Solvable two dimensional time-dependent non-Hermitian quantum systems with infinite dimensional Hilbert space in the broken PT-regime

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Abstract: We provide exact analytical solutions for a two dimensional explicitly time-dependent non-Hermitian quantum system. While the time-independent variant of the model studied is in the broken PT-symmetric phase for the entire range of the model parameters, and has therefore a partially complex energy eigenspectrum, its time-dependent version has real energy expectation values at all times. In our solution procedure we compare the two equivalent approaches of directly solving the time-dependent Dyson equation with one employing the Lewis-Riesenfeld method of invariants. We conclude that the latter approach simplifies the solution procedure due to the fact that the invariants of the non-Hermitian and Hermitian system are related to each other in a pseudo-Hermitian fashion, which in turn does not hold for their corresponding time-dependent Hamiltonians. Thus constructing invariants and subsequently using the pseudo-Hermiticity relation between them allows to compute the Dyson map and to solve the Dyson equation indirectly. In this way one can bypass to solve nonlinear differential equations, such as the dissipative Ermakov-Pinney equation emerging in our and many other systems.

1. Introduction

In the context of non-Hermitian time-independent quantum mechanics many systems are known to posses real spectra in a certain parameter regime that becomes spontaneously broken when some coupling constants are driven beyond the exceptional point [1, 2, 3, 4]. Unlike their optical analogues [5, 6, 7], where the spontaneously broken regime is of great interest, in quantum mechanics this regime is usually discarded on grounds of being nonphysical since it leads inevitably to infinite growth in energy due to the fact that the energy eigenvalues emerge as complex conjugate pairs. In [8] we demonstrated that the introduction of an explicit time-dependence into a non-Hermitian Hamiltonian can make the spontaneously broken $\mathcal{PT}$-regime physically meaningful. The reason for this phenomenon is that the energy operator becomes modified due an additional term related to the Dyson operator and hence its expectation values can become real. Here we extend the previous analysis of the broken $\mathcal{PT}$-regime from a one dimensional two-level system [8] to a two-dimensional system with infinite Hilbert space.

In addition, we show that technically it is simpler to employ Lewis-Riesenfeld invariants [9] instead of directly solving the time-dependent Dyson map or the time-dependent quasi-Hermiticity relation. All approaches are of course equivalent, but the invariant method splits the problem into several more treatable steps. In particular, it can be viewed as reformulating the nonpseudo-Hermitian relation for the Hamiltonians involved, i.e. the time-dependent Dyson relation, into a pseudo-Hermitian relation for the corresponding invariants. The latter quantities are well studied in the time-independent setting and are far easier to solve as they do not involve derivatives with respect to time. Loosely speaking the time-derivative in the time-dependent Dyson relation acting on the Dyson map has been split up into the two time-derivatives acting on the invariants ensuring their conservation. Besides this aspect related to the technicalities associated to the solution procedure we also provide the first explicitly solved time-dependent system in higher dimensions.

Our manuscript is organized as follows: In section 2 we recall the key equations that determine the Dyson map and hence the metric operator. In section 3 we introduce our two-dimensional model. As first we demonstrate how it may be solved in a time–independent setting. Subsequently we determine the time-dependent Dyson map in two alternative ways, comparing the direct and the Lewis-Riesenfeld method. In addition, we compute the analytical solutions to the time-dependent Schrödinger equation and use them to evaluate instantaneous energy expectation values. Our conclusions are stated in section 4.

2. Time-dependent Dyson equation versus Lewis-Riesenfeld invariants

The central object to compute in the study non-Hermitian Hamiltonian systems is the metric operator ρ that can be expressed in terms of the Dyson operator η as $\rho = \eta^\dagger \eta$. Unlike as in the time-independent scenario a non-Hermitian Hamiltonian $H(t) \neq H^\dagger(t)$ can no longer be related to a Hermitian counterpart $h(t) = h^\dagger(t)$ in a pseudo-Hermitian way, that is via a similarity transformation, but instead the two Hamiltonians are related to each other by means of the time-dependent Dyson relation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t). \tag{2.1}$$

When the Hamiltonian $h(t)$ is observable, this relation implies immediately that the Hamiltonian $H(t)$ is not observable [10, 11, 12, 13] as the latter is not a self-adjoint operator with regard to the standard or modified inner product. The Hamiltonians are understood to be the operators governing the time-evolution of the systems satisfying the time-dependent Schrödinger equations

$$\mathcal{H}(t)\Psi_{\mathcal{H}}(t) = i\hbar\partial_t\Psi_{\mathcal{H}}(t), \qquad \text{for } \mathcal{H} = h, H. \tag{2.2}$$

The Hamiltonian is only identical to the observable energy operator in the Hermitian case, but different in the non-Hermitian setting where it has to be modified to

$$\tilde{H}(t) := \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t). \tag{2.3}$$

The two wavefunctions in (2.2) are related to each other by the Dyson map

$$\Psi_h(t) = \eta(t)\Psi_H(t). \tag{2.4}$$

Besides the time-dependent Dyson relation also the time-dependent quasi-Hermiticity relation is then modified, by acquiring an additional derivative term in the metric operator

$$H^\dagger(t)\rho(t) - \rho(t)H(t) = i\hbar\partial_t\rho(t). \tag{2.5}$$

It was demonstrated [13, 14, 15, 16, 8] that the equations (2.1) and (2.5) can be directly solved consistently for $\eta(t)$ and $\rho(t)$, respectively. Alternatively, but completely equivalent, one may also employ the standard Lewis-Riesenfeld approach [9] of computing invariants as argued in [17, 18]. This approach requires to compute the two conserved time-dependent invariants $I_h(t)$ and $I_H(t)$, i.e. $dI_h/dt = dI_H/dt = 0$, from the evolution equations

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar\left[I_{\mathcal{H}}(t), \mathcal{H}(t)\right] = 0, \qquad \text{for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger. \tag{2.6}$$

Using these two equations together with the Dyson relation (2.1) it is straightforward to derive that the two invariants are simply related by a similarity transformation

$$I_h(t) = \eta(t)I_H(t)\eta^{-1}(t). \tag{2.7}$$

Since the invariant I_h is Hermitian, the invariant I_H is its pseudo-Hermitian counterpart. When I_h and I_H have been constructed, (2.7) is a much easier equation to solve for $\eta(t)$, than directly the Dyson relation (2.1). At this point one has therefore also obtained the metric operator simply by $\rho = \eta^\dagger\eta$. Next one may also employ the invariants to construct the time-dependent eigenstates from the standard equations [9]

$$I_{\mathcal{H}}(t)\left|\phi_{\mathcal{H}}(t)\right\rangle = \Lambda\left|\phi_{\mathcal{H}}(t)\right\rangle, \qquad \left|\Psi_{\mathcal{H}}(t)\right\rangle = e^{i\hbar\alpha(t)}\left|\phi_{\mathcal{H}}(t)\right\rangle, \tag{2.8}$$

$$\dot{\alpha} = \left\langle\phi_{\mathcal{H}}(t)\right| i\hbar\partial_t - \mathcal{H}(t)\left|\phi_{\mathcal{H}}(t)\right\rangle, \qquad \dot{\Lambda} = 0 \tag{2.9}$$

for $\mathcal{H} = h$ and $\mathcal{H} = H$. Below we compare the two approaches and conclude that even though the approach using invariants is more lengthy, it dissects the original problem into several easier smaller steps when compared to solving the Dyson equation directly. Of course both approaches are equivalent and must lead to the same solutions for $\eta(t)$, as we also demonstrate.

In what follows we set $\hbar = 1$.

3. 2D systems with infinite Hilbert space in the broken $\mathcal{PT}$-regime

3.1 Two dimensional time-independent models

We set up our model by considering at first a $\mathcal{PT}$-symmetric system that we then slightly modify by going from a model with partially broken $\mathcal{PT}$-symmetry to one with completely broken $\mathcal{PT}$-symmetry[1]. We commence with one of the simplest options for a two-dimensional non-Hermitian system by coupling two harmonic oscillators with a non-Hermitian coupling term in space

$$H_{xy} = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\left(\Omega_x^2 x^2 + \Omega_y^2 y^2\right) + i\kappa xy, \qquad m, \kappa, \Omega_x, \Omega_y \in \mathbb{R}. \tag{3.1}$$

This non-Hermitian Hamiltonian is symmetric with regard to the antilinear transformations [19] $\mathcal{PT}_\pm : x \to \pm x$, $y \to \mp y$, $p_x \to \mp p_x$, $p_y \to \pm p_y$, $i \to -i$, i.e. $[\mathcal{PT}_\pm, H_{xy}] = 0$. Using standard techniques from $\mathcal{PT}$-symmetric/quasi-Hermitian quantum mechanics [1, 2, 3], it can be decoupled easily into two harmonic oscillators

$$h_{xy} = \eta H_{xy} \eta^{-1} = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right), \tag{3.2}$$

by a simple rotation using the angular momentum operator $L_z = xp_y - yp_x$ in the Dyson map $\eta = e^{\theta L_z}$ and constraining the parameters involved as

$$\omega_x^2 = \frac{\Omega_x^2 \cosh^2\theta + \Omega_y^2 \sinh^2\theta}{\cosh 2\theta}, \quad \omega_y^2 = \frac{\Omega_x^2 \sinh^2\theta + \Omega_y^2 \cosh^2\theta}{\cosh 2\theta}, \quad \tanh 2\theta = \frac{2\kappa}{m\left(\Omega_y^2 - \Omega_x^2\right)}. \tag{3.3}$$

By the last equation in (3.3) it follows that one has to restrict $|\kappa| \leq m\left(\Omega_y^2 - \Omega_x^2\right)/2$ for this transformation to be meaningful. Thus as long as the Dyson map is well defined, i.e. the constraint holds, the energy eigenspectra

$$E_{n,m} = \left(n + \frac{1}{2}\right)\omega_x + \left(m + \frac{1}{2}\right)\omega_y. \tag{3.4}$$

of h and H are identical and real. The restriction on κ is the same as the one found in [20, 21], where the decoupling of H to h was realized by an explicit coordinate transformation instead of the Dyson map. In fact, identifying the parameter k in [20] as $k = \cosh 2\theta$, and somewhat similarly in [21], the coordinate transformation becomes a rotation realized by the similarity transformation acting on the coordinates and the momenta, i.e. we obtain $H \to h$ with the coordinate transformation

$$v \to \quad \eta v \eta^{-1} = \begin{pmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{pmatrix} v, \qquad \text{for } v = \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} p_x \\ p_y \end{pmatrix}. \tag{3.5}$$

Such a scenario is mostly well understood and in analogy to the case studied in [8], solving the time-dependent Dyson equation for $\eta(t)$ will allow to make sense of the regime for $\kappa \to \kappa(t)$ beyond the exceptional point.

[1] We use here the standard terminology, referring to the situation $[\mathcal{PT}, H] = 0$, $\mathcal{PT}\phi_H = \phi_H$ as $\mathcal{PT}$-symmetric, $[\mathcal{PT}, H] = 0$, $\mathcal{PT}\phi_H \neq \phi_H$ as spontaneously broken $\mathcal{PT}$-symmetric and $[\mathcal{PT}, H] \neq 0$, $\mathcal{PT}\phi_H \neq \phi_H$ as completely broken.

Let us now slightly modify the model above by modifying some of the constants and by adding a term that also couples the two harmonic oscillator Hamiltonians in the momenta

$$H_{xyp} = \frac{a}{2}\left(p_x^2 + x^2\right) + \frac{b}{2}\left(p_y^2 + y^2\right) + i\frac{\lambda}{2}\left(xy + p_x p_y\right), \qquad a, b, \lambda \in \mathbb{R}. \tag{3.6}$$

Clearly this Hamiltonian is also symmetric with regard to the same antilinear symmetry as H_{xy}, i.e. we have $[\mathcal{PT}_\pm, H_{xyp}] = 0$. Thus we expect the eigenvalues to be real or to be grouped in pairs of complex conjugates when the symmetry is broken for the wavefunctions.

It is convenient to express this Hamiltonian in a more generic algebraic fashion as

$$H_K = aK_1 + bK_2 + i\lambda K_3, \tag{3.7}$$

where we defined Lie algebraic generators

$$K_1 = \frac{1}{2}\left(p_x^2 + x^2\right), \quad K_2 = \frac{1}{2}\left(p_y^2 + y^2\right), \quad K_3 = \frac{1}{2}\left(xy + p_x p_y\right), \quad K_4 = \frac{1}{2}\left(xp_y - yp_x\right). \tag{3.8}$$

Besides the generators already appearing in the Hamiltonian we added one more generator, $K_4 = L_z/2$, to ensure the closure of the algebra, i.e. we have

$$\begin{array}{lll} [K_1, K_2] = 0, & [K_1, K_3] = iK_4, & [K_1, K_4] = -iK_3, \\ [K_2, K_3] = -iK_4, & [K_2, K_4] = iK_3, & [K_3, K_4] = i(K_1 - K_2)/2. \end{array} \tag{3.9}$$

Notice that $K_i^\dagger = K_i$ for $i = 1, \ldots, 4$. In what follows we mostly use the algebraic formulation so that our results also hold for representations different from (3.8). We report that the Hamiltonian H_{xy} in (3.1) requires at least a ten dimensional Lie algebra when demanding xy to be one of the Lie algebraic generators, which is the reason we consider first the more compactly expressible Hamiltonian H_{xyp}.

Using the same form of the Dyson map $\eta = e^{\theta L_z}$ as above, albeit with $\theta = \operatorname{arctanh}[\lambda/(b-a)]$, this Hamiltonian is decoupled into

$$h_K = \eta H_K \eta^{-1} = \frac{1}{2}(a+b)\left(K_1 + K_2\right) + \frac{1}{2}\sqrt{(a-b)^2 - \lambda^2}\left(K_1 - K_2\right), \tag{3.10}$$

for $|\lambda| < |a-b|$. So clearly for $a = b$ we are in the spontaneously broken $\mathcal{PT}$-regime[2]. That choice is in addition very convenient as it allows for a systematic construction of the eigenvalue spectrum of $H_K(b=a)$. Since the following commutators vanish $[H_K(b=a), K_1 + K_2] = [H_K(b=a), K_3] = [K_1 + K_2, K_3] = 0$, one simply needs to search for simultaneous eigenstates of K_3 and $K_1 + K_2$ to determine the eigenstates if $H_K(b=a)$, due to Schur's lemma. Indeed for the representation (3.8) we obtain for $H_K(b=a)$ the eigenstates

$$\varphi_{n,m}(x,y) = \frac{e^{-\frac{x^2}{2} - \frac{y^2}{2}}}{2^{n+m}\sqrt{n!m!\pi}} \left[\sum_{k=0}^{n} \binom{n}{k} H_k(x) H_{n-k}(y)\right] \left[\sum_{l=0}^{m} (-1)^l \binom{m}{l} H_l(y) H_{m-l}(x)\right], \tag{3.11}$$

[2] We use here the standard terminology, referring to the situation $[\mathcal{PT}, H] = 0$, $\mathcal{PT}\phi_{\mathcal{H}} = \phi_{\mathcal{H}}$ as $\mathcal{PT}$-symmetric, $[\mathcal{PT}, H] = 0$, $\mathcal{PT}\phi_{\mathcal{H}} \neq \phi_{\mathcal{H}}$ as spontaneously broken $\mathcal{PT}$-symmetry and $[\mathcal{PT}, H] \neq 0$, $\mathcal{PT}\phi_{\mathcal{H}} \neq \phi_{\mathcal{H}}$.

with corresponding eigenenergies

$$E_{n,m} = E_{m,n}^{*} = a(1+n+m) + i\frac{\lambda}{2}(n-m). \tag{3.12}$$

Here $H_n(x)$ denotes the n-th Hermite polynom in x. The states are orthonormal with regard to the standard inner product $\langle \varphi_{n,m} | \varphi_{n',m'} \rangle = \delta_{n,n'}\delta_{m,m'}$. The reality of the subspectrum with $n = m$ is explained by the fact that the $\mathcal{PT}_{\pm}$-symmetry is preserved, i.e. we can verify that $\mathcal{PT}_{\pm}\, \varphi_{n,n} = \varphi_{n,n}$. However, when $n \neq m$ the $\mathcal{PT}_{\pm}$-symmetry is spontaneously broken and the eigenvalues occur in complex conjugate pairs.

Hence this Hamiltonian should be discarded as nonphysical in the time-independent regime, but we shall see that it becomes physically acceptable when the parameters a and λ are taken to be explicitly time-dependent.

3.2 A solvable 2D time-dependent Hamiltonian in the broken $\mathcal{PT}$-regime

We solve now the explicitly time-dependent non-Hermitian Hamiltonian

$$H(t) = \frac{a(t)}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + i\frac{\lambda(t)}{2}\left(xy + p_x p_y\right), \qquad a(t), \lambda(t) \in \mathbb{R}. \tag{3.13}$$

According to the above discussion, the instantaneous eigenvalue spectrum of $H(t)$ belongs to the spontaneously broken $\mathcal{PT}$-regime.

3.2.1 The time-dependent Dyson equation

Let us now compute the right hand side of the time-dependent Dyson relation (2.1). For that purpose we assume that the Dyson map is an element of the group associated to the algebra (3.9) and take it to be of the form

$$\eta(t) = \prod\nolimits_{i=1}^{4} e^{\gamma_i(t) K_i}, \qquad \gamma_i \in \mathbb{R}. \tag{3.14}$$

As η is not a unitary operator by definition, we have taken the γ_i to be real to avoid irrelevant phases. Using now (3.14) and (3.13) in (2.1), the right hand side will be Hermitian if and only if

$$\gamma_1 = \gamma_2 = q_1, \quad \dot{\gamma}_3 = -\lambda \cosh\gamma_4, \quad \dot{\gamma}_4 = \lambda \tanh\gamma_3 \sinh\gamma_4, \tag{3.15}$$

for some real constant $q_1 \in \mathbb{R}$. The Hermitian Hamiltonian results to

$$h(t) = a(t)\left(K_1 + K_2\right) + \frac{\lambda(t)}{2}\frac{\sinh\gamma_4}{\cosh\gamma_3}\left(K_1 - K_2\right). \tag{3.16}$$

For the representation (3.8) these are simply two decoupled harmonic oscillators with time-dependent coefficients. The energy operator $\tilde{H}$ as defined in equation (2.3) becomes

$$\tilde{H}(t) = a(t)\left(K_1 + K_2\right) + \frac{\lambda(t)}{4}\sinh(2\gamma_4)\left(K_1 - K_2\right) - i\lambda(t)\left(\sinh^2\gamma_4 K_3 - \sinh\gamma_4 \tanh\gamma_3 K_4\right). \tag{3.17}$$

The constraining relations (3.15) may be solved directly for γ_3 and γ_4, but not in a straightforward manner. We eliminate λ and dt from the last two equations in (3.15), so that $d\gamma_4 = -\tanh\gamma_3 \tanh\gamma_4 d\gamma_3$, hence obtaining γ_4 as a function of γ_3

$$\gamma_4 = \operatorname{arcsinh}(\kappa \operatorname{sech}\gamma_3) \tag{3.18}$$

with integration constant κ. Defining $\chi(t) := \cosh\gamma_3$ we use (3.15) and (3.18) to derive that the central equation that needs to be satisfied is the Ermakov-Pinney equation [22, 23] with a dissipative term

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} - \lambda^2\chi = \frac{\kappa^2\lambda^2}{\chi^3}. \tag{3.19}$$

This equation is ubiquitous in the context of solving time-dependent Hermitian systems, even in the Hermitian setting, see e.g. [24]. While some solutions to this equation are known, we demonstrate here that solving this nonlinear differential equation can be completely bypassed when employing Lewis-Riesenfeld invariants instead and computing η from the pseudo-Hermiticity relation (2.7) for the invariants instead.

3.2.2 The time-dependent Dyson map from pseudo-Hermiticity

It is natural to assume that the invariants I_H, I_h as well as the Hermitian Hamiltonian $h(t)$ lie in the same algebra as the non-Hermitian Hamiltonian $H(t)$. Furthermore we note that $I_h(t)$ needs to be Hermitian, so that we make the Ansätze

$$I_H(t) = \sum_{i=1}^{4} \alpha_i(t) K_i, \qquad I_h(t) = \sum_{i=1}^{4} \beta_i(t) K_i, \qquad h(t) = \sum_{i=1}^{4} b_i(t) K_i, \tag{3.20}$$

with $\alpha_i = \alpha_i^r + i\alpha_i^i \in \mathbb{C}$, $b_i, \beta_i, \alpha_i^r, \alpha_i^i \in \mathbb{R}$.

The Lewis-Riesenfeld invariant $I_H(t)$: Substituting the expressions for $I_H(t)$ and $H(t)$ into the equation in (2.6) and reading off the coefficients of the generators K_i we obtain the four constraints

$$\dot{\alpha}_1 = \frac{i}{2}\lambda\alpha_4, \quad \dot{\alpha}_2 = -\frac{i}{2}\lambda\alpha_4, \quad \dot{\alpha}_3 = 0, \quad \dot{\alpha}_4 = i\lambda(\alpha_2 - \alpha_1). \tag{3.21}$$

These equations are easily solved by

$$\alpha_1 = \frac{c_1}{2} + c_3 \cosh\left[c_4 - \int_0^t \lambda(s)ds\right], \;\; \alpha_2 = c_1 - \alpha_1, \;\; \alpha_3 = c_2, \;\; \alpha_4 = 2ic_3 \sinh\left[c_4 - \int_0^t \lambda(s)ds\right], \tag{3.22}$$

with complex integration constants $c_i = c_i^r + ic_i^i$, $c_i^r, c_i^i \in \mathbb{R}$. At this point we have two options, we may either compute directly the invariant $I_h(t)$ for the Hamiltonian $h(t)$ as given in (3.16) by using the evolution equation (2.6) or the similarity relation (2.7) instead.

The Lewis-Riesenfeld invariant $I_h(t)$: Denoting the coefficients of K_1 and K_2 in (3.16) by $b_1(t)$ and $b_2(t)$, respectively, as defined in the expansion for generic $h(t)$ in (3.20), the relation for the invariants (2.6) leads to the constraints

$$\dot{\beta}_1 = 0, \quad \dot{\beta}_2 = 0, \quad \dot{\beta}_3 = \beta_4(b_2 - b_1), \quad \dot{\beta}_4 = \beta_3(b_1 - b_2). \tag{3.23}$$

These four coupled first order differential equations are easily solved by

$$\beta_1 = c_5, \quad \beta_2 = c_6, \quad \beta_3 = c_7 \cos\left[c_8 - \int_0^t (b_1 - b_2)ds\right], \quad \beta_4 = -c_7 \sin\left[c_8 - \int_0^t (b_1 - b_2)ds\right]. \tag{3.24}$$

Next we invoke the pseudo-Hermiticity relation for the invariants (2.7).

Relating $I_H(t)$ and $I_h(t)$: So far we have treated the Hermitian and non-Hermitian systems separately. Next we relate them using the Ansätze (3.14) for $\eta(t)$ and (3.20) for the invariants in the expression (2.7). We obtain eight equations by reading off the coefficients and separating the result into real and imaginary parts. We can solve the resulting equations for the real functions

$$\beta_1 = \frac{1}{2}\left[\alpha_1^r + \alpha_2^r - \alpha_4^i \sinh\gamma_3 + \alpha_3^i \sinh\gamma_4 \cosh\gamma_3 + (\alpha_1^r - \alpha_2^r)\cosh\gamma_3 \cosh\gamma_4\right], \tag{3.25}$$

$$\beta_2 = \frac{1}{2}\left[\alpha_1^r + \alpha_2^r + \alpha_4^i \sinh\gamma_3 - \alpha_3^i \sinh\gamma_4 \cosh\gamma_3 - (\alpha_1^r - \alpha_2^r)\cosh\gamma_3 \cosh\gamma_4\right], \tag{3.26}$$

$$\beta_3 = \left(\alpha_2^i - \alpha_1^i\right)\sinh\gamma_4 + \alpha_3^r \cosh\gamma_4, \tag{3.27}$$

$$\beta_4 = \left[\left(\alpha_1^i - \alpha_2^i\right)\cosh\gamma_4 - \alpha_3^r \sinh\gamma_4\right]\sinh\gamma_3 + \alpha_4^r \cosh\gamma_3 \tag{3.28}$$

with the additional constraints

$$\alpha_1^i + \alpha_2^i = 0, \qquad \alpha_3^r\alpha_3^i + \alpha_4^r\alpha_4^i = 2\alpha_1^i(\alpha_2^r - \alpha_1^r), \tag{3.29}$$

$$\tanh\gamma_3 = \frac{\alpha_4^i}{\sqrt{(\alpha_1^r - \alpha_2^r)^2 - (\alpha_3^i)^2}}, \qquad \tanh\gamma_4 = \frac{\alpha_3^i}{\alpha_2^r - \alpha_1^r} . \tag{3.30}$$

We also used here $\gamma_1 = \gamma_2$.

Next we compare our solutions in (3.22), (3.24) and (3.25)-(3.30). First we use the expressions for the α_i from (3.22) in (3.25)-(3.30). The constraints (3.29) imply that $c_1^i = 0$ and $4c_3^r c_3^i = -c_2^r c_2^i$ so that the time-dependent coefficients in the Hermitian invariant I_h result to

$$\beta_1 = \frac{c_1^r}{2} \pm \frac{1}{2}\sqrt{4(c_3^r)^2 - (c_2^i)^2}, \tag{3.31}$$

$$\beta_2 = \frac{c_1^r}{2} \pm \frac{1}{2}\sqrt{4(c_3^r)^2 - (c_2^i)^2}, \tag{3.32}$$

$$\beta_3 = \pm\frac{c_2^r}{2c_3^r}\frac{\left[4(c_3^r)^2 - (c_2^i)^2\right]}{\sqrt{4(c_3^r)^2 - (c_2^i)^2 \operatorname{sech}^2\left[c_4^r - \int_0^t \lambda(s)ds\right]}}, \tag{3.33}$$

$$\beta_4 = \pm\frac{c_2^r c_2^i}{2c_3^r}\sqrt{\frac{4(c_3^r)^2 - (c_2^i)^2}{4(c_3^r)^2 - (c_2^i)^2 \operatorname{sech}^2\left[c_4^r - \int_0^t \lambda(s)ds\right]}} \tanh\left[c_4^r - \int_0^t \lambda(s)ds\right], \tag{3.34}$$

with the constraint $2|c_3^r| > |c_2^i|$. These expressions need to match with those computed directly in (3.25). It is clear how to identify the constants c_5 and c_6 in (3.24) when comparing to (3.31) and (3.32). Less obvious is the comparison between the β_3 and β_4. Reading off b_1 and b_2 from (3.16) and using (3.30), we compute

$$\int_0^t (b_1 - b_2)ds = \arctan\left[\frac{c_2^i}{\sqrt{4(c_3^r)^2 - (c_2^i)^2}} \tanh\left[c_4^r - \int_0^t \lambda(s)ds\right]\right]. \tag{3.35}$$

Setting next the constants $c_8 = 0$, $c_7 = \pm c_2^r\sqrt{4(c_3^r)^2 - (c_2^i)^2}/(2c_3^r)$ the solution in (3.24) matches indeed with (3.33) and (3.34).

We can now assemble our expressions for η by using the results for γ_3 and γ_4 from (3.30) together with the expressions in (3.22) obtaining

$$\gamma_3 = \arctan\left[\frac{\tanh\left[q_2 - \int_0^t \lambda(s)ds\right]}{\sqrt{1 - q_3^2 \operatorname{sech}\left[q_2 - \int_0^t \lambda(s)ds\right]^2}}\right], \tag{3.36}$$

$$\gamma_4 = -\operatorname{arccot}\left[\frac{1}{q_3}\cosh\left[q_2 - \int_0^t \lambda(s)ds\right]\right], \tag{3.37}$$

with the identification $q_2 = c_4^r$ and $q_3 = c_2^i/(2c_3^r)$.

We convince ourselves that the function

$$\chi(t) = \cosh\gamma_3 = \sqrt{\frac{\cosh^2\left[q_2 - \int_0^t \lambda(s)ds\right] - q_3^2}{1 - q_3^2}} \tag{3.38}$$

computed with γ_3 as given in (3.36) does indeed satisfy the dissipative Ermakov-Pinney equation (3.19) when identifying the constants as $\kappa = q_3/\sqrt{1 - q_3^2}$. We also express the Hamiltonian (3.16) explicitly as

$$h(t) = f_+(t)K_1 + f_-(t)K_2 \quad \text{with } f_\pm(t) = a(t) \pm \frac{q_3\sqrt{1 - q_3^2}\lambda(t)}{1 + \cosh\left[2q_2 - 2\int_0^t \lambda(s)ds\right] - 2q_3^2}, \tag{3.39}$$

which is evidently Hermitian for $|q_3| < 1$.

3.2.3 Eigenstates, phases and instantaneous energy expectation values

We note that the computation of the Dyson map did not require the knowledge of any eigenstates, neither when using Lewis-Riesenfeld invariants nor in the directly approach of solving the time-dependent Dyson relation. This also means that so far we have not solved the time-dependent Schrödinger equation nor did we use the eigenstate equations (2.8) and (2.9). Let us therefore carry out the final step and determine all eigenstates, including relevant phases, and use them to evaluate the energy expectation values.

The exact solution to the time-dependent Schrödinger equation for the harmonic oscillator with time-dependent mass and frequency is well known for twenty years [25]. Since the Hamiltonian $h(t)$ in (3.39) are two decoupled harmonic oscillators it suffices to consider the Hamiltonian $\tilde{h}(t)$[25] $a(t)K_1$, with $a(t)$ being any real function of t. Adapting the solution of [25] to our notation and situation, it reads

$$\tilde{\varphi}_n(x,t) = \frac{e^{i\alpha_n(t)}}{\sqrt{\varkappa(t)}} \exp\left[\left(\frac{i}{a(t)}\frac{\dot{\varkappa}(t)}{\varkappa(t)} - \frac{1}{\varkappa^2(t)}\right)\frac{x^2}{2}\right] H_n\left[\frac{x}{\varkappa(t)}\right], \tag{3.40}$$

with phase

$$\alpha_n(t) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{a(s)}{\varkappa^2(s)}ds, \tag{3.41}$$

and $\varkappa(t)$ being restricted to the dissipative Ermakov-Pinney equation

$$\ddot{\varkappa} - \frac{\dot{a}}{a}\dot{\varkappa} + a^2\varkappa = \frac{a^2}{\varkappa^3}. \tag{3.42}$$

Thus while we could bypass to solve this equation in the form of (3.19) for the determination of η when it involved λ, it has re-emerged for the computation of the eigenstates involving a with a different sign in front of the last term on the left hand side. Using the wavefunction (3.40) we compute here the expectation value for K_1 and a normalization factor

$$\langle\tilde{\varphi}_n(x,t)|\, K_1 \,|\tilde{\varphi}_m(x,t)\rangle = 2^{n-2}n!(2n+1)\sqrt{\pi}\frac{a^2(1+\varkappa^4)+\varkappa^2\dot{\varkappa}^2}{a^2\varkappa^2}\delta_{n,m}, \tag{3.43}$$

$$\langle\tilde{\varphi}_n(x,t)\,|\tilde{\varphi}_n(x,t)\rangle = 2^n n!\sqrt{\pi} := N. \tag{3.44}$$

Next we notice that the expectation value (3.43) does not depend on time

$$\frac{d}{dt}\left[\frac{a^2(1+\varkappa^4)+\varkappa^2\dot{\varkappa}^2}{a^2\varkappa^2}\right] = \frac{2\dot{\varkappa}}{a^2}\left(\ddot{\varkappa} - \frac{\dot{a}}{a}\dot{\varkappa} + a^2\varkappa - \frac{a^2}{\varkappa^3}\right) = 0. \tag{3.45}$$

by recognizing in (3.45) one of the factors as the Ermakov-Pinney equation in the form (3.42). It is clear that this constant will dependent on the explicit solution for (3.42). So for definiteness we compute it by adapting the solution (3.38) to account for the aforementioned different sign

$$\varkappa(t) = \sqrt{\tilde{\kappa}\cos\left[2\int_0^t a(s)ds\right] + \sqrt{1+\tilde{\kappa}^2}}, \tag{3.46}$$

with integration constant $\tilde{\kappa}$. For this solution we calculate

$$\frac{a^2(1+\varkappa^4)+\varkappa^2\dot{\varkappa}^2}{a^2\varkappa^2} = 2\sqrt{1+\tilde{\kappa}^2}. \tag{3.47}$$

Thus for the normalized wavefunction $\hat{\varphi}_n(x,t) = \tilde{\varphi}_m(x,t)/\sqrt{N}$ involving the solution (3.46) we find

$$\langle\hat{\varphi}_n(x,t)|\, K_1 \,|\hat{\varphi}_m(x,t)\rangle = \left(n+\frac{1}{2}\right)\sqrt{1+\tilde{\kappa}^2}\delta_{n,m}. \tag{3.48}$$

Having established the solution for one time-dependent harmonic oscillator, the solution for the time-dependent Schrödinger equation for the Hermitian Hamiltonian $h(t)$ in (3.39) is simply

$$\Psi_h^{n,m}(x,y,t) = \hat{\varphi}_n^+(x,t)\hat{\varphi}_m^-(y,t) \tag{3.49}$$

when the notation replacing $a \to f^{\pm}$, $\varkappa \to \varkappa_{\pm}$, $\tilde{\kappa} \to \tilde{\kappa}_{\pm}$ and $\alpha_n \to \alpha_n^{\pm}$ in an obvious manner. We have now assembled all the information needed to compute the instantaneous energy expectation values

$$\begin{aligned} E^{n,m}(t) &= \left\langle \Psi_h^{n,m}(t) \right| h(t) \left| \Psi_h^{n,m}(t) \right\rangle = \left\langle \Psi_H^{n,m}(t) \right| \rho(t)\tilde{H}(t) \left| \Psi_H^{n,m}(t) \right\rangle \\ &= f_+(t)\left(n+\frac{1}{2}\right)\sqrt{1+\tilde{\kappa}_+^2} + f_-(t)\left(m+\frac{1}{2}\right)\sqrt{1+\tilde{\kappa}_-^2}, \end{aligned} \tag{3.50}$$

with constants $\kappa_{\pm}$. It is clear that this expectation value is real for any given time-dependent fields $a(t)$, $\lambda(t) \in \mathbb{R}$ and constants $\tilde{\kappa}_{\pm} \in \mathbb{R}$, $|q_3| < 1$. Hence, we have explicitly shown that one can draw the same conclusion as in the one-dimensional case [8], that a time-independent non-Hermitian Hamiltonian in the spontaneous spontaneously broken $\mathcal{PT}$-regime becomes physically meaningful in the time-dependent setting.

4. Conclusions

We have presented the first higher dimensional solution of the time-dependent Dyson relation (2.1) relating a non-Hermitian and a Hermitian Hamiltonian system with infinite dimensional Hilbert space. As for the one dimensional case studied in [8], we have demonstrated that the time-independent non-Hermitian system in the spontaneously broken $\mathcal{PT}$-regime becomes physically meaningful when including an explicit time-dependence into the parameters of the model and allowing the metric operator also to be time-dependent. The energy operator (2.3) has perfectly well-defined real expectation values (3.50).

Technically we have compared two equivalent solution procedures, solving the time-dependent Dyson relation directly for the Dyson map or alternatively computing Lewis-Riesenfeld invariants first and subsequently constructing the Dyson map from the similarity relation that related the Hermitian and non-Hermitian invariants. The latter approach was found to be simpler as the similarity relation is far easier than the differential version (2.1). The price one pays in this approach is that one needs to compute the two invariants first. However, the differential equations for these quantities turned out to be easier than the (2.1). In particular, it was possible to entirely bypass the dissipative Ermakov-Pinney equation in the computation of $\eta(t)$. Nonetheless, this ubiquitous equation re-emerged in the evaluation of the eigenfunctions involving different time-dependent fields and with a changed sign.

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