A quantum probability explanation for violations of 'rational' decision theory

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Summary

Two experimental tasks in psychology, the two stage gambling game and the prisoner's dilemma game, show that people violate the sure thing principle of decision theory. These paradoxical findings have resisted explanation by classic decision theory for over a decade. A *quantum* probability model, based on a Hilbert space representation and Schrödinger's equation, provides a simple and elegant explanation for this behaviour. The quantum model is compared to an equivalent Markov model and it is shown that the latter is unable to account for violations of the sure thing principle. Accordingly, it is argued that quantum probability provides a better framework for modelling human decision making.

Key words: prisoner's dilemma, quantum probability, decision making, cognitive science

Cognitive science is concerned with providing formal, computational descriptions for various aspects of cognition. Over the last few decades, several frameworks have been thoroughly examined, such as formal logic (e.g., Evans, Newstead, & Byrne, 1991), information theory (e.g., Chater, 1999), classical (Bayesian) probability (e.g., Tenenbaum & Griffiths, 2001), neural networks (Rumelhart & McClelland, 1986), and a range of formal, symbolic systems (e.g., Anderson, Matessa, & Lebiere, 1997). Being able to establish an advantage of one computational approach over another is clearly a fundamental issue for cognitive scientists. Two criteria are needed to achieve this goal: one is to establish a striking empirical finding that provides a strong theoretical challenge, and the second is to provide a rigorous mathematical argument that a theoretical class fails to meet this challenge. This article reviews findings that challenge the classical (Bayesian) probability approach to cognition, and proposes to exchange this with a more generalized (quantum) probability approach.

The empirical challenge is provided by two experimental tasks in decision making, the prisoner's dilemma and the two-stage gambling task, which have had an enormous influence in cognitive psychology (and economics—there are over 31,000 citations to Tversky's work, one of the researchers who first studied these tasks; e.g., Shafir & Tversky, 1992; Tversky & Kahneman, 1983; Tversky & Shafir, 1992). These experimental tasks are important because they show a violation of a fundamental law of classic (Bayesian) probability theory which, when applied to human decision making, is called the 'sure thing' principle (Savage, 1954).

The sure thing principle (Savage, 1954) is fundamental to classic decision theory: If you prefer action A over B under state of the world *X*, and you also prefer A over B under the complementary state ~*X*, then you should prefer A over B when the state is unknown. This principle was tested by Tversky and Shafir (1992) in a simple two-stage gambling experiment: participants were told that they had just played a gamble (even chance to win $200 or lose $100), and then they were asked to choose whether to play the same gamble a second time. In one condition, they knew they won the first play; in a second condition, they knew they lost the first play; and in a third condition, they *did not know* the outcome. Surprisingly, the results violated the sure thing principle: following a win/ loss, participants chose to play again on 69% / 59% respectively of the trials; but when the outcome was unknown, they only chose to play again on 36% of the trials. This preference reversal was observed at the individual level of analysis with real money at stake.

Similar results were obtained using a two person prisoner's dilemma game with payoffs defined for each player as in Table 1.The Nash equilibrium in standard game theory is for both parties to defect. Three conditions are used to test the sure thing principle: In an 'unknown' condition, you act without knowing your opponent's action; in the 'known defect' condition, you know your opponent will defect before you act; and in the 'known cooperate' condition, you know your opponent will cooperate before you act. According to the sure thing principle, if you prefer to defect, regardless of whether you know your opponent will defect or cooperate, then you should prefer to defect even when your opponent's action is unknown.

---------------------- Table 1 ----------------------

Once again, people frequently violate the sure thing principle (Shafir & Tversky, 1992) – many players defected knowing the opponent defected and knowing the opponent cooperated, but they switched and decided to cooperate when they didn't know the opponent's action. This preference reversal by many players caused the proportion of defections for the unknown condition to fall below the proportions observed under the known conditions. This key finding of Shafir and Tversky (1992) has been replicated in several subsequent studies (Busemeyer, Matthew, & Wang, 2006; Croson, 1999; Li & Taplin, 2002, Table 2).

Note that prisoner's dilemma is a task that has attracted widespread attention not just from decision making scientists. For example, it has been intensely studied in the context of how altruistic and cooperative behaviour can arise in both humans and animals (e.g., Axelrod & Hamilton, 1981; Kefi et al., 2007; Stephens et al., 2002). Violations of the sure thing principle specifically have not been demonstrated in animal cognition. However, both Waite (2001) and Shafir (2001) showed that *transitivity*, another fundamental aspect of classical probability theory, can be violated in animal preference choice (with gray jays and bees, respectively). Also, it turns out that a core element of our model for human decision making in prisoner's dilemma has an analogue in animal cognition, raising the possibility that such a model may be applicable to animal cognition as well.

---------------------- Table 2 ----------------------

We present an alternative probabilistic framework for modelling decision making, based on quantum probability. Why consider a quantum probability model for decision making? The original motivation for the development of quantum mechanics in physics was to explain findings that seemed paradoxical from a classical point of view. Similarly, paradoxical findings in psychology have made a growing number of researchers seek explanations that make use of the quantum formalism in cognitive situations. For example, Aerts and Aerts (1994; see also Khrennikov, 2004; La Mura, in press; Mogiliansky et al., in press) modelled incompatibility and interference effects that arise in human preference judgements. Gabora and Aerts (2002; see also Aerts & Gabora, 2005a, 2005b; Aerts, Broekaert & Gabora, in press) modelled puzzling findings found in human reasoning with conceptual combinations. Bordley (1998; see also Franco, in press) modelled paradoxical results obtained with human probability judgments. Van Rijsbergen (2004; see also Bruza, Widdows, & Wood, 2006; Widdows, 2006) showed that classical logic does not appear the right type of logic when dealing with classes of objects and a more appropriate representation for classes is possible with mathematical tools from quantum theory. Such approaches can be labelled 'geometric' (cf. Aerts & Aerts, 1994) in that they utilize the geometric properties of Hilbert space representations and the measurement principles of quantum theory, but not the dynamical aspects of quantum theory (time evolution with Schrödinger's equation).

A much smaller number of applications have attempted to apply quantum dynamics to cognition. For example, Atmanspacher, Romer, and Walach (2002) modelled oscillations in human perception of impossible figures. Aerts, Broekaerst and Smets (2004) modelled how a human observer alternates between perceiving the statements in a liar's paradox situation as false and true. Busemeyer, Wang, and Townsend (2006) presented a quantum model for signal detection type of human decision processes. Our proposal builds on this latter work (and particularly that of Busemeyer et al., 2006). We were interested in a model which would have implications for the time course of a decision, as well as accurately predicting choice probabilities in the prisoner's dilemma and two-stage gambling task. A novel aspect of our proposal is that we attempt to derive a relevant Hamiltonian *a priori*, on the basis of the psychological parameters of the decision making situation.

Finally, note that the goal of models such as the above is to formulate a mathematical framework for understanding the behavioural findings from cognition and decision making. This objective must be distinguished from related ones, such as constructing new game strategies using quantum game theory (Eisert, Wilkens & Lewenstein, 1999), modelling the biology of the brain using quantum mechanics (Hameroff & Penrose, 1996; Pribram, 1993), or developing new algorithms for quantum computation (Nielsen & Chuang, 2000).

The violation of the sure thing principle readily suggests that a classic probability model for Tversky and Shafir's results will fail. We go beyond intuition and develop a standard Markov model for the two-stage gambling task and prisoner's dilemma, to prove that this model can never account for the empirical findings. In this way we motivate a more general model, based on *quantum probability*. Researchers have recently successfully explored applications of the quantum mechanics formalism outside physics (for example, notably computer science. e.g., Grover, 1997). Explorations of how the quantum principles could apply in psychology have been slow, partly because of a confusion of whether such attempts implicate a statement that the operation of the brain is quantum mechanical (e.g., Hameroff & Penrose, 1996). This could be the case or not (cf. Marr, 1982), but it is *not* the issue at stake: Rather, we are asking whether quantum probability could provide an appropriate mathematical framework for understanding/ modelling certain behavioural aspects of cognition. Key problems in such an endeavour are (a) what is an appropriate Hilbert space representation of the task, (b) what is the psychological motivation for the corresponding Hamiltonian, and (c) what is the meaning of time evolution in this context? We address all these problems in our quantum probability model of decision making in the prisoner's dilemma task and the two-stage gambling task. The model is described with respect to prisoner's dilemma task, but extension to the two-stage gambling task is straightforward.

Step 1: Representation of beliefs and actions.

The prisoner's dilemma game involves a set of four mutually exclusive and exhaustive outcomes Ω ={B_DA_D, B_DA_C, B_CA_D, B_CA_C}, where B_iA_j represents your belief that your opponent will take the *i*-th action, but you intend to take the *j*-th action (*D*=Defect; *C*=Cooperate). For both the Markov and quantum models, we assume that the probabilities of the four outcomes can be computed from a 4 × 1 state vector $\psi = \begin{bmatrix} \psi_{DD} \\ \psi_{DC} \\ \psi_{CD} \\ \psi_{CC} \end{bmatrix}$. For the Markov model, ψ_{ij} = Pr[observe belief *i* and action *j*], with $\sum_i \sum_j \psi_{ij} = 1$. For the quantum model, ψ_{ij} is an amplitude, so that $|\psi_{ij}|^2$ = Pr[observe belief *i* and action *j*], with $\sum_i \sum_j |\psi_{ij}|^2 = 1$. For both models, we assume the individual begins the decision process in an uninformed state: $\psi_0 = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for the Markov model and $\psi_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for the quantum model, .

Step 2: Inferences based on prior information.

Information about the opponent's action changes the initial state ψ_0 into a new state ψ_1. If the opponent is known to defect, the state changes to $\psi_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ for the Markov model, and $\psi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ for the quantum model; similarly, if the opponent is known to cooperate, the state changes to $\psi_1 = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ for the Markov model and $\psi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ for the quantum model. If no information is provided then the state remains unchanged so that $\psi_1 = \psi_0$

Step 3: Strategies based on payoffs.

The decision maker must evaluate the payoffs in order to select an action, which changes the previous state ψ_1 into a final state ψ_2. We assume that the time evolution of the initial state to the final state corresponds to the thought process leading to a decision.

For the Markov model, we can model this change using a Kolmogorov forward equation $\frac{d\psi}{dt} = K_A \cdot \psi$, which has a solution $\psi_2(t) = e^{t \cdot K_A} \cdot \psi_1$. The matrix $T(t) = e^{t \cdot K_A}$ is a transition matrix, with $T_{ij}(t)$ = Pr[transiting to state *i* from state *j* during time period *t*]. The transition matrix satisfies $\sum_i T_{ij} = 1$ to guarantee that ψ_2 sums to unity. Initially, we assume

$$K_A = \begin{bmatrix} K_{Ad} & 0 \\ 0 & K_{Ac} \end{bmatrix}, \text{ where } K_{Ai} = \begin{bmatrix} -1 & \mu_i \\ 1 & -\mu_i \end{bmatrix}. \qquad (1a)$$

The intensity matrix K_A transforms the state probabilities to favour either defection or cooperation, depending on the parameters μ_d or μ_c, which correspond to your gain if you defect, depending on whether you believe the opponent will defect or cooperate, respectively; these parameters depend on the payoffs associated with different actions, and will be considered shortly. The intensity matrix requires $K_{ij} > 0$ for $i \neq j$ and $\sum_i K_{ij} = 0$ to guarantee that $e^{t \cdot K_A}$ is a transition matrix.

For the quantum model, the time evolution is determined by Schrödinger's equation $i \cdot \frac{d\psi}{dt} = H_A \cdot \psi$ with solution $\psi_2(t) = e^{-i \cdot t \cdot H_A} \cdot \psi_1$. The matrix $U(t) = e^{-i \cdot t \cdot H_A}$ is unitary with $\left|U_{ij}(t)\right|^2 =$ Pr[transiting to state *i* from state *j* during time period *t*]. This matrix must satisfy $U^\dagger U = I$ (identity matrix) to guarantee that ψ_2 retains unit length. Initially, we assume that

$$H_A = \begin{bmatrix} H_{Ad} & 0 \\ 0 & H_{Ac} \end{bmatrix}, \text{ where } H_{Ai} = \frac{1}{\sqrt{1+\mu_i^2}} \begin{bmatrix} \mu_i & 1 \\ 1 & -\mu_i \end{bmatrix}, \qquad (1b)$$

The Hamiltonian H_A rotates the state to favor either defection or cooperation, depending on the parameters μ_d or μ_c, which (as before) correspond to your gain if you defect, depending on whether you believe the opponent will defect or cooperate, respectively. The Hamiltonian must be Hermitian ($H_A{}^{\dagger} = H_A$) to guarantee that U is unitary.

For both models, the parameter μ_i is assumed to be a monotonically increasing utility function of the differences in the payoffs for each of your actions, depending on the opponent's action: $\mu_d = u(x_{DD} – x_{DC})$ and $\mu_c = u(x_{CD} – x_{CC})$, where x_{ij} is the payoff you receive if your opponent takes action i and you take action j. For example, given the payoffs in Table 1, $u_{DD} = x(10,10)$, $u_{DC} = x(25,5)$, $u_{CD} = x(5,25)$, and $u_{CC} = x(20,20)$. Assuming that utility is determined solely by your own payoffs, then $\mu_d = u(x_{DD} – x_{DC}) = u(5) = \mu = u(x_{CD} – x_{CC}) = \mu_c$. In other words, typically, μ_i can be set equal to the difference in the payoffs (possibly multiplied by a constant, scaling factor), although more complex utility functions can be assumed.

For both models, a decision corresponds to a measurement of the state $\psi_2(t)$. For the Markov model, Pr[you defect] = Pr[D] = ($\psi_{DD} + \psi_{CD}$); similarly, for the quantum model, Pr[you defect] = Pr[D] = ($|\psi_{DD}|^2 + |\psi_{CD}|^2$).

Inserting Equation 1a into the Kolmogorov equation and solving for $\psi_2(t)$ yields the following probability when the opponent's action is known:

$$\Pr[D] = \left(\frac{\mu}{1+\mu}\right) \cdot \left(1 - e^{-(\mu+1)\cdot t}\right) + \frac{e^{-(\mu+1)\cdot t}}{2}.$$

This probability gradually grows monotonically from $\left(\frac{1}{2}\right)$ at t=0 to $\left(\frac{\mu}{1+\mu}\right)$ as t → ∞. The behaviour of the quantum model is more complex. Inserting Equation 1b into the Schrödinger equation and solving for $\psi_2(t)$ yields:

$$\Pr\,[D] = \left(\frac{1}{2} + \frac{\mu}{1+\,\mu^2} \cdot \sin\,(\tfrac{\pi}{2} \cdot t)^2\right).$$

For $-1 < \mu < +1$, this probability increases across time from $\left(\frac{1}{2}\right)$ at t=0 to $\left(\frac{1}{2}+\frac{\mu}{1+\mu^2}\right)$ at t=1, and subsequently it oscillates between the minimum and maximum values. Empirically, choice probabilities in laboratory-based, decision making tasks monotonically increase across time (for short decision times, see Diederich & Busemeyer, 2006), and so a reasonable approach for fitting the model is to assume that a decision is reached within the interval ($0 < t < 1$) for the quantum model (t=1 would correspond to around 2s for such tasks).

Equations 1a, 1b produce reasonable choice models when the action of the opponent is known. However, when the opponent's action is unknown, both models predict that the probability of defection is the average of the two known cases, which fails to explain the violations of the sure thing principle. The K_A and H_A components of each model can be understood as the 'rational' components of each model, whereby the decision maker is simply assumed to try to maximize utility. In the next section we introduce a component in each model for describing an *additional* influence in the decision making process, which can lead to decisions which *do not* maximize expected utility (and so could lead to violations of the sure thing principle). These two components in each model have to be separate since in many cases the behaviour of decision makers can be explained (just) by an urge to maximize expected utility. The difference between the Markov model and the quantum one relates to how the two components are combined with each other. Importantly, we prove that the Markov model still cannot produce the violations of the sure thing principle even when this second, non-rational component is added. Only the quantum model explains the result.

Step 4: Strategies based on evaluations of both beliefs and payoffs.

To explain violations of the sure thing principle, we introduce the idea of cognitive dissonance (Festinger, 1957). People tend to change their beliefs to be consistent with their actions. In the case of the prisoner dilemma game, this motivates a change of beliefs about what the opponent will do in a direction that is consistent with the person's intended action. In other words, if a player chooses to cooperate he/ she would *tend* to think that the other player will cooperate as well. The reduction of cognitive dissonance is an intriguing, and extensively supported, cognitive process. It has been shown with monkeys (Egan, Santos, & Bloom, 2007), suggesting that the applicability of our model might extend to such animals. Shafir and Tversky (1992) explained it in terms of a personal bias for 'wishful thinking' and Chater, Vlaev, and Grinberg (2008) by considering a statistical approach based on Simpson's paradox (specifically, Chater et al. showed that, in a prisoner's dilemma game, the propensity to cooperate or defect would depend on assumptions about what the opponent would do, given whether the parameters of the game would encourage cooperation or defection). Such approaches may not be incompatible (for example, wishful thinking may have an underlying statistical explanation).

Although cognitive dissonance tendencies can be implemented in both the Markov and the quantum model, we shall see that it does not help the Markov model, and only the quantum model explains the sure thing principle violations.

For the Markov model, an intensity matrix that produces a change of 'wishful thinking' is

$$K_B = \left(\begin{bmatrix} -1 & 0 & +\gamma & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & +\gamma & 0 & -1 \end{bmatrix} \right). \qquad (2a)$$

The first/second matrix changes beliefs about the opponent toward defection/cooperation when you plan to defect/cooperate, respectively.

Note that

$$\frac{d}{dt}\cdot\begin{bmatrix}\psi_{DD}\\ \psi_{DC}\\ \psi_{CD}\\ \psi_{CC}\end{bmatrix}=\begin{bmatrix}-1 & 0 & +\gamma & 0\\ 0 & -\gamma & 0 & +1\\ +1 & 0 & -\gamma & 0\\ 0 & +\gamma & 0 & -1\end{bmatrix}\cdot\begin{bmatrix}\psi_{DD}\\ \psi_{DC}\\ \psi_{CD}\\ \psi_{CC}\end{bmatrix}=\begin{bmatrix}\gamma\cdot\psi_{CD}-\psi_{DD}\\ \psi_{CC}-\gamma\cdot\psi_{DC}\\ \psi_{DD}-\gamma\cdot\psi_{CD}\\ \gamma\cdot\psi_{DC}-\cdot\psi_{CC}\end{bmatrix}.$$

If $\gamma>1$, then the rate of increase for first and last rows is greater than the middle rows, leading to an increase in the probabilities that beliefs and actions agree. For example, setting $\gamma=10$ at $t=1$ produces $e^{t\cdot K_B}\cdot\psi_0=\begin{bmatrix}.45\\ .05\\ .05\\ .45\end{bmatrix}$, where it can be seen that beliefs tend to match actions, achieving a reduction of cognitive dissonance. For the quantum model, a Hamiltonian that produces this change is

$$H_B=\frac{-\gamma}{\sqrt{2}}\cdot\left(\begin{bmatrix}+1 & 0 & +1 & 0\\ 0 & 0 & 0 & 0\\ +1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & -1 & 0 & +1\\ 0 & 0 & 0 & 0\\ 0 & +1 & 0 & +1\end{bmatrix}\right). \tag{2b}$$

The first/second matrix rotates beliefs about the opponent toward defection/cooperation when you plan to defect/cooperate, respectively. Note that

$$\frac{d}{dt}\cdot\begin{bmatrix}\psi_{DD}\\ \psi_{DC}\\ \psi_{CD}\\ \psi_{CC}\end{bmatrix}=-i\cdot\frac{-\gamma}{\sqrt{2}}\cdot\begin{bmatrix}+1 & 0 & +1 & 0\\ 0 & -1 & 0 & +1\\ +1 & 0 & -1 & 0\\ 0 & +1 & 0 & +1\end{bmatrix}\cdot\begin{bmatrix}\psi_{DD}\\ \psi_{DC}\\ \psi_{CD}\\ \psi_{CC}\end{bmatrix}=i\cdot\frac{\gamma}{\sqrt{2}}\cdot\begin{bmatrix}\psi_{DD}+\psi_{CD}\\ \psi_{CC}-\psi_{DC}\\ \psi_{DD}-\psi_{CD}\\ \psi_{DC}+\psi_{CC}\end{bmatrix}.$$

If $\gamma>0$, then the rate of increase for the first and last rows is greater than the middle rows, so that, as before, there is an increase in the amplitudes for the states in which beliefs and actions agree. For example, setting $\gamma=1$ at $t=\pi/2$ produces $e^{-i\cdot t\cdot H_B}\cdot\psi_0$ which results in a vector of squared magnitudes equal to $\begin{bmatrix}.50\\ .00\\ .00\\ .50\end{bmatrix}$. In both the Markov and the quantum model, we can see that the above intensity matrix/ Hamiltonian respectively, makes beliefs and actions correlated.

By itself, Equation 2 is an inadequate description of behaviour in prisoner's dilemma, because it cannot explain how preferences vary with payoffs. We need to combine Equations 1 and 2 to produce an intensity matrix $K_C = K_A + K_B$ or a Hamiltonian $H_C = H_A + H_B$ so that the time evolution of the initial state to the final state reflects the influences of both the payoffs and the process of wishful thinking. Note that in this combined model, both beliefs and actions are evolving simultaneously and in parallel.

Accordingly, we suggest that the final state is determined by $\psi_2 = e^{t \cdot K_C} \cdot \psi_1$ for the Markov model and $\psi_2 = e^{-i \cdot t \cdot H_C} \cdot \psi_1$ for the quantum model. Each model has two free parameters: one changes the actions using Equation 1 and depends on payoffs (i.e., μ), and another which corresponds to a psychological bias to assume the opponent will act as we do with Equation 2 (i.e., γ).

Model Predictions.

For the Markov model, probabilities for the unknown state can be related to probabilities for the known states by expressing the initial unknown state as a probability *mixture* of the two initial known states:

$$\psi_2 = e^{t \cdot K_C} \cdot \tfrac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = e^{t \cdot K_C} \cdot \tfrac{1}{2}\left(\tfrac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right) = \tfrac{1}{2}\left(e^{t \cdot K_C} \cdot \tfrac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + e^{t \cdot K_C} \cdot \tfrac{1}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right).$$

We see that the state probabilities in the unknown case *must* equal the average of the state probabilities for the two known cases. Therefore, the Markov model fails to reproduce the violations of the sure thing principle, regardless of what parameters, time point, initial state, or intensity matrix we use. This conclusion reflects the fundamental fact that the Markov model obeys the law of total probability, which mathematically restricts the unknown state to remain a weighted average of the two known states. Note that this failure of the Markov model occurs even when we include the cognitive dissonance tendencies in the model.

For the quantum model, the amplitudes for the unknown state can also be related to the amplitudes for the two known states:

$$\psi_2 = e^{-i\cdot t\cdot H_C}\cdot\tfrac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = e^{-i\cdot t\cdot H_C}\cdot\tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+\tfrac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right) = \tfrac{1}{\sqrt{2}}e^{-i\cdot t\cdot H_C}\cdot\tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+e^{-i\cdot t\cdot H_C}\cdot\tfrac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}.$$

We see that the amplitudes in the unknown case equal the *superposition* of the amplitudes for the two known cases. However, *here* is precisely where the quantum model departs from the Markov model: probabilities are obtained from the *squared magnitudes* of the amplitudes. This last computation produces interference effects that can cause the unknown probabilities to deviate from the average of the known probabilities.

We initially fit the parameters of the quantum model at t=1·(π/2), which is the time when the choice probabilities produced by Equation 1b first reach their maximum. Note that the quantum model predicts that choice probabilities will oscillate with time. However, in modelling results from the prisoner's dilemma task, one needs to consider, first, that such tasks are very simple and, second, that respondents in such experiments are typically paid for their participation so that they are motivated (and indeed sometimes requested) to respond quickly. Both these considerations suggest that a decision will be made (the state vector collapses) as early as one course of action emerges as advantageous (Diederich & Busemeyer, 2006). Regarding the two-stage gambling task, setting μ = .59 and γ = 1.74 produces probabilities for choosing to gamble equal to (.68, .58, .37) for the (known win, known loss, unknown) conditions, respectively.

These predictions closely match the observed results (.69, .59, .36, from Tversky & Shafir, 1992). For the prisoner's dilemma game, setting μ =.51 and γ =2.09. produces probabilities for defection equal to (.81, .65, .57) for the (known defect, known cooperate, unknown) conditions, respectively. Again, model predictions closely reproduce the average pattern (.84, .66, .55) in Table 2. Figure 1 shows that model predictions are fairly robust as parameter values vary. An interference effect appears after t=.75· (π/2) and is evident across a large section of the parameter space. Finally, note that we can relax the assumption that participants will make a decision at the same time point (we thank a reviewer for this observation). To allow for this possibility, we assumed a gamma distribution of decision times with a range from t = .50·(π/2) to t=2·(π/2) and a mode at t=1·(π/2). We refitted the model using the mean choice probability averaged over this distribution, and this produced very similar predicted results: (.77, .64, .58) for the known defect, known cooperate, and unknown conditions, respectively, in prisoner's dilemma (with μ = .47, and γ = 2.10).

-------------Figure 1-------------

Classic probability theory has been widely applied in understanding human choice behaviour. Accordingly, one can naturally wonder whether it is possible to salvage the Markov model. First recall that the classic Markov model fails even when we allow for cognitive dissonance effects in this model. Second, the analyses above hold for *any* initial state, ψ_0, and *any* intensity matrix, K (not just the ones used above to motivate the model), but they are based on two main assumptions: The *same* initial state and the *same* intensity matrix are used across both known and unknown conditions. However, we can relax even these assumptions. Even if the initial state is not the same across conditions, the Markov model must predict that the marginal probability of defecting in the unknown condition (whatever mixture is used) is a convex combination of the two probabilities conditioned on the known action of the opponent. This prediction is violated in the data of Table 2. Furthermore, even if we change intensity matrices across conditions (using the K_A intensity matrix for known conditions and using the K_C matrix for the unknown condition), the Markov model continues to satisfy the law of total probability because this change has absolutely no effect on the predicted probability of defection (the K_B matrix does not change the defection rate). Thus our tests of the Markov model are very robust.

Concluding Comments

In this work we considered empirical results which have been a focal point in the controversy over whether classic probability theory is an appropriate framework for modelling cognition or not. Tversky, Shafir, Kahneman and colleagues have argued that the cognitive system is generally sensitive to environmental statistics, but is also routinely influenced by heuristics and biases which can violate the prescription from probability theory (Tversky & Kahneman, 1983; Tversky & Shafir, 1992; cf. Gigerenzer, 1996). This position has had a massive influence not only in psychology, but also in management sciences and economics, collimating to a Nobel Prize award to Kahneman. Moreover, findings such as the violation of the sure thing principle in Prisoner's Dilemma has led researchers to raise fundamental questions about the nature of human cognition (for example, what does it mean to be rational? Oaksford & Chater, 1994).

In this work, we adopted a different approach from the heuristics and biases one advocated by Tversky, Kahneman, and Shafir. We propose that human cognition can and should be modelled within a probabilistic framework, but classic probability theory is too restrictive to fully describe human cognition. Accordingly, we explored a model based on quantum probability, which can subsume classic probability, as a special case. The main problems in developing a convincing cognitive quantum probability model are to determine an appropriate Hilbert space and Hamiltonian. We attempted to present a satisfactory prescriptive approach to dealing with these problems and so encourage the development of other quantum probability models in cognitive science. For example, the Hamiltonian is derived directly from the parameters of the problem (e.g., the payoffs associated with different actions) and known general principles of cognition (e.g., reducing cognitive dissonance). Importantly, our model works: it is able to account for violations of the sure thing principle in prisoner's dilemma and the two-stage gambling task and leads to close fits to empirical data.

Quantum probability provides a promising framework for modelling human decision making. First, we can think of the set of basis states as a set of preference orders over actions. According to a Markov process, an individual is committed to exactly one preference order at any moment in time, although it can change from time to time. According to a quantum process, an individual experiences a superposition of all these orders, and at any moment the person remains uncommitted to any specific order. This is an intriguing perspective on human cognition, which may shed light on the functional role of different modes of memory and learning (cf. Atallah, Frank, & O'Reilly, 2004). Second, Schrödinger's equation predicts a periodic oscillation of the propensity to perform one action (assuming that the decision maker can be persuaded to extend the decision time beyond the first cycle of the process), which is broadly analogous to the electroencephalography (EEG) signals recorded from participants engaged in choice tasks (cf. Haggard & Eimer, 1999). Only after preferences are revealed does the process collapse onto exactly one preference order. This contrasts with time development in the Markov model, whereby the system monotonically converges to its final state. Third, quantum probability models allow interference effects which can make the probability of the disjunction of two events to be lower than the probability of either event individually (see also Khrennikov, 2004). Such interference effects are ubiquitous in psychology, but incompatible with Markov models, which are constrained by classic probability laws. Fourth, 'back to back' measurements on the same decision will produce the same result in a quantum system (because of state reduction), which agrees with what people do (Atmanspacher, Filk, & Romer, 2004). However, 'back to back' choices remain probabilistic in classic random utility models, which is not what people do. Finally, recent results in computer science show quantum computation to be fundamentally faster compared to classic computation, for certain problems (Nielsen & Chuang, 2000). Perhaps the success of human cognition can be partly explained by its use of quantum principles.

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Electronic supplementary material. Additional proofs.

1. Rational for choice of compatible (commuting observable) over incompatible (non commuting observable) representation of measurements. .

The prisoner dilemma experiment involves two different types of measurements: one is a measure of the belief about what an opponent will do (the opponent will defect or cooperate); the second is a measure of the preference for an action to take (decide to defect or cooperate). According to quantum theory, two measurements can be compatible (commuting observables) or incompatible (non commuting observables). We chose not to treat them as compatible for the following empirical reason. If the two measures are incompatible, then the transition matrix

$$T = \begin{bmatrix} \Pr\left[Defect\ Action|Defect\ \ Belief\right] & \Pr\left[Defect\ Action|Cooperate\ \ Belief\right] \\ \Pr\left[Cooperate\ Action|Defect\ \ Belief\right] & \Pr\left[Defect\ Action|Cooperate\ Belief\right] \end{bmatrix}$$

must be doubly stochastic (Peres, 1995, p. 33). This implies that Pr[Defect Action | Cooperate Belief] = Pr[Cooperate action | Defect Belief] = 1 – Pr[Defect action | Defect Belief] which is strongly violated in Table 2. The above transition matrix does not have to be doubly stochastic for the compatible representation.

The fact that the two observables (my belief of the opponent’s action, my intended action) in our model are compatible implies that they can be measured simultaneously, which is intuitively reasonable. Note that interference effects in our model arise from the way the initial state vector evolves in time (through Schrödinger’s equation). In cases in which two observables are *incompatible*, interference effects can arise from the order in which they are measured against a given state vector (Mogiliansky et al., in press, describe an example related to the noncommutativity in preferences and Franco, in press, explains the conjunction fallacy in a similar way).

2. Application of the quantum model to the two stage gambling paradigm.

In this application, we define the four basis states as $\{|B_W A_P\rangle, |B_W A_N\rangle, |B_L A_P\rangle, |B_L A_N\rangle\}$ where $|B_i A_j\rangle$ represents the state where you believe that outcome *i* (W = win, L = lose) occurred on the previous round, and you choose action *j* (P = play or N = not play) on the next round. The parameter μ represents the difference between the expected utility of the gamble and the utility of the sure amount. The parameter γ rotates the beliefs so that if you plan to play again, then your beliefs rotate toward winning; if you do not plan to play again, then your beliefs rotate toward losing.

3. Proof that the Markov and quantum models predict that the probabilities for the unknown case are equal to the average of the probabilities for the known cases when only Equations 1a and 1b are used to generate the final probabilities.

We prove this for the quantum model. The same argument applies for the Markov model. The state after step 3 for the unknown case equals

$$\psi_2 = e^{-i\cdot t\cdot H_A} \cdot \psi_0 = e^{-i\cdot t\cdot \begin{bmatrix} H_{Ad} & 0 \\ 0 & H_{Ac} \end{bmatrix}} \cdot \psi_0 = \begin{bmatrix} e^{-i\cdot t\cdot H_{Ad}} & \mathbf{0} \\ 0 & e^{-i\cdot t\cdot H_{Ac}} \end{bmatrix} \cdot \psi_0$$

$$= \begin{bmatrix} e^{-i\cdot t\cdot H_{Ad}} & 0 \\ 0 & e^{-i\cdot t\cdot H_{Ac}} \end{bmatrix} \cdot \begin{matrix} .5 \\ .5 \\ .5 \\ .5 \end{matrix}$$

$$= \begin{bmatrix} e^{-i\cdot t\cdot H_{Ad}} & 0 \\ 0 & e^{-i\cdot t\cdot H_{Ac}} \end{bmatrix} \cdot \left(\begin{bmatrix} \begin{pmatrix} .5 \\ .5 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} + \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} .5 \\ .5 \end{pmatrix} \end{bmatrix} \right)$$

$$= \begin{bmatrix} e^{-i\cdot t\cdot H_{Ad}} \begin{pmatrix} .5 \\ .5 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} + \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ e^{-i\cdot t\cdot H_{Ac}} \begin{pmatrix} .5 \\ .5 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{.5} \\ 0 \end{bmatrix} \otimes e^{-i\cdot t\cdot H_{Ad}} \cdot \begin{bmatrix} \sqrt{.5} \\ \sqrt{.5} \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{.5} \end{bmatrix} \otimes e^{-i\cdot t\cdot H_{Ac}} \cdot \begin{bmatrix} \sqrt{.5} \\ \sqrt{.5} \end{bmatrix} = \sqrt{.5} \left(\begin{bmatrix} \psi_D \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_C \end{bmatrix} \right) \ .$$

Note that the tensor product separation of the state vector involves a part of what I believe the opponent will do $\otimes$ what I intend to do myself. Also, observe that $\begin{bmatrix} \psi_D \\ 0 \end{bmatrix}$ is the state after step 3 given that the opponent is known to defect, and $\begin{bmatrix} 0 \\ \psi_C \end{bmatrix}$ is the state after step 3 given that the opponent is known to cooperate.

In order to compute the probability of me defecting from this state vector, we need to apply the operator $I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. This operator leaves unchanged the part of the state vector corresponding to my belief of what the opponent does and collapses the part of the state vector corresponding to my intended action along the eigenstate for defecting. Accordingly, the probability of defecting for the unknown state equals

$$P[D] = (.5) \left| \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} \psi_D \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_C \end{bmatrix} \right) \right|^2$$

$$= (.5) \left| \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} \psi_D \\ 0 \end{bmatrix} + \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ \psi_C \end{bmatrix} \right|^2.$$

Note that $\left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} \psi_D \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_{DD} \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ \psi_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \psi_{CD} \\ 0 \end{bmatrix}$.

These two vectors are orthogonal and so

$$P[D] = (.5)\left|\left(I \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)\cdot\left(\begin{bmatrix}\psi_D\\0\end{bmatrix}\right)\right|^2 + (.5)\left|\left(I \otimes \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)\cdot\left(\begin{bmatrix}0\\\psi_C\end{bmatrix}\right)\right|^2$$

$$= (.5)|\psi_{DD}|^2 + (.5)|\psi_{CD}|^2.$$

The last expression equals the equal weight average of the probability of defecting given the opponent is known to defect ($|\psi_{DD}|^2$) and the probability of defecting given the opponent is known to cooperate ($|\psi_{CD}|^2$).

4. Proof that the Markov model always fails to explain the violations of the sure thing principle, no matter what initial state is used for the unknown case and for any intensity matrix.

Define $T = e^{t\cdot K}$ for the Markov model, where *K* is an *arbitrary* intensity matrix. Define $\psi_0 = \begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{bmatrix} = p\cdot\begin{bmatrix}1\\0\end{bmatrix}\otimes\psi_D + (1-p)\cdot\begin{bmatrix}0\\1\end{bmatrix}\otimes\ \psi_C$ to be an arbitrary initial state where $p = (\psi_1+\psi_2)$ and $(1-p) = (\psi_3+\psi_4)$ and $\psi_D = \frac{1}{\psi_1+\psi_2}\cdot\begin{bmatrix}\psi_1\\\psi_2\end{bmatrix}$ and $\psi_C = \frac{1}{\psi_3+\psi_4}\cdot\begin{bmatrix}\psi_3\\\psi_4\end{bmatrix}$. In the unknown case, the final state equals $\psi_2 = T\cdot\psi_0 = T\cdot\left(p\cdot\begin{bmatrix}1\\0\end{bmatrix}\otimes\psi_D + (1-p)\cdot\begin{bmatrix}0\\1\end{bmatrix}\otimes\psi_C\right) = p\cdot T\cdot\begin{bmatrix}1\\0\end{bmatrix}\otimes\ \psi_D + (1-p)\cdot T\cdot\begin{bmatrix}1\\0\end{bmatrix}\otimes\psi_C$. The two parts, $T\cdot\begin{bmatrix}1\\0\end{bmatrix}\otimes\psi_D$ and $T\cdot\begin{bmatrix}0\\1\end{bmatrix}\otimes\psi_C$, determine the probabilities of defection when you know the opponent's decision. Thus the probability of defection in the unknown case is a weighted average or convex combination of the two known cases with weights equal to *p* and (1-*p*).

5. Proof that the quantum model produces interference that violates the law of total probability.

On the one hand, if the opponent is known to defect, then the state after step 3 equals

$$\psi_{2,D} = e^{-i \cdot t \cdot H_C} \cdot \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix};$$

on the other hand, if the opponent is known to cooperate, then the state after step 3 equals

$$\psi_{2,C} = e^{-i \cdot t \cdot H_C} \cdot \tfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then for the unknown case, the state after step 3 equals

$$\psi_{2,U} = e^{-i \cdot t \cdot H_C} \cdot \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix} = \sqrt{.5} \cdot e^{-i \cdot t \cdot H_C} \cdot \left(\begin{bmatrix} \sqrt{.5} \\ \sqrt{.5} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sqrt{.5} \\ \sqrt{.5} \end{bmatrix} \right)$$

$$= \left(\sqrt{.5} \cdot e^{-i \cdot t \cdot H_C} \begin{bmatrix} \sqrt{.5} \\ \sqrt{.5} \\ 0 \\ 0 \end{bmatrix} + \sqrt{.5} \cdot e^{-i \cdot t \cdot H_C} \begin{bmatrix} 0 \\ 0 \\ \sqrt{.5} \\ \sqrt{.5} \end{bmatrix} \right) = \sqrt{.5} \cdot \psi_{2,D} + \sqrt{.5} \cdot \psi_{2,C}.$$

The probability of defecting in the two known cases equal

$$P[D|opponent\ defects] = \left| \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \psi_{2,D} \right|^2 = |\phi_D|^2$$

$$P[D|opponent\ cooperates] = \left| \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \psi_{2,C} \right|^2 = |\phi_C|^2$$

The probability of defecting in the unknown case equals

$$P[D|unknown] = (.5) \left| \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \psi_{2,D} + \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \psi_{2,C} \right|^2$$

$$= (.5) \cdot |\phi_D + \phi_C|^2 = (.5) \cdot \left[|\phi_D|^2 + |\phi_C|^2 + \left({\phi_D}^{\dagger} \phi_C + {\phi_C}^{\dagger} \phi_D \right) \right],$$

where the last term in soft brackets is the interference term, which is non zero because the two vectors are ***not*** orthogonal.

6. Proof that if we use K_A for the two known conditions, and we use K_C only for the unknown condition, then the Markov model still obeys the law of total probability and fails to explain the violations of the sure thing principle.

According to the Markov model, the probability of choosing defection is determined by adding the first and third row of $\psi(t)$ as follows: $\Pr[D] = [1 \quad 0 \quad 1 \quad 0] \cdot \psi(t) = J \cdot \psi(t)$, where J = [1 0 1 0]. For the unknown condition, this probability changes across time according to the differential equation

$$\frac{d}{dt} J \cdot \psi = J \cdot \frac{d}{dt} \psi = J \cdot K_C \cdot \psi = J \cdot (K_A + K_B) \cdot \psi$$

$$= J \cdot K_A \cdot \psi + J \cdot K_B \cdot \psi = J \cdot K_A \cdot \psi + 0 \,,$$

because $J \cdot K_B = 0$ (this is obvious in the expression immediately below Eq 2a). Thus K_B has no effect on this probability. This implies that the probability of defection in the unknown case equals

$$\Pr[D|unknown] = \int_0^t J \cdot \frac{d}{dt} \psi = J \cdot \int_0^t \frac{d}{dt} \psi = J \cdot \int_0^t K_A \cdot \psi = J \cdot e^{K_A \cdot t} \cdot \psi_0.$$

Now we can always define $\psi_0 = \left\{ p \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \psi_D + (1-p) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \ \psi_C \right\}$ by setting $p = (\psi_1 + \psi_2)$ and $(1-p) = (\psi_3 + \psi_4)$ and $\psi_D = \frac{1}{\psi_1 + \psi_2} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ and $\psi_C = \frac{1}{\psi_3 + \psi_4} \cdot \begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix}$. Thus

$$J \cdot e^{K_A \cdot t} \cdot \psi_0 = J \cdot e^{K_A \cdot t} \cdot \left\{ p \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \psi_D + (1-p) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \ \psi_C \right\},$$

$$= J \cdot e^{K_A \cdot t} \cdot p \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \psi_D + J \cdot e^{K_A \cdot t} \cdot (1-p) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \ \psi_C$$

$$= p \cdot \Pr[D|known\ Defect] + (1-p) \cdot \Pr\ [D|known\ cooperate].$$

Thus if we use only the K_A intensity matrix for the known cases, and we use the combined K_C intensity for the unknown case, then this does not change the probabilities of defection and the law of total probability must be maintained. The belief intensity matrix K_B only changes the beliefs but does not affect the total probability of defection.

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Figure captions.

Figure 1. The probability of defection under the unknown condition minus the average for the two known conditions, at six time points (note that time incorporates a factor of pi/2). Negative values (blue) typically indicate an interference effect in the predicted direction.

Running head: Quantum probability

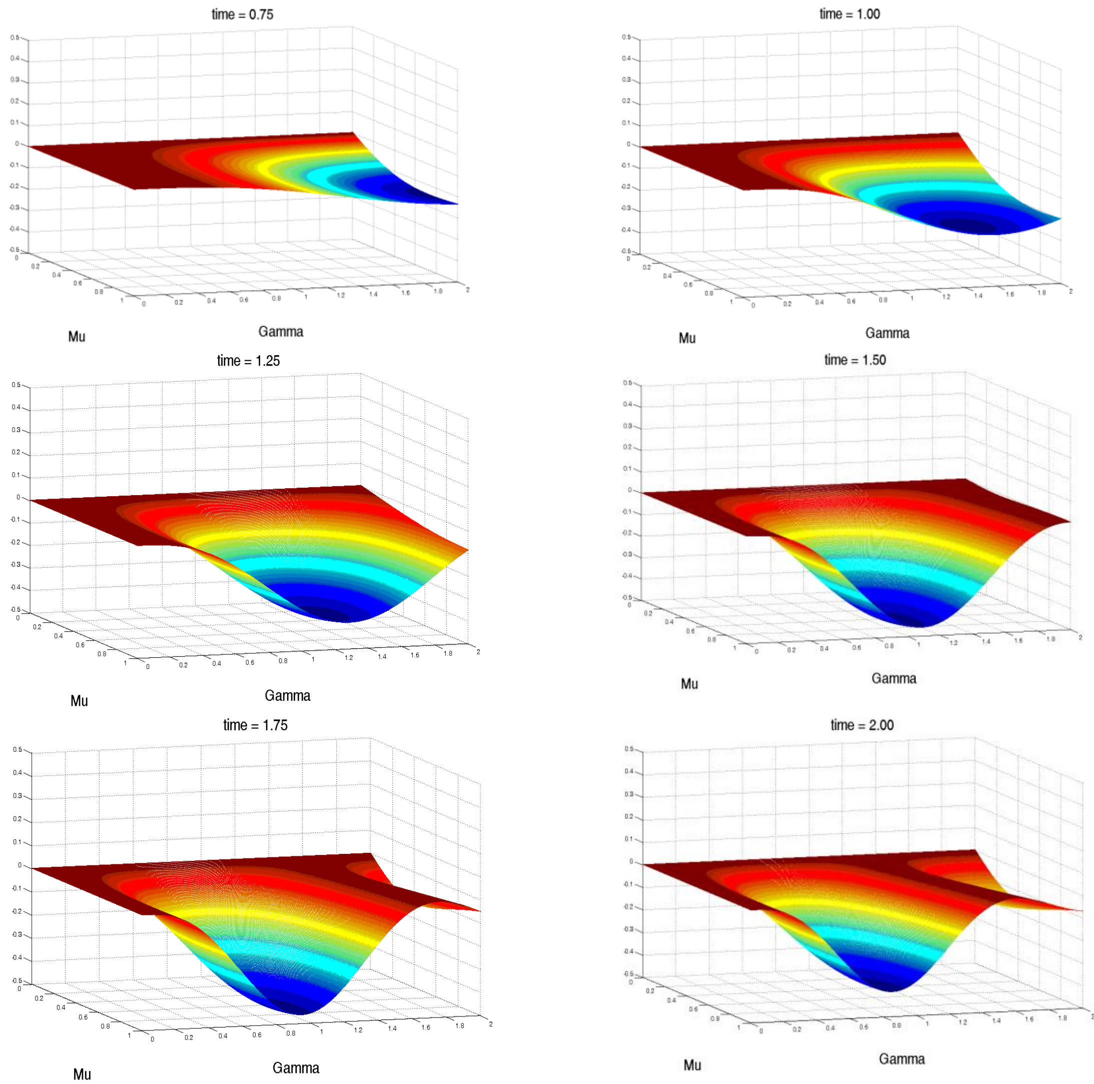
Table 1. Example payoff matrix for prisoner's dilemma game.

	You Defect	You Cooperate
Other Defects	Other: 10 You: 10	Other: 25 You: 5
Other Cooperates	Other: 5 You: 25	Other: 20 You: 20

Table 2. Empirically observed proportion of defections different conditions in the prisoner's dilemma game

Study	Known to Defect	Known to Cooperate	Unknown
Shafir & Tversky, 1992	97	84	63
Croson, 1999 (Avg. of first two experiments)	67	32	30
Li, Taplan, 2002	83	66	60
Busemeyer, Matthew, Wang, 2005	91	84	66
Average	84	66	55
Q Model	81	65	57

Figure 1


time = 0.75
Mu
Gamma
time = 1.00
Mu
Gamma
time = 1.25
Mu
Gamma
time = 1.50
Mu
Gamma
time = 1.75
Mu
Gamma
time = 2.00
Mu
Gamma

SUPPLEMENTARY MATERIAL – ALTERNATIVE PROOFS

3. Proof that the Markov and quantum models predict that the probabilities for the unknown case are equal to the average of the probabilities for the known cases when only Equations 1a and 1b are used to generate the final probabilities.

We prove this for the quantum model. The same argument applies for the Markov model. The state after step 3 for the unknown case equals

$$\psi_2 = e^{-i\cdot t\cdot H_A}\cdot\psi_0 = e^{-i\cdot t\cdot\begin{bmatrix}H_{Ad} & 0\\ 0 & H_{Ac}\end{bmatrix}}\cdot\psi_0 = \begin{bmatrix}e^{-i\cdot t\cdot H_{Ad}} & \mathbf{0}\\ 0 & e^{-i\cdot t\cdot H_{Ac}}\end{bmatrix}\cdot\psi_0$$

$$= \begin{bmatrix}e^{-i\cdot t\cdot H_{Ad}} & 0\\ 0 & e^{-i\cdot t\cdot H_{Ac}}\end{bmatrix}\cdot\begin{bmatrix}.5\\ .5\\ .5\\ .5\end{bmatrix}$$

$$= \begin{bmatrix}e^{-i\cdot t\cdot H_{Ad}} & 0\\ 0 & e^{-i\cdot t\cdot H_{Ac}}\end{bmatrix}\cdot\left(\begin{bmatrix}\begin{pmatrix}.5\\ .5\end{pmatrix}\\ \begin{pmatrix}0\\ 0\end{pmatrix}\end{bmatrix}+\begin{bmatrix}\begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}.5\\ .5\end{pmatrix}\end{bmatrix}\right)$$

The above is a direct sum decomposition of the full state space into the state space for Knowing Other D and the state space for Knowing Other C (under all circumstances, this is possible before time evolution).

Ignoring cognitive dissonance, the unitary operator corresponding to the thought process is completely reducible and can be written as a direct sum of its ***restriction* to the subspaces Knowing Other D and Knowing Other C.**

So, basically, we have:
$\widehat{U}_{Other\ D}\oplus\widehat{U}_{Other\ C}\ (\psi_{other\ D},\psi_{other\ C}) = (\widehat{U}_{Other\ D}\psi_{other\ D},\widehat{U}_{Other\ C}\psi_{other\ C})$

if you set psi = [sqrt(.5) sqrt(.5)]' then psi tensor psi equals psy0 = [.5 .5 . 5 .5]. But our U (above) cannot be written as a tensor product.

$$= \begin{bmatrix} e^{-i \cdot t \cdot H_{Ad}} \begin{pmatrix} .5 \\ .5 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} + \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ e^{-i \cdot t \cdot H_{Ac}} \begin{pmatrix} .5 \\ .5 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{.5} \\ 0 \end{bmatrix} \otimes e^{-i \cdot t \cdot H_{Ad}} \cdot \begin{bmatrix} \sqrt{.5} \\ \sqrt{.5} \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{.5} \end{bmatrix} \otimes e^{-i \cdot t \cdot H_{Ac}} \cdot \begin{bmatrix} \sqrt{.5} \\ \sqrt{.5} \end{bmatrix} = \sqrt{.5} \left(\begin{bmatrix} \psi_D \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_C \end{bmatrix} \right) .$$

Note that the initial state vector could be written as $\psi_{what\ the\ Other\ does} \otimes \psi_{what\ I\ do}$
In fact, this formulation is implied in the above equations (highlighted green, from the paper), since the overall state is expressed as a tensor product in this sense:
$\psi_{what\ the\ Other\ does} \otimes \psi_{what\ I\ do}$

What is slightly confusing is that, even through the overall state can be written as a tensor product, the operator U *cannot* be written as a tensor product (even in the simple case in which there is no cognitive dissonance). By contrast, the U can be written as a direct sum.

So, this is more a presentation point rather than anything else. Basically, starting from:

$$= \begin{bmatrix} e^{-i \cdot t \cdot H_{Ad}} & 0 \\ 0 & e^{-i \cdot t \cdot H_{Ac}} \end{bmatrix} \cdot \left(\begin{bmatrix} \begin{pmatrix} .5 \\ .5 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} + \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} .5 \\ .5 \end{pmatrix} \end{bmatrix} \right)$$

We want to reach:

$$\sqrt{.5} \left(\begin{bmatrix} \psi_D \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_C \end{bmatrix} \right)$$

So, as to show that the final (time-evolved) state in the Unknown (general) situation is a convex combination of the final states when the Other D or the Other C.
(When there is no cognitive dissonance.)

My point is that if we recognize that the equation in yellow can be decomposed into a direct sum $\widehat{U}_{Other\ D} \oplus \widehat{U}_{Other\ C}\ (\psi_{other\ D}, \psi_{other\ C})$, then it follows **immediately** that the final, time-evolved state, can also be written as a direct sum $(\widehat{U}_{Other\ D} \psi_{other\ D}, \widehat{U}_{Other\ C} \psi_{other\ C})$.

That is, the additional steps involving the tensor product expression (equation in green) are *not* needed (and have led to some confusion...)

Do you agree with all this?

Now, in trying to express the model in more abstract form, we know:

1) Without cognitive dissonance, time evolution can be written as: $\hat{U}_{Other\ D} \oplus \hat{U}_{Other\ C}\ (\psi_{other\ D}, \psi_{other\ C})$. This implies that in the Unknown case, the amplitudes relating to knowledge that the Other D evolve independently from the amplitudes relating to knowledge that the Other C.
2) Now, with cognitive dissonance we have that $\hat{U} = \hat{U}_{Other\ D} \oplus \hat{U}_{Other\ C} + other\ terms$ (this is how we constructed $\hat{U}$).
3) We also know that with cognitive dissonance the time-evolved state can *no longer* be written as a direct sum decomposition.

The question is whether we can express these *other terms* in a more abstract form, although it is not immediately obvious how this can be possible.

Note that the tensor product separation of the state vector involves a part of what I believe the opponent will do $\otimes$ what I intend to do myself. Also, observe that $\begin{bmatrix} \psi_D \\ 0 \end{bmatrix}$ is the state after step 3 given that the opponent is known to defect, and $\begin{bmatrix} 0 \\ \psi_C \end{bmatrix}$ is the state after step 3 given that the opponent is known to cooperate.

In order to compute the probability of me defecting from this state vector, we need to apply the operator $I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Here we write the state vector as $\psi_{what\ the\ Other\ does} \otimes \psi_{what\ I\ do}$ noting that both components are normalized. Then we have:

$$I \otimes M_D(\psi_{what\ the\ Other\ does} \otimes \psi_{what\ I\ do}) = \psi_{what\ the\ Other\ does} \otimes M_D \psi_{what\ I\ do}$$

The amplitude of this vector is the probability of me D:

$$\begin{aligned} |\psi_{what\ the\ Other\ does} \otimes M_D \psi_{what\ I\ do}|^2 \\ &= \langle \psi_{what\ the\ Other\ does} \otimes M_D \psi_{what\ I\ do} | \psi_{what\ the\ Other\ does} \otimes M_D \psi_{what\ I\ do} \rangle \\ &= \langle \psi_{what\ the\ Other\ does} | \psi_{what\ the\ Other\ does} \rangle \langle M_D \psi_{what\ I\ do} | M_D \psi_{what\ I\ do} \rangle \\ &= \langle M_D \psi_{what\ I\ do} | M_D \psi_{what\ I\ do} \rangle \end{aligned}$$

But in terms of the computations we make, we can't separate out the state vector like this: $\psi_{what\ the\ Other\ does} \otimes \psi_{what\ I\ do}$

So, what we do is compute $\langle M_D \psi_{y full\ state\ vector} | M_D \psi_{full\ state\ vector} \rangle = |DD|^2 + |CD|^2$

This operator leaves unchanged the part of the state vector corresponding to my belief of what the opponent does and collapses the part of the state vector corresponding to my intended action along the eigenstate for defecting. Accordingly, the probability of defecting for the unknown state equals

$$P[D] = (.5)\left|\left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} \psi_D \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_C \end{bmatrix}\right)\right|^2$$

The reason why this is slightly confusing is that you are employing a tensor product operator of the form $I \otimes M_D$ to a direct sum decomposition of the state vector of the form: $(\psi_{other\ D}, \psi_{other\ C})$

$$= (.5)\left|\left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} \psi_D \\ 0 \end{bmatrix} + \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} 0 \\ \psi_C \end{bmatrix}\right|^2.$$

Note that $\left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} \psi_D \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_{DD} \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} 0 \\ \psi_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \psi_{CD} \\ 0 \end{bmatrix}$.

These two vectors are orthogonal and so

$$P[D] = (.5)\left|\left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} \psi_D \\ 0 \end{bmatrix}\right)\right|^2 + (.5)\left|\left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 0 \\ \psi_C \end{bmatrix}\right)\right|^2$$

$$= (.5)|\psi_{DD}|^2 + (.5)|\psi_{CD}|^2.$$

An alternative way to express this is by taking advantage of the possible direct sum decomposition of the time-evolved state, without cognitive dissonance, which is: $(\hat{U}_{Other\ D} \psi_{other\ D}, \hat{U}_{Other\ C} \psi_{other\ C})$

Then, if I am interested in me D, I need to pick out the relevant state in both subspaces, so that I have to apply the operator $M_D \oplus M_D$ (which in itself can be written as a direct sum).

Then,

$$\Pr(Me\ D|Unknown) = |M_D \oplus M_D(\hat{U}_{Other\ D}\psi_{other\ D}, \hat{U}_{Other\ C}\psi_{other\ C})|^2$$
$$= |(M_D\hat{U}_{Other\ D}\psi_{other\ D}, M_D\hat{U}_{Other\ C}\psi_{other\ C})|^2$$
$$= \langle (M_D\hat{U}_{Other\ D}\psi_{other\ D}, M_D\hat{U}_{Other\ C}\psi_{other\ C}) | (M_D\hat{U}_{Other\ D}\psi_{other\ D}, M_D\hat{U}_{Other\ C}\psi_{other\ C}) \rangle$$
$$= \langle M_D\hat{U}_{Other\ D}\psi_{other\ D} | M_D\hat{U}_{Other\ D}\psi_{other\ D} \rangle_{Other\ D}$$
$$+ \langle M_D\hat{U}_{Other\ C}\psi_{other\ C} | M_D\hat{U}_{Other\ C}\psi_{other\ C} \rangle_{Other\ C} = |DD|^2 + |CD|^2$$

(where any coefficients are implied in the vectors).

As before, the whole point of avoiding the tensor products is that the general state vector is naturally written as a direct sum of the subspaces for the Other D and the Other C, and the *U* time evolution operator can be written as a direct sum of its restriction to these subspaces.

The last expression equals the equal weight average of the probability of defecting given the opponent is known to defect ($|\psi_{DD}|^2$) and the probability of defecting given the opponent is known to cooperate ($|\psi_{CD}|^2$).

5. Proof that the quantum model produces interference that violates the law of total probability.

On the one hand, if the opponent is known to defect, then the state after step 3 equals

$$\psi_{2,D} = e^{-i \cdot t \cdot H_C} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix};$$

on the other hand, if the opponent is known to cooperate, then the state after step 3 equals

$$\psi_{2,C} = e^{-i \cdot t \cdot H_C} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then for the unknown case, the state after step 3 equals

$$\psi_{2,U} = e^{-i\cdot t\cdot H_C}\cdot\begin{bmatrix}.5\\.5\\.5\\.5\end{bmatrix} = \sqrt{.5}\cdot e^{-i\cdot t\cdot H_C}\cdot\left(\begin{bmatrix}\sqrt{.5}\\\sqrt{.5}\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\\sqrt{.5}\\\sqrt{.5}\end{bmatrix}\right)$$

$$= \left(\sqrt{.5}\cdot e^{-i\cdot t\cdot H_C}\begin{bmatrix}\sqrt{.5}\\\sqrt{.5}\\0\\0\end{bmatrix}+\sqrt{.5}\cdot e^{-i\cdot t\cdot H_C}\begin{bmatrix}0\\0\\\sqrt{.5}\\\sqrt{.5}\end{bmatrix}\right) = \sqrt{.5}\cdot\psi_{2,D}+\sqrt{.5}\cdot\psi_{2,C}.$$

The probability of defecting in the two known cases equal

$$P[D|opponent\ defects] = \left|\left(I\otimes\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)\cdot\psi_{2,D}\right|^2 = |\phi_D|^2$$

$$P[D|opponent\ cooperates] = \left|\left(I\otimes\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)\cdot\psi_{2,C}\right|^2 = |\phi_C|^2$$

This demonstration hinges on the fact that $|\phi_D|^2$ and $|\phi_C|^2$ are no longer orthogonal if we allow for cognitive dissonance. As before, I wanted to express this without the tensor product and see whether the resulting formulation might be simpler:

So, *with* cognitive dissonance, we have:
$\{\widehat{U}_{Other\ D}\oplus\widehat{U}_{Other\ C} + other\ terms\}(\psi_{other\ D},\psi_{other\ C})$

The above can be written as
$T\cdot\psi_{other\ D} + T\cdot\psi_{other\ C} = \phi_D + \phi_C$
noting that while the initial state vector could be written as a direct sum, this is *not* the case for the time-evolved state vector and, relatedly, that ϕ_D , ϕ_C would in general not be orthogonal. That is, $\phi_D = T\cdot\psi_{other\ D}$ will have non-zero components for all possible projections.

Then, the probability of me D in the unknown case is given by $|M_D(\phi_D+\phi_C)|^2$, which leads to the same result that you obtained. [This will lead to cross terms.]

As before, the only *possible* advantage of expressing things in the above way is that we employ the natural (for our problem) direct sum decomposition of the state vector into the subspaces corresponding to Other D and Other C.

(Note that without cognitive dissonance the cross terms are of the form: $\langle M_D \hat{U} \psi_{other\ D} | M_D \hat{U} \psi_{other\ C} \rangle$, whereby $\langle \hat{U} \psi_{other\ D} | \hat{U} \psi_{other\ C} \rangle = 0$ and *we know* that $\langle M_D \hat{U} \psi_{other\ D} | M_D \hat{U} \psi_{other\ C} \rangle = 0$, by virtue of fact that these projections exist in different subspaces.)

The probability of defecting in the unknown case equals

$$P[D|unknown] = (.5) \left| \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \psi_{2,D} + \left(I \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \psi_{2,C} \right|^2$$

$$= (.5) \cdot |\phi_D + \phi_C|^2 = (.5) \cdot \left[|\phi_D|^2 + |\phi_C|^2 + \left({\phi_D}^{\dagger} \phi_C + {\phi_C}^{\dagger} \phi_D \right) \right],$$

where the last term in soft brackets is the interference term, which is non zero because the two vectors are ***not*** orthogonal.