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Exact computation of one-loop correction to energy of spinning folded string in $AdS_5 \times S^5$

M. Beccaria[a,1], G. V. Dunne[b,2], V. Forini[c,3] , M. Pawellek[d,4] and A. A. Tseytlin[e,5]

[a] *Physics Department, Salento University and INFN, 73100 Lecce, Italy*

[b] *Department of Physics, University of Connecticut, Storrs CT 06269-3046, USA*

[c] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut Am Mühlenberg 1, D-14476 Potsdam, Germany*

[d] *Institut für Theoretische Physik III, Universität Erlangen-Nürnberg, Staudtstr.7, D-91058 Erlangen, Germany*

[e] *The Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.*

Abstract

We consider the 1-loop correction to the energy of folded spinning string solution in the AdS_3 part of $AdS_5 \times S^5$. The classical string solution is expressed in terms of elliptic functions so an explicit computation of the corresponding fluctuation determinants for generic values of the spin appears to be a non-trivial problem. We show how it can be solved exactly by using the static gauge expression for the string partition function (which we demonstrate to be equivalent to the conformal gauge one) and observing that all the corresponding second order fluctuation operators can be put into the standard (single-gap) Lamé form. We systematically derive the small spin and large spin expansions of the resulting expression for the string energy and comment on some of their applications.

[1] beccaria@le.infn.it
[2] dunne@phys.uconn.edu
[3] forini@aei.mpg.de
[4] Michael.Pawellek@physik.uni-erlangen.de
[5] Also at Lebedev Institute, Moscow. tseytlin@imperial.ac.uk

1 Introduction

Classical string solutions in $AdS_5 \times S^5$, or non-topological solitons of the string sigma model, play an important guiding role in the study of gauge-string duality. One of the basic examples is the folded spinning string in AdS [1, 2]. The classical string energy is a non-trivial function of the spin, interpolating between the flat-space regime, $E \sim \sqrt{S} + ...$, for small spin and the scaling AdS regime, $E = S + a \ln S + ...$, for large spin. The dual gauge-theory interpretation of the latter was suggested in [1] and since then was used and verified in many papers.

The form of this spinning string solution is determined by an elliptic sn function (a solution of the sinh-Gordon equation). Computing the quantum correction to its energy in $AdS_5 \times S^5$ string theory is thus a non-trivial problem, first addressed in [3]. In [3] the 1-loop correction to the energy $E_1(S)$ was expressed in terms of determinants of the bosonic and fermionic fluctuation operators with elliptic-function potentials. It was explicitly computed only in the large-spin limit when the solution simplifies drastically (elliptic function potentials become constant). Recently, attempts were made to compute the first few leading terms in $E_1(S)$ in the small S [9] and the large S [10] expansions.

The aim of the present paper is to solve the problem addressed in [3], i.e. to present the general analytic expression for the 1-loop correction $E_1(S)$ for an arbitrary value of the spin. This would enable us to systematically expand E_1 in the small S or large S limits.

The study of the large S expansion of string energy is important for several reasons, e.g., (i) for comparison with the Bethe ansatz predictions (see, e.g., [4, 5, 6, 7]); (ii) for further verification of the reciprocity property at strong coupling [8, 10]; (iii) for understanding the on-set of finite size (exponential or "wrapping") corrections in the anomalous dimension of the corresponding twist 2 gauge theory operator (cf. [11, 12, 13]) and the problem of orders of large-spin/large-coupling limits. The study of the small S expansion may shed light on quantum corrections to quantum string states or "short" operators [9, 14].

It is also interesting to compare the explicit form of the 1-loop string correction derived directly from the string theory action with the expression coming out of the approach based on classical integrability of the string sigma model [15, 16]. The two should match in general (see [17] and refs. there) but detailed comparison may teach us important lessons about the workings of the integrability in the case of cylindrical world-sheet topology.

We shall start in section 2 with a summary of the basic relations for the classical spinning string solution of [1, 2]. Then in section 3 we shall review the approach of [3] to the computation of the 1-loop correction E_1 to the string energy. In addition to having elliptic function potentials in the quadratic fluctuation operators, a complication of the conformal gauge expression for the 1-loop partition function of the string sigma model expanded near this solution is a mixing of the three AdS_3 modes. This mixing is absent in the static gauge [3], and we go beyond the discussion in [3] by arguing that the conformal-gauge and the static-gauge expressions are indeed equivalent (in particular, the string correction in the static gauge is also UV finite). This allows us to use the static gauge expression for E_1 in which all 8+8 bosonic and fermionic fluctuation modes are decoupled as a starting point of our investigation.

A further crucial observation made in section 3.3 is that all second-order fluctuation operators in this stationary soliton problem can be put into the standard single-gap Lamé ordinary differential operator form on a circle. As discussed in section 4, this allows us to compute their determinants in an explicit way. In section 4.1 we review several equivalent forms of the general expression for the determinant of the second-order ordinary differential operator $\mathcal{O} = -\partial_x^2 + V(x)$ on a circle: in terms of the discriminant, in terms of the quasi-momentum, in terms of the ζ-function or resolvent. In section 4.2 we specify these relations to the case of the Lamé potential $V = 2k^2 \mathrm{sn}^2(x|k^2)$. Then in section 4.3 we apply this formalism to the case of the fluctuation operators whose determinants appear in the string 1-loop correction E_1.

In section 5 we first demonstrate that the resulting expression for E_1 is UV finite as expected. We then check the equivalence between the conformal gauge and the static gauge expressions for E_1 by numerically evaluating the "mixed" conformal gauge determinant and comparing it with its static gauge counterpart that we found analytically. We also plot E_1 and compare it with its large spin and small spin asymptotics derived analytically in sections 6 and 7. In section 6 we also check the reciprocity constraints on the few leading terms in the large spin expansion of the energy.

Some concluding remarks are made in section 8. One natural extension that we plan to address in the future [18] is to repeat the analysis of the present paper in the case of the (S, J) folded string solution with an extra orbital momentum in S^5 [3] . This problem is more complicated in that even the static gauge expression for the fluctuation Lagrangian has now two mixed fluctuations and thus the standard expressions for the Lamé operator determinant cannot be directly applied.

There are several Appendices containing notation and technical details. In Appendix A we summarise the basic definitions for the elliptic functions, and describe the Landen transformation used to convert certain fluctuation operators to the Lamé form. Appendix B describes the Gel'fand-Yaglom numerical method for computing determinants of second-order differential operators, including the case of coupled operators. Appendix C contains the details of the relevant elliptic function expansions needed for studying the small spin and the large spin limits, while in Appendix D we evaluate the leading correction due to the (exponentially suppressed) contributions that we neglect in the main calculation. In Appendix E we consider an alternative approach to the expansion of the one-loop energy in the large spin limit. Appendix F relates our exact results to the perturbative expansion of the associated determinants.

2 Review of folded spinning string solution in AdS_3

The folded spinning string in AdS_3 space

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2 \tag{2.1}$$

is a classical closed string solution given by [1]

$$t = \kappa\,\tau, \qquad \phi = \omega\,\tau, \qquad \rho = \rho(\sigma) = \rho(\sigma + 2\pi), \tag{2.2}$$

where κ, ω are constant parameters. The equation of motion in conformal gauge[6] and its solution with initial condition $\rho(0) = 0$ are [7]

$$\rho'^2 = \kappa^2 \cosh^2\rho - w^2 \sinh^2\rho, \tag{2.3}$$

$$\sinh\rho(\sigma) = \frac{k}{\sqrt{1-k^2}}\,\mathrm{cn}(\omega\,\sigma + \mathbb{K}\,|k^2)\ , \qquad \rho'(\sigma) = \kappa\,\mathrm{sn}(\omega\,\sigma + \mathbb{K}\,|\,k^2)\ , \tag{2.4}$$

where $\mathbb{K} \equiv \mathbb{K}(k^2)$ is the *complete elliptic integral of the first kind* [33], with elliptic modulus given by $k \equiv \frac{\kappa}{\omega}$.[8] Here ρ varies from 0 to its maximal value ρ_0, which is related to the useful parameter η or k by

$$\coth^2\rho_0 = \frac{\omega^2}{\kappa^2} \equiv 1 + \eta \equiv \frac{1}{k^2}\ . \tag{2.5}$$

The periodicity implies an extra condition for the parameters

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^{\rho_0} \frac{d\rho}{\sqrt{\kappa^2\,\cosh^2\rho - \omega^2\,\sinh^2\rho}} \tag{2.6}$$

integrating which one finds (see 2.5)

$$\kappa = \frac{2\,k}{\pi}\,\mathbb{K}, \qquad \omega = \frac{2}{\pi}\,\mathbb{K}. \tag{2.7}$$

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[6] We use Minkowski signature in both target space and world sheet, so that in conformal gauge $\sqrt{-g}\,g^{ab} = \eta^{ab} = \mathrm{diag}(-1,1)$.

[7] To construct the full (2π periodic) folded closed string solution one should glue together four such functions $\rho(\sigma)$ on $\frac{\pi}{2}$ intervals and cover the full $0 \le \sigma \le 2\pi$ interval.

[8] See Appendix A for notation. We adopt here the Abramowitz-Mathematica notation for the modulus of the elliptic functions.

The corresponding induced 2-d metric on the (τ, σ) cylinder and its curvature are

$$g_{ab} = \rho'^2(\sigma)\, \eta_{ab} \ , \qquad R^{(2)} = -\frac{\partial_\sigma^2 \ln \rho'^2}{\rho'^2} = -2 + \frac{2\,\kappa^2\,\omega^2}{\rho'^4} \tag{2.8}$$

The two conserved momenta conjugate to t and ϕ are the classical energy and the spin

$$E_0 = \sqrt{\lambda}\,\kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2\rho \equiv \sqrt{\lambda}\,\mathcal{E}, \qquad S = \sqrt{\lambda}\,\omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2\rho \equiv \sqrt{\lambda}\,\mathcal{S} \tag{2.9}$$

Using (2.3) and (2.6) we get the following explicit expressions in terms of the complete elliptic integrals $\mathbb{K} = \mathbb{K}(k^2)$ and $\mathbb{E} = \mathbb{E}(k^2)$ (see Appendix A)

$$\mathcal{E}_0 = \frac{2}{\pi}\,\frac{k}{1-k^2}\,\mathbb{E} \ , \tag{2.10}$$

$$\mathcal{S} = \frac{2}{\pi}\Big(\frac{1}{1-k^2}\,\mathbb{E} - \mathbb{K}\Big) \ . \tag{2.11}$$

To find the energy in terms of the spin one is to solve for k (or η) in terms of $\mathcal{S}$ and then substitute it into the expression for the energy $\mathcal{E}$. This can be easily done in the two limiting cases:

(i) large spin or long string limit: $\rho_0 \to \infty$, i.e. $\eta \to 0$ or $k \to 1$

(ii) small spin or short string limit: $\rho_0 \to 0$, i.e. $\eta \to \infty$ or $k \to 0$

In the "long string" limit when the string's ends are close to the boundary of AdS_5, the spin is automatically large and the parameter η is expanded around zero as

$$\eta = \frac{2}{\mathcal{S}} - \frac{\ln(8\pi\mathcal{S}) - 3}{\pi^2\mathcal{S}^2} + \ldots \ , \qquad \eta \ll 1 \tag{2.12}$$

Substituting this in (2.10) one obtains for the energy the well known logarithmic behavior [1, 3, 10]

$$\mathcal{E}_0 = \mathcal{S} + \frac{\ln(8\pi\mathcal{S}) - 1}{\pi} + \frac{\ln(8\pi\mathcal{S}) - 1}{2\pi^2\mathcal{S}} + \ldots \ , \qquad \mathcal{S} \gg 1 \ , \tag{2.13}$$

where the leading $\ln \mathcal{S}$ term is governed by the so-called "scaling function" (cusp anomaly) and the subleading ones can be shown to obey non-trivial reciprocity relations [8, 10].

In the "short string" limit, when the string is rotating in the small central ($\rho = 0$) region of AdS_3, the spin is small and the parameter η is large

$$\frac{1}{\eta} = 2\,\mathcal{S} - \frac{1}{2}\,\mathcal{S}^2 + \frac{7}{8}\,\mathcal{S}^3 - \frac{117}{64}\,\mathcal{S}^4 + \ldots \ , \qquad \eta \gg 1 \tag{2.14}$$

This results in the usual flat-space Regge relation [1, 3, 9]

$$\mathcal{E}_0 = \sqrt{2\,\mathcal{S}}\Big(1 + \frac{3}{8}\,\mathcal{S} + \ldots\Big) \ , \qquad \mathcal{S} \ll 1. \tag{2.15}$$

These small and large spin expansions of the classical energy $\mathcal{E}_0$ are shown in Figure 1, compared to the exact relation. A similar plot for the one-loop correction is provided below, see Figure 8.

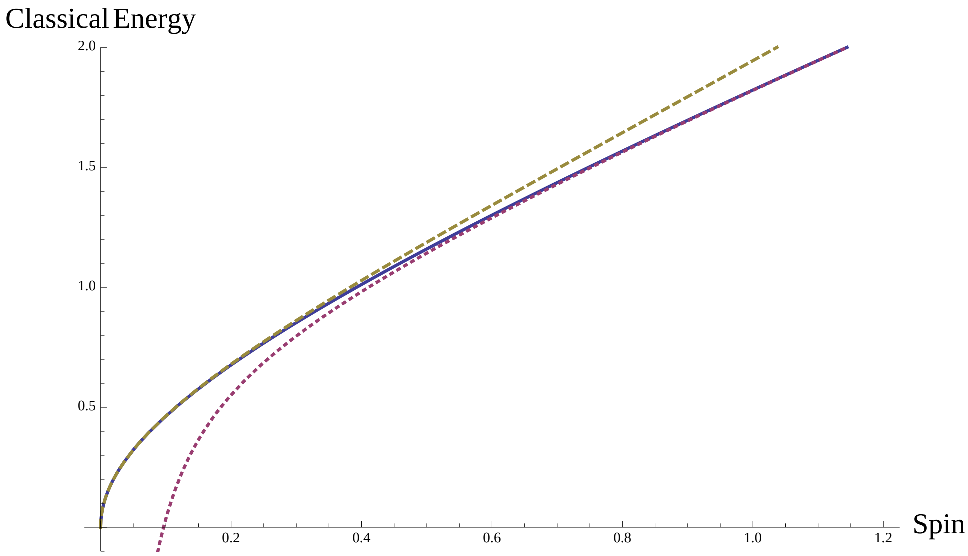

Figure 1: Plot (blue, solid curve) of the classical energy $\mathcal{E}_0$ as a function of the spin $\mathcal{S}$, compared with the large spin expansion (red, dotted curve) in (2.13), and the small spin expansion (gold, dashed curve) in (2.15).

3 One-loop correction to the spinning string energy

As discussed in [3], one can compute the leading quantum correction to the energy of this solution by expanding the action to quadratic order in fluctuations near the classical solution

$$\widetilde{I} = -\frac{\sqrt{\lambda}}{4\,\pi}\int d\tau \int_0^{2\,\pi} d\sigma\,(\widetilde{\mathcal{L}}_B + \widetilde{\mathcal{L}}_F) \tag{3.1}$$

and computing the corresponding partition function expressed in terms of determinants of the quadratic fluctuation operators. Then (switching to the euclidean time $\tau \to i\,\tau$) the 1-loop correction to the energy can be found from the 2d effective action Γ by dividing over the time interval ($t = \kappa\,\tau$)

$$E_1 = \frac{\Gamma}{\kappa\,\mathcal{T}}, \qquad \mathcal{T} \equiv \int d\tau \to \infty, \qquad \Gamma = -\ln Z \tag{3.2}$$

where Z is given by the ratio of the fermionic and bosonic determinants.

Since the above rigid spinning string solution is stationary, the coefficients in the fluctuation Lagrangian do not depend on τ. Then the relevant 2-d functional determinants may be reduced to 1-d determinants as in

$$\ln\det[-\partial_\sigma^2 - \partial_\tau^2 + M^2(\sigma)] = \mathcal{T}\int_{-\infty}^{+\infty}\frac{d\Omega}{2\pi}\;\ln\det[-\partial_\sigma^2 + \Omega^2 + M^2(\sigma)] \tag{3.3}$$

i.e. we may introduce $\tilde{\Gamma}$ defined by

$$\Gamma = \mathcal{T} \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi}\, \tilde{\Gamma}\ . \tag{3.4}$$

3.1 Conformal gauge

Following [3] one may use either the conformal gauge or the static gauge to compute the fluctuation Lagrangian and thus the corresponding 1-loop partition function. The bosonic fluctuation Lagrangian reads

$$\begin{aligned}\tilde{\mathcal{L}}_B^{(\rm conf)} &= -\partial_a \tilde{t}\partial^a \tilde{t} - \mu_t^2 \tilde{t}^2 + \partial_a\tilde{\phi}\partial^a\tilde{\phi} + \mu_\phi^2\tilde{\phi}^2 + \partial_a\tilde{\rho}\partial^a\tilde{\rho} + \mu_\rho^2\tilde{\rho}^2 + \\ &\quad + 4\,\tilde{\rho}(\kappa \sinh\rho\ \partial_0\tilde{t} - \omega\ \cosh\rho\ \partial_0\tilde{\phi}) +\ \partial_a\beta_u\partial^a\beta_u + \mu_\beta^2\beta_u^2 + \partial_a\zeta_s\partial^a\zeta_s\ ,\end{aligned} \tag{3.5}$$

$$\mu_t^2 = 2\rho'^2 - \kappa^2, \qquad \mu_\phi^2 = 2\rho'^2 - \omega^2, \qquad \mu_\rho^2 = 2\rho'^2 - \omega^2 - \kappa^2, \qquad \mu_\beta^2 = 2\rho'^2. \tag{3.6}$$

Here β_u $(u = 1, 2)$ are the two AdS_5 fluctuations transverse to the AdS_3 subspace in which the string is moving, while ζ_s $(s = 1, ..., 5)$ are fluctuations in S^5. The three AdS_3 fields $(\tilde{t},\ \tilde{\rho},\ \tilde{\phi})$ are coupled so that the corresponding 1-d determinant in (3.3) will involve the following 3×3 matrix differential operator acting on the 3 fields $X = (\tilde{t},\ \tilde{\rho},\ \tilde{\phi})$ (after $\tau \to i\tau,\ \partial_\tau \to i\Omega$)

$$\mathcal{O}_{t\rho\phi} = \begin{pmatrix} \partial_\sigma^2 - \Omega^2 - 2\rho'^2 + \kappa^2 & 2\Omega\,\kappa\ \sinh\rho & 0 \\ -2\Omega\,\kappa\ \sinh\rho & -\partial_\sigma^2 + \Omega^2 + 2\rho'^2 - \omega^2 & 2\Omega\,\omega\ \cosh\rho \\ 0 & -2\Omega\,\omega\ \cosh\rho & -\partial_\sigma^2 + \Omega^2 + 2\rho'^2 - \omega^2 - \kappa^2 \end{pmatrix} \tag{3.7}$$

In addition to the coefficients being dependent on σ according to (2.4), this mixing makes finding the determinant of this operator a non-trivial problem. Taking into account the contribution of the two massless conformal gauge ghosts,[9] the bosonic contribution to $\tilde{\Gamma} = \tilde{\Gamma}_B + \tilde{\Gamma}_F$ in (3.4) may be written then as

$$\tilde{\Gamma}_B^{(\rm conf)} = \frac{1}{2}\Big(\ln\det\mathcal{O}_{t\rho\phi} + 2\ln\det\mathcal{O}_\beta + 3\ln\det\mathcal{O}_0\Big)\ , \tag{3.8}$$

$$\mathcal{O}_\beta = -\partial_\sigma^2 + \Omega^2 + 2\rho'^2\ , \qquad \mathcal{O}_0 = -\partial_\sigma^2 + \Omega^2\ . \tag{3.9}$$

The fermionic part of the quadratic fluctuation Lagrangian can be put into the form [3]

$$\tilde{\mathcal{L}}_F = 2i(\bar{\Psi}\gamma^a\partial_a\Psi - i\mu_F\bar{\Psi}\gamma_3\Psi)\ , \qquad \mu_F = \rho'\ , \tag{3.10}$$

where γ_a are 2-d gamma matrices (times a unit 8×8 matrix) and $\gamma_3 = \mathrm{diag}(I, -I)$. It may be interpreted as describing a system of 4+4 2-d Majorana fermions with σ-dependent masses

[9] Here we are implicitly assuming that fluctuation determinants are defined with flat rather than (in general curved, for $\rho' \neq$ const) induced 2d metric, see related discussion in [21, 3].

$\pm\rho'$. Squaring the corresponding Dirac operator, the fermionic contribution to the 2-d effective action $\tilde{\Gamma}$ in (3.4) can be written as (see also [9])

$$\tilde{\Gamma}_F = -\frac{1}{2}\Big(4\ln\det\mathcal{O}_{\psi_+} + 4\ln\det\mathcal{O}_{\psi_-}\Big) \ , \tag{3.11}$$

$$\mathcal{O}_{\psi_\pm} \equiv -\partial_\sigma^2 + \Omega^2 + \mu^2_{\psi_\pm} \ , \qquad \mu^2_{\psi_\pm} = \pm\mu'_{_F} + \mu^2_{_F} = \pm\rho'' + \rho'^2 \ . \tag{3.12}$$

3.2 Static gauge

Another approach considered in [3] was to start with the Nambu action and use the same classical solution but impose the static gauge on quantum fluctuations: $\tilde{t} = \tilde{\rho} = 0$. [10] In this case the remaining AdS_3 mode $\tilde{\phi}$ is decoupled and is described by

$$\tilde{\mathcal{L}}^{(\rm stat)}_\phi = \partial_a\tilde{\phi}\partial^a\tilde{\phi} + \bar{\mu}^2_\phi\tilde{\phi}^2 \ , \tag{3.13}$$

$$\bar{\mu}^2_\phi = 2\rho'^2 + \frac{2\kappa^2\omega^2}{\rho'^2} \ . \tag{3.14}$$

Then the static gauge analog of (3.8) takes the form (the masses of other modes are the same as in the conformal gauge but there is no ghost determinant contribution)

$$\tilde{\Gamma}^{(\rm stat)}_B = \frac{1}{2}\Big(\ln\det\mathcal{O}_\phi + 2\ln\det\mathcal{O}_\beta + 5\ln\det\mathcal{O}_0\Big) \ , \tag{3.15}$$

$$\mathcal{O}_\phi = -\partial_\sigma^2 + \Omega^2 + 2\rho'^2 + \frac{2\kappa^2\omega^2}{\rho'^2} \ , \tag{3.16}$$

while the fermionic contribution to 1-loop partition function is the same as in (3.12).

The advantage of the static gauge expression for the effective action is that here all fluctuation modes are decoupled and are described by elliptic differential operators of the same type, $-\partial_\sigma^2 + V(\sigma)$. On general grounds, one may expect to find the same expression for the on-shell 1-loop partition function in the two gauges.[11] In this case one should get the following relation between the determinants of the conformal-gauge operator $\mathcal{O}_{t\rho\phi}$ in (3.7) and the static-gauge operator $\mathcal{O}_\phi$ (3.16)

$$\det\mathcal{O}_{t\rho\phi} = \det\mathcal{O}_\phi \ (\det\mathcal{O}_0)^2 \ , \tag{3.17}$$

where $\mathcal{O}_0$ is the massless operator in (3.9).[12] A concern about this equality was raised in [3] based on the fact that while the conformal gauge 1-loop partition function is UV finite [21], the

[10] The classical solution of section 2 is also the solution of the Nambu action as the induced metric is conformally flat. We may define the static gauge by the condition that τ and σ are such that t and ρ have their classical values, i.e. do not fluctuate.

[11] For example, in the case of string theory in flat target space the static-gauge Nambu and conformal-gauge Polyakov 1-loop partition functions (with nontrivial boundary conditions on a disc) are indeed the same [20].

[12] Here we assume that one of the massless decoupled modes is time-like, like time mode in $\mathcal{O}_{t\rho\phi}$.

static gauge one apparently contains an extra divergent term. Indeed, observing that the 1-loop logarithmic UV divergence of $\ln Z$ is given by the sum of mass-squared terms and that $\bar\mu_\phi^2$ in (3.14) may be written in terms of the curvature of the induced metric (2.8) as $\bar\mu_\phi^2 = \sqrt{-g}(4 + R^{(2)})$, one finds that this extra divergence is proportional to $\int d\tau d\sigma\ \sqrt{-g}R^{(2)}$. This is proportional to the Euler number of the world surface, so one may suggest [3] that it may be cancelled by the contribution of some extra "topological" factor representing the ratio of measures in the Polyakov and Nambu path integrals.

However, this extra divergence actually vanishes in the case of the cylindrical world sheet appropriate for computing the correction to the energy of a closed string state: since $\sqrt{-g}R^{(2)}$ is a total divergence, as long as the induced metric and thus its curvature are defined to be periodic in σ, the integral over σ should vanish.[13]

As we shall explicitly show in Section 5 below, the effective action in the static gauge given by the sum of (3.15) and (3.11) is indeed UV finite. Moreover, we shall also verify the relation (3.17), i.e. demonstrate the equivalence of the conformal gauge and static gauge results for the finite 1-loop correction to folded string energy. In [3] this equivalence was seen only in the long-string (infinite-spin) limit when the solution (2.2) approaches the following asymptotic solution [19] ($\omega \to \kappa \gg 1$)

$$t = \kappa\tau\ , \qquad \phi = \kappa\tau\ , \qquad \rho = \kappa\sigma\ , \qquad \kappa = \frac{1}{\pi}\ln \mathcal{S} \gg 1\ , \tag{3.18}$$

for which $\rho' = \kappa$=const, $R^{(2)} = 0$ and the relation (3.17) can be easily checked.

Proving (3.17) analytically for any κ by direct approach appears to be non-trivial. One indirect way to demonstrate (3.17) is to notice that since the corresponding quadratic fluctuation operators appear in the linearized (near folded string solution) form of the string equations of motion in the two gauges, one may be able to relate these operators by relating the two sets of equations.

The conformal-gauge equations for small fluctuations following from (3.5)

$$(\partial_\tau^2 - \partial_\sigma^2)\,\tilde t + \mu_t^2\,\tilde t + 2\,\kappa\ \sinh\rho\,\partial_\tau\tilde\rho = 0 \tag{3.19}$$

$$(\partial_\tau^2 - \partial_\sigma^2)\,\tilde\rho + \mu_\rho^2\,\tilde\rho + 2\,(\kappa\ \sinh\rho\,\partial_\tau\tilde t - \omega\ \cosh\rho\,\partial_\tau\tilde\phi) = 0 \tag{3.20}$$

$$(\partial_\tau^2 - \partial_\sigma^2)\,\tilde\phi + \mu_\phi^2\,\tilde\phi + 2\,\omega\ \cosh\rho\,\partial_\tau\tilde\rho = 0\ , \tag{3.21}$$

[13]More explicitly, here the relevant integral is $\int_0^{2\pi} d\sigma(\ln\rho')'' = [(\ln\rho')']_0^{2\pi} = [\frac{\rho''}{\rho'}]_0^{\pi/2} + [\frac{\rho''}{\rho'}]_{\pi/2}^{\pi} + [\frac{\rho''}{\rho'}]_{\pi}^{3/2\pi} + [\frac{\rho''}{\rho'}]_{3/2\pi}^{2\pi}$, where $\rho'' = (\kappa^2 - \omega^2)\sinh\rho\cosh\rho$, $\rho' = \pm\sqrt{\kappa^2\cosh^2\rho - \omega^2\sinh^2\rho}$. While there is an apparent singularity at the turning points were $\rho' = 0$, this integral should vanish. One may consider using a suitable regularization of the turning points to make this vanishing manifest.

should be supplemented with the conformal gauge conditions (Virasoro constraints)

$$-\kappa\,\cosh^2\rho\,\partial_\tau\,\tilde t+(\omega^2-\kappa^2)\,\sinh\rho\,\cosh\rho\,\tilde\rho+\rho'\,\partial_\sigma\,\tilde\rho+\omega\,\sinh^2\rho\,\partial_\tau\,\tilde\phi=0 \qquad (3.22)$$

$$-\kappa\,\cosh^2\rho\,\partial_\sigma\,\tilde t+\omega\,\sinh^2\rho\,\partial_\sigma\tilde\phi+\rho'\,\partial_\tau\,\tilde\rho=0\ . \qquad (3.23)$$

The latter should allow one, in principle, to eliminate the two modes (say $\tilde t$ and $\tilde\phi$) in terms of the third one ($\tilde\rho$), getting an effective equation for the latter. Since the ρ-background does not depend on τ and since the above equations are linear we may do this elimination at the Fourier mode level, i.e. replacing $\tilde t\to e^{i\,\Omega\,\tau}\bar t(\sigma)$, $\tilde\phi\to e^{i\,\Omega\,\tau}\bar\phi(\sigma)$, $\tilde\rho\to e^{i\,\Omega\,\tau}\bar\rho(\sigma)$. Then (3.22),(3.23) imply (changing to euclidean time notation, i.e. $\Omega\to i\Omega$)

$$\bar t \;=\; \frac{\sinh\rho}{2\,\kappa\,\Omega}\,\Big(\partial_\sigma^2-2\rho'\,\partial_\sigma+\kappa^2-\omega^2-\Omega^2\Big)\bar\rho\ , \qquad (3.24)$$

$$\bar\phi \;=\; -\frac{\cosh\rho}{2\,\omega\,\Omega}\,\Big(\partial_\sigma^2+2\rho'\,\partial_\sigma-\kappa^2+\omega^2-\Omega^2\Big)\bar\rho\ . \qquad (3.25)$$

Substituting this into the equations of motion (3.19)-(3.21) we find that one of them is satisfied automatically while the other two become equivalent to the following fourth-order differential equation for $\tilde\rho$, i.e. $\mathcal{O}^{(4)}\bar\rho=0$, where

$$\mathcal{O}^{(4)}\equiv\partial_\sigma^4+2(\omega^2+\kappa^2-\Omega^2-4\rho'^2)\,\partial_\sigma^2-8\rho'\rho''\,\partial_\sigma+\kappa^4+(\Omega^2+\omega^2)^2+2\kappa^2(\Omega^2-\omega^2)\ . \qquad (3.26)$$

Remarkably, this operator can be factorized as a product of two second-order operators as follows:

$$\mathcal{O}^{(4)}=\mathcal{O}_1\cdot\mathcal{O}_2\ , \qquad \mathcal{O}_1=(\rho')^{-1}\ \mathcal{O}_\phi\ \rho'\ , \qquad \mathcal{O}_2=\rho'\ \mathcal{O}_0\ (\rho')^{-1}\ , \qquad (3.27)$$

where $\mathcal{O}_\phi$ and $\mathcal{O}_0$ are the same as the static-gauge operator in (3.16) and the massless mode operator in (3.9), respectively. The algebraic ρ' and $(\rho')^{-1}$ factors may be attributed to a change of normalization of the corresponding fluctuations.[14] This way we see how the static gauge operator $\mathcal{O}_\phi$ emerges from the mixed conformal gauge fluctuation operator, i.e. provides support for the relation (3.17).

In Section 5 below we shall verify that the effective action in the static gauge is indeed UV finite, and we demonstrate the equivalence (3.17) between the conformal gauge and the static gauge results for the finite 1-loop correction to the folded string energy by computing the corresponding functional determinants.

[14] A similar discussion could be given at the level of path integral with the conformal gauge condition accounted for by two delta-functions (as appropriate if one starts with the Nambu path integral). The step analogous to (3.24),(3.25) would then produce an extra det$\mathcal{O}_0$ factor as required for balance of degrees of freedom. For a discussion of the equivalence of conformal gauge and static gauge partition functions in a simpler case of a homogeneous string solution see also [23].

3.3 Lamé form of the second-order fluctuation operators

To summarize, the simplest starting point for computing the 1-loop correction to the folded string energy is thus its representation in terms of the 1-loop effective action in the static gauge given by the sum of (3.15) and (3.11), i.e. is expressed in terms of determinants of the following three types of operators defined on periodic functions $f = (\beta, \phi, \psi_\pm)$

$$\mathcal{O}_f = -\partial_\sigma^2 + V_f(\sigma) + \Omega^2 \ , \qquad\qquad f(\sigma) = f(\sigma + 2\pi) \ , \tag{3.28}$$

where (using the form of the classical solution (2.4))

$$V_\beta = 2\rho'^2 \ = 2\kappa^2 \,\mathrm{sn}^2(\bar\sigma \,|\, k^2) \ , \qquad \bar\sigma \equiv \omega\,\sigma + \mathbb{K} \ , \tag{3.29}$$

$$V_\phi = 2\rho'^2 + \frac{2\kappa^2\,\omega^2}{\rho'^2} \ = 2\kappa^2 \,\mathrm{sn}^2(\bar\sigma \,|\, k^2) + 2\,\omega^2 \,\mathrm{ns}^2(\bar\sigma \,|\, k^2) \ , \tag{3.30}$$

$$V_{\psi_\pm} = \rho'^2 \pm \rho'' \ = \kappa^2 \,\mathrm{sn}^2(\bar\sigma \,|\, k^2) \pm \kappa\,\omega \,\mathrm{cn}(\bar\sigma \,|\, k^2)\,\mathrm{dn}(\bar\sigma \,|\, k^2) \ . \tag{3.31}$$

These potentials are plotted in Figures 2, 3, 4 where we have chosen four particular values of the elliptic modulus $k = \frac{\kappa}{\omega}$=0.5, 0.9, 0.99, 0.999.

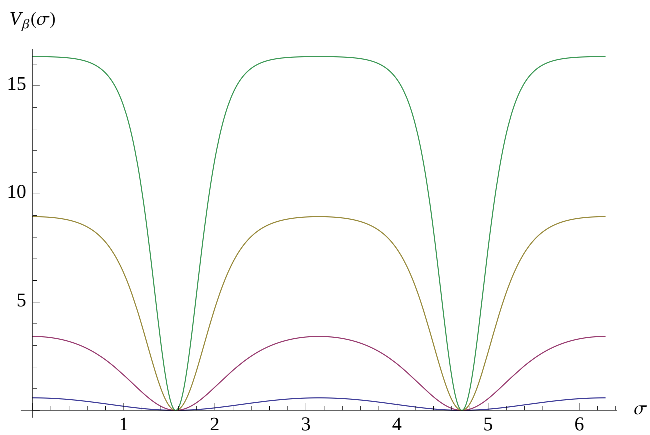

Figure 2: Potential V_β in (3.29), for $k = 0.5, 0.9, 0.99$ and 0.999, from bottom to top.

It is convenient to introduce the rescaled spatial variable (cf. (2.7))

$$x \equiv \omega\,\sigma = \frac{2\,\mathbb{K}}{\pi}\,\sigma \ , \tag{3.32}$$

and write $\mathcal{O}_\beta$ as (we ignore a trivial overall constant factor)

$$\mathcal{O}_\beta = -\partial_x^2 + 2k^2 \,\mathrm{sn}^2(x + \mathbb{K} \,|\, k^2) + \frac{\pi^2\,\Omega^2}{4\,\mathbb{K}^2} \ , \tag{3.33}$$

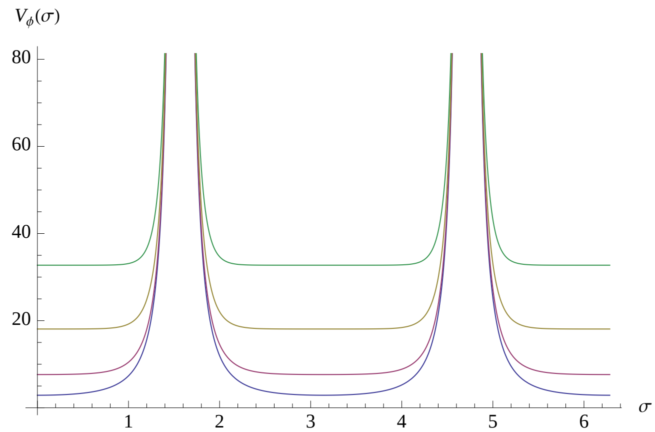

Figure 3: Potential V_ϕ in (3.30), for $k = 0.5, 0.9, 0.99$ and 0.999, from bottom to top. Note that singularities appear at the turning points where $\sigma = \left(n + \frac{1}{2}\right)\pi$.

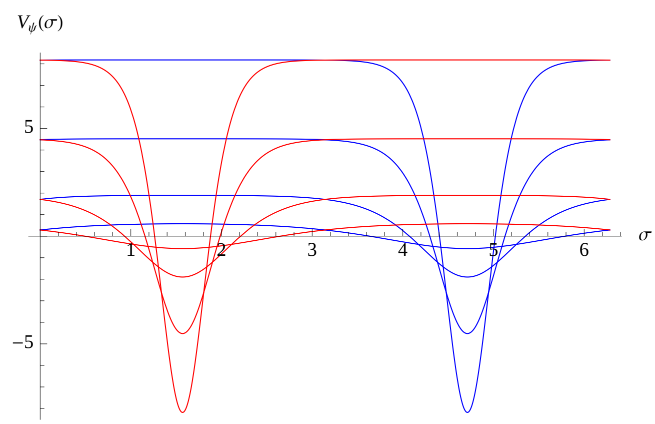

Figure 4: Potentials V_{ψ_+} (red) and V_{ψ_-} (blue) in (3.31) , for $k = 0.5, 0.9, 0.99$ and 0.999, from bottom to top. Note that $V_\pm(\sigma)$ are identical in from, but are displaced from one another by a half-period π; they are *self-isospectral* [27].

which is now defined on the periodic functions $\beta(x) = \beta(x + 4\mathbb{K})$. The expression in (3.33) is recognized as being a *Lamé* differential operator in the *single-gap* form (which will be reviewed in the next section).

Remarkably, all other fluctuation operators entering the effective action, i.e. $\mathcal{O}_\phi$ and $\mathcal{O}_{\psi_\pm}$, whose structure is apparently much more involved, can also be cast into the single-gap Lamé form. Their transformation has several steps involving rescaling the coordinate and the elliptic modulus a special way, see (A.18)-(A.21) for $\mathcal{O}_\phi$, and (A.22)-(A.31) for $\mathcal{O}_{\psi_\pm}$.

We can summarize the results as follows: each (static gauge) fluctuation operator is a *single-gap Lamé operator* with the following periodic eigenvalue problem

$$\Big[-\partial_x^2 + 2\,\bar{k}^2\,\mathrm{sn}^2(x\,|\,\bar{k}^2) + \bar{\Omega}^2 \Big]\, f_\Lambda(x) = \Lambda\, f_\Lambda(x)\ , \qquad f_\Lambda(x) = f_\Lambda(x+L), \tag{3.34}$$

where x is a rescaled σ variable with period L and $\bar{k}$ and $\bar{\Omega}$ are rescaled modulus and euclidean frequency in (3.29)-(3.31), namely,

(a) for the bosonic operator $\mathcal{O}_\beta$:

$$x = \frac{2\,\mathbb{K}}{\pi}\,\sigma + \mathbb{K}, \qquad \bar{k} = k\ , \qquad \bar{\Omega}^2 = \Big(\frac{\pi\,\Omega}{2\,\mathbb{K}}\Big)^2, \qquad L = 4\mathbb{K} \tag{3.35}$$

(b) for the bosonic operator $\mathcal{O}_\phi$: the elliptic modulus is $\tilde{k}^2 = \frac{4k}{(1+k)^2}$ and

$$x = \frac{2\,\widetilde{\mathbb{K}}}{\pi}\,\sigma + i\,\widetilde{\mathbb{K}}'\,, \qquad \bar{k} = \tilde{k} \equiv \frac{2\sqrt{k}}{1+k}\ , \qquad \bar{\Omega}^2 = \Big(\frac{\pi\,\Omega}{2\,\widetilde{\mathbb{K}}}\Big)^2 + \tilde{k}^2, \qquad L = 4\widetilde{\mathbb{K}} \tag{3.36}$$

(c) for the fermionic operators $\mathcal{O}_{\psi_\pm}$:

$$x = \begin{cases} \frac{\widetilde{\mathbb{K}}}{\pi}\,\sigma + \frac{\widetilde{\mathbb{K}}}{2}, & \text{for } \psi_+ \\ \frac{\widetilde{\mathbb{K}}}{\pi}\,\sigma + \frac{3\,\widetilde{\mathbb{K}}}{2}, & \text{for } \psi_- \end{cases}\ , \quad \bar{k} = \tilde{k} \equiv \frac{2\sqrt{k}}{1+k}\ , \quad \bar{\Omega}^2 = \Big(\frac{\pi\,\Omega}{\widetilde{\mathbb{K}}}\Big)^2 + \tilde{k}^2, \quad L = 2\widetilde{\mathbb{K}} \tag{3.37}$$

Here $\widetilde{\mathbb{K}} \equiv \mathbb{K}(\tilde{k}^2)$, and $k' \equiv \sqrt{1-k^2}$, see Appendix A for notation and details.

As we shall discuss in the next section, the remarkable feature of this Lamé spectral problem (3.34) is that it can be solved *exactly*, and hence the corresponding determinant can be computed *analytically* (with a result that is independent of constant shifts in the coordinates).

4 Determinants of single-gap Lamé operators

Below we shall first review the method that allows one to compute the determinant of a single-gap Lamé operator without having to solve the corresponding spectral problem explicitly. We will then apply this technique to the computation of determinants of the fluctuation operators discussed in the previous section.

4.1 Floquet theory of determinants of 2-nd order one-dimensional operators

Consider the following eigenvalue problem for an ordinary differential operator, $\mathcal{O} = -\partial_x^2 + V(x)$, with a periodic potential

$$\big[-\partial_x^2 + V(x)\big]\, f(x) = \Lambda\, f(x)\ , \qquad V(x+L) = V(x)\ . \tag{4.1}$$

For either periodic or antiperiodic boundary conditions on $f(x)$, we find a discrete spectrum of eigenvalues $\{\Lambda_n\}$, and the associated determinant is then formally given by $\mathrm{Det}\mathcal{O} = \prod_n \Lambda_n$.

Given a general potential $V(x)$ it is of course difficult to find the eigenvalues, and even given the eigenvalues, the infinite product must be regulated. Both difficulties can be overcome in the following way. Consider two independent solutions $f_{1,2}(x;\Lambda)$ to (4.1) satisfying the conditions

$$\begin{aligned} f_1(0;\Lambda) &= 1\ , \qquad & f_1'(0;\Lambda) &= 0\ , \\ f_2(0;\Lambda) &= 0\ , \qquad & f_2'(0;\Lambda) &= 1\ , \end{aligned} \tag{4.2}$$

where $f' = \partial_x f$. Then the *discriminant* $\Delta(\Lambda)$ of the operator $\mathcal{O}$ is defined as [34]

$$\Delta(\Lambda) = f_1(L;\Lambda) + f_2'(L;\Lambda) \tag{4.3}$$

The periodic and the antiperiodic eigenvalues are given by the following (in general transcendental) equations:

$$\Delta(\Lambda) = \begin{cases} +2 & \text{(periodic)} \\ -2 & \text{(antiperiodic)} \end{cases} \tag{4.4}$$

Remarkably, the determinant can be computed without knowing these eigenvalues explicitly. Indeed, the Hill determinant, i.e. the ratio of determinants with non-zero V and $V = 0$ has a simple expression in terms of the discriminant [34]:

$$\frac{\det[-\partial_x^2 + V(x) - \Lambda]}{\det[-\partial_x^2 - \Lambda]} = \frac{\Delta(\Lambda) - 2}{-4\,\sin^2(L\sqrt{\Lambda}/2)} \qquad \text{(periodic)} \tag{4.5}$$

$$\frac{\det[-\partial_x^2 + V(x) - \Lambda]}{\det[-\partial_x^2 - \Lambda]} = \frac{\Delta(\Lambda) + 2}{4\,\cos^2(L\sqrt{\Lambda}/2)} \qquad \text{(antiperiodic)} \tag{4.6}$$

In what follows we shall always assume that determinants we consider are normalized to the trivial free determinant $\det[-\partial_x^2]$, and thus omit the resulting $\Lambda-$ and V-independent overall constant (such constants will cancel in the string partition function due to balance of the degrees of freedom). Then we may write the above relations simply as

$$\det{}_{P,AP}[-\partial_x^2 + V(x) - \Lambda] = \begin{cases} \Delta(\Lambda) - 2 & \text{(periodic)} \\ \Delta(\Lambda) + 2 & \text{(antiperiodic)} \end{cases} \tag{4.7}$$

It is useful to relate this representation for the determinant to a familiar physical notion of "quasi-momentum". By the Floquet/Bloch theory [34], the equation (4.1) has two independent solutions of the form $f_\pm(x) = e^{\pm i\, p(\Lambda)\, x}\, \chi_\pm(x)$, where $\chi_\pm(x)$ are periodic, so that under translation through one period the Bloch solutions $f_\pm(x)$ change by a phase

$$f_\pm(x+L) = e^{\pm i p(\Lambda)\, L}\;\; f_\pm(x) \tag{4.8}$$

where, by definition, $p(\Lambda)$ is the "quasi-momentum". Then $\Delta(\Lambda) = 2\cos(L\, p(\Lambda))$, and we can re-write (4.7) in terms of the quasi-momentum as follows [34]:

$$\det{}_{P,AP}[-\partial_x^2 + V(x) - \Lambda] = \begin{cases} -4\sin^2\left(\frac{L}{2}\, p(\Lambda)\right) & \text{(periodic)} \\ +4\cos^2\left(\frac{L}{2}\, p(\Lambda)\right) & \text{(antiperiodic)} \end{cases} \tag{4.9}$$

Thus, knowing the quasi-momentum $p(\Lambda)$ amounts to knowing the discriminant and also the determinant.

Another interesting and useful relation is the link between the determinant and the discriminant through the contour integral representation for the spectral zeta function. For definiteness, let us consider the case of the periodic boundary conditions. Then the spectral zeta function is

$$\zeta(s) = \frac{1}{2\pi i}\int_\gamma \mathrm{d}\Lambda\, \Lambda^{-s}\, \frac{\partial}{\partial \Lambda} \ln\left[\Delta(\Lambda) - 2\right] = \frac{1}{2\pi i}\int_\gamma \mathrm{d}\Lambda\, \Lambda^{-s}\;\, R(\Lambda), \tag{4.10}$$

where the resolvent

$$R(\Lambda) = \frac{\Delta'(\Lambda)}{\Delta(\Lambda) - 2} \tag{4.11}$$

has simple poles exactly at the values of Λ corresponding to the points of the periodic spectrum. The contour γ in (4.10) runs counter-clockwise above and below the positive real axis enclosing all poles of the resolvent. Wrapping the contour along the branch cut along the negative real line gives [25]

$$\zeta(s) = -\frac{\sin(\pi s)}{\pi}\int_0^\infty \mathrm{d}\Lambda\, \Lambda^{-s}\;\, R(-\Lambda). \tag{4.12}$$

According to the zeta function definition of the functional determinant

$$\det\left[-\partial_x^2 + V(x)\right] = e^{-\zeta'(0)} \tag{4.13}$$

to compute the determinant we need to know

$$-\zeta'(0) = -\int_0^\infty \mathrm{d}\Lambda \frac{\partial}{\partial\Lambda} \ln\left[\Delta(-\Lambda) - 2\right] = \ln\frac{[\Delta(0) - 2]}{[\Delta(-\infty) - 2]} \tag{4.14}$$

Here we subtracted the divergent term $\ln[\Delta(-\infty) - 2]$ by assuming that we again divide by a "free" reference determinant. Then finally

$$\det{}_P\left[-\partial_x^2 + V(x)\right] = \Delta(0) - 2 \tag{4.15}$$

Shifting the potential by a constant $-\Lambda$, we reproduce the representation in (4.7).

The important feature of the above expressions is that the determinants can be calculated in closed form without computing any of the eigenvalues. There is yet another way to compute the determinants, known as the Gel'fand-Yaglom method [30, 38], which for periodic systems reduces essentially to a numerical evaluation of the discriminant giving the determinants via (4.7). This method is described in Appendix B, where we also consider systems of coupled equations, which will be important for demonstrating explicitly the equivalence of the computation in the conformal and static gauges.

It is useful to illustrate the above general relations on the simple example of constant potential

$$V(x) = m^2, \qquad x \in (0, L) \ . \tag{4.16}$$

Then the two independent solutions in (4.2) are $f_1(x;\Lambda) = \cosh(\sqrt{m^2-\Lambda}\,x)$ and $f_2(x;\Lambda) = \sinh(\sqrt{m^2-\Lambda}\,x)/\sqrt{m^2-\Lambda}$. Therefore, the discriminant (4.3) and the determinants (4.9) are

$$\Delta(\Lambda) = 2\cosh(L\sqrt{m^2-\Lambda}) \ , \tag{4.17}$$

$$\det{}_{P,AP}(-\partial_x^2 + m^2 - \Lambda) = \begin{cases} 4\sinh^2\left(\frac{L}{2}\sqrt{m^2-\Lambda}\right) \\ 4\cosh^2\left(\frac{L}{2}\sqrt{m^2-\Lambda}\right) \end{cases} \tag{4.18}$$

The quasi-momentum in (4.8) here is $p(\Lambda) = \sqrt{\Lambda - m^2}$, so these relations are consistent with (4.9). Furthermore, in this case we know the explicit eigenvalues ($n \in \mathbb{Z}$):

$$\Lambda_n = \begin{cases} m^2 + \left(\frac{2n\pi}{L}\right)^2 & \text{(periodic)} \\ m^2 + \left(\frac{(2n+1)\pi}{L}\right)^2 & \text{(antiperiodic)} \end{cases} \tag{4.19}$$

Then the expressions in (4.18) also follow from the infinite product representations for the sinh and cosh functions, combined with zeta function regularization.

4.2 Case of single-gap Lamé potential $V(x) = 2k^2\,\mathrm{sn}^2(x\,|\,k^2)$

The important example which is our main interest here is provided by the single-gap Lamé operator in (3.34), i.e.

$$\Big[-\partial_x^2 + \ 2k^2\,\mathrm{sn}^2(x\,|\,k^2)\Big]\, f(x) = \Lambda\, f(x). \tag{4.20}$$

The two independent Bloch solutions of (4.20) here are [33]

$$f_\pm(x) = \frac{H(x\pm\alpha)}{\Theta(x)}\, e^{\mp x\, Z(\alpha)} \ , \tag{4.21}$$

where H, Θ, Z are the Jacobi Eta, Theta and Zeta functions defined in (A.10), and $\alpha = \alpha(\Lambda)$ is given implicitly by

$$\mathrm{sn}(\alpha \,|\, k^2) = \sqrt{\frac{1 + k^2 - \Lambda}{k^2}} \ . \tag{4.22}$$

Using the period properties of the Jacobi functions (A.12) we see that

$$f_\pm(x + 2\mathbb{K}) = -f_\pm(x)\, e^{\mp 2\,\mathbb{K}\, Z(\alpha)} \ \equiv \ f_\pm(x)\, e^{2i\,\mathbb{K}\, p(\alpha)} \ , \tag{4.23}$$

which defines the quasi-momentum as

$$p(\Lambda) = i\, Z(\alpha \,|\, k^2) + \frac{\pi}{2\,\mathbb{K}} \ . \tag{4.24}$$

Therefore, from (4.9) we immediately find analytic expressions for the determinants. Assuming the period is $L = 2\mathbb{K}$, we find

$$\det{}^{(L=2\mathbb{K})}_{P,AP} \Big[-\partial_x^2 + 2k^2\, \mathrm{sn}^2(x \,|\, k^2) - \Lambda \Big] = \begin{cases} -4 \cosh^2[\mathbb{K}\, Z(\alpha | k^2)] \\ -4 \sinh^2[\mathbb{K}\, Z(\alpha | k^2)] \end{cases} \tag{4.25}$$

where the relation between Λ and α is given by (4.22). On the other hand, for the period $L = 4\mathbb{K}$, we find

$$\det{}^{(L=4\mathbb{K})}_{P} \Big[-\partial_x^2 + 2k^2\, \mathrm{sn}^2(x \,|\, k^2) - \Lambda \Big] = 4 \sinh^2[2\,\mathbb{K}\, Z(\alpha | k^2)] \tag{4.26}$$

Note that this is the same as the product of the periodic and antiperiodic determinants (4.25) with the period $2\mathbb{K}$, as it should be.

The periodic potential in (4.20) has the special property that its band spectrum has only a single gap, and is known therefore as a one-gap potential, as illustrated in Fig. 5. The spectrum has three band edges (which are also the lowest eigenvalues of the periodic spectrum of the problem on the interval $4\mathbb{K}$):

$$\Lambda_1 = k^2 \ , \qquad \Lambda_2 = 1 \ , \qquad \Lambda_3 = 1 + k^2 \ . \tag{4.27}$$

One can rewrite the relation between Λ and α in (4.22) in terms of the band edges as follows

$$k\, \mathrm{sn}(\alpha; k^2) = \sqrt{\frac{1}{2}\,(\Lambda_1 + \Lambda_2 + \Lambda_3) - \Lambda}. \tag{4.28}$$

The resolvent is

$$R(\Lambda) \ = \ \frac{d}{d\Lambda}\, \ln\left[\Delta(\Lambda) - 2\right] = L\, \frac{dp}{d\Lambda}\, \cot\Big[\frac{L}{2}\, p(\Lambda)\Big] \ . \tag{4.29}$$

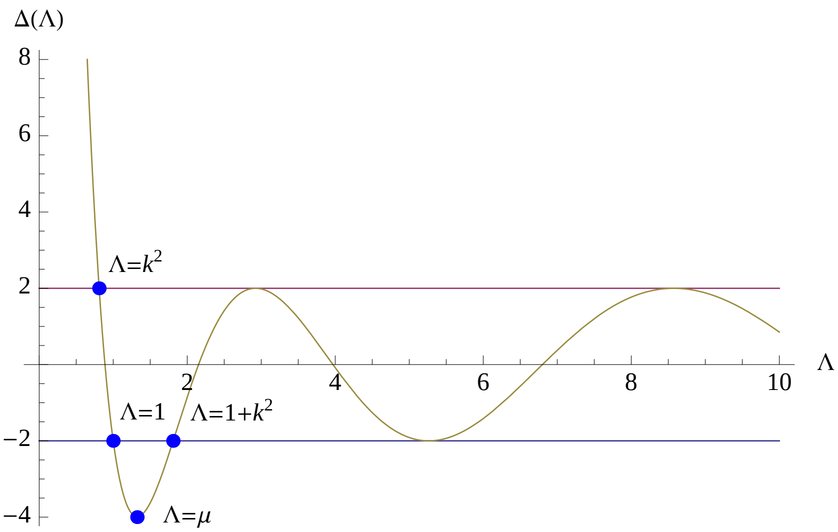

Figure 5: The discriminant $\Delta(\Lambda)$ for the Lamé potential $V(x) = 2\,k^2\,\mathrm{sn}^2(x\,|\,k^2)$ with $k = 0.9$. The three band edges occur at the points $\Lambda_1, \Lambda_2, \Lambda_3$ where $\Delta(\Lambda)$ cuts the lines ± 2, while the remainder of the periodic/antiperiodic spectrum consists of points where $\Delta(\Lambda)$ touches the lines ± 2.

We can also express $dp/d\Lambda$ simply in terms of the band edges:

$$\frac{dp}{d\Lambda} = i\,\frac{\Lambda - \mu}{2\sqrt{(\Lambda_1 - \Lambda)(\Lambda_2 - \Lambda)(\Lambda_3 - \Lambda)}} \tag{4.30}$$

where

$$\mu = \frac{1}{2}\Big(\Lambda_1 + \Lambda_2 + \Lambda_3 - \langle V\rangle\Big)\ , \qquad \langle V\rangle \equiv \frac{1}{L}\int_0^L V(x)\,dx \tag{4.31}$$

To see this, note that from (4.8)

$$\frac{dp}{d\Lambda} = \frac{dp}{d\alpha}\frac{d\alpha}{d\Lambda} = i\,\frac{dZ(\alpha\,|\,k^2)}{d\alpha}\,\frac{d\alpha}{d\Lambda} = i\left(1 - k^2\mathrm{sn}^2(\alpha\,|\,k^2) - \frac{\mathbb{E}(k^2)}{\mathbb{K}(k^2)}\right)\frac{d\alpha}{d\Lambda}$$

where we have used the definition (A.16) of the Zeta function. Also, from (4.28) we have

$$\frac{d\alpha}{d\Lambda} = \frac{1}{2k^2\,\mathrm{dn}(\alpha\,|\,k^2)\,\mathrm{cn}(\alpha\,|\,k^2)\,\mathrm{sn}(\alpha\,|\,k^2)} = \frac{1}{2\sqrt{(\Lambda_1 - \Lambda)(\Lambda_2 - \Lambda)(\Lambda_3 - \Lambda)}}.$$

Finally, for the potential $V(x) = 2k^2\,\mathrm{sn}^2(x\,|\,k^2)$ we find for $L = 2\mathbb{K}$

$$\langle V\rangle = \frac{1}{L}\int_0^L dx\,2k^2\,\mathrm{sn}^2(x\,|\,k^2) = 2\Big(1 - \frac{\mathbb{E}(k^2)}{\mathbb{K}(k^2)}\Big)\ , \tag{4.32}$$

and the same for $L = 4\mathbb{K}$. Thus, taking into account (4.27), we get

$$\mu = k^2 + \frac{\mathbb{E}(k^2)}{\mathbb{K}(k^2)}\ . \tag{4.33}$$

	β	ϕ	fermions
$\bar\Lambda_1 = \Omega_1^2 + \Omega^2$	$\kappa^2 + \Omega^2$	Ω^2	Ω^2
$\bar\Lambda_2 = \Omega_2^2 + \Omega^2$	$\omega^2 + \Omega^2$	$(\omega-\kappa)^2 + \Omega^2$	$\frac{1}{4}(\omega-\kappa)^2 + \Omega^2$
$\bar\Lambda_3 = \Omega_3^2 + \Omega^2$	$\omega^2 + \kappa^2 + \Omega^2$	$(\omega+\kappa)^2 + \Omega^2$	$\frac{1}{4}(\omega+\kappa)^2 + \Omega^2$
$\bar\mu$	$\kappa^2 + \omega^2 \frac{\mathbb{E}}{\mathbb{K}}$	$-(\omega^2-\kappa^2) + 2\omega^2 \frac{\mathbb{E}}{\mathbb{K}}$	$-\frac{1}{4}(\omega^2-\kappa^2) + \frac{\omega^2}{2} \frac{\mathbb{E}}{\mathbb{K}}$

Table 1: The lowest (analytically known) eigenvalues of the fluctuation operators

Thus we obtain a compact expression for the resolvent of the single gap Lamé potential with period L in terms of the quasi-momentum $p(\Lambda)$ and the band edges as

$$R(\Lambda) = \frac{L}{2} \frac{\Lambda - \mu}{\sqrt{(\Lambda_1 - \Lambda)(\Lambda_2 - \Lambda)(\Lambda_3 - \Lambda)}} \coth\left(\frac{L\, p(\Lambda)}{2i}\right) . \tag{4.34}$$

4.3 Results for determinants of static-gauge fluctuation operators

Let us now apply the above results to the case of the fluctuation operators defined by (3.34)-(3.37). The results in Eqs. (4.22) and (4.25)-(4.26) are actually all that we need in order to write down *exact* analytic expressions for the determinants of these operators. The analytically known eigenvalues or band edges can be obtained from (4.27) with the appropriate shifts ($\Lambda_i \to \Lambda_i - \bar\Omega^2$) and rescalings, and an analogous procedure applies to the corresponding resolvents in (4.34).

The results can be summarized as follows.

(a) for the β *operator*, in view of (3.35), the determinant reads

$$\det\, \mathcal{O}_\beta(\Omega) = 4 \sinh^2 \left[2\,\mathbb{K}\, Z(\alpha_\beta \,|\, k^2) \right] \ , \tag{4.35}$$

$$\mathrm{sn}(\alpha_\beta; k^2) = \frac{\sqrt{1 + k^2 + (\frac{\pi\,\Omega}{2\mathbb{K}})^2}}{k} \ . \tag{4.36}$$

The band edges are obtained from (4.27) by shifting and rescaling

$$\bar\Lambda_i = \left(\frac{2\mathbb{K}}{\pi}\right)^2 (\Lambda_i + \bar\Omega^2) \equiv \Omega_i^2 + \Omega^2 \ , \qquad \bar\Omega^2 = \left(\frac{\pi\,\Omega}{2\,\mathbb{K}}\right)^2 \tag{4.37}$$

where the rescaled Λ_i have been defined as "characteristic frequencies" Ω_i^2. One thus gets the eigenvalues in the first column of Table 1 that can now be re-expressed in terms of the parameters of the classical solution

$$\begin{aligned}\{\bar\Lambda_1,\bar\Lambda_2,\bar\Lambda_3\} &= \Big(\frac{2\mathbb{K}}{\pi}\Big)^2\Big\{k^2+\bar\Omega^2, 1+\bar\Omega^2, 1+k^2+\bar\Omega^2\Big\}\\ &\equiv \Big\{\kappa^2+\Omega^2,\ \omega^2+\Omega^2,\ \kappa^2+\omega^2+\Omega^2\Big\}. \end{aligned}\tag{4.38}$$

(b) For the ϕ *operator* in (3.36) we have $\tilde k^2=\frac{4k}{(k+1)^2}$, and thus

$$\det\mathcal{O}_\phi(\Omega)=4\sinh^2\big[2\,\widetilde{\mathbb{K}}\,Z(\alpha_\phi\,|\,\tilde k^2)\big]\ ,\tag{4.39}$$

$$\mathrm{sn}(\alpha_\phi\,|\,\tilde k^2)=\frac{\sqrt{1+(\frac{\pi\,\Omega}{2\widetilde{\mathbb{K}}})^2}}{\tilde k}\ .\tag{4.40}$$

The band edges are obtained from (4.27) by shifting and rescaling

$$\bar\Lambda_i=\Big(\frac{2\widetilde{\mathbb{K}}}{\pi}\Big)^2(\Lambda_i+\bar\Omega^2)\equiv\Omega_i^2+\Omega^2\ ,\qquad\qquad \bar\Omega^2=\Big(\frac{\pi\,\Omega}{2\,\widetilde{\mathbb{K}}}\Big)^2+\tilde k^2\ ,\tag{4.41}$$

getting thus the eigenvalues in the second column of Table 1.

(c) For the $\psi_\pm$ *operators* in (3.37) the elliptic parameter is $\tilde k^2=\frac{4k}{(k+1)^2}$, and we get

$$\det\mathcal{O}_\psi(\Omega)=-4\cosh^2\big[\widetilde{\mathbb{K}}\,Z(\alpha_\psi\,|\,\tilde k^2)\big]\ ,\tag{4.42}$$

$$\mathrm{sn}(\alpha_\psi\,|\,\tilde k^2)=\frac{\sqrt{1+(\frac{\pi\,\Omega}{\widetilde{\mathbb{K}}})^2}}{\tilde k}\ .\tag{4.43}$$

Since the determinant is independent of constant shifts of coordinates like the one in (3.37) (see, e.g., Appendix B) the expressions for the determinants of $\mathcal{O}_{\psi_-}$ and $\mathcal{O}_{\psi_+}$ are the same and therefore we will not distinguish them in what follows. The band edges follow from (4.27)

$$\bar\Lambda_i=\Big(\frac{\widetilde{\mathbb{K}}}{\pi}\Big)^2(\Lambda_i+\bar\Omega^2)\equiv\Omega_i^2+\Omega^2,\qquad\qquad \bar\Omega^2=\Big(\frac{\pi\,\Omega}{\widetilde{\mathbb{K}}}\Big)^2+\tilde k^2\tag{4.44}$$

5 Exact expression for one-loop correction to string energy

As follows from the above discussion (see (3.2),(3.15),(3.11),(4.35),(4.39),(4.42)) the 1-loop correction to the energy of the folded spinning string may be written as

$$E_1=-\frac{1}{4\pi\kappa}\int_{-\infty}^{\infty}d\Omega\ \ln\frac{\det^8\mathcal{O}_\psi}{\det\mathcal{O}_\phi\ \det^2\mathcal{O}_\beta\ \det^5\mathcal{O}_0},\tag{5.1}$$

where $\kappa = \frac{2k}{\pi}\mathbb{K}$, $\mathbb{K} = \mathbb{K}(k^2)$ (see (2.7)) and the determinants as functions of Ω have the following explicit expressions[15]

$$\det\mathcal{O}_\beta = 4\sinh^2\left[2\,\mathbb{K}\,Z(\alpha_\beta\,|\,k^2)\right] \qquad \text{where} \qquad \mathrm{sn}(\alpha_\beta\,|\,k^2) = \frac{\sqrt{1+k^2+(\frac{\pi\,\Omega}{2\mathbb{K}})^2}}{k} \tag{5.2}$$

$$\det\mathcal{O}_\phi = 4\sinh^2\left[2\,\widetilde{\mathbb{K}}\,Z(\alpha_\phi\,|\,\tilde{k}^2)\right] \qquad \text{where} \qquad \mathrm{sn}(\alpha_\phi\,|\,\tilde{k}^2) = \frac{\sqrt{1+(\frac{\pi\,\Omega}{2\widetilde{\mathbb{K}}})^2}}{\tilde{k}} \tag{5.3}$$

$$\det\mathcal{O}_\psi = -4\cosh^2\left[\widetilde{\mathbb{K}}\,Z(\alpha_\psi\,|\,\tilde{k}^2)\right] \qquad \text{where} \qquad \mathrm{sn}(\alpha_\psi\,|\,\tilde{k}^2) = \frac{\sqrt{1+(\frac{\pi\,\Omega}{\widetilde{\mathbb{K}}})^2}}{\tilde{k}} \tag{5.4}$$

$$\det\mathcal{O}_0 = 4\sinh^2\left[\pi\,\Omega\right] \tag{5.5}$$

The computation of E_1 is thus reduced to inverting the transcendental equations for $\alpha_\beta, \alpha_\phi, \alpha_\phi$, finding the corresponding values of Z-function (A.15). The integral is then a function of $k = \frac{\kappa}{\omega}$ and Ω.[16] Doing the integral over Ω we then end up with a function of k only or the spin (2.11). It is straightforward to evaluate the Ω integral numerically, as discussed below.

5.1 UV finiteness

Let us first check that the resulting expression for E_1 is indeed UV finite, i.e. the integral over Ω is convergent at infinity. The large Ω behavior of the determinant factors in (5.1) can most easily be extracted from the general large Ω behavior of the associated resolvents. Changing variable from Λ to $-\Omega^2$, we define

$$\mathcal{R}(\Omega) \equiv -2\,\Omega\,R(-\Omega^2) \tag{5.6}$$

Then we find from (4.34) that the general structure of the expansion is

$$\mathcal{R}(\Omega) = r_0 + \frac{r_1}{\Omega^2} + \frac{r_2}{\Omega^4} + \mathcal{O}(\Omega^{-6})\ , \qquad \Omega \to \infty \tag{5.7}$$

$$r_0 = 2\pi\ , \qquad r_1 = 2\pi\left[\bar{\mu} - \frac{1}{2}(\bar{\Lambda}_1 + \bar{\Lambda}_2 + \bar{\Lambda}_3)\right] = 2\pi\,\langle V\rangle \tag{5.8}$$

Therefore, the large Ω behavior of the log determinant is

$$\ln\det\mathcal{O} = r_0\,\Omega - \frac{r_1}{\Omega} + \mathcal{O}(\Omega^{-3}), \qquad \Omega \to \infty \tag{5.9}$$

[15]The determinant of the massless operator $\mathcal{O}_0$ is found by taking the regularized infinite product of its eigenvalues $\lambda_n = n^2 + \Omega^2$.

[16]Note that in the limit when $k = 0$, i.e. potentials vanish all determinants take the same value as $\det\mathcal{O}_0$, i.e. E_1 in (5.1) vanishes.

Using the corresponding values of $\bar{\mu}$ and $\mathcal{L}_i$ of the three non trivial fluctuation modes given in Table 1, we find [we have also used the elliptic identities (A.28)–(A.29)]:

$$\begin{aligned}
\ln \det\mathcal{O}_\beta &= 2\pi\,\Omega + 4\,\omega\,(\mathbb{K}-\mathbb{E})\,\Omega^{-1} + \mathcal{O}(\Omega^{-3}), &(5.10)\\
\ln \det\mathcal{O}_\phi &= 2\pi\,\Omega + 8\,\omega\,(\mathbb{K}-\mathbb{E})\,\Omega^{-1} + \mathcal{O}(\Omega^{-3}), &(5.11)\\
\ln \det\mathcal{O}_\psi &= 2\pi\,\Omega + 2\,\omega\,(\mathbb{K}-\mathbb{E})\,\Omega^{-1} + \mathcal{O}(\Omega^{-3})\ , &(5.12)\\
\ln \det\mathcal{O}_0 &= 2\pi\,\Omega + \mathcal{O}(\Omega^{-3})\ . &(5.13)
\end{aligned}$$

The leading (quadratically divergent) terms cancel in (5.1) due to the balance of world-sheet degrees of freedom in (5.1). The subleading (logarithmically divergent) terms also cancel in the combination appearing in (5.1), $(2\times 4 + 8 - 8\times 2)(\mathbb{K}-\mathbb{E})\,\Omega^{-1} = 0$. We thus confirm that the static gauge result for the 1-loop energy is indeed UV finite, as was argued in section 3.2.

5.2 Equivalence between the static gauge and conformal gauge results

To check the equivalence between the static gauge and conformal gauge results one needs to verify the factorization relation (3.17). This can be done numerically, as follows. To evaluate the left hand side of (3.17) we used the Gel'fand-Yaglom method (for details, see Appendix B) to compute numerically the determinant of the operator $\mathcal{O}_{t\rho\phi}$ in (3.7) as a function of Ω for various values of k. The right hand side of (3.17) can be computed directly using the expression for the determinant of $\mathcal{O}_\phi$ found above (5.11). We find perfect agreement. In Figure 6 we have plotted the expressions on both sides of (3.17) as functions of Ω for $k = \frac{1}{\sqrt{10}}$. Similar agreement is found for any k.

5.3 General form of the 1-loop correction E_1

Going back to the complete expression for E_1 in (5.1) it is useful, in order to safely expand in one of the interesting limits analyzed below, to separate there the contributions of the massless modes of $\mathcal{O}_0$ (i.e. Ω^2) and the lowest eigenvalues ($\bar{\Lambda}_1$ in Table 1) of $\mathcal{O}_\beta, \mathcal{O}_\phi$ and $\mathcal{O}_\psi$. Then we get

$$E_1 = -\frac{1}{4\pi\kappa}\int_{-\infty}^{\infty} d\Omega\,\left[\ln\frac{(\det'\mathcal{O}_\psi)^8}{\det'\mathcal{O}_\phi\ (\det'\mathcal{O}_\beta)^2\ (\det'\mathcal{O}_0)^5} + h(\Omega)\right] \tag{5.14}$$

where

$$\begin{aligned}
\det'\mathcal{O}_{\beta,\phi,\psi} &\equiv \frac{\det\mathcal{O}_{\beta,\phi,\psi}}{\bar{\Lambda}_1}\quad ,\quad \det'\mathcal{O}_0 \equiv \frac{\det\mathcal{O}_0}{\Omega^2} &(5.15)\\
h(\Omega) &= 2\ln(\Omega^2) - 2\ln(\Omega^2+\kappa^2)\ . &(5.16)
\end{aligned}$$

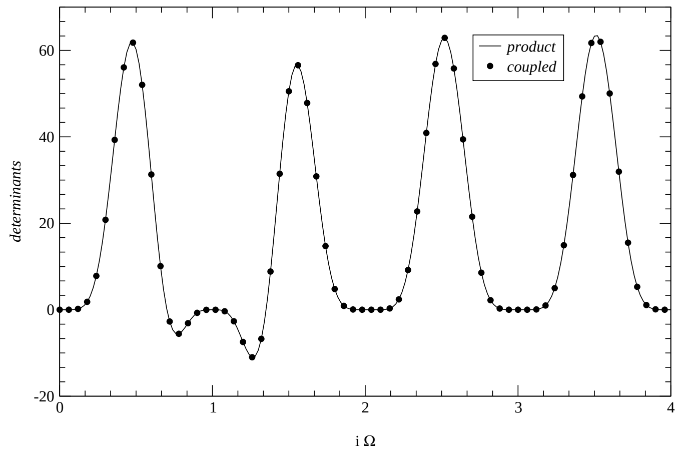

Figure 6: Comparison between the left-hand-side and the right-hand-side of Eq.(3.17), for $k = \frac{1}{\sqrt{10}}$. The circles represent the numerical Gel'fand-Yaglom result for the determinant $\det\mathcal{O}_{t\rho\phi}$ of the three coupled fluctuations in conformal gauge, while the solid line is a plot of the corresponding analytic static gauge expression, given by the product of the determinant (5.3) for the massive fluctuation ϕ and the square of the determinant (5.5) of a massless mode. To emphasize the precision of the agreement we have plotted the oscillatory form, as a function of $i\,\Omega$.

Using that

$$\int_{-\infty}^{\infty} d\Omega \; h(\Omega^2) = -4\,\pi\,\kappa \tag{5.17}$$

the one-loop correction to the energy (5.1) takes the form

$$E_1 = 1 - \frac{1}{4\pi\kappa}\int_{-\infty}^{\infty} \mathrm{d}\Omega \;\; \ln\frac{(\det{}'\mathcal{O}_\psi)^8}{\det{}'\mathcal{O}_\phi \; (\det{}'\mathcal{O}_\beta)^2 \; (\det{}'\mathcal{O}_0)^5} \; . \tag{5.18}$$

This expression is straightforward to evaluate numerically for various values of k or the spin $\mathcal{S}$ in (2.11), and thus to plot E_1.

To gain more analytic control over the form of E_1 as a function of spin $\mathcal{S}$ we may consider the expansion of it in the large spin ("long string" or $k \to 1$) limit or in the small spin ("short string" or $k \to 0$) limit. This will be done in detail in the following two sections 6 and 7 respectively. In figure 7 we presented together the results – the plots of $E_1(k)$ found analytically in the large spin expansion (right-most green curve) and in the small spin expansion (left-most red curve) and also the plot of the exact E_1 found numerically from (5.18) (blue curve connecting the two asymptotic ones). As one can see, already the few leading terms in the two respective analytic expansions give a very good approximation to the exact result.

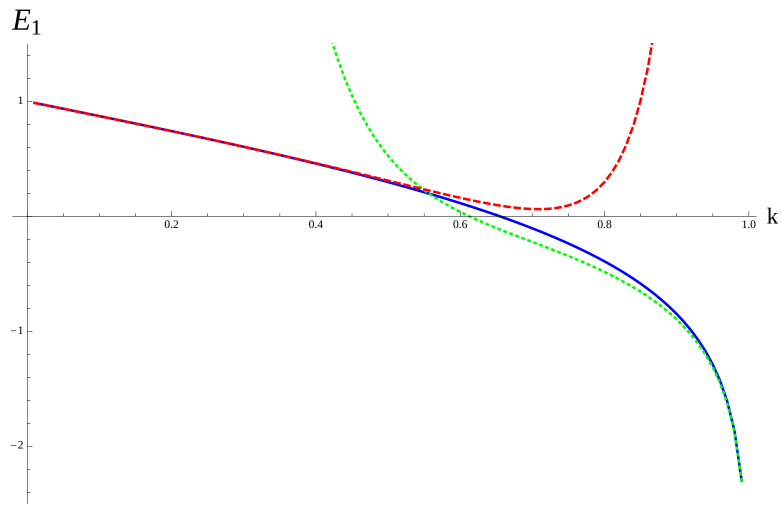

Figure 7: Plots of E_1 as a function of k: the blue, solid, curve is found numerically from the exact expression (5.18) for generic values of k; the green, dotted, curve is found from an analytic expansion in the $k \to 1$ or large spin limit, using the first two terms in (6.36); the red, dashed, curve is found from an analytic expansion in the $k \to 0$ or small spin limit, using the first two terms in (7.7). The agreement is excellent in both extreme limits.

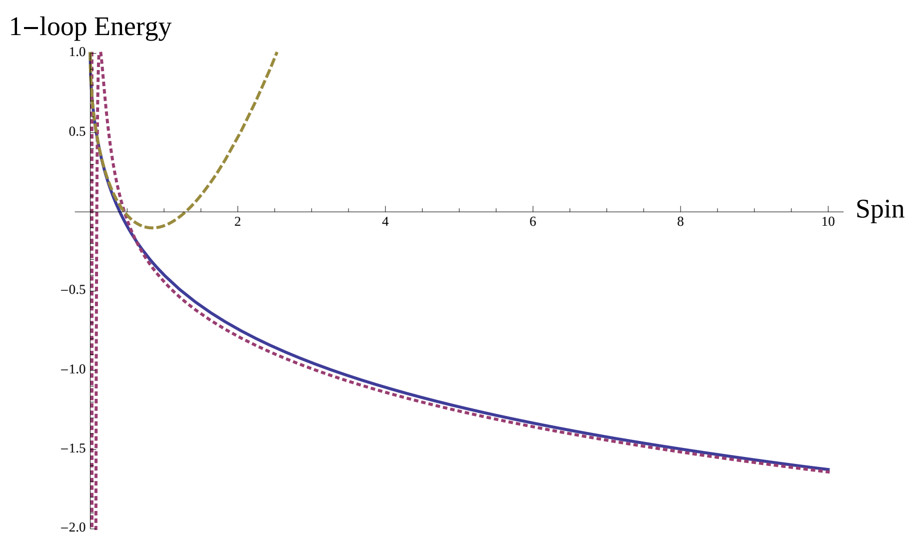

Figure 8: Plots of E_1 as a function of the classical spin $\mathcal{S}$. The solid, blue curve is the exact result, compared with the red (dotted) curve representing the large spin expansion, see (6.41), and with the gold (dashed) curve representing the small spin expansion found below in (7.11)-(7.12).

6 Large spin expansion

This limit (see (2.12)) is defined as $k \to 1$ or, equivalently, $\eta \to 0$ in (2.5).

6.1 Leading order

In this subsection we will compute the leading term in $k \to 1$ expansion of E_1 in (5.1) and also comment on first exponential subleading terms.

At the leading order the classical solution is approximated by

$$\rho' \approx \kappa_0 \ , \qquad \rho'' \approx 0 \ , \qquad \omega \approx \kappa \approx \kappa_0 \ , \qquad \kappa_0 = \frac{1}{\pi} \ln \frac{16}{\eta} \to \infty \ . \tag{6.1}$$

If we use these limiting expressions directly in the fluctuation operators then we conclude that their potential terms become constant

$$\mathcal{O}_{\beta,0} = -\partial_\sigma^2 + 2\kappa_0^2 + \Omega^2 \ , \qquad \mathcal{O}_{\phi,0} = -\partial_\sigma^2 + 4\kappa_0^2 + \Omega^2 \ , \qquad \mathcal{O}_{\psi_\pm,0} = -\partial_\sigma^2 + \kappa_0^2 + \Omega^2 \tag{6.2}$$

and thus we find from (5.1) [3]

$$E_1^{(0)} = \frac{1}{2\kappa_0} \sum_{n=-\infty}^{\infty} \Big[\sqrt{n^2 + 4\kappa_0^2} + 2\sqrt{n^2 + 2\kappa_0^2} + 5\sqrt{n^2} - 8\sqrt{n^2 + \kappa_0^2} \Big] \tag{6.3}$$

where we performed the integration over Ω before commuting the determinants defined on a unit circle. Using the Euler-MacLaurin formula to transform the sum into an integral one finds

$$E_1^{(0)} = \frac{1}{\kappa_0} \Big[-3\,\kappa_0^2 \ \ln 2 \ - \frac{5}{12} + \mathcal{O}(e^{-2\pi\kappa_0}) \Big] \ , \qquad \kappa_0 \to \infty \ , \tag{6.4}$$

where the leading term is the result of [3] and the subleading term appeared in [22].

Let us now see what we get if we start instead with the exact expressions for the determinants (5.2)-(5.4). Using the expressions collected in Appendix C (see (C.8)-(C.10)), we get at the leading order

$$2\mathbb{K}(k^2)\, Z(\alpha_\beta \,|\, k^2) \ \approx \ \pi\,\kappa_0\, x \qquad \text{with} \qquad x = \sqrt{2 + \frac{\Omega^2}{\kappa_0^2}} \ , \tag{6.5}$$

$$2\mathbb{K}(\tilde{k}^2)\, Z(\alpha_\phi \,|\, \tilde{k}^2) \ \approx \ \pi\,\kappa_0\, y \qquad \text{with} \qquad y = \sqrt{4 + \frac{\Omega^2}{\kappa_0^2}} \ , \tag{6.6}$$

$$\mathbb{K}(\tilde{k}^2)\, Z(\alpha_\psi \,|\, \tilde{k}^2) \ \approx \ \pi\,\kappa_0\, z \qquad \text{with} \qquad z = \sqrt{1 + \frac{\Omega^2}{\kappa_0^2}} \ , \tag{6.7}$$

which would give, once substituted into (5.2)-(5.4), the following expressions for the determinants

$$\det\mathcal{O}_\beta(\Omega^2) \ \approx \ 4\sinh^2[\pi\,\kappa_0\, x] \tag{6.8}$$

$$\det\mathcal{O}_\phi(\Omega^2) \ \approx \ 4\sinh^2[\pi\,\kappa_0\, y] \tag{6.9}$$

$$\det\mathcal{O}_\psi(\Omega^2) \ \approx \ -4\cosh^2[\pi\,\kappa_0\, z] \tag{6.10}$$

Integrating logarithms of (6.8)-(6.10) over Ω one gets the result that may be represented also as

$$E_1 \approx \tilde{E}_1^{(0)} = \frac{1}{2\kappa_0} \sum_{n=-\infty}^{\infty} \left[\sqrt{n^2 + 4\kappa_0^2} + 2\sqrt{n^2 + 2\kappa_0^2} + 5\sqrt{n^2} - 8\sqrt{(n+\tfrac{1}{2})^2 + \kappa_0^2} \right] . \tag{6.11}$$

Here the shift $n \to n + \frac{1}{2}$ in the fermionic contribution is due to the $\cosh^2$ instead of $\sinh^2$ form of the determinant (6.10).[17] While this shift does not affect the result for the two leading terms in (6.4), it formally changes the form of the subleading corrections (which should not, however, be trusted in the approximation used to arrive at (6.11)).

Indeed, there is of course no contradiction as the approximation used to derive (6.4) was supposed to be valid only for the leading term in large κ_0 expansion, i.e. the expressions for the subleading terms should not be trusted a priori. Still, let us briefly comment on the exponential corrections to the first two leading terms in (6.4) comparing what follows from (6.3) to what follows from (6.11). As was found in [29] using ζ-function regularization of the sums in (6.3)

$$E_1^{(0)} = -3\kappa_0 \ln 2 - \frac{5}{12\kappa_0} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \left[K_1(4\pi n\kappa_0) + \sqrt{2}K_1(2\sqrt{2}\pi n\kappa_0) - 4K_1(2\pi n\kappa_0) \right], \tag{6.12}$$

where K_1 is the Bessel function of the second type

$$\int_m^\infty \mathrm{d}x \sqrt{x^2 - m^2} e^{-2\pi k x} = \frac{m}{2\pi k} K_1(2\pi k m). \tag{6.13}$$

The K_1 terms represent the exponential corrections since

$$K_1(y) \to \sqrt{\frac{\pi}{2y}} e^{-y} \left[1 + \mathcal{O}(y^{-1})\right] , \qquad y \to \infty. \tag{6.14}$$

Repeating the same computation in the case of (6.11) one finds that

$$\tilde{E}_1^{(0)} = E_1^{(0)} + \frac{4}{\pi} \sum_{n=1}^{\infty} [(-1)^n - 1] K_1(2\pi n\kappa_0) . \tag{6.15}$$

6.2 Beyond the leading order

To find subleading corrections in large κ let us add and subtract the leading order contribution (6.4) from the expression (5.1):

$$E_1 = \frac{1}{\kappa}[-3\kappa_0^2 \ln 2 - \frac{5}{12} + \mathcal{O}(\eta^2)] + E_1^{(sub)} , \qquad \kappa_0 \to \infty \tag{6.16}$$

$$E_1^{(sub)} = -\frac{\kappa_0}{4\pi\kappa} \int_{-\infty}^{\infty} \mathrm{d}\bar{\Omega} \ \ln \frac{\mathcal{D}_\psi^8}{\mathcal{D}_\beta^2 \mathcal{D}_\phi}. \tag{6.17}$$

[17] This shift may be formally interpreted by saying that fermions have antiperiodic boundary conditions, so that $\ln\det(-\partial_\sigma^2 + \Omega^2 + \kappa_0^2) = \sum_{n=-\infty}^{+\infty} \ln[(n+\frac{1}{2})^2 + \omega^2 + \kappa_0^2]$. This interpretation is more of a curiousity and should not be taken literally as this expression was derived in the large κ_0 limit where the distinction between the periodic and antiperiodic fermion boundary conditions is not actually visible.

Here we defined

$$\mathcal{D}_\beta = \frac{\det\mathcal{O}_\beta}{\det\mathcal{O}_{\beta,0}}, \qquad \mathcal{D}_\phi = \frac{\det\mathcal{O}_\phi}{\det\mathcal{O}_{\phi,0}}, \qquad \mathcal{D}_\psi = \frac{\det\mathcal{O}_\psi}{\det\mathcal{O}_{\psi,0}} \tag{6.18}$$

and introduced $\bar{\Omega} = \frac{\Omega}{\kappa_0}$ which is the argument the integrand according to (6.5)-(6.7).

Expanding the arguments of the determinants, one finds (see (C.8)-(C.10))

$$\begin{aligned} 2\mathbb{K}\, Z(\alpha_\beta \,|\, k^2) &\approx \pi\,\kappa_0\, x - 2\tanh^{-1} x && (6.19)\\ 2\widetilde{\mathbb{K}}\, Z(\alpha_\phi \,|\, \tilde{k}^2) &\approx \pi\,\kappa_0\, y - 2\,\tanh^{-1} \frac{y}{2} && (6.20)\\ \widetilde{\mathbb{K}}\, Z(\alpha_\psi \,|\, \tilde{k}^2) &\approx \pi\,\kappa_0\, z - \tanh^{-1} z && (6.21) \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{D}_\beta &= \frac{\sinh^2[2\mathbb{K}\, Z(\alpha_\beta \,|\, k^2)]}{\sinh^2[\pi\,\kappa_0\, x]} \approx \Big[\frac{x^2+1}{x^2-1} - \frac{2x}{x^2-1}\, \coth(\pi\,\kappa_0\, x)\Big]^2 && (6.22)\\ \mathcal{D}_\phi &= \frac{\sinh^2[2\widetilde{\mathbb{K}}\, Z(\alpha_\phi \,|\, \tilde{k}^2)]}{\sinh^2[\pi\,\kappa_0\, y]} \approx \Big[\frac{y^2+4}{y^2-4} - \frac{4\,y}{y^2-4}\, \coth(\pi\,\kappa_0\, y)\Big]^2 && (6.23)\\ \mathcal{D}_\psi &= \frac{\cosh^2[\widetilde{\mathbb{K}}\, Z(\alpha_\psi \,|\, \tilde{k}^2)]}{\cosh^2[\pi\,\kappa_0\, z]} \approx \frac{1}{1-z^2}\big[1 - z\,\tanh(\pi\,\kappa_0\, z)\big]^2 && (6.24) \end{aligned}$$

Neglecting the tanh and coth terms in the square brackets for large κ_0 one finds that the second contribution in (6.16), first contribution at next-to-leading order, results in

$$\begin{aligned} E_1^{(sub)} &\approx -\frac{\kappa_0}{4\pi\kappa} \int_{-\infty}^{+\infty} \mathrm{d}\bar{\Omega}\, \ln\Big[\Big(\frac{1-\sqrt{1+\bar{\Omega}^2}}{1+\sqrt{1+\bar{\Omega}^2}}\Big)^8 \Big(\frac{1-\sqrt{2+\bar{\Omega}^2}}{1+\sqrt{2+\bar{\Omega}^2}}\Big)^{-4} \Big(\frac{2-\sqrt{4+\bar{\Omega}^2}}{2+\sqrt{4+\bar{\Omega}^2}}\Big)^{-1}\Big] \\ &= 1 + \frac{6}{\pi}\, \ln 2\,, \qquad\qquad \kappa_0 \to \infty\,, \end{aligned} \tag{6.25}$$

where we set $\kappa \approx \kappa_0$. The same result for this subleading coefficient was found in [24] using the integrability (algebraic curve) approach (see also [10] and [7]). As discussed in [10] this correction should be due to the near turning point contribution that is lost in the naive approach that treats the potential terms perturbatively.

Proceeding to the next order $\sim \eta = k^{-2} - 1$, the evaluation of the various functional determinant ratios gives

$$\begin{aligned} \mathcal{D}_\beta &= \frac{(x-1)^2}{(x+1)^2}\, \Big[1 + \frac{2}{x}\,\eta - \frac{\eta}{\pi\,\kappa_0}\, \frac{2\,(x^2-2)}{x\,(x^2-1)} + \mathcal{O}(\eta^2)\Big]\,, && (6.26)\\ \mathcal{D}_\phi &= \frac{(y-2)^2}{(y+2)^2}\, \Big[1 + \frac{4}{y}\,\eta - \frac{\eta}{\pi\,\kappa_0}\, \frac{4}{y} + \mathcal{O}(\eta^2)\Big]\,, && (6.27)\\ \mathcal{D}_\psi &= \frac{1-z}{1+z}\, \Big[1 + \frac{1}{z}\,\eta - \frac{\eta}{\pi\,\kappa_0}\, \frac{1}{z} + \mathcal{O}(\eta^2)\Big]\,, && (6.28) \end{aligned}$$

so that

$$\ln \frac{\mathcal{D}_\psi^8}{\mathcal{D}_\beta^2\, \mathcal{D}_\phi} = e(\bar\Omega) + \eta \left[f(\bar\Omega) + \frac{g(\bar\Omega)}{\pi\, \kappa_0} \right] + \dots \,, \tag{6.29}$$

where $e(\bar\Omega)$ is the integrand in (6.25). The functions $f(\bar\Omega)$ and $g(\bar\Omega)$ are

$$f(\bar\Omega) = \frac{8}{\sqrt{1+\bar\Omega^2}} - \frac{4}{\sqrt{2+\bar\Omega^2}} - \frac{4}{\sqrt{4+\bar\Omega^2}} \tag{6.30}$$

$$g(\bar\Omega) = -\frac{8}{\sqrt{1+\bar\Omega^2}} + \frac{4\,\bar\Omega^2}{(1+\bar\Omega^2)\sqrt{2+\bar\Omega^2}} + \frac{4}{\sqrt{4+\bar\Omega^2}} \tag{6.31}$$

and their integrals take the values

$$\int_{-\infty}^{+\infty} d\bar\Omega\ f(\bar\Omega) = 12\ \ln\, 2 \,, \qquad \int_{-\infty}^{+\infty} d\bar\Omega\ g(\bar\Omega) = -2\,(\pi + 6\ \ln\, 2)\ . \tag{6.32}$$

We conclude that to order η the large κ expansion of the 1-loop energy reads

$$E_1 = \frac{1}{\kappa}\Big[-3\,\kappa_0^2\ \ln 2\ - \frac{5}{12} + \big(1 + \frac{6}{\pi}\ \ln 2\big)\,\kappa_0 - \frac{1}{\pi}\big[3\,\kappa_0\ \ln 2\ - \frac{1}{2}\,\big(1+\frac{6}{\pi}\ \ln\, 2\big)\big]\ \eta + \mathcal{O}(\eta^2)\Big] \,, \qquad \kappa_0 \to \infty\ . \tag{6.33}$$

Here we did not expanded explicitly the overall factor of $\frac{1}{\kappa}$,

$$\frac{1}{\kappa} = \frac{1}{\kappa_0}\Big[1 + \frac{1}{4}(1 - \frac{2}{\pi\kappa_0})\eta + \mathcal{O}(\eta^2)\Big] \,, \qquad \eta = 16 e^{-\pi\kappa_0} \to 0\ . \tag{6.34}$$

The coefficient of the leading η correction is in agreement with the one found in [10], while the next $\frac{\eta}{\kappa_0}$ term is a new result.

In going to higher than first orders of expansion in η there is a potential problem of accounting for the contributions of terms like $\coth(\pi\kappa_0 x) - 1$ in (6.22)- (6.24) we have dropped above. For example,

$$e^{-2\kappa_0\pi z} \sim \Big(\frac{\eta}{16}\Big)^2\Big[1 - \frac{\pi\Omega^2}{\kappa_0} + \frac{\pi^2\Omega^4}{2\kappa_0^2} + \frac{3\pi^4\Omega^4 - 2\pi^6\Omega^6}{12\kappa_0^3\pi^3} + \dots\Big] \,, \tag{6.35}$$

and similar terms arise also in the expansion of the reference determinants, see (6.14). Such terms need to be resummed. and, while there is the possibility that all such terms may cancel, this is not clear at the moment. In Appendix D we present the evaluation of the leading large κ_0 correction to the one-loop energy due to these contributions, while in Appendix E we consider a different type of expansion in the $k \to 1$ limit.

Ignoring this complication, we have found that the one-loop energy has the following structure of large spin expansion

$$E_1 = \frac{\kappa_0}{\kappa}\Big[\big(c_{01}\,\kappa_0 + c_{00} + \frac{c_{0,-1}}{\kappa_0}\big) + \big(c_{11}\,\kappa_0 + c_{10} + \frac{c_{1,-1}}{\kappa_0}\big)\eta + \big(c_{21}\,\kappa_0 + c_{20} + \frac{c_{2,-1}}{\kappa_0}\big)\eta^2 + \big(c_{31}\,\kappa_0 + c_{30} + \frac{c_{3,-1}}{\kappa_0}\big)\eta^3 + \mathcal{O}(\eta^4)\Big] \,, \tag{6.36}$$

where the explicit values are

$$c_{01} = -3\ln 2\ , \qquad c_{00} = 1+\frac{6}{\pi}\ln 2 \qquad c_{0,-1} = -\frac{5}{12}\ , \tag{6.37}$$

$$c_{11} = 0\ , \qquad c_{10} = -\frac{3}{\pi}\ln 2 \qquad c_{1,-1} = \frac{1}{2\pi}+\frac{3\ \ln 2}{\pi^2}\ , \tag{6.38}$$

$$c_{21} = -\frac{\pi}{32}-\frac{3}{32}\ln 2\ , \qquad c_{20} = \frac{1}{16}+\frac{39\ln 2}{32\pi}\ , \qquad c_{2,-1} = -\frac{13}{64\pi}-\frac{63\ln 2}{32\pi^2}\ , \tag{6.39}$$

$$c_{31} = \frac{\pi}{32}+\frac{3}{32}\ln 2\ , \qquad c_{30} = -\frac{3}{32}-\frac{13\ln 2}{16\pi}\ , \qquad c_{3,-1} = \frac{29}{192\pi}+\frac{85\ln 2}{64\pi^2}\ . \tag{6.40}$$

For completeness, we report here the first few orders in the large spin expansion of the 1-loop energy as found using (2.12) in (6.36)

$$E_1 = -\frac{3\ln 2}{\pi}\ln\bar{\mathcal{S}}+\frac{\pi+6\ln 2}{\pi}-\frac{5\pi}{12\ln\bar{\mathcal{S}}}-\frac{1}{\bar{\mathcal{S}}}\Big[\frac{24\ln 2}{\pi}\ln\bar{\mathcal{S}}-\frac{4\pi+36\ln 2}{\pi}+\frac{5\pi}{3\ln^2\bar{\mathcal{S}}}\Big]+\mathcal{O}\left(\frac{1}{\bar{\mathcal{S}}^2}\right)$$
$$\bar{\mathcal{S}} = 8\,\pi\,\mathcal{S}, \qquad \mathcal{S}\gg 1 \tag{6.41}$$

6.3 Test of reciprocity

With the expressions (6.36)-(6.40) at hand, we are able to the confirm and extend the analysis of [10], in which the reciprocity relations between the coefficients in large spin expansion of the energy (or twist 2 anomalous dimension at strong coupling) [8, 35, 36] were checked up to order η.

To do this one needs to determine the functions[18]

$$\Delta(\mathcal{S}) = \Delta_0+\frac{1}{\sqrt{\lambda}}\Delta_1+...\ , \qquad \Delta_0=\mathcal{E}_0(\mathcal{S})-\mathcal{S}, \qquad \Delta_1=\mathcal{E}_1(\mathcal{S}), \tag{6.42}$$

as functions of the spin $\mathcal{S}$, which as at the classical level is obtained by replacing the parameter η with its expansion $\eta=\eta(\mathcal{S})$ in terms of the spin (2.14). One is then to compute the function $\mathcal{P}$ defined by

$$\Delta(\mathcal{S}) \;=\; \mathcal{P}(\mathcal{S}+\frac{1}{2}\,\Delta(\mathcal{S}))\ . \tag{6.43}$$

The test of reciprocity amounts to the check of parity of $\mathcal{P}(\mathcal{S})$ under $\mathcal{S}\to-\mathcal{S}$. Solving the functional equation in (6.43) as

$$\mathcal{P}(\mathcal{S}) = \sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{1}{2}\frac{d}{d\mathcal{S}}\right)^{k-1}\big[\Delta(\mathcal{S})\big]^k\ , \tag{6.44}$$

and expanding the function $\mathcal{P}$ in $\frac{1}{\sqrt{\lambda}}$

$$\mathcal{P} = \mathcal{P}_0+\frac{1}{\sqrt{\lambda}}\mathcal{P}_1+\cdots\ , \tag{6.45}$$

[18] $\mathcal{E}_0$ and $\mathcal{E}_1$ are the classical and the 1-loop energies rescaled by a factor string tension.

one finds

$$\mathcal{P}_0(\mathcal{S}) = \sum_{k=1}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}\frac{d}{d\mathcal{S}}\right)^{k-1} [\Delta_0(\mathcal{S})]^k, \tag{6.46}$$

$$\mathcal{P}_1(\mathcal{S}) = \sum_{k=1}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}\frac{d}{d\mathcal{S}}\right)^{k-1} [k\,\Delta_0(\mathcal{S})^{k-1}\,\Delta_1(\mathcal{S})]. \tag{6.47}$$

Working out $\mathcal{P}_1$ and looking at all terms which are odd under $\mathcal{S} \to -\mathcal{S}$ we find that they vanish if the following reciprocity constraints hold

$$c_{10} = \frac{1}{\pi}\,c_{01}\ , \qquad c_{1,-1} = \frac{1}{2\pi}\,c_{00}\ , \qquad c_{31} = -c_{21}\ , \tag{6.48}$$

$$c_{30} = -c_{20} - \frac{1}{6\pi}c_{01} + \frac{1}{\pi}\,c_{21}\ , \tag{6.49}$$

$$c_{3,-1} = -c_{2,-1} + \frac{1}{4\pi^2}\,c_{01} - \frac{1}{12\pi}c_{00} + \frac{1}{2\pi}\,c_{20}. \tag{6.50}$$

As usual, the coefficients of terms with odd powers of $\eta = \frac{2}{\mathcal{S}} + ...$ in (2.12) are determined by coefficients of terms with even powers of η. Using the list of explicit coefficients found above (6.37)-(6.40), we find that these relations are indeed satisfied.

7 Small spin expansion

The small spin or short string limit [9, 14] is realized by sending $\eta \to \infty$ or $k \to 0$ (see section 2).

The general expansion of the determinants, see (C.13)-(C.18), has the form

$$\det \mathcal{O}_f = D_f^{(0)}(\Omega) + \frac{1}{\eta}\,D_f^{(1)}(\Omega) + \frac{1}{\eta^2}\,D_f^{(2)}(\Omega) + \cdots\,, \qquad f = (\beta,\phi,\psi)\ , \tag{7.1}$$

where

$$D_\beta^{(0)}(\Omega) = D_\phi^{(0)}(\Omega) = D_\psi^{(0)}(\Omega) = 4\,\sinh^2(\pi\,\Omega), \tag{7.2}$$

$$D_\beta^{(1)}(\Omega) = \frac{2\pi\sinh(2\pi\Omega)}{\Omega}, \quad D_\phi^{(1)}(\Omega) = \frac{4\pi\Omega\sinh(2\pi\Omega)}{\Omega^2+1}, \quad D_\psi^{(1)}(\Omega) = \frac{4\pi\Omega\sinh(2\pi\Omega)}{4\Omega^2+1},$$

$$D_\beta^{(2)}(\Omega) = \frac{\pi^2\cosh(2\pi\Omega)}{\Omega^2} - \frac{\pi\left(3\Omega^4+6\Omega^2+2\right)\sinh(2\pi\Omega)}{4\left(\Omega^5+\Omega^3\right)}$$

$$D_\phi^{(2)}(\Omega) = \frac{4\pi^2\Omega^2\cosh(2\pi\Omega)}{\left(\Omega^2+1\right)^2} - \frac{\pi\Omega\left(3\left(\Omega^2+4\right)\Omega^2+1\right)\sinh(2\pi\Omega)}{2\left(\Omega^2+1\right)^3},$$

$$D_\psi^{(2)}(\Omega) = -\frac{\pi\Omega\left(48\left(\Omega^4+\Omega^2\right)+1\right)\sinh(2\pi\Omega)}{2\left(4\Omega^2+1\right)^3} + \frac{4\pi^2\Omega^2\cosh(2\pi\Omega)}{\left(4\Omega^2+1\right)^2}. \tag{7.3}$$

The first correction to the quantity entering the effective action (5.1) is

$$\ln\frac{\det^8\mathcal{O}_\psi}{\det\mathcal{O}_\phi\,\det^2\mathcal{O}_\beta\,\det^5\mathcal{O}_0} = \frac{1}{\eta}\,\frac{2\pi\left(2\Omega^2-1\right)\coth(\pi\Omega)}{\Omega\left(\Omega^2+1\right)\left(4\Omega^2+1\right)} + \mathcal{O}\left(\frac{1}{\eta^2}\right), \tag{7.4}$$

which is integrable at $\Omega \to \infty$ but has a pole at $\Omega = 0$, i.e. produces an IR divergence. Such an IR effect disappears by integrating separately the lowest eigenvalues (see Table 1), which, in fact, behave as zero modes around $\Omega \sim 0$ (in the case of the β fluctuation this only happens in the short string limit $\eta \to \infty$)

$$\bar{\Lambda}_1^{(\beta)} = \Omega^2 + \frac{1}{\eta} + \cdots, \qquad \bar{\Lambda}_1^{(\phi)} = \Omega^2, \qquad \bar{\Lambda}_1^{(\psi)} = \Omega^2. \tag{7.5}$$

This is equivalent to use the definition (5.18) for the 1-loop correction to the energy. Indeed, with the definition (5.15) the quantity one is to evaluate

$$\ln \frac{(\det{}'\mathcal{O}_\psi)^8}{\det{}'\mathcal{O}_\phi\,(\det{}'\mathcal{O}_\beta)^2\,(\det{}'\mathcal{O}_0)^5} = \frac{1}{\eta}\,\frac{2\left[4\Omega^4 + 5\Omega^2 + \pi\left(2\Omega^2-1\right)\Omega\coth(\pi\Omega) + 1\right]}{\Omega^2\left(4\Omega^4+5\Omega^2+1\right)} + \ldots \tag{7.6}$$

is now finite and can be integrated to give $2\pi(8\ln 2 - 3)$. On the other hand, the contribution of the lowest eigenvalues has been shown to give a finite number at (5.17) at all orders in $1/\eta$.

Going to one further order in the large η expansions of the determinants and adding all together one finds for the expansion of the 1-loop energy (5.18)

$$\begin{aligned} E_1 &= 1 - \frac{1}{4\pi\kappa}\int_{-\infty}^{\infty} d\Omega\ \ln\frac{(\det{}'\mathcal{O}_\psi)^8}{(\det{}'\mathcal{O}_\beta)^2\det{}'\mathcal{O}_\phi\,(\det{}'\mathcal{O}_0)^5} \\ &= 1 + \frac{1}{\kappa}\Big[\big(\tfrac{3}{2} - 4\ln 2\big)\eta^{-1} - \big(1 - \tfrac{3}{2}\ln 2 - \tfrac{3}{8}\zeta(3)\big)\eta^{-2} \\ &\qquad - \big(-\tfrac{27}{16} + \tfrac{7}{4}\ln 2 + \tfrac{9}{32}\zeta(3) + \tfrac{15}{32}\zeta(5)\big)\eta^{-3} + \mathcal{O}(\eta^4)\Big]\ . \end{aligned} \tag{7.7}$$

Here we did not expand explicitly the factor

$$\frac{1}{\kappa} = \sqrt{\eta}\left[1 + \frac{1}{4}\eta^{-1} + \mathcal{O}(\eta^{-2})\right]\ . \tag{7.8}$$

Substituting the expansion of η in terms of the spin (2.14), we can finally obtain the following small spin expansion of the 1-loop correction to the energy

$$E_1 = E_1^{(\mathrm{an})} + E_1^{(\mathrm{nan})}\ , \tag{7.9}$$

$$E_1^{(\mathrm{an})} = \sqrt{2\,\mathcal{S}}\Big([\tfrac{3}{2} - 4\ln 2] + [-\tfrac{23}{16} + \tfrac{3}{2}\ln 2 + \tfrac{3}{4}\zeta(3)]\,\mathcal{S} \tag{7.10}$$

$$+[\tfrac{689}{256} - \tfrac{63}{32}\ln 2 - \tfrac{15}{32}\zeta(3) - \tfrac{15}{16}\zeta(5)]\,\mathcal{S}^2 + \mathcal{O}(\mathcal{S}^3)\Big)\ , \tag{7.11}$$

$$E_1^{(\mathrm{nan})} = 1 + \mathcal{O}(\mathcal{S}),\qquad . \tag{7.12}$$

We have separated E_1, as in [14], into an "analytic" part (with $\mathcal{S}$-dependence similar to the classical energy (2.15)) and a "non-analytic" part, containing "would-be IR singular" contributions of the lowest eigenvalues.

We conclude that the procedure adopted in this paper leads to the same structure of the small spin expansion of the 1-loop energy as found in [9]. The coefficients of the transcendental terms proportional to $\ln 2$ and $\zeta(3)$ in (7.11) are exactly the same as in [9] (see eq. (4.37) there). The coefficients of the rational terms are, however, different.

Let us note that a separate treatment of the zero-mode contribution in [37] led also to a different result for (7.12) (cited in eq. (3.60) in [14]). Refs. [9, 37] used the standard "near-flat-space" perturbation theory treatment of the determinants in the conformal gauge. A disagreement with our present results is apparently due to the prescription adopted in [9, 37] for the projecting out the zero mode contributions. As discussed in Appendix F, a somewhat different prescription would lead to the same result as the one that one obtains using the small spin perturbation theory for the static gauge determinants.

8 Conclusions

In this paper we have found an exact expression for the one-loop correction to the energy of the folded string spinning in the AdS_3 part of $AdS_5 \times S^5$. The main technical advance is that we have shown that all the fluctuation operators, in the static gauge, have the single-gap Lamé form. As a result, their determinants can be computed in a closed form. We have verified explicitly that, as expected, the one-loop energy correction is the same in the static and the conformal gauges, even though the structure of the two fluctuation determinant ratios appears to be quite different.

The analytic expressions for the fluctuation determinants permitted us to carry out improved expansions in the small and large spin limit; the latter allowed us to verify that the reciprocity relations continue to be satisfied at strong coupling. Perhaps more importantly, our demonstration that the semi-classical fluctuation problem is governed by simple finite-gap operators gives a new perspective on the role of integrable systems in the analysis of quantum corrections in such string models. In fact, finite-gap fluctuation operators are naturally described in terms of algebraic curves of Riemann surfaces associated with the finite-gap spectrum [39], making the connection with the classical integrability (algebraic curve) approach of [15, 16] explicit.

The integrability approach to semi-classical quantization relies on the classical integrable structure of the theory. The investigation of the monodromy of the Lax connection [40] for the $AdS_5 \times S^5$ superstring action leads to the derivation of a spectral curve for any solution of the classical string equations of motion [15, 41]. This is an example of the general finite gap description of classically integrable theories [42] which, reformulated in terms of a Riemann-Hilbert problem, leads to certain integral equations for each finite-gap curve associated to a classical solution. The same finite-gap integral equations happen to appear in the continuum limit of the (discrete) algebraic Bethe Ansatz equations [43].

Starting with the classical algebraic curve describing a particular solution one can develop a semiclassical quantization [16, 44] by deforming the cuts defining the algebraic curve (adding extra roots) [45, 46]. Fluctuations are then perturbations of the cuts, and the one-loop correction to the energy is given as usual by the sum of the energy shifts (or characteristic frequencies) due to these fluctuations. Alternatively, one may try to guess the quantum extension of the classical finite gap integral equations, having as guiding principle the gauge theory information implying a description in terms of an asymptotic Bethe Ansatz [43]. Improved by the phase [47, 48] extracted from the 1-loop string data of [49], the Bethe Ansatz result for the 1-loop correction to string energy was shown [50] to agree, for a generic classical superstring solution, with the approach based on extracting the characteristic frequencies by perturbing the algebraic curve. This general equivalence was recently extended to include also the exponentially suppressed finite size effects with the asymptotic Bethe Ansatz starting point replaced by an appropriate Thermodynamic Bethe Ansatz (see [17] and references therein).

Comparing this integrability approach to the one of the present paper, notice that even if we did not explicitly refer to the classical integrability of the string sigma model, we "rediscovered" the integrability at the one-loop level via the connection with the integrable, finite-gap, Lamé equation.

In addition to stimulating the study of detailed relation between the two approaches at the 1-loop level [18], the findings of the present papers have a methodological merit of explicitly illustrating on a rather important and non-trivial example of how the integrability of the $AdS_5 \times S^5$ superstring sigma model is extended from the classical to the semiclassical one-loop level.

This connection is, of course, not surprising from a general perspective: given a set of integrable classical equations, the linear problem for small fluctuations near a given solution is found by considering a small variation of the original non-linear equations and should thus be essentially controlled by the original classical integrable structure. However, the technical details of such connection may be quite intricate. The small fluctuation problem is, in general, described by a complicated coupled set of linear differential equations, i.e. by a matrix differential operator, while the standard examples of integrable spectral problems involve 2-nd order ordinary differential operators with their integrability related to a special type of their potential terms. The general study of which kind of integrable matrix differential operator spectral problems are associated to non-linear string sigma model type classical equations appears to be an interesting open problem.

The extension of our present results to the case of spinning folded string with a non-zero angular momentum in S^5 is currently under investigation [18].

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Appendix A: Relevant elliptic function properties and identities

Complete elliptic integrals

The *complete elliptic integrals of the first and second kind* are defined as functions of their modulus k^2 as follows

$$\mathbb{K}(k^2) = \mathbb{K} = \int_0^{\frac{\pi}{2}} d\theta\ (1 - k^2\ \sin^2\theta)^{-1/2}\,, \qquad \mathbb{E}(k^2) = \mathbb{E} = \int_0^{\frac{\pi}{2}} d\theta\ (1 - k^2\ \sin^2\theta)^{1/2} \quad \text{(A.1)}$$

One also defines the complementary modulus

$$k'^2 = 1 - k^2 \qquad \text{and} \qquad \mathbb{K}'(k^2) = \mathbb{K}' = \mathbb{K}(1-k^2) = \int_0^{\frac{\pi}{2}} d\theta\ (1 - k'^2\ \sin^2\theta)^{-1/2}\,. \quad \text{(A.2)}$$

Jacobi elliptic functions

Defining the *Jacobi amplitude* as

$$\varphi = \mathrm{am}(u\,|\,k^2), \quad \text{where} \quad u = \int_0^{\varphi} d\theta\ (1 - k'^2\ \sin^2\theta)^{-1/2} \quad \text{(A.3)}$$

the *Jacobi elliptic functions* $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ defined by

$$\mathrm{sn}(u\,|\,k^2) = \sin\varphi, \qquad \mathrm{cn}(u\,|\,k^2) = \cos\varphi, \qquad \mathrm{dn}(u\,|\,k^2) = (1 - k^2\ \sin^2\varphi)^{1/2} \quad \text{(A.4)}$$

are doubly periodic functions of u, with real-valued periods that are either $2\mathbb{K}$ (dn) or $4\mathbb{K}$ (sn and cn) and purely imaginary periods that are either $2i\,\mathbb{K}'$ (sn) or $4i\,\mathbb{K}'$ (cn and dn). The fundamental period-parallelogram for the Jacobi elliptic functions is, therefore, the rectangle with corners at $(0, 4\mathbb{K}, 4i\mathbb{K}', 4\mathbb{K} + 4i\mathbb{K}')$, where zero occur for real values of u (at $2\mathbb{K}$ and $4\mathbb{K}$) while singularities occur for imaginary values of u (at $i\,\mathbb{K}'$ and $3i\mathbb{K}'$).

Other Jacobian elliptic functions useful for us are

$$\mathrm{cd}(u\,|\,k^2) = \frac{\mathrm{cn}(u\,|\,k^2)}{\mathrm{dn}(u\,|\,k^2)}, \qquad \mathrm{sd}(u\,|\,k^2) = \frac{\mathrm{sn}(u\,|\,k^2)}{\mathrm{dn}(u\,|\,k^2)} \tag{A.5}$$

$$\mathrm{ns}(u\,|\,k^2) = \frac{1}{\mathrm{sn}(u\,|\,k^2)}, \qquad \mathrm{nd}(u\,|\,k^2) = \frac{1}{\mathrm{dn}(u\,|\,k^2)} \tag{A.6}$$

Useful relations between the squares of the functions are

$$-\mathrm{dn}^2(u\,|\,k^2) + k'^2 = -k^2\,\mathrm{cn}^2(u\,|\,k^2) = k^2\,\mathrm{sn}^2(u\,|\,k^2) - k^2 \tag{A.7}$$

$$-k'^2\,\mathrm{nd}(u\,|\,k^2) + k'^2 = -k^2\,k'^2\,\mathrm{sd}^2(u\,|\,k^2) = k^2\,\mathrm{cd}(u\,|\,k^2) - k^2. \tag{A.8}$$

A useful representation for $\mathrm{sn}(u\,|\,k^2)$ is

$$\mathrm{sn}(u\,|\,k^2) = \frac{\pi}{2\mathbb{K}'}\sum_{n=-\infty}^{\infty}(-1)^n\tanh\Big(\frac{\pi}{2\mathbb{K}'}\,(u - 2\,n\,\mathbb{K})\Big) \tag{A.9}$$

Jacobi Eta, Theta and Zeta functions

The *Jacobi* H, Θ *and* Z functions are defined as follows in terms of the Jacobi ϑ functions

$$H(u\,|\,k^2) = \vartheta_1\Big(\frac{\pi\,u}{2\,\mathbb{K}}, q\Big), \qquad \Theta(u\,|\,k^2) = \vartheta_4\Big(\frac{\pi\,u}{2\,\mathbb{K}}, q\Big), \qquad Z(u\,|\,k^2) = \frac{\pi}{2\,\mathbb{K}}\,\frac{\vartheta_4'(\frac{\pi\,u}{2\,\mathbb{K}}, q)}{\vartheta_4(\frac{\pi\,u}{2\,\mathbb{K}}, q)} \tag{A.10}$$

where

$$q = q(k^2) = \exp\Big(-\pi\frac{\mathbb{K}'}{\mathbb{K}}\Big). \tag{A.11}$$

Useful periodicities for them are

$$H(u + 2\mathbb{K}\,|\,k^2) = -H(u\,|\,k^2), \tag{A.12}$$

$$\Theta(u + 2\mathbb{K}\,|\,k^2) = \Theta(u\,|\,k^2), \tag{A.13}$$

$$Z(u + 2\mathbb{K}\,|\,k^2) = Z(u\,|\,k^2) \tag{A.14}$$

Useful representations for $Z(u\,|\,k^2)$ are the integral representation

$$Z(\mathrm{sn}^{-1}(y|k^2)\,|\,k^2) = \int_0^y \mathrm{d}t\left[\sqrt{\frac{1-k^2t^2}{1-t^2}} - \frac{\mathbb{E}(k^2)}{\mathbb{K}(k^2)}\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\right] \tag{A.15}$$

and

$$\mathbb{Z}(\alpha; k^2) = \int_0^\alpha du\;\mathrm{dn}^2(u; k^2)\; - \frac{\mathbb{E}(k^2)}{\mathbb{K}(k^2)}\,\alpha\;. \tag{A.16}$$

We also recall the following series representation

$$Z(u\,|\,k^2) = -\frac{\pi}{2\mathbb{K}'\,\mathbb{K}}u + \frac{\pi}{2\mathbb{K}'}\sum_{n=-\infty}^{\infty}\tanh\Big(\frac{\pi}{2\mathbb{K}'}\,(u - 2\,n\,\mathbb{K})\Big) \tag{A.17}$$

Landen transformations useful for folded string fluctuation operators

(a) Bosonic fluctuation ϕ

Consider the bosonic fluctuation (3.28)-(3.30)

$$\mathcal{O}_\phi = -\partial_\sigma^2 + 2\omega^2\, k^2\, \mathrm{sn}^2(\omega\sigma + \mathbb{K}\,|\,k^2) + 2\omega^2\, \mathrm{ns}^2(\omega\sigma + \mathbb{K}\,|\,k^2)$$

In the potential in $\mathcal{O}_\phi$ one can use

$$\mathrm{sn}^2(u\,|\,k^2) = \frac{1-\mathrm{cn}(2u\,|\,k^2)}{1+\mathrm{dn}(2u\,|\,k^2)}, \qquad \mathrm{ns}(u\,|\,k^2) = i\,\mathrm{sn}(u + i\mathbb{K}'\,|\,k^2) \tag{A.18}$$

and perform the Landen transformation

$$\mathrm{sn}((1+\tilde{k}')u\,|\,k^2) = (1+\tilde{k}')\mathrm{sn}(u\,|\,\tilde{k}^2)\,\mathrm{cd}(u\,|\,\tilde{k}^2) \tag{A.19}$$

with $\tilde{k}$ defined in (A.25). Rescaling the variable

$$\bar{\sigma} = \frac{2\omega\sigma}{1+\tilde{k}'} \tag{A.20}$$

one obtains the single-gap Lamé operator

$$\mathcal{O}_\phi \;=\; -\partial_z^2 + 2\,\tilde{k}^2\,\mathrm{sn}^2(z + i\,\widetilde{\mathbb{K}}'\,|\,\tilde{k}^2) - \tilde{k}^2 + \frac{\pi^2\,\Omega^2}{4\,\widetilde{\mathbb{K}}^2} \tag{A.21}$$

with periodicity $\phi(z) = \phi(z + 4\widetilde{\mathbb{K}})$ in the rescaled variable $z = \frac{2\widetilde{\mathbb{K}}}{\pi}\sigma$. Notice that the imaginary shift of x makes the potential singular, as it must be from the original form (3.30), where singularities are manifest at $\sigma = \left(n+\frac{1}{2}\right)\pi$, $n \in \mathbf{N}$. Such imaginary part, however, does not affect the discussion leading to the determinant expression. [19]

(b) Fermionic fluctuations

Consider the fermionic fluctuations (3.28)-(3.31)

$$\mathcal{O}_{\psi_\pm} = -\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} + \kappa^2 \mathrm{sn}^2(\omega\sigma + \mathbb{K}\,|\,k^2) \pm \kappa\,\omega\,\mathrm{cn}(\omega\sigma + \mathbb{K}\,|\,k^2)\,\mathrm{dn}(\omega\sigma + \mathbb{K}\,|\,k^2) + \Omega^2$$

One can use the Landen transformation

$$\mathrm{sn}((1+\tilde{k}')u\,|\,k^2) \;=\; (1+\tilde{k}')\,\mathrm{sn}(u\,|\,\tilde{k}^2)\,\mathrm{cd}(u\,|\,\tilde{k}^2), \tag{A.22}$$

$$\mathrm{cn}((1+\tilde{k}')u\,|\,k^2) \;=\; \mathrm{nd}(u\,|\,\tilde{k}^2) - (1+\tilde{k}')\mathrm{sn}(u\,|\,\tilde{k}^2)\mathrm{sd}(u\,|\,\tilde{k}^2), \tag{A.23}$$

$$\mathrm{dn}((1+\tilde{k}')u\,|\,k^2) \;=\; \mathrm{nd}(u\,|\,\tilde{k}^2) - (1-\tilde{k}')\mathrm{sn}(u\,|\,\tilde{k}^2)\mathrm{sd}(u\,|\,\tilde{k}^2), \tag{A.24}$$

[19] In particular, it does not affect the monodromy of the potential.

where

$$k=\frac{1-\tilde{k}'}{1+\tilde{k}'}\ \Leftrightarrow\ \tilde{k}^2=\frac{4k}{(1+k)^2} \tag{A.25}$$

and the relations of the parameters with the new modulus $\tilde{k}$ are

$$\kappa=\frac{1-\tilde{k}'}{1+\tilde{k}'}\omega\ \Longrightarrow \tilde{k}'=\frac{\omega-\kappa}{\omega+\kappa},\qquad \tilde{k}^2=\frac{4\kappa\omega}{(\omega+\kappa)^2}. \tag{A.26}$$

Rescaling then the variable

$$y=\frac{\omega\sigma}{1+\tilde{k}'}=\frac{\widetilde{\mathbb{K}}\,\sigma}{\pi} \tag{A.27}$$

and exploiting the relations

$$\mathbb{K}(k^2)=\frac{1+\tilde{k}'}{2}\mathbb{K}(\tilde{k}^2),\qquad \mathbb{E}(k^2)=\frac{1}{1+\tilde{k}'}\left(\mathbb{E}(\tilde{k}^2)+\tilde{k}'\mathbb{K}(\tilde{k}^2)\right) \tag{A.28}$$

and

$$\frac{\mathbb{E}(k^2)}{\mathbb{K}(k^2)}=\frac{2}{(1+\tilde{k}')^2}\Big[\frac{\mathbb{E}(\tilde{k}^2)}{\mathbb{K}(\tilde{k}^2)}+\tilde{k}'^2\Big]\ , \tag{A.29}$$

one obtains two single-gap Lamé operators

$$\mathcal{O}_{\psi_+}\ =\ -\partial_y^2+2\,\tilde{k}^2\mathrm{sn}^2(y+\frac{\widetilde{\mathbb{K}}}{2}\,|\,\tilde{k}^2)-\tilde{k}^2+\frac{\pi^2\,\Omega^2}{\widetilde{\mathbb{K}}^2} \tag{A.30}$$

$$\mathcal{O}_{\psi_-}\ =\ -\partial_y^2+2\,\tilde{k}^2\mathrm{sn}^2(y+\frac{3\,\widetilde{\mathbb{K}}}{2})\,|\,\tilde{k}^2)-\tilde{k}^2+\frac{\pi^2\,\Omega^2}{\widetilde{\mathbb{K}}^2} \tag{A.31}$$

where the new elliptic parameter is $\tilde{k}^2=\frac{4\,k}{(1+k)^2}$, $\widetilde{\mathbb{K}}=\mathbb{K}(\tilde{k}^2)$, and the periodicity $\psi_\pm(y)=\psi_\pm(y+2\widetilde{\mathbb{K}})$ is in the new variable $y=\frac{\widetilde{\mathbb{K}}}{\pi}\sigma$.

Appendix B: Determinant via Gel'fand-Yaglom method

For a periodic potential, we can compute the determinant via the discriminant as in (4.7). In certain special cases, the discriminant can be found exactly because we know the explicit solutions $f_{1,2}$ in (4.2). This is the case, for example, for the constant potential $V(x)=m^2$, and also for the single-gap Lamé potential $V(x)=2k^2\,\mathrm{sn}^2(x|k^2)$. But in our comparison between the static gauge and conformal gauge we will need the determinant for *coupled* operators, and here we can use the Gel'fand-Yaglom theorem. To introduce this, we first state it for uncoupled operators, and then for coupled operators.

For the uncoupled equation (4.1) with periodic or antiperiodic boundary conditions, the Gel'fand Yaglom method [30, 38] is simply an evaluation of the discriminant. That is, we numerically [or analytically, in special cases] evaluate the discriminant, using initial value boundary conditions (4.2) for the two independent functions $f_{1,2}(x;\Lambda)$. Note that here Λ is just a parameter, so we are solving a homogeneous problem, with initial value conditions, which is numerically trivial. In fact, we can specify the initial conditions at any arbitrary point $\bar{x}$, and simply evolve through one period to evaluate the discriminant. That is, we can take initial conditions

$$\begin{aligned} f_1(\bar{x};\Lambda) = 1 \qquad &; \qquad f_1'(\bar{x};\Lambda) = 0 \\ f_2(\bar{x};\Lambda) = 0 \qquad &; \qquad f_2'(\bar{x};\Lambda) = 1 \end{aligned} \tag{B.1}$$

Then the Gelf'and-Yaglom theorem states that the determinant with period L is

$$\begin{aligned} \det_P^{\rm GY}(\Lambda) &= \Delta(\Lambda) - 2 \\ &= f_1(\bar{x}+L;\Lambda) + f_2'(\bar{x}+L;\Lambda) - 2 \end{aligned} \tag{B.2}$$

We can illustrate this method for the single-gap Lamé system, by taking two linear combinations of f_+ and f_- in (4.23)

$$f_1(x) = m_1\, f_+(x) + m_2\, f_-(x) \quad , \quad f_2(x) = n_1\, f_+(x) + n_2\, f_-(x) \tag{B.3}$$

such that the (B.1) are satisfied and α is given by (4.22). One finds

$$m_1 = -\frac{f_-'(\bar{x})}{D(\bar{x})} \quad , \quad m_1 = \frac{f_+'(\bar{x})}{D(\bar{x})} \quad ; \quad n_1 = \frac{f_-(\bar{x})}{D(\bar{x})} \quad , \quad n_2 = -\frac{f_+(\bar{x})}{D(\bar{x})} \tag{B.4}$$

where

$$D(\bar{x}) = f_+'(\bar{x})\, f_-(\bar{x}) - f_-'(\bar{x})\, f_+(\bar{x}) \tag{B.5}$$

Exploiting the monodromy

$$f_\pm(x+2\mathbb{K}) = -f_\pm(x)\, e^{\mp 2\,\mathbb{K}\, Z(\alpha|k^2)} \quad , \quad f_\pm'(x+2\mathbb{K}) = -f_\pm'(x)\, e^{\mp 2\,\mathbb{K}\, Z(\alpha|k^2)}, \tag{B.6}$$

it is then easy to check that the expression for the determinant (B.2) yields

$${\rm Det}_{\rm P} = 2\cosh[4\mathbb{K}\, Z(\alpha|k^2)] - 2 = 4\sinh^2\left[2\mathbb{K}\, Z(\alpha|k^2)\right] \tag{B.7}$$

as before (4.26). Note the important observation that this result is *independent of the initial point* $\bar{x}$. A specific example of the linear combinations (B.3) satisfying (B.1) at the initial point $\bar{x} = 0$ are the solutions [31]

$$f_1(x) = f_+(x) - f_-(x) \quad , \quad f_2(x) = \frac{{\rm sn}(\alpha\,|\,k^2)}{{\rm cn}(\alpha\,|\,k^2)\,{\rm dn}(\alpha\,|\,k^2)}\Big(f_+(x) + f_-(x)\Big). \tag{B.8}$$

For a system of coupled equations, the Gel'fand-Yaglom theorem generalizes in a straightforward manner. Consider (4.1) with $V(x)$ now an $n \times n$ matrix, and $f(x)$ an n-component column vector. Then we define $2n$ independent solutions $f_1^{(a)}(x;\Lambda)$ and $f_2^{(a)}(x;\Lambda)$, for $a = 1, 2, \ldots n$, with the initial conditions expressed as a $2n \times 2n$ matrix:

$$\begin{pmatrix} f_1^{(1)}(0;\Lambda) & \ldots & f_1^{(n)}(0;\Lambda) & f_1^{(1)\prime}(0;\Lambda) & \ldots & f_1^{(n)\prime}(0;\Lambda) \\ f_2^{(1)}(0;\Lambda) & \ldots & f_2^{(n)}(0;\Lambda) & f_2^{(1)\prime}(0;\Lambda) & \ldots & f_2^{(n)\prime}(0;\Lambda) \end{pmatrix} = \mathbb{I}_{2n\times 2n} \tag{B.9}$$

Then the Gel'fand-Yaglom theorem states that the infinite dimensional determinant can be expressed as a finite dimensional determinant:

$$\begin{aligned} &\det{}_P\left[-\partial_x^2 + V(x) - \Lambda\right] = \\ &= -\det{}_{2n\times 2n}\left[\mathbb{I} - \begin{pmatrix} f_1^{(1)}(L;\Lambda) & \ldots & f_1^{(n)}(L;\Lambda) & f_1^{(1)\prime}(L;\Lambda) & \ldots & f_1^{(n)\prime}(L;\Lambda) \\ f_2^{(1)}(L;\Lambda) & \ldots & f_2^{(n)}(L;\Lambda) & f_2^{(1)\prime}(L;\Lambda) & \ldots & f_2^{(n)\prime}(L;\Lambda) \end{pmatrix}\right] \end{aligned} \tag{B.10}$$

Again, this is completely straightforward to evaluate numerically. This was used to evaluate the determinant of the coupled operators in the conformal gauge example discussed in Section 5.

Appendix C: Relevant expansions of elliptic and Jacobi functions

Expansions of $\mathbb{K}$ and $\mathbb{E}$ $(k \to 1)$

The expansion of the complete elliptic integrals in k'^2 for $k \to 1$ reads as follows (with $L = \ln 4/k'$)

$$\mathbb{K}(k^2) = L + \tfrac{1}{4}(L-1)\,k'^2 + \tfrac{9}{64}\left(L - \tfrac{7}{6}\right)k'^4 + \tfrac{25}{256}\left(L - \tfrac{37}{30}\right)k'^6 + \ldots \tag{C.1}$$

$$\begin{aligned} \mathbb{E}(k^2) &= 1 + \tfrac{1}{2}\left(L - \tfrac{1}{2}\right)k'^2 + \tfrac{3}{16}\left(L - \tfrac{13}{12}\right)k'^4 + \tfrac{15}{128}\left(L - \tfrac{6}{5}\right)k'^6 \\ &\quad + \tfrac{175}{2048}\left(L - \tfrac{1051}{840}\right)k'^8 + \ldots \end{aligned} \tag{C.2}$$

Also

$$\begin{aligned} \frac{\mathbb{E}(k^2)}{\mathbb{K}(k^2)} &= \frac{1}{L} + \left(\tfrac{1}{2} - \tfrac{1}{2L} + \tfrac{1}{4L^2}\right)k'^2 + \left(\tfrac{1}{16} - \tfrac{3}{32L} - \tfrac{3}{128L^2} + \tfrac{1}{16L^3}\right)k'^4 \\ &\quad + \left(\tfrac{1}{32} - \tfrac{3}{64L} - \tfrac{11}{768L^2} + \tfrac{5}{256L^3} + \tfrac{1}{64L^4}\right)k'^6 + \ldots \end{aligned} \tag{C.3}$$

This gives

$$\kappa = \kappa_0 - \tfrac{1}{4\pi}(\pi\kappa_0 - 2)\eta + \tfrac{9}{64\pi}\left(\pi\kappa_0 - \tfrac{7}{3}\right)\eta^2 - \tfrac{25}{256\pi}\left(\pi\kappa_0 - \tfrac{37}{15}\right)\eta^3 + \ldots \tag{C.4}$$

Expansion of Jacobi Zeta function Z $(k \to 1)$

Consider the integral representation of the Jacobi Zeta function (A.15)

$$f(y) = Z(\mathrm{sn}^{-1}(y|k^2)\,|\,k^2) \tag{C.5}$$

We find for the asymptotics for $k \to 1$ (setting $L = \ln \frac{4}{k'}$)

$$\begin{aligned} f(y) = {} & y - \frac{1}{L}\int_0^y \frac{\mathrm{d}t}{1-t^2} - \frac{1}{2}k'^2 y + \frac{1}{2L}k'^2 \int_0^y \frac{\mathrm{d}t}{(1-t^2)^2} - \frac{1}{4L^2}k'^2 \int_0^y \frac{\mathrm{d}t}{1-t^2} + \\ & + k'^4 \int_0^y \mathrm{d}t \left[-\frac{1}{16}\frac{1-5t^2+2t^4}{(1-t^2)^2} - \frac{1}{32L}\frac{-3+14t^2+t^4}{(1-t^2)^3} + \frac{1}{128L^2}\frac{3+13t^2}{(1-t^2)^2} - \frac{1}{16L^3}\frac{1}{1-t^2} \right] \end{aligned} \tag{C.6}$$

which gives

$$\begin{aligned} f(y) = {} & y - \frac{1}{L}\mathrm{artanh}(y) + \frac{1}{2}k'^2 \left[-y + \frac{1}{2L}\left(\frac{y}{1-y^2} + \mathrm{artanh}(y)\right) - \frac{1}{2L^2}\mathrm{artanh}(y) \right] \\ & + \frac{1}{16}k'^4 \left[-\frac{y(1-2y^2)}{1-y^2} + \frac{1}{4L}\left(\frac{y(1-7y^2)}{(1-y^2)^2} + 5\mathrm{artanh}(y) \right) + \frac{1}{8L^2}\left(\frac{8y}{1-y^2} - 5\mathrm{artanh}(y) \right) \right. \\ & \left. - \frac{1}{L^3}\mathrm{artanh}(y) \right] + \frac{1}{32}k'^6 \left[-\frac{y(1-2y^2+2y^4)}{(1-y^2)^2} + \frac{1}{8L}\left(\frac{y(3-20y^2+57y^4)}{3(1-y^2)^3} + 11\mathrm{artanh}(y) \right) \right. \\ & \left. + \frac{1}{48L^2}\left(\frac{9y(5-9y^2)}{(1-y^2)^2} - 23\mathrm{artanh}(y) \right) - \frac{1}{8L^3}\left(\frac{4y}{1-y^2} - 9\mathrm{artanh}(y) \right) - \frac{1}{2L^4}\mathrm{artanh}(y) \right] + \ldots \end{aligned} \tag{C.7}$$

Using the results above, one can read off the expansions of the relevant quantities appearing in the static gauge fluctuation determinants

$$\begin{aligned} 2\mathbb{K}(k^2)\, Z(\alpha_\beta\,|\,k^2) \sim {} & \kappa_0\pi x - 2\tanh^{-1} x + \eta\, \frac{\left(\kappa_0\pi x^2 - x^2 - \kappa_0\pi + 2\right)}{\kappa_0\pi x\,(x^2-1)} + \\ & + \frac{\eta^2}{32\kappa_0^2\pi^2 x^3\,(x^2-1)^2} \Big[\kappa_0^3\pi^3 x^6 - 13\kappa_0^2\pi^2 x^6 + 21\kappa_0\pi x^6 - 4\kappa_0^3\pi^3 x^4 + 30\kappa_0^2\pi^2 x^4 + \\ & -71\kappa_0\pi x^4 + 32x^4 + 3\kappa_0^3\pi^3 x^2 - 19\kappa_0^2\pi^2 x^2 + 66\kappa_0\pi x^2 - 80x^2 - 16\kappa_0\pi + 32 \Big] + \ldots \end{aligned} \tag{C.8}$$

$$\begin{aligned} 2\mathbb{K}(\tilde{k}^2)\, Z(\alpha_\phi\,|\,\tilde{k^2}) \sim {} & \kappa_0\pi y - 2\tanh^{-1}\frac{y}{2} + \eta\, \frac{2(\kappa_0\pi - 1)}{\kappa_0\pi y} + \frac{\eta^2}{16\pi^2 y^3 \omega^2}\big(\kappa_0^3\pi^3 y^4 - 13\kappa_0^2\pi^2 y^4 + \\ & + 21\kappa_0\pi y^4 - 4\kappa_0^3\pi^3 y^2 + 56\kappa_0^2\pi^2 y^2 - 116\kappa_0\pi y^2 + 32y^2 + 128\kappa_0\pi - 128 \big) + \ldots \end{aligned} \tag{C.9}$$

$$\begin{aligned} \mathbb{K}(\bar{\nu})\, Z(\tilde{\alpha}) \sim {} & \kappa_0\pi z - \tanh^{-1} z + \eta\, \frac{(\kappa_0\pi - 1)}{2\kappa_0\pi z} + \frac{\eta^2}{64\kappa_0^2\pi^2 z^3\,(z^2-1)}\big(\kappa_0^3\pi^3 z^4 - 13\kappa_0^2\pi^2 z^4 + \\ & + 21\kappa_0\pi z^4 - \kappa_0^3\pi^3 z^2 + 14\kappa_0^2\pi^2 z^2 - 29\kappa_0\pi z^2 + 8z^2 + 8\kappa_0\pi - 8 \big) + \ldots \end{aligned} \tag{C.10}$$

where x, y, z are defined in (6.5)-(6.6)-(6.7).

Expansions of $\mathbb{K}$ and $\mathbb{E}$ $(k \to 0)$

The first few orders of the $\kappa \to 0$ expansions for the elliptic integrals $\mathbb{K}$ and $\mathcal{E}$ read

$$\mathbb{K}(k^2) = \frac{\pi}{2} + \frac{\pi}{8}k^2 + \frac{9\pi}{128}k^4 + \frac{25\pi}{512}k^6 + \frac{1225\pi}{32768}k^8 + O\left(k^9\right) \tag{C.11}$$

$$\mathbb{E}(k^2) = \frac{\pi}{2} - \frac{\pi}{8}k^2 - \frac{3\pi}{128}k^4 - \frac{5\pi}{512}k^6 - \frac{175\pi}{32768}k^8 + O\left(k^9\right) \tag{C.12}$$

Expansion of Jacobi Zeta function Z $(k \to 0)$

As efficient way to evaluate the small spin expansion ($k \to 0$, $\eta \to \infty$) presented in Section 7 is to first compute the expansion of $\partial Z(\alpha \,|\, k^2)/\partial\Omega$, where the dependence of Z on Ω is via α as solution of the (5.2)-(5.4), and then perform an indefinite integration over Ω.

Using (A.16) valid for $0 < \alpha < \mathbb{K}$, after some straightforward manipulations one can write

$$\frac{\partial\, Z(\alpha \,|\, k^2)}{\partial\Omega} = \frac{1}{\sqrt{1-\mathrm{sn}^2(\alpha \,|\, k^2)}\,\sqrt{1-k^2\,\mathrm{sn}^2(\alpha \,|\, k^2)}}\,\frac{\partial\,\mathrm{sn}(\alpha \,|\, k^2)}{\partial\Omega}\left[1-k^2\,\mathrm{sn}(\alpha \,|\, k^2)-\frac{\mathbb{E}}{\mathbb{K}}\right] \tag{C.13}$$

Considering the determinant for beta modes

$$\det\mathcal{O}_\beta = 4\sinh^2\left[2\mathbb{K}Z(\alpha|k^2)\right], \qquad \mathrm{sn}(\alpha|k^2) = \frac{1}{k}\sqrt{1+k^2+\left(\frac{\pi\Omega}{2\mathbb{K}}\right)^2} \tag{C.14}$$

one can see, that $\mathrm{sn}(\alpha \,|\, k^2) > 1$. So before proceeding with the short string expansion, one needs the following transformation:

$$\alpha = \beta + \mathbb{K} + i\mathbb{K}' \tag{C.15}$$

which gives

$$\mathrm{sn}(\beta|k^2) = \sqrt{\frac{k^2+\left(\frac{\pi\Omega}{2\mathbb{K}}\right)^2}{1+\left(\frac{\pi\Omega}{2\mathbb{K}}\right)^2}} < 1 \tag{C.16}$$

This affects the determinant in the following way:

$$\det\mathcal{O}_\beta = 4\sinh^2\left[2\mathbb{K}Z(\beta|k^2) - 2\mathbb{K}\frac{\mathrm{sn}(\beta|k^2)\mathrm{dn}(\beta|k^2)}{\mathrm{cn}(\beta|k^2)} - i\pi\right] \tag{C.17}$$

Substituting the explicit expression for $\mathrm{sn}(\beta \,|\, k^2)$ given by (C.16) into (C.13), expanding then for large η and integrating back in Ω, one can easily evaluate the expansions for the relevant determinants. In the case of the fluctuation β it is finally

$$\begin{aligned}\det\mathcal{O}_\beta \equiv\ & 4\sinh^2[2\,\mathbb{K}Z(\alpha_\beta \,|\, k^2)] = 4\sinh^2(\pi\Omega) + \frac{1}{\eta}\,\frac{2\pi\sinh(2\pi\Omega)}{\Omega} + \\ & + \frac{1}{\eta^2}\left[\frac{\pi^2\cosh(2\pi\Omega)}{\Omega^2} - \frac{\pi\left(3\Omega^4+6\Omega^2+2\right)\sinh(2\pi\Omega)}{4\left(\Omega^5+\Omega^3\right)}\right] + O\left(\frac{1}{\eta^3}\right)\,. \end{aligned} \tag{C.18}$$

Applying this approach to the fermion determinant

$$\det \mathcal{O}_\psi = -4\cosh^2\left[\tilde{\mathbb{K}} Z(\alpha|\tilde{k}^2)\right], \qquad \mathrm{sn}(\alpha|\tilde{k}^2) = \frac{1}{\tilde{k}}\sqrt{1+\left(\frac{\pi\Omega}{\tilde{\mathbb{K}}}\right)^2} \tag{C.19}$$

by using $\alpha = \beta + \tilde{\mathbb{K}} + i\tilde{\mathbb{K}}'$ with

$$\mathrm{sn}(\beta|\tilde{k}^2) = \sqrt{\frac{\left(\frac{\pi\Omega}{\tilde{\mathbb{K}}}\right)^2}{1-\tilde{k}^2+\left(\frac{\pi\Omega}{\tilde{\mathbb{K}}}\right)^2}} \tag{C.20}$$

gives

$$\begin{aligned}\det \mathcal{O}_\psi &= -4\cosh^2\left[\tilde{\mathbb{K}} Z(\beta|\tilde{k}^2) - \tilde{\mathbb{K}}\,\frac{\mathrm{sn}(\beta|\tilde{k}^2)\mathrm{dn}(\beta|\tilde{k}^2)}{\mathrm{cn}(\beta|\tilde{k}^2)} - \frac{i\pi}{2}\right] \\ &= 4\sinh^2\left[\tilde{\mathbb{K}} Z(\beta|\tilde{k}^2) - \tilde{\mathbb{K}}\frac{\mathrm{sn}(\beta|\tilde{k}^2)\mathrm{dn}(\beta|\tilde{k}^2)}{\mathrm{cn}(\beta|\tilde{k}^2)}\right],\end{aligned} \tag{C.21}$$

where the term $i\pi/2$ flips the cosh to sinh.The expansion of this expression in $1/\eta$ gives

$$\det \mathcal{O}_\psi(\Omega) = D_\psi^{(0)}(\Omega) + \frac{1}{\eta} D_\psi^{(1)}(\Omega) + \frac{1}{\eta^2} D_\psi^{(2)}(\Omega) + \ldots \tag{C.22}$$

with

$$D_\psi^{(0)}(\Omega) = 4\sinh(\pi\Omega), \tag{C.23}$$

$$D_\psi^{(1)}(\Omega) = \frac{4\pi\Omega\sinh(2\pi\Omega)}{1+4\Omega^2}, \tag{C.24}$$

$$D_\psi^{(2)}(\Omega) = \frac{4\pi^2\Omega^2\cosh(2\pi\Omega)}{(1+4\Omega^2)^2} - \frac{\pi\Omega(48\Omega^2(1+\Omega^2)+1)\sinh(2\pi\Omega)}{2(1+4\Omega^2)^3} \tag{C.25}$$

Appendix D: Exponentially suppressed contributions

As explained below (6.24) and around (6.35), in performing the large spin expansion on the exact determinants we systematically adopted an approximation based on the replacement $\tanh(\cdots) \to 1$. The neglected terms are exponential in the large quantity κ_0 and give back powers of η, see (6.35). Lacking a better complete control of this approximation, we present in this Appendix the evaluation of the leading large κ_0 correction to the one-loop energy due to the above replacement. It is clear that such leading correction come indeed from the fermion determinant, that has the following leading order in the formal small η, *i.e.* large spin, expansion

$$\mathcal{D}_{\psi,\mathrm{LO}} = \frac{1}{1-z^2}\,\left[1 - z\,\tanh(\pi\,\kappa_0\,z)\right]^2, \qquad z = \sqrt{1+\frac{\Omega^2}{\kappa_0^2}}. \tag{D.1}$$

and which we treated in the following approximated way

$$\mathcal{D}_{\psi,\text{LO}}^{\text{approx}} = \frac{1}{1-z^2}\,(1-z)^2 = \frac{1-z}{1+z}. \tag{D.2}$$

The effect of this approximation in the one-loop energy is

$$\Delta E_1 = -\frac{1}{4\,\pi\,\kappa}\cdot 8\cdot \int_{-\infty}^{\infty} d\Omega\,(\ln \mathcal{D}_{\psi,\text{LO}} - \ln \mathcal{D}_{\psi,\text{LO}}^{\text{approx}}) = -\frac{4\,\kappa_0}{\pi\,\kappa}\int_0^{\infty} d\overline{\Omega}\,F(\overline{\Omega};\kappa_0), \tag{D.3}$$

where

$$F(\overline{\Omega};\kappa_0) = \ln\left(\frac{1-\sqrt{1+\overline{\Omega}^2}\,\tanh(\pi\,\kappa_0\,\sqrt{1+\overline{\Omega}^2})}{1-\sqrt{1+\overline{\Omega}^2}}\right)^2. \tag{D.4}$$

The function $F(\overline{\Omega};\kappa_0)$ has the generic shape shown in Fig. (9). The point where it goes to $-\infty$ is where the numerator inside the logarithm vanishes. This happens at approximately

$$\overline{\Omega}^* \simeq 2\,e^{-\pi\,\kappa_0}. \tag{D.5}$$

The large κ_0 analysis must be done carefully since the two regions $\overline{\Omega} < \overline{\Omega}^*$ and $\overline{\Omega} > \overline{\Omega}^*$ contribute the above integrals with opposite signs and large cancellations.

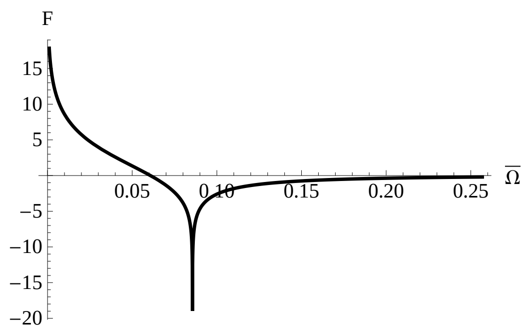

Figure 9: Shape of the function $F(\overline{\Omega};\kappa_0)$ at $\kappa_0 = 1$.

The final result is the estimate

$$\int_0^{\infty} d\overline{\Omega}\,F(\overline{\Omega};\kappa_0) = 8\,\pi\,\sqrt{\kappa_0}\,e^{-2\,\pi\,\kappa_0} + \cdots\ , \tag{D.6}$$

whose accuracy is shown in Fig. (10). In terms of the one-loop energy, it is

$$\Delta E_1 = -32\,\frac{\kappa_0^{3/2}}{\kappa}\,e^{-2\,\pi\,\kappa_0} + \cdots\ = \mathcal{O}(\eta^2\,\ln^{1/2}\eta). \tag{D.7}$$

The peculiar half-integer exponent of $\ln\eta$ suggest that a systematic resummation of these corrections is needed possibly taking into account mixing with similar terms coming from the

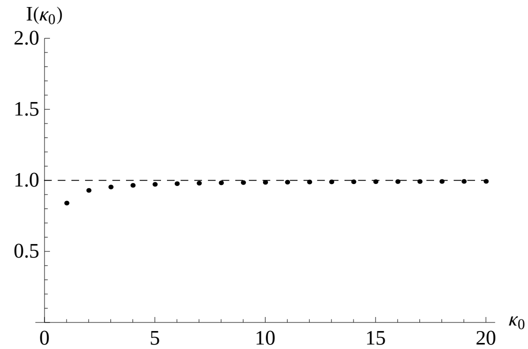

Figure 10: Numerical evaluation of the integral $I(\kappa_0) = \frac{1}{8\,\pi\,\sqrt{\kappa_0}}\, e^{2\pi\kappa_0} \int_0^\infty d\overline{\Omega}\, F(\overline{\Omega};\kappa_0)$ and comparison with the leading analytic prediction 1 (dashed line).

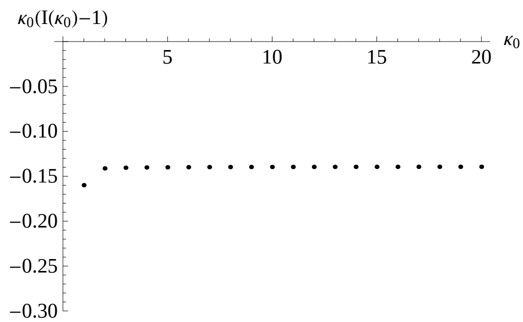

Figure 11: Numerical evaluation of the quantity $\kappa_0(I(\kappa_0) - 1)$.

expansion of Bessel functions K_1 in Eqs. (6.12)-(6.15). We did not evaluate the next correction, but Fig. (11) suggest that

$$\int_0^\infty d\overline{\Omega}\, F(\overline{\Omega};\kappa_0) = \left(8\,\pi\,\sqrt{\kappa_0} + \frac{a}{\sqrt{\kappa_0}} + \cdots\right) e^{-2\,\pi\,\kappa_0} + \cdots\ , \tag{D.8}$$

for some constant a. This term has the same form as the Bessel K_1 corrections.

Appendix E: Large spin limit with a different approximation

The analysis of the large spin limit in the Section 6, leading to the large spin limit expansion (6.36), agrees extremely well with the exact numerical dependence of the one-loop energy on the elliptic parameter k, as can be seen from Figure 7. However, this expansion neglected exponential terms in the $k \to 1$ limit, as explained in Section 6.1, and the significance of these neglected terms at smaller values of k is not clear. In this section we consider another type of approximation, that leads to a different type of large spin expansion of the one-loop energy.

Start from the effective action

$$\Gamma = -\frac{\mathcal{T}}{4\pi}\int_{\mathbb{R}} d\Omega\, \log\frac{\det^8\mathcal{O}_\psi}{\det\mathcal{O}_\phi\,\det^2\mathcal{O}_\beta\,\det^5\mathcal{O}_0}, \tag{E.1}$$

and integrate by parts (we set also $\mathcal{T}=1$)

$$\begin{aligned}\Gamma &= \frac{1}{4\pi}\int_{\mathbb{R}} d\Omega\,\Omega\,\partial_\Omega\,[8\,\log\det\mathcal{O}_\psi - \log\det\mathcal{O}_\phi - 2\,\log\det\mathcal{O}_\beta - 5\,\log\det\mathcal{O}_0] \\ &\equiv 8\,\Gamma_\psi - \Gamma_\phi - 2\,\Gamma_\beta - 5\,\Gamma_0.\end{aligned} \tag{E.2}$$

We now systematically apply the approximate substitutions

$$\log(4\,\sinh^2 x)\to 2|x| \quad , \qquad \log(4\,\cosh^2 x)\to 2x \quad , \tag{E.3}$$

which also correspond to neglecting exponential terms, but now the approximation is made **before** integrating over Ω. Then, the effective action can be computed exactly, *i.e.* without any leftover integral. As an example, let us consider the contribution from the ϕ mode. We convert the integral over Ω into an integral over the spectral parameter α_ϕ as follows:

$$\begin{aligned}\Gamma_\phi &= \frac{1}{4\pi}\int_{-\infty}^{\infty} d\Omega\,\Omega\,\partial_\Omega\,\log(4\,\sinh^2(2\,\widetilde{\mathbb{K}}\,Z(\alpha_\phi\,|\,\widetilde{k}^2)) \\ &\simeq \frac{2\,\widetilde{\mathbb{K}}}{\pi}\int_0^{\infty} d\Omega\,\Omega\,\partial_\Omega Z(\alpha_\phi\,|\,\widetilde{k}^2)) \\ &= \frac{2\,\widetilde{\mathbb{K}}}{\pi}\int_{\alpha_0}^{i\,\widetilde{\mathbb{K}}'} d\alpha\,\Omega\,\partial_\alpha Z(\alpha_\phi\,|\,\widetilde{k}^2) \\ &= \left(\frac{2\,\widetilde{\mathbb{K}}}{\pi}\right)^2\int_{\alpha_0}^{i\,\widetilde{\mathbb{K}}'} d\alpha\,\widetilde{\Omega}\,\left[\mathrm{dn}^2(\alpha\,|\,\widetilde{k}^2) - \frac{\widetilde{\mathbb{E}}}{\widetilde{\mathbb{K}}}\right],\end{aligned} \tag{E.4}$$

where

$$\widetilde{\Omega} = \sqrt{-\mathrm{dn}^2(\alpha\,|\,\widetilde{k}^2)}, \tag{E.5}$$

and α_0 is the value associated with $\Omega = 0$. The spectral parameter α_ϕ takes values along the straight line joining $\widetilde{\mathbb{K}} + i\widetilde{\mathbb{K}}'$ to $i\widetilde{\mathbb{K}}'$, so it is convenient to write

$$\alpha = \widetilde{\mathbb{K}} + i\,\widetilde{\mathbb{K}}' - \beta, \tag{E.6}$$

and use the identity

$$\mathrm{dn}^2(\widetilde{\mathbb{K}} + i\,\widetilde{\mathbb{K}}' - \beta\,|\,\widetilde{k}^2) = (\widetilde{k}^2 - 1)\,\mathrm{sc}^2(\beta\,|\,\widetilde{k}^2). \tag{E.7}$$

We find

$$\Gamma_\phi \simeq \widetilde{k}'\left(\frac{2\,\widetilde{\mathbb{K}}}{\pi}\right)^2\int_0^{\widetilde{\mathbb{K}}} d\beta\,\mathrm{sc}(\beta\,|\,\widetilde{k}^2)\,\left[\frac{\widetilde{\mathbb{E}}}{\widetilde{\mathbb{K}}} + \widetilde{k}^{2\prime}\,\mathrm{sc}^2(\beta\,|\,\widetilde{k}^2)\right]. \tag{E.8}$$

A similar treatment leads to

$$\Gamma_\psi \simeq \frac{1}{4}\,\Gamma_\phi \quad , \tag{E.9}$$

within this approximation, and the following contribution from the β modes

$$\Gamma_\beta \;\simeq\; \left(\frac{2\,\mathbb{K}}{\pi}\right)^2 \int_{\mathrm{sn}^{-1}(k\,|\,k^2)}^{\mathbb{K}} d\beta\,\sqrt{(1-k^2)\,\mathrm{nc}^2(\beta\,|\,k^2)-1}\;\left[\frac{\mathbb{E}}{\mathbb{K}}+(1-k^2)\,\mathrm{sc}^2(\beta\,|\,k^2)\right]. \tag{E.10}$$

It is convenient to introduce the variables

$$s_\phi = \mathrm{sn}(\beta\,|\,\widetilde{k}^2), \qquad s_\beta = \mathrm{sn}(\beta\,|\,k^2). \tag{E.11}$$

The integrals can be computed in closed-form, and after a long calculation, one finds

$$\Gamma_\phi \;=\; -\frac{(k-1)^2}{\pi^2}\,\mathbb{K}^2\,\frac{1}{s_\phi-1}+\frac{4(\mathbb{K}-\mathbb{E})\mathbb{K}}{\pi^2}\,\log(1-s_\phi)+\Gamma_\phi^{\mathrm{finite}}, \tag{E.12}$$

$$\Gamma_\beta \;=\; \frac{k^2-1}{\pi^2}\,\mathbb{K}^2\,\frac{1}{s_\beta-1}+\frac{2(\mathbb{K}-\mathbb{E})\mathbb{K}}{\pi^2}\,\log(1-s_\beta)+\Gamma_\beta^{\mathrm{finite}}. \tag{E.13}$$

The integrals are divergent at $s_\phi, s_\beta \to 1$, which is simply the individual UV divergence. Introducing a cut-off $\Omega_{\max}$ with

$$s_{\beta,\max} \;=\; 1-\frac{2\,\mathbb{K}^2(k^2)}{\pi^2\,\Omega_{\max}^2}\,(1-k^2)+\cdots\,, \tag{E.14}$$

$$s_{\phi,\max} \;=\; 1-\frac{2\,\widetilde{\mathbb{K}}^2(\widetilde{k}^2)}{\pi^2\,\Omega_{\max}^2}\,(1-\widetilde{k}^2)+\cdots\,, \tag{E.15}$$

$$s_{\psi,\max} \;=\; 1-\frac{\widetilde{\mathbb{K}}^2(k^2)}{2\,\pi^2\,\Omega_{\max}^2}\,(1-\widetilde{k}^2)+\cdots\,. \tag{E.16}$$

one checks that the pole cancels against the free field contribution which is

$$\begin{aligned}\Gamma_0 \;&=\; \frac{1}{4\,\pi}\int_{-\Omega_{\max}}^{\Omega_{\max}} d\Omega\,\Omega\,\partial_\Omega\,\log(4\,\sinh^2(\pi\,\Omega))\\ &\simeq\; \int_0^{\Omega_{\max}} d\Omega\,\Omega = \frac{1}{2}\,\Omega^2_{\max.}\end{aligned} \tag{E.17}$$

The logarithmic divergence cancels in the sum of the various mode contributions. The full result is quite compact and reads

$$\begin{aligned}\Gamma \;\simeq\; &\frac{\mathbb{K}(k^2)}{\pi^2\mathbb{K}(k^4)}\Big[4\mathbb{K}(k^2)\mathbb{E}(k^4)\,\mathrm{Im}\,\mathbb{F}\Big(\arcsin\frac{1}{k}\,|\,k^4\Big)+\mathbb{K}(k^2)\Big(\pi+2\mathbb{K}(k^4)\,(k^2-2\ln k-1-6\ln 2)\Big)+\\ &+4\mathbb{K}(k^4)\mathbb{E}(k^2)\left(\ln(8k)-(1+k^2)\,\mathrm{Im}\,\mathbb{F}\Big(\arcsin\frac{1}{k}\,|\,k^4\Big)\right)\Big].\end{aligned} \tag{E.18}$$

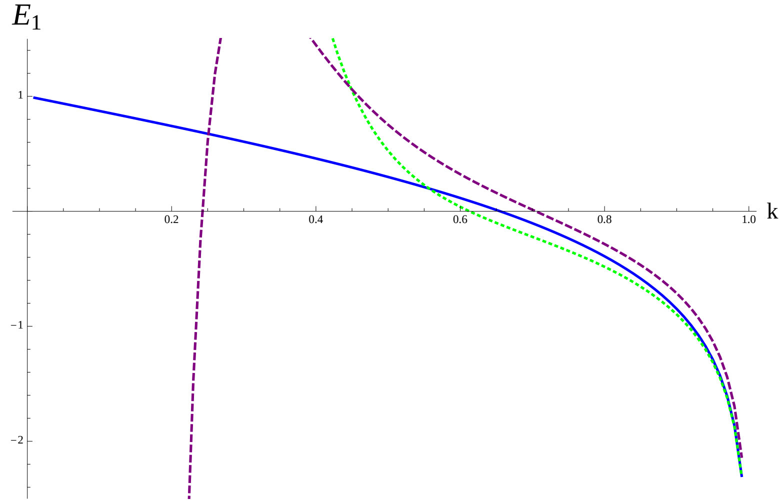

Figure 12: Plots of E_1 as a function of k: the blue (solid) curve is found numerically from the exact expression (5.18) for generic values of k; the green (dotted) curve is found from an analytic expansion in the $k \to 1$ or large spin limit, using the first two terms in (6.36); the purple (dashed) curve is the first two terms in the alternative $k \to 1$ expansion (E.19).

Since this compact expression is written in closed-form in terms of elliptic functions, its $k \to 1$ expansion can be computed and the result to order $\mathcal{O}(\eta^4)$ is

$$
\begin{aligned}
\Gamma \;\simeq\;& \kappa_0\left(1+\frac{6\log 2}{\pi}\right)-3\kappa_0^2\log 2+\\
&+\left[\kappa_0\left(-\frac{1}{\pi}-\frac{3\log 2}{\pi}\right)+\frac{1}{2\pi}+\frac{3\log 2}{\pi^2}\right]\eta+\\
&+\left[\kappa_0^2\left(-\frac{\pi}{32}-\frac{3\log 2}{32}\right)+\kappa_0\left(\frac{1}{16}+\frac{1}{2\pi}+\frac{39\log 2}{32\pi}\right)-\frac{13}{64\pi}-\frac{1}{2\pi^2}-\frac{63\log 2}{32\pi^2}\right]\eta^2+\\
&+\left[\kappa_0^2\left(\frac{1}{96}+\frac{\pi}{32}+\frac{3\log 2}{32}\right)+\kappa_0\left(-\frac{3}{32}-\frac{1}{3\pi}-\frac{13\log 2}{16\pi}\right)+\frac{29}{192\pi}+\frac{29}{64\pi^2}+\frac{85\log 2}{64\pi^2}\right]\eta^3+\\
&+\left[\kappa_0^2\left(-\frac{1}{64}-\frac{115\pi}{4096}-\frac{693\log 2}{8192}\right)+\kappa_0\left(\frac{51}{512}+\frac{25}{96\pi}+\frac{10263\log 2}{16384\pi}\right)+\right.\\
&\left.-\frac{4397}{32768\pi}-\frac{149}{384\pi^2}-\frac{16403\log 2}{16384\pi^2}\right]\eta^4+\cdots
\end{aligned}
\tag{E.19}
$$

Recalling that $E_1 = \Gamma/\kappa$, we see that this expansion is very similar to, but not precisely the same as, the $k \to 1$ expansion found in (6.36). A comparison of these two different approximations is presented in Figure 12. Both agree with the exact result at large spin, but we see that the expansion in (6.36) provides a better approximation as $k \to 1$. Nevertheless, the approximation considered in this appendix may be of interest as it provides a closed-form expression.

Appendix F: Static gauge determinants in perturbation theory

We repeat here an evaluation of the determinants in the short string limit adopting the standard perturbation theory method of [9].

The basic idea is to compute

$$\begin{aligned} \ln\det(-\partial^2+\Omega^2+V) &= \ln\det(-\partial^2+\Omega^2)+\mathrm{Tr}\Big(\frac{1}{-\partial^2+\Omega^2}\,V\Big) \\ &\quad -\frac{1}{2}\mathrm{Tr}\Big(\frac{1}{-\partial^2+\Omega^2}\,V\,\frac{1}{-\partial^2+\Omega^2}\,V\Big)+\cdots\ , \end{aligned} \tag{F.1}$$

evaluating the traces on the following basis $|n\rangle = \frac{1}{\sqrt{2\pi}}\,e^{i\,n\,\sigma}$, $n\in\mathbb{Z}$, $\sigma\in[0,2\pi]$. In all cases, we define

$$V_{n,m} = \frac{1}{2\,\pi}\int_0^{2\pi} d\sigma e^{i\,(m-n)\,\sigma}\,V(\sigma) \tag{F.2}$$

and also

$$\ln\det\mathcal{O}_f = \ln\det(-\partial^2+\Omega^2)+X_f, \qquad\qquad f=(\beta,\phi,\psi). \tag{F.3}$$

For the β *mode*, we have (see (3.33) and the rescaling (3.35))

$$V_\beta = \Big(\frac{2\,\mathbb{K}(k^2)}{\pi}\Big)^2\,2\,k^2\,\mathrm{sn}^2\big(\frac{2\,\mathbb{K}(k^2)}{\pi}\,\sigma\,|\,k^2\big) = \frac{2}{\eta}\sin^2\sigma+\cdots\ , \tag{F.4}$$

and the non zero matrix elements

$$V_{n,n} = \frac{1}{\eta}, \qquad V_{n,n\pm 2} = -\frac{1}{2\eta}. \tag{F.5}$$

Thus

$$X_\beta = \frac{1}{\eta}\sum_{n\in\mathbb{Z}}\frac{1}{n^2+\Omega^2}. \tag{F.6}$$

For the ϕ *mode*, we have (see (A.21) and the rescaling (3.36) [20])

$$V_\phi = \Big(\frac{2\,\mathbb{K}(\tilde{k}^2)}{\pi}\Big)^2\,\Big[2\,k^2\,\mathrm{sn}^2\big(\frac{2\,\mathbb{K}(\tilde{k}^2)}{\pi}\,\sigma\,|\,\tilde{k}^2\big)-\tilde{k}^2\Big] = -\frac{4}{\sqrt{\eta}}\cos(2\sigma)+\frac{4}{\eta}\sin^2(2\sigma)+\cdots\ , \tag{F.7}$$

and the non zero matrix elements

$$V_{n,n} = \frac{2}{\eta}, \qquad V_{n,n\pm 2} = \frac{2}{\sqrt{\eta}}, \qquad V_{n,n\pm 4} = -\frac{1}{\eta}. \tag{F.8}$$

[20] The imaginary shift in the argument of the elliptic sine in (A.21) is irrelevant for the determinant calculation since it does not change the monodromy. For the check here proposed it is useful to consider the analytically continued potential (F.7).

Thus

$$X_\phi = \frac{2}{\eta}\sum_{n\in\mathbb{Z}} \frac{1}{n^2+\Omega^2}\left(1-\frac{1}{(n+2)^2+\Omega^2}-\frac{1}{(n-2)^2+\Omega^2}\right). \tag{F.9}$$

For the ψ *mode* (see (A.30)-(A.31) and (3.37)), we have

$$V_\psi = -\frac{1}{\sqrt{\eta}}\cos\sigma + \frac{1}{\eta}\sin^2\sigma + \cdots\ , \tag{F.10}$$

and the non zero matrix elements

$$V_{n,n} = \frac{1}{2\eta}, \qquad V_{n,n\pm 1} = -\frac{1}{2\sqrt{\eta}}, \qquad V_{n,n\pm 2} = -\frac{1}{4\eta}. \tag{F.11}$$

Thus

$$X_\psi = \frac{1}{2\eta}\sum_{n\in\mathbb{Z}} \frac{1}{n^2+\Omega^2}\left(1-\frac{1}{4}\frac{1}{(n+1)^2+\Omega^2}-\frac{1}{4}\frac{1}{(n-1)^2+\Omega^2}\right). \tag{F.12}$$

For the combination, we have

$$\begin{aligned} 8\,X_\psi - 2\,X_\beta - X_\phi \ &= \ -\frac{1}{\eta}\sum_{n\in\mathbb{Z}} \frac{1}{n^2+\Omega^2}\left(\frac{1}{(n+1)^2+\Omega^2}+\frac{1}{(n-1)^2+\Omega^2}\right)+ \\ &\quad +\frac{2}{\eta}\sum_{n\in\mathbb{Z}} \frac{1}{n^2+\Omega^2}\left(\frac{1}{(n+2)^2+\Omega^2}+\frac{1}{(n-2)^2+\Omega^2}\right). \end{aligned}$$

Evaluating the infinite sums over n, we find

$$8\,X_\psi - 2\,X_\beta - X_\phi = \frac{1}{\eta}\,\frac{2\pi\left(2\Omega^2-1\right)\coth(\pi\Omega)}{\Omega\,(\Omega^2+1)\,(4\Omega^2+1)}. \tag{F.13}$$

This is the same as Eq. (7.4) showing that the old-fashioned way of calculation is in perfect agreement with the the procedure adopted in this paper.

Comparing now this with the calculation in conformal gauge of [9], one can see that the difference is due to the second order contribution of the $1/\sqrt{\eta}$ term from the ϕ field. It is

$$\frac{2}{\eta}\sum_{n\in\mathbb{Z}} \frac{1}{n^2+\Omega^2}\left(\frac{1}{(n+2)^2+\Omega^2}+\frac{1}{(n-2)^2+\Omega^2}\right) = \frac{2\pi\coth(\pi\Omega)}{\Omega\,(\Omega^2+1)}. \tag{F.14}$$

Now, reconsider Eq. (3.32) of [9] which reads

$$\int_{\mathbb{R}} d\Omega \sum_{n\in\mathbb{Z}}\left[\frac{2}{n^2+\Omega^2}-\frac{i\,\Omega}{n^2+(\Omega+i)^2}+\frac{i\,\Omega}{n^2+(\Omega-i)^2}\right] = 0, \tag{F.15}$$

upon doing a shift in Ω in the last two terms. Actually, one could perform first the sum over modes of the above integrand thus getting

$$\sum_{n\in\mathbb{Z}}\left[\frac{2}{n^2+\Omega^2}-\frac{i\,\Omega}{n^2+(\Omega+i)^2}+\frac{i\,\Omega}{n^2+(\Omega-i)^2}\right]=\frac{2\pi\coth(\pi\Omega)}{\Omega\,(\Omega^2+1)} \tag{F.16}$$

This means that avoiding the shifts in Eq. (3.32) of [9] one recovers full equality with the static gauge.

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