Gauge invariant finite size spectrum of the giant magnon

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Abstract

It is shown that the finite size corrections to the spectrum of the giant magnon solution of classical string theory, computed using the uniform light-cone gauge, are gauge invariant and have physical meaning. This is seen in two ways: from a general argument where the single magnon is made gauge invariant by putting it on an orbifold as a wrapped state obeying the level matching condition as well as all other constraints, and by an explicit calculation where it is shown that physical quantum numbers do not depend on the uniform light-cone gauge parameter. The resulting finite size effects are exponentially small in the R-charge and the exponent (but not the prefactor) agrees with gauge theory computations using the integrable Hubbard model.

The problem of computing conformal dimensions in planar $\mathcal{N}=4$ Yang-Mills theory has a beautiful analog as a spin chain which is thought to be integrable [1][2][3][4]. In the limit of large R-charge J, the dynamics of the chain are greatly simplified and, in the context of integrability, can be viewed as magnons which propagate on an infinite line and interact with each other with a factorized S-matrix [5]. The string theory dual of the magnon, called the giant magnon was found by Hofman and Maldacena [6] and has attracted a significant amount of attention [7]-[29]. Being a solution of classical string theory, by AdS/CFT duality, it is relevant to the limit of large 't Hooft coupling λ as well as large J.

Integrability suggests a dispersion relation for a single magnon [30]

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_{mag}}{2}} \tag{1}$$

This is confirmed at lower orders in Yang-Mills theory, is predicted by current integrability *ansatze* and it also agrees with the energy of the giant magnon at the limit of infinite λ. There is a question as to whether, in between these limits, λ could be replaced by a function of λ that has this strong and weak coupling behavior. Speculation using the fact that the single giant magnon state is a BPS state of a certain modified version of the supersymmetry algebra [31][21] and therefore could be protected by quantum corrections suggests that (1) is indeed exact, in the infinite volume limit $J \to \infty$.

An interesting question is whether there are finite size corrections to the magnon spectrum when J is finite. This has been studied in various contexts [32] - [36][10][11][23]. Leading finite J corrections to the classical giant magnon were computed in a beautiful paper, Ref. [10]. A striking result was an apparent dependence of all but the leading order on the uniform light-cone gauge parameter. The authors came to the conclusion that the finite size corrections were not gauge invariant. The reason why worldsheet reparameterization invariance is suspect is that one of the Virasoro constraints, the level matching condition, is modified.

In this Letter, we shall re-examine this issue. We revisit the explicit computation in the uniform light-cone gauge [37] which was presented in Ref. [10]. We shall differ in the conclusion: we find that the finite size spectrum is independent of the gauge parameter. The cancelation of the gauge parameter, which we shall find explicitly, is intricate. Our reason for suspecting it at all is that, with very little modification of the classical string theory analysis, rather than finding the giant magnon as a state of closed string theory on $R^1 \times S^2$ where the angle coordinate is left open, $\Delta\phi = p_{mag}$, we can consider a wrapped closed string on an orbifold $R^1 \times S^2/Z_M$, where the orbifold identification is $\phi \sim \phi + 2\pi\frac{m}{M}$. The wrapped closed string obeys all of the Virasoro constraints and therefore should be gauge invariant. If we identify p_{mag} with $2\pi\frac{m}{M}$, there is virtually no difference between the mathematical problems of finding the classical giant magnon on the un-orbifold and the classical wrapped string on the orbifold. Therefore we would conclude that the giant magnon spectrum cannot depend on the gauge parameter either.

The gauge theory dual of the orbifold theory is well known [38]. It is an $\mathcal{N}=2$ quiver gauge theory that is obtained from $\mathcal{N}=4$ by a standard orbifold construction [39]. The scalar fields Z and Φ which make a single magnon operator in $\mathcal{N}=4$ Yang-Mills theory, $....ZZZ\Phi ZZZ...$ are dissected by orbifolding to a set of bi-fundamental $Z\to\{A_1,...,A_M\}$ and adjoint $\Phi\to\{\Phi_1,...,\Phi_M\}$ scalars in a quiver gauge theory. The $\frac{1}{2}$-BPS state analogous to $\mathrm{Tr}Z^J$ is $\mathrm{Tr}(A_1...A_M)^k$. Here $J=kM$ obeys the quantization rule that one would expect for angular momentum on the orbifold (compared to $J\sim$ integer on the non-orbifold).

There exists a single impurity one-"magnon" state,

$$\sum_{I=1}^{M} e^{2\pi i\frac{m}{M}I}\mathrm{Tr}[A_1...A_{I-1}\Phi_I A_I...A_M(A_1...A_M)^{k-1}]$$

with magnon momentum $p_{mag}=2\pi\frac{m}{M}$. Its spectrum can be computed in perturbation theory [38, 40, 36] and agrees with (1) at least up to two loops (even for finite J). There is also a version of the BMN limit[38] which agrees with the analogous limit of (1). Under AdS/CFT duality, the magnon momentum is dual to string wrapping number (m) and at strong coupling this magnon becomes a classical wrapped string on the orbifold, which we have argued is mathematically identical to the giant magnon and therefore also has spectrum matching the large λ limit of (1). In the string dual, finding a relationship between energy and wrapping number of the string yields a prediction of the strong coupling limit of the energy-momentum dispersion relation of the gauge theory magnon.

The study of this orbifold one-magnon state is interesting in its own right as a twisted state of a spin chain[40, 41, 42, 43]. Here, we have considered it only to illustrate the fact that there is a sensible one-magnon state which is dual to the wrapped string. As far as it is known, its dispersion relation is identical to (1).

We now turn to the classical giant magnon. For convenience of the reader, we will follow the conventions and notation of Ref. [10]. The giant magnon lives on a $R^1\times S^2$ subspace of $AdS_5\times S^5$. The metric of $R^1\times S^2$ is

$$ds^2=\sqrt{\lambda}\alpha' G_{MN}x^M x^N=\sqrt{\lambda}\alpha'\left[-dt^2+(1-z^2)d\phi^2+\frac{1}{1-z^2}dz^2\right]\ , \tag{2}$$

The string action is

$$S=-\frac{\sqrt{\lambda}}{4\pi}\int_{-r}^{r}d\sigma d\tau\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN}, \tag{3}$$

with $-r\le\sigma\le r$, $h_{\alpha\beta}$ the world-sheet metric and $X^M=\{t,\phi,z\}$. Conjugate momenta are

$$p_M=\frac{2\pi}{\sqrt{\lambda}}\frac{\delta S}{\delta\dot{X}^M}=-\sqrt{-h}h^{0\beta}\partial_\beta X^N G_{MN} \tag{4}$$

and the phase space action is

$$S=\frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r}d\sigma d\tau\left(p_M\dot{X}^M+\frac{h^{01}}{h^{00}}C_1+\frac{1}{2\sqrt{-h}h^{00}}C_2\right) \tag{5}$$

where the world-sheet metric forms Lagrange multipliers enforcing Virasoro constraints,

$$C_1=p_M X'^M=0,\qquad C_2=G^{MN}p_M p_N+X'^M X'^N G_{MN}=0 \tag{6}$$

Translations along t and ϕ are isometries resulting in Noether charges

$$E = -\frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_t, \qquad J = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_\phi \tag{7}$$

We will use the light-cone coordinates and momenta

$$x_- = \phi - t \qquad x_+ = (1-a)t + a\phi \tag{8}$$
$$p_- = p_\phi + p_t \qquad p_+ = (1-a)p_\phi - ap_t \tag{9}$$

where a is a parameter in the range $0 \leq a \leq 1$. The light-cone charges are

$$P_- = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_- = J - E, \qquad P_+ = \frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r} d\sigma p_+ = (1-a)J + aE \tag{10}$$

In light-cone-gauge, x_+ will be identified with world-sheet time. There is a subtlety here. When $a > 0$, x_+ in (8) contains ϕ which has boundary conditions depending on the specific problem that we want to consider and which must be carefully taken into account. For a closed string, ϕ is a periodic variable, $\phi \sim \phi + 2\pi m$. For the magnon,

$$\phi(\tau, r) - \phi(\tau, -r) = p_{mag} \tag{11}$$

Consider a Z_M orbifold of S^2 where the action of the orbifold group is $\phi \sim \phi + \frac{2\pi}{M}$. [1] The coordinate of an (m-times) wrapped string must obey $\phi(\tau, r) - \phi(\tau, -r) = 2\pi\frac{m}{M}$. The analogy between the orbifold and the giant magnon identifies p_{mag} with $2\pi\frac{m}{M}$. With this identification, the following analysis is identical when applied to either case.

In the uniform light-cone gauge, a conformal transformation is used to set

$$x_+ = \tau + a\frac{p_{mag}}{2r}\sigma \equiv \tau + aA\sigma \quad \text{and} \quad p_+ = 1, \tag{12}$$

where the σ-dependent part of x_+ is necessary to satisfy (11) and we shall denote $A \equiv \frac{p_{mag}}{2r}$. Retaining and dealing with the term with $aA\sigma$ is essentially the only difference between the remainder of the following and the analogous development in Ref. [10]. We shall find that it plays an important role in making the spectrum gauge invariant.

Consistency of the gauge choice (12) requires

$$2r = \frac{2\pi}{\sqrt{\lambda}}P_+ \equiv \int_{-r}^{r} d\sigma p_+ = \frac{2\pi}{\sqrt{\lambda}}\left(J + a(E - J)\right) \tag{13}$$

In addition, the Virasoro constraint C_1 implies

$$\int_{-r}^{r} d\sigma \left(p_+ x_-' + p_- x_+' + p_z z'\right) = 0 \quad \rightarrow \quad p_{mag} = -\int_{-r}^{r} d\sigma (aAp_- + p_z z') = \int_{-r}^{r} d\sigma x_-' \tag{14}$$

[1] Generally, the orbifold identification also acts on other angles on the $S^5 \subset AdS_5 \times S^5$. Different choices leave different amounts of residual supersymmetry [38, 44, 45].

Note that this is the level matching condition that is implied by the Virasoro constraints. We emphasize that it is important to keep A on the right-hand-side with $A = \frac{p_{mag}}{2r} \neq 0$.

Solving the Virasoro constraint C_1 for x'_- and substituting x'_- in C_2 provides a quadratic equation for p_-, whose solution is

$$p_-(p_z, z, z') = \frac{1-(1-a)z^2}{1-2a-(1-a)^2 z^2} + \frac{aA\,(1-z^2)\,z' p_z}{1-(1-z^2)\,a^2 A^2}$$
$$- \frac{\sqrt{1+(1-z^2)[(1-2a-(1-a)^2 z^2)p_z^2 - a^2 A^2]}\sqrt{(1-z^2)[1-(1-z^2)a^2A^2]+[1-2a-(1-a)^2z^2]z'^2}}{[1-2a-(1-a)^2z^2][1-(1-z^2)a^2A^2]} \tag{15}$$

Then the action is

$$S = \frac{\sqrt{\lambda}}{2\pi}\int d\tau \int_{-r}^{r} d\sigma \left[p_z \dot{z} + p_-(p_z, z, z')\right] \tag{16}$$

It is convenient to return to the coordinate space description of the system. The Hamilton equation of motion $\dot{z} = -\frac{\partial}{\partial p_z} p_-(p_z, z, z')$ yields

$$p_z = \frac{\frac{\dot{z}}{1-z^2} + aA(z' - aA\dot{z})}{\sqrt{(1-z^2) + [1-2a-(1-a)^2z^2]\left[(z'-aA\dot{z})^2 - \frac{\dot{z}^2}{1-z^2}\right]}} \tag{17}$$

Substituting, we get

$$S = \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \left[\frac{1-(1-a)z^2}{1-2a-(1-a)^2z^2} - \frac{\sqrt{(1-z^2) + [1-2a-(1-a)^2z^2]\left[(z'-aA\dot{z})^2 - \frac{\dot{z}^2}{1-z^2}\right]}}{1-2a-(1-a)^2z^2}\right] \tag{18}$$

The giant magnon is a left-moving soliton

$$z(\tau, \sigma) = z(\eta) \quad \text{where} \quad \eta = \sigma - v x_+ = \sigma - v(\tau + aA\sigma)$$

Note the appearance of the wrapping number in the solution (which could always be absorbed by re-scaling v). With this *ansätz*,

$$z' = (1-aAv)\partial_\eta z\ , \quad \dot{z} = -v\partial_\eta z\ , \quad z' - aA\dot{z} = \partial_\eta z \equiv \partial z$$

and the reduced Lagrangian is

$$L = \left[\frac{1-(1-a)z^2}{1-2a-(1-a)^2z^2} - \frac{\sqrt{(1-z^2) + [1-2a-(1-a)^2z^2]\left(1-\frac{v^2}{1-z^2}\right)(\partial z)^2}}{1-2a-(1-a)^2z^2}\right] \tag{19}$$

There is a constant of motion, corresponding to the symmetry of the reduced system under translations of η,

$$\frac{\omega-1}{1-a+a\omega}=\partial z\frac{\partial}{\partial(\partial z)}L-L=\frac{1}{1-2a-(1-a)^2z^2}\Big[-1+(1-a)z^2 + \frac{1-z^2}{\sqrt{(1-z^2)+[1-2a-(1-a)^2z^2]\left(1-\frac{v^2}{1-z^2}\right)(\partial z)^2}}\Big] \tag{20}$$

where we have set it equal to the judiciously chosen a-dependent constant introduced in Ref. [10]. We can solve this equation as

$$(\partial z)^2=\frac{(1-z^2)^2}{\left[(1-a)\left(1-z^2+\frac{a}{(1-a)\omega}\right)\right]^2}\frac{z^2-1+\frac{1}{\omega^2}}{1-v^2-z^2} \tag{21}$$

The parameters

$$z_{min}=\sqrt{1-\frac{1}{\omega^2}},\qquad z_{max}=\sqrt{1-v^2} \tag{22}$$

are turning points at which the ∂z vanishes or diverges, respectively. On the solution (21), $-p_-$ and p_z read

$$-p_-=\Big\{\Big[1-z^2-\Big(1-z^2+v^2(\omega-1)\Big)\omega\Big]+a(1-v^2-z^2)(\omega-1)\Big(1-(1-z^2)\omega\Big) -aAv(1-z^2)\Big(1-(1-z^2)\omega\Big)\Big\}\frac{1}{(1-v^2-z^2)\left[\omega(1-a)(1-z^2)+a\right]\left[1+a(\omega-1)\right]} \tag{23}$$

$$|p_z|=\frac{\omega\left[v-aA(1-z^2)\right]}{(1-z^2)(1-a+a\omega)}\frac{\sqrt{z^2-z_{min}^2}}{\sqrt{z_{max}^2-z^2}} \tag{24}$$

We can use Eqs. (21),(23) and (24) to evaluate

$$\frac{\pi}{\sqrt{\lambda}}P_+ = r=\int_0^r d\sigma=\int_{z_{min}}^{z_{max}}\frac{dz}{|z'|}=\frac{1}{1-aAv}\int_{z_{min}}^{z_{max}}\frac{dz}{|\partial z|} \tag{25}$$

$$E-J = -\frac{\sqrt{\lambda}}{2\pi}\int_{-r}^{r}d\sigma p_-=-\frac{\sqrt{\lambda}}{\pi(1-aAv)}\int_{z_{min}}^{z_{max}}\frac{p_-}{|\partial z|}dz \tag{26}$$

$$p_{mag} = -aA\int_{-r}^{r}p_-d\sigma-\int_{-r}^{r}p_zz'd\sigma=-\frac{2aA}{1-aAv}\int_{z_{min}}^{z_{max}}dz\frac{p_-}{|\partial z|}+2\int_{z_{min}}^{z_{max}}dz|p_z| \tag{27}$$

These expressions have a complicated dependence on the gauge parameter. For $A=0$ they coincide with the results for the physical quantities found in [10]. For a hint as to how the parameter will cancel here, consider (25) and recall that $A=\frac{\sqrt{\lambda}p_{mag}}{2\pi P_+}$ and

$P_+ = J + a(E-J)$. Multiplying (25) by a factor of $(1-aAv)$, we find that it can be written as an equation that is linear in a:

$$\tilde{J} + a\left(\tilde{E} - \tilde{J} - p_{mag}v\right) = 2\sqrt{1-v^2}(K(\eta) - E(\eta)) + \\ +2a\left\{\frac{K(\eta) - \Pi\left(\eta - \frac{\eta}{v^2};\eta\right)}{\omega\sqrt{1-v^2}} + \sqrt{1-v^2}\left(E(\eta) - K(\eta)\right)\right\} \tag{28}$$

where $K(\eta)$, $E(\eta)$, $\Pi\left(\eta - \frac{\eta}{v^2};\eta\right)$ are elliptic functions (see the appendix), $\eta = 1 - z_{min}^2/z_{max}^2$, with $z_{min/max}$ defined in (22) and $(J, E, P_+) = \frac{\sqrt{\lambda}}{2\pi}(\tilde{J}, \tilde{E}, \tilde{P}_+)$. If we anticipate that the parameters are a-independent, we can identify

$$\tilde{J} = 2\sqrt{1-v^2}(K(\eta) - E(\eta)) \tag{29}$$

$$\tilde{E} - \tilde{J} = vp_{mag} + 2\frac{K(\eta) - \Pi\left(\eta - \frac{\eta}{v^2};\eta\right)}{\omega\sqrt{1-v^2}} + 2\sqrt{1-v^2}\left(E(\eta) - K(\eta)\right). \tag{30}$$

This will turn out to be correct, *however*, it is too early to make this conclusion as we do not yet know whether ω and v in (28) depend on a. For this we need more information. The remaining equations (26) and (27) can be presented as

$$(\tilde{E} - \tilde{J})\int_{z_{min}}^{z_{max}} \frac{dz}{|\partial z|} = -\tilde{P}_+ \int_{z_{min}}^{z_{max}} \frac{dz}{|\partial z|}p_- \quad , \quad p_{mag}\tilde{J} = \tilde{P}_+ \cdot 2\int_{z_{min}}^{z_{max}} dz|p_z| \tag{31}$$

where the integrals are

$$\begin{aligned} -\int_{z_{min}}^{z_{max}} \frac{dz}{|\partial z|}p_- &= \sqrt{1-v^2}E(\eta) - (\omega - 1)\frac{\left[1 - a + a\omega + \omega v^2(1-a)\right]K(\eta) + a\Pi\left(\eta - \frac{\eta}{v^2},\eta\right)}{\omega(1-a+a\omega)\sqrt{1-v^2}} \\ &- \frac{av\omega}{(1-a+a\omega)}\frac{p_{mag}}{\tilde{P}_+}\left[\sqrt{1-v^2}E(\eta) - \frac{K(\eta)}{\sqrt{1-v^2}}\left(1 - \frac{1}{\omega^2}\right)\right] \end{aligned} \tag{32}$$

$$\begin{aligned} 2\int_{z_{min}}^{z_{max}} dz|p_z| &= -2\frac{\omega^2 v^2 K(\eta) - \Pi\left(\eta - \frac{\eta}{v^2},\eta\right)}{v\omega(1-a+a\omega)\sqrt{1-v^2}} \\ &- \frac{2a\omega}{(1-a+a\omega)}\frac{p_{mag}}{\tilde{P}_+}\left[\sqrt{1-v^2}E(\eta) - \frac{K(\eta)}{\sqrt{1-v^2}}\left(1 - \frac{1}{\omega^2}\right)\right] \end{aligned} \tag{33}$$

With these integrals and recalling the definition of $P_+ = J + a\left(E - J\right)$, (31) can be re-organized as equations which also turn out to be linear in a

$$\tilde{J}\left[\frac{K(\eta)\left(-1 + v^2\omega^2\right) + \left(K(\eta) - E(\eta)\right)\left(1 - v^2\right)\omega}{\omega\sqrt{1-v^2}}\right] + \left(\tilde{E} - \tilde{J}\right)\left(K(\eta) - E(\eta)\right)\sqrt{1-v^2} \\ + \frac{a}{\omega\sqrt{1-v^2}}\left\{vp_{mag}\left[K(\eta)\left(1-\omega^2\right) + \left(1 - v^2\right)\omega^2 E(\eta)\right]\right.$$

$$+\tilde{J}\,(1-\omega)\left[K(\eta)\left(1-\omega+\omega v^2\right)+E(\eta)\left(1-v^2\right)\omega-\Pi\left(\eta-\frac{\eta}{v^2},\eta\right)\right]$$
$$+\left(\tilde{E}-\tilde{J}\right)\left[-\Pi\left(\eta-\frac{\eta}{v^2},\eta\right)+\omega\,(1-\omega)\left(1-v^2\right)E(\eta)+\omega\left(-1+v^2+\omega\right)K(\eta)\right]\Bigg\}=0 \tag{34}$$

$$\left[p_{mag}+2\frac{\omega^2v^2K(\eta)-\Pi\left(\eta-\frac{\eta}{v^2},\eta\right)}{v\omega\sqrt{1-v^2}}\right]\tilde{J}=-a\Bigg\{(\omega-1)\tilde{J}p_{mag}$$
$$+2(\tilde{E}-\tilde{J})\frac{\omega^2v^2K(\eta)-\Pi\left(\eta-\frac{\eta}{v^2},\eta\right)}{v\omega\sqrt{1-v^2}}+2p_{mag}\omega\left[\sqrt{1-v^2}E(\eta)-\frac{K(\eta)}{\sqrt{1-v^2}}\left(1-\frac{1}{\omega^2}\right)\right]\Bigg\} \tag{35}$$

Now, if we assume that the only a-dependence in these equations occurs in the explicit linear terms, i.e. that $\tilde{E}$, $\tilde{J}$, p_{mag}, v and ω are a-independent, (28), (34) and (35) constitute six equations which must be solved by three variables. Indeed, if we solve the a-independent parts by

$$p_{mag}=-2\frac{\omega^2v^2K(\eta)-\Pi\left(\eta-\frac{\eta}{v^2},\eta\right)}{v\omega\sqrt{1-v^2}} \tag{36}$$

$$J=\frac{\sqrt{\lambda}}{2\pi}\,\sqrt{1-v^2}\,\left[\,K(\eta)-E(\eta)\,\right] \tag{37}$$

$$E-J=\frac{\sqrt{\lambda}}{2\pi}\,\left[\sqrt{1-v^2}E(\eta)+(1-\omega)\frac{(1+v^2\omega)K(\eta)}{\omega\sqrt{1-v^2}}\right]. \tag{38}$$

the a-dependent parts are solved identically and we have found a solution. It agrees with our initial guess, Eqs. (29), (30).

The result is complete cancelation of the gauge parameter a from the physical quantities (36), (37) and (38). To use this solution, two of these equations, for example (36) and (37) should be used to relate the parameters v and ω to p_{mag} and J. The third (38) then determines the equation for the spectrum, resulting in $E-J$ as a function of J and p_{mag}. In the large J limit, this can be done explicitly using an asymptotic expansion of the elliptic functions. The result is the formula which is the $a=0$ limit of the one quoted in Ref. [10],

$$E-J=\frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p_{mag}}{2}\right|\left[1-\left[4\sin^2\frac{p_{mag}}{2}\right]\left[e^{-2-2\pi J/\sqrt{\lambda}|\sin\frac{p_{mag}}{2}|}\right]-\right.$$
$$\left[8\sin^2\frac{p_{mag}}{2}\right]\left[6\cos^2\frac{p_{mag}}{2}+\frac{1}{2}+\left(\frac{2\pi J}{\sqrt{\lambda}|\sin\frac{p_{mag}}{2}|}\right)\left(6\cos^2\frac{p_{mag}}{2}-1\right)\right.$$
$$\left.\left.+\left(\frac{2\pi J}{\sqrt{\lambda}\sin\frac{p_{mag}}{2}}\right)^2\left(\cos^2\frac{p_{mag}}{2}\right)\right]\left[e^{-2-2\pi J/\sqrt{\lambda}|\sin\frac{p_{mag}}{2}|}\right]^2+\ldots\right] \tag{39}$$

The finite size corrections are exponentially small. The exponent $2\pi J/\sqrt{\lambda}|\sin\frac{p_{mag}}{2}|$ has a nice physical interpretation as the ratio of the size of the spin chain, J to the size of the magnon.

It is interesting to compare this result with a computation of the finite size corrections to single magnon energy on the gauge theory side. This has been done [10], for example using the Hubbard model, which at one time was a candidate for the effective theory for integrable $\mathcal{N}=4$ Yang-Mills in the SU(2) sector, but is now known to disagree at and beyond four loop order [46, 47]

The one magnon spectrum in Hubbard model is given by [35]

$$(E-J)_{\text{Hubbard}} = \frac{\sqrt{\lambda}}{\pi}\sin\left(\frac{p_{mag}}{2}\right)\cosh\beta \quad \text{where} \quad \sinh\beta = \frac{\pi\tanh(\beta L)}{\sqrt{\lambda}\sin\left(\frac{p_{mag}}{2}\right)} \tag{40}$$

The total number of sites is $L=J+1$. From this expression, we can find an asymptotic expansion in large J,

$$(E-J)_{\text{Hubbard}} = \frac{\sqrt{\lambda}}{\pi}\sin\left(\frac{p_{mag}}{2}\right)\left[1-\frac{2\pi^2}{\lambda\sin^2\left(\frac{p_{mag}}{2}\right)}e^{-\frac{2\pi J}{\sqrt{\lambda}\sin\left(\frac{p_{mag}}{2}\right)}}+O(e^{-2\frac{2\pi J}{\sqrt{\lambda}\sin\left(\frac{p_{mag}}{2}\right)}})\right] \tag{41}$$

where $0<p_{mag}<\pi$. The exponent that governs finite size corrections is the same both in the Hubbard model (41) and the giant magnon (39), but the prefactors multiplying the exponentials contain different powers of λ. The former is an intriguing consistency of AdS/CFT, the latter is expected and consistent with the already known fact that the Hubbard model does not describe $\mathcal{N}=4$ Yang-Mills beyond a few orders of λ.

Finally, coming back to the orbifold, the large λ limit of finite size corrections to the single magnon state are predicted by (39) with $p_{mag}=2\pi\frac{m}{M}$. It would be interesting to understand the origin of finite size corrections on the gauge theory side. It is reasonable to think that any exponential finite size correction could only begin at a high order, where wrapping interactions [30][48][49] come into play. In fact, once J is fixed, these corrections appear non-perturbative $\sim\exp(-1/\sqrt{\lambda})$ in Yang-Mills theory, similar to D-brane, or D-instanton contributions in string theory. It would be interesting to study this further.

Appendix: Elliptic functions The following useful identities are needed for the integrations

$$\begin{aligned}
\int_{z_{min}}^{z_{max}} dz\frac{1}{\sqrt{z^2-z_{min}^2}\sqrt{z_{max}^2-z^2}} &= \frac{1}{z_{max}}K(\eta) \\
\int_{z_{min}}^{z_{max}} dz\frac{z^2}{\sqrt{z^2-z_{min}^2}\sqrt{z_{max}^2-z^2}} &= z_{max}E(\eta) \\
\int_{z_{min}}^{z_{max}} dz\frac{1}{(1-z^2)\sqrt{z^2-z_{min}^2}\sqrt{z_{max}^2-z^2}} &= \frac{1}{z_{max}(1-z_{max}^2)}\Pi\left(\frac{z_{max}^2-z_{min}^2}{z_{max}^2-1};\eta\right)
\end{aligned} \tag{42}$$

where $\eta = 1 - \frac{z_{min}^2}{z_{max}^2}$.
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