooth Semi-Martingale (SSM) (see Jensen (4)), this notion is considered as unpredictable noise term with zero-mean value resulting from the subtraction of a stochastic process, and a smoothly increasing process.

Definition 2.7.1. (Martingale) (4) An integrable $\mathbb{F}$-adapted process $X = (X_t)$, $t \in \mathbb{R}_+$, is called a martingale if

$$X_t = E[X_s|\mathcal{F}_t]$$

for all $s \geq t$, $s, t \in \mathbb{R}_+$. A super-martingale is defined in the some way, except that above equality is replaced by

$$X_t \geq E[X_s|\mathcal{F}_t],$$

and a sub-martingale is defined with above equality being replaced by

$$X_t \leq E[X_s|\mathcal{F}_t].$$

With taking expectation of both sides of the (in)equality it is followed $E(X_t) = (\geq, \leq)E(X_s)$ which state

- A martingale is 'constant' on average, and models a fair game
- Sub-martingale (Super-martingale) is increasing (decreasing) on average.
- X is a sub-martingale (super-martingale) if $(-X)$ is a super-martingale (sub-martingale).

Example 2.7.1. *Let X be an integrable $\mathcal{F}$-adapted process. Suppose that the increments $X_t - X_s$ are independent of $\mathcal{F}_s$ for all $t > s$, $s, t \in \mathbb{R}_+$. If $X_0 = 0$ and the increments $X_t - X_s$ follow a poisson distribution with mean $t - s$ for $t > s$, then X is a Poisson process. Now X is a sub-martingale (or, increasing on average) because of*

$$E(X_t|\mathcal{F}_s) = X_s + E(X_t - X_s|\mathcal{F}_s) = X_s + (t - s) \geq X_s$$

On the other hand we have

$$E(X_t - t|\mathcal{F}_s) = X_s - s$$

that means $(X_t - t)$ is a martingale M_t:

$$X_t = t + M_t$$

As seen the martingale term $X_t - t$ is subtraction of a sub-martingale i.e. X_t and an increasing process t known as compensator (systematic term) of the sub-martingale process X_t. Such representation simply results from Doob-Meyer Decomposition Theorem.

Theorem 2.7.2. *(Doob-Meyer Decomposition) (5) Let X be a right continuous non-negative $\mathbb{F}$ sub-martingale. Then there exists a right continuous martingale M and an non-decreasing right continuous predictable process Λ such that $E(\Lambda_t) < \infty$ and*

$$X_t = M_t + \Lambda_t \quad a.s.$$

for any $t \geq 0$. If $\Lambda(0) = 0$ a.s. then this decomposition is a.s. unique, i.e. if $X_t = M_t^* + \Lambda_t^*$ for any $t \geq 0$ with $\Lambda^*(0) = 0$, then for any $t \geq 0$,

$$p(M_t^* \neq M_t) = p(\Lambda_t^* \neq \Lambda_t).$$

The process Λ is called the compensator of the submartingale X.

Following a slightly weakened version of the Doob Meyer Decomposition Theorem termed as Smooth semi-martingale (SSM) representation is introduced. SSM representation which plays a key role to set up the maintenance models (see Chapter 4,5), allow the process to be decomposed into a drift part and an additive random fluctuation described by a martingale.

Definition 2.7.2. (Smooth Semi-Martingale) (4)

A stochastic process $Z = (Z_t), t \in \mathbb{R}_+$, is called a smooth semi-martingale (SSM) if it has a decomposition of the form

$$Z_t = Z_0 + \int_0^t f_s ds + M_t,$$

where $f = (f_t)$,$t \in \mathbb{R}_+$, is a progressively measurable stochastic process with

$$E\int_0^t |f_s|\, ds < \infty \quad \forall t \in \mathbb{R}_+$$

and $E\,|Z_0| < \infty$ and $M = (M_t) \in \mathcal{M}_0$ where $\mathcal{M}_0$ denotes the class of cadlag [1] martingales with $M_0 = 0$. Short notation: $Z = (f, M)$.

[1] A process with almost right continuous and left-limited paths

So, because the martingale term is a mathematical model of a fair game with constant expectation function $E(M_t) = E(M_0) = 0$, the stochastic process Z_t, $t \in \mathbb{R}_+$ can be considered as a diffusion, which varies randomly around the regression term with expected value zero.

Theorem 2.7.3. *(Smooth Semi-Martingale) (4) Let $Z = (Z_t), t \in \mathbb{R}_+$, be a stochastic process on the probability space $(\Omega, \mathcal{F}, P)$, adapted to the filtration $\mathcal{F}$. If C_1, C_2 and C_3 hold true, then Z is an SSM with representation $Z = (f, M)$, where f is the limit defined in C_1 and M is an $\mathbb{F}$-martingale given by*

$$M_t = Z_t - Z_0 - \int_0^t f_s ds.$$

And $C_1, C_2,$ and C_3 with assumption of

$$D(t,h) = h^{-1}E[Z_{t+h} - Z_t|\mathcal{F}_t], \quad t, h \in \mathbb{R}_+$$

are

C_1. For all $t, h \in \mathbb{R}_+$ versions of the conditional expectation $E[Z_{t+h}|\mathcal{F}_t]$ exist such that the limit

$$f_t = \lim_{h \to 0^+} D(t,h)$$

exists $P-a.s.$ for all $t \in \mathbb{R}_+$ and $(f_t), t \in \mathbb{R}_+$, is $\mathcal{F}$-progressively measurable with $E \int_0^t |f_s|\, ds < \infty$ for all $t \in \mathbb{R}_+$.

C_2. For all $t \in \mathbb{R}_+, (hD(t,h)), h \in \mathbb{R}_+$, has P-a.s. paths, which are absolutely continuous.

C_3. For all $t \in \mathbb{R}_+$, a constant $c > 0$ exists such that $\{D(t,h) : 0 < h \leq c\}$ is uniformly integrable.

One of the simplest example of a process with an SSM representation is the Poisson process $(N_t), t \in \mathbb{R}_+$ with constant rate $\lambda > 0$. To determine (SSM) representation of (N_t), using independent and stationary increment property of (N_t), it can be simply shown that

$$M_t = N_t - \lambda t$$

is a martingale term with respect to filtration $\mathcal{F}_t^N = \sigma(N_s : 0 < s \leq t)$. On the other hand from the condition C_1 of the Theorem 2.7.3 the compensator of (N_t) is

$$\begin{aligned} f_t &= \lim_{h \to 0^+} \frac{D(t,h)}{h} \\ &= \lim_{h \to 0^+} \frac{E(N_{t+h} - N_t | \mathcal{F}_t^N)}{h} = \lim_{h \to 0^+} \frac{E(N_{t+h} - N_t)}{h} \\ &= \lim_{h \to 0^+} \frac{E(N_h)}{h} = \lambda. \end{aligned} \tag{2.7.1}$$

Therefore, since the conditions $C_1 - C_3$ are satisfied with $f_t = \lambda$, N_t admits following (SSM) representation

$$N_t = \lambda t + M_t$$

The Figures 2.5 and 2.6 show an illustration of the counting process N_t, its compensator and the martingale term M_t for $n = 51$ identically independent lifetimes which are simulated of Poisson distribution with hazard $\lambda = 1$.

In the next section a general definition of intensity of a counting process N_t is given.

2.8 Stochastic Intensity

Definition 2.8.1. (Stochastic Intensity) (10)

Let N_t be a point process adapted to some filtration $\mathcal{F}_t$, and let λ_t be a nonnegative $\mathcal{F}_t$-progressive process such that for all $t \geq 0$

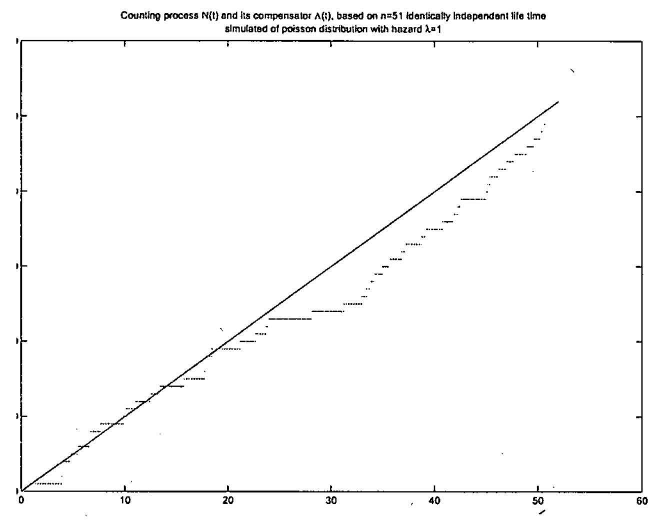

Figure 2.5: Counting process N_t and its compensator Λ_t, based on $n = 51$ identically independent lifetime simulated of Poisson distribution with hazard $\lambda = 1$

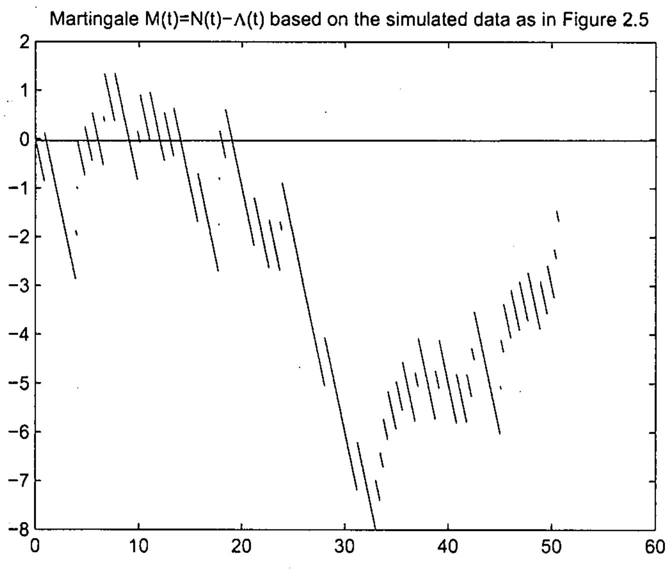

Figure 2.6: Martingale $M_t = N_t - \Lambda_t$ based on the simulated data

$$\int_0^t \lambda_s ds < \infty \quad P - a.s.$$

If for all nonnegative $\mathcal{F}_t$-predictable processes C_t, the equality

$$E\left[\int_0^\infty C_s dN_s\right] = E\left[\int_0^\infty C_s \lambda_s ds\right] \tag{2.8.1}$$

is verified, then we say that N_t admits the $\mathcal{F}_t$-intensity λ_t. Also, λ_t is so-called the compensator of N_t with respect to $\mathcal{F}$.

Next example shows how the general definition of the stochastic intensity is used as a key tool to solve an optimal stopping problem.

Example 2.8.1. *Let N_t be a Poisson process with intensity λ and let T be a fixed time (the terminal time). Let ζ be the class of all $\mathbb{F}_t^N$-stopping time τ bounded by T. Find $\tau^* \in \zeta$ such that $E[N_{\tau^*}(T - \tau^*)] \geq E[N_\tau(T - \tau)]$ for all $\tau \in \zeta$. Note that N_t can be interpreted as follows: it is the counting process of a flow of goods, or items, entering a warehouse. For any given item sojourning a time x in the warehouse, a fee proportional to x must be paid. At time T, the owner of the goods will take them back to the factory or dump them, so that there are no storage expenses after T. At an intermediary time τ, the owner of the goods has the option of removing the N_t items presenting in the warehouse, thus saving an amount of money proportional to $N_\tau(T - \tau)$. The time τ is chosen according to the observation of N_t and cannot anticipate on the future observations, therefore it has to be a $\mathbb{F}_t^N$-stopping time.*

Integration by parts follows

$$N_\tau(T - \tau) = \int_0^\tau (T - s) dN_s - \int_0^\tau N_s ds.$$

Since the intensity of N_t is λ from definition 2.8.1 with substituting the predictable process $C_t = 1(t \leq \tau)(T - t)$ we have

$$E\left[\int_0^{\tau}(T-s)dN_s\right] = E\left[\int_0^{\tau}(T-s)\lambda ds\right]$$

So,

$$E[N_\tau(T-\tau)] = E\left[\int_0^{\tau}\left(\lambda(T-s) - N_s\right)ds\right].$$

The integrand $\lambda(T-t) - N_t$ that can be realized as the saving rate per unite of time is decreasing and takes the value $\lambda T > 0$ at time $t = 0$. Therefore a reasonable candidate for an optimal $\mathbb{F}_t^N$-stopping time (τ^*) is the first time at which the integrand crosses the horizontal line zero, or equivalently,

$$\tau^* = \inf\left\{t : \lambda(T-t) - N_t \leq 0\right\}$$

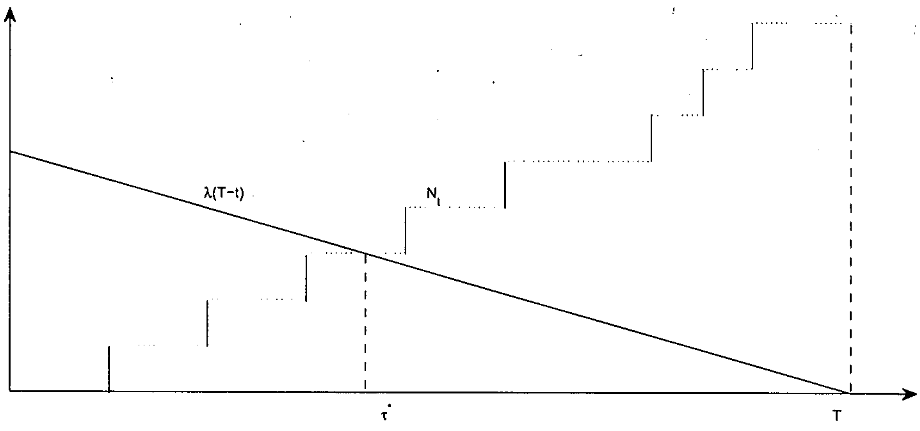

Figure 2.7: The Optimal Stopping Time τ^*

The Figure 2.7 shows an illustration of the stopping time τ^*. As noted in addition to the conditions $C_1 - C_3$, the general definition of intensity process (2.8.1) is an alternative method to determine the compensator of a point process.

2.9 Intensity Process Models

In this section to model failure process of systems subject to repair and maintenance actions, a class of intensity process models characterized by repair and maintenance factors is presented. It will be shown the repair and maintenance factors known as multiplicative, additive or age reduction factors through incorporating into failure intensity function play a main role to adjust failure process of systems. Before introducing intensity process models, let have following example.

Example 2.9.1. *Suppose the system under study is a repairable system with lifetime distribution* $F(t), t \in \mathbb{R}_+$ *and corresponding baseline failure rate* $r(t) = \frac{1}{1-F(t)} \frac{dF(t)}{dt}$. *We introduce the nonnegative measure* A_n *as repair random variables which represent the virtual age or effective age of the system just after* n^{th} *repair and maintenance action:*

$$A_n = A_{n-1} + \xi_n(T_n - T_{n-1}) \quad \xi_n \in [0,1], \quad (n \geq 1)$$

where $T_0 = 0$, $A_0 = 0$ *and* ξ_n *denoting the repair degree reflects the effect of repair just after* n^{th} *repair time. To be more precise, if the system is replaced by new one at repair times (as good as new) then* ξ_n (A_n) *take zero value, also* $\xi_n > 0$ $(A_n > 0)$ *can be interpreted as the virtual age following* n^{th} *repair. Now, let* $\mathbb{F}^{\mathbb{N}}$ *be* σ*-algebra generated by repair point process* N_t *and the sequence of nonnegative random variables* A_n *be adapted to* $F^N_{T_n}$. *It can be shown that (see (4, p. 55)) the* $\mathbb{F}^{\mathbb{N}}$*-intensity of* N *is given by*

$$\lambda_t = \sum_{n=0}^{\infty} r(t - T_n + A_n) I(T_n < t \leq T_{n+1}) \quad A_0 = T_0 = 0. \tag{2.9.1}$$

Where T_n *refers to* n^{th} *repair tome. In particular case, let*

- $\xi_n = 0$ *for all* $n \in N$, *then* N *recalls the renewal process with inter-arrival time distribution* F

- $\xi_n = 1$ *for all* $n \in N$. *Then* N *represents minimal repairs with intensity* $r(t)$.

As seen the repair degree ξ_n simply acts as an age reduction factor where through incorporating into baseline failure rate adjusts the age of the system.

2.9.1 Proportional Intensity Model (PIM)

Based on (PIM) well known as Cox model (14, p. 55-66), the intensity process is the multiplication of a deterministic baseline intensity function λ_0 dependent only on the age of the system and a positive function Ψ formulated in terms of the time dependent covariates values. Hence, if $z(t)$ is the vector of covariates values, the (PIM) can be expressed as

$$\lambda(t) = \lambda_0(t)\Psi(z(t)), \tag{2.9.2}$$

where usually $\Psi(z(t))$ is represented as $\exp(\gamma z(t))$ that γ is a vector of possible parameters. Besides, the trend of the baseline intensity function can be formulated as the power-law form, that is, $\lambda_0(t) = \alpha\beta t^{\beta-1}(\alpha > 0, \beta \in \mathbb{R})$, the log-linear form $\lambda_0(t) = \alpha\beta^t(\alpha, \beta > 0)$, and the constant form $\lambda_0(t) = \alpha(\alpha > 0)$.

2.9.2 Percy Model (Partial Repair)

Percy (62), to model the failure behavior of the repairable systems, considered an extension of the non-homogeneous Poisson process (NHPP). According to the Percy's approach the baseline intensity function is improved at each corrective maintenance by some measures well-known as intensity scaling factors. Theses factors are applied as tools to reflect the improving and deteriorating trend of the system over the repair process. This model is stated as

$$\lambda(t) = \lambda_0(t) \prod_{i=1}^{N(t)} s_i, \tag{2.9.3}$$

Where $s_i > 0$ denote intensity scaling factors with constant values and $\lambda_0(t)$ is the baseline intensity function. In specific case, it can be assumed that s_i take equal values i.e., $s_1 = s_2 = ... = s_{N(t)} = \sigma$ for some unknown parameter σ.

2.9.3 Generalized Proportional Intensity Model (GPIM)

Generalized proportional intensity model (60), which is a generalization of both models i.e. prentice (63), percy et.al. model (62) and also (PIM), offers much potential for maintenance decision making not only by incorporating covariates, and the aging factor of the system, but also through introducing intensity scaling factors r_i and s_i being respectively effects of the preventive and corrective maintenance actions over the repair process. According to the generalized version of the percy model we have

$$\lambda(t) = \lambda_0(t) \left\{ \prod_{i=1}^{M(t)} r_i \right\} \left\{ \prod_{i=1}^{N(t)} s_i \right\} \exp(\gamma z(t)), \tag{2.9.4}$$

where $M(t)$ and $N(t)$ refer to the number of Preventive Maintenance (PM) and (CM) actions respectively in the time interval $(0, t]$. In especial case, let s_i take constant values σ for $i = 1, 2, ..., N(t)$ and also $r_i = \rho$ for $i = 1, 2, ..., M(t)$. Then the equation (2.9.4) reduces to

$$\lambda(t) = \lambda_0(t) \rho^{M(t)} \sigma^{N(t)} \exp(\gamma z(t)), \tag{2.9.5}$$

Percy (60) with a numerical example, to analysis oil refinery pump data, shows that the intensity process with log-linear baseline intensity function in viewpoint of the estimated log likelihood is more admissible than that with constant or power law form. Furthermore, it is shown that the estimated intensity scaling factors corresponding to (PM)

($\hat{\rho}$) is less than 1 which is an indicative of the key role of (PM) in reducing the failure intensity and consequently substantial savings. Also, Percy (61) with a new approach, to make a more flexible maintenance model to (GPIM) in viewpoint of adjusting the failure intensity, introduces a new model based on random scaling factors, deterministic scaling factors which are the specific functions of i and j or t_i and t_j referring the (PM) and (CM) times respectively, and also random variables with evolving means (combined scaling). For example, in the random scaling case a suitable random form corresponding to (CM) might assume conditionally independent exponential random variables

$$S_j|\sigma \sim \exp(\sigma) \tag{2.9.6}$$

For $j = 1, 2, ...N(t)$ with $\sigma > 0$. Moreover, in such case conditionally independent gamma or log-normal random variables are the other options that might be considered as stochastic scaling factors. In deterministic sense, when intensity scaling factors (S_j) follow a increasing trend over the repair process, we have

$$S_j = \frac{t_j}{t_j + \sigma}, \tag{2.9.7}$$

where $\sigma > 0$. According to the equation (2.9.7), the successive repairs have decreasing multiplicative effect over the intensity function. If S_j are given by

$$S_j = \frac{\sigma}{\sigma + t_j}, \tag{2.9.8}$$

then successive repairs have decreasing multiplicative effect over the intensity function. Note that for both equations (2.9.7), (2.9.8), $0 < S_j < 1$ is an indicative of the effective role of (CM) in reducing the failure likelihood of the system. Finally, in the combined scaling case, which may be the most realistic form in practice, intensity scaling factors corresponding to (CM) might be distributed exponentially with mean increasing over time, namely,

$$S_j|\sigma, t_j \sim \exp(\sigma_j), \tag{2.9.9}$$

Such that $\sigma_j = \frac{t_j}{\sigma + t_j}$ for $j = 1, 2, ..., N(t)$ with $\sigma > 0$. In practice, to fit (GPIM) based on presented intensity scaling factors, Percy (61) uses the oil pump data $A - D$ gathered over a period of nearly seven years of the oil refinery. Each of these pumps perform the same function but the working conditions are different. With assumption of the log-linear form for baseline intensity function, the deterministic sense $s_j = \theta \exp(-j\sigma) + \varphi$ for (CM), and also the constant (PM) scaling factor, it is shown that the estimated value of $\hat{\rho}$ is less than one for pumps A and D indicating (PM) has a basic role to reduce the failure intensity. On the other hand, the estimated value $\hat{\rho}$ for the other pumps is greater than one that means to avoid unnecessary expenses, decreasing the frequency of (PM) is recommended.

2.9.4 Age Reduction Models

In this section we introduce some intensity process models well-known as age reduction models. As noted before, to reflect the effect of the repair at intervention times, intensity scaling factors in a multiplicative way have adjusted the failure rate. But, what distinguishes the age reduction models from the others is that the age reduction factors directly influence the global time in an multiplicative or additive manner. For instance, based on the multiplicative and additive type (see (24), (28)), the failure intensity may be stated as

$$\lambda(t) = \lambda_0(t \prod_{i=1}^{n} s_i), \tag{2.9.10}$$

and,

$$\lambda(t) = \lambda_0(t - \sum_{i=1}^{n} s_i). \tag{2.9.11}$$

respectively where constant values s_i are named age reduction factors, and $\lambda_0(t)$ is the baseline failure intensity.

The other maintenance model which put in this class is known as Kijima type repair models (38). Assume that t_k and ξ_k are the time of k^{th} event and the degree of the repair at that time respectively so that $0 \leq \xi_k \leq 1$ and $k \geq 1$. Based on Kijima type 1 model we have

$$\lambda_{k+1}(t) = \lambda(v_k + t - t_k), \quad t_k \leq t < t_{k+1}, \quad k \geq 0 \tag{2.9.12}$$

that $\lambda_{k+1}(t)$ and $v_k = v_{k-1} + \xi_k(t_k - t_{k-1})$ are considered as the intensity of the failure and the virtual age of the system just after k^{th} repair. The expression (2.9.12) shows at k^{th} intervention, irrespective of existing damage up to the last cycle, just the damage created over k^{th} sojourn is removed. But, on the basis of the Kijima type 2 repair model the virtual age is represented as

$$v_k = \xi_k \left(v_{k-1} + (t_k - t_{k-1})\right).$$

As noted, based on the Kijima type 2 repair model the repair and maintenance action removes entire damage created up to the k^{th} sojourn. The process defined by

$$v(t, \xi_k, k = 1, 2, ...) = t - t_k + v_k,$$

for $t_k \leq t < t_{k+1}, k \geq 0$ is called the virtual age process.

2.9.5 Modeling the Intensity Function Based on Repair and Maintenance Indicators (RMI)

In this section, using repair and maintenance indicators (RMI) presented by Jardine (46), some intensity process models are studied. Theses repair and maintenance indicators acting as intensity scaling factors or age reduction factors can be ideally incorporated into the failure intensity to adjust failure intensity of the system at repair times.

Assume that a system consists of k parts, and $N_i(t)(i = 1, 2, ..., k)$ are counting processes that count the number of interventions/repairs of the part i in time interval $[0, t]$. Moreover, assume that $T_{N_i}(t)$ refer to $N_i(t)^{th}$ intervention time for the part i. If the weight w_i is considered as the importance of i^{th} part within the system, then the RMI_a is formulated as

$$RMI_a(t) = \sum_{i=1}^{k} w_i s_i(t) \tag{2.9.13}$$

so that,

$$s_i(t) = t - T_{N_i(t)} + e_i(t) \tag{2.9.14}$$

where $e_i(t)$ is the accumulated operating time, or the virtual age on the i^{th} part replaced at the moment of $N_i(t)^{th}$ repair and maintenance action. So, if t is treated as the global age of the system, $RMI_a(t)$ could be considered as the virtual age of the system. Subject to the RMI's structure it is clear that $0 \leq RMI_a(t) \leq t$ where it reaches the lower bound if and only if all parts are replaced by new one $\big(e_i(T_{N_i(t)}) = 0\big)$ and $RMI_a = t$ when all parts are replaced by ones as old as the whole system $\big(e_i(t) = T_{N_i(t)}\big)$.

The other type of RMI which reflects both repair action and ageing process of the system is the repair and maintenance indicator type b i.e. RMI_b. This indicator which can be considered as the state of the system at each moment of time is influenced by both global time t and the virtual age of the system measured by RMI_a. More precisely, according to the definition of Jardine (46), we have

$$RMI_b(t) = 1 - \frac{j(t)}{\big[1 + g(t)\big]^{RMI_a(t)}}, \tag{2.9.15}$$

where $j(t) = j(t, \theta)$ is a non-increasing function in t, $g(t) = g(t, \theta)$ is a non-decreasing function in t, $j(0) = 1, j(\infty) = c, 0 \leq c \leq 1, g(0) = 0, g(\infty) = \infty$, and θ is a vector of parameters. As noted $RMI_b(t)$ is an increasing function of t such that $0 \leq RMI_b \leq 1$

where $RMI_b(t) = 0(1)$ denotes the best (worst) state of the system. Finally, the repair and maintenance indicator type c, RMI_c, in terms of RMI_a is defined as

$$RMI_c(t) = 1 - \frac{RMI_a(t)}{RMI_a(t-0)}, \tag{2.9.16}$$

which $RMI_a(t-0)$ is the left limit of the RMI_a. So, $1 - RMI_c(t)$ can be interpreted as ratio of the virtual age of the system just after repair time $RMI_a(t)$ and the virtual age of the system at repair time $RMI_a(t-0)$. It is easy to see the greater values of $RMI_c(t)$ imply the better quality of the repair. For example, if all parts are replaced by new ones then $RMI_a(t) = 0$, and $RMI_a(t-0) > 0$, so $RMI_c(t)$ takes value one. If we have minimal repair corresponding to each part then $RMI_a(t) = RMI_a(t-0)$ that is $RMI_c(t) = 0$. Obviously, $0 \leq RMI_c(t) \leq 1$ if just a number of parts are replaced, or the repair action can not have a significant effect to reduce the accumulated operating time of the parts.

Some Intensity Process Models Based on RMI

- **Incorporating the (RMI) as an Additional Covariate**

In order to reflect the effect of the repair at intervention time, let RMI_a as a repair covariate is incorporated into the exponential multiplicative factor of the intensity function that is

$$\lambda(t) = \lambda_0(t)\exp[\gamma z(t) + \partial RMI_a(t)], \tag{2.9.17}$$

where $\partial > 0$ denotes the regression parameter. From the equation (2.9.17) we have $\lambda_{nr}(t) \leq \lambda(t) \leq \lambda_{nr}(t)\exp(\partial t)$ where $\lambda_{nr}(t) = \lambda_0(t)\exp(\gamma z(t))$ is a function of both the global age of the system, and the virtual age process $RMI_a(t)$. So, the term $\exp(\partial RMI_a(t))$ can be considered as the repair covariate expressing the history of RMA

during the repair process. As seen in special case when all new parts are used at RMA time the intensity function reduces to $\lambda(t) = \lambda_{nr}(t)$ which is Cox model (15), if all parts are restored to their conditions just before repair and maintenance action (minimal repair), then the failure intensity increases to $\lambda_{nr}(t)\exp(\partial t)$.

- **Incorporating the (RMI) as a Non-exponential Multiplicative Factor**

The RMI's role as a non-exponential multiplicative factor can be compared with the function of intensity scaling factors in the Percy model (60). In both models the aim followed is to reflect the repair effect on the failure intensity. If RMI_b is considered as a multiplicative factor, then

$$\lambda(t) = \lambda_{nr}(t).RMI_b(t), \tag{2.9.18}$$

Now let $j(t) = 1$, and all parts are replaced by new ones then the failure intensity can take zero value.

- **Incorporating the (RMI) Through the Virtual Age (VA) Concept**

To reflect the effect of repair and maintenance action and adjust the intensity of failure RMI_a can be incorporating into the baseline intensity function of the PIM as an age reduction factor. In other words,

$$\lambda(t) = \lambda_0(RMI_a(t))\exp\left(\gamma z(t)\right), \tag{2.9.19}$$

As seen the effect of repair and maintenance action is reflected in the intensity process through shifting the time origin in the baseline intensity function. On the other hand, the values of the RMI_b's varies between zero and one, and the failure intensity can be expressed in terms of the VA concept. In such sense, the global age of the system reduces through incorporation of the RMI_b into the baseline intensity function, that is,

$$\lambda(t) = \lambda_0\left(tRMI_b(t)\right)\exp\left(\gamma z(t)\right), \tag{2.9.20}$$

As shown, the virtual concept can be easily incorporated into the baseline intensity function by changing the system's age. Also, on the basis of Kijima's repair model we can write

$$\lambda(t) = \lambda_0(V(t))\exp\left(\gamma z(t)\right), \tag{2.9.21}$$

Where $V(t)$ denotes the Kijima's virtual age in terms of the repair degree ξ_n (see section 2.9.4). Because the RMI_c expresses the quality of the repair, it can be taken into account as an ideal measure instead of the repair degree ξ_n. More precisely, $\xi_n = 1 - RMI_c(t_n)$ which t_n refers to the n^{th} RMA time.

Chapter 3

Stochastic Process Models

3.1 Introduction

This chapter is assigned to provide an overview of some stochastic notions which play a fundamental role to model the occurrence of events whose intensity is driven by a stochastic process, a phenomenon which is most common in application. Consider a manufacturing system (e.g. robot) whose failure intensity depends on type of part made (see section 3.2), or a deteriorating system whose inspection frequency is associated with the system state. In both cases, the intensity of occurrence of failure and inspection is determined by type of part made and the state of the system respectively which are unknown over time. To tackle this kind of problem which is typically raised in modelling repair and maintenance process of systems operating in a stochastic environment, Cox process (18), alternative Cox process (10), or Markov modulated Poisson process (4) so called minimal repair process (MRP) (4) are asset to be used.

The next section describes the Cox process known as a Doubly Stochastic Poisson Process. In this section we study a generalized of Poisson process with non-deterministic intensity, a random variable which is known at the time origin. More precisely, the intensity function λ_t is a stochastic process which is measurable at initial time ($\mathcal{F}_0$-measurable) where $\mathcal{F}_0 = \sigma(\lambda_s, s \in \mathbb{R}_+)$. In the next section, an alternative definition of Cox process which is more adapted for modelling occurrence of events is presented. In the modified version, the stochastic intensity λ_t is not measurable at initial time, but it is adapted to the filtration $\mathcal{F}_t = \sigma(\lambda_s : 0 \leq s \leq t)$. In section 4 we give a brief description of a Markov modulated Poisson process in which intensity of occurrence of events $\lambda_t = f(t, Y_t)$ is linked by a stochastic process driven by Markov process Y_t. Finally, subject to the stopping time notion a class of stochastic processes including (alternative) Cox process and Markov modulated Poisson process which is remarked by minimal repair process (MRP) is presented.

3.2 Cox Process

A Cox Process, also known as a Doubly Stochastic Poisson process, or Conditional Poisson process is a stochastic process which can be described as a non-homogeneous Poisson process with stochastic intensity function. Extending the discussion of the Poisson process from both applied and theoretical viewpoint can be found in Cox and Lewis (16) and Cramer (17). Before formulating the structure of the Cox process, let us have a brief look at its application in financial mathematics, especially in insurance modeling. As known, in insurance modeling, the Poisson process plays a main role to build up the claim arrival process. Clearly, in the case that the claims depend on the intensity of natural disasters (e.g. flood, windstorm, hail, earthquake) to model the claim arrival process the Poisson process whose intensity is deterministic does not meet our desire. For instance, in insurance modeling one of the criteria used to determine the effect of catastrophic events, is the intensity function, or the shot noise process. This process is useful to measure the frequency, magnitude and the time period needed to determine the effect of catastrophic events. As time goes on in the shot noise process the frequency of claims decreases and this trend carries on till another catastrophic event which will lead to a jump in the claim intensity. Therefore, the claim intensity function can be treated as the stochastic measure that generates the number of claims resulting from catastrophic events. To overcome this deficiency, that is, the deterministic motion of the intensity D. Cox (16) presented the Doubly stochastic Poisson process. What distinguishes this process from other point processes (e.g., Poisson process) is the flexibility of the intensity function in that not only it can be considered as a function of the time but also its structure, which is mostly modeled by a driving process, allows the intensity of the events arrival process to have a stochastic behavior. Roughly speaking, the Cox process can be realized as a two steps randomization procedure. In the first step one draws at random the trajectory of a "driving process", say Y_t, and once the whole trajectories are selected, one matches a Poisson process of intensity $f(t, Y_t)$ to these trajectories. To get more insight into the Cox process consider an example presented by D.P. Heyman and M.J. Sobel (27). Assume that X is a random variable with distribution $F_\gamma(x)$ that depends on parameter a γ. Suppose that γ is a random variable with distribution G. If X' is treated as a random variable indexed by the parameter γ, then the distribution function of X' can be stated as

$$p(X' \leq x) = \int F_\gamma(x)G(d\gamma)$$

So, X' is X with its parameter γ randomized by G. For example, if X is distributed normally with mean value γ, then X' is a normal variable with a randomized mean or in generalized case if the counting process N_t is distributed by Poisson process with stochastic process γ_t, then N_t' is a Cox process with stochastic mean value. Now, to have an example let the stochastic process $(U_t)_{t\geq 0}$ be realized as the type of the part made by a robot at time t in a manufacturing plant, and the counting process N_t denote the number of the robots' failures up to time t so that when it is used on a specific job of type u, the number of failures is modeled by a Poisson process with constant rate $\lambda(u)$. In particular, as the production schedule is specified by a non-random function u_t, denoting the type of part being produced at time t, the counting process N_t is modeled by a non-homogeneous Poisson process with deterministic measure

$$\Lambda_t = \int_0^t \lambda(u_s)ds, \;\; t \geq 0.$$

In general, if the production scheme is a stochastic process $\{U_t; t \geq 0\}$ which is not influenced by failures, then the failure process N_t is a Cox process:

$$\Lambda_t = \int_0^t \lambda(U_s)ds, \;\; t \geq 0.$$

In such sense Λ_t is a randomized case of the previous one and can be interpreted as the stochastic intensity of a non-homogeneous Poisson process. Note that if the production schedule at origin of the time is specified then it is said that the intensity function is measurable at origin of time.

The following definition is based on Bremaud (Point Processes and Queues (10)).

Definition 3.2.1. (Doubly Stochastic or Conditional Poisson Process) (10)

Let N_t be a point process adapted to a filtration $\mathcal{F}_t$, and let t be a nonnegative measurable process. Suppose that λ_t is $\mathcal{F}_0$- measurable $\forall t \geq 0$ and

$$\int_0^t \lambda_s ds < \infty \;\; P-a.s., \; t \geq 0, \tag{3.2.1}$$

where $\mathcal{F}_0 = \sigma\left\{\lambda_t : t \geq 0\right\}$.

If for all $0 \leq s \leq t$ and all $u \in R$ the term

$$\exp\left\{(e^{iu}-1)\int_s^t \lambda_v dv\right\},$$

denotes the characteristic function of the conditional distribution $(N_t - N_s)$ given filtration $\mathcal{F}_s$, that is,

$$E[e^{iu(N_t-N_s)}|\mathcal{F}_s] = \exp\left\{(e^{iu}-1)\int_s^t \lambda_v dv\right\}, \tag{3.2.2}$$

Then N_t is called a $(P, \mathcal{F}_t)$-doubly stochastic Poisson process or a $(P, \mathcal{F}_t)$-conditional Poisson process with the stochastic intensity λ_t.

The above assumptions imply $(N_t - N_s)$ for all $0 \leq s \leq t$ is P–independent of $\mathcal{F}_s$ given $\mathcal{F}_0$. The above result comes out of this fact that the right hand side of (3.2.2) could be stochastic just through the $\mathcal{F}_0$-measurable intensity λ_t. Also, since right hand side of (3.2.2) denotes the characteristic function of the conditional distribution $(N_t - N_s)$ given filtration $\mathcal{F}_s$, then for all $0 \leq s \leq t$ and all $k \geq 0$ it follows

$$p(N_t - N_s|\mathcal{F}_s) = \frac{e^{-\int_s^t \lambda_u du}(\int_s^t \lambda_u du)^k}{k!} \tag{3.2.3}$$

Now in the sequel let us focus on some special cases of the Cox process. If

Case1: λ_t is deterministic, or more precisely, it is treated as the non-randomized function $\lambda_t \equiv \lambda(t)$, then N_t is said to be a $(P, \mathcal{F}_t)$-Poisson process, in addition

Case2: $\mathcal{F}_t \equiv \mathcal{F}_t^N$, then the $(P, \mathcal{F}_t)$-doubly stochastic Poisson process reduces to the non-homogeneous Poisson process,

Case3: $\lambda(t) \equiv \lambda$ for all $t \geq 0$, then it reduces to the homogeneous Poisson process,

Case4: $\lambda_t = \Lambda$ for all $t \geq 0$, where Λ is non-negative $\mathcal{F}_0$-measurable random variable, then N_t is called a homogeneous doubly stochastic Poisson process,

Case5: λ_t is formulated as $\lambda_t = f(t, Y_t)$ for some appropriately measurable non-negative function f and for some measurable process Y_t adapted to $\mathcal{F}_0$ that $\mathcal{F}_\infty^Y \subseteq \mathcal{F}_0$, then N_t driven by the driving process, or environmental process Y_t is called a doubly stochastic

Poisson process.

Following we will give an alternative definition of the Cox process presented by A. Dassios and J. Jang (18) on which the driving process Y_t is not necessarily measurable at origin of time what is the most common in application. For example, let the process under study be a sequence of repair and maintenance actions so that the intensity of occurrence of repairs depends on the process state, more precisely $\lambda_t = f(t, X_t)$ where X_t denotes the system state at time t. In such case since the flow of the process over inter-arrival time is influenced by some environmental factors, obviously the whole trajectories of the driving process i.e., X_t are unknown at origin of the time, and consequently, Cox process is not efficient to model above maintenance process.

3.3 Alternative Cox Process

Definition 3.3.1. (Alternative Cox Process) (18) Let $(\Omega, \mathcal{F}, P)$ be a probability space with information structure given by $\mathcal{F} = \{\mathcal{F}_t; t \in [0, T]\}$. Let N_t be a point process adapted to $\mathcal{F}$, and λ_t denote a non-negative process adapted to $\mathcal{F}$ such that

$$\int_0^t \lambda_s ds < \infty \quad a.s.$$

If for all $0 \leq s \leq t$, and $u \in R$

$$E\left\{e^i u(N_t - N_s)|\mathcal{F}_t^\lambda\right\} = \exp\left\{(e^{iu} - 1)\int_s^t \lambda_v dv\right\} \tag{3.3.1}$$

then N_t is called a $\mathcal{F}_t$-doubly stochastic Poisson process with intensity λ_t where

$$\mathcal{F}_t^\lambda = \sigma\left\{\lambda_s : s \leq t\right\}$$

Clearly, above equation gives us

$$p(N_t - N_s|\mathcal{F}_s^\lambda) = \frac{e^{-\int_s^t \lambda_u du}(\int_s^t \lambda_u du)^k}{k!} \tag{3.3.2}$$

and

$$
\begin{aligned}
p\left\{\tau_k > t|\lambda_s; t_{k-1} \leq s \leq t_k\right\}) &= p\left\{N_{t_k} - N_{t_{k-1}} = 0|\lambda_s; t_{k-1} \leq s \leq t_k\right\} \\
&= \exp\left(-\int_{t_{k-1}}^{t_k} \lambda_s ds\right)
\end{aligned}
\tag{3.3.3}
$$

where τ_k denotes the inter-arrival time between the $(k-1)^{th}$, and k^{th} time point. Also, under alternative Cox process definition, the survival probability is equal to

$$
p(\tau_1 > t|\lambda_0) = E\left\{\exp\left(-\int_0^t \lambda_u du\right)|\lambda_0\right\} = E(e^{-\Lambda_t}|\lambda_0)
$$

where $\Lambda_t = \int_0^t \lambda_s ds$ and $\tau_1 = \inf\left\{t : N_t = 1|N_0 = 0\right\}$ denotes the first jump arrival time of the Cox process N_t. Also, from (3.3.3) it is easy to show that

$$
E(\theta^{N_t - N_s}) = E\left\{e^{-(1-\theta)(\Lambda_t - \Lambda_s)}\right\}.
$$

The above equation states that the evaluation of distribution of N_t is equivalent to finding the distribution of Λ_t such that probability generating function of N_t yields the moment generating function of Λ_t and vice versa.

Now to get insight into alternative Cox process, let the process under study is a repair and maintenance process in such away that the intensity of occurrence of repairs depends on the system state X_t controlled by a homogeneous Markov process. More precisely, the intensity of the process is driven by U-values Markov process X_t as $\lambda_t = f(t, X_t)$. Thus the filtration generated by the intensity measure λ_t is equivalent to the filtration $\mathcal{F}_t^X$ where

$$
\mathcal{F}_t^X = \sigma\left\{X_s : 0 \leq s \leq t\right\}
$$

and from (3.3.1) we have

$$
\begin{aligned}
E\left\{e^{iu(N_t - N_s)}|\mathcal{F}_t^X\right\} &= \exp\left\{(e^{iu} - 1)\int_s^t f(v, X_v) dv\right\} \\
&= \exp\left\{(e^{iu} - 1)\int_s^t \sum_u \varphi_v(u) f(v, u) dv\right\}
\end{aligned}
\tag{3.3.4}
$$

Where $\varphi_v(u)$ refers to the stochastic indicator of the system state at time $v \geq 0$ i.e., $\varphi_v(u) = I(X_v = u)$. In special case given a known value of the system state $\varphi_v(u) \equiv \varphi_v$ the alternative Cox process reduces to a $\mathcal{F}_t$-non-homogeneous Poisson process. Note that the left hand side of the above equation is stochastic just through the intensity function, or the system state X_t. If the stochastic part, that is, φ_v can be estimated by sub-filtration $\mathcal{A}_t \subset \mathcal{F}_t^X$ including the observed history of the maintenance process then we have an estimated version of the alternative Cox process as

$$E\left\{e^{iu(N_t - N_s)}|\mathcal{A}_t\right\} = \exp\left\{(e^{iu} - 1)\int_s^t \sum_u \hat{\varphi}_v(u) f(v,u) dv\right\}$$

where $\hat{\varphi}_v(u) = E(\varphi_v(u)|\mathcal{A}_v) = p(X_v = u|\mathcal{A}_v)\ \forall v \geq 0$, and $u \in U$. In such case the alternative Cox process reduces to $\mathcal{A}_t$-Alternative Cox process.

In the next section we study a special case of the alternative Cox process known as Markov-Modulated Poisson process.

3.4 Markov Modulated Poisson Process

As mentioned above the intensity function of the Alternative Cox process can be driven by a stochastic process, that is, $\lambda_t = f(t, Y_t)$. In particular case, let the flow of the driving process Y_t over time is controlled by a homogeneous Markov process. In such sense the alternative Cox process reduces to the repair model well-known as the Markov-Modulated Poisson process. Also, the presented repair model can be treated as a generalized version of the Poisson process whose intensity is indexed by driving process $Y_t = i$, $i \in E\{1, 2, ..., m\}$ varying randomly. This case is common in application. For instance, Jensen (4) applies Markov Modulated Poisson process to model intensity of occurrence of minimal repairs of a deteriorating system subject to failure. Jensen models the intensity of minimal repairs events N_t by an (stochastic) function μ_{X_t} which is driven by the Markov process X_t describing the state of the system. More precisely, it is assumed that N_t admits the following $\mathcal{F}$-SSM representation:

$$N_t = \int_0^t \mu_{X_s} ds + M_t, \quad t \in \mathbb{R}_+,$$

where $0 < \mu_i < \infty$, $i \in S = \{1, 2, ..., N\}$ and M_t is an $\mathcal{F}_t$-martingale.

3.5 Minimal Repair Process

In this section we focus on considering the some repair models in view point of the Minimal repair Process concept. To clarify this notion informally, let us restrict ourselves to a simple case of the minimal process, that is, the basic statistical minimal repair model. As known on the basis of the statistical minimal repair model not only the age or more precisely, the failure intensity of the system as a result of minimal repairs leaves unchanged, but also the inter-arrival failure intensity, corresponding to the non-homogeneous Markov process, is time dependent deterministic. To get an insight into minimal repair concept, let the system under study be a complex system such as T.V. consisting a great number of components. It is expected after replacing a single tube in the T.V. set, the set as a whole will be prone to failure before the tube fails. Therefore, with respect to what mentioned above, the time points of repair and maintenance actions based on the information level, obtained through the intensity measure, are not identifiable, or in terms of the stopping time notion the minimal repair times denoted by $\{T_n\}$ $(\forall n > 0)$ are not measurable with respect to σ-algebra generated by the intensity function $\mathcal{F}_t^\lambda$. In other words,

$$\{T_n \leq t\} = \{\omega \in \Omega : T_n(\omega) \leq t\} \notin \mathcal{F}_t^\lambda \ \forall t \in R$$

So, the stopping times $\{T_n\}$ ($\forall n > 0$) can be determined under one filtration but not under the finer one. In the case that the system at failure times T_n ($n \geq 1$) is replaced by new one, then in terms of the hazard rate function of inter-arrival times which is $r(u)$, $u \in (0, T_n - T_{n-1})$ where $T_0 = 0$ the failure intensity of the system can be represented as

$$\lambda_t = \sum_{n \geq 0} r(t - T_n) I(T_n \leq t < T_{n+1})$$

It is easy to see that the information generated by the intensity measure λ_t up to time t includes the number of repair actions N_t where can be represented as follows

$$N_t = |\{s \in R_+ : 0 < s \leq t, \lambda_s = \lambda_0\}|$$

and it follows that the renewal process does not put in the class of the minimal repair process. As seen minimal repair models are simply characterized by level of information generated by intensity function. If intervention times T_n ($n > 0$) are not identifiable (measurable) with respect to λ_t then the repair model is minimal.
Following to characterize minimal repair models, a definition of minimal repair model presented by Aven and Jensen (4) is given.

Definition 3.5.1. Let (T_n), $n \in N$ be a point process with integrable counting process N and corresponding F-intensity λ. suppose that $\mathcal{F}^\lambda = (\mathcal{F}_t^\lambda)$, $t \in R_+$, is the filtration generated by λ: $\mathcal{F}_t^\lambda = \sigma(\lambda_s : 0 \leq s \leq t)$. Then the point process (T_n) is called a minimal repair process (MRP) if none of the variables T_n, $n \in N$, for which $P(T_n < \infty) > 0$ is an $\mathcal{F}^\lambda$-stopping time, i.e., for all $n \in N$ with $P(T_n < \infty) > 0$ there exists $t \in R_+$ such that $\{T_n\} \leq t \notin \mathcal{F}_t^\lambda$.

Above definition comes from this fact that just after minimal repairs deterioration level and consequently the failure intensity of the system at stopping times $\{T_n\}_{n>0}$ leaves unchanged. So, since the system undergoing minimal repairs can be treated identical in law to one of the same age which has not undergone any failure or repair, the minimal repair times $\{T_n\}_{n>0}$ with respect to the failure intensity trend of the system i.e. λ_t are not identifiable.

In the sequel, we will have a study on Cox process, the Alternative Cox process, and also Markov modulated Poisson process in viewpoint of the minimal repair process notion.

As mentioned before, the intensity function of the Cox process is set up at origin of the time, or more generally the stochastic process λ_t is $\mathcal{F}_0$-adapted for all $t \in R$ i.e., $\mathcal{F}_0 = \sigma(\lambda_s; s \in R_+)$. So, since the trajectories of the driving process are identifiable before starting the process, it means we say the process at time points T_n $(n > 0)$ is restarted to the state in which the events have occurred. To get a sense of minimal repair processes (MRP), let us have following example. As before suppose that the production scheme of a robot, operating in a manufacturing plant, is set at initial time. So, under this assumption the production process is not affected by robot failures and consequently, the failure intensity of the robot, marked by the job type under which it is working at failure time points, leaves unchanged. In other words, if $\mathcal{F}_t$ denotes the history of the process up to time t then

$$\mathcal{F}_t = \mathcal{F}_0 \vee \sigma(N_s, 0 \leq s \leq t)$$

Also we have

$$\mathcal{F}_t^\lambda = \sigma(\lambda_s : 0 \leq s \leq t) \subset \mathcal{F}_0$$

and $\{T_n\}$ are not $\mathcal{F}_0$-adapted. So, it follows $\{T_n\}$ $(\forall n > 0)$ are no $\mathcal{F}_\lambda$-stopping time, or equivalently, the Cox process is an (MRP).

Also, the Markov-modulated Poisson process can be considered as a minimal repair process if just after each repair and maintenance action the process is restored to the state in which the event (e.g. failure) has occurred. In such case the intensity measure indexed by the stochastic process does not include any information about event time points $\{T_n\}$. More precisely, time points of events under filtration $\mathcal{F}_t^\lambda$ for all $t > 0$ are not measurable, namely,

$$\{T_n \leq t\} = \{\omega : T_n(\omega) \leq t\} \notin \mathcal{F}_t^\lambda = \sigma\{Y_s : 0 \leq s \leq t\}$$

Where Y_t $(t \geq 0)$ is a stochastic process steered by the homogeneous Markov process. However, it is necessary to be pointed out, in the case that the process at some time points as a result of repair and maintenance action is adjusted then the (MRP) property for Markov modulated Poisson process is not satisfied because

$$N_t = \int_{0+}^t I(\lambda_s < \lambda_{s-})ds = \sum_{n>0} I(T_n \leq t)$$

Where N_t counts the number of repairs up to time t.

Following theorem gives another characterization of an minimal repair model (MRP).

Theorem 3.5.1. *Assume that $P(T_n < \infty) = 1$ for all $n \in N$ and that there exist version of conditional probabilities $F_t(n) = E[I(T_n \leq t)|\mathcal{F}_t^\lambda]$ such that for each $n \in N$ $(F_t(n))$, $t \in R_+$ is an ($\mathcal{F}^\lambda$-progressive) stochastic process.*

(i) Then the point process (T_n) is an (MRP) if and only if for each $n \in N$ there exists some $t \in R_+$ such that

$$P(0 < F_t(n) < 1) > 0.$$

(ii) If furthermore $(F_t) = (F_t(1))$ has P-a.s. continuous paths of bounded variation on finite intervals, then

$$1 - F_t = \exp\left\{-\int_0^t \lambda_s ds\right\}.$$

Chapter 4

Optimal Maintenance Policies for Stochastically Deteriorating Systems Subject to Bivariate Stochastic Process

4.1 Introduction

The maintenance decision policy for a repairable system has aroused great attention. Bergman (8) studies an optimal replacement problem with a non-decreasing damage process. He shows that the optimal replacement policy is a control limit policy with respect to the damage process. Makis and Jardine (31) address an optimal replacement problem for a deteriorating system subject to random failure. The proportional hazard model (see Cox (13)) has been used to describe the failure rate of the system which is function of both the system age and a stochastic (damage) process. It is shown the optimal replacement policy is a control limit rule with respect to the proportional hazard process. Newby and Dagg (56) in a maintenance decision framework tackle the problem of determining optimal inspection and maintenance policies for a system with perfect repair whose performance is described by a stochastic process. With the same approach as Newby and Dagg (56), Newby and Barker (52) under both periodic and non-periodic inspection policy present an extension of decision models which use the first hitting time of a critical level as a definition of failure. The inspection and maintenance policy is determined by the crossing of a critical threshold by an aggregate performance measure. To extend the model to non-periodic inspection policy, they apply a scheduling function $m(x)$ (Grall et al. (1)) which determines the time to the next inspection based on the observed system state x. Recently, using the extended proportional hazards model (EPHM), You, Li and Meng (49) develop two component-level control-limit preventive maintenance (PM) policies for systems subject to the joint effect of partial recovery PM acts (imperfect PM acts) and variable operational conditions.

In this chapter under both periodic and non-periodic inspection policy we present a new approach to maintenance optimization of a stochastically deteriorating system which is subject to repair and maintenance. The state of the system is determined by the failure probability measure $\bar{R}_t^{(X,V)}$, $t \in \mathbb{R}_+$ described by a general stochastic process (damage process) X with monotone paths and a virtual age process V induced by repair. The structure of the optimal maintenance strategy is formed under periodic and non-periodic inspection policy. Under non-periodic inspection policy, by using repair alert model time to next $((n+1)^{th})$ inspection $\tau_{n+1} \equiv \tau(x_n, v_n; \beta)$ $(n \geq 0)$ is formulated to provide an inspection schedule based on the known bivariate state process $(X_n, V_n) = (x_n, v_n)$ updated just after n^{th} repair. The damage state process is revealed by inspections at non-periodic times $X_{T_{n+1}} = X_{n+1} \mapsto x_{n+1}$. At inspection time T_{n+1} the decision to perform $(n+1)^{th}$ repair action is taken with respect to the failure state process $\bar{R}_{\tau_{n+1}}^{(x_{n+1},v_n)}$ characterized by the known bivariate state process (x_{n+1}, v_n) just before repair, inter-arrival inspection time $\tau_{n+1} = \tau(x_n, v_n; \beta)$ and decision thresholds ξ_r, ξ_f which respectively refer to the preventive partial repair and replacement rule. The decision maker has disposition to adjust the virtual age process V (imperfect repair), leave it unchanged (minimal repair) or replace the system by new one (perfect repair). The critical threshold , ξ_r, is used as definition of partial repair action. If the system state process $\bar{R}_t^{(X,V)}$ crosses the boundary ξ_r a partial repair is made. The acceptance performance of the process is limited by the critical level ξ_f, $(0 < \xi_r < \xi_f < 1)$. The threshold ξ_f is the level at which failure and replacement occur. The replacement action (renewal) is determined by the first hitting time to the failure threshold ξ_f. The problem is to minimize the long-run average cost subject to the preventive maintenance decision rules ξ_r and ξ_f and the repair alert parameter β. Because the model presented allows replacement if the system state crosses ξ_f, the replacement cycles constitute a renewal process. This embedded renewal process is used to derive expressions for the long-run average cost based on the repair alert parameter β and decision rules ξ_r, ξ_f. To demonstrate the use of this maintenance policy in practical applications, using Gamma process describing evolution of damage process X, an analytical method applied for non-periodic inspection policy is presented. The simple change to the model $\tau_n \mapsto \tau$ allows us to represent a maintenance policy under periodic inspection strategy. To demonstrate the use of this maintenance model in practical applications, employing the modified analytical method, a solution to the optimal decision rules ξ_r^*, ξ_f^* and the period of inspection τ^* is derived.

4.2 Model

Let $X = \{X_t : t \geq 0\}$ be a stochastic process with monotone paths which can influence time to failure of the system. The process X is a damage process which reflects the effect of the operating environment on the system. The state of the damage process X is determined by inspections which occur at times $\{T_n : n = 0, 1, 2, ...\}$, $T_0 = 0$ with inter-arrival times $\tau_{n+1} = T_{n+1} - T_n$, $(n \geq 0)$. The only available information is given by the history of the damage and virtual age process at inspection times just after repair action, i.e.

$$\mathcal{A}_n = \sigma\left\{(X_{T_k}, V_{T_k}) : k = 1, ..., n\right\}.$$

where X_{T_k} and V_{T_k} denote the damage process and virtual age process just after k^{th} repair action. Since the values of X are known only in some discrete points of time T_n $(n \geq 0)$, with the same approach as Makis and Jardine (31) we approximate the stochastic process $\{X_t, t \geq 0\}$ by the right continuous jump process $\{X_t^*, t \geq 0\}$ which increases by jumps at intervention times, otherwise is constant. More precisely,

$$X_n = X_t^* I(T_n \leq t < T_{n+1}).$$

This assumption implies that if f is a function defined on $\mathbb{R}_+^2$, then $f(X_t, V_t)$ for $t \in [T_n, T_{n+1})$ is measurable with respect to the filtration $\mathcal{A}_n$. In other words,

$$E[f(X_t, V_t)|\mathcal{A}_n] = f(X_t, V_t) = f(X_n, V_n + t - T_n) \tag{4.2.1}$$

The link between the lifetime indicator process $Z_t = I(T \leq t)$ adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)$ $(\mathcal{A}_n \subset F_t)$ and the bivariate state process (X_t, V_t) is modeled by the proportional intensity model (PIM) (see Cox (13))

$$\lambda(t, X_t, V_t) = \lambda_0(V_t)\psi(X_t), \quad t \geq 0 \tag{4.2.2}$$

which is product of a baseline intensity λ_0 dependent on the age of the system defined by the virtual age process $V = \{V_t : t \geq 0\}$ and a positive and increasing function ψ dependent on the stochastic process X. More precisely, it is assumed that Z_t admits the following semi-martingale representation:

$$Z_t = \int_0^t I(T > s)\lambda_0(V_s)\psi(X_s)ds + M_t,$$

where M_t is an $\mathcal{F}_t$ martingale. Obviously Z is the counting process corresponding to the simple point process (T_n^*) with $T = T_1^*$ and $T_n^* = \infty$ for $n \geq 2$. Processes of this kind are a realization of Cox processes or conditional Poisson processes if $\lambda(t, X_t, V_t)$ is $\mathcal{F}_0$ measurable for all $t \in \mathbb{R}_+$ (see Serfozo (67) and Bremaud (10)). Using the projection theorem (see Aven and Jensen (4)) we obtain for $T_n \leq t < T_{n+1}$,

$$\begin{aligned} E[Z_t|\mathcal{A}_n] &= 1 - \bar{F}_t^{(X_t,V_t)} \\ &= \int_0^t \mu_s^\lambda(X_s, V_s)ds + \bar{M}_t \end{aligned} \tag{4.2.3}$$

where $\bar{M}_t$ is an $\mathcal{A}_n$ martingale, for $t \in [T_n, T_{n+1})$,

$$\bar{F}_t^{(X_t,V_t)} = E[I(T > t)|\mathcal{A}_n] = P(T > t|\mathcal{A}_n)$$

and

$$\mu_t^\lambda(X_t, V_t) = E[I(T > t)\lambda_0(V_t)\psi(X_t)|\mathcal{A}_n].$$

Since $\lambda_0(V_t)\psi(X_t)$, $t \in [T_n, T_{n+1})$, is measurable with respect to the filtration $\mathcal{A}_n$ (see equation (4.2.1)), it follows

$$\mu_t^\lambda(X_t, V_t) = \lambda_0(V_t)\psi(X_t)E[I(T > s)|\mathcal{A}_n] \tag{4.2.4}$$

$$= \lambda_0(V_t)\psi(X_t)\bar{F}_t^{(X_t,V_t)} \tag{4.2.5}$$

By substituting the intensity measure (4.2.4) into the equation (4.2.3) we get

$$F_t^{(X_t,V_t)} = \int_0^t \lambda_0(V_s)\psi(X_s)\bar{F}_s^{(X_s,V_s)}ds + \bar{M}_t \tag{4.2.6}$$

From equation (4.2.6) it is easy to see that for $t \in [T_n, T_{n+1})$, $(n \geq 0)$

$$\begin{aligned} F_t^{(X_t,V_t)} &= \sum_{k=0}^{n-1} \psi(X_k) \int_{V_k}^{V_k+\tau_{k+1}} \lambda_0(u) \bar{F}_{u+l_k}^{(X_k,u)} du \\ &+ \psi(X_n) \int_{V_n}^{V_n+(t-T_n)} \lambda_0(u) \bar{F}_{u+l_n}^{(X_n,u)} du + \bar{M}_t \end{aligned}$$

that $l_n = T_n - V_n$. Using the fact that $F_t^{(X_t,V_t)}$ on $\{T_n \leq t < T_{n+1}\}$ has continuous paths of bounded variation, it follows that $\bar{M}_t = 0$ and

$$\begin{aligned} F_t^{(X_t,V_t)} &= \psi(X_n) \int_{V_n}^{V_n+(t-T_n)} \lambda_0(u) \bar{F}_{u+l_n}^{(X_n,u)} du \\ &+ F_{T_n}^{(X_n,V_n)} \end{aligned} \tag{4.2.7}$$

Simply, it can be shown that the solution of the resulting integral equation (4.2.7) for $t \in [T_n, T_{n+1})$ is

$$\begin{aligned} \bar{F}_t^{(X_t,V_t)} &= \bar{F}_{T_n}^{(X_n,V_n)} \\ &\times \exp\left(-\psi(X_n) \int_{V_n}^{V_n+(t-T_n)} \lambda_0(u) du \right) \end{aligned} \tag{4.2.8}$$

or, for $u \in [0, \tau_{n+1})$

$$\begin{aligned} \frac{\bar{F}_{T_n+u}^{(X_t,V_t)}}{\bar{F}_{T_n}^{(X_n,V_n)}} &= P\left(T > T_n + u | T > T_n, \mathcal{A}_n\right) \\ &= \exp(-\psi(X_n)[\Lambda_0(V_n + u) - \Lambda(V_n)]) \\ &= R_u^{(X_n,V_n)} = 1 - \bar{R}_u^{(X_n,V_n)}, \end{aligned} \tag{4.2.9}$$

where $\Lambda_0(u) = \int_0^u \lambda_0(s) ds$. The failure state process (4.2.9) can be viewed as a sequence of survival functions updated by each intervention. From equation (4.2.9) it simply follows for $T_n \leq t < T_{n+1}$, $(n \geq 0)$

$$\lambda(t, X_t, V_t) = \psi(X_n) \lambda_0(t - T_n + V_n),$$

which is a sequence of truncated hazard rates updated by virtual age V_n induced by repair action.

Using the observed damage state process X_{n+1} at inspection time T_{n+1}, the decision to perform repair is made at T_{n+1} based on the $\mathcal{A}_{n+1}^{-}$-adapted failure state process $\bar{R}_{T_{n+1}}^{(X_{n+1},V_n)}$ where $\mathcal{A}_{n+1}^{-}$ $(n \geq 0)$ includes the history of damage and virtual age processes up to time T_{n+1}, just before $(n+1)^{th}$ repair action, i.e.,

$$\mathcal{A}_{n+1}^{-} = \mathcal{A}_n \vee (X_{n+1}).$$

The decision maker with respect to repair and failure threshold ξ_r, ξ_f, $(0 < \xi_r < \xi_f < 1)$ and the system failure state process $\bar{R}_{T_{n+1}}^{(X_{n+1},V_n)}$ has disposition to adjust the virtual age of the system (imperfect repair), return the system to the 'good as new' state (perfect repair), or restore the system to its condition just prior to inspection (minimal repair). The first failure and replacement time of the system $T_{\xi_f}^{(x,v)}$ is defined as the first inspection time the failure process $\bar{R}_{T_{n+1}}^{(X_{n+1},V_n)}$, $(n \geq 0)$ reaches or exceeds a given threshold ξ_f $(0 < \xi_f < 1)$:

$$T_{\xi_f}^{(x,v)} = \inf\left\{T_{n+1} : \bar{R}_{T_{n+1}}^{(X_{n+1},V_n)} \geq \xi_f | (X_0, V_0) = (x, v)\right\}$$

Thus, the perfect repair is decided according to whether at $(n+1)^{th}$ inspection time the system has failed

$$\bar{R}_{T_{n+1}}^{(X_{n+1},V_n)} \geq \xi_f$$

or, is still working

$$\bar{R}_{T_{n+1}}^{(X_{n+1},V_n)} < \xi_f.$$

The imperfect repair mechanism based on the failure state process $\bar{R}_{T_{n+1}}^{(X_{n+1},V_n)}$, $(n \geq 0)$ and the repair and failure threshold ξ_r, ξ_f $(0 < \xi_r < \xi_f < 1)$ is as follows: If at $(n+1)^{th}$ inspection time $\bar{R}_{T_{n+1}}^{(X_{n+1},V_n)} < \xi_r$ $(n \geq 0)$ then the system is restored to its condition just prior to inspection (minimal repair), and the updated bivariate state process just after minimal repair i.e. $(X_{n+1}, V_n + \tau_{n+1})$ is incorporated into the equation (4.2.9) to evaluate the failure state of the system for $u \in [0, \tau_{n+2})$. If

$$\xi_r \leq \bar{R}_{\tau_{n+1}}^{(X_{n+1}, V_n)} < \xi_f$$

an imperfect repair just after $(n+1)^{th}$ inspection time is performed. The repair action updates the virtual age of the system $V_n + \tau_{n+1} \mapsto V_{n+1}$ and

$$\bar{R}_{\tau_{n+1}}^{(X_{n+1}, V_n)} \mapsto \bar{R}_{v_{n+1}^*}^{(X_{n+1}, V_n)}$$

where $v_{n+1}^* = V_{n+1} - V_n$ s.t. $v_{n+1}^* < \tau_{n+1}$. By incorporating the post repair bivariate state process $(X_{n+1}, V_n + v_{n+1}^*)$ into the equation (4.2.9), an estimate of the failure state of the system for $u \in [0, \tau_{n+2})$ is provided.

The improvement of the system via the imperfect repair is reflected in an age reduction factor (repair degree) which serves to adjust the virtual age of the system in a Kijima's type I manner (see Kahle (33)). In Kijima's type I model it is assumed that the repair action could remove damage created in the last sojourn. Precisely speaking, the virtual age of the system just after n^{th} repair action, V_n, is determined by $V_n = V_{n-1} + \xi_n \tau_n$, $(n \geq 1)$ where $\tau_n = T_n - T_{n-1}$ and ξ_n, $(0 \leq \xi_n \leq 1)$ denotes the repair degree at n^{th} inspection event.

As noted such repair can reset the virtual age of the system to that of a partially repaired system if $0 < \xi_n < 1$ or restore the system to its condition just before inspection (minimal repair) if $\xi_n = 1$. So, by using this fact given that

$$\xi_n = \xi \quad s.t. \quad \xi \in (0,1), \quad \forall n \geq 1 \tag{4.2.10}$$

the virtual age of the system just after n^{th} repair action with respect to the failure process at n^{th} inspection time $\bar{R}_{\tau_n}^{(X_n, V_{n-1})}$ and decision thresholds ξ_r, ξ_f can be expressed as

$$V_n = \begin{cases} V_{n-1} + \xi\tau_n, & \text{if } \xi_r \leq \bar{R}_{\tau_n}^{(X_n,V_{n-1})} < \xi_f; \\ V_{n-1} + \tau_n, & \text{if } \bar{R}_{\tau_n}^{(X_n,V_{n-1})} < \xi_r. \end{cases}$$

where the parameter ξ denotes the degree of imperfect repair. The condition (4.2.10) ensures that over the replacement cycle the system gradually deteriorates and becomes obsolete with time. By using the monotone property of ψ, the virtual age V_n can be represented as

$$V_n = \begin{cases} V_{n-1} + \xi\tau_n, & \text{if } \psi_{\xi_r}^{(\tau_n,v_{n-1})} \leq X_n < \psi_{\xi_f}^{(\tau_n,v_{n-1})}; \\ V_{n-1} + \tau_n, & \text{if } X_n < \psi_{\xi_r}^{(\tau_n,v_{n-1})}. \end{cases}$$

where

$$\psi_{\xi_{r(f)}}^{(\tau_n,v_{n-1})} = \psi^{-1}\left\{\frac{-\ln(1-\xi_{r(f)})}{\Lambda_0(\tau_n + V_{n-1}) - \Lambda_0(V_{n-1})}\right\}.$$

Briefly, the decision and action process are built up in following way: starting from state process $(X_{n-1}, V_{n-1}) = (x_{n-1}, v_{n-1})$ $(n \geq 1)$ evaluated just after $(n-1)^{th}$ repair, the n^{th} inspection is made at T_n. Inspection at time T_n reveals $X_n \mapsto x_n$. To perform a repair, the decision maker subject to the decision thresholds ξ_r, ξ_f and the failure state process $\bar{R}_{\tau_n}^{(x_n,v_{n-1})}$ driven by the known bivariate state process $(X_n, V_{n-1}) = (x_n, v_{n-1})$ either restore the system to its condition just prior to inspection that is $(x_n, v_{n-1} + \tau_n)$, adjust the system's virtual age $v_{n-1} + \tau_n \mapsto v_{n-1} + \xi\tau_n$, or return the system to the regeneration state $(X_0, V_0) = (x, v)$. The repair action determines the virtual age of the system at T_n i.e. $V_n = v_n$:

$$(x_n, v_n) = \begin{cases} \left(x_n, v_n^1\right), & \text{if } \bar{R}_{\tau_n}^{(x_n,v_{n-1})} < \xi_r; \\ \left(x_n, v_n^{\xi}\right), & \text{if } \xi_r \leq \bar{R}_{\tau_n}^{(x_n,n-1)} < \xi_f; \\ (x, v), & \text{if } \xi_f \leq \bar{R}_{\tau_n}^{(x_n,v_{n-1})}. \end{cases}$$

or, equivalently,

$$(x_n, v_n) = \begin{cases} \left(x_n, v_n^1\right), & \text{if } x_n < \psi_{\xi_r}^{(\tau_n,v_{n-1})}; \\ \left(x_n, v_n^{\xi}\right), & \text{if } \psi_{\xi_r}^{(\tau_n,v_{n-1})} \leq x_n < \psi_{\xi_f}^{(\tau_n,v_{n-1})}; \\ (x, v), & \text{if } \psi_{\xi_f}^{(\tau_n,v_{n-1})} \leq x_n. \end{cases}$$

where $v_n^1 = v_{n-1} + \tau_n$ and $v_n^\xi = v_{n-1} + \xi\tau_n$. If renewal does not take place at inspection time T_n, i.e.

$$\bar{R}_{\tau_n}^{(x_n,v_{n-1})} < \xi_f \quad or, \quad x_n < \psi_{\xi_f}^{(\tau_n,v_{n-1})},$$

then starting from state $(X_n, V_n) = (x_n, v_n)$, similarly as above the $(n+1)^{th}$ decision and action process are carried out at T_{n+1} and this process continues. As noted, the decision at inspection time based on last virtual age and current observed damage state determines a repair action and resulting action updates the virtual age. This series of decision and action events makes a sequence of bivariate state process (x_n, v_n), $(n \geq 1)$. Specifically, let the process starts in state $(X_0, V_0) = (x, v)$, and decisions can be made at periodic times $k\tau$, $(k = 1, 2, ...)$. Then, the virtual age of the system just after first repair action at time τ, $V_\tau^{(x,v)}$, is

$$V_\tau^{(x,v)} = \begin{cases} v + \xi\tau, & \text{if } a \leq X_\tau < b; \\ v + \tau, & \text{if } X_\tau < a. \end{cases}$$

where $a = \psi_{\xi_r}^{(\tau,v)}$ and $b = \psi_{\xi_f}^{(\tau,v)}$.

In the following section under periodic inspection policy an expression for the long-run average cost based on the decision thresholds ξ_r, ξ_f and the period of inspection τ is obtained.

4.3 Long-run average cost given periodic inspections policy

4.3.1 Expected cost per cycle

Let $C_\tau^{(x,v)}$ denote the cost of repair and maintenance actions per cycle starting from initial state value $(X_0, V_0) = (x, v)$: the system is instantaneously replaced by a new one at cost C_f and each (partial or minimal) repair and maintenance action incurs a cost determined by a random cost function $C_\tau^v(V_\tau)$, i.e.

$$C_\tau^v(V_\tau) = \begin{cases} C + C_\tau^v(v+\xi\tau), & \text{if } a \leq X_\tau < b; \\ C + C_\tau^v(v+\tau), & \text{if } X_\tau < a. \end{cases}$$

where the bounded cost measures C, $C_\tau^v(v+\tau)$, $C_\tau^v(v+\xi\tau)$ respectively denote the inspection cost, the minimal repair cost and imperfect repair cost to adjust the system age $v+\tau \mapsto v+\xi\tau$, $(0<\xi<1)$. It is assumed that the imperfect repair cost $C_\tau^v(v+\xi\tau)$ is a non-increasing function of the reduction factor ξ. That means, better repair induces higher cost of repair. Then a renewal type argument yields

$$C_\tau^{(x,v)} = C_f I\left(X_\tau \geq \psi_{\xi_f}^{(\tau,v)}\right) + \left(C_\tau^v(V_\tau) + C_\tau^{(X_\tau,V_\tau)}\right) I\left(X_\tau < \psi_{\xi_f}^{(\tau,v)}\right) \tag{4.3.1}$$

Or,

$$\begin{aligned} C_\tau^{(x,v)} &= \left(C_\tau^{(X_\tau,v+\xi\tau)} - C_\tau^{(X_\tau,v+\tau)}\right) I\left(\psi_{\xi_r}^{(\tau,v)} \leq X_\tau < \psi_{\xi_f}^{(\tau,v)}\right) \\ &\quad + C_f I\left(X_\tau \geq \psi_{\xi_f}^{(\tau,v)}\right) + \bar{C}_\tau(v;\xi) I\left(\psi_{\xi_r}^{(\tau,v)} \leq X_\tau < \psi_{\xi_f}^{(\tau,v)}\right) \\ &\quad + \left(C + C_\tau^v(v+\tau)\right) I\left(X_\tau < \psi_{\xi_f}^{(\tau,v)}\right) + C_\tau^{(X_\tau,v+\tau)} I\left(X_\tau < \psi_{\xi_f}^{(\tau,v)}\right) \end{aligned} \tag{4.3.2}$$

where $I(\cdot)$ is the indicator function and

$$\bar{C}_\tau(v;\xi) = C_\tau^v(v+\xi\tau) - C_\tau^v(v+\tau).$$

Let $T_{\psi_{\xi_{r(f)}}^{(\tau,v)}}^{(x,v)}$ denote the first time the damage process X_t, $t \in \mathbb{R}_+$ crosses the imperfect repair (failure) limit $\psi_{\xi_{r(f)}}^{(\tau,v)}$:

$$T_{\psi_{\xi_{r(f)}}^{(\tau,v)}}^{(x,v)} = \inf\left\{t : X_t \geq \psi_{\xi_{r(f)}}^{(\tau,v)} | (X_0, V_0) = (x,v)\right\}$$

then, using the fact that

$$T_{\psi_{\xi_{r(f)}}^{(\tau,v)}}^{(x,v)} \leq \tau \Leftrightarrow X_\tau \geq \psi_{\xi_{r(f)}}^{(\tau,v)}$$

$C_\tau^{(x,v)}$ can be represented as

$$\begin{aligned} C_\tau^{(x,v)} &= \left(C_\tau^{(X_\tau,v+\xi\tau)} - C_\tau^{(X_\tau,v+\tau)}\right) I\left(\psi_{\xi_r}^{(\tau,v)} \le X_\tau < \psi_{\xi_f}^{(\tau,v)}\right) \\ &+ C_f I\left(T^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}} \le \tau\right) + \bar{C}_r(v;\xi) I\left(T^{(x,v)}_{\psi_{\xi_r}^{(\tau,v)}} \le \tau < T^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}}\right) \\ &+ (C + C_r^v(v+\tau))\, I\left(T^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}} > \tau\right) + C_\tau^{(X_\tau,v+\tau)} I\left(X_\tau < \psi_{\xi_f}^{(\tau,v)}\right) \end{aligned} \tag{4.3.3}$$

Without loss of generality assume that the damage process X_t is a continuous-time process with continuous space state. The expected cost per cycle $\mu_{C_\tau^{(x,v)}}$ is

$$\begin{aligned} \mu_{C_\tau^{(x,v)}} &= (C + C_r^v(v+\tau))\, \bar{F}_{\psi_{\xi_f}^{(\tau,v)}}(\tau) \\ &+ C_f F_{\psi_{\xi_f}^{(\tau,v)}}(\tau) + \bar{C}_r(v;\xi)\left(\bar{F}_{\psi_{\xi_f}^{(\tau,v)}}(\tau) - \bar{F}_{\psi_{\xi_r}^{(\tau,v)}}(\tau)\right) \\ &+ \int_0^{\psi_{\xi_f}^{(\tau,v)}} \mu_{C_\tau^{(y,v+\tau)}} f_\tau(y|x)dy + \int_{\psi_{\xi_r}^{(\tau,v)}}^{\psi_{\xi_f}^{(\tau,v)}} \bar{\mu}_{C_\tau}(y;\xi) f_\tau(y|x)dy \end{aligned} \tag{4.3.4}$$

where

$$\bar{\mu}_{C_\tau}(y;\xi) = \mu_{C_\tau^{(y,v+\xi\tau)}} - \mu_{C_\tau^{(y,v+\tau)}},$$

and $f_\tau(y|x)$ is the transition density of the damage process X_t from $X_0 = x$ to $X_\tau = y$ and $F_{\psi_{\xi_r}^{(\tau,v)}}$, $F_{\psi_{\xi_f}^{(\tau,v)}}$ that $F = 1 - \bar{F}$ denote the distribution functions of the stopping times $T^{(x,v)}_{\psi_{\xi_r}^{(\tau,v)}}$, $T^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}}$ respectively.

4.3.2 Expected length per cycle

The expected length of a cycle is obtained similarly. The length of a cycle, $L_\tau^{(x,v)}$, is

$$L_\tau^{(x,v)} = T^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}} I\left(X_\tau \ge \psi_{\xi_f}^{(\tau,v)}\right) + \left(\tau + L_\tau^{(X_\tau,V_\tau)}\right) I\left(X_\tau < \psi_{\xi_f}^{(\tau,v)}\right) \tag{4.3.5}$$

But,

$$L_\tau^{(X_\tau,V_\tau)} = \begin{cases} L_\tau^{(X_\tau,v+\xi\tau)}, & \text{if } a \le X_\tau < b; \\ L_\tau^{(X_\tau,v+\tau)}, & \text{if } X_\tau < a. \end{cases}$$

Thus,

$$\begin{aligned} L_\tau^{(x,v)} &= T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}} I\left(X_\tau \geq \psi^{(\tau,v)}_{\xi_f}\right) + \left(\tau + \mathrm{L}_\tau^{(X_\tau, v+\tau)}\right) I\left(X_\tau < \psi^{(\tau,v)}_{\xi_f}\right) \\ &+ \left(\mathrm{L}_\tau^{(X_\tau, v+\xi\tau)} - \mathrm{L}_\tau^{(X_\tau, v+\tau)}\right) I\left(\psi^{(\tau,v)}_{\xi_r} \leq X_\tau < \psi^{(\tau,v)}_{\xi_f}\right) \end{aligned} \tag{4.3.6}$$

Or, equivalently

$$\begin{aligned} L_\tau^{(x,v)} &= T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}} I\left(T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}} \leq \tau\right) + \tau I\left(T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}} > \tau\right) + \mathrm{L}_\tau^{(X_\tau, v+\tau)} I\left(X_\tau < \psi^{(\tau,v)}_{\xi_f}\right) \\ &+ \left(\mathrm{L}_\tau^{(X_\tau, v+\xi\tau)} - \mathrm{L}_\tau^{(X_\tau, v+\tau)}\right) I\left(\psi^{(\tau,v)}_{\xi_r} \leq X_\tau < \psi^{(\tau,v)}_{\xi_f}\right) \end{aligned} \tag{4.3.7}$$

The expected length per cycle, $\mu_{L_\tau^{(x,v)}}$, is

$$\begin{aligned} \mu_{L_\tau^{(x,v)}} &= \int_0^\tau \bar{F}_{\psi^{(\tau,v)}_{\xi_f}}(u)du \\ &+ \int_{\psi^{(\tau,v)}_{\xi_r}}^{\psi^{(\tau,v)}_{\xi_f}} \bar{\mu}_{L_r}(y;\xi) f_\tau(y|x)dy + \int_0^{\psi^{(\tau,v)}_{\xi_f}} \mu_{L_\tau^{(y,v+\tau)}} f_\tau(y|x)dy. \end{aligned} \tag{4.3.8}$$

where

$$\bar{\mu}_{L_r}(y;\xi) = \mu_{L_\tau^{(y,v+\xi\tau)}} - \mu_{L_\tau^{(y,v+\tau)}}.$$

4.3.3 Long-run average cost

Now, the problem is to minimize the long-run average cost per unit time subject to the period of inspection τ and the preventive partial repair and the preventive replacement rule ξ_r, ξ_f.

Let $\mu_\tau^{(x,v)}$ be the long-run average cost per unit time given the start state $(X_0, V_0) = (x, v)$. Because given the assumptions (4.2.10) the virtual age process V_n is increasing in the number of both minimal and imperfect repair events, i.e.

$$V_{n+1} > V_n, \quad n \geq 1$$

the failure process $\bar{R}_{\tau}^{(X_{n+1},V_n)}$ tends to 1 as $n \to \infty$. This property subject to the failure threshold ξ_f that $0 < \xi_f < 1$ implies the existence of regeneration time points $T_{\xi_f}^{(x,v)}$:

$$T_{\xi_f}^{(x,v)} = \inf \left\{ T_{n+1} : \bar{R}_{\tau}^{(X_{n+1},V_n)} \geq \xi_f | (X_0, V_0) = (x, v) \right\}, \quad n \geq 0$$

Since, the sequence of failure and replacement times $T_{\xi_f}^{(x,v)}$ forms a regenerative process, the inter-arrival time between two consecutive replacements is a regenerative cycle. Theses regeneration cycles form an embedded renewal process. As noted our process is based on a renewal reward argument with policy $\{T_1 < T_2 < \dots < T_n < T_{n+1} < \dots\}$ with possibly irregularly spaced inspections. In the case that the catastrophic failure of the system occurs within the n^{th} interval, $T \in [T_n, T_{n+1})$, because the failure probability of the system is large enough, $\bar{R}_{t-T_n}^{(X_n,V_n)} \to 1$ for $t \in [T_n, T_{n+1})$, the reliability threshold ξ_f is achieved at subsequent inspection time T_{n+1} after system failure time $T \in [T_n, T_{n+1})$ and renewal (replacement) takes place. The replacement instants in both the periodic and non-periodic (see section 4.6) define the embedded renewal process. This implies that modelling includes the probability of failing within interval. This approach to failure modelling has been addressed by Newby and Dagg (56) and Newby and Barker (55). According to the standard renewal reward theorem (see Ross (66)), we have

$$\mu_{\tau}^{(x,v)} = \frac{\mu_{C_{\tau}^{(x,v)}}}{\mu_{L_{\tau}^{(x,v)}}}, \tag{4.3.9}$$

where regarding equations (4.3.4) and (4.3.8), expected cost per cycle $\mu_{C_{\tau}^{(x,v)}}$ and expected length per cycle $\mu_{L_{\tau}^{(x,v)}}$ are solution of following integral equation:

$$\begin{aligned} \mu_{C_{\tau}^{(x,v)}} &= g_{C_{\tau}^{(x,v)}} + \int_{\psi_{\xi_r}^{(\tau,v)}}^{\psi_{\xi_f}^{(\tau,v)}} \bar{\mu}_{C_{\tau}}(y;\xi) f_{\tau}(y|x) dy \\ &+ \int_{x}^{\psi_{\xi_f}^{(\tau,v)}} \mu_{C_{\tau}^{(y,v+\tau)}} f_{\tau}(y|x) dy \end{aligned} \tag{4.3.10}$$

where

$$
\begin{aligned}
g_{C_\tau^{(x,v)}} &= (C + C_r^v(v+\tau))\,\bar{F}^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}}(\tau) \\
&+ C_f F^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}}(\tau) + \bar{C}_r(v;\xi)\left(\bar{F}^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}}(\tau) - F^{(x,v)}_{\psi^{(\tau,v)}_{\xi_r}}(\tau)\right)
\end{aligned}
\tag{4.3.11}
$$

and

$$
\begin{aligned}
\mu_{L_\tau^{(x,v)}} &= h_{L_\tau^{(x,v)}} + \int_{\psi^{(\tau,v)}_{\xi_r}}^{\psi^{(\tau,v)}_{\xi_f}} \bar{\mu}_{L_r}(y;\xi) f_\tau(y|x)dy \\
&+ \int_x^{\psi^{(\tau,v)}_{\xi_f}} \mu_{L_\tau^{(y,v+\tau)}} f_\tau(y|x)dy
\end{aligned}
\tag{4.3.12}
$$

that

$$
h_{L_\tau^{(x,v)}} = \int_0^\tau \bar{F}^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}}(u)du = E\big(T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}} | T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}} < \tau\big) + \tau \bar{F}^{(x,v)}_{\psi^{(\tau,v)}_{\xi_f}}(\tau). \tag{4.3.13}
$$

Starting in state $(X_0, V_0) = (0; 0)$, the optimal period of inspection and repair and failure threshold can be determined as

$$
(\tau^*, \xi_r^*, \xi_f^*) = \underset{(\tau,\xi_r,\xi_f)\in\mathbb{R}_+\times[0,\xi_f)\times[0,\infty)}{\arg\min} \left\{\mu_\tau^{(0,0)}\right\}.
$$

Thus, the optimal maintenance policy characterized by the optimization of the long-run average cost per unit time will lead to an optimal inspection policy τ^* and preventive maintenance rules -repair/replacement policy- ξ_r^* and ξ_f^*.

4.3.4 The long average cost under minimal repair and replacement policy

In particular case, let the virtual age of the system just after partial repair leave unchanged (minimal repair) or precisely speaking, parameter ξ which reflects the degree of partial repair be large enough: $\xi \to 1$. Given above assumption which restricts the repair action space to just minimal repair and replacement, the integral equations (4.3.10) and (4.3.12) respectively reduce to the following integral equations

$$\mu_{C_\tau^{(x,v)}} = g_{C_\tau^{(x,v)}} + \int_x^{\psi_{\xi_f}^{(\tau,v)}} \mu_{C_\tau^{(y,v+\tau)}} f_\tau(y|x)dy \tag{4.3.14}$$

and

$$\mu_{L_\tau^{(x,v)}} = h_{L_\tau^{(x,v)}} + \int_x^{\psi_{\xi_f}^{(\tau,v)}} \mu_{L_\tau^{(y,v+\tau)}} f_\tau(y|x)dy \tag{4.3.15}$$

with corresponding long-run average cost per unit time

$$\mu_{\tau(x,v)} = \frac{g_{C_\tau^{(x,v)}} + \int_x^{\psi_{\xi_f}^{(\tau,v)}} \mu_{C_\tau^{(y,v+\tau)}} f_\tau(y|x)dy}{h_{L_\tau^{(x,v)}} + \int_x^{\psi_{\xi_f}^{(\tau,v)}} \mu_{L_\tau^{(y,v+\tau)}} f_\tau(y|x)dy} \tag{4.3.16}$$

where

$$g_{C_\tau^{(x,v)}} = C_f + (C + C_\tau^v(v+\tau) - C_f)\,\bar{F}^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}}(\tau)$$

and

$$h_{L_\tau^{(x,v)}} = \int_0^\tau \bar{F}^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}}(u)du.$$

So, the optimal maintenance policy characterized by the optimization of the long-run average cost per unit time will lead to an optimal inspection policy τ^* and replacement policy ξ_f^*:

$$\left(\tau^*, \xi_f^*\right) = \underset{(\tau,\xi_f)\in\mathbb{R}_+\times[0,\infty)}{\arg\min} \left\{\mu_\tau^{(0,0)}\right\}.$$

4.3.5 The long average cost under partial repair and replacement policy

In the case that the partial repair limit is small enough: $\psi_{\xi_r} \to x$, the action space reduces to the partial repair and replacement. Provided this assumption the integral equations (4.3.10) and (4.3.12) respectively reduce to

$$\mu_{C_\tau^{(x,v)}} = g_{C_\tau^{(x,v)}} + \int_x^{\psi_{\xi_f}^{(\tau,v)}} \mu_{C_\tau^{(y,v+\xi\tau)}} f_\tau(y|x)dy \tag{4.3.17}$$

and

$$\mu_{L_\tau^{(x,v)}} = h_{L_\tau^{(x,v)}} + \int_x^{\psi_{\xi_f}^{(\tau,v)}} \mu_{L_\tau^{(y,v+\xi\tau)}} f_\tau(y|x)dy \tag{4.3.18}$$

with corresponding long-run average cost function

$$\mu_\tau^{(x,v)} = \frac{g_{C_\tau^{(x,v)}} + \int_x^{\psi_{\xi_f}^{(\tau,v)}} \mu_{C_\tau^{(y,v+\xi\tau)}} f_\tau(y|x)dy}{h_{L_\tau^{(x,v)}} + \int_x^{\psi_{\xi_f}^{(\tau,v)}} \mu_{L_\tau^{(y,v+\xi\tau)}} f_\tau(y|x)dy} \tag{4.3.19}$$

where

$$g_{C_\tau^{(x,v)}} = C_f + (C + C_r^v(v+\xi\tau) - C_f)\,\bar{F}^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}}(\tau)$$

and

$$h_{L_\tau^{(x,v)}} = \int_0^\tau \bar{F}^{(x,v)}_{\psi_{\xi_f}^{(\tau,v)}}(u)du.$$

As above the optimal maintenance policy is characterized by the optimization of the long-run average cost per unit time with respect to inspection time τ and replacement threshold ξ_f:

$$(\tau^*, \xi_f^*) = \underset{(\tau,\xi_f)\in\mathbb{R}_+\times[0,\infty)}{\arg\min} \left\{\mu_\tau^{(0,0)}\right\}.$$

Clearly, if the repair degree ξ is large enough: $\xi \to 1$, then the long-run average cost (4.3.19) approaches to the equation (4.3.16).

In the following section, an analytical method to solve the maintenance optimization problem is proposed.

4.4 A numerical iteration algorithm to solving optimization problem

To find a solution to optimization problem of $\mu_\tau^{(x,v)}$ (see equation (4.3.9)), let $\phi_c(x,v)$ and $\phi_l(x,v)$ respectively denote the expected cost and expected length per cycle given initial state $(X_0, V_0) = (x, v)$. It is easy to see from equations (4.3.10) and (4.3.12), both $\phi_c(x,v)$ and $\phi_l(x,v)$ are solution of following integral equation:

$$\phi(x,v) = f(x,v) + \int_{\psi_{\xi_r}^{(\tau,v)}}^{\psi_{\xi_f}^{(\tau,v)}} f_\tau(y|x)\phi(y, v+\xi\tau)dy + \int_{x}^{\psi_{\xi_r}^{(\tau,v)}} f_\tau(y|x)\phi(y, v+\tau)dy \quad (4.4.1)$$

where $f(x,v)$ refers to $g_{C_\tau^{(x,v)}}$ or $h_{L_\tau^{(x,v)}}$.

To present a solution to the integral equation (4.4.1), let (x_k, v_k) $(k = 0, 1, ...)$ imply the state of the process just after k^{th} inspection. Given starting state $(X_k, V_k) = (x_k, v_k)$, from equation (4.4.1) the $\phi(x_k, v_k)$ can be expressed as

$$\phi^{(k)}(x_k, v_k) = \phi_1^{(k)} + \phi_2^{(k)} + \phi_3^{(k)}, \quad (4.4.2)$$

with

$$\begin{aligned} \phi_1^{(k)} &= f(x_k, v_k), \\ \phi_2^{(k)} &= \int_{\psi_{\xi_r}^{(\tau,v_k)}}^{\psi_{\xi_f}^{(\tau,v_k)}} f_\tau(y|x_k)\phi^{(k+1)}(y, \underbrace{v_k+\xi\tau}_{v_{k+1}})dy, \\ \phi_3^{(k)} &= \int_{x_k}^{\psi_{\xi_r}^{(\tau,v_k)}} f_\tau(y|x_k)\phi^{(k+1)}(y, \underbrace{v_k+\tau}_{v_{k+1}})dy, \end{aligned}$$

where $\phi(x_k, v_k) = \phi^{(k)}(x_k, v_k)$ and

$$v_{k+1} = \begin{cases} v_k + \tau, & x_k < x_{k+1} < \psi_{\xi_r}^{(\tau,v_k)}; \\ v_k + \xi\tau, & \psi_{\xi_r}^{(\tau,v_k)} \le x_{k+1} < \psi_{\xi_f}^{(\tau,v_k)}. \end{cases} \quad (4.4.3)$$

By conditioning on the value of the damage process at $(k+1)^{th}$ inspection time, $X_{k+1} = x_{k+1}$, the equation (4.4.2) can be represented as

$$\phi(x_k, v_k) = \begin{cases} \phi_1^{(k)}, & \psi_{\xi_f}^{(\tau,v_k)} \leq x_{k+1}; \\ \phi_{12}^{(k)}, & \psi_{\xi_r}^{(\tau,v_k)} \leq x_{k+1} < \psi_{\xi_f}^{(\tau,v_k)}; \\ \phi_{13}^{(k)}, & x_k < x_{k+1} < \psi_{\xi_r}^{(\tau,v_k)}. \end{cases} \tag{4.4.4}$$

where $\phi_{12}^{(k)} = \phi_1^{(k)} + \phi_2^{(k)}$ and $\phi_{13}^{(k)} = \phi_1^{(k)} + \phi_3^{(k)}$. Assume that $m-1$ $(m \geq 1)$ and $n-m$ $(n \geq m)$ respectively denote the number of minimal and partial repair(s) and T_m and T_n are the first time the damage process X_t $(t \in \mathbb{R}_+)$ reaches or exceeds a given partial repair and failure threshold $\psi_{\xi_r}^{(\tau,v_t)}$ and $\psi_{\xi_f}^{(\tau,v_t)}$ respectively:

$$T_m = \inf\left\{t \in \mathbb{R}_+ : X_t \geq \psi_{\xi_r}^{(\tau,v_t)}\right\} = \inf\left\{T_k : X_{k\tau} \geq \psi_{\xi_r}^{(\tau,v_{k-1})}\right\}, \tag{4.4.5}$$

$$T_n = \inf\left\{t \in \mathbb{R}_+ : X_t \geq \psi_{\xi_f}^{(\tau,v_t)}\right\} = \inf\left\{T_k : X_{k\tau} \geq \psi_{\xi_f}^{(\tau,v_{k-1})}\right\}, \tag{4.4.6}$$

where $n-1$ $(n \geq 1)$ denote the whole number of minimal and partial repairs. Given starting state $(X_{n-1}, V_{n-1}) = (x_{n-1}, v_{n-1})$, from equations (4.4.2) and (4.4.6), it is easy to see that

$$\phi^{(n-1)}(x_{n-1}, v_{n-1}) = f(x_{n-1}, v_{n-1}); \qquad n-1 \geq m, \tag{4.4.7}$$

Given that just after $(n-2)^{th}$ inspection a partial repair is performed i.e. $m \leq n-2$, since $v_{n-1} = v_{n-2} + \xi\tau$ (see equation (4.4.3)), from equation (4.4.4) we have

$$\begin{aligned} \phi^{(n-2)}(x_{n-2}, v_{n-2}) = f(x_{n-2}, & v_{n-2}) \\ & + \int_{\psi_{\xi_r}^{(\tau,v_{n-2})}}^{\psi_{\xi_f}^{(\tau,v_{n-2})}} f_\tau(y|x_{n-2})\phi^{(n-1)}(y, \underbrace{v_{n-2} + \xi\tau}_{v_{n-1}})dy \end{aligned} \tag{4.4.8}$$

But, in terms of initial condition (4.4.7), equation (4.4.8) can be represented as

$$\phi^{(n-2)}(x_{n-2}, v_{n-2}) = f(x_{n-2}, v_{n-2}) + \int_{\psi_{\xi_r}^{(\tau,v_{n-2})}}^{\psi_{\xi_f}^{(\tau,v_{n-2})}} f_\tau(y|x_{n-2})f(y, v_{n-1})dy \tag{4.4.9}$$

Thus, recursively $\phi^{(n-i)}(x_{n-i}, v_{n-i})$ for $1 \le i \le n$ can be calculated as

$$\phi(x_{n-i}, v_{n-i}) = \begin{cases} \phi_1^{(n-i)}, & i = 1; \\ \phi_{12}^{(n-i)}, & 2 \le i \le n-m+1; \\ \phi_{13}^{(n-i)}, & n-m+2 \le i \le n. \end{cases} \tag{4.4.10}$$

where

$$v_{n-i+1} = \begin{cases} v_{n-i} + \xi\tau, & 2 \le i \le n-m+1; \\ v_{n-i} + \tau, & n-m+2 \le i \le n. \end{cases} \tag{4.4.11}$$

As seen, given starting state $(X_0, V_0) = (x, v)$, equation (4.4.10) recursively gives a solution to both expected cost $\phi_c(x, v)$ and expected length per cycle $\phi_l(x, v)$ which are function of control parameters:

$$\mu_\tau^{(x,v)} = \frac{\phi_c(x, v)}{\phi_l(x, v)}.$$

From recursive relation (4.4.10) it is easy to see that both $\phi_c(x, v)$ and $\phi_c(x, v)$ are bounded. This comes from the fact that for $\xi_r \in (0, \xi_f)$, $\xi_f \in (0, 1)$ integrands

$$f_\tau(y|x_{n-i})\phi^{(n-i+1)}(y, v_{n-i+1}), \qquad 2 \le i \le n$$

are continuous on the closed, bounded intervals

$$\left[x_{n-i}, \psi_{\xi_r}^{(\tau, v_{n-i})}\right], \quad 2 \le i \le n-m+1$$

and

$$\left[\psi_{\xi_r}^{(\tau, v_{n-i})}, \psi_{\xi_f}^{(\tau, v_{n-i})}\right], \quad n-m+2 \le i \le n$$

and functions $f_c(x_{n-i}, v_{n-i})$ and $f_l(x_{n-i}, v_{n-i})$ corresponding to $\phi_c(x, v)$ and $\phi_l(x, v)$ $\forall i \in \{1 : n\}$ are bounded, i.e.

$$0 < f_c(x_{n-i}, v_{n-i}) \le M_c^{n-i}$$

where $M_c^{n-i} = C + C_f + C_r^{v_{n-i}}(v_{n-i} + \xi\tau)$ and for $c_0 \in (0, \tau)$

$$0 < c_0\Big(\bar{F}_{\psi_{\xi_f}^{(\tau,v_{n-i})}}(c_0) - \bar{F}_{\psi_{\xi_f}^{(\tau,v_{n-i})}}(\tau)\Big) \leq f_l(x_{n-i}, v_{n-i}) \leq \tau,$$

which implies that the long-run average cost per unit time is bounded:

$$0 < \mu_\tau^{(x,v)} < \infty.$$

In the next section, using the recursive procedures (4.4.10), numerically such a preventive maintenance policy subject to the decision thresholds ξ_r, ξ_f and the period of inspection τ is illustrated. Numerical result is based on Gamma process describing damage state process X_t $(t \geq 0)$.

4.5 Optimizing model

4.5.1 Deterioration model based on Gamma process

Let the damage process X_t be described by a stationary Gamma process with shape parameter $\gamma > 0$ and scale parameter δ:

$$X_t - X_s \sim G(\gamma(t-s), \delta)$$

and

$$E(X_t - X_s) = \frac{\gamma(t) - \gamma(s)}{\delta}$$

where shape parameter is linear in t, $\gamma(t) = \gamma \times t$.

The transition probability density function $f_\tau(y|x)$, the density of X_τ given $X_0 = x$, is the gamma density

$$f_\tau(y|x) = \frac{\delta^{\gamma\tau}(y-x)^{\gamma\tau-1}e^{-\delta(y-x)}}{\Gamma(\gamma\tau)} \tag{4.5.1}$$

If the state of the process at initial time is $(X_0, V_0) = (x, v)$, the cumulative distribution of the hitting time $T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_{r(f)}}}$ of the partial repair (failure) barrier at $\psi^{(\tau,v)}_{\xi_{r(f)}}$ is

$$F^{(x,v)}_{\psi^{(\tau,v)}_{\xi_{r(f)}}}(t) = P(T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_{r(f)}}} \le t) = P(X_t > \psi^{(\tau,v)}_{\xi_{r(f)}} | X_0 = x) = \int_{\psi^{(\tau,v)}_{\xi_{r(f)}}}^{\infty} f_\beta(y|x)dy. \tag{4.5.2}$$

The distribution of $T^{(x,v)}_{\psi^{(\tau,v)}_{\xi_{r(f)}}}$ can be expressed as ratio of an incomplete gamma function:

$$F^{(x,v)}_{\psi^{(\tau,v)}_{\xi_{r(f)}}}(t) = \frac{\Gamma(\gamma t; \delta(\psi^{(\tau,v)}_{\xi_{r(f)}} - x))}{\Gamma(\gamma t)} \tag{4.5.3}$$

where $\Gamma(\gamma; x_0)$ is an incomplete gamma function as

$$\Gamma(\gamma; x_0) = \int_{x_0}^{\infty} t^{\gamma-1} e^{-t} dt\,.$$

From stationary and independent increments property of Gamma process it is easy to show that an smooth semi-martingale representation of X_t is

$$\begin{aligned} X_t &= X_0 + \int_0^t \frac{\gamma}{\delta} ds + M_t \\ &= x + \frac{\gamma}{\delta} t + M_t, \end{aligned} \tag{4.5.4}$$

which M_t is an $\mathcal{F}$-martingale.

Let $T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}$ $(0 \le k \le m-1)$ denote the partial repair stopping time given starting state (X_k, V_k). Using equation (4.5.4) an $\mathcal{A}_k$-SSM representation of the damage process at stopping time $T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}$ is

$$E\left(X_{T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}} \middle| \mathcal{A}_k \right) = E\left(X_k | \mathcal{A}_k\right) + E\left(\int_0^{T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}} \frac{\gamma}{\delta} ds \middle| \mathcal{A}_k \right) + E\left(M_{T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}} \middle| \mathcal{A}_k \right) \tag{4.5.5}$$

Applying the optional sampling theorem (see Aven and Jensen (4)) to the $\mathcal{A}_k$-martingale term

$$\hat{M}_{T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}} = E\left(M_{T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}} \middle| \mathcal{A}_k \right)$$

we have

$$E\left(X_{T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}}\middle|\mathcal{A}_k\right) = E\left(X_k|\mathcal{A}_k\right) + E\left(\int_0^{T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}} \frac{\gamma}{\delta}ds\middle|\mathcal{A}_k\right) \tag{4.5.6}$$

Since X_k and $\psi^{(\tau,v_k)}_{\xi_r}$ are measurable with respect to $\mathcal{A}_k$ it follows

$$\begin{aligned}\psi^{(\tau,v_k)}_{\xi_r} &= E\left(\psi^{(\tau,v_k)}_{\xi_r}\middle|\mathcal{A}_k\right) = E\left(X_{T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}}\middle|\mathcal{A}_k\right)\\ &= X_k + \frac{\gamma}{\delta}E\left(T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}\middle|\mathcal{A}_k\right)\end{aligned} \tag{4.5.7}$$

From equation (4.5.7), a random measure of the mean hitting time to partial repair threshold $\psi^{(\tau,v_k)}_{\xi_r}$ given starting state (X_k, V_k) $(0 \leq k \leq m-1)$ is

$$\begin{aligned}\mu^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}} = E\left(T^{(x_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}}\middle|\mathcal{A}_k\right) &= \frac{\delta}{\gamma}\left(\psi^{(\tau,v_k)}_{\xi_r} - X_k\right)\\ &= \frac{\delta}{\gamma}\left(\frac{-\ln(1-\xi_r)}{\Lambda_0(\tau+V_k)-\Lambda_0(V_k)} - X_k\right).\end{aligned} \tag{4.5.8}$$

Since,

$$\mu_k = E(X_k) = \frac{\gamma}{\delta}k\tau,$$

starting in state $(X_k, V_k) = (x_k, v_k)$ an estimate of the mean time to partial repair can be evaluated by

$$\hat{\mu}^{(\mu_k,v_k)}_{\psi^{(\tau,v_k)}_{\xi_r}} = \frac{\delta}{\gamma}\left(\frac{-\ln(1-\xi_r)}{\Lambda_0(\tau+v_k)-\Lambda_0(v_k)} - \mu_k\right) \tag{4.5.9}$$

Because $T_{m-1} = (m-1)\tau$ is the last inspection time before the damage process reaches or exceeds the partial repair barrier $\psi^{(\tau,v_{m-1})}_{\xi_r}$ (see equation (4.4.5)), we have

$$\hat{\mu}^{(\mu_{m-1},v_{m-1})}_{\psi^{(\tau,v_{m-1})}_{\xi_r}} \leq \tau \tag{4.5.10}$$

Or, equivalently

$$\frac{-\frac{\delta}{\tau\gamma}\ln(1-\xi_r)}{\Lambda_0(\tau+v_{m-1})-\Lambda_0(v_{m-1})} \leq m \tag{4.5.11}$$

So, the minimum number of inspections required to exceed the partial repair threshold is

$$m = \left\lfloor \frac{-\frac{\delta}{\tau\gamma}\ln(1-\xi_r)}{\Lambda_0(\tau+v_{m-1})-\Lambda_0(v_{m-1})} \right\rfloor + 1. \tag{4.5.12}$$

where $\lfloor\cdot\rfloor$ is the floor function.

With the same argument as above, starting in state (X_k, V_k) for $m \le k \le n-1$, mean time to reach failure threshold $\psi_{\xi_f}^{(\tau,v_k)}$ is

$$\begin{aligned}\mu_{\psi_{\xi_f}^{(\tau,v_k)}}^{(x_k,v_k)} = E\left(T_{\psi_{\xi_f}^{(\tau,v_k)}}^{(x_k,v_k)}\middle|\mathcal{A}_k\right) &= \frac{\delta}{\gamma}\left(\psi_{\xi_f}^{(\tau,v_k)} - X_k\right)\\ &= \frac{\delta}{\gamma}\left(\frac{-\ln(1-\xi_f)}{\Lambda_0(\tau+V_k)-\Lambda_0(V_k)} - X_k\right)\end{aligned} \tag{4.5.13}$$

where for $m \le k \le n-1$,

$$V_k = (m-1)\tau + (k-m+1)\xi\tau$$

is the virtual age of the system just after k^{th} inspection.

Since,

$$\mu_k = E(X_k) = \frac{\gamma}{\delta}k\tau,$$

starting in state $(X_k, V_k) = (x_k, v_k)$ an estimate of the mean time to failure is

$$\hat{\mu}_{\psi_{\xi_f}^{(\tau,v_k)}}^{(\mu_k,v_k)} = \frac{\delta}{\gamma}\left(\frac{-\ln(1-\xi_f)}{\Lambda_0(\tau+v_k)-\Lambda_0(v_k)} - \mu_k\right) \tag{4.5.14}$$

Clearly, for $k = n-1$, using equation (4.4.6), we have

$$\hat{\mu}_{\psi_{\xi_f}^{(\tau,v_{n-1})}}^{(\mu_{n-1},v_{n-1})} \le \tau$$

Or, equivalently

$$\frac{-\frac{\delta}{\tau\gamma}\ln(1-\xi_f)}{\Lambda_0(\tau+v_{n-1})-\Lambda_0(v_{n-1})} \le n \tag{4.5.15}$$

Thus, the minimum number of inspections required to exceed the failure threshold is

$$n = \left\lfloor \frac{-\frac{\delta}{\tau\gamma}\ln(1-\xi_f)}{\Lambda_0(\tau+v_{n-1})-\Lambda_0(v_{n-1})} \right\rfloor + 1. \tag{4.5.16}$$

In particular, to get an evolution of the mean hitting time subject to state process (μ_k, v_k), let $\tau = 0.32$, $(\xi_r, \xi_f) = (0.3, 0.8)$, $\xi = 0.8$, $\psi(x) = x$, $\lambda_0(t) = t$ and $\delta = 2\gamma$. Using equation (4.5.12) and (4.5.16), the least number of inspections to exceed partial repair and failure threshold are $m = 5$ and $n = 11$. Figure 4.1 illustrates an evolution of mean hitting time to partial repair threshold $\xi_r = 0.3$ (right sub-figure) and failure threshold $\xi_f = 0.8$ (left sub-figure) given bivariate state process (μ_k, v_k), for $k = 0, 1, .., 10$. As shown the mean time to hit the partial repair and failure threshold $\psi_{\xi_{r(f)}}^{(\tau, v_k)}$ decreases as a function of number of inspections and bivariate state process (μ_k, v_k).

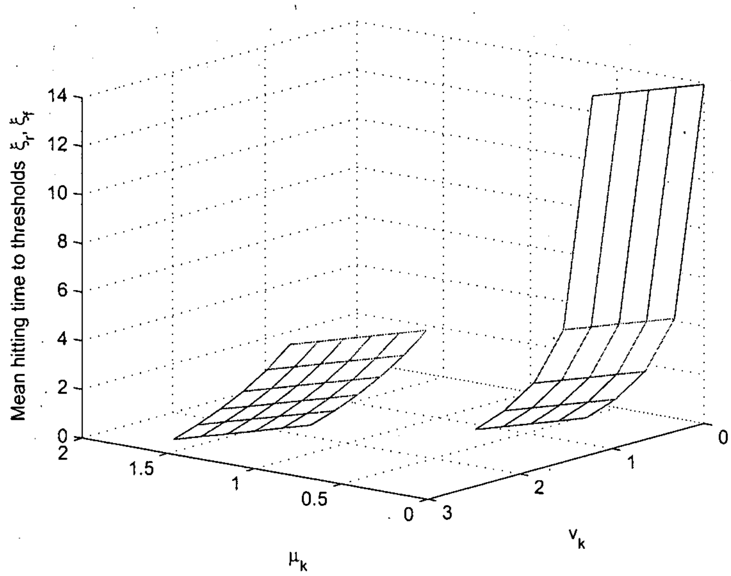

Figure 4.1: Mean hitting time to partial repair and failure threshold $\xi_r = 0.3$, $\xi_f = 0.8$ as a function of bivariate state process (μ_k, v_k), given $\tau = 0.32$, $\xi = 0.8$ and $(m, n) = (5, 11)$.

4.5.2 Maintenance optimization methodology

Following subject to the system parameters (ξ_r, ξ_f) and the period of inspection time τ a solution to optimization problem of the long-run average cost per unit time $\mu_\tau^{(0,0)}$ is proposed. For fixed number of inspections (m_0, n_0) $(1 \leq m_0 \leq n_0)$, let

$$\mu_\tau^{(0,0)} = \mu_{\tau(m_0,n_0)}^{(0,0)}(\underline{\xi}(m_0, n_0)),$$

where

$$\underline{\xi}(m_0, n_0) = (\xi_r(m_0, n_0), \xi_f(m_0, n_0)).$$

Applying recursive relation (4.4.10), an optimal solution to the period of inspection $\tau^*(m_0, n_0)$ and the repair and failure threshold parameters $\underline{\xi}^*(m_0, n_0)$ is obtained such that

$$(\tau^*(m_0, n_0), \underline{\xi}^*(m_0, n_0)) = \underset{\underline{\xi}^\tau(m_0,n_0)\in S^\tau}{\arg\min} \ \mu_\tau^{(0,0)} \tag{4.5.17}$$

that $\underline{\xi}^\tau(m_0, n_0) = (\tau(m_0, n_0), \underline{\xi}(m_0, n_0))$ and $S^\tau = \mathbb{R}_+ \times [0, \xi_f(m_0, n_0)) \times [0, \infty)$.

If for fixed number m_0 and n_0, $\tau^*(m_0, n_0)$ and $\xi^*_{r(f)}(m_0, n_0)$ do not satisfy the equations (4.5.12) and (4.5.16) i.e., m_0 or n_0 greater (less) than the optimal inspection frequencies $L_{\xi^*_{r(f)}}$ where

$$L_{\xi_r^*} = \left\lfloor \frac{-\frac{\delta}{\gamma\tau^*(m_0,n_0)} \ln(1 - \xi_r^*(m_0, n_0))}{\Lambda_0(\tau^*(m_0, n_0) + v_{m_0-1}) - \Lambda_0(v_{m_0-1})} \right\rfloor + 1$$

and

$$L_{\xi_f^*} = \left\lfloor \frac{-\frac{\delta}{\gamma\tau^*(m_0,n_0)} \ln(1 - \xi_f^*(m_0, n_0))}{\Lambda_0(\tau^*(m_0, n_0) + v_{n_0-1}) - \Lambda_0(v_{n_0-1})} \right\rfloor + 1.$$

an optimal solution to the system parameters

$$\tau(m_0 - 1, n_0), \xi_{r(f)}(m_0 - 1, n_0)$$

$$((\tau(m_0 + 1, n_0), \xi_{r(f)}(m_0 + 1, n_0)))$$

or

$$\tau(m_0, n_0 - 1), \xi_{r(f)}(m_0, n_0 - 1)$$

$$\big((\tau(m_0, n_0 + 1), \xi_{r(f)}(m_0, n_0 + 1))\big)$$

is derived. Using above exploration method, the optimum number of inspections (m_0^*, n_0^*) which are required to exceed partial repair and failure threshold are determined such that $(m_0^*, n_0^*) = (L_{\xi_r^*}, L_{\xi_f^*})$. The optimal inspection frequency (m_0^*, n_0^*) give a solution to the optimal inspection time τ^* and repair and failure threshold ξ_r^* and ξ_f^* such that

$$(\tau^*, \xi_r^*, \xi_f^*) \equiv (\tau^*(m_0^*, n_0^*), \underline{\xi}^*(m_0^*, n_0^*))$$

and

$$(\tau^*(m_0^*, n_0^*), \xi_r^*(m_0^*, n_0^*), \xi_f^*(m_0^*, n_0^*)) = \underset{\underline{\xi}^\tau(m_0,n_0)\in S^\tau}{\arg\min}\ \mu_\tau^{(0,0)},$$

for all $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$.

Following a numerical example is provided to illustrate the proposed maintenance model.

4.5.3 Numerical results

To optimize the model with respect to the system parameters, let $\psi(x) = x$ and $\lambda(t) = t$. The choice for the maintenance costs, degrading and maintenance model's parameters are

$$(C, C_r^v(v + \tau), C_r^v(v + \xi\tau), C_f) = (20, 60, 80, 150),$$

$\delta = 2\gamma$ and $\xi = 0.8$. Using the optimization method proposed above, the optimal system parameters (τ, ξ_r, ξ_f) for different inspection frequency values (m_0, n_0) have been derived (see Table 4.1). As illustrated given inspection frequencies $(m_0, n_0) = (1, 3)$, the optimal parameters which satisfy the equations (4.5.12) and (4.5.16) i.e. $(m_0^*, n_0^*) = (L_{\xi_r^*}, L_{\xi_f^*})$ are $(\tau^*, \xi_r^*, \xi_f^*) = (\tau^*(1, 3), \underline{\xi}^*(1, 3)) = (1, 0.2, 0.73)$ with corresponding optimal expected

(m_0, n_0)	$\left(\tau^*, \xi_r^*, \xi_f^*\right)$	$\mu_\tau^{(0,0)^*}$	$\left(L_{\xi_r^*}, L_{\xi_f^*}\right)$
$(2,2)$	$(2, 0.9, 0.91)$	42.08	$(1,1)$
$(1,2)$	$(1.5, 0.5, 0.99)$	126.51	$(1,3)$
$(2,3)$	$(1, 0.3, 0.78)$	82.89	$(1,2)$
$(3,3)$	$(1, 0.7, 0.85)$	79.79	$(1,2)$
(1,3)	**(1,0.2,0.73)**	**100.00**	**(1,3)**

Table 4.1: Optimal system parameters and inspection frequencies for different (m_0, n_0)

cost per unit time $\mu_\tau^{(0,0)^*} = 100$. The optimum values $(m_0^*, n_0^*) = (1,3)$ give a solution to the optimum inspection frequency $n_0^* = 3$ and the number of minimal and partial repairs which are $m_0^* - 1 = 0$ and $n_0^* - m_0^* = 2$ respectively. The optimal system parameters provide the basic partial repair and replacement decision rule subject to the failure state and the damage process. The optimal partial repair and failure decision rule

$$\begin{aligned} T_{\xi_r^*}^{(0,0)} &= \inf\left\{n\tau^* : R_{\tau^*}^{(X_n, v_{n-1})} \leq 0.8\right\} \\ &= \inf\left\{n\tau^* : X_n \geq \psi_{\xi_r^*}^{(\tau^*, v_{n-1})}\right\} \end{aligned} \tag{4.5.18}$$

and

$$\begin{aligned} T_{\xi_f^*}^{(0,0)} &= \inf\left\{n\tau^* : R_{\tau^*}^{(X_n, v_{n-1})} \leq 0.27\right\} \\ &= \inf\left\{n\tau^* : X_n \geq \psi_{\xi_f^*}^{(\tau^*, v_{n-1})}\right\} \end{aligned} \tag{4.5.19}$$

respectively are used just after each inspection instant: if $\bar{R}_{\tau^*}^{(X_n, v_{n-1})} \in (0.2, 0.73)$ partial repair should take place; otherwise operation with minimal repairs continues, if $R_{\tau^*}^{(X_n, v_{n-1})} \geq 0.8$, replacement occurs if $R_{\tau^*}^{(X_n, v_{n-1})} \leq 0.27$. Equivalently, the decision process can be made subject to the damage state process X_n and the optimal decision thresholds $\psi_{0.2}^{(\tau^*, v_{n-1})}$ and $\psi_{0.73}^{(\tau^*, v_{n-1})}$: if $X_n \in (\psi_{0.2}^{(\tau^*, v_{n-1})}, \psi_{0.73}^{(\tau^*, v_{n-1})})$ partial repair should take place; otherwise operation with minimal repairs continues, if $X_n < \psi_{0.2}^{(\tau^*, v_{n-1})}$, replacement occurs if $X_n \geq \psi_{0.73}^{(\tau^*, v_{n-1})}$.

To get an insight into the decision and action process, using the estimate of the damage state process $\mu_n^* = \frac{\gamma}{\delta} n\tau^*$, an evolution of the failure state process as the function of bivariate state process just before repair (μ_n, v_n^-) i.e. $R_{\tau^*}^{(\mu_n, v_{n-1})}$ and just after repair (μ_n, v_n) i.e. $R_{v_n - v_{n-1}}^{(\mu_n, v_{n-1})}$ given optimal system parameters $(\tau^*, \xi_r^*, \xi_f^*) = (1, 0.2, 0.73)$ has been illustrated (see Table 4.2) where $v_n - v_{n-1} = 0.8\tau^*$. From equations (4.5.18) and (4.5.19) it is easy to see that the optimal decision times to perform partial repair and replacement are $T_{\xi_r^*}^{(0,0)} = \tau^*$ and $T_{\xi_f^*}^{(0,0)} = 3\tau^*$ respectively.

n	(μ_n, v_n^-)	$R_{\tau^*}^{(\mu_n, v_{n-1})}$	(μ_n, v_n)	$R_{v_n - v_{n-1}}^{(\mu_n, v_{n-1})}$
1	$(0.5, 1)$	$0.778 < 1 - \xi_r^*$	$(0.5, 0.8)$	0.852
2	$(1, 1.8)$	$0.2725 > 1 - \xi_f^*$	$(1, 2.6)$	0.38
3	$(1.5, 2.6)$	$0.04 < 1 - \xi_f^*$	–	–

Table 4.2: An illustration of decision process and action process subject to the failure state process given optimal system parameters $(\tau^*, \xi_r^*, \xi_f^*) = (1, 0.2, 0.73)$

Table 4.3 summarizes the decision process subject to the estimated damage state process μ_n^* at n^{th} inspection time given optimal system parameters $(\tau^*, \xi_r^*, \xi_f^*) = (1, 0.2, 0.73)$ and optimal decision thresholds

$$\psi_{\xi_{r(f)}^*}^{(\tau^*, v_{n-1})} = \psi^{-1}\left\{\frac{-\ln(1 - \xi_{r(f)}^*)}{\Lambda_0(\tau^* + v_{n-1}) - \Lambda_0(v_{n-1})}\right\}.$$

n	$(\psi_{\xi_r^*}^{(\tau^*, v_{n-1})}, \psi_{\xi_f^*}^{(\tau^*, v_{n-1})})$	μ_n
1	$(0.45, 2.62)$	$0.5 \in (0.45, 2.62)$
2	$(0.17, 1.0072)$	$1 \in (0.17, 1.0072)$
3	$(0.1062, 0.6235)$	$0.6235 < 1.5$

Table 4.3: An illustration of decision process subject to the estimated damage state process μ_n^* given the optimal decision thresholds

In the following section under non-periodic inspection policy an expression for the long-run average cost per unit time is derived. Using the repair alert model presented by Lindqvist (44), we introduce a scheduling function $\tau_{n+1} = \tau(x_n, v_n, \beta)$ which determines the time to the next inspection based on the bivariate state process $(X_n, V_n) = (x_n, v_n)$ $(n \geq 0)$ and repair alert parameter β.

4.6 Optimal decision policy with non-periodic inspection

Lindqvist (44) presents a repair alert model for a repairable system which is subject to failure and possible preventive maintenance (PM). The Lindqvist model under a set of assumptions (see Definition 4.6.1) defines a so-called repair alert function which describes the alertness of the maintenance crew as a function of time. The repair alert model is used to schedule non-periodic inspection times.

Definition 4.6.1. (Lindqvist et al. 2006) Let the random variables Y and Z denote failure time and preventive maintenance time of a repairable system. The pair (Y,Z) of life variables satisfies the requirements of the repair alert model provided the following two conditions both hold:

- The event $\{Z < Y\}$ is stochastically independent of Y (i.e. Z is a random signs censoring of Y). That means the event that the failure of the component is preceded by PM, is not influenced by the time Y at which the component fails or would have failed without PM, or the conditional probability

  $$q = P(Z < Y|Y = y)$$

  does not depend on the value of y.

- There exists an increasing function G with G(0) = 0 such that for all $y > 0$,

$$P(Z \leq z | Z < Y, Y = y) = \frac{G(z)}{G(y)}, \quad 0 < z < y,.$$

The function G is called the cumulative repair alert function. Its derivative g is called the repair alert function.

The second part of the above definition means that, given that there would be a failure at time $Y = y$ and the maintenance crew will perform a PM before that time (i.e. $Z < Y$), the conditional density of the time Z of the PM is proportional to the repair alert function g. Lindqvist shows that given the increasing repair alert function G, which reflects the reaction of the maintenance crew as a function of time, mean time to preventive maintenance before the system failure is

$$E(Z|Z < Y) = E(Y) - E[\frac{M(Y)}{G(Y)}]$$

where $M(y) = \int_0^y G(t)dt$. In particular case when the cumulative repair alert function $G(t) = t^{\beta}$, it simply results that the mean time to preventive maintenance before the system failure is proportional to mean time to potential failure of the system. In other words,

$$E(Z|Z < Y) = \frac{\beta}{1+\beta} E(Y) \tag{4.6.1}$$

Following to find an expression for long-run average cost under non-periodic inspection policy, using the result of repair alert model (see equation (4.6.1)), imposing some conditions on the sequence of preventive maintenance times τ_n and elapsed system lifetime $\{Y_n = T - T_{n-1}\}$ $(n \geq 1)$ between $(n-1)^{\text{th}}$ and n^{th} inspection are required:

- Given partial information $\mathcal{A}_n$ the potential failure times just after $(n-1)^{\text{th}}$ preventive maintenance i.e. $\{Y_n = T - T_{n-1}\}$ $(T \geq T_{n-1})$ $(n \geq 1)$ are a random signs censoring of preventive maintenance times (τ_n) $(n \geq 1)$. That means, the n^{th} preventive maintenance event, $\{\tau_n < Y_n\}$, are stochastically independent of the potential failure time (Y_n), and

- An increasing function G defined on $[0, \infty)$ with $G(0) = 0$ exist such that for all $y_n > 0$ $(n \geq 1)$ and $0 < z_n \leq y_n$,

$$P(\tau_n \leq z_n | \tau_n < Y_n, Y_n = y_n, \mathcal{A}_n) = \frac{G(z_n)}{G(y_n)},$$

In Following example which is a simple generalization of the example given in Lindqvist (43) we shows that provided some assumptions, the pair (Y_n, τ_n) $(n \geq 1)$ of life variables simply satisfies the requirements of the repair alert model.

Example 4.6.1. *Models which satisfy the repair-alert model assumptions can be constructed by imitating the derivation of conditionally conjugate prior distributions in Bayesian analysis; the use of the normal-inverse gamma in normal models illustrates the approach (9, Section **5.2** & Appendix A). Let* (Y_n, τ_n) $(n \geq 1)$ *be a pair of life variable just after* $(n-1)$ *preventive maintenance at inspection time* T_n *with conditional joint density parameterized by* $0 < q_n < 1$,

$$f_{(Y_n,\tau_n)}(y_n, z_n | \mathcal{A}_n) = \frac{q_n}{y_n^{\beta}} \beta z_n^{\beta-1} \psi(X_n) \lambda_0(y_n + V_n) \exp\left(-\psi(X_n)[\Lambda_0(y_n + V_n) - \Lambda_0(V_n)]\right) ,$$

where $y_n > 0$, $0 < z_n < \frac{y_n}{q_n}$. *From equation (4.2.9) the marginal distribution of* Y_n *given* $\mathcal{A}_n$ *is the exponential distribution with density*

$$f_{Y_n}(y_n | \mathcal{A}_n) = \psi(X_n) \lambda_0(y_n + V_n) \exp\left(-\psi(X_n)[\Lambda_0(y_n + V_n) - \Lambda_0(V_n)]\right) ,$$

while the conditional distribution of τ_n *given* $Y_n = y_n$ *and* $\mathcal{A}_n$ *is the power distribution with density*

$$f_{\tau_n}(z_n | Y_n = y_n, \mathcal{A}_n) = \left\{ \begin{array}{ll} \frac{q_n}{y_n^{\beta}} (\beta z_n^{\beta-1}), & 0 < z_n < \frac{y_n}{q_n^{1/\beta}}; \\ 0, & \textit{Otherwise.} \end{array} \right\} .$$

From this we obtain $P(\tau_n < Y_n | Y_n = y_n, \mathcal{A}_n) = q_n$ *for all* $y_n > 0$ *and* $n \geq 1$. *That means, the event* $\{\tau_n < Y_n\}$ *is independent of* Y_n *and the first assumption of repair alert model (random sings censoring of* (Y_n, τ_n)*) is satisfied. Since the system deteriorates over time it is assumed that for* $n \geq 1$, $(q_{n+1} < q_n)$. *This assumption implies that the probability of occurrence of preventive maintenance before system failure time is decreasing function of inspection events. The following calculation shows that the second assumption of repair alert model holds as well. Let* $0 < z_n < y_n$ *for* $n \geq 1$. *Then*

$$\begin{aligned} P(\tau_n \leq z_n | \tau_n < Y_n, Y_n = y_n, \mathcal{A}_n) &= \frac{P(\tau_n \leq z_n, \tau_n < Y_n | Y_n = y_n, \mathcal{A}_n)}{P(\tau_n \leq Y_n | Y_n = y_n, \mathcal{A}_n)} \\ &= \frac{P(\tau_n \leq z_n | Y_n = y_n, \mathcal{A}_n)}{q_n} \\ &= \frac{q_n (z_n / y_n)^\beta}{q_n} = (\frac{z_n}{y_n})^\beta \end{aligned} \tag{4.6.2}$$

which denotes the second assumption of the repair alert model with increasing cumulative repair alert function $G(t) = t^\beta$.

Provided assumptions of Definition 4.6.1, applying equations (4.2.9) and (4.6.1), the mean time to the first inspection given $(X_0, V_0) = (x, v)$ can be calculated as

$$\begin{aligned} \tau(x, v; \beta) &= \frac{\beta}{1+\beta} E(T) \\ &= \frac{\beta}{1+\beta} \int_0^\infty \exp\left(-\psi(x)[\Lambda_0(v+t) - \Lambda_0(v)]\right) dt \end{aligned} \tag{4.6.3}$$

Let the system state just after n^{th} inspection is (X_n, V_n). Then, by applying equation (4.2.9) the mean time between n^{th} and $(n+1)^{\text{th}}$ inspection given $\mathcal{A}_n$ is

$$\begin{aligned} \tau(X_n, V_n; \beta) &= \frac{\beta}{1+\beta} E(T - T_n | T > T_n, \mathcal{A}_n) \\ &= \frac{\beta}{1+\beta} \int_0^\infty \exp\left(-\psi(X_n)[\Lambda_0(t+V_n) - \Lambda_0(V_n)]\right) dt, \quad n \geq 1 \end{aligned} \tag{4.6.4}$$

As shown, given repair alert parameter β the subsequent inspection ($(n+1)^{\text{th}}$ inspection) is scheduled only by using the last state just after n^{th} repair, i.e. (X_n, V_n). Thus, subject to the equations (4.6.3) and (4.6.4) the sequence of inspections can be planed in following way: starting from initial state $(X_0, V_0) = (x, v)$, with probability q_1 the first preventive maintenance is performed before potential failure time $U_1 = T$, at a time which for given $U_1 = u_1$ is distributed as power. Using scheduling function (4.6.3) the first inspection is made at scheduled time $\tau(x, v; \beta)$. Inspection at $\tau(x, v; \beta)$ reveals $X_{\tau(x,v;\beta)}$. The decision maker subject to the decision thresholds $\psi_{\xi_r}^{(\beta,v)}$, $\psi_{\xi_f}^{(\beta,v)}$ and $X_{\tau(x,v;\beta)}$ either restores the system to its condition just prior to inspection that is $\left(X_{\tau(x,v;\beta)}, v + \tau(x, v; \beta)\right)$, adjusts the system's virtual age $v + \tau(x, v; \beta) \mapsto v + \xi\tau(x, v; \beta)$, or returns the system to the regeneration state (x, v), i.e.

$$\left(X_{\tau(x,v;\beta)}, V_{\tau(x,v;\beta)}^{(x,v)}\right) = \begin{cases} \left(X_{\tau(x,v;\beta)}, v + \tau(x, v; \beta)\right), & \text{if } X_{\tau(x,v;\beta)} < \psi_{\xi_r}^{(\beta,v)}; \\ \left(X_{\tau(x,v;\beta)}, v + \xi\tau(x, v; \beta)\right), & \text{if } \psi_{\xi_r}^{(\beta,v)} \le X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}; \\ (x, v), & \text{if } \psi_{\xi_f}^{(\beta,v)} \le X_{\tau(x,v;\beta)}. \end{cases}$$

where $\psi_{\xi_r}^{(\beta,v)} = \psi^{-1}\left\{\frac{-\ln(1-\xi_r)}{\Lambda_0(\tau(x,v;\beta)+v)-\Lambda_0(v)}\right\}$ and $\psi_{\xi_f}^{(\beta,v)} = \psi^{-1}\left\{\frac{-\ln(1-\xi_f)}{\Lambda_0(\tau(x,v;\beta)+v)-\Lambda_0(v)}\right\}$. If renewal does not take place at first inspection time $\tau(x, v; \beta)$, i.e. $X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}$, then with probability q_2 $(q_2 < q_1)$ the second PM is performed before potential failure time $U_2 = T - T_1$. By using the scheduling function (4.6.4) next inspection is determined with respect to the state $(X_1, V_1) = (X_{\tau(x,v;\beta)}, V_{\tau(x,v;\beta)}^{(x,v)})$ and above process continues. This series of decision and action events subject to the state process (X, V) makes a sequence of inspection times.

In the particular case, let $\lambda_0(t) = t$. From equation (4.6.4) it follows

$$\tau(X_n, V_n; \beta) = \frac{\beta}{1+\beta} \exp\left(\frac{V_n^2\psi(X_n)}{2}\right) \sqrt{\frac{2\pi}{\psi(X_n)}} \left[1 - \Phi\left(V_n\sqrt{\psi(X_n)}\right)\right], \quad n \ge 0 \tag{4.6.5}$$

Where Φ denotes the standard normal distribution function. Applying the equation (4.6.5), given $\lambda_0(t) = t$, $\beta = 1$ and $\psi(x) = x$ an evolution of expected time to next inspection,

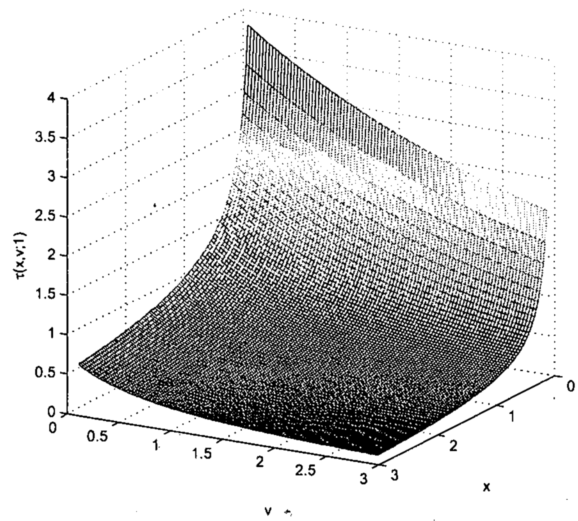

Figure 4.2: An evolution of expected time to inspection given the start state $(X_0, V_0) = (x, v)$, for $\psi(x) = x$, $\lambda_0(t) = t$ and $\beta = 1$.

i.e. $\tau(x, v; 1)$ subject to the starting state $(X_0, V_0) = (x, v)$, $(x, v > 0)$ is illustrated (see Figure (5.2)):

$$\tau(x, v; 1) = \frac{1}{2} \exp\left(\frac{xv^2}{2}\right) \sqrt{\frac{2\pi}{x}} \left[1 - \Phi\left(v\sqrt{x}\right)\right]$$

As shown, mean time to next inspection is a non-increasing function in both x (damage) and v (virtual age) process. That means, with increasing degree of deterioration of the system the reaction of the maintenance crew to make an inspection increases.

In the following section under non-periodic inspection policy an expression for the long-run average cost per unit time is derived. The time to inspection event is formulated by the sequence of non-periodic inspection times $\tau(X_n, V_n; \beta)$ $(n \geq 0)$ which can be evaluated by the equations (4.6.3) and (4.6.4).

4.7 Long-run average cost given non-periodic inspections policy

4.7.1 Expected cost per cycle

Let $C_\beta^{(x,v)}$ and $\tau(x,v;\beta)$ respectively denote the cost of repair and maintenance actions per cycle and the mean time to the first inspection starting from initial state value $(X_0, V_0) = (x, v)$: the system is instantaneously replaced by a new one at cost C_f and each (partial or minimal) repair and maintenance action incurs a cost determined by a random cost function $C_r^v(V_\tau)$, i.e.

$$C_r^v(V_{\tau(x,v;\beta)}) = \begin{cases} C + C_r^v(v + \xi\tau(x,v;\beta)), & \text{if } \psi_{\xi_r}^{(\beta,v)} \leq X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}; \\ C + C_r^v(v + \tau(x,v;\beta)), & \text{if } X_{\tau(x,v;\beta)} < \psi_{\xi_r}^{(\beta,v)}. \end{cases}$$

where the bounded cost measures C, $C_r^v(v+\tau)$, $C_r^v(v + \xi\tau(x,v;\beta))$ respectively denote the inspection cost, the minimal repair cost and imperfect repair cost to adjust the system age $v + \tau \mapsto v + \xi\tau(x,v;\beta)$, $(0 < \xi < 1)$.

It is assumed that the imperfect repair cost $C_r^v(v+\xi\tau(x,v;\beta))$ is a non-increasing function of the reduction factor ξ. Then a renewal type argument yields

$$\begin{aligned} C_\beta^{(x,v)} &= C_f I\left(X_{\tau(x,v;\beta)} \geq \psi_{\xi_f}^{(\beta,v)}\right) \\ &\quad + \left(C_r^v(V_{\tau(x,v;\beta)}) + C_\beta^{(X_{\tau(x,v;\beta)}, V_{\tau(x,v;\beta)})}\right) I\left(X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) \end{aligned} \tag{4.7.1}$$

Or,

$$\begin{aligned} C_\beta^{(x,v)} &= \bar{C}_\beta(v;\xi) I\left(\psi_{\xi_r}^{(\beta,v)} \leq X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) + C_\beta^{(X_{\tau(x,v;\beta)}, v+\tau(x,v;\beta))} \\ &\quad .I\left(X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) + \bar{C}_r(v;\xi,\beta) I\left(\psi_{\xi_r}^{(\beta,v)} \leq X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) \\ &\quad + \left(C + C_r^v(v + \tau(x,v;\beta))\right) I\left(X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) + C_f I\left(X_{\tau(x,v;\beta)} \geq \psi_{\xi_f}^{(\beta,v)}\right) \end{aligned} \tag{4.7.2}$$

where

$$\bar{C}_r(v;\xi,\beta) = C_r^v(v+\xi\tau(x,v;\beta)) - C_r^v(v+\tau(x,v;\beta)),$$

$$\bar{C}_\beta(v;\xi) = C_\beta^{(X_{\tau(x,v;\beta)},v+\xi\tau(x,v;\beta))} - C_\beta^{(X_{\tau(x,v;\beta)},v+\tau(x,v;\beta))}.$$

and $I(\cdot)$ is the indicator function. Let the stopping time(s) $T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_{r(f)}}}$ be defined as

$$T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_{r(f)}}} = \inf\left\{t : X_t \geq \psi^{(\beta,v)}_{\xi_{r(f)}} | (X_0,V_0) = (x,v)\right\}$$

Because

$$T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_{r(f)}}} \leq \tau(x,v;\beta) \Leftrightarrow X_{\tau(x,v;\beta)} \geq \psi^{(\beta,v)}_{\xi_{r(f)}}$$

the cost per cycle $C_\beta^{(x,v)}$ can be built up as

$$\begin{aligned} C_\beta^{(x,v)} &= \bar{C}_\beta(v;\xi) I\left(\psi^{(\beta,v)}_{\xi_r} \leq X_{\tau(x,v;\beta)} < \psi^{(\beta,v)}_{\xi_f}\right) \\ &+ (C + C_r^v(v+\tau(x,v;\beta)))\, I\left(T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_f}} > \tau(x,v;\beta)\right) \\ &+ C_f I\left(T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_f}} \leq \tau(x,v;\beta)\right) + C_\beta^{\left(X_{\tau(x,v;\beta)},v+\tau(x,v;\beta)\right)} \\ &.I\left(X_{\tau(x,v;\beta)} < \psi^{(\beta,v)}_{\xi_f}\right) + \bar{C}_r(v;\xi,\beta) I\left(T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_r}} \leq \tau(x,v;\beta) < T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_f}}\right) \end{aligned} \tag{4.7.3}$$

that $\tau(x,v;\beta)$ is derived from the equation (4.6.4). By taking expectation from both sides of the equation (4.7.3), the expected cost per cycle $\mu_{C_\beta^{(x,v)}}$ is

$$\begin{aligned} \mu_{C_\beta^{(x,v)}} &= g_{C_\tau^{(x,v)}} + \int_0^{\psi^{(\beta,v)}_{\xi_f}} \mu_{C_\beta^{(y,v+\tau(x,v;\beta))}} f_\beta(y|x)dy \\ &+ \int_{\psi^{(\beta,v)}_{\xi_r}}^{\psi^{(\beta,v)}_{\xi_f}} \bar{\mu}_{C_\beta}(y;v,\xi) f_\beta(y|x)dy \end{aligned} \tag{4.7.4}$$

where $\bar{\mu}_{C_\beta}(y;v,\xi) = \mu_{C_\beta^{(y,v+\xi\tau(x,v;\beta))}} - \mu_{C_\beta^{(y,v+\tau(x,v;\beta))}}$ and

$$\begin{aligned} g_{C_\beta^{(x,v)}} &= C_f F_{\psi^{(\beta,v)}_{\xi_f}}(\tau(x,v;\beta)) \\ &+ (C + C_r^v(v+\tau(x,v;\beta)))\, \bar{F}_{\psi^{(\beta,v)}_{\xi_f}}(\tau(x,v;\beta)) \\ &+ \bar{C}_r(v;\xi,\beta)\left(\bar{F}_{\psi^{(\beta,v)}_{\xi_f}}(\tau(x,v;\beta)) - \bar{F}_{\psi^{(\beta,v)}_{\xi_r}}(\tau(x,v;\beta))\right) \end{aligned} \tag{4.7.5}$$

and $f_\beta(y|x)$ is the transition density of the damage process X_t from $X_0 = x$ to $X_{\tau(x,v;\beta)} = y$.

4.7.2 Expected length per cycle

The expected length of a cycle is obtained similarly. The length of a cycle, $L_\tau^{(x,v)}$, is

$$\begin{aligned} L_\beta^{(x,v)} &= T^{(x,v)}_{\psi_{\xi_f}^{(\beta,v)}} I\left(X_{\tau(x,v;\beta)} \geq \psi_{\xi_f}^{(\beta,v)}\right) \\ &+ \left(\tau(x,v;\beta) + L_\beta^{(X_{\tau(x,v;\beta)},V_{\tau(x,v;\beta)})}\right) I\left(X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) \end{aligned} \tag{4.7.6}$$

But,

$$L_\beta^{(X_{\tau(x,v;\beta)},V_{\tau(x,v;\beta)})} = \begin{cases} L_\beta^{(X_{\tau(x,v;\beta)},v+\xi\tau(x,v;\beta))}, & \text{if } \psi_{\xi_r}^{(\beta,v)} \leq X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}; \\ L_\beta^{(X_{\tau(x,v;\beta)},v+\tau(x,v;\beta))}, & \text{if } X_{\tau(x,v;\beta)} < \psi_{\xi_r}^{(\beta,v)}. \end{cases}$$

Thus,

$$\begin{aligned} L_\beta^{(x;v)} &= \left(\tau(x,v;\beta) + \text{Ł}_\beta^{(X_{\tau(x,v;\beta)},v+\tau(x,v;\beta))}\right) I\left(X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) \\ &+ T^{(x,v)}_{\psi_{\xi_f}^{(\beta,v)}} I\left(X_{\tau(x,v;\beta)} \geq \psi_{\xi_f}^{(\beta,v)}\right) + \bar{L}_\beta(v;\xi) I\left(\psi_{\xi_r}^{(\beta,v)} \leq X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) \end{aligned} \tag{4.7.7}$$

Or, equivalently

$$\begin{aligned} L_\beta^{(x,v)} &= \bar{L}_\beta(v;\xi) I\left(\psi_{\xi_r}^{(\beta,v)} \leq X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) \\ &+ \left(\text{Ł}_\beta^{(X_{\tau(x,v;\beta)},v+\tau(x,v;\beta))}\right) I\left(X_{\tau(x,v;\beta)} < \psi_{\xi_f}^{(\beta,v)}\right) \\ &+ T_{\xi_f}^{(x,v)} I\left(T_{\xi_f}^{(x,v)} \leq \tau(x,v;\beta)\right) + \tau(x,v;\beta) I\left(T_{\xi_f}^{(x,v)} > \tau(x,v;\beta)\right) \end{aligned} \tag{4.7.8}$$

where

$$\bar{L}_\beta(v;\xi) = \text{Ł}_\beta^{(X_{\tau(x,v;\beta)},v+\xi\tau(x,v;\beta))} - \text{Ł}_\beta^{(X_{\tau(x,v;\beta)},v+\tau(x,v;\beta))}.$$

The expected length per cycle $\mu_{L_\beta^{(x,v)}}$ is

$$\begin{aligned} \mu_{L_\beta^{(x,v)}} &= h_{L_\beta^{(x,v)}} + \int_0^{\psi_{\xi_f}^{(\beta,v)}} \mu_{L_\beta^{(y,v+\xi\tau(x,v;\beta))}} f_\beta(y|x)dy \\ &+ \int_{\psi_{\xi_r}^{(\beta,v)}}^{\psi_{\xi_f}^{(\beta,v)}} \bar{\mu}_{L_\beta}(y;v,\xi) f_\beta(y|x)dy \end{aligned} \tag{4.7.9}$$

where

$$\bar{\mu}_{L_\beta}(y;v,\xi)=\mu_{L_\beta^{(y,v+\xi\tau(x,v;\beta))}}-\mu_{L_\beta^{(y,v+\tau(x,v;\beta))}}$$

and

$$h_{L_\beta^{(x,v)}}=\int_0^{\tau(x,v;\beta)}\bar{F}_{\psi_{\xi_f}^{(\beta,v)}}(u)du. \tag{4.7.10}$$

4.7.3 Long-run average cost per unit time

Now, the problem is to minimize the long-run average cost per unit time subject to the repair alert parameter β and the preventive partial repair and the preventive replacement rule ξ_r, ξ_f.

Let $\mu_\beta^{(x,v)}$ be the long-run average cost per unit time given the start state $(X_0,V_0)=(x,v)$. Because given the assumptions (4.2.10) the virtual age process V_n is increasing in the number of both minimal and imperfect repair events, i.e.

$$V_{n+1}>V_n,\quad n\geq 1$$

the failure process $\bar{R}_t^{(X_n,V_n)}$ tends to 1 as $n\to\infty$. This property subject to the failure threshold ξ_f that $0<\xi_f<1$ implies the existence of regeneration time points $T_{\xi_f}^{(x,v)}$:

$$T_{\xi_f}^{(x,v)}=\inf\left\{T_{n+1}:\bar{R}_{\tau_{n+1}}^{(X_{n+1},V_n)}\geq\xi_f|(X_0,V_0)=(x,v)\right\},\quad n\geq 0$$

where $\tau_{n+1}=\tau(x_n,v_n;\beta)$. Since, the sequence of failure and replacement times $T_{\xi_f}^{(x,v)}$ forms a regenerative process, the inter-arrival time between two consecutive replacements is a regenerative cycle. Theses regeneration cycles form an embedded renewal process. Then according to the standard renewal reward theorem (see Ross (66)), we have

$$\mu_\beta^{(x,v)}=\frac{\mu_{C_\beta^{(x,v)}}}{\mu_{L_\beta^{(x,v)}}} \tag{4.7.11}$$

where the expected cost per cycle $\mu_{C_\tau^{(x,v)}}$ and expected length per cycle $\mu_{L_\tau^{(x,v)}}$ are solution of the integral equations (4.7.4) and (4.7.9).

Starting in state $(X_0, V_0) = (0,0)$, an optimal measure of repair alert parameter and repair and failure threshold can be determined as

$$(\beta^*, \xi_r^*, \xi_f^*) = \underset{(\beta,\xi_r,\xi_f)\in\mathbb{R}_+\times[0,\xi_f)\times[0,\infty)}{\arg\min} \left\{ \mu_\beta^{(0,0)} \right\}.$$

Thus, the optimal maintenance policy characterized by the optimization of the long-run average cost per unit time will lead to an optimal repair alert parameter β^* and the preventive maintenance rules ξ_r^* and ξ_f^*.

4.8 A numerical iteration algorithm to solving optimization problem

To find a solution to optimization problem of $\mu_\beta^{(x,v)}$ (see equation (4.7.11)), let $\phi_c(x,v)$ and $\phi_l(x,v)$ respectively denote the expected cost and expected length per cycle given initial state $(X_0, V_0) = (x,v)$. It is easy to see from equations (4.7.4) and (4.7.9), both $\phi_c(x,v)$ and $\phi_l(x,v)$ are solution of following integral equation:

$$\begin{aligned}\phi(x,v) = f(x,v) &+ \int_x^{\psi_{\xi_r}^{(\beta,v)}} f_\beta(y|x)\phi(y, v+\tau(x,v;\beta))dy \\ &+ \int_{\psi_{\xi_r}^{(\beta,v)}}^{\psi_{\xi_f}^{(\beta,v)}} f_\beta(y|x)\phi(y, v+\xi\tau(x,v;\beta))dy\end{aligned} \tag{4.8.1}$$

where $f(x,v)$ refers to $g_{C_\beta^{(x,v)}}$ or $h_{L_\beta^{(x,v)}}$.

To present a solution to the integral equation (4.8.1), let (x_k, v_k) $(k = 0, 1, ...)$ imply the state of the process just after k^{th} repair. Given starting state $(X_k, V_k) = (x_k, v_k)$, from equation (4.8.1) the $\phi(x_k, v_k)$ can be expressed as

$$\phi^{(k)}(x_k, v_k) = \phi_1^{(k)} + \phi_2^{(k)} + \phi_3^{(k)} \tag{4.8.2}$$

with

$$\phi_1^{(k)} = f(x_k, v_k),$$

$$\phi_2^{(k)} = \int_{\psi_{\xi_r}^{(\beta,v_k)}}^{\psi_{\xi_f}^{(\beta,v_k)}} f_\beta(y|x_k)\phi^{(k+1)}(y, \underbrace{v_k + \xi\tau(x_k, v_k; \beta)}_{v_{k+1}})dy,$$

$$\phi_3^{(k)} = \int_{x_k}^{\psi_{\xi_r}^{(\beta,v_k)}} f_\beta(y|x_k)\phi^{(k+1)}(y, \underbrace{v_k + \tau(x_k, v_k; \beta)}_{v_{k+1}})dy,$$

where $\phi(x_k, v_k) = \phi^{(k)}(x_k, v_k)$ and

$$v_{k+1} = \begin{cases} v_k + \tau(x_k, v_k; \beta), & x_k < x_{k+1} < \psi_{\xi_r}^{(\beta,v_k)}; \\ v_k + \xi\tau(x_k, v_k; \beta), & \psi_{\xi_r}^{(\beta,v_k)} \leq x_{k+1} < \psi_{\xi_f}^{(\beta,v_k)}. \end{cases} \tag{4.8.3}$$

By conditioning on the value of the damage process at $(k+1)^{th}$ inspection time, $X_{k+1} = x_{k+1}$, the equation (4.8.3) can be represented as

$$\phi(x_k, v_k) = \begin{cases} \phi_1^{(k)}, & \psi_{\xi_f}^{(\beta,v_k)} \leq x_{k+1}; \\ \phi_{12}^{(k)}, & \psi_{\xi_r}^{(\beta,v_k)} \leq x_{k+1} < \psi_{\xi_f}^{(\beta,v_k)}; \\ \phi_{13}^{(k)}, & x_k < x_{k+1} < \psi_{\xi_r}^{(\beta,v_k)}. \end{cases} \tag{4.8.4}$$

where $\phi_{12}^{(k)} = \phi_1^{(k)} + \phi_2^{(k)}$ and $\phi_{13}^{(k)} = \phi_1^{(k)} + \phi_3^{(k)}$. Assume that $m-1$ $(m \geq 1)$ and $n-m$ $(n \geq m)$ respectively denote the number of minimal and partial repair(s) and T_m and T_n are the first time the damage process X_t $(t \in \mathbb{R}_+)$ reaches or exceeds a given partial repair and failure threshold $\psi_{\xi_r}^{(\beta,v_t)}$ and $\psi_{\xi_f}^{(\beta,v_t)}$ respectively:

$$T_m = \inf\left\{t \in \mathbb{R}_+ : X_t \geq \psi_{\xi_r}^{(\beta,v_t)}\right\} = \inf\left\{T_k : X_{T_k} \geq \psi_{\xi_r}^{(\beta,v_{k-1})}\right\}, \tag{4.8.5}$$

$$T_n = \inf\left\{t \in \mathbb{R}_+ : X_t \geq \psi_{\xi_f}^{(\beta,v_t)}\right\} = \inf\left\{T_k : X_{T_k} \geq \psi_{\xi_f}^{(\beta,v_{k-1})}\right\}, \tag{4.8.6}$$

where $n-1$ $(n \geq 1)$ denote the whole number of minimal and partial repairs and

$$T_k = \sum_{i=1}^{k} \tau(x_i, v_i; \beta).$$

Given starting state $(X_{n-1}, V_{n-1}) = (x_{n-1}, v_{n-1})$, from equations (4.8.4) and (4.8.6), it is easy to see that

$$\phi^{(n-1)}(x_{n-1}, v_{n-1}) = f(x_{n-1}, v_{n-1}); \qquad n-1 \geq m, \tag{4.8.7}$$

Given that just after $(n-2)^{th}$ inspection a partial repair is performed i.e. $m \leq n-2$, since $v_{n-1} = v_{n-2} + \xi\tau(x_{n-2}, v_{n-2}; \beta)$ (see equation (4.8.3)), from equation (4.8.4) we have

$$\begin{aligned}\phi^{(n-2)}(x_{n-2}, v_{n-2}) &= f(x_{n-2}, v_{n-2}) \\ &+ \int_{\psi_{\xi_r}^{(\beta,v_{n-2})}}^{\psi_{\xi_f}^{(\beta,v_{n-2})}} f_\beta(y|x_{n-2})\phi^{(n-1)}(y, \underbrace{v_{n-2} + \xi\tau(x_{n-2}, v_{n-2}; \beta)}_{v_{n-1}})dy\end{aligned} \tag{4.8.8}$$

But, in terms of initial condition (4.8.7), equation (4.8.8) can be represented as

$$\phi^{(n-2)}(x_{n-2}, v_{n-2}) = f(x_{n-2}, v_{n-2}) + \int_{\psi_{\xi_r}^{(\beta,v_{n-2})}}^{\psi_{\xi_f}^{(\beta,v_{n-2})}} f_\beta(y|x_{n-2})f(y, v_{n-1})dy \tag{4.8.9}$$

Thus, recursively $\phi^{(n-i)}(x_{n-i}, v_{n-i})$ for $1 \leq i \leq n$ can be calculated as

$$\phi(x_{n-i}, v_{n-i}) = \begin{cases} \phi_1^{(n-i)}, & i = 1; \\ \phi_{12}^{(n-i)}, & 2 \leq i \leq n-m+1; \\ \phi_{13}^{(n-i)}, & n-m+2 \leq i \leq n. \end{cases} \tag{4.8.10}$$

where

$$v_{n-i+1} = \begin{cases} v_{n-i} + \xi\tau(x_{n-i}, v_{n-i}; \beta), & 2 \leq i \leq n-m+1; \\ v_{n-i} + \tau(x_{n-i}, v_{n-i}; \beta), & n-m+2 \leq i \leq n. \end{cases} \tag{4.8.11}$$

As seen, given starting state $(X_0, V_0) = (x, v)$, equation (4.8.10) recursively gives a solution to both expected cost $\phi_c(x, v)$ and expected length per cycle $\phi_l(x, v)$ which are function of control parameters:

$$\mu_\beta^{(x,v)} = \frac{\phi_{C_\beta^{(x,v)}}}{\phi_{L_\beta^{(x,v)}}}.$$

From recursive relation (4.8.10) it is easy to see that both $\phi_{C_\beta^{(x,v)}}$ and $\phi_{L_\beta^{(x,v)}}$ are bounded. This comes from the fact that for $\xi_r \in (0, \xi_f)$, $\xi_f \in (0,1)$ integrands

$$f_\beta(y|x_{n-i})\phi^{(n-i+1)}(y, v_{n-i+1}), \qquad 2 \le i \le n$$

are continuous on the closed, bounded intervals $\left[x_{n-i}, \psi_{\xi_r}^{(\beta, v_{n-i})}\right]$ $(2 \le i \le n-m+1)$ and $\left[\psi_{\xi_r}^{(\beta,v_{n-i})}, \psi_{\xi_f}^{(\beta,v_{n-i})}\right]$ $(n-m+2 \le i \le n)$ and functions $f_c(x_{n-i}, v_{n-i})$ and $f_l(x_{n-i}, v_{n-i})$ corresponding to $\phi_{C_\beta^{(x,v)}}$ and $\phi_{L_\beta^{(x,v)}}$ for all $1 \le i \le n$ are bounded, i.e.

$$0 < f_c(x_{n-i}, v_{n-i}) \le M_c^{n-i}$$

where $M_c^{n-i} = C + C_f + C_r^{v_{n-i}}(v_{n-i} + \xi\tau(x_{n-i}, v_{n-i}; \beta))$,

$$\begin{aligned} 0 < c_i\Big(\bar{F}_{\psi_{\xi_f}^{(\beta,v_{n-i})}}(c_0) - \bar{F}_{\psi_{\xi_f}^{(\beta,v_{n-i})}}(\tau(x_{n-i}, v_{n-i}; \beta))\Big) \\ \le f_l(x_{n-i}, v_{n-i}) \le \tau(x_{n-i}, v_{n-i}; \beta) < \infty, \end{aligned} \tag{4.8.12}$$

where $c_i \in (0, \tau(x_{n-i}, v_{n-i}; \beta))$. This implies that the long-run average cost per unit time is bounded:

$$0 < \mu_\beta^{(x,v)} < \infty.$$

In next section based on Gamma process describing the damage process X_t, $t \in \mathbb{R}_+$, using the numerical integration algorithms (4.8.10), a solution to the optimization problem (4.7.11) is proposed.

4.9 Deterioration model based on Gamma process

Let the damage process X_t be described by a stationary Gamma process with shape parameter $\gamma > 0$ and scale parameter δ:

$$X_t - X_s \sim G(\gamma(t-s), \delta)$$

and

$$E(X_t - X_s) = \frac{\gamma(t) - \gamma(s)}{\delta}$$

where shape parameter is linear in t, $\gamma(t) = \gamma \times t$.

The transition probability density function $f_\beta(y|x)$, the density of $X_{\tau(x,v;\beta)}$ given $(X_0, V_0) = (x, v)$, is the gamma density

$$f_\beta(y|x) = \frac{\delta^{\gamma\tau(x,v;\beta)}(y-x)^{\gamma\tau(x,v;\beta)-1}e^{-\delta(y-x)}}{\Gamma(\gamma\tau(x,v;\beta))} \tag{4.9.1}$$

If the state of the process at initial time is $(X_0, V_0) = (x, v)$, the cumulative distribution of the hitting time $T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_{r(f)}}}$ of the partial repair (failure) barrier at $\psi^{(\beta,v)}_{\xi_{r(f)}}$ is

$$F^{(x,v)}_{\psi^{(\beta,v)}_{\xi_{r(f)}}}(t) = P(T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_{r(f)}}} \leq t) = P(X_t > \psi^{(\beta,v)}_{\xi_{r(f)}}|X_0 = x) = \int_{\psi^{(\beta,v)}_{\xi_{r(f)}}}^{\infty} f_\beta(y|x)dy. \tag{4.9.2}$$

The distribution of $T^{(x,v)}_{\psi^{(\beta,v)}_{\xi_{r(f)}}}$ can be expressed as ratio of an incomplete gamma function:

$$F^{(x,v)}_{\psi^{(\beta,v)}_{\xi_{r(f)}}}(t) = \frac{\Gamma(\gamma t; \delta(\psi^{(\beta,v)}_{\xi_{r(f)}} - x))}{\Gamma(\gamma t)} \tag{4.9.3}$$

where $\Gamma(\gamma; x_0)$ is an incomplete gamma function as

$$\Gamma(\gamma; x_0) = \int_{x_0}^{\infty} t^{\gamma-1}e^{-t}dt\,.$$

From stationary and independent increments property of Gamma process it is easy to show that an smooth semi-martingale representation of X_t is

$$\begin{aligned} X_t &= X_0 + \int_0^t \frac{\gamma}{\delta}ds + M_t \\ &= x + \frac{\gamma}{\delta}t + M_t \end{aligned} \tag{4.9.4}$$

which M_t is an $\mathcal{F}$-martingale.

Let $T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}$ $(0 \leq k \leq m-1)$ denote the partial repair stopping time given starting state

(X_k, V_k). Using equation (4.9.4) an $\mathcal{A}_k$-SSM representation of the damage process at stopping time $T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}$ is

$$E\left(X_{T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}}\middle|\mathcal{A}_k\right) = E\left(X_k|\mathcal{A}_k\right) + E\left(\int_0^{T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}} \frac{\gamma}{\delta} ds\middle|\mathcal{A}_k\right) + E\left(M_{T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}}\middle|\mathcal{A}_k\right) \quad (4.9.5)$$

Applying the optional sampling theorem to the $\mathcal{A}_k$-martingale term

$$\hat{M}_{T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}} = E\left(M_{T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}}\middle|\mathcal{A}_k\right)$$

we have

$$E\left(X_{T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}}\middle|\mathcal{A}_k\right) = E\left(X_k|\mathcal{A}_k\right) + E\left(\int_0^{T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}} \frac{\gamma}{\delta} ds\middle|\mathcal{A}_k\right) \quad (4.9.6)$$

Since X_k and $\psi^{(\beta,v_k)}_{\xi_r}$ are measurable with respect to $\mathcal{A}_k$ it follows

$$\begin{aligned} \psi^{(\beta,v_k)}_{\xi_r} &= E\left(\psi^{(\beta,v_k)}_{\xi_r}\middle|\mathcal{A}_k\right) = E\left(X_{T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}}\middle|\mathcal{A}_k\right) \\ &= X_k + \frac{\gamma}{\delta} E\left(T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}\middle|\mathcal{A}_k\right) \end{aligned} \quad (4.9.7)$$

From equation (4.9.7), a random measure of the mean hitting time to partial repair threshold $\psi^{(\beta,v_k)}_{\xi_r}$ given starting state (X_k, V_k) $(0 \leq k \leq m-1)$ is

$$\begin{aligned} \mu^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}} = E\left(T^{(x_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}}\middle|\mathcal{A}_k\right) &= \frac{\delta}{\gamma}\left(\psi^{(\beta,v_k)}_{\xi_r} - X_k\right) \\ &= \frac{\delta}{\gamma}\left(\frac{-\ln(1-\xi_r)}{\Lambda_0(\tau(x_k, v_k; \beta) + V_k) - \Lambda_0(V_k)} - X_k\right) \end{aligned} \quad (4.9.8)$$

Since $\mu_k = E(X_k) = \frac{\gamma}{\delta} k\tau$, starting in state $(X_k, V_k) = (x_k, v_k)$ an estimate of the mean time to partial repair can be measured by

$$\hat{\mu}^{(\mu_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_r}} = \frac{\delta}{\gamma}\left(\frac{-\ln(1-\xi_r)}{\Lambda_0(\tau(x_k, v_k; \beta) + V_k) - \Lambda_0(V_k)} - \mu_k\right) \quad (4.9.9)$$

Because

$$T_{m-1} = \sum_{i=1}^{m-1} \tau(x_i, v_i; \beta)$$

is the last inspection time before the damage process reaches or exceeds the partial repair barrier $\psi_{\xi_r}^{(\beta, v_{m-1})}$ (see equation (4.8.5)), we have

$$\hat{\mu}_{\psi_{\xi_r}^{(\beta, v_{m-1})}}^{(\mu_{m-1}, v_{m-1})} \leq \tau(x_{m-1}, v_{m-1}; \beta) \tag{4.9.10}$$

Let $\tau(\mu_{m-1}, v_{m-1}; \beta)$ be an estimate of $\tau(x_{m-1}, v_{m-1}; \beta)$. Then from equation (4.9.10) we have

$$\hat{\mu}_{\psi_{\xi_r}^{(\beta, v_{m-1})}}^{(\mu_{m-1}, v_{m-1})} \leq \tau(\mu_{m-1}, v_{m-1}; \beta) \tag{4.9.11}$$

Or, equivalently

$$\frac{-\frac{\delta}{\gamma\tau(\mu_{m-1}, v_{m-1}; \beta)} \ln(1 - \xi_r)}{\Lambda_0(\tau(\mu_{m-1}, v_{m-1}; \beta) + v_{m-1}) - \Lambda_0(v_{m-1})} \leq m \tag{4.9.12}$$

So, the minimum number of inspections required to exceed failure threshold is

$$m = \left\lfloor \frac{-\frac{\delta}{\gamma\tau(\mu_{m-1}, v_{m-1}; \beta)} \ln(1 - \xi_r)}{\Lambda_0(\tau(\mu_{m-1}, v_{m-1}; \beta) + v_{m-1}) - \Lambda_0(v_{m-1})} \right\rfloor + 1 \tag{4.9.13}$$

where $\lfloor \cdot \rfloor$ is the floor function.

With the same argument as above, starting in state (X_k, V_k) for $m \leq k \leq n-1$, mean time to reach failure threshold $\psi_{\xi_f}^{(\tau, v_k)}$ is

$$\begin{aligned} \mu_{\psi_{\xi_f}^{(\beta, v_k)}}^{(x_k, v_k)} = E\left(T_{\psi_{\xi_f}^{(\beta, v_k)}}^{(x_k, v_k)} \middle| A_k \right) &= \frac{\delta}{\gamma} \left(\psi_{\xi_f}^{(\beta, v_k)} - X_k \right) \\ &= \frac{\delta}{\gamma} \left(\frac{-\ln(1 - \xi_f)}{\Lambda_0(\tau(x_k, v_k; \beta) + V_k) - \Lambda_0(V_k)} - X_k \right) \end{aligned} \tag{4.9.14}$$

where for $m \leq k \leq n-1$,

$$V_k = (m-1)\tau + (k - m + 1)\xi\tau$$

is the virtual age of the system just after k^{th} repair, or $(k-m+1)^{th}$ partial repair action. Since $\mu_k = E(X_k) = \frac{\gamma}{\delta}k\tau$, we have

$$\hat{\mu}^{(\mu_k,v_k)}_{\psi^{(\beta,v_k)}_{\xi_f}} = \frac{\delta}{\gamma}\left(\frac{-\ln(1-\xi_f)}{\Lambda_0(\tau(\mu_k,v_k;\beta)+V_k)-\Lambda_0(V_k)} - \mu_k\right) \tag{4.9.15}$$

Clearly, for $k = n-1$, using equation (4.8.6), it follows

$$\hat{\mu}^{(\mu_{n-1},v_{n-1})}_{\psi^{(\beta,v_{n-1})}_{\xi_f}} \leq \tau(\mu_{n-1}, v_{n-1};\beta)$$

Or, equivalently

$$\frac{-\frac{\delta}{\gamma\tau(\mu_{n-1},v_{n-1};\beta)}\ln(1-\xi_f)}{\Lambda_0(\tau(\mu_{n-1},v_{n-1};\beta)+v_{n-1})-\Lambda_0(v_{n-1})} \leq n \tag{4.9.16}$$

So, the minimum number of inspections required to exceed failure threshold is

$$n = \left\lfloor \frac{-\frac{\delta}{\gamma\tau(\mu_{n-1},v_{n-1};\beta)}\ln(1-\xi_f)}{\Lambda_0(\tau(\mu_{n-1},v_{n-1};\beta)+v_{n-1})-\Lambda_0(v_{n-1})} \right\rfloor + 1 \tag{4.9.17}$$

4.10 Maintenance optimization methodology

Following subject to the system parameters (ξ_r, ξ_f) and the repair alert parameter β a solution to optimization problem of the long-run average cost per unit time $\mu^{(0,0)}_\beta$ is proposed. For fixed number of inspections (m_0, n_0) $(1 \leq m_0 \leq n_0)$, let

$$\mu^{(0,0)}_\beta = \mu^{(0,0)}_{\beta(m_0,n_0)}(\underline{\xi}(m_0,n_0)),$$

where

$$\underline{\xi}(m_0,n_0) = (\xi_r(m_0,n_0), \xi_f(m_0,n_0)).$$

Applying recursive relation (4.8.10), an optimal solution to the repair alert parameter $\beta^*(m_0,n_0)$ and the repair and failure threshold parameters $\underline{\xi}^*(m_0,n_0)$ is obtained such that

$$(\beta^*(m_0,n_0), \underline{\xi}^*(m_0,n_0)) = \underset{\underline{\xi}^\beta(m_0,n_0)\in S^\beta}{\arg\min}\ \mu^{(0,0)}_\beta \tag{4.10.1}$$

that $\underline{\xi}^{\beta}(m_0, n_0) = (\beta(m_0, n_0), \underline{\xi}(m_0, n_0))$ and $S^{\beta} = \mathbb{R}_+ \times [0, \xi_f(m_0, n_0)) \times [0, \infty)$.

If for fixed number m_0 and n_0, $\beta^*(m_0, n_0)$ and $\xi^*_{r(f)}(m_0, n_0)$ do not satisfy the equations (4.9.13) ((4.9.17)) i.e., m_0 or n_0 greater (less) than the optimal inspection frequencies $U_{\xi^*_{r(f)}}$ where

$$U_{\xi^*_r} = \left\lfloor \frac{-\frac{\delta}{\gamma\tau(\mu_{m_0-1}, v_{m_0-1}; \beta^*(m_0,n_0))} \ln(1 - \xi^*_r(m_0, n_0))}{\Lambda_0(\tau(\mu_{m_0-1}, v_{m_0-1}; \beta^*(m_0, n_0)) + v_{m_0-1}) - \Lambda_0(v_{m_0-1})} \right\rfloor + 1,$$

and

$$U_{\xi^*_f} = \left\lfloor \frac{-\frac{\delta}{\gamma\tau(\mu_{n_0-1}, v_{n_0-1}; \beta^*(m_0,n_0))} \ln(1 - \xi^*_f(m_0, n_0))}{\Lambda_0(\tau(\mu_{n_0-1}, v_{n_0-1}; \beta^*(m_0, n_0)) + v_{n_0-1}) - \Lambda_0(v_{n_0-1})} \right\rfloor + 1$$

using the recursive equation (4.8.10), an optimal solution to the system parameters

$$\beta(m_0 - 1, n_0), \xi_{r(f)}(m_0 - 1, n_0)$$

$$((\beta(m_0 + 1, n_0), \xi_{r(f)}(m_0 + 1, n_0)))$$

or

$$\beta(m_0, n_0 - 1), \xi_{r(f)}(m_0, n_0 - 1)$$

$$((\beta(m_0, n_0 + 1), \xi_{r(f)}(m_0, n_0 + 1)))$$

is derived. Using above exploration method, the optimum number of inspections (m^*_0, n^*_0) which are required to exceed partial repair and failure threshold are determined such that $(m^*_0, n^*_0) = (U_{\xi^*_r}, U_{\xi^*_f})$. The optimal inspection frequency (m^*_0, n^*_0) give a solution to the optimal repair alert parameter β^* and repair and failure threshold ξ^*_r and ξ^*_f such that

$$(\beta^*, \xi^*_r, \xi^*_f) \equiv (\beta^*(m^*_0, n^*_0), \underline{\xi}^*(m^*_0, n^*_0)) \tag{4.10.2}$$

and

$$(\beta^*(m^*_0, n^*_0), \xi^*_r(m^*_0), \xi^*_f(n^*_0)) = \underset{\underline{\xi}^{\beta}(m_0,n_0) \in S^{\beta}}{\arg\min} \; \mu^{(0,0)}_{\beta}, \tag{4.10.3}$$

for all $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$.

Remark 4.10.1. The Maintenance model presented here outlines a repair approach which can be simply extended by incorporating modified bivariate state process (X_{V_t}, V_t) into failure distribution function

$$(X_{V_t}, V_t) \mapsto R_t^{(X_{V_t}, V_t)} .$$

where X_{V_t} denotes the damage process controlled by the virtual age process V_t. The modified setting provides a generalized approach to the current repair policy in the sense that the system state can be reseted at repair times by adjusting not only system age $t \searrow V_t$, but also damage process $X_t \searrow X_{V_t}$ which is induced by the repair degree (time shift factor) ξ.

4.11 Conclusion

In this chapter we have shown how to formulate an inspection and maintenance model with in a very general form for both periodic and non-periodic inspection policies. The standard renewal-reward argument coupled with the identification of regeneration points provides the basis for the analysis of the process. the solution of a numerical example illustrates how the approach can be applied in a particular case. Optimization with respect to various system parameters has been demonstrated so that different scenarios can be explored.

The model outlines a systematics approach and structure which can be applied to both periodic and non-periodic inspection policy. Using the repair alert model, the extension to non-periodic policy shares many features with the periodic policy. The model has potential to provide a flexible approach to the repair by incorporating modified bivariate state process. The model shows the feasibility of this programme.

Chapter 5

Optimal Maintenance Scheduling for a Manufacturing System Subject to Deterioration

5.1 Introduction

Nowadays one of challenges facing the manufacturing industry characterized by heavy production of items is to optimize revenue from manufacturing systems whose resulting output measures are not identifiable. Examples of output (product) measures include defective items produced and products quality. These unidentifiable measures associated with output which cause the underestimating quality incur an invisible quality cost and consequently overestimation of revenue. This implies that the true measure of revenue over production process is unknown. Here to overcome this inefficiency and get a precise measure of revenue, a state dependent measure of revenue is presented. More precisely, since product quality is usually a function of system deterioration state, revenue from system is linked by physical (damage) state of the system described by unobservable stochastic damage process X_t. This approach is based on the realistic assumption that operation of the system in either one of the normal state ($X_t = 1$) or degraded state

$(X_t = 2)$ generates income, which is higher in the normal state. But, due to the fact that the quality of resulting output is subject to system deterioration state, to optimize revenue from system, an appropriate maintenance strategy for such systems is essential. Insufficient maintenance leads to an increase in the number of defective items, low product quality and low maintenance cost; excessive maintenance results in a reduction in the proportion of defective items, high product quality and high maintenance cost. Therefore to optimize revenue an optimum maintenance policy which balances the amount of maintenance is required. Here by joint determination of optimal inspection and repair policy, using intensity control model, a more realistic approach to maintenance optimization of manufacturing systems resulting from optimal control the failure rate of the system (system state) is presented. More Specifically, the maintenance optimization model is presented in the following setting:

The maintenance model is given partial information. That means the state and true quality of production process from system is unknown and only available information is given by the history of inspection times. This assumption implies that revenue from system due to invisible quality costs is not measurable from production process. It is supposed that revenue is inferred from system state which is unobservable and can be estimated from partial information. This is the case which is common when revenue from production process due to invisible costs (e.g. defective items costs, internal inefficiencies costs and hidden quality costs) is not easy to measure (see (26),(42)). Examples of hidden quality costs include the cost to society and goodwill which may cause the underestimating of quality. The cycle begins with the system in the as-good-as-new (stable) condition producing items of high or perfect quality ("in-control" state). At some random point in time the system state described by its failure rate may shift to an out-of-control state (quality shift), characterized by a lower revenue and higher proneness to failure. The lower revenue may be due to the system malfunction resulting in the production of items which are defective or of substandard (lower) quality. Since the system state (or, production process) is subject to deterioration resulting from environmental factors, inspection and repair are scheduled to prevent system breakdown and increase the reliability of the system which leads to improving the product quality and increasing revenue from system. To detect and rectify any minor defects which may eventually cause complete breakdown of the system, the maintenance crew inspects the system from the beginning of the production run, at random times T_n $(n > 0)$ during a production run of T time units. These inspections incur cost in terms of materials and wages. Also, the maintenance crew has disposition to perform repair. In this concept, it is assumed that after performing repair, the damage incurred by environmental factors during the time (or, departing failure rate process from in-control state to out-of-control state) is adjusted proportional to the repair level. The model is based on the realistic assumption that greater level of inspection and repair leads to the greater inspection and repair cost. But there is a greater chance that potential breakdown will be detected by inspections and the system performs more reliable and efficient resulting in better quality of output or more revenue. Thus, the maintenance procedure is faced with the dilemma of either excessive repair and inspection which leads to more revenue and repair and inspection cost or insufficient repair and inspection which results in less revenue and repair and inspection cost. In this case a balance would be required between the costs of the various possible degree of repair and inspection and revenue from system. In this paper using modeling intensity control given partial information we determine an optimal repair and inspection policy which gives a correct balance between the frequency of inspection repair level and the resulting output.

Before proceeding to the model development, under above maintenance policy, the underlying process of the model is formulated as follows:

The underlying deteriorating (system state) process is described by the proportional intensity model (PIM) (see (13)) dependent on the damage process and the system age. That means the time to shift to the out-of-control state from in-control (stable) state of the system is distributed continuous random variable. It is assumed that the deterioration rate of the system is increasing in both system age and damage process. The non-homogeneous Markov process is used to describe the underlying physical state process (damage process) which reflects the effect of the operating environment on the system. The evolution of damage process is characterized by two physical states, a normal state (state 1) and degraded state (state 2). The Weibull/generalized Pareto distribution used in accelerated failure time (AFT) model is proposed to model the transition rate of the damage process. More precisely it is assumed that the time until transition to the degraded state is a Weibull/two parameters generalized Pareto distributed random variable with time dependent transition rate $q_{12}(t)$. The model based on the assumption that the transition rate is non-decreasing in time which incurs the transition from normal state to degraded state of damage process and consequently time to shift to the out-of- control state of the production process increases over time. So, as time goes on the process deteriorates resulting in the production of items which are defective or of substandard quality (less revenue). Since the system is subject to deterioration, to prevent the system breakdown, it is inspected and repaired. Inspections are made according to a modulated Poisson process (59) with a stochastic intensity function. The intensity of occurrence of inspections is linked by the unobservable damage process X_t. It is assumed that inspection intensity increases in X_t. This setting ensures that not only the frequency of inspections is to be time dependent, but also the inspection intensity when operating in the degraded state with system age t ($t \geq 0$) is to be larger or equal to the inspection intensity in the normal state with the same system age. Due to fact that, the production process is a function of the system state, to raise the system reliability resulting in improving productivity of the system and increasing revenue, the maintenance crew has disposition to repair the system. The effect of repair which leads to adjusting departing deterioration rate process from in-control state to the out-of-control state is reflected by repair degree process (control process) $u \in (0, 1]$ through incorporating into transition rate $q_{12}(t) \rightarrow q_{12}^{u}(t)$. It is shown the repair degree process by incorporating into Weibull/generalized Pareto distribution as the scale parameter provides an accelerated failure time (PH)/(AFT) model. The repair degree is either partial ($u \in (0, 1)$) returning the system to a state which may not be perfect, or minimal ($u = 1$) which returns the system to a functioning state but equivalent to the state it was in just before the repair. Since, the underlying damage process X_t is not observable, by projection on observed history, the above partial information control problem is converted into a complete information problem. This results in an estimate for the indicator function of the state $i \in S = \{1, 2\}$ of the damage process X_t ($t \geq 0$) i.e.

$$\varphi(n, t; i) \rightarrow \hat{\varphi}(n, t; i)$$

$\varphi(n, t; i) = I(X_t = i)I(T_n \leq t < T_{n+1})$, where through transition rate function $q_{12}^{u}(t)$, is influenced by the repair degree (control) process u ($\hat{\varphi}(n, t; i) \rightarrow \hat{\varphi}^{u}(n, t; i)$). It is easy to see that, since the underlying deteriorating process describing the system state depends on the damage process X_t, given partial information the resulting output (revenue) from system through time dependent repair and inspection factor (RI_F) i.e. $\hat{\varphi}^{u}(n, t; i)$ is controlled by both repair factor u_t and inspection frequency factor N_t.

As mentioned before, the model is based on the realistic assumption that the greater level of repair and inspection, the greater repair and maintenance cost. That means, the small (large) values of the repair degree process u leads to an increase (a decrease) in both maintenance (repair and inspection) cost and revenue. So, to keep a correct balance between revenue from system and maintenance costs, choosing an optimal control strategy u_t^* is essential. By using an intensity control model (see Bremaud (10)) adapted to partial information, a solution to the optimal maintenance scheduling problem is derived. Also, it is shown that the optimal replacement policy resulting in an optimal production run length is a control limit policy; that is, the system should be replaced if the failure intensity of the system reaches an optimal critical threshold.

5.2 Modelling Deterioration (Linear Transition rate)

Consider a manufacture system whose resulting output quality (or revenue) is subject to system state. At the start of the production cycle, the system is in an "in-control" (stable) state, producing items of acceptable quality. After producing for some period of time since the system deteriorates as time goes on, the system state may shift to an "out-of-control" state which results in the production of items which are defective or of substandard quality. From that point on, it is realistic to be assumed that revenue from system when operating in the out-of-control state is less than that as working in an in-control state. We suppose that the system state is influenced by the system age and the physical state (damage) process X_t $(t \geq 0)$ with state space $S = \{1, 2\}$ which reflects the effect of the operating environment on the manufacturing system. To model the system state, we consider the proportional intensity model (PIM) (13) which is product of a baseline failure rate $\lambda_0(\cdot)$ dependent on the age of the system and a positive function

$\psi(\cdot)$ dependent on the values of the damage process X_t. More precisely,

$$\lambda(t, X_t) = \lambda_0(t)\psi(X_t)\,. \tag{5.2.1}$$

where functions $\lambda_0(t)$ and $\psi(x)$ are non-decreasing, which means that the system deteriorates with age and the failure rate of the system is non-decreasing function of the damage process X_t. The transition between physical states is driven by a non-homogeneous Markov process whose sojourn time in normal state (state 1) is described by a Weibull distribution

$$\begin{aligned} F(t) = P(T_d \le t) &= 1 - \exp\left\{-\int_0^t q_{12}(v)dv\right\}, \quad t > 0 \\ &= 1 - \exp\left(\frac{-t^2}{2}\right) \end{aligned} \tag{5.2.2}$$

where T_d is the passage time from state 1 to state 2,

$$T_d = \inf_{t \in \mathbb{R}^+} \{t \,|X_t = 2\}\,.$$

This implies that the transition rate from normal state (state 1) to degraded state (state 2) is a linear function of t as

$$q_{12}(t) = t, \quad 0 \le t < \infty$$

The rate $q_{12}(t)$ is non-decreasing on $[0, \infty)$ which means the intensity of leaving the state one increases with time.

5.3 Modelling Maintenance

It is assumed that the manufacturing system is subject to repair and inspection. To model repair let $\{u_t : t \in \mathbb{R}_+\}$ be the control process with state (or action) space $\mathcal{U} = (c, 1) \cup \{1\}$, $(c \in (0, 1))$ where $u_t = u$ represents the decision at time t to perform a repair with repair degree u. The controller can influence the productivity of the system by adjusting the deterioration level of the system (partial repair) ($c < u < 1$), or with a minimal repair ($u = 1$) the system continues in operation with no change in performance. The repair cost is non-increasing in control values $u \in \mathcal{U}$, small values of control (repair degree) process u are more expensive than large values. The repair actions are assumed to impact the failure intensity by adjusting the intensity of leaving state one $q_{12}(t) \mapsto q^u_{12}(t)$ reflected by the repair degree (control) process $u \in (0, 1]$.

Example 5.3.1. *Suppose that $q^u_1(t) = u.t$ and $\bar{F}^u(t)$, $m^u(t)$ denote the waiting time distribution and mean residual waiting time in the state one associated with the control process u_t. Clearly,*

$$\bar{F}^u(t) = \exp(-\frac{ut^2}{2}), \tag{5.3.1}$$

and

$$m^u(t) = \frac{\int_t^\infty \bar{F}^u(v)dv}{\bar{F}^u(t)} = \sqrt{\frac{2\pi}{u}}\exp(\frac{ut^2}{2})[1 - \phi(\sqrt{u}t)], \tag{5.3.2}$$

By using the equation (5.3.8) an evolution of $m^u(t)$ is illustrated (see Figure 5.1). As shown at fixed time t with decreasing the (repair degree) control value $u : 1 \mapsto \{0.5, 0.1\}$ the mean residual waiting time in the state one (good state) increases that means smaller value of the control process u_t leads to the less deterioration of the system. It is clear that the control U_t determines a stochastic order in the sense that

$$\mathbf{P}^u [T_d > t] \geq \mathbf{P}^v [T_d > t] \tag{5.3.3}$$

when $u \leq v$. The system is inspected from time to time. The inspection includes detection and correction of minor defects is done by trying to prevent system downtime, major breakdown or failure of the system. Because the model is given the partial information including only the history of inspection times, to take decision based on the

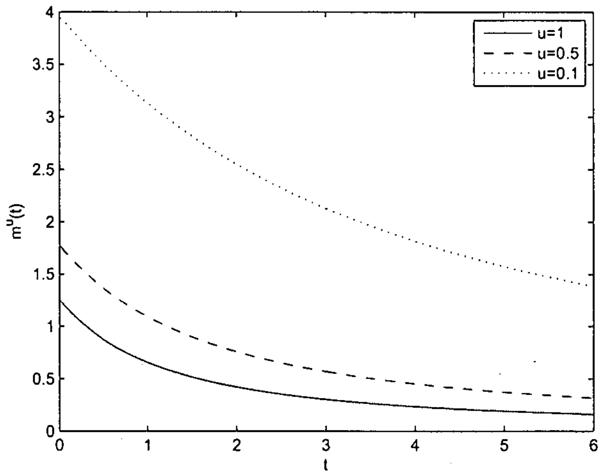

Figure 5.1: An evolution of the mean residual waiting times in the state one given $q_1^u(t) = u.t$ and control values $u = 0.1, 0.5, 1$.

system state X_t $(t \geq 0)$ for scheduling subsequent inspection time, the real state of the system at current inspection time is not observable. Here to schedule inspection times based on the system state, beautifully the intensity of occurrence of inspection is linked by the system state, γ_{X_t}, where $\gamma_1 \leq \gamma_2$. This setting provides an ideal condition under which the intensity of occurrence of inspections is influenced by the system state process X_t in such a way that the inspection intensity when operating in the degraded state is larger than the inspection intensity in the normal state, or equivalently; as the failure rate dependent on the system state increases the frequency of inspections increases. The stochastic intensity modulated by the driven process X_t implies distribution of time inspection as Markov modulated Poisson process. To schedule inspection times the mean time to the next inspection as the time to inspect is used. Given partial information $\mathcal{F}_t^N = \sigma\{N_s : 0 \leq s \leq t\}$, including the history of inspections times, the mean time to the next inspection provides an schedule function $\hat{\eta}_{n+1}$ $(n \geq 0)$ (see equation 5.3.8) characterized by the frequency of inspections $N_t = n$.

To formulate the underlying inspection process, let the stochastic process

$$N_t = \sum_{n \geq 1} I_{\{T_n \leq t\}}$$

defined on a measurable space $(\Omega, \mathcal{F})$ denote the total number of inspections until time t. The inspection of the system is performed from the beginning of the production run according to a modulated Poisson process (59) at random times $T_1 < T_2 < \ldots, \lim_{n\to\infty} T_n = \infty$ (nonexplosive). The modulation is done via the environmental (damage) process X_t with the state space $S = \{1, 2\}$ where X_t describes the state of the damage process at time t. The rate of occurrence of inspections at time t is modeled by γ_{X_t}. It is assumed that while the state of the damage process is $i \in S$ inspections occur according to an ordinary Poisson process with intensity γ_i. Thus, the number of inspection until time t (N_t) can be expressed in a smooth semi-martingale form (see Aven and Jensen (4)) as

$$\begin{aligned} N_t &= \int_0^t \gamma_{X_s} ds + M_t \\ &= \int_0^t \sum_{i=1}^{2} \gamma_i \varphi(s; i) ds + M_t, \quad M \in \mathcal{M}_0, \end{aligned} \tag{5.3.4}$$

where $0 < \gamma_1 < \gamma_2 < \infty$, $\varphi(t; i)$ is indicator function of the state process X_t at time t and $\mathcal{M}_0$ refers to the class of $\mathcal{F}$-martingales (with $M_0 = 0$). This setting provides an ideal condition under which the intensity of occurrence of inspections is influenced (modulated) by the damage (environmental) process in such away that the inspection intensity when operating in the degraded state with system age t ($t \geq 0$) is larger than the inspection intensity in the normal state with the same system age.

Let the distribution of the $(n + 1)^{th}$ inter-arrival inspection time $V_{n+1} = T_{n+1} - T_n$, ($n \geq 0$) ($T_0 = 0$) be modeled by $\bar{F}_n(v)$ adapted to the observed information

$$\mathcal{F}_t^N = \sigma \{N_s : 0 \leq s \leq t\}$$

including the history of inspection events. In other words,

$$\begin{aligned} \bar{F}_n(v) &= p(V_{n+1} \geq v|\hat{\gamma}_t(n)) \\ &= \exp\left(-\int_{T_n}^{T_n+v} \sum_{i\in S} \gamma_i \hat{\varphi}(n,t;i)dt\right) \end{aligned} \tag{5.3.5}$$

where $\hat{\gamma}_t(n) = \hat{\gamma}(n,t)$ and $\hat{\varphi}(n,t;i)$, $(i \in S)$ denote the inspection intensity and the probability measure of the physical state (damage) process X_t (see section 5.4) over inter-arrival inspection times: $T_n \leq t < T_{n+1}$, $(n \geq 0)$ given partial information $\mathcal{F}_t^N$ respectively. More precisely,

$$\hat{\gamma}_t(n) = E\left(\gamma_t(n)|\mathcal{F}_t^N\right)$$

where $\gamma_t(n) = \gamma_{X_t} I(T_n \leq t < T_{n+1})$ and

$$\hat{\varphi}(n,t;i) = E\left(\varphi(n,t;i)|\mathcal{F}_t^N\right)$$

that

$$\varphi(n,t;i) = I(X_t = i)I(T_n \leq t < T_{n+1}).$$

In following sections, incorporating the repair degree (control) process u_t into the probability measure $\hat{\varphi}(n,t;i) \to \hat{\varphi}^u(n,t;i)$, it will be shown how $\hat{\varphi}^u(n,t;i)$ serves as repair and inspection factor (RI_F) to influence both underlying inspection intensity and deteriorating process.

In particular case $n = 0$ equation (5.3.5) reduces to the first inspection time law.

$$\begin{aligned} \bar{F}_0(v) &= p(V_1 \geq v|\hat{\gamma}_t(0)) \\ &= \exp\left(-\int_0^v \sum_{i\in S} \gamma_i \hat{\varphi}(n,t;i)dt\right). \end{aligned} \tag{5.3.6}$$

To get the mean times between inspection let η_{n+1} denote the expected value of $(n+1)^{th}$ interval between inspections. Then from equation (5.3.5)

$$\eta_{n+1} = \int_0^\infty \bar{F}_n(v)dv$$
$$= \int_0^\infty \exp\left(-\int_{T_n}^{T_n+v} \sum_{i\in S} \gamma_i \hat{\varphi}(n,t;i)dt\right) dv \tag{5.3.7}$$

Because the integral term depends in (5.3.7) on the inspection time $T_n (n \geq 0)$, η_{n+1} is not measurable. To settle this problem, an estimated version of η_{n+1} for $n \geq 1$ is presented

$$\hat{\eta}_{n+1} = \int_0^\infty \hat{\bar{F}}_n(v)dv$$
$$= \int_0^\infty \exp\left(-\int_{\mu_n}^{\mu_n+v} \sum_{i\in S} \gamma_i \hat{\varphi}(n,t;i)dt\right) dv \tag{5.3.8}$$

where,

$$\mu_1 = \int_0^\infty \hat{\bar{F}}_0(v)dv$$
$$= \int_0^\infty \exp\left(-\int_0^v \sum_{i\in S} \gamma_i \hat{\varphi}(n,t;i)dt\right) dv \tag{5.3.9}$$

and $\hat{\mu}_{n+1}$ denotes the $(n+1)^{th}$ expected inspection time

$$\mu_{n+1} = E(T_{n+1}) = \sum_{k=1}^{n+1} \hat{\eta}_{n+1}\,.$$

Next the filtering theorem provides an estimate of the damage process X_t given partial information $\mathcal{F}^N$.

5.4 Damage Process X Given Partial Information

The goal of filtering is to estimate the stochastic process X_t based on all observations up to the moment. In the argument below, Bremaud's treatment of filtering is applied as a key tool to estimate the underlying damage process X_t based on history of inspection events $\mathcal{F}_t^N$. Bremaud (10), deals with estimating the position of stochastic processes by the technique of filtering. Using quadratic criterion, filtering theorem provides a recursive estimate of the quantity of interest based on the point process observations $\mathcal{F}_t^N$. To get an estimate of the unobservable state process X_t and then convert partial information control problem into a complete information problem, let

$$\hat{\varphi}_t(i) = P(X_t = i|\mathcal{F}_t^N), \quad (i \in S)$$

which is the probability of the state i, $(i \in S)$ at time t given partial information

$$\mathcal{F}_t^N = \sigma\left\{N_s : 0 \leq s \leq t\right\}.$$

Using a Corollary of the filtering theorem (10)[Chapter IV: R4 Result], the filter $\hat{\varphi}_t(j)$ with respect to the point process observation N_t is given by

$$\begin{aligned}\hat{\varphi}_t(j) =& \hat{\varphi}_0(j)\\ &\ldots + \int_0^t \left(\sum_{i\in S} \hat{\varphi}_s(i) q_{ij}(s)\right) ds \\ &\ldots + \int_0^t \left(-\hat{\varphi}_{s^-}(j) + \frac{\gamma_j \hat{\varphi}_{s^-}(j)}{\sum_{i=1}^m \gamma_i \hat{\varphi}_{s^-}(i)}\right)\left(dN_s - \sum_{i=1}^m \gamma_i \hat{\varphi}_{s^-}(i) ds\right),\end{aligned} \tag{5.4.1}$$

Or equivalently, the equation (5.4.1) after some calculations can be reformulated as

$$\begin{aligned}\hat{\varphi}_t(j) =& \hat{\varphi}_0(j)\\ &\ldots + \int_0^t \left(\sum_{i\in S} \hat{\varphi}_s(i)\left\{q_{ij}(s) + \hat{\varphi}_s(j)(\gamma_i - \gamma_j)\right\}\right) ds \\ &\ldots + \sum_{n\geq 1}\left(-\hat{\varphi}_{T_n^-}(j) + \frac{\gamma_j \hat{\varphi}_{T_n^-}(j)}{\sum_{i=1}^m \gamma_i \hat{\varphi}_{T_n^-}(i)} I_{\{T_n \geq t\}}\right),\end{aligned} \tag{5.4.2}$$

Since over the inspection intervals $T_n \leq t < T_{n+1}$ the increment $dN_t = 0$, from equation (5.4.1) the estimator $\hat{\varphi}_t(i)$ of $\varphi_t(i)$ illustrated by

$$\hat{\varphi}(n, t; j) = \hat{\varphi}_t(j) I(T_n \leq t < T_{n+1})$$

can be expressed as

$$\begin{aligned}\hat{\varphi}(n,t;j) &= \hat{\varphi}(n,T_n;j) \\ &+ \int_{T_n}^{t}\left(\sum_{i\in S}\hat{\varphi}(n,s;i)\left\{q_{ij}(s)+\hat{\varphi}(n,s;j)(\gamma_i-\gamma_j)\right\}\right)ds\end{aligned} \tag{5.4.3}$$

and at inspection times T_n $(n \geq 1)$ we have $dN_{T_n} = 1$. Using the equation (5.4.1), an estimate of the damage indicator process at inspection times is

$$\hat{\varphi}_{T_n}(j) = \frac{\gamma_j \hat{\varphi}_{T_n^-}(j)}{\sum_{i\in S}\gamma_i \hat{\varphi}_{T_n^-}(i)}, \quad j \in S, \tag{5.4.4}$$

where $\hat{\varphi}_{t^-}(j)$ refers to the left limit.

To get an explicit solution of $\hat{\varphi}_t(i)$, let the transition rate matrix of the damage process X_t be as

$$Q(t) = \begin{pmatrix} -q_1(t) & q_{12}(t) \\ 0 & 0 \end{pmatrix}$$

where for $0 \leq t < \infty$, $q_{12}(t) = t$.

The integral equation (5.4.3) can be solved by taking the derivative of each side and solving the resulting differential equation. Using the differential equation, an explicit solution of $\hat{\varphi}(n,t;1)$ is given by

$$\hat{\varphi}(n,t;1) = \frac{\exp\left(-\int(\bar{\gamma}+q_1(t))dt\right)}{A(n,t)}, \tag{5.4.5}$$

where

$$A(n,t) = \frac{\exp\left[-\int\left(\bar{\gamma}+q_1(t)\right)dt\right]_{t=T_n}}{1-\hat{\varphi}_{T_n}(2)} - \int_{T_n}^{t}\bar{\gamma}\exp\left(-\int_0^v\left(\bar{\gamma}+q_1(s)\right)ds\right)dv$$

and $\bar{\gamma} = \gamma_1 - \gamma_2$. From the equation (5.4.5) it can be simply shown that given $q_1(t) = t$, the probability of state one $\hat{\varphi}(n,t;1)$ for $t \in [T_n, T_{n+1})$, $(n \geq 0)$ is represented as

$$\begin{aligned}\hat{\varphi}(n,t;1) &= \left\{\frac{\exp\left(-(\bar{\gamma}T_n + \frac{T_n^2}{2})\right)}{1-\hat{\varphi}_{T_n}(2)} - \sqrt{2\pi}\bar{\gamma}\exp(\frac{\bar{\gamma}^2}{2})[\phi(t+\bar{\gamma}) - \phi(T_n+\bar{\gamma})]\right\}^{-1} \\ &\times \exp\left(-(\bar{\gamma}+\frac{t^2}{2})\right)\end{aligned} \tag{5.4.6}$$

where T_n for $n \geq 0$ ($T_0 = 0$) denote n^{th} inspection time. To consider the effect of control process $u \in \mathcal{U}$ on $\hat{\varphi}(n,t;1)$, in particular case let $n = 0$ and $q_1^u(t) = ut$. Then from (5.4.6) for $t \in [0, T_1)$ we get

$$\begin{aligned} \hat{\varphi}^u(0,t;1) = &\left\{ \bar{\gamma}\sqrt{\frac{2\pi}{u}} \exp(\frac{\bar{\gamma}^2}{2u}) \left[\phi\left(\sqrt{u}t + \frac{\bar{\gamma}}{\sqrt{u}}\right) - \phi\left(\frac{\bar{\gamma}}{\sqrt{u}}\right)\right]\right\}^{-1} \\ &\times \exp\left(-(\bar{\gamma}t + \frac{ut^2}{2})\right), \quad 0 \leq t < T_1 \end{aligned} \tag{5.4.7}$$

Where T_1 denotes the time to the fist inspection event. Following an evolution of the probability of state one $\hat{\varphi}^u(0,t;1)$ given control (repair degree) values $u = 0.1, 0.5, 1$ is illustrated (see Figure 5.2). The stochastic ordering of the sojourn time distributions determined by the repair degree values $u : 1 \mapsto \{0.5, 0.1\}$ is clear to see. From equation (5.8.4) it is easy to see that both time dependent inspection frequency $N_t = n$ and repair process u_t have a dynamic relationship with the probability function of damage process $\hat{\varphi}^u(n,t;1)$ ($i \in S$). As repair is performed and the degree of repair u_t decreases

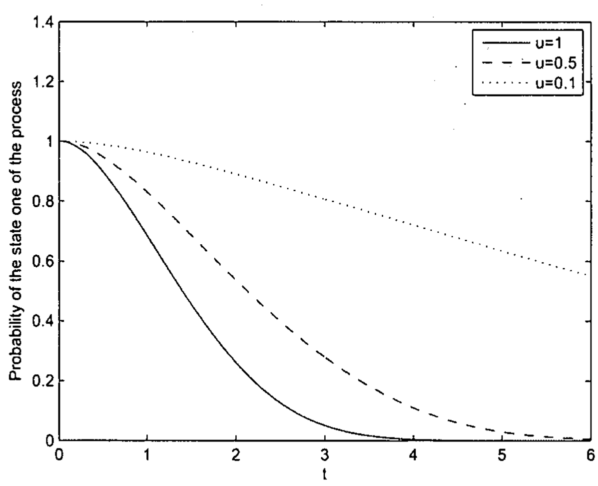

Figure 5.2: An evolution of the probability of the state one of the process given $q_1^u(t) = u.t$, $\lambda_{21} = 1$ and the control values u=0.1,0.5,1

the probability of the sojourn time in normal state (state 1) $\hat{\varphi}^u(n,t;1)$ proportional to the level of repair increases. This occurs by removing damage of the system created over time. Also, changes in inspection frequency $N_t = n$ affect the $\hat{\varphi}^u(n,t;1)$. The function $\hat{\varphi}^u(n,t;i)$ $(i \in S)$ is driven by bivariate stochastic process (u_t, N_t) which are repair and inspection frequency factor respectively is remarked by (RI_F for short). It is clear that the function $\hat{\varphi}^u(n,t;i)$ $(i \in S)$ influenced by bivariate repair and inspection factor (N_t, u_t) provides a superiority to previous works in which either the frequency of inspection is not time dependent or the inspection factor (I_F) is not affected by the repair factor.

To get an insight into impact of repair on RI_F an evolution of the probability of being in normal state (state 1), $\hat{\varphi}^u(0,t;1)$, with control (repair degree) values $u = 0.1, 0.5, 1$ is illustrated (see Figure 5.2). As seen, the control process u_t as described by (5.3.3) determines a stochastic ordering of the sojourn time distribution in the sense that a decline in the repair degree $v \searrow u$ $(u, v \in \mathcal{U})$ results in a decrease in intensity of departing damage process X_t from normal state (state 1) to the degraded state (state 2). More precisely, for $0 \leq t < T_1$ and $u \leq v$,

$$P^u\left[T_d > t|\mathcal{F}_t^N\right] \geq P^v\left[T_d > t|\mathcal{F}_t^N\right], \tag{5.4.8}$$

where T_d denotes the sojourn time in state 1.

Following we show that given partial information $\mathcal{F}_t^N$, the repair and inspection factor (RI_F) $\hat{\varphi}(n,t;i)$ serves as a key tool to reflect the effect of repairs on the underlying deteriorating process. From equation (5.2.1) it is easy to see that the failure rate dependent on the unobservable (damage) process X_t is not measurable given partial information $\mathcal{F}_t^N$. To get an estimate of the underlying deteriorating process with respect to the partial information $\mathcal{F}_t^N$ let, $\lambda(n,t)$ denote the failure rate of the system over n^{th} inter-arrival inspection time i.e.

$$\lambda(n,t) = \lambda(t, X_t) I(T_n \leq t < T_{n+1})$$

By projection on partial information $\mathcal{F}_t^N$, since inspection times (T_n) $(n \geq 0)$ are measurable subject to $\mathcal{F}_t^N$, it simply follows that

$$\hat{\lambda}(n,t) = E\left(\lambda(n,t)|\mathcal{F}_t^N\right) = \sum_{i \in S} \lambda(n,t,i)\hat{\varphi}(n,t;i) I(T_n \leq t < T_{n+1})$$

where $\hat{\lambda}(n,t)$ is an $\mathcal{F}_t^N$ adapted measure of $\lambda(n,t)$. By incorporating repair degree process u into the probability measure $\hat{\varphi}(n,t;i) \mapsto \hat{\varphi}^u(n,t;i)$ $(i \in S)$ it is easy to see that the underlying deteriorating process is influenced by the repair action $\hat{\lambda}(n,t) \mapsto \hat{\lambda}(n,t;u)$:

$$\hat{\lambda}(n,t;u) = \sum_{i \in S} \lambda(n,t,i)\hat{\varphi}^u(n,t;i) I(T_n \leq t < T_{n+1})$$

Hence, the repair resets the intensity of failure proportional to the damage. By removing damage accumulated over time, such repairs can reset the system failure rate (system state) to somewhere between that of a partially restored system if $u \in (0,1)$ and a minimally repaired system if $u = 1$. That means, repair degree process u_t determines a stochastic ordering of the failure rate in the sense that

$$\hat{\lambda}(n,t;u) < \hat{\lambda}(n,t;v)$$

when $u < v$ $(u, v \in \mathcal{U})$.

5.5 Modelling Intensity Control

5.5.1 Control, Cost Structure

Let $\mathcal{U}$ be the set of measurable processes u_t, $t \in [0,T]$ adapted to the partial information $\mathcal{F}_t^N$ and taking values in $(0,1]$. To each control $u \in \mathcal{U}$, we associate a probability P_u on

$(\Omega, \mathcal{F})$ such that frequency of inspections N_t admits the $(P_u, \mathcal{F}_t)$-intensity $\gamma_t(u)$. Each inspection incurs an instantaneous cost $K - C$ or K $(0 < C < K)$ dependent on the damage process is either in the normal state (state 1) or in degraded state (state 2). More precisely, with perfect information $\mathcal{F}_t$ the inspection cost at time t, k_t is $k_t = K - C\varphi(t;1)$ where $\varphi(t;1)$ is the indicator function of the state one. The repair action per unit of time with repair degree $u \in \mathcal{U}$ incurs a repair cost $k_\varepsilon(t,u) = \varepsilon(1-u)t$ $(\varepsilon > 0)$ which is a decreasing (increasing) function of the repair degree process $u \in \mathcal{U}$ (scale parameter ε). As noted, the repair cost per unit of time

$$k'_\varepsilon(t,u) = \frac{dk_\varepsilon(t,u)}{dt} = \epsilon(1-u)$$

is proportional to the scale parameter ϵ and it varies somewhere between zero if the system is restored to a minimally repaired system $(u = 1)$ and ϵ if the system is restored to a perfectly repaired system (but not as good as new system) (as $u \to 0$). So, ϵ value determines the "scale" or dispersion of the repair cost that is

$$0 \leq k'_\varepsilon(t,u) < \epsilon$$

If ϵ is large, then the repair cost domain will be more spread out; if ϵ is small then it will be more concentrated.

The model is based on the assumption that the the true measure of revenue over production process is unknown. This is the case which is common when resulting output measures including defective items produced and product quality is not identifiable. These unidentifiable measures associated with output cause the overestimating the quality and incur an invisible quality cost and consequent overestimation of revenue. To overcome this inefficiency and get a precise measure of revenue, since product quality is a function of system deterioration state, the revenue from system is linked by unobservable physical state (damage) process X_t. More precisely it is assumed that revenue from system per unit of time is damage state dependent on μ_{X_t} with $\mu_2 < \mu_1$. These conditions imply that, operation in either one of the physical states, normal state or degraded state, generates revenue which is higher in normal state. It is assumed that the cost incurred on replacement at the terminal time T is dependent on the physical state as $\phi^u_{X_T}$ $(\phi_1 < \phi_2)$. To each $u \in \mathcal{U}$ there is a performance measure $J(u)$:

$$J(u) = E_u \left[\int_0^T \left(\mu_{X_t} - k_\varepsilon(t,u) \right) - \sum_{n>0} k_{T_n} - \phi_{\dot{X}_T} \right] < \infty \,. \tag{5.5.1}$$

To convert the partial information control problem into a complete-information problem, let

$$\hat{\varphi}^u_t(i) = P_u \left(X_t = i | \mathcal{F}^N_t \right)$$

(see equation (5.8.2)). Then the $(P_u, \mathcal{F}^N_t)$-intensity of N_t is

$$\hat{\gamma}_t(u) = E_u \left(\gamma_{X_t} | \mathcal{F}^N_t \right) = \sum_{i=1}^{2} \gamma_i \hat{\varphi}^u_t(i)$$

Given partial information $\mathcal{F}^N_t$, an $\mathcal{F}^N_t$ adapted measure of inspection cost and the final cost are given by

$$k(t, N_t, u) = K - C\hat{\varphi}(N_t, t; 1)$$

and

$$\hat{\phi}_t(u) = \sum_{i=1}^{2} \phi_i \hat{\varphi}^u_t(i)$$

respectively. Since $k(t, N_t, u)$ is $\mathcal{F}^N_t$-predictable the integration theorem (see Bremaud (10)) gives the $\mathcal{F}^N_t$ measure of the value function

$$
\begin{aligned}
\hat{J}(u) &= E_u\bigg[\int_0^T \bigg(\sum_{i\in I}\mu_i\hat{\varphi}_t^u(i) - k_\varepsilon(t,u) - k(t,N_t,u)\hat{\gamma}_t(u)\bigg)\,dt - \sum_{i\in S}\phi_i\hat{\varphi}_T^u(i)\bigg] \\
&= E_u\bigg[\int_0^T \bigg(\sum_{i\in S}\mu_i\hat{\varphi}_t^u(i) - k_\varepsilon(t,u) - k(t,N_t,u)\sum_{i\in S}\gamma_i\hat{\varphi}_t^u(i)\bigg)\,dt \\
&\qquad \ldots - \sum_{i\in S}\phi_i\hat{\varphi}_T^u(i)\bigg]
\end{aligned}
\tag{5.5.2}
$$

The control problem above is Markovian with respect to N_t. In a Markovian control problem the value function $\hat{J}(u)$ is path independent and $\hat{J}(u)$ the corresponding control process u is of the form $f(t,N_t)$ where $f(t,n)$ for each $n \in N_+$ is an $\mathbb{R}$-valued measurable *deterministic* function. The problem can now be re-formulated as a deterministic Hamilton-Jacobi problem over a finite time horizon $\{u_t^* \equiv u_t^*(t,n) : u_t^* \in \mathcal{U}\}$:

$$
\hat{J}(u^*) = \sup_{u\in\mathcal{U}} \hat{J}(u)\,.
$$

The required result is found in Bremaud (10)[Chapter VII: Corollary C2].

C2 Corollary *Suppose that $\hat{\gamma}(t,n,u)$, $\hat{\varphi}_t^u(i) = \hat{\varphi}(t,n,u)(i)$ $(i \in S)$ and $k(t,n,u)$ do not depend on ω, and that there exists for each $n \in N_+$ a function $V(t,n)$ such that*

$$
\begin{aligned}
&\frac{\partial V(t,n)}{\partial t} + \sup_{u\in U_t}\bigg\{\hat{\gamma}(t,n,u)\left[V(t,n) - V(t,n-1) - k(t,n,u)\right] \\
&\qquad\qquad \ldots + \sum_{i\in S}\mu_i\hat{\varphi}(t,n,u)(i) - k_\varepsilon(t,u)\bigg\} = 0,
\end{aligned}
\tag{5.5.3}
$$

$$
V(T,n) = \inf_{u\in\mathcal{U}} \phi(T,n,u)
$$

where

$$
\phi(T,n,u) = \sum_{i\in S}\phi_i\hat{\varphi}_T^u(i)\,.
$$

Suppose also that there exists for each $n \in N_+$ *a measurable* $\mathbb{R}_+$ *valued function* $u^*(t,n)$ *such that*

$$u^*(t,n) \in \mathcal{U}, \quad t \in [0,T],$$

and

$$\begin{aligned} u^*(t,n) = \underset{u \in U_t}{argmax} \Big\{ & \hat{\gamma}(t,n,u)\,[V(t,n) - V(t,n-1) - k(t,n,u)] \\ & \ldots + \sum_{i \in S} \mu_i \hat{\varphi}(t,n,u)(i) - k_\varepsilon(t,u) \Big\}, \end{aligned} \tag{5.5.4}$$

Then u_t^* *defined by*

$$u_t^*(\omega) = u^*(t, N_t(\omega)) \tag{5.5.5}$$

for $\omega \in F_t^N$ *is an optimal solution.*

In the next section, using Hamilton-Jacobi equations (5.5.3), a numerical solution to the optimal (repair degree) control problem $\{u_t^* : t \in [0,T]\}$ and optimal inspection frequency of the system are derived. Finally, given u_t^* an optimal decision rule for production run length of the system is obtained.

5.6 Numerical example

Using corollary **C2** the optimality condition for the control problem is

$$\begin{aligned} & \frac{\partial V}{\partial t}(t,n) + \sup_{u \in U_t} \Big\{ C\bar{\gamma}(\hat{\varphi}(t,n,u)(1))^2 \\ & + \hat{\varphi}(t,n,u)(1)\,[(\mu_1 - \mu_2) + C\gamma_2 + (V(t,n) - V(t,n-1) - K)\,\bar{\gamma}] \\ & + \lambda_2\,[V(t,n) - V(t,n-1) - K] - k_\varepsilon(t,u) + \mu_2 \Big\} = 0, \end{aligned} \tag{5.6.1}$$

$$\phi(T,n,u) = \sum_{i \in S} \phi_i \hat{\varphi}_T^u(i). \tag{5.6.2}$$

The optimal control process $u^*(t,n)$ is given by

$$u^*(t,n) = \underset{u \in \mathcal{U}_t}{argmax} \Big\{ C\bar{\gamma} \left(\hat{\varphi}(t,n,u)(1)\right)^2 + \hat{\varphi}(t,n,u)(1) \left[\left((\mu_1 - \mu_2) + C\gamma_2 \right) \left(V(t,n) - V(t,n-1) - K \right) \bar{\gamma} \right] + \lambda_2 \left[V(t,n) - V(t,n-1) - K \right] - k_\varepsilon(t,u) + \mu_2 \Big\} \tag{5.6.3}$$

Numerical values are chosen for the parameters to illustrate the solution of the ordinary differential equation (5.6.1) and the determination of the corresponding optimal control process $u^*_{(t,n)}$, $(n \geq 0)$. Let $T = 15$, $K = 2$, $C = 1$, $\mu_1 = 2$, $\mu_2 = 1$, $\gamma_2 = 2$, $\gamma_1 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\alpha = 2$, $\beta = \sqrt{2}$, $\phi_1 = 1$, $\phi_2 = 2$ and $\varepsilon = 0.15$. By using equation (5.3.8) to estimate mean inspection times $\hat{\mu_n}$, $n \geq 0$ and the Euler method with step size $h = 0.1$ an evolution of the optimal expected revenue $V(t,n)$ (see Figure 5.3) for $0 \leq n \leq 12$ are derived.

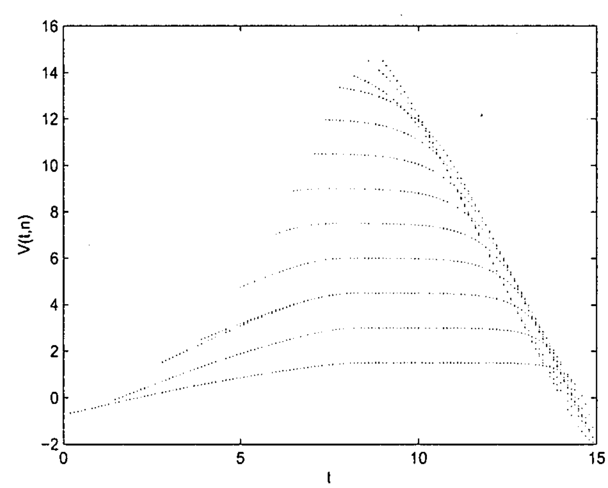

Figure 5.3: An evolution of the optimal expected revenue $V(t,n), t \in [0,15], (0 \leq n \leq 12)$

As illustrated (see figure 5.3), the optimal expected revenue $V(t,n)$ for $t \in [0, 8.63]$ is non-decreasing in the number of inspections, at 11^{th} inspection the revenue reaches to the maximum value, then for $t \in (8.63, 15]$ it follows a decreasing trend.

Also, corresponding to n^{th} inspection event ($n = 0, 1, 2, ...12$), the Table 5.1 illustrates a sequence of the expected revenue V_n^*, expected inspection times $\hat{\mu}_n$, mean time between inspections (MTBI) $\Delta\hat{\mu}_n$, and the optimal control process $u^*_{(t,n)}$ where

$$V_n^* = \max_{\hat{\mu}_n \leq t \leq 15} V(t,n), \quad 0 \leq n \leq 12$$

As seen, the optimal control sequence $u^*_{(t,n)}$ for $n = 0, 1, ..., 4$ take the boundary values $\{0.1, 1\}$ of the constraint set $[0.1, 1]$, and from fifth inspection on i.e. ($5 \leq n \leq 12$), $u^*_{(t,n)}$ chooses just the upper endpoint 1. The second column of the Table describes an evolution of the expected revenue V_n^* which is concave in the number of inspections n. As shown, V_n^* takes the maximum value $V^* = \max_n V_n^* = V_{11}^* = 14.8487$, then for ($n \geq 12$) follows an decreasing trend. So, the sequence of V_n^* not only gives us a solution to the optimal (repair degree) control process u_t^* and optimal inspection problem which is the optimum frequency of inspections, but also provides a solution to the optimal run length of the system. More precisely, the optimum maintenance policy includes an optimal stopping rule to replace the system at 11^{th} inspection event which is $T^* = \hat{\mu}_{11} = 8.63$ and a sequence of the optimal inspection times $\hat{\mu}_n$ for $n = 1, 2, ..., 11$ driven by the optimal control process u_t^*, $t \in [0, 8.63]$ which is bang-bang in the sense that it takes the boundary values $\{0.1, 1\}$ that is

$$u_t^* = 0.1I(0 \leq t \leq 5.88) + I(5.88 < t \leq 8.63).$$

Following, a sequence of the mean time between inspections $\Delta\hat{\mu}_n = \hat{\mu}_n - \hat{\mu}_{n-1}$ ($0 \leq n \leq 12$)($\hat{\mu}_0 = 0$) is shown (see Figure 5.5). As expected, mean time between inspections decreases as the number of inspections increases. This results from inspection intensity $\hat{\gamma}(n, t, u^*)$ which is increasing in the number of inspections n and time t (see Figure 5.4).

n	V_n^*	$\hat{\mu}_n$	$\Delta\hat{\mu}_n$	$u^*_{(t,n)}$
0	1.5	0	–	$0.1I(0 \leq t < 8.8) + I(8.8 \leq t < 15)$
1	3	1.36	1.36	$0.1I(1.36 \leq t < 8.4) + I(8.4 \leq t < 15)$
2	4.4998	2.63	1.269	$0.1I(2.63 \leq t < 8.4) + I(8.4 \leq t < 15)$
3	5.9992	3.79	1.165	$0.1I(3.79 \leq t < 8.3) + I(8.3 \leq t < 15)$
4	7.4974	4.87	1.081	$0.1I(4.87 \leq t < 7.9) + I(7.9 \leq t < 15)$
5	8.9937	5.88	1.017	$I(5.88 \leq t < 15)$
6	10.4878	6.36	0.48	$I(6.36 \leq t < 15)$
7	10.9757	6.83	0.452	$I(6.83 \leq t < 15)$
8	11.9662	7.28	0.451	$I(7.28 \leq t < 15)$
9	13.3437	7.73	0.451	$I(7.73 \leq t < 15)$
10	13.8464	8.18	0.451	$I(8.18 \leq t < 15)$
11	**14.4887**	**8.63**	**0.451**	$I(\mathbf{8.63} \leq t < 15)$
12	14.2907	9.087	0.45	$I(9.08 \leq t < 15)$

Table 5.1: An evolution of optimal expected revenue V^*, mean inspection times $\hat{\mu}$ and mean time between inspections $\Delta\hat{\mu}$ given the optimal control process u^* and linear transition rate

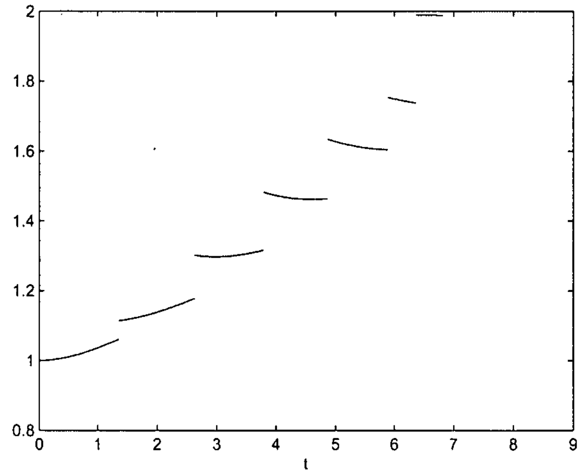

Figure 5.4: An evolution of the inspection intensity $\hat{\gamma}(n, t, u^*)$ given the optimal control process u_t^*

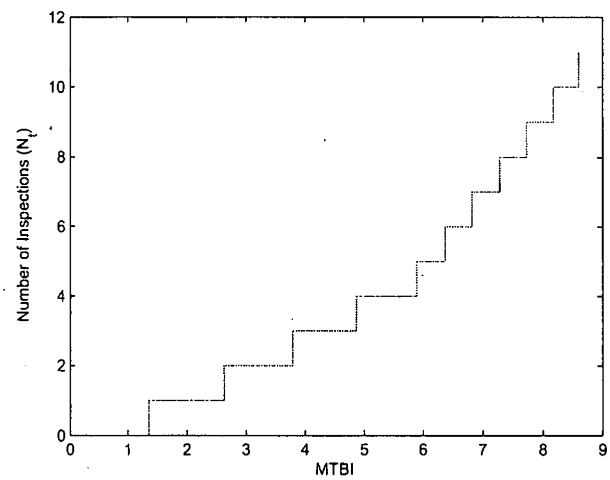

Figure 5.5: An evolution of the mean time between inspections (MTBI) given the optimal control process u_t^*

So, with respect to the optimal control process u_t^*, $t \in [0, 8.63]$ and results above, the optimal maintenance schedule is established as follows: from initial time 0 to 5.88 unit of time the system is repaired with repair degree $u^* = 0.1$. The system is inspected at optimum scheduled times $\{1.36, 2.63, 3.79, 4.87, 5.88\}$. Just after $t = 5.88$ to $t = 8.63$ the system is repaired minimally and inspections takes places at scheduled times $\{6.36, 6.83, 7.28, 7.73, 8.18\}$. So, for 2.75 unit of operating time the deterioration level of the system leaves unchanged. At 8.63 unit of time of operation (optimal production run length) the system is renewed.

Finally, to have a realization of the prediction of the system failure, let $\psi(x) = x$ and the baseline function be distributed Weibull with intensity function

$$\lambda_0(t) = \frac{\alpha t^{\alpha-1}}{\beta^{\alpha}} = t \quad t \geq 0$$

Finally, figure 5.6 beautifully give us a rule to optimal run length of the system based on the failure intensity $\hat{\lambda}(n,t,u^*)$ that is

$$T^* = \inf\left\{t \geq 0 : \hat{\lambda}(n,t,u^*) \geq c, \hat{\mu}_n \leq t < \hat{\mu}_{n+1}\right\}, (n \geq 0) \tag{5.6.4}$$

where c denotes the optimum threshold deterioration level at which the system is replaced. Subject to the optimal replacement time $T^* = 8.63$, the threshold level is given by

$$c = \hat{\lambda}(10, 8.63, u^*) = 17.26.$$

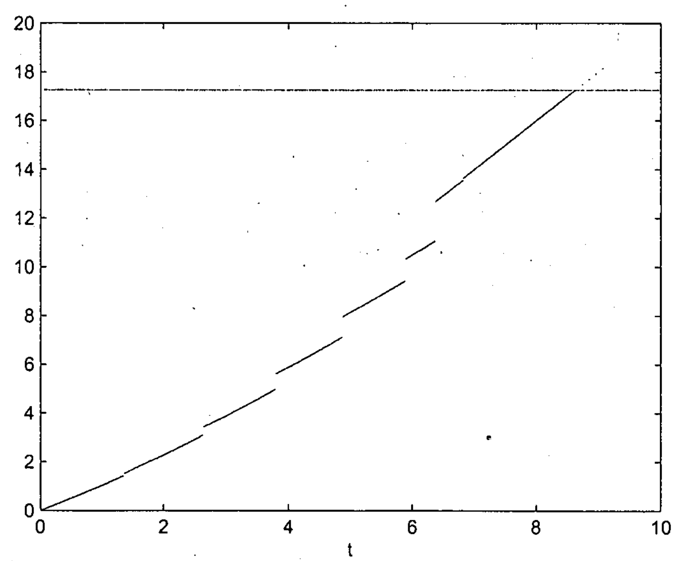

Figure 5.6: An evolution of failure intensity $\hat{\lambda}(n,t,u^*)$ given the optimal control process u_t^*

5.7 The Model (Nonlinear Transition Rate)

As above let the physical state of the system is described by a stochastic process $X = \{X_t, \ t \geq 0\}$. with the state space $S = \{1, 2\}$. It is assumed that the transition between states is driven by a non-homogeneous Markov process whose sojourn time in state one is described by a two-parameter generalized Pareto distribution

$$F(t) = 1 - \left(\frac{b}{at+b}\right)^{1+\frac{1}{a}}, \quad t \geq 0, \quad 0 \leq t < -\frac{b}{a}. \tag{5.7.1}$$

The transition intensity from state one (normal state) to state 2 (degraded state) is

$$q_{12}(t) = \frac{a+1}{at+b}, \quad 0 \leq t < -\frac{b}{a}$$

with parameters $-1 < a < 0$ and $b > 0$. The rate $q_{12}(t)$ is non-decreasing on $[0, -\frac{b}{a})$ with escape at $t = -\frac{b}{a}$.

The repair actions change the sojourn distribution in state one as in (5.7.1)

$$F^u(t) = 1 - \left(\frac{b}{aut+b}\right)^{1+\frac{1}{a}}, \quad t \geq 0 \tag{5.7.2}$$

The transition rate is

$$q_{12}^u(t) = \frac{u(a+1)}{aut+b} = u\, q_{12}(ut)$$

Given parameters values $a = -0.25$ and $b = 1$, following an evolution of the first passage time distribution of the damage process X_t is illustrated. As shown, with decreasing control (repair degree) values $u : 1 \mapsto \{0.8, 0.4\}$ intensity of leaving state one (normal state) decreases.

5.8 Damage Process X Given Partial Information

As shown before (see equation (5.4.5)), using the differential equation, an explicit solution of $\hat{\varphi}(n,t;1)$ is given by

$$\hat{\varphi}(n,t;1) = \frac{\exp\left(-\int(\bar{\gamma} + q_1(t))dt\right)}{A(n,t)}, \tag{5.8.1}$$

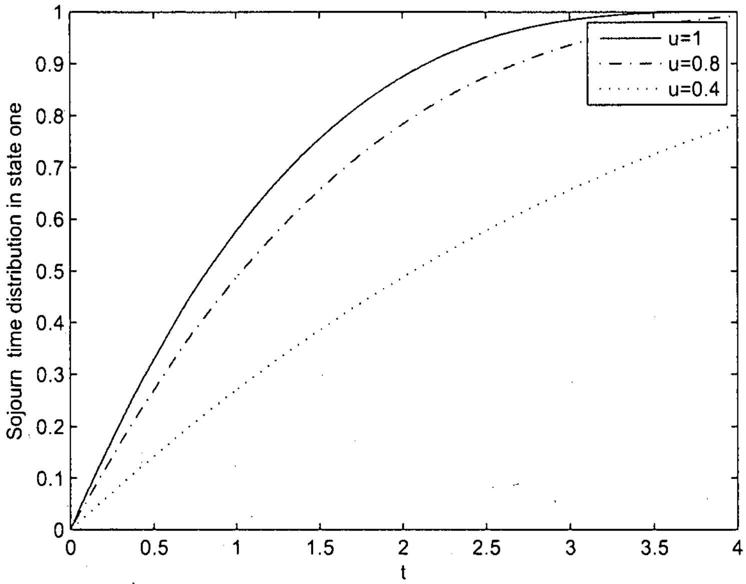

Figure 5.7: An evolution of sojourn time distribution of the damage process X_t in state one given control (repair degree) values $u = 0.4, 0.8, 1$ and parameters $a = -0.25, b = 1$.

where

$$A(n,t) = \frac{\exp\left[-\int (\bar{\gamma} + q_1(t))\,dt\right]_{t=T_n}}{1 - \hat{\varphi}_{T_n}(2)} - \int_{T_n}^{t} \bar{\gamma} \exp\left(-\int_0^v (\bar{\gamma} + q_1(s))\,ds\right) dv$$

and $\bar{\gamma} = \gamma_1 - \gamma_2$. Clearly, in terms of the sojourn time distribution in state one

$$\bar{F}(t) = \left(\frac{b}{at+b}\right)^{1+\frac{1}{a}}$$

The probability $\hat{\varphi}(n,t;1)$ can be expressed as

$$\begin{aligned}\hat{\varphi}(n,t;1) &= \exp(-\bar{\gamma}t)\,\bar{F}(t) \\ &\times \left\{\frac{\exp(-\bar{\gamma}T_n)\,\bar{F}(T_n)}{1 - \hat{\varphi}_{T_n}(2)} - \int_{T_n}^{t} \bar{\gamma}\exp(-\bar{\gamma}v)\,\bar{F}(v)dv\right\}^{-1}.\end{aligned} \tag{5.8.2}$$

Given $a = -0.25, b = 1$, from equation (5.8.2) it can be shown that the probability of being in state one

$\hat{\varphi}(n,t;1)$ for $t \in [T_n, T_{n+1})$, $(n \geq 0)$ is

$$\begin{aligned}
\hat{\varphi}(n,t;1) = &\left(1-\frac{1}{4}t\right)^3 \exp(t)\times \\
&\dots\left\{\frac{\exp(T_n)(1-\frac{1}{4}t_n)^3}{1-\hat{\varphi}_{T_n}(2)} + \exp(t)\right. \\
&\dots - \frac{3}{4}\left[t\exp(t) - \exp(t)\right] + \frac{3}{16}\left[\exp(t)t^2 - 2t\exp(t) + 2\exp(t)\right] \\
&\dots - \frac{1}{64}\left[t^3\exp(t) - 3t^2\exp(t) + 6t\exp(t) - 6\exp(t)\right] \\
&\dots - \exp(T_n) - \frac{3}{4}\left[T_n\exp(T_n) - \exp(T_n)\right] \\
&\dots + \frac{3}{16}\left[T_n^2\exp(T_n) - 2T_n\exp(T_n) + 2\exp(T_n)\right] \\
&\dots \left. - \frac{1}{64}\left[T_n^3\exp(T_n) - 3T_n^2\exp(T_n) + 6T_n\exp(T_n) - 6\exp(T_n)\right]\right\}^{-1};
\end{aligned} \tag{5.8.3}$$

where T_n for $n \geq 0$ $(T_0 = 0)$ denote n^{th} inspection time. Let $\hat{\varphi}^u(n,t;1)$ denote the probability of the state (damage) process X_t adjusted by the repair degree process $u \in \mathcal{U}$. $(\bar{F}(t) \mapsto \bar{F}^u(t))$. From (5.8.2) we have

$$\begin{aligned}
\hat{\varphi}^u(n,t;1) = &\exp\left(-\bar{\gamma}t\right)\bar{F}^u(t)\times \\
&\left\{\frac{\exp\left(-\bar{\gamma}T_n\right)\bar{F}^u(T_n)}{1-\hat{\varphi}^u_{T_n}(2)} - \int_{T_n}^{t}\bar{\gamma}\exp\left(-\bar{\gamma}v\right)\bar{F}^u(v)dv\right\}^{-1}
\end{aligned} \tag{5.8.4}$$

In particular case, let $n = 0$. Then equation (5.8.4) for $t \in [0, T_1]$ $(0 < T_1 < 4)$ reduces to

$$\begin{aligned}
\hat{\varphi}^u(0,t;1) = &\exp(t)\left(1-\frac{1}{4}u\,t\right)^3\times \\
&\dots\left\{\exp(t) - \frac{3}{4}u\left[t\exp(t) - \exp(t)\right]\right. \\
&\dots + \frac{3}{16}u^2\left[t^2\exp(t) - 2t\exp(t) + 2\exp(t)\right] \\
&\dots - \frac{1}{64}u^3\left[t^3\exp(t) - 3t^2\exp(t) + 6t\exp(t) - 6\exp(t)\right] \\
&\dots \left. - \frac{3}{4}u - \frac{6}{16}u^2 - \frac{6}{64}u^3\right\}^{-1};
\end{aligned} \tag{5.8.5}$$

The evolution of the probability of being in state one; $\hat{\varphi}^u(0,t;1)$, with control (repair degree) values $u = 0.4, 0.8, 1$ is illustrated in Figure 5.8. The stochastic ordering of the sojourn time distributions determined by the repair degree values $u : 1 \mapsto \{0.8, 0.4\}$ is clear to see

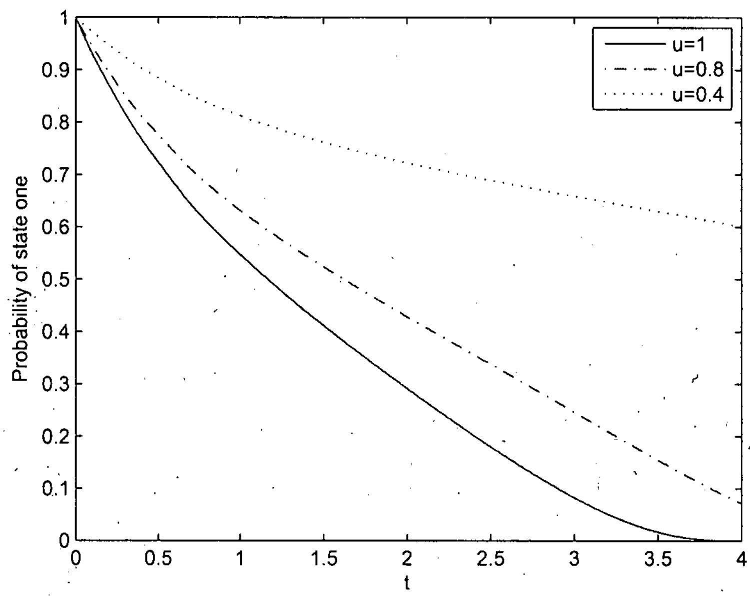

Figure 5.8: An evolution of probability of state one $\hat{\varphi}^u(0,t,1)$ given control (repair degree) values $u = 0.4, 0.8, 1$.

5.9 Numerical Example

To obtain an optimal repair degree strategy $\{u_t^* : t \in [0,T]\}$ which maximizes revenue from the system over a fixed time period $T = \frac{-b}{a}$, let $K = 2$, $C = 1$, $\mu_1 = 1.5$, $\mu_2 = 1$, $\gamma_2 = 2$, $\gamma_1 = 1$, λ_1, $\lambda_2 = 2$, $\phi_1 = 1$ and $\phi_2 = 2$, $a = -0.25$, $b = 1$ and $\varepsilon = 0.65$. By using equation (5.3.5) to estimate mean inspection times $\hat{\mu_n}$, $n \geq 0$ and the Euler method with step size $h = 0.1$ an evolution of the optimal expected revenue $V(t,n)$ (see Figure 5.9) and optimal control (repair degree) process (see Figure 5.10) for $0 \leq n \leq 3$ are derived. As illustrated in Figure 5.10, the optimal expected revenue $V(t,n)$ for $t \in [0, 1.76]$ is non-

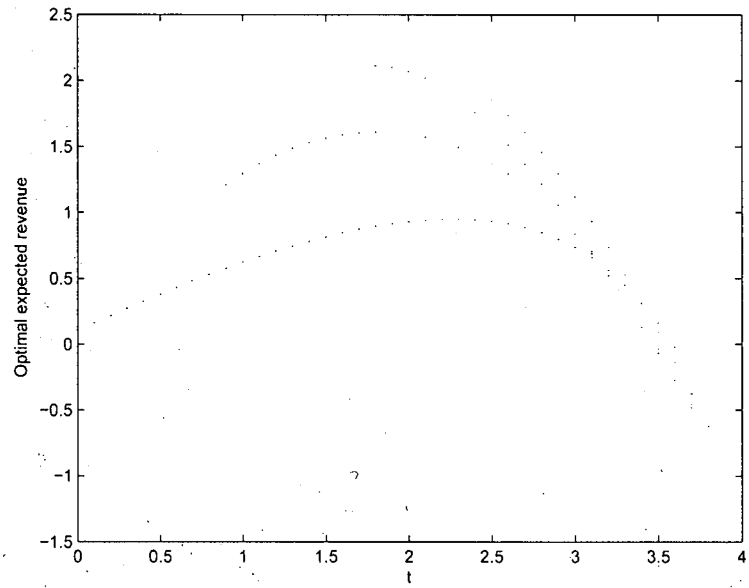

Figure 5.9: An evolution of optimal expected revenue over finite time $[0, 4]$.

decreasing in the number of inspections, at second inspection the revenue reaches to the maximum value, then for $t \in (1.76, 4]$ it follows a decreasing trend. Also, corresponding to n^{th} inspection event ($n = 0, 1, 2, 3$), an evolution of the optimal expected revenue V_n^*, expected inspection times $\hat{\mu}_n$ and mean time between inspections (MTBI) $\Delta\hat{\mu}_n$ are shown (see Table 5.2) where

$$V_n^* = \max_{\hat{\mu}_n \leq t \leq 4} V(t,n), \quad 0 \leq n \leq 3$$

The second column of the table describes an evolution of the optimal expected revenue V_n^* which is concave in the number of inspections n. As shown, V_n^* achieves to its maximum value at second inspection, that is, $V^* = \max_n V_n^* = 2.1144 \; \forall n \geq 0$, then for

n	V_n^*	$\hat{\mu}_n$	$\Delta\hat{\mu}_n$
0	0.9462	0	–
1	1.5906	0.9	0.9
2	**2.1144**	**1.76**	**0.86**
3	1.8527	2.45	0.69

Table 5.2: An evolution of optimal expected revenue V^*, mean inspection times $\hat{\mu}$ and mean time between inspections $\Delta\hat{\mu}$ given the optimal control process u^* and nonlinear transition rate

$(n > 2)$ follows an decreasing trend. So, the sequence of V_n^* not only gives us a solution to the optimal (repair degree) control process u_t^*, $t \in [0, 1.76]$ that is

$$u_t^* = \begin{cases} 0.1, & 0 < t \leq 0.9; \\ 1, & 0.9 < t \leq 1.3; \\ 0.85, & 1.3 < t \leq 1.4; \\ 0.71, & 1.4 < t \leq 1.5; \\ 0.6, & 1.5 < t \leq 1.6; \\ 0.5, & 1.6 < t \leq 1.7; \\ 0.43, & 1.7 < t \leq 1.76; \\ 1, & 1.76 < t \leq 1.8. \end{cases}$$

but also provides a solution to optimal inspection frequency $\hat{\mu}_n$ $(n = 1, 2, 3)$ and the optimal run length problem of the system which is $T^* = \hat{\mu}_2 = 1.76$.

Also, in the third and forth column of the table a sequence of the mean inspection time $\hat{\mu}_n$ and mean time between inspections $\Delta\hat{\mu}_n = \hat{\mu}_n - \hat{\mu}_{n-1}$ $(n = 1, 2, 3)$ $(\hat{\mu}_0 = 0)$ are shown. As expected, mean time between inspections is decreasing in the number of inspections. This follows from inspection intensity $\hat{\gamma}(n, t, u^*)$ (see Figure5.11) which is increasing in number of inspections n. As shown, intensity of inspection over

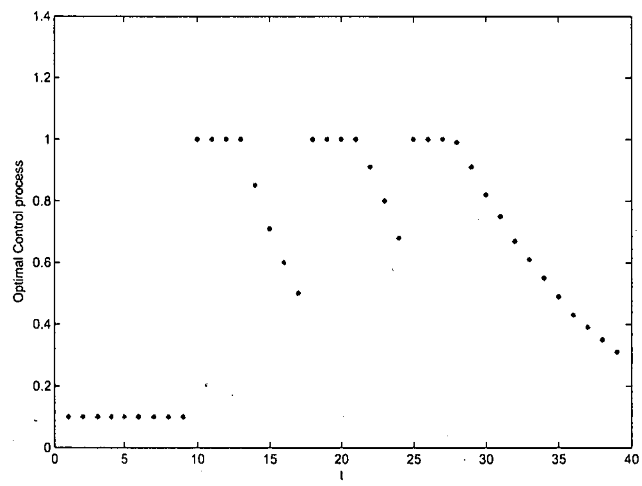

Figure 5.10: An evolution of optimal control process u^* given $\mathcal{F}^N$.

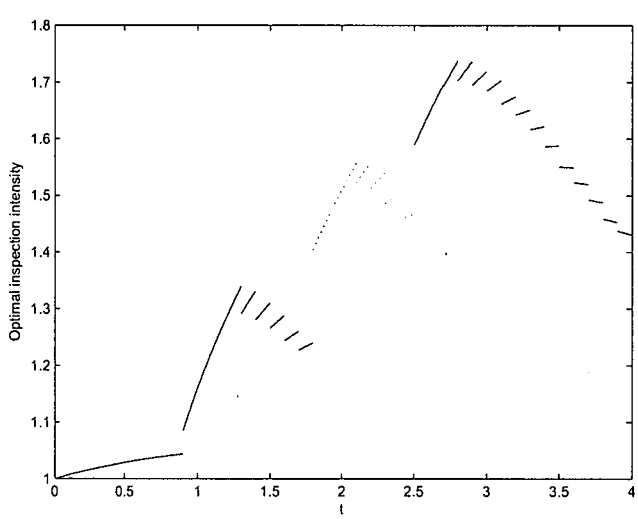

Figure 5.11: An evolution of optimal inspection intensity $\hat{\gamma}(n, t, u^*)$ given $\mathcal{F}^N$.

inter-arrival inspection times as a result of decreasing trend of repair degree process u_t^*, $(t \in [\hat{\mu}_n, \hat{\mu}_{n+1}))$, $(n = 0, 1, 2)$ is non-increasing. So, with respect to the optimal control process u_t^*, $t \in [0, 1.76]$ and the table illustrated above, the optimal maintenance schedule is established as follows: from initial time 0 to 0.9 unit of time the system is repaired with repair degree $u^* = 0.1$. The first scheduled inspection of the system occurs just after 0.9 unit of time. Over time interval $(0.9, 1.3]$ the system is repaired minimally. That means, for 0.4 unit of operating time the deterioration level of the system leaves unchanged. For $t \in (1.3, 1.76]$ the system is repaired partially with repair degree u_t^* which is decreasing in time. At $T^* = 1.76$ unit of time of operation (optimal production run length) the system is renewed. As illustrated in Figure 5.12 given optimal control process u_t^*, the optimal replacement policy, which results in an optimal production run length, is a control limit policy with respect to the failure rate (system state) process $\hat{\lambda}(n, t, u^*)$. That means, the replacement is performed when $\hat{\lambda}(n, t, u^*)$ first reaches the optimal critical limit c, where

$$\begin{aligned} T^* &= \inf\left\{t : \hat{\lambda}(n, t, u^*) \geq c = 2.5267\right\} \\ &= 1.76 \end{aligned} \tag{5.9.1}$$

It is easy to see that the optimization problem of inspection frequency and production run length/replacement policy investigated here is a two steps process optimization. First we optimized the control (repair degree) process u^*, then given u^* solutions to the optimal inspection intensity $\hat{\gamma}(n, t, u^*)$ and optimal production run length/replacement policy have been obtained.

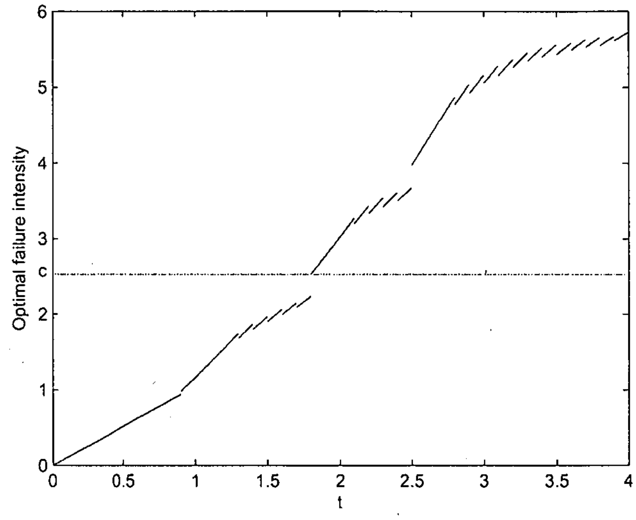

Figure 5.12: An evolution of optimal failure intensity $\hat{\lambda}(n,t,u^*)$ given $\mathcal{F}^N$.

5.10 Conclusion

In this chapter using optimal intensity control modeling, given both linear and non-linear transition rate $q_{12}(t)$ of damage process X, a solution to the maintenance scheduling problem of a manufacturing system subject to deterioration is obtained. The modeling rests on assumptions that inspections do not impact on the failure characteristics of the system, the process is adapted to partial information and the resulting output (revenue) from system is subject to the system state influenced by repair action and deterioration process. The formulation allows the application of a standard for Markov control (10) to be exploited through the control of the intensity of the underlying physical wear process. The optimum policy determined gives a correct balance between revenue from system and maintenance costs.

Chapter 6

Conclusion and Further Works

6.1 Summery and Conclusion

6.1.1 The optimal intensity control problem

In chapter 5 we addressed the problem of maintenance scheduling a manufacturing system subject to deterioration. The model investigated rests on realistic assumption that resulting output (production process) from system is subject to the state of the system. Ideal state of manufacturing performance results in more products (revenue); System malfunction which arises from system deterioration leads to significant proportion of defective products. It is assumed that the manufacturing system is subject to maintenance (repair and inspection): insufficient maintenance leads to an increase in the number of defective items, low profit and low maintenance cost; excessive maintenance results reduces the proportion of defective items, high profit and high maintenance cost. Repair actions are reflected through incorporating control (repair degree) process $u_t \in \mathcal{U}$ as scale parameter of AFT and PH model (see Newby (51)) into transition rate of damage process X_t. To balance the amount of maintenance (repair and inspection) and optimize revenue from system, using optimal intensity control model (see Bremuad (10)), an optimal control (repair degree) policy under both proportional hazard (PH) model (linear transition rate) and accelerated failure time (AFT) (non-linear transition rate) is derived. Also, to keep a correct balance between revenue from system and inspection frequency of system given optimal control process u_t^*, a solution to optimal inspection intensity and corresponding optimum sequence of inspection times (optimum inspection schedule) is obtained. An illustrative numerical example under both AFT and PH model was provided. Given the optimal control process u_t^* which is solution of Hamilton Jacobi equation, results of the model provides a realistic inspection policies for systems which are subject to deterioration. Under both maintenance models a decreasing sequence of the inspection intervals is derived. Besides, to get an insight into the prediction of system failure, using the proportional intensity process driven by u_t^*, an evolution of failure intensity of the system was illustrated. Finally, an optimal production run length T^* was determined. In sum, our model presented in Chapter 5 is a novel approach to maintenance optimization which through modeling intensity control provides a solution to the optimal repair policy problem of manufacturing systems subject to deterioration.

6.2 Decision Modeling for Stochastically Deteriorating Systems

In chapter 4 we presented a new approach to decision modeling of stochastically deteriorating systems whose state is determined by bivariate process (X, V): X refers to damage process and V denotes the virtual age of the system. The problem is to determine optimal repair rule ξ_r and replacement rule ξ_f under both periodic and non-periodic inspection policy in such away that the long-run average cost per unit of time is minimized. In preference to current decision models such as (56) and (55), using Virtual age process and repair alert model, expressions for the long-run average cost under both periodic and non-periodic inspections policy are obtained. To optimize the long-run average cost per unit time subject to system parameters an algorithm applied for both periodic and non-periodic inspection policy was proposed. The model investigated above can be extended by formulating it in both semi-parametric and parametric framework. The degradation process of the system X_t can be modeled by some semi-parametric process such as (non)-homogeneous Markov process. Also, the parametric processes such as Gamma process , Maximum process which are monotone in time are suggested to describe the degradation process of the system. Furthermore, to demonstrate the use of this maintenance model in practical application, using the presented analytical method, providing a numerical example for non-periodic inspection policy is worthwhile.

6.3 Further Works

In this section we propose some maintenance models to be further studied in the future:

- Modelling optimal intensity control subject to bivariate control and virtual age process or/and multivariate point process
- Decision modelling for stochastically deteriorating systems subject to bivariate state process (n, z)

In following section briefly a generalized approach to maintenance scheduling problem of chapter 5 is presented. Then, to tackle an optimal control problem of a system which is subject to repair and maintenance actions (RMAs) at inspection times, a generalized case of the intensity control model (see Chapter 5) is presented. In generalized model, the flow of the process which consists of random jumps resulting from RMAs, and continuous motion between consecutive jumps is controlled in such away that the maximum expected value arising from continuous revenue, jump and terminal cost is derived. In other words, the model outlined above is a new approach to the controlled Piecewise Deterministic Markov Process (PDP) (see Bremuad (19)) indexed by bivariate processes (u_t, V_t) $(t \geq 0)$ denoting the control process and the virtual age process respectively where through driving inspection intensity of system and the change of the time origin control the motion of the PDP between jumps. The objective is to control both continuous deterministic motion and the random jumps of the processes so that the expected value is maximized. Finally, in the last section, using Marked point process (see Aven and Jensen (4)), with the same approach as Chapter 4 a new decision model for maintenance scheduling deteriorating systems is proposed.

6.4 Modeling optimal intensity control subject to bivariate control and virtual age process or/and multivariate point process

The maintenance optimization model proposed in chapter 5 has potential to consider the optimal maintenance scheduling problem of a variety of systems (or system components) which are subject to repair and inspections. This arises from the intensity control model set up (see Bremuad (10)) which can simply provides solution for optimal intensity control of a multivariate counting process $N_t = \big(N_1(t), N_2(t), ..., N_k(t)\big)$. In such case, the problem of controlling the intensity of univariate point process is generalized to multivariate point process case which $N_i(t)$ $(i = 1, 2, ..., k)$ refers to the number of inspections of system type i (or i^{th} component of the system).

In sequel, a new approach to the controlled piecewise deterministic Markov process (PDP) (see Almudevar (2) and Dempster (20)) is presented. A piecewise deterministic process (PDP) introduced by Davis (19) is a continuous-time homogeneous Markov process (x_t, P_x). The trajectories of piecewise deterministic Markov processes are solution of an ordinary differential equation

$$\frac{dx(t)}{dt} = f(x_t)$$

with possible random jumps between different integral curves. In the interior E_0 of the state space $E \subset \mathbb{R}$, the jump intensity is given by a non-negative real valued function while if the boundary E_δ of the state space E is attained a jump occurs immediately. The distribution of new initial point after a jump is determined by the probability measures q_x or p_x dependent on whether the jump started from a state $x \in E_0$ or $x \in E_\delta$. More precisely, if $\beta := \inf\{t : x_{t^-} \neq x_{t^+}\}$ implies the first jump time of the process, then

$$P_x(\beta < t) = \lambda(x).t + o(t),$$

$$P_x(x_{\beta^+} \in d\xi | x_{\beta^+} \neq x_{\beta^-} = x) = q_x(d\xi)$$

if $x \in E_0$,

$$P_x(\beta = 0) = 1,$$

$$P_x(x_{\beta^+} \in d\xi | x_{\beta^+} \neq x_{\beta^-} = x) = p_x(d\xi)$$

if $x \in E_\delta$. As shown, the quadruple (f, λ, q, p) determines the flow of the PDP (x, P_x). The problem is to determine an optimal control of random jumps of the processes in such away that the optimum expected value of the performance functional consisting of continuous, jump and terminal cost is derived.

Here, using maintenance model presented in chapter 5, we introduce a new controlled Piecewise Deterministic Process (PDP) in which in contrast to PDP model the paths between consecutive jumps are determined by the deterministic probability measure $\hat{\varphi}$ adapted to partial information $\mathcal{F}^N$ (see chapter 5). More precisely, new controlled piecewise deterministic Markov process is characterized by the quadruple

$$(\hat{\varphi}(u,v), \gamma(u,v), q_v^u, p_v^u)$$

where $\mathcal{F}^N$ adapted measure $\hat{\varphi}(u,v)$ indexed by processes (u_t, V_t) $(t \geq 0)$ determines the trajectories of the PDP. Both continuous deterministic motion and the random jumps of the process are controlled by incorporating the control process u_t and the virtual age process V_t into transition rate q_{12} and standardized normal distribution ϕ of probability measure $\hat{\varphi}_t \mapsto \hat{\varphi}_t^{(u,v)}$. The jump mechanism is determined by two further functions, the jump rate $\gamma(u,v)$ and the transition measures p_v^u or q_v^u where respectively maps the virtual age space $v_{t-} = v \in E_0$ or $v_{t-} = v \in E_\delta$ into the set of probability measures $\mathcal{P}(E_0)$. in such case, time to the first jump is

$$\beta := \inf\{t : v_{t-} \neq v_{t+}\}$$

An evolution of the process $\hat{\varphi}_t$ (see chapter 6) controlled by the bivariate process (u_t, v_t) is illustrated (see Figure 6.1). As shown, trajectories of the new controlled PDP is influenced by the bivariate process (u_t, V_t): u_t adjusts damage process intensity and V_t changes the time origin.

$$\hat{\varphi}_t^{(u,v)} = \left\{1 - \bar{\gamma}\sqrt{\frac{2\pi}{u}}\exp(\frac{\bar{\gamma}^2}{2u})\left[\phi\left(\sqrt{u}(V_t + \frac{\bar{\gamma}}{u})\right) - \phi\left(\frac{\bar{\gamma}}{\sqrt{u}}\right)\right]\right\}^{-1} \times \exp\left(-(\bar{\gamma}t + \frac{ut^2}{2})\right) \tag{6.4.1}$$

In generalized case, the flow of the process which will consist of random jumps resulting from repair and maintenance action, and continuous deterministic motion between consecutive jumps is controlled in such away that the maximum expected value including continuous revenue, jump and terminal cost is derived.

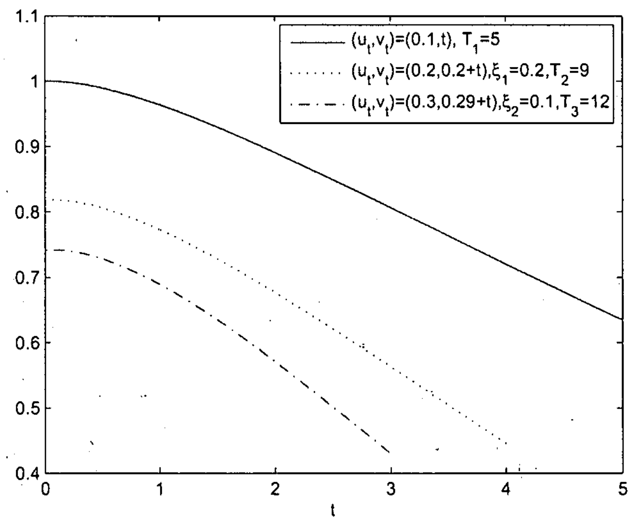

Figure 6.1: An evolution of the deterministic flow of the process $\hat{\varphi}_t$ driven by the control process u_t, the virtual age process $V_t = V_{T_n} + t$ $(0 \leq t < T_{n+1} - T_n)$ $(0 \leq n \leq 2)$ corresponding to n^{th} jump time T_n with the repair degree $\xi_n(0 \leq \xi_n \leq 1)$ and $V_{T_{n+1}} = V_{T_n} + \xi_n \phi(T_{n+1} - T_n)$.

6.5 Decision modeling for deteriorating systems subject to bivariate state process (n, z)

6.5.1 Model

We introduce a decision model for a repairable system subject to stochastic deterioration. It is assumed that the system deteriorates continuously over time and can fail at any instant. The model is given the partial information. The problem is converted into the complete information pattern by projecting on the observed history of the process. The deterioration flow (or, failure rate) of the system in a stochastic manner is measured by the Proportional Intensity Model (PIM) λ_t^f ($t \in \mathcal{R}_+$) which is the product of a baseline failure rate $\lambda_0^f(t)$ dependent on the age of the system and a positive function ψ which describes the effect of environmental factors in which the system is operating. To model the impact of the operating environment on the system the underlying stochastic non-homogeneous Markov process X_t is used. More precisely,

$$\lambda_t^f = \lambda_t^f(t)\psi(X_t), \tag{6.5.1}$$

To model the mechanism of the repair and maintenance process (see Figure 6.2)

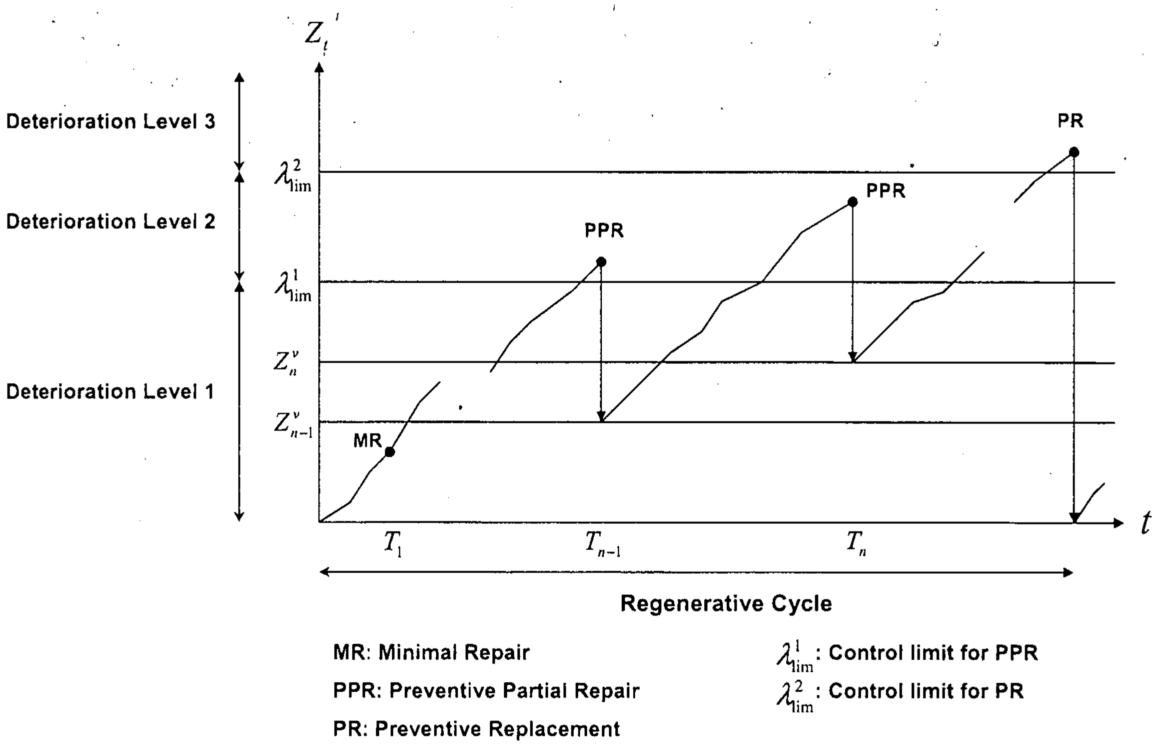

Figure 6.2: An evolution of the maintenance process subject to stochastic deterioration

we take advantage of the *Marked Point Process* $(T_n, D_n)_{n\geq 1}$ in which the point process T_n refer to the sequential non-periodic inspection times, determined by the stochastic intensity measure $\gamma_t = \gamma(t, X_t)$, and the Marks $D_n = (Y_n, W_n)$ denote pairs of random variables, where

$$Y_n = \Lambda_n - \Lambda_{n-1} = \int_{T_{n-1}}^{T_n} \lambda_s^f ds \tag{6.5.2}$$

(where $T_0 = 0$) represents the amount of accumulated deterioration over n^{th} inter-arrival inspection times and W_n equals (0)1 or 2 according to whether the accumulated deterioration defined by

$$Z_t = \sum_{n=1}^{N(t,S_v)} Y_n \tag{6.5.3}$$

(doesn't) reach the preventive partial repair threshold λ_{lim}^1 or the preventive replacement control limit λ_{lim}^2 so that $\lambda_{\text{lim}}^1 < \lambda_{\text{lim}}^2$. Also, associated with above control values let the T_{lim}^i (i=1,2) be $\mathcal{F}$-stopping times:

$$\begin{aligned} T_{\text{lim}}^i &= \inf\left\{ t \in \mathbb{R}_+ : \sum_{k=1}^{N(t,S_v)} Y_k \geq \lambda_{\text{lim}}^i, i = 1,2 \right\} \\ &= \inf\left\{ T_n : \sum_{k=1}^{n} Y_k \geq \lambda_{\text{lim}}^i, i = 1,2 \right\} \end{aligned} \tag{6.5.4}$$

and

$$\mathcal{F}_n = \sigma\left\{(T_i, D_i), X_{T_i}, i = 1, 2, ..., n\right\}$$

So, the marks D_n take values in $S_v = \mathbb{R}_+ \times \{0, 1, 2\}$. We define the associated counting process $N(t, \mathbb{R}_+ \times \{0, 1, 2\})$ with corresponding intensity function $\gamma_t(\mathbb{R}_+ \times \{i\})$ which denotes the number of times that the deterioration level of the system described by Z_t are observed in minimal repair region $[0, \lambda_{\text{lim}}^1)$, partial repair region $[\lambda_{\text{lim}}^1, \lambda_{\text{lim}}^2)$ and replacement region $[\lambda_{\text{lim}}^2, \infty)$.

To configure the decision and action process it is assumed that the action space includes three kinds of actions: (i) minimal repair action if the deterioration level of the system varies over range $[0, \lambda^1_{\text{lim}})$ (Deterioration level 1), (ii) the Preventive partial repair action if the deterioration level of the system varies over range $[\lambda^1_{\text{lim}}, \lambda^2_{\text{lim}})$. In such case the accumulated deterioration value through adjusting the virtual age of the system reduces to the level $Z^v_n = Z(V_n)$ where $V_n = V_{n-1} + \xi(T_n - T_{n-1})$, $0 < \xi < 1$, $(n \geq 1)$ denotes the virtual age of the system just after n^{th} repair action and (iii) the preventive replacement action if the deterioration level of the system exceeds the threshold value λ^2_{lim} (Deterioration level 3). With respect to the action space above, the state of the process is depicted by the pair $(N(t, S_v), Z^v_t)$: the system is in state $(N(t, S_v), Z^v_t)$ if just after $N(t, S_v)^{th}$ repair the accumulated deterioration is Z^v_t. Thus the state space is

$$S = N \times \mathbb{R}_+ \text{ where } N = \{0, 1, 2, ...\}, \mathbb{R}_+ = [0, \infty).$$

Using the renewal argument, as Chapter 4 expressions for long-run average cost per unit of time under both periodic and non-periodic inspection times can be derived. The maintenance process is optimized subject to the decision thresholds λ^1_{lim} and λ^2_{lim}, that respectively refer to the preventive partial repair and the preventive maintenance rule. This thresholds configuration, characterized by the minimization of long run average cost per unit of time, leads to an optimal decision rule -repair/replacement policy- with non-periodic/periodic inspection times.

In this thesis the power and flexibility of the martingale approach has shown how many maintenance and repair models can be built up. The martingale coupled with the study of regenerative processes as illustrated played a key role in stochastically modeling and maintenance optimization of systems which are subject to deterioration.

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