THE KORTEWEG-DE VRIES

EQUATION AND ITS HOMOLOGUES :

A COMPARATIVE ANALYSIS.II :

BY

HAMID MOUSTAFA EL-SHERBINY

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TO

RAMI, my son,

RANDA, my daughter,

and

my wife

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DECLARATION

I hereby declare that this thesis, or any part of it, must not be copied before a period of one year from submission, but after this period I grant powers of discretion to the University Librarian to allow this thesis to be copied in whole or in part without further reference to the author.

H.M. EL-SHERBINY

ABSTRACT

The Korteweg-de Vries equation (KdV) is a partial differential equation which has some remarkable mathematical properties. Furthermore, it also appears as a useful model in a great many physical situations. Thus, although it was originally obtained as an approximation in fluid dynamics, it was reinterpreted as a canonical field theory for weakly dispersive and weakly nonlinear systems. This reinterpretation led to the hypothesis that the properties of the KdV could be understood in terms of a balance between the competing effects of dispersion and nonlinearity. Alternatives to the KdV were proposed on the basis that their dispersive properties were physically and mathematically preferable to those of the KdV.

The use of dispersion, which is a linear concept, as a criterion for predicting the properties of these nonlinear equations was examined in an earlier thesis by Abbas. By introducing a general class of equations which includes the KdV and all its proposed alternatives as special cases, Abbas investigated in detail the predictions based on the dispersion relation and compared them with the actual properties of the equation, particularly in regard to the existence of solitary waves. He found little correlation and some contradictions and concluded that the idea of a balance between nonlinearity and dispersion is not useful way of understanding these equations. It is clear, therefore, that we must develop other criteria to obtain this understanding.

In this thesis we continue this investigation by looking at other properties of the class of equations introduced by Abbas which are relevant to the KdV. The general question which we are considering is whether the properties of the KdV are unique in this class and if so how can we decide this a priori, i.e., from the equation and its elementary solutions. A prerequisite for tackling this problem is to establish whether the embedding of the KdV in this class is reasonable, i.e., that these equations can indeed be considered as homologues of the KdV. Thus, it is necessary to establish well-posedness, the existence of solitary wave and other elementary solutions and the existence of other properties such as, for example, conservation laws. These are the specific questions that we consider in this thesis.

To make the thesis self-contained we begin with a comprehensive review of the KdV and its main alternative, the regularized long wave equation, together with the work of Abbas. This comprises the first part of the thesis and puts our own contribution in its proper perspective.

The second part of the thesis contains our own contribution and begins with a completion of the analysis of solitary waves begun by Abbas. We next partition the general class into five equivalence classes and establish well-posedness for three of them and existence for a fourth. Finally, we show that all equations

have at least two conservation laws, some of the equations have at most three conservation laws. These results enable us to conclude that this class of equations is a reasonable one in which to investigate the question referred to above.

The thesis ends with a résume and suggests avenues for continuing this investigation.

CHAPTER ONE

INTRODUCTION

1.1 General Perspective

The last twenty five years have seen a great deal of progress in the theory of nonlinear partial differential equations. This has come about for three reasons. Firstly, the use of functional analysis has advanced the understanding of well-posedness in the subject. By posing the problem in a Banach space or Hilbert space and using weak topologies on these spaces, techniques such as a priori inequalities together with fixed point theorems, contraction maps and sequencing can be used to establish the existence of weak solutions for a large variety of equations. Uniqueness and regularity can then be proved separately to establish the existence of classical solutions. The point here is that it is easier to prove the existence of weak solutions and then establish regularity rather than do it both together. Secondly, the advent of high speed computers with large processing capability has allowed efficient numerical procedures to be developed to obtain detailed quantitative solutions of complicated equations such as the Navier-Stokes and Einstein equations. It has also allowed numerical experimentation to become a standard tool of mathematical investigation by providing large-scale simulations. Thirdly, in the area of exact solutions and reduction to quadratures, i.e., integrability, there has been a significant advance with the discovery of a class of nonlinear partial differential equations which can be transformed to a linear integral equation via an associated linear eigenvalue problem. This procedure is usually referred to as the inverse scattering method.

In this thesis we are concerned with this third aspect and in particular with the properties of the equations in this class, which we refer to as the integrable class. Since one of our major questions concerns well-posedness, we shall use functional analytic techniques. On the other hand we make no use whatsoever of numerical methods.

1.2 The integrable class

The class of equations for which the inverse scattering method applies includes equations such as the well-known Korteweg-de Vries equation (KdV)

$$u_t + u_x + uu_x + u_{xxx} = 0 \qquad (1.1)$$

and the sine-Gordon equation (SGE)

$$\phi_{xx} - \phi_{tt} = \sin \phi. \qquad (1.2)$$

Both of these equations arise in many applications and hence are not merely pathological examples. Typical applications include shallow water wave theory, plasma waves, ion-acoustic, the anharmonic lattic, ..., etc. Furthermore, from the construction of the equations it is clear that the KdV and SGE can be considered as particular nonlinear extensions of the undirectional wave equation and one dimensional wave equation respectively. This view point will be taken up later for the KdV.

It turned out that these equations have a number of very interesting properties apart from the fact that they can be linearized. These are as follows. (1) They each have a family of solitary travelling wave solutions which move with different speeds. This contrasts with the linear case where all solitary waves have the same speed. (2) They each have a new class of exact solutions called multisolitons. A multisoliton is a nonlinear combination of solitary waves which decomposes asymptotically as $|t| \to \infty$ into a linear combination. (3) Each equation has an infinite number of local conservation laws. These lead to an infinite set of conserved functionals within a class of solutions which includes the multisolitons. (4) Each equation has an auto-Bäcklund transformation. By definition this is a map of one solution to another. The usual form of the map is in terms of a system of partial differential equations.

The question now arises as to whether all these properties are connected. We look at this question from the following point of view: if an equation has soliton solutions does it follow that (1) it is solvable by the inverse scattering method, (2) it has an infinite number of conservation laws, (3) there exists a Bäcklund transformation. The reason for doing this is because of the following observations. Firstly, there is no systematic way of getting the associated linear eigenvalue problem for inverse scattering from the nonlinear partial differential equation. It has to be guessed. Secondly, although there are systematic ways of getting conservation laws and Bäcklund transformations, the methods are tedious and may not always work on arbitrary differential equations. The situation is similar in regard to finding soliton solutions. Clearly, if we are going to solve this problem then it is necessary to first establish whether an equation has soliton solutions or not. To put it more specifically, we would like to know when a solitary travelling wave solution of a nonlinear partial differential equation is also a soliton. That is, we would like to know what properties this wave and the equation must have to guarantee the existence of multisoliton solutions. This study was initiated by Abbas [1] and this thesis is a continuation of it. In order to put our contribution in perspective we now turn to a discussion of the KdV, its properties and the work of Abbas.

1.3 The KdV as a field theory

The KdV equation is a nonlinear partial differential equation of evolution type in one space and one time dimension. The equation (1.1) was first derived in 1895 in the study of shallow water waves by Korteweg and de Vries [10] to demonstrate that it could support a solitary wave.

Inspite of this earlier derivation of the equation it was neglected for about 70 years until 1964 when Broer, in his study of the interaction of nonlinearity and dispersion in wave propagation [5] suggested that the equation could be approached from the point of view of a field theory. The properties of this field are thought to be obtained from nonlinear and dispersive effects corresponding to the terms uu_x and u_{xxx} respectively. Since these terms appear additively in the equation their interaction will be observed only in the solution space. Hence the general scheme proposed by Broer is to write the field equation as a structural perturbation

$$u_t + u_x + N(u) + D(u) = 0 \tag{1.3}$$

of the basic unidirectional linear nondispersive equation

$$u_t + u_x = 0, \tag{1.4}$$

where N and D are the nonlinear and dispersive perturbations respectively and can be generated from physical considerations.

This scheme has the following advantages:

(i) Since the zero order approximation implies that $\frac{\partial}{\partial x} + \frac{\partial}{\partial t} = 0$, then the equality $\frac{\partial}{\partial x} = -\frac{\partial}{\partial t}$ can be used to construct alternatives to the field equation by changing N or D or both. For example, the equality is used to change u_{xxx} in the KdV equation to $-u_{xxt}$ to establish an alternative to the KdV equation [3].

(ii) The terms in the field equation can be considered independently so that physical and mathematical information can be introduced through the dispersive terms or nonlinear terms or both.

Hence, this interpretation of the KdV equation led to the belief that its properties could be understood in terms of a balance between the nonlinear and dispersive effects.

1.4 Properties of the KdV

The field theoretic interpretation outlined above awakened the interest of the investigators to look again at the properties of the KdV. One of the fascinating discoveries was made in 1965 when Zabusky and Kruskal [16] found by numerical experiments that the KdV solitary wave $u = 3C\text{sech}^2 \frac{\sqrt{C}}{2}[x - (1+c)t]$ is a soliton and that these solitons are remarkably stable. This was the first use of the term soliton. Following this discovery attention was paid by many others to investigate its mathematical properties. Among these properties we present the following:

(1) The equation has multisoliton solutions. This behaviour was first observed numerically by Zabusky and Kruskal [16] in the case of two solitons and the analytic expression for the general case was given by Hirota [8] and Wadati and Toda [13]. For example the two soliton is given by

$$u = 72 \frac{[3 + 4 \cosh (2x-8t) + \cosh (4x-64t)]}{[3 \cosh (x-28t) + \cosh (3x-36t)]^2} . \qquad (1.5)$$

Asymptotically (as $|t| \to \infty$) this decomposes into the two solitary waves

$u_1 = 12 \operatorname{sech}^2 [(x-4t) + \delta_1]$ and

$u_2 = 48 \operatorname{sech}^2 [2(x-16t) + \delta_2]$, with δ_1 and δ_2 constants.

(2) The equation has an infinite number of independent local conservation laws [11] by which it is meant that the KdV can be expressed in a form $\frac{\partial}{\partial t} T_i + \frac{\partial}{\partial x} X_i = 0$, where $(T_i)_{i=1}^{\infty}$ and $(X_i)_{i=1}^{\infty}$ are polynomials in x, t, u and derivatives of u. For example, the KdV itself can be written in the form:

$$(u)_t + (u + \frac{u^2}{2} + u_{xx})_x = 0 \to T_1 = u \text{ and } X_1 = u + \frac{u^2}{2} + u_{xx}.$$

Also multiplying the KdV by u gives

$$(\frac{u^2}{2})_t + (\frac{u^2}{2} + \frac{u^3}{3} + uu_{xx} - \frac{u_x^2}{2})_x = 0 \to T_2 = \frac{u^2}{2},$$

$$X_2 = \frac{u^2}{2} + \frac{u^3}{3} + uu_{xx} - \frac{u_x^2}{2} .$$

This property was exploited by Bona and Smith [4] to derive a priori estimates of the solutions in proving well-posedness.

(3) The KdV was the prototype for developing the inverse scattering method. The solution $u(x,t)$ of the KdV equation can be represented by the potential of the linear Schrödinger equation

$$\phi_{xx} + (\lambda - u(x,t))\phi = 0. \tag{1.6}$$

This forms the associated eigenvalue problem. Hence, the exact solution $u(x,t)$ of the KdV equation is obtained from the usual inverse scattering method in terms of the solution of the Gelfand-Levitan integral equation [7]. Note that (i) the method can only be applied to certain classes of initial data and (ii) the multisolitons can be obtained in explicit form.

(4) There exists a Bäcklund transformation [14] for the KdV as follows:

$$\frac{\partial}{\partial x}(w_1+w_2) = -\alpha_1^2 + \frac{1}{2}(w_1-w_2)^2,$$

$$\frac{\partial}{\partial t}(w_1+w_2) = -(w_1-w_2)\frac{\partial^2}{\partial x^2}(w_1-w_2) + 2(u_1^2 + u_1u_2 + u_2^2),$$

where $u_i = -\dfrac{\partial w_i}{\partial x}$, $i = 1, 2$ are solutions of the KdV.

For example starting with $u_1 = 0$ gives the single soliton (or solitary waves).

1.5 Alternatives to the KdV and the general class

There are many alternatives to the KdV equation generated according to the Broer scheme. Most of these alternatives have been proposed on the basis that their mathematical and physical properties are preferable to those of the KdV. The mathematical basis used to construct these alternatives was investigated by Abbas [1] who concentrated his study on the solitary wave solutions.

In the first place, analysing separately the effects of nonlinearity and dispersion on selected initial profiles, Abbas found that for the KdV equation the effect of nonlinearity is insignificant, but that the sech^2 solitary wave profile disperses more slowly than the others. Moreover, synthesising the KdV and comparing the properties of its solitary waves with the prediction of its component parts, he found several contradictions. He concluded that the Broer hypothesis is not valid on the KdV equation.

Next, the general belief that existence of solitary waves is due to a balance between nonlinearity and dispersion was tested. For this context, Abbas considered the general class of third order equations with quadratic nonlinearities, i.e.,

$$u_t + u_x + a_1 u u_x + a_2 u u_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0 \tag{1.7}$$

where $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, 6)$. This contains the KdV and some proposed alternatives such as the regularized long wave equation (RLW) [3]

$$u_t + u_x + u u_x - u_{xxt} = 0 \tag{1.8}$$

and Joseph & Egri equation (J.E) [9]

$$u_t + u_x + u u_x + u_{xtt} = 0. \tag{1.9}$$

Solitary wave solutions with $sech^2$ profiles are shown to exist for a wide variety of dispersion relations. However, Abbas also showed the existence of a formally nondispersive subclass of the general class which has stable solitary waves. This clearly contradicts the belief that the formation and properties of solitary waves can be understood in terms of a balance between nonlinearity and dispersion.

Consequently he concluded that dispersion is not necessary for the existence of solitary waves [1] and [2].

Since dispersion is not a useful criterion in understanding the properties of the KdV equation, the question arises as to whether it is possible to develop other criteria for such understanding. In order to attack this problem it is convenient to establish a well-defined class which includes the KdV and its alternatives and has solitary wave solutions and to investigate the extent to which this class has the properties of the KdV which were listed above. This is the specific problem that we look at in this thesis and our strategy and contribution is described below.

1.6 Properties of the general class

We investigate the general class of equations (1.7) since it has solitary wave solutions and can be thought of as forming a neighbourhood of the KdV in the space of coefficients. Our contribution is in two main areas. Firstly, we examine the well-posedness of the general class (1.7) for certain prescribed data. Secondly, we look at the number of conservation laws of the general class. The well-posedness is examined as follows:

(1) We prove that the general class can be reduced, for a certain class of data, to a system of first order partial differential equations. This reduction is then used to classify the problem into two subclasses which we call the nonsingular and singular subclasses.

(2) We use the method of characteristics on the nonsingular class and find that well-posedness is ensured for certain data.

(3) For the singular class, i.e., the class in which the method of characteristics fails, we find that this class can be reduced to four equivalence classes, namely, the KdV and RLW classes and two others which we refer to as W_{54} and W_{53}. (The names of these classes is characterized by their dispersive terms).

(4) Finally, we consider the well-posedness of these singular classes and provide some theorems which are necessary for their well-posedness.

This is the first part of our original contribution.

The second subject is to examine the conservation law property on the general class (1.7). We use elementary operations to derive the first two conservation laws. We then establish a necessary condition for the existence of a third conservation law, namely, coupling of the coefficients that if $\frac{a_1}{a_2}$ is a root of the cublic equation $a_3 - a_4\lambda + a_5\lambda^2 - a_6\lambda^3 = 0$, then the corresponding subset has a third conservation law. This condition is then used to classify the problem into four equivalence classes, which are the same as those in (3) above in the simple sense, i.e., no uu_t term is present. We now turn to study the conservation laws of these classes separately. We find that unless the equation is in the KdV class in the simple sense, then it has only three conservation laws. Finally, we turn to the case in which the coupling coefficients condition is invalid and show that such equations have at least two conservation laws.

This is the second part of our original contribution.

1.7 Summary of contents

The thesis is organized as follows: Together with this introduction it consists of eight chapters, five appendices and a bibliography.

Chapters 2, 3 and 4 review the general class, the KdV equation and the RLW equation, respectively. Chapters 5, 6 and 7 contain our own contributions. Chapter 8 contains our conclusions. A summary of the chapters is given below.

In chapter 2 a review of the general class of equations is presented. In the first section we study the existence of solitary wave and periodic wave solutions. The second section is devoted to the study of the linear stability of these solitary waves. A general classification in terms of these solitary waves is presented in the third section. This is followed by a conclusion of the work in this chapter.

In chapter 3 a review of the mathematical properties of the KdV equation is introduced. In the first section we state the existence theorem of the solitary and periodic wave solutions. In the second section the linear stability theorem is stated. The third section is devoted to the study of the inverse scattering method. This method is then used to find the N-soliton solution of the equation which is presented in the fourth section. This is followed by the investigation of the local conservation laws of the equation together with a proof of the existence of an infinite number of such conservation laws in the fifth section. Next a relationship between those conservation laws and the inverse scattering method is presented. Finally, the well-posedness of the corresponding initial value problem i.e. existence, uniqueness and continuous dependence of the solution on the initial data is discussed briefly leaving the technical proofs to appendix A.

In chapter 4 we review the mathematical properties of the regularized long wave (RLW) equation. This chapter begins with the existence of solitary and periodic wave solutions in the first section. This is followed by the linear stability theorem which is stated in the second section. The third section is devoted to the investigation of the conservation laws of the equation. Next, the conditional stability of the solitary wave solution is presented together with outline proof of the theorem. Finally, we establish the well-posedness of the corresponding initial value problem in detail.

In chapter 5 we complete the classification derived in chapter 2 on the existence of solitary waves for the general class. Next we define the class $W \cap S$ and study some of its properties. In the third section we show that the general class can be reduced to four equivalence classes by defining solutions to be equivalent if they are connected by nonsingular linear transformations. Next, we show how this reduction simplifies the well-posedness of the general class of equations. This is followed by a section devoted to showing that the reduction is then shown to preserve the existence of solitary waves as well. Finally, the existence of multisolitons in the KdV equivalence class is discussed.

In chapter 6, the well-posedness of the general class of equations is presented. This chapter begins with the reduction to a semi-linear system of first order partial differential equations and the characteristics are established in the second section. In the third section the normal form of the system is obtained. This is followed by introducing the method of characteristics together with an illustration using the linear wave equation. The well-posedness classification into nonsingular and singular classes is then presented in the fifth section. Next, the well-posedness of the nonsingular class is investigated. First, an integral formula for the nonsingular class is established followed by the proof of the uniqueness of the solutions is then shown by using the method of characteristics followed by the proof of the continuous dependence of the solution on the given data. Finally, the well-posedness of the singular class is investigated. Reduction into the four equivalence classes is used to simplify this ivestigation. This is followed by some applications to illustrate the above technique.

In chapter 7, the conservation laws of the general class are discussed. In the first section we use elementary operations to derive the first two conservation laws and establish the coupling coefficients condition which is necessary for deriving the third one. This condiiton is then used in the next section to classify the problem into four classes. This is followed by a section devoted to introducing the general formalism for the existence of conservation laws. This formalism is used in the next section to prove that the class W_{54} (in the simple sense) has only three conservation laws and to show that W_{53} (in the simple sense) has also three conservation laws. Finally, we turn to the case in which the coupling coefficients condition is invalid and provide an example to show that there exist at least two conservation laws.

Chapter 8 contains our concluding remarks and list some questions for future investigation.

CHAPTER TWO

THE GENERAL CLASS OF EQUATIONS

Consider the set of evolution equations, defined by

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0 \quad (2.1)$$

where $a_i (i = 1,2,\ldots,6)$ are real numbers and $u(x,t)$ is a real scalar field defined for all $(x,t) \in R^2$. This class, which was first defined by Abbas [1], includes the KdV equation and some of its alternatives.

In this chapter we shall concentrate on the study of the existence and properties of solitary waves which have received considerable attention in the above quoted reference.

2.1 Existence of solitary wave solutions

The solitary waves are special cases of the travelling waves and the latter are obtained by transforming the evolution equation (2.1) to the frame of reference in which the waves appear stationary (rest frame). This is achieved by using the transformation

$$x \to x - (1+c)t, \; t \to t \; \text{ and } \; u(x,t) \to v(x,t), \quad (2.1.1)$$

Then (2.1) reduces to

$$v_t - cv_x + [a_1 - a_2(1+c)]vv_x + a_2 vv_t + [a_3 - a_4(1+c) + a_5(1+c)^2$$

$$- a_6(1+c)^3]v_{xxx} + [a_4 - 2a_5(1+c) - 3a_6(1+c)^2]v_{xxt}$$

$$+ [a_5 - 3a_6(1+c)]v_{xtt} + a_6 v_{ttt} = 0. \quad (2.1.2)$$

Since the travelling waves appear stationary in this frame, then all the t - derivatives should vanish and (2.1.2) reduces to the ordinary differential equation

$$-cv' + [a_1-a_2(1+c)]vv' + [a_3-a_4(1+c)+a_5(1+c)^2-a_6(1+c)^3]v''' = 0, \tag{2.1.3}$$

or, for simplicity,

$$-v' + \alpha vv' + \beta v''' = 0, \tag{2.1.4}$$

where $\alpha c = a_1-a_2(1+c)$, $\beta c = a_3-a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3$ and the prime stands for the total x - derivatives.

Integrating (2.1.4) once with respect to x gives

$$-v + \frac{\alpha}{2}v^2 + \beta v'' + A_1 = 0. \tag{2.1.5}$$

Multiplying (2.1.5) by v', integration again with respect to x is possible yielding

$$A_1 v + A_2 - \frac{v^2}{2} + \frac{\alpha}{6}v^3 + \frac{\beta}{2}(v')^2 = 0$$

i.e.

$$\frac{3\beta}{\alpha}(v')^2 = -v^3 + \frac{3}{\alpha}v^2 + \frac{6}{\alpha}A_1 v + \frac{6}{\alpha}A_2, \tag{2.1.6}$$

where A_1 and A_2 are constants of integrations. Using the substitutions

$$x \to \xi = \sqrt{\frac{\alpha}{12\beta}}\,x \quad \text{and} \quad v \to w = \frac{1}{\alpha} - v, \tag{2.1.7}$$

where α and β have the same sign, then (2.1.6) reduces to the simple form

$$(w')^2 = 4w^3 - k_2 w - k_3, \tag{2.1.8}$$

where $k_2 = \frac{12}{\alpha^2}[1+2A_1\alpha]$ and $k_3 = \frac{-4}{\alpha^3}[2+6A_1\alpha+6A_2\alpha^2]$.

Equation (2.1.8) has the general solution which is the weierstrassion elliptic function

$$q(\xi) = r_1 + (r_3-r_2)\,\mathrm{cn}^2\,(\lambda\xi;k)\,, \qquad (2.1.9)$$

where r_1, r_2 and r_3 are the roots of the cubic equation

$$4r^3 - k_2 r - k_3 = 0 \qquad (2.1.10)$$

and cn is the Jacobian elliptic cosine amplitude with modulus

$$k,\; k^2 = \frac{r_2 - r_3}{r_1 - r_3} \text{ and } \lambda^2 = r_1 - r_3\,.$$

If $k^2 \neq 1$, cn $(\lambda\xi;k)$ is periodic. If $k^2 = 1$, then the solitary wave occurs and (2.1.9) reduces to

$$q(\xi) = r_2 + (r_3-r_2)\,\mathrm{sech}^2\lambda\xi. \qquad (2.1.11)$$

Since for this case (2.1.10) has two identical roots, then its discriminant should vanish yielding the necessary condition,

$$k_2^3 = 27k_3^2 \qquad (2.1.12)$$

for the existence of solitary waves. From the expressions for k_2 and k_3, (2.1.12) leads to

$$27\left[\frac{-4}{\alpha^3}(2+6A_1\alpha+6A_2\alpha^2)\right]^2 = \left[\frac{12}{\alpha^2}(1+A_1\alpha)\right]^3.$$

Thus (2.1.12) can only be true for all $\alpha \neq 0$ if $A_1 = A_2 = 0$ which implies that the boundary conditions of the solitary waves are v, $\frac{dv}{dx}$ and $\frac{d^2v}{dx^2} \to 0$ as $|x| \to \infty$.

Furthermore, $k_2 = \frac{12}{\alpha^2}$ and $k_3 = \frac{-8}{\alpha^3}$. Thus (2.1.10) reduces to

$4r^3 - \frac{12}{\alpha^2} r + \frac{8}{\alpha^2} = 0 \quad \Rightarrow \quad r_1 = r_2 = \frac{1}{\alpha}$ and $r_3 = \frac{-2}{\alpha}$. Hence

$\lambda^2 = \frac{3}{\alpha}$ and (2.1.11) becomes

$$q(\xi) = \frac{1}{\alpha} - \frac{3}{\alpha} \operatorname{sech}^2 \left(\frac{3}{\alpha}\right)^{\frac{1}{2}} \xi. \qquad (2.1.13)$$

But $q(\xi) = w = \frac{1}{\alpha} - v(x,t)$, thus

$$v(x,t) = \frac{1}{\alpha} - \left[\frac{1}{\alpha} - \frac{3}{\alpha} \operatorname{sech}^2\left(\frac{3}{\alpha}\right)^{\frac{1}{2}} \xi\right] = \frac{3}{\alpha} \operatorname{sech}^2\left(\frac{3}{\alpha}\right)^{\frac{1}{2}} \xi. \qquad (2.1.14)$$

Using the expression for ξ and the transformation (2.1.1), then the solitary wave solutions of (2.1) are

$$u_s(x,t) = \frac{3}{\alpha} \operatorname{sech}^2 \left\{\frac{1}{2\sqrt{\beta}}[x - (1+c)t]\right\} \qquad (2.1.15)$$

The above results are summarized as follows:

Theorem 2.1

(i) All the equations (2.1) have periodic waves which are Weierstrassian elliptic functions with possible constraints on the parameters to keep the solutions real.

(ii) The necessary condition for the existence of real solitary waves is that $\beta(a_3, a_4, a_5, a_6) > 0$ and they all have the sech^2 profile.

(iii) The solitary waves have necessarily the boundary conditions $u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial u^2} \to 0$ as $|x| \to \infty$.

Furthermore, the work of Abbas [1] gives the following corollary:

Corollary (2.2.1)

The nonlinearity is the dominant term in producing the sech^2

profile whereas the dispersion only effects the width of this profile. □

2.2 Linear stability

Having obtained the existence of solitary waves, we turn our attention to whether these solitary waves are stable or not. We shall not go into the proof of stability for the general case but merely illustrate this proof briefly for the non-dispersive class of equations [2],

$$u_t + u_x + a_1 u u_x + a_2 u u_t + a_3 u_{xxx} + a_4 u_{xxt} = 0, \quad a_3 = a_4 < 0 \tag{2.2.1}$$

which have the solitary waves

$$u_s(x,t) = \frac{3}{\alpha} \operatorname{sech}^2 \frac{1}{2\sqrt{\beta}} [x - (1+c)t] \tag{2.2.2}$$

where $\alpha c = a_1 - a_2(1+c), \quad \beta = -a_3 = -a_4 > 0.$ (2.2.3)

Definition 2.1

The solitary wave solution $u_s(x,t)$ of (2.2.1) is said to be stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$||v(\xi,0) - u_s(\xi)|| \leq \delta \Rightarrow ||v(\xi,t) - u_s(\xi)|| < \epsilon$

for a suitable norm $||.||$ for all $t > 0$, where $v(\xi,t)$ is a solution of (2.2.1) in the rest frame of the solitary wave, i.e. $\xi = x - (1+c)t, t$.

To implement the above definition in its linearized form, let

$$v(\xi,t) = u_s(\xi) + \epsilon g(\xi,t) \tag{2.2.4}$$

be the solution of (2.2.1) in the rest frame, i.e.

$$v_t - c v_\xi + \alpha c v v_\xi + a_2 v v_t + \beta c v_{\xi\xi\xi} + a_4 v_{\xi\xi t} = 0 , \tag{2.2.5}$$

by working to the first order in ϵ, g must satisfy

$$g_t - cg_\xi + \alpha c[g_\xi u_s + g\frac{du_s}{d\xi}] + a_2 u_s g_t + \beta c g_{\xi\xi\xi} + a_4 g_{\xi\xi t} = 0. \tag{2.2.6}$$

Hence g, being the solution of the linear equation (2.2.6) can be expanded in terms of the elementary solutions of (2.2.6), i.e.

$$g(\xi,t) = \Sigma\, f_d(\xi,w_d)e^{iw_d t} + \int f_c(\xi,w_c)e^{iw_c t}dw_c \tag{2.2.7}$$

where the summation is over the discrete spectrum of (2.2.6) with $w_d = 0$ and $w_d = m_d + in_d$, $(n_d \neq 0)$, and the integral is over the continuous spectrum with $w_c \in R$; $(w_c \neq 0)$. Now, since the continuous component consists of periodic travelling waves which are bounded for all time t, then w_c is real and the waves are consequently stable. Thus the main concern is with the discrete components only. The frequencies of these components are usually complex, hence $g(\xi,t)$ reduces to

$$g(\xi,t) = \Sigma\, [f_d(\xi,m_d+in_d)e^{im_d t}]e^{-n_d t} + \int f_c(\xi,w_c)e^{iw_c t}dw_c. \tag{2.2.8}$$

Hence the problem of determining the linear stability of (2.2.1) is reduced to that of finding the discrete frequency spectrum of (2.2.6), since for the non-zero frequency, $w_d \neq 0$, either (i) $n_d > 0$, the waves are stable wherever the discrete components die away as t increase leaving small oscillations remaining, or (ii) $n_d < 0$, the discrete components grow as t increases in an unbounded manner and the solitary wave is consequently unstable. But the zero frequency corresponds to some sort of stability, i.e. "neutral stability", and g, being now time independent, moves the solitary wave to a slightly different position in the rest frame, since u_s modulo phase shift is the unique static solution of (2.2.1) in the rest frame and

$$v(\xi) = u_s(\xi+\epsilon) \simeq u_s(\xi) + \epsilon u_s'(\xi) + O(\epsilon^2). \tag{2.2.9}$$

Now, to find the discrete spectrum which corresponds to the non-zero frequencies of (2.2.6) it is convenient to make use of the substitution

$$g(\xi,t) = f(\xi)e^{iwt} \tag{2.2.10}$$

to reduce (2.2.6) into the ordinary differential equation

$$\beta c \frac{d^3 f}{d\xi^3} + \alpha c[u_s \frac{df}{d\xi} + f \frac{du_s}{d\xi}] - c \frac{df}{d\xi} + iw [a_4 \frac{d^2 f}{d\xi^2} + a_2 u_s f + f] = 0. \tag{2.2.11}$$

Further $f_d \to 0$ as $|\xi| \to \infty$, since $v(\xi,t)$, being required in the same class as u_s, satisfies; v, $\frac{dv}{d\xi}$ and $\frac{d^2 v}{d\xi^2} \to 0$ as $|\xi| \to \infty$.

Equation (2.2.11) has the following properties:

(i) the asymptotic solutions of (2.2.11), being the solutions of constant coefficients differential equation are unique.

(ii) the equation possesses a symmetry property.

(iii) the equation can be arranged in the matrix form

$$\frac{d}{ds}\begin{bmatrix} f_1 \\ f_1' \\ f_1'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{iw}{\beta c} & \frac{1}{\beta} & \frac{iwa_4}{\beta} \end{bmatrix} \begin{bmatrix} f_1 \\ f_1' \\ f_1'' \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{\beta c}(\alpha c+iwa_2)u_s & \frac{-\alpha u_s}{\beta} & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_1' \\ f_1'' \end{bmatrix} \tag{2.2.12}$$

where f_1 satisfies (2.2.11).

Since $u_s \to 0$ as $|\xi| \to \infty$, the last term in (2.2.12) vanishes as $|\xi| \to \infty$ and at this limit (2.2.12) reduces to

$$\frac{d}{d\xi}\begin{bmatrix} f_1 \\ f_1' \\ f_1'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{iw}{\beta c} & \frac{1}{\beta} & \frac{iwa_4}{\beta} \end{bmatrix}\begin{bmatrix} f_1 \\ f_1' \\ f_1'' \end{bmatrix} \tag{2.2.13}$$

or the single equation

$$\beta c \frac{d^3 f}{d\xi^3} - iw\frac{d^2 f}{d\xi^2} - c\frac{df}{d\xi} + iw = 0, \quad a_3 = a_4 = -\beta. \tag{2.2.14}$$

This equation has clearly the solutions

$$f_d(\xi, w_d) = \begin{cases} a_+ e^{\lambda_1|\xi|} + b_+ e^{\lambda_2|\xi|} + c_+ e^{\lambda_3|\xi|} & \text{as} \quad \xi \to \infty \\ a_- e^{-\lambda_1|\xi|} + b_- e^{-\lambda_2|\xi|} + c_- e^{-\lambda_3|\xi|} & \text{as} \quad \xi \to -\infty \end{cases}$$

$$\tilde{f}_d(\xi, -w_d) = \begin{cases} a_+ e^{-\lambda_1|\xi|} + b_+ e^{-\lambda_2|\xi|} + c_+ e^{-\lambda_3|\xi|} & \text{as} \quad \xi \to \infty \\ a_- e^{\lambda_1|\xi|} + b_- e^{\lambda_2|\xi|} + c_- e^{\lambda_3|\xi|} & \text{as} \quad \xi \to -\infty \end{cases}$$

where λ_1, λ_2 and λ_3 are the roots of the cubic equation
$\beta c\lambda^3 - iw\beta\lambda^2 - c\lambda + iw = 0$
i.e. $\lambda_1 = -\lambda_2 = \frac{1}{\sqrt{\beta}}$ and $\lambda_3 = \frac{iw}{c}$. Now the boundary conditions require that the eigenfunctions tend asymptotically to zero as $|\xi| \to \infty$, but since there is no choice of the constants $a_\pm$, $b_\pm$ and $c_\pm$ for which this occurs simultaneously then the values $w \neq 0$ are excluded. Hence $w = 0$ is the only discrete eigenvalue. Hence the perturbed solutions (2.2.4) are
$v(\xi, t) = u_s(\xi+\epsilon)$,
and correspond to translations of solitary waves, i.e. the solitary waves of (2.2.1) are neutrally stable.

2.3 Classification

A general classification of the equations (2.1) by means of their solitary waves can be obtained by considering the properties of the width parameter β, where

$$\beta c = a_3 - a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3 \qquad (2.3.1)$$

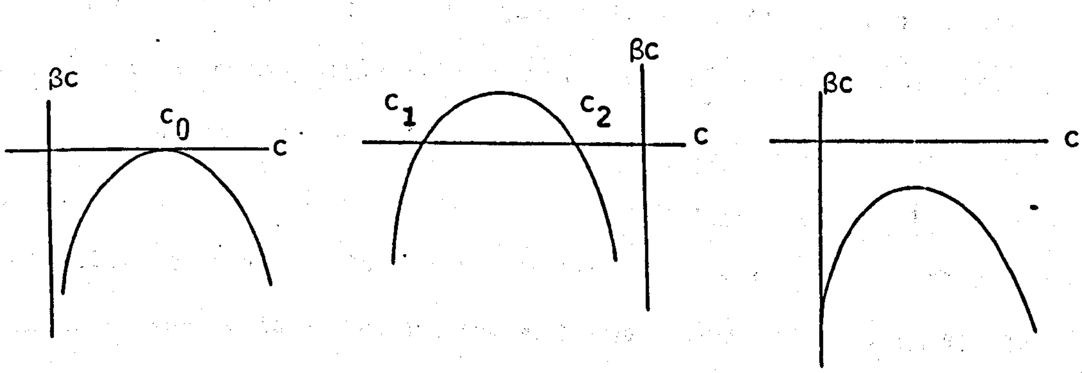
which should be positive from Theorem (2.1). This classification has been done in [1] for the simple cases as follows:

(A) $a_6 = 0,\ \ a_5 \neq 0,\ \ c > 0.$

In this case $\beta c = a_3 - a_4(1+c) + a_5(1+c)^2.$ (2.3.2)

Hence two cases should be considered

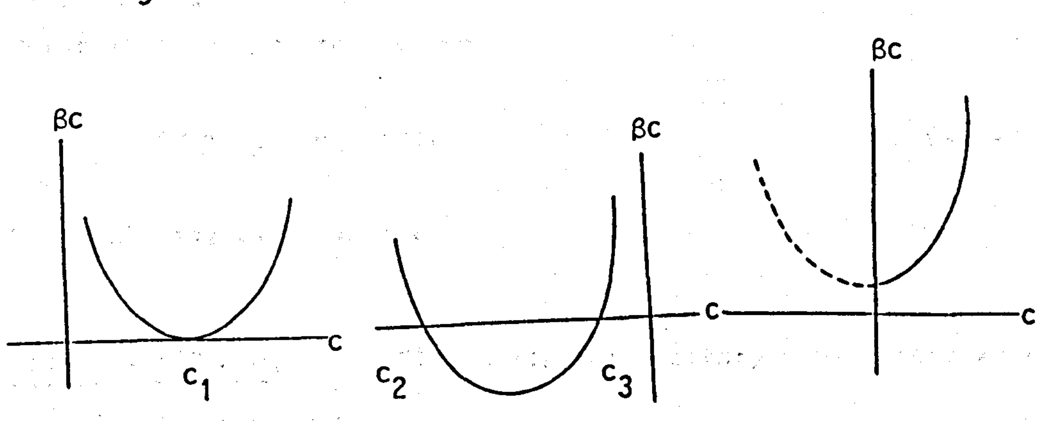
A(i) $a_5 < 0$, βc has three possible graphs



A(i)1 A(i)2 A(i)3

In the two cases A(i)1 and A(i)3, there would be no solitary waves, whilst in the case A(i)2 the solitary wave does not exist if the roots c_1 and c_2 of the quadratic (2.3.2) are negative i.e. $c_1 < c_2 < 0$. If $0 < c_1 < c_2$, the solitary wave would exist only inside the interval (c_1,c_2) and if $c_1 < 0 < c_2$ it exists inside the interval $(0,c_2)$, without loss of generality.

A(ii) $a_5 > 0$, three graphs of βc are possible.



A(ii)1 A(ii)2 A(ii)3

In the case A(ii)1 which corresponds to the condition $a_4^2 - 4a_3a_5 = 0$, if $c_1 < 0 \rightarrow c \in (0, \infty)$ in which the solitary wave would exist. Whenever $c_1 > 0 \rightarrow c \in (0, c_1) \cup (c_1, \infty)$ in which the solitary wave exists, where c_1 is the root of the quadratic (2.3.2).

In the case A(ii)2, i.e. the quadratic (2.3.2) has two distinct real root, corresponding to the condition $a_4^2 - 4a_3a_5 > 0$, if c_2, c_3 denote to these roots and $c_2 < c_3$ (without loss of generality), then $0 < c_2 < c_3 \rightarrow c \in (0, c_2)\ (c_3, \infty)$ for which the solitary waves would exist. Whilst if $c_2 < c_3 < 0$, then the solitary wave exists everywhere and the speed is unbounded. But if $c_2 < 0 < c_3$, then this case reduces to A(ii)1.

In the case A(ii)3, there is no real roots of the quadratic (2.3.2), corresponding to the condition $a_4^2 - 4a_3a_5 < 0$. The solitary wave which corresponds to this case exists everywhere and the speed is unbounded.

The results for $c < 0$ follow from the above by reversing the direction of the c-axis and interchanging the interpretation of the figures.

(B) $a_6 = 0 = a_5, \quad c > 0$.

For this case βc reduces to

$$\beta c = a_3 - a_4(1+c). \qquad (2.3.3)$$

The following cases arise:

B(i) $a_3 > 0 \geq a_4$ (the KdV case), the solitary wave exists where the speeds are unbounded above.

B(ii) $a_3 > a_4 > 0$, the solitary wave would exist if $\beta c > 0$,

i.e. $c < \frac{a_3}{a_4} - 1$,

B(iii) $a_3 < a_4 < 0$, the solitary wave exists if $c > \frac{a_3}{a_4} - 1 > 0$

i.e. bounded from below.

B(iv) $a_3 = 0$, i.e. $\beta c = - a_4(1+c)$, then if $a_4 > 0$, there would be no solitary waves. Whilst if $a_4 < 0$, the solitary wave exists and the speed is unbounded.

Thus the results from this classification show that for quadratic nonlinearities and third order dispersive terms, solitary waves, where they exist, have the sech^2 form. Also these results indicate that there is a variety of equations which have solitary wave solutions where all the equations have the same nonlinearity but different dispersion terms.

Corollary (2.3.1)

The linear equation is an unreliable indicator of the properties of the full nonlinear equation.

The proof has been carried out by Abbas [1] by analysing the linear part of (2.1) in terms of the dispersion relation $w(k)$,

$$a_6 w^3 - a_5 k w^2 + (a_4 k^2 - 1) w + (k - a_3 k^3) = 0 \qquad (2.3.4)$$

where $u(x,t) = A(k)\exp[(ikx - w(k)t)]$, $k > 0$, is the fundamental solution of the linear part of (2.1). Then by ordering these dispersion relations from being single-valued and real to many-valued and complex and comparing with the results from the classification above, the solitary wave solutions of the corresponding nonlinear equations exist in all cases.

2.4 Conclusion

In this chapter we have studied the existence of solitary wave solutions and their stability under linear perturbations. It was demonstrated that the solitary wave solutions exist for a wide variety of dispersion relations. Furthermore it was shown that stable solitary wave solutions exist for formally non-dispersive equations. Thus the main conclusion is that dispersion is not necessary for the existence of unique and stable solitary wave solutions of the KdV alternatives.

This uselessness of the dispersion to predict the properties of the KdV alternatives leads us to discuss whether other criteria can be found for such predictions. For this concept it is convenient to present a comparative study for the alternatives of the KdV in terms of the properties which comes from the KdV theory, such as:

(1) well-posedness (2) solitary waves (3) soliton
(4) conservation laws (5) linearization by inverse method.

The theory of an evolution equation is said to be complete if all the information about the above properties are confirmed either positively or negatively.

In the next chapter the theory of the KdV equation shall be presented in detail.

CHAPTER THREE

REVIEW OF THE KORTEWEG-DE VRIES EQUATION

This chapter is devoted to a review of some mathematical properties of the KdV equation. The results of chapter 2 are used to prove that the equations has a solitary wave which is linearly stable. We, then, turn to the linearization of the equation by the inverse scattering method and obtain the N-soliton solution. We go on to show that the equation has an infinite number of conservation laws, and, finally that it is well-posed.

3.1 Existence of solitary and periodic wave solutions

Consider the Korteweg-de Vries equation (KdV) in the form

$$u_t + u_x + uu_x + u_{xxx} = 0. \qquad (3.1.1)$$

Then both the solitary wave and periodic wave (cnoidal wave) solutions are obtained by choosing $a_6 = a_5 = a_4 = a_2 = 0$ and $a_1 = a_3 = 1$ in the proof of theorem 2.1. Hence the proof of the following theorem is clearly obtained.

Theorem 3.1

The KdV equation has two different types of travelling wave solutions, namely

(1) Periodic waves (cnoidal waves), given by

$$u(x,t) = r_1 + (r_3-r_2)\mathrm{cn}^2\{\tfrac{1}{12}\sqrt{r_3-r_1}\,[x-(1+c)t];K\} \qquad (3.1.2)$$

where, $K^2 = (r_3-r_2)/(r_3-r_1)$, $r_1 < r_2 < r_3$ are the roots of the equation

$r^3 - 3cr^2 - 6A_1r - 6A_2 = 0$, A_1, A_2 are constants of integrations and cn is the Jacobian elliptic cosine amplitude with modulus K

(ii) Solitary waves, given by

$$u_s(x,t) = 3c\,\mathrm{sech}^2 \cdot \frac{\sqrt{c}}{2}[x-(1+c)t], \quad c > 0. \;\Box \qquad (3.1.3)$$

3.2 Stability of the solitary wave solution

In a sense similar to that used in the proof of the linear stability of the solitary wave solutions of the general class 2.1, the following theorem is proved

Theorem 3.2

The solitary wave solution of the KdV equation is stable under linear perturbations. □

3.3 Inverse scattering method

This method, which was first discovered by Gardner, Green, Kruskal and Miura [10], provides a procedure for solving the initial value problem of the KdV equation and is applicable to initial data that vanishes rapidly as $|x| \to \infty$.

The initial value problem, considered in this section and in the next one, is

$$u_t - 6uu_x + u_{xxx} = 0 \qquad -\infty < x < \infty, \quad t > 0, \qquad (3.3.1)$$

$$u(x,0) = g(x) \qquad (3.3.2)$$

where, $g(x)$ satisfies the two conditions

$$\text{(i)} \quad \sum_{r=0}^{4} \int_{-\infty}^{\infty} \left|\frac{d^r g}{dx^r}\right|^2 dx < \infty, \qquad \text{(ii)} \quad \int_{-\infty}^{\infty} (1+|x|)g(x)\,dx < \infty$$

$$(3.3.3)$$

where the first condition guarantees the existence of a classical solution of the KdV equation [7], whilst the second condition guarantees the existence of a solution of the eigenvalue problem, stated below, [9].

Lemma 3.3.1

If v is a solution of the modified KdV equation in the form

$$v_t - 6v^2v_x + v_{xxx} = 0 \qquad (3.3.4)$$

then

$$u = v^2 + v_x \qquad (3.3.5)$$

is a solution of the KdV equation (3.3.1). □

If we take u to be known then (3.3.5) is a Riccati equation in v and can be linearized by making use of the transformation,

$$v = \frac{\phi_x}{\phi} . \qquad (3.3.6)$$

Hence (3.3.5) reduces to

$$\phi_{xx} - u\phi = 0$$

which, without loss of generality, can be replaced by

$$\phi_{xx} - (u-\lambda)\phi = 0. \qquad (3.3.7)$$

Equation (3.3.7) is time independent Schrödinger equation with potential u, energy level λ and wave function ϕ.

The inverse scattering problem is to determine u from a knowledge of its scattering data, i.e. discrete eigenvalues, the normalizing coefficients of the corresponding eigenfunctions, and the reflection coefficients (reflection and transmission coefficients occur when a wave sent in from ∞ interacts with a potential and some is reflected back and the rest transmitted through).

For the continuous spectrum, since $u \to o$ as $|x| \to \infty$, the asymptotic behaviour of the eigenfunctions that corresponds to the set of all positive eigenvalues $\lambda = k^2$ may be written as

$$\phi(k,x;t) = \begin{cases} \exp\{-ikx\} + b(k,t)\exp\{ikx\} & x \to +\infty \\ \\ a(k,t)\exp\{-ikx\} & x \to -\infty, \end{cases} \tag{3.3.8}$$

where b is the reflection coefficient and a is the transmission coefficient. The conservation of energy is expressed by $|a|^2 + |b|^2 = 1$. Assuming these scattering data are known, the problem has been studied by many people [13], [15], [30], ... and it has been solved by writing

$$u(x) = -2 \frac{d}{dx} K(x,x) \tag{3.3.9}$$

where K satisfies the Gel'fand-Levitan integral equation

$$K(x,y) + B(x+y) + \int_x^\infty B(y+z)K(x,z)dz = 0 \tag{3.3.10}$$

and the Kernel B is given by

$$B(\xi) = \sum_{m=1}^{N} c_m^2 \exp\{-k_m \xi\} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k)\exp\{ik\xi\}dk. \tag{3.3.11}$$

In the above calculations t entered as a parameter. But if we take into account the dependence of u on t and consequently the dependence of the eigenvalues, reflection and transmission coefficients and the wave function on t, then, to determine $u(x,t)$, the KdV solution, we have to know the quantities k_m, c_m and $b(k,t)$. These are given by the following theorem,

Theorem 3.3 [10], [20]

If $u(x,t)$ evolves according to the KdV equation (3.3.1) and $u(x,o) \to 0$ as $|x| \to \infty$, the following relations are satisfied

(i) each discrete eigenvalue λ_m of (3.3.7) is constant

(ii) $b(k,t) = b(k,0) \exp\{8ik^3t\}$

(iii) $c_m(t) = c_m(0) \exp\{4k_m^3 t\}$

(iv) $a(k,t) = a(k,0)$

where $c_m(0)$, $b(k,0)$ and $a(k,0)$ are determined from the initial data of the KdV equation $u(x,0) = g(x)$. □

Hence the solution of the KdV equation is given by (3.3.9), and theorem 3.3., having the form:

$$u(x,t) = -2 \frac{d}{dx} K(x,x;t) \tag{3.3.12}$$

where K satisfies the integral equation (3.3.10) such that

$$B(\xi,t) = \sum_{n=1}^{N} c_n^2 \exp\{8k_n^3 t - k_n\xi\}$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k,0) \exp\{i(8k^3t - k\xi)\}dk. \tag{3.3.13}$$

3.4 N-soliton solution

Having discussed the solitary wave solution and the inverse scattering method in the above sections we turn to the study of the N-soliton solution. The solitary wave of any nonlinear evolution equation is called a soliton if there exist solutions for this equation which approach a linear superposition of its solitary waves as $|t| \to \infty$. Zabusky [33] claimed that if the initial condition $u(x,0)$ of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (3.4.1)$$

satisfies the condition $\int_{-\infty}^{\infty} u(x,0) > 0$, then at least one soliton emerges from this initial disturbance. On the other hand Segur [25] extended the inequality of Bargmann [2] and found an upper bound for N, namely

$$N \leq 1 + \int_{-\infty}^{\infty} |x| g(x) dx, \quad \text{where} \quad g(x) = \begin{cases} u(x,0), & u(x,0) > 0 \\ 0 \quad , & u(x,0) < 0 \end{cases}.$$

The interaction between solitary waves for the KdV equation (3.4.1) was first observed, numerically, by Zabusky and Kruskal [34]. They showed that if two solitary waves placed on the real line, the taller to the left of the shorter at $t = 0$, are travelling, then after a sufficient time passed away, they overlap, interact and the taller overtakes the shorter and they both regain this original shapes and velocities. The only change is that a phase shift occurs. Lax [18] discussed the same phenomena analytically and confirmed Zabusky and Kruskal's observations.

The exact solution for the case of multiple collision of N solitons with different amplitudes was first found by Hirota [13]. The proof of such solutions is found in many references [10], [13], [20].

This proof depends on the fact that the N-soliton solutions have zero reflection coefficients. With this fact the Gelfand-Levitan integral equation (3.3.10) reduces to

$$K(x,y) + \sum_{m=1}^{N} c_m^2 \exp\{-k_m(x+y)\} + \sum_{m=1}^{N} c_m^2 \int_x^{\infty} \exp\{-k_m(Z+y)\} K(x,Z)dZ \qquad (3.4.2)$$

where $c_m = c_m(t) = c_m(0) \exp\{k_m^3 t\}$.

In order to remove y-dependence from (3.4.2) we must take

$$K(x,y) = -\sum_{m=1}^{N} c_m \phi_m(x) \exp\{-k_m y\} \qquad (3.4.3)$$

where c_m have been introduced so that the ϕ_m turn out to be normalized eigenfunctions of the Schrödinger equation

$$\phi_{xx} - (k_m^2+u)\phi = 0 \qquad (3.4.4)$$

we now substitute (3.4.3) in (3.4.2) and separately equate the coefficients of $\exp\{-k_m y\}$ to zero. We get the following N linear equations in ϕ

$$\phi_m(x) + \sum_{n=1}^{N} c_m c_n \phi_n(x) \exp \frac{\{-(k_m + k_n)x\}}{k_m + k_n} = c_m \exp\{-k_m x\},$$

$$m = 1, 2, \ldots, N \qquad (3.4.5)$$

which can be re-written in the form

$$(I + C)\Phi = E \qquad (3.4.6)$$

where I is the unit matrix of order N,

$$C = [C_{mn}] = c_m c_n \left[\frac{\exp\{-(k_m + k_n)x}{k_m + k_n} \right] \qquad (3.4.7)$$

is a $N \times N$ matrix and

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} c_1 \exp\{-k_1 x\} \\ c_2 \exp\{-k_2 x\} \\ \vdots \\ c_N \exp\{-k_N x\} \end{bmatrix} \qquad (3.4.8)$$

are column matrices.

A sufficient condition under which (3.4.6) has a unique solution is that C is positive definite. This is true since the quadratic form corresponding to C is

$$\sum_{m=1}^{N} \sum_{n=1}^{N} c_m c_n \frac{\exp\{-(k_m + k_n)x\}}{k_m + k_n} x_m x_n = \int_x^{\infty} dz \, [\sum_{m=1}^{N} c_m \exp(-k_m z) x_m]^2$$

which is positive. Now we note that

$$0 < \det C = (\prod_{m=1}^{N} c_m^2) \exp\{-2(\sum_{m=1}^{N} k_m)x\} \det \frac{1}{k_m + k_n} \qquad (3.4.9)$$

so that $\det \frac{1}{k_m + k_n} > 0$.

From (3.4.9) it is clear that C can be written as $\det C = \alpha \exp\{-\beta x\}$, α and β are positive. Then by expanding along the nth column we have

$$\Delta = \det (I+C) = \sum_{m=1}^{N} (\delta_{mn} + c_m c_n \frac{\exp\{-(k_m + k_n)x\}}{k_m + k_n}) \Omega_{mn} \qquad (3.4.10)$$

where Ω_{mn} is the co-factor of the coefficient matrix $I+C$.

Using Cramer's rule to solve (3.4.6), then

$$\phi_m = \frac{1}{\Delta} \sum_{m=1}^{N} c_n \exp\{-k_m x\} Q_{mn} . \qquad (3.4.11)$$

Replacing y by x in the expression of $K(x,y)$ in (3.4.3) and using (3.4.11), we have

$$K(x,x) = -\sum_{m=1}^{N} c_m \phi_m \exp\{-k_m x\} = -\frac{1}{\Delta} \sum_{m=1}^{N} \sum_{n=1}^{N} c_m c_n \exp\{-(k_m + k_n)\} Q_{mn}$$

$$= \frac{1}{\Delta} \frac{d}{dx} \Delta = \frac{d}{dx} \ln \Delta . \qquad (3.4.12)$$

Substituting (3.4.12) in (3.3.12)

$$u(x,t) = -2 \frac{d}{dx} K(x,x) = -2 \frac{d^2}{dx^2} \ln \Delta$$

$$= -2 \frac{d^2}{dx^2} \ln \{\det (I+C)\} \qquad (3.4.13)$$

which is a solution of the KdV equation corresponding to a reflectionless potential where this reflectionless initial condition remains reflectionless from theorem 3.3.

The asymptotic analysis carried out by Gardner, Green, Kruskal and Miura [10] for the solution of the KdV equation showed that it represents some finite number of (interacting) solitons with nothing else present. This is summarized by the following theorem:

<u>Theorem 3.4</u> [10]

If u is a reflectionless solution of the KdV equation, then as $|t| \to \infty$ each eigenvalue $\lambda_p = -k_p^2$ has associated with it a solution which approaches the solitary wave form

$$\lim_{\xi \text{ fixed}} u = \begin{cases} -2k_p^2 \operatorname{sech}^2 [k_p(x - 4k_p^2 t) - \delta_p] , & t \to \infty \\ -2k_p^2 \operatorname{sech}^2 [k_p(x - 4k_p^2 t) - \bar{\delta}_p] , & t \to -\infty \end{cases} \quad (3.4.14)$$

where $\xi = x - 4k_p^2 t$, $p = 1, 2, \ldots, N$ and,

$$\delta_p - \bar{\delta}_p = \frac{1}{k_p}\left[\sum_{m=1}^{p-1} \log \frac{k_m - k_p}{k_m + k_p} - \sum_{m=p+1}^{N} \log \frac{k_p - k_m}{k_p + k_m}\right] \quad (3.4.15)$$

i.e. (3.4.15) implies that the total phase shift is the sum of the phase shifts in isolated pairwise interaction with every other soliton. □

3.5 Local conservation laws

A local conservation law associated with a given equation is expressed by an equation of the form $\frac{\partial}{\partial t} T + \frac{\partial}{\partial x} X = 0$, where X and T are functions of x and t and the various derivatives of u.

In this section, the conservation laws for the C^∞-solutions of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 , \quad (3.5.1)$$

where u together with all its x-derivatives vanish as $|x| \to \infty$, are established.

In fact it is historically known that Korteweg and de-Vries [17] themselves derived their model in a conserved form. The first conservation law of (3.5.1) is obtained by re-writing the equation in the form,

$$\frac{\partial}{\partial t}[u] + \frac{\partial}{\partial x}[u_{xx} - 3u^2] = 0. \qquad (3.5.2)$$

The second conservation law is obtained by multiplying (3.5.1) by u and arranging the resulting equation in the form

$$\frac{\partial}{\partial t}[\frac{u^2}{2}] + \frac{\partial}{\partial x}[-2u^3 + uu_{xx} - \frac{u_x^2}{2}] = 0. \qquad (3.5.3)$$

Multiplying (3.5.1) by $u^2 - \frac{1}{3}u_{xx}$, then the third conservation law can be established in the form

$$\frac{\partial}{\partial t}[u^3 - \frac{1}{6}u_x^2] + \frac{\partial}{\partial x}[\frac{1}{6}u_{xx}^2 + u^2u_{xx} + \frac{1}{3}u_xu_t - \frac{3}{2}u^4] = 0.$$

(3.5.4)

In fact, the existence of an infinite number of such conservation laws has been found by Miura et al [21]. These conservation laws are used to derive a priori estimates of the solution of the KdV equation as shall be seen in section 3.7.

Theorem 3.5

There exist an infinite number of polynomial conservation laws for the C^∞-solution of the KdV equation (3.5.1). □

The proof of this theorem is outlined as follows:

Consider the Miura transformations,

$$u = v^2 + v_x \qquad (3.5.5)$$

which couples the modified KdV equation

$$v_t - 6v^2v_x + v_{xxx} = 0 \qquad (3.5.6)$$

with the KdV equation (3.5.1) as in lemma 3.3.1. By making use of the transformation

$$x' = x + \frac{3}{2\epsilon^2}\,t, \quad t' = t, \quad u(x,t) = u'(x',t') + \frac{1}{4\epsilon^2} \text{ and}$$

$$v(x,t) = \epsilon w(x',t') + \frac{1}{2\epsilon} \tag{3.5.7}$$

where the specific dependence on the arbitrary parameter ϵ has been chosen to get the desired results below, (3.5.1) gives

$$0 = u_t - 6uu_x + u_{xxx}$$

$$= (2v + \frac{\partial}{\partial x})\,[v_t - 6v^2v_x + v_{xxx}] \qquad \text{(by using (3.5.5))}$$

$$= (1 + \epsilon\frac{\partial}{\partial x} + 2\epsilon^2 w)\,[w_t - 6(w + \epsilon^2w^2)w_x + w_{xxx}]. \tag{3.5.8}$$

(by dropping all primes and using (3.5.7)].

Now, inserting (3.5.7) into (3.5.5) and dropping all primes, then

$$u = w + \epsilon w_x + \epsilon^2 w^2. \tag{3.5.9}$$

By solving (3.5.9), recursively, w can be determined in the form of a formal power series in ϵ with coefficients which are functions of u and x-derivatives of u, i.e.,

$$w(x,t;\epsilon) = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots.$$

$$= u - \epsilon u_x - \epsilon^2(u^2 - u_{xx}) + \ldots \quad . \tag{3.5.10}$$

Using (3.5.10) and (3.5.8), then the expression in the square brackets of (3.5.8) must vanish to all orders in ϵ since we are dealing only with formal series, i.e.

$$w_t - 6(w + \epsilon^2w^2)w_x + w_{xxx} = 0 \tag{3.5.11}$$

(to all orders in ϵ).

Equation (3.5.11) can be re-written, to all powers in ϵ, in the form

$$\frac{\partial}{\partial t} w + \frac{\partial}{\partial x} [-3w^2 - 2\epsilon^2 w^3 + w_{xx}] = 0. \qquad (3.5.12)$$

Thus, the coefficient of each power of ϵ is a conservation law for the KdV equation (3.5.1). This leads to the existence of an infinite number of conservation laws of the KdV equation since (3.5.11) does not depend on ϵ and it can be shown that the coefficients of the even power of ϵ gives nontrivial conservation laws whereas the coefficients of the odd powers of ϵ are trivial conservation laws [21].

The above theorem shows that the KdV equation has an infinite number of local conservation law . The constants of motion are derived by integrating each conservation laws with respect to x between $x = -\infty$ to $x = \infty$ and using the assumption that u vanishes rapidly together with all its x-derivatives as $|x| \to \infty$ e.g.

$$F_1 = \int_{-\infty}^{\infty} u dx, \quad F_2 = \frac{1}{2} \int_{-\infty}^{\infty} u^2 dx, \quad F_3 = \int_{-\infty}^{\infty} (u^3 - \frac{1}{6} u_x^2) dx, \; \ldots.$$

3.6 Relationship between the inverse method and conservation laws

In the last section, the eigenvalue problem played a distinguished part to prove the existence of an infinite number of conservation laws. This proof was via the use of the Miura transformation. This result provides a clue to the relationship between inverse method and the conservation laws for a broad class of nonlinear evolution equations which was systematically discussed by Ablowitz et al [1] and contains the KdV, the sine-Gordon and other equations. A general method for deriving conservation laws from the inverse method was provided by Konno et al [16] and Wadati et al [29] as in the following:

Theorem 3.6

The conservation laws of the KdV equation (3.5.1) can be obtained from the inverse method. □

The proof of this theorem is in [16].

3.7 Well-posedness

The theory of existence and uniqueness of the solution of the KdV equation began with Sjoberg [26]. He showed that for periodic data with three L^2 derivatives, the initial value problem of the KdV equation with this data has a solution, but he did not consider the continuous dependence of the solution on the data. Temam [27] has used the method of regularization by adding the term ϵu_{xxxx} to the KdV equation to get some properties. Then, by letting $\epsilon \to 0$, a weak solution to the KdV equation corresponding to periodic initial data has been shown to exist. However no claim was made to extend the initial value problem to the inifite interval by this method. Furthermore there was no consideration of the continuous dependence of the solution on the initial data. Up to the present the problem has been studied by many others [8], [28], ... Among those people Bona and Smith [7] used the method of regularization to prove that the initial value problem,

$$u_t + uu_x + u_{xxx} = 0 \quad , \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = g(x) \tag{3.7.1}$$

is well-posed. Since the uniqueness is easier to prove and it had been done by, for example Sjoberg [26], we begin the theory of well-posedeness by the uniqueness of the the solution of (3.7.1).

Theorem 3.7 (uniqueness)

If the initial value problem (3.7.1) has a solution, then this solution is unique. □

Proof

Let u and v be two solutions of the initial value problem (3.7.1) and $w = u - v$. Then w satisfies the initial value problem,

$$\begin{aligned} &w_t + \frac{1}{2}[(u+v)w]_x + w_{xxx} = 0 \\ &w(x,0) = 0. \end{aligned} \tag{3.7.2}$$

Multiplying the first equation in (3.7.2) by w, we have

$$ww_t + \frac{1}{2} w\,[\,(u+v)w]_x + ww_{xxx} = 0. \tag{3.7.3}$$

Integrating (3.7.3) from $x = -\infty$ to $x = \infty$, then if w, w_x and w_{xx} vanish as $|x| \to \infty$ (this will be confirmed in the existence proof), (3.7.3) reduces to

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty} w^2(x,t) &= -\frac{1}{2}\int_{-\infty}^{\infty}(u_x + v_x)w^2dx \\ &\le \frac{1}{2}\sup_{x\in R}|u_x + v_x|\int_{-\infty}^{\infty} w^2dx\,. \end{aligned} \tag{3.7.4}$$

If $c = \sup_{x\in R}|u_x + v_x|$, then

$$\frac{d}{dt}F(w) \le \frac{1}{2}\,C\,F(w) \tag{3.7.5}$$

where $F(w) \leq \int_{-\infty}^{\infty} w^2 dx$. Hence (3.7.5) gives

$$F(w) = F(0) \exp \frac{1}{2} ct$$

$$= 0 \quad (\text{since } F(0) = 0)$$

i.e. $w = 0$ almost everywhere,

i.e. $u \equiv v$ and the solution of (3.7.1) is unique.

We turn now to prove the existence. The proof is very long and complicated. Before outlining this proof, we define the function spaces which are used in this proof:

Definition 3.7.1

(i) $L^2(R) = \{u(x): \int_{-\infty}^{\infty} (u)^2 dx < \infty\}$, $||u||_{L^2(R)} = \{\int_{-\infty}^{\infty} u(x)^2 dx\}^{\frac{1}{2}}$

(ii) $H^S(R)$ [Sobolev space of order S] = $\{u(x): u \in L^2(R)$ and

$\frac{d^k u}{dx^k} \in L^2(R)$, $1 \leq k \leq s\}$, and

$$||u(x,t)||_{H^S(R)} = \sum_{k=0}^{S} \int_{-\infty}^{\infty} \left|\frac{d^k u}{dx^k}\right|^2 dx = \sum_{k=0}^{S} ||\frac{d^k u}{dx^k}||_{L^2(R)}$$

(iii) $\mathcal{H}_T^S \equiv C(0, T; H^S) = \{u, u: R \times T \to R,$ for each $t \in [0,T]$, $u(\cdot,t) \in H^S$ and the mapping $u: [0,T] \to H^S$ is continuous and bounded$\}$,

$$||u||_{\mathcal{H}_T^S} = \sup_{0 \leq t \leq T} ||u(\cdot,t)||_{H^S}$$

(iv) $\mathcal{H}_T^{s,k} = \{u(x,t) \in \mathcal{H}_T^{s} : \dfrac{\partial^i u}{\partial t^i} \in H_T^s, \; 0 \le i \le k\},$

$$||u||_{\mathcal{H}_T^{s,k}} = \sup_{0 \le t \le T} \sup_{0 \le i \le k} \left|\left| \frac{\partial^i u(x,t)}{\partial t^i} \right|\right|_{H^s} .$$

Now, leaving the technical proofs to Appendix (A) the proof for the existence of the solution of (3.7.1) is summarized as follow:

(a) <u>Regularization of the KdV</u>

The KdV equation in (3.7.1) is regularized by adding the term $-\epsilon u_{xxt}$ so that (3.7.1) becomes

$$u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0$$
$$u(x,0) = g(x) \quad , \qquad -\infty < x < \infty \text{ and } \epsilon \in (0,1]. \tag{3.7.6}$$

By making use of the transformation,

$$x \to \xi = \epsilon^{\frac{1}{2}}(x-t), \quad t \to \tau = \epsilon^{\frac{3}{2}}t \text{ and } u \to v(\xi,\tau) = \epsilon u(x,t),$$
$$\epsilon \in (0,1] \tag{3.7.7}$$

equation (3.7.6) transforms to

$$v_\tau + v_\xi + vv_\xi - v_{\xi\xi\tau} = 0$$
$$v(\xi,0) = h(\xi) = \epsilon\, g(\epsilon^{\frac{1}{2}}x) . \tag{3.7.8}$$

The initial value problem (3.7.8) was proposed by Benjamin et al [4] as an alternative to the KdV equation. The exact theory for (3.7.8) has been provided by [4] and shall be discussed in the next chapter. Hence for a fixed ϵ both $u(x,0)$ and $v(\xi,0)$ are in the same function class and the following lemma can be clearly introduced.

Lemma 3.7.1 [7]

(1) If $g \in H^k$, $k \geq 2$, then there exists a unique solution $u(x,t)$ to the regularized KdV equation (RKdV) (3.7.6), $u \in \mathcal{H}_T^k$ for any finite $T > 0$ and $\frac{\partial^p u}{\partial t^p} \in \mathcal{H}_T^{k-p}$, $0 \leq p \leq k$.

(2) If $g \in H^\infty$, then there exists a unique solution $u(x,t)$ to (3.7.6) which together with all its derivatives lies in $\mathcal{H}_T = \mathcal{H}_T^0$ for all finite T.

Our purpose is to let $\epsilon \to 0$, but at this limit the transformation (3.7.7) is singular and the bounds of u cannot be obtained in terms of the bounds of v. Hence the bounds of u should be obtained by its own and this is done in the next part

(b) A priori bounds for solutions of RKdV

Lemma 3.7.2 [see appendix (A)]

Let $g \in H^\infty$, then the solution u of (3.7.6) satisfies

(1) $||u||_{L^2} \leq ||g||_{H^1}$

(2) $||u||_{H^1} \leq \xi(||g||_{H^1})$ independently of ϵ, $0 < \epsilon \leq 1$, where $\xi: R^+ \to R^+$ is continuous monotone increasing function with $\xi(0) = 0$

(3) For $T > 0$, there exists $\epsilon_0 = \epsilon_0(T, ||g||_{H^3})$ such that $||u||_{H^2} \leq \xi_1(||g||_{H^3})$ independently of $t \in [0,T]$, where $\xi_1: R^+ \to R^+$ is a continuous monotone increasing function, $\xi_1(0) = 0$, $0 < \epsilon \leq \epsilon_0$

(4) For any $T > 0$ and $K \geq 3$ and ϵ_0 as in (3), $u(x,t)$ is bounded in $\mathcal{H}_T^k$ with bounds depending only on T, ϵ_0, $||g||_{H^k}$ and $\epsilon^{\frac{1}{2}} ||g||_{H^{k+1}}$

(5) $u(x,t)$ is bounded in $\mathcal{H}_T^{k,1}$ independently of $\epsilon \leq \epsilon_0$ for all $k,1$; $T > 0$

(c) <u>Regularization of the initial data</u>

The data $g(x) = u(x,0)$ is now regularized by convolution with a smooth function ϕ to obtain the regularized data

$$\hat{g}_\epsilon(k) = \phi(\epsilon^{\frac{1}{6}}k)\hat{g}(k)$$

where $\hat{f}$ is the Fourier transformation of f, ϕ is an even C^∞-funtion satisfies $0 \leq \phi < 1$ such that $\phi(0) = 1$ and $\psi(x) = 1 - \phi(x)$ has a zero of infinite order and $\phi \to 0$, exponentially, as $|x| \to \infty$ [e.g. $\phi = \exp\{-x^2\exp(-\frac{1}{x^2})\}$]. Then $g_\epsilon \in H^\infty$.

Using this regularization of the data and the result of part (a), then the initial value problem

$$u_t + uu_x + u_{xxx} - \epsilon u_{xxt}$$

$$u(x,0) = g_\epsilon(x) \tag{3.7.9}$$

has a unique C^∞-solution u_ϵ, which lies together with all its derivatives in $\mathcal{H}_T$ for all finite $T > 0$.

(d) <u>Small ϵ consideration</u>

Since to each ϵ, $0 < \epsilon \leq 1$ there is a unique solution of

(3.7.9), then the behaviour of this solution as $\epsilon \to 0$ must be considered. This is summarized in the following

Lemma 3.7.3 [see the Appendix (A)]

Let $g \in H^k$, $k \geq 3$ and g_ϵ as in (c), then as $\epsilon \to 0$

(1) $||g_\epsilon||_{H^{k+j}} = O(\epsilon^{-\frac{1}{6}j})$, $j = 1, 2, \ldots$. Uniformly on bounded subset of H^k

(2) $||g - g_\epsilon||_{H^{k-j}} = o(\epsilon^{\frac{1}{6}j})$ for $j = 1, 2, \ldots$. Uniformly on compact subset of H^k

(3) $||g - g_\epsilon||_{H^k} = o(1)$. Uniformly on compact subset of H^k

(4) u_ϵ is bounded in $\mathcal{H}_T^k$ independently of sufficiently small ϵ for each finite $T > 0$. Morever $\epsilon^{\frac{1}{6}m} u_\epsilon$ is bounded in $\mathcal{H}_T^{k+m}$ independently of sufficiently small ϵ for each finite $T > 0$ and $m \geq 1$

(5) $\frac{\partial}{\partial t} u_\epsilon$ is bounded in $\mathcal{H}_T^{k-3}$ and $\epsilon^{\frac{1}{6}m} \partial_x^{k+m-3} \partial_t u_\epsilon$ is bounded in $\mathcal{H}_T$ independently of sufficiently small ϵ, for all finite $T > 0$ and $m = 1, 2, \ldots, 5$, (where $\partial_t = \frac{\partial}{\partial t}$).□

(e) Sequencing to a weak solution of the KdV

Using the results in (d), then the final part in this procedures is summarized in the following lemma whose proof is in [7]

Lemma 3.7.4

Let u_ϵ be the solution of (3.7.9) where $g \in H^k$, $k \geq 3$, then

(1) $\{u_\epsilon\}$ is Cauchy sequence in $\mathcal{H}^k_T$ as $\epsilon \to o$

(2) $(u_t(x,t;\epsilon)$ is Cauchy sequence in $\mathcal{H}^{k-3}_T$ as $\epsilon \to o$. □

The above procedures can effectively used to prove the following:

Theorem 3.8 (Existence)

Let $g \in H^k$, $k \geq 3$. Then there exists a unique solution $u(x,t)$ of the initial value problem (3.7.1) which lies in $\mathcal{H}^k_T$ for all finite $T > 0$. □

Proof

The uniqueness of the solution is guaranteed by theorem 3.7. To prove the existence let g_ϵ be the regularization of g as in (c) and u_ϵ be the solution of the regularized KdV equation (3.7.9), i.e. to each ϵ, $0 < \epsilon \leq 1$ there exists a solution of (3.7.9). Using lemma 3.7.4 (1) these solutions form a Cauchy sequence in $\mathcal{H}^k_T$, $k \geq 3$ for any finite $T > 0$. Since $\mathcal{H}^k_T$ is a is a Banach space, then as $\epsilon \to o$, $u_\epsilon \to u$ in $\mathcal{H}^k_T$. Similarly, using lemma 3.7.4 (2), $\partial_t u_\epsilon \to v \in \mathcal{H}^{k-3}_T$. This leads to

$$\frac{\partial}{\partial x} u_\epsilon^2 \to \frac{\partial}{\partial x} u^2 \text{ in } \mathcal{H}^{k-1}_T \text{ and } \frac{\partial^3 u_\epsilon}{\partial x^3} \to \frac{\partial^3 u}{\partial x^3} \text{ in } \mathcal{H}^{k-3}_T \text{ as } \epsilon \to o.$$

Furthermore, $\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t} u_\epsilon$ is bounded in $\mathcal{H}^{k-5}_T$, so that,

$$\epsilon \frac{\partial^3}{\partial x^2 \partial t} u_\epsilon \to o \text{ as } \epsilon \to o \text{ (at least in the sense of distribution).}$$

Since $u_\epsilon \to u$ in $\mathcal{H}^k_T$, then $u_\epsilon \to u$ as $\epsilon \to o$ (in the sense of distributions). Thus

$\frac{\partial}{\partial t} u_\epsilon \to \frac{\partial}{\partial t} u = v$ (in the sense of distributions) as $\epsilon \to 0$, where u is the solution of the KdV equation.

Since the choice of T was arbitrary, i.e. if T becomes large enough, ϵ can be chosen small enough such that the results in part (b) hold. Then the solution can be extended over any T and the solution exists globally which completes the proof.

We consider now the continuity of the solution with respect to the initial data. For this, let $\mathcal{X}_{s,T}$ be the space defined by $\mathcal{X}_{s,T} = \mathcal{X}_T^s \cap \mathcal{X}_T^{s-3} \cap \mathcal{X}_T^{s-6} \cap \ \ldots$, and let $F: H^s \to \mathcal{X}_{s,T}$, be the mapping which assigns to each $g \in H^s$ the unique solution of (3.7.1), then with this notation the following theorem is introduced:

<u>Theorem 3.8</u> [7]

Let $T > 0$ be given and let F be the restriction to the time interval [0,T] of the map assigning to $g \in H^k$, $k \geq 3$ the unique global solution of (3.7.1). The F is continuous. □

3.8 Conclusion

In this chapter we have presented a review of the mathematical properties of the KdV. We have shown that the equation has the following properties:

(a) is well-posed (b) has solitary wave and N-soliton solutions (c) can be linearized by the inverse scattering method and (d) has infinite number of conservation laws.

Hence the thoery of the KdV equation is complete in the sense given in the last chapter.

CHAPTER FOUR

THE REGULARIZED LONG WAVE EQUATION

The regularized long wave (RLW) equation was first obtained by Peregrine [17] to describe the development of an undular bore, i.e., a smooth solitary wave that is observed to propagate in shallow water channels, and later by Benjamin et al [5] to describe approximately the unidirectional propagation of long waves in certain dispersive systems. Under the same approximations which lead to the KdV equation the RLW equation is derived in the form [5]:

$$u_t + u_x + uu_x - u_{xxt} = 0. \qquad (4.1)$$

In this chapter a review of the mathematical properties of the model (4.1) is presented.

4.1 Existence of solitary and periodic wave solutions

Like the KdV, the RLW has bounded travelling wave solutions which are either solitary waves or periodic waves.

Hence if we choose $a_6 = a_5 = a_3 = a_2 = 0$ and $a_1 = - a_4 = 1$ in theorem (2.1) the following theorem can be proved.

Theorem 4.1

The RLW equation has two classes of solutions namely:

(i) periodic solutions

$$u(x,t) = r_2 + (r_3-r_2)\,cn^2\,(\lambda\xi;k) \qquad (4.1.1)$$

where cn is the Jacobian elliptic cosine amplitued with modulus k, $\lambda^2 = r_1 - r_3$, $k^2 = (r_2-r_3)/(r_1-r_3)$, and r_1, r_2, r_3 are the roots of the cubic equation

$$4r^3 - 12c^2[1 + \frac{3A}{c}]r + 8c^3[1 - \frac{3A}{c} + \frac{3B}{c^2}] = 0, \text{ A, B are constants;}$$

(ii) solitary wave solutions

$$u_s(x,t) = 3c \operatorname{sech}^2\frac{1}{2}\sqrt{\frac{c}{1+c}}[x - (1+c)t], \quad c > 0. \quad \square \qquad (4.1.2)$$

4.2 Linear stability of the solitary wave solutions

Having proved the existence of solitary wave solutions, the question about the stability of these solutions arises and in this section we consider linear stability. If the solution $u(x,t)$ of (4.1) is approximated by

$$u = u_s(\xi) + \epsilon g(\xi,t), \quad \xi = x - (1+c)t \qquad (4.2.1)$$

then by using the asymptotic analysis method as in section 2.2, we can prove the following:

Theorem 4.2

The solitary wave solution of the RLW equation is linearly stable. $\square$

There is another type of stability (conditional stability) which makes use of the conservation laws of the equation.

4.3 Conservation laws

In this section we consider another important property of the RLW, the existence of a number of independent conservation laws.

Replacing u by $-1-u$, the RLW equation reduces to

$$u_t - uu_x - u_{xxt} = 0 \qquad (4.3.1)$$

which can be re-written in a conserved form :

$$\frac{\partial}{\partial t}[u] + \frac{\partial}{\partial x}[-\frac{u^2}{2} - u_{xt}] = 0 \qquad (4.3.2)$$

This is the first conservation law.

Multiplying (4.3.1) by u, the resulting equation can be re-written in the form :

$$\frac{\partial}{\partial t}[\frac{u^2}{2} + \frac{u_x^2}{2}] + \frac{\partial}{\partial x}[-\frac{u^3}{3} - uu_{xt}] = 0 \qquad (4.3.3)$$

which is a second conservation law.

The third conservation law is obtained by multiplying (4.3.1) by u^2. Hence the resulting equation has the form:

$$\frac{\partial}{\partial t}[\frac{u^3}{3}] = \frac{\partial}{\partial x}[\frac{u^4}{4} + u^2u_{xt}] - 2uu_xu_{xt}$$

$$= \frac{\partial}{\partial x}[\frac{u^4}{4} + u^2u_{xt}] - 2u_{xt}[u_t - u_{xxt}]$$

$$= \frac{\partial}{\partial x}[\frac{u^4}{4} + u^2u_{xt}] - \frac{\partial}{\partial x}[u_t^2 - u_{xt}^2], \quad \text{i.e.}$$

$$\frac{\partial}{\partial t}[\frac{u^3}{3}] + \frac{\partial}{\partial x}[u_t^2 - u_{xt}^2 - u^2u_{xt} - \frac{u^4}{4}] = 0 \qquad (4.3.4)$$

which is the third conservation law.

Hence (4.3.2), (4.3.3) and (4.3.4) are three independent conservation laws for the RLW (4.3.1). The corresponding functionals (constants of motion) are obtained by integrating the above equations and using the assumption u, u_x and $u_{xt} \to 0$ as $|x| \to \infty$, then the three functionals are respectively;

$$F_1(u) = \int_{-\infty}^{\infty} u dx, \quad F_2(u) = \int_{-\infty}^{\infty} (\frac{u^2}{2} + \frac{u_x^2}{2}) dx \quad \text{and} \quad F_3(u) = \int_{-\infty}^{\infty} u^3 dx$$

Olver [16] showed that the RLW has no other conserved densities depending on x, u, u_x, u_{xx}, ... than those stated above. This is summarised in the following theorem:

Theorem 4.3

The only nontrivial independent conservation laws of (4.3.1) are (4.3.2), (4.3.3) and (4.3.4). □

The method of the proof is based on a comprehensive algebraic machinery for use in the investigation of conservation laws of partial differential equations, and a nice presentation of the Olver's proof was given by Abbas [1].

In spite of the fact that the RLW has only three conservation laws, there might exist other conserved densities depend on t and the t-derivatives of u and u_x. Duzhin & Tsujishita [10] have discussed this possibility by using the method which is based on the calculation of a certain part of the Vinogradov spectral sequence [19] and the universal operator $\ell_F = -D_x^2 D_t + D_x - D_t$, where D is the total differential operator defined on the algebra of x, t, u and the various derivatives of u. They have proved that the conjugate ℓ_F^* of the operator ℓ_F has a finite dimensional kernel generated by three elements 1, u, $\frac{1}{2}(u^2 + u_{xt})$. The conjugate is given by $\ell_F^* = D_x^2 D_t - D_x + D_t$, and by using the relationship between Ker ℓ_F^* and the space of conservation laws, the following theorem can be proved:

Theorem 4.4

The dimension of the space of conservation laws of the RLW (4.1) is not greater than three. □

4.4 Conditional stability

In section 4.2, the stability of solitary wave solution of the RLW equation, under linear perturbation, was considered. In this section the linearity assumption is not specifically included. Morever, the stability of solitary waves is discriminated in respect of shape. This is achieved by a device entailing the definition of certain quotient space, as shall be seen in definition 4.4.1 below.

To study this type of stability, consider the initial value problem for the RLW equation

$$u_t + u_x + uu_x - u_{xxt} = 0\,, \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = g(x) \tag{4.4.1}$$

which has the solitary wave solution

$$u_s(x,t) = 3c\ \mathrm{sech}^2 \tfrac{1}{2} \left[\frac{c}{1+c}\right]^{\frac{1}{2}} [x - (1+c)t] = \phi(x-\bar{c}t), \tag{4.4.2}$$

where $\bar{c} = 1+c$.

Now, the stability of ϕ means that if u is made close to ϕ at $t = 0$ then u will remain close for all t. For this purpose, some precise measure of distance between u and ϕ must be specified. This metric is a functional depending on pairs of functions defined on the whole real axis and evaluated on the two solutions u and ϕ of (4.4.1). This metric is generally a function of t and not constant.

Thus to establish the stability of (4.4.2), the following assumptions, made by Benjamin [4], are necessary:

(1) The solutions of (4.4.1) are c^{∞}-functions all of whose derivatives vanish rapidly as $|x| \to \infty$.

(2) The initial value $g(x)$ and the solitary wave $\phi(x)$ are close to each other.

(3) The solution $u(x,t)$ exists for the considered class of $g(x)$ and has the required smoothness properties (this is guaranteed by the well-posedness theory of (4.4.1) which is given in the next section).

(4) The two functionals:

$$E(u) = \int_{-\infty}^{\infty} (u^2+u_x^2)dx = \text{constant and}$$

$$M(u) = \int_{-\infty}^{\infty} (u^2+\frac{1}{3}u^3)dx = \text{constant} \tag{4.4.3}$$

for solutions of (4.4.1) with required restrictions on asymptotic values are used.

(5) A device can be found to concentrate the proof on the stability of the shape of the wave. This is clear from the following definition of the metric used to measure closeness.

Definition 4.4.1

Let $f, g \in H^1$ and H^1/G be the quotient space, where G is the translation group in R, i.e., $G_y f(x) = f(x+y)$, $y \in R$. Now, define

$$d(f,g) = \inf_{y \in R} \|f(x+y) - g(x)\|_{H^1(R)} . \tag{4.4.4}$$

Then d is a pseudo-metric on H^1 and a proper metric on the quotient space H^1/G.

Using the above points, (1) to (5), Benjamin was able to prove a stability theorem. However his proof contained some restrictions and unjustified assumptions. These were improved and corrected later by Bona [6].

The final theorem, whose proof is provided by Bona [6] is as follows:

Theorem 4.5

Let $\epsilon > 0$ be given. Then there exists $\delta > 0$, such that if $g \in H^2$, u is the solution of (4.4.1) and

$\|g - \phi\|_{H^1} < \delta$, then

$d(u,\phi) \leq \epsilon$. □

4.5 Well-posedness

Consider the initial value problem

$$u_t + u_x + uu_x - u_{xxt} = 0 , \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = g(x). \tag{4.5.1}$$

Re-writing the differential equation in (4.5.1) in the form

$$(1 - \frac{\partial^2}{\partial x^2})u_t = -\frac{\partial}{\partial x}[u(x,t) + \frac{1}{2}u^2(x,t)] \qquad (4.5.2)$$

it can be regarded as an ordinary differential equation in u_t. If we consider the boundary conditions $u(x,t) \to 0$ as $|x| \to \infty$, the Green's function of the differential operator, i.e.,

$$G(x,\xi) = \begin{cases} A(\xi)e^{x} + B(\xi)e^{-x}, & x < \xi \\ e(\xi)e^{x} + D(\xi)e^{-x}, & x > \xi \end{cases}$$

is reduced, by using the boundary condition, to

$$G(x,\xi) = \begin{cases} A(\xi)e^{x} & x < \xi \\ D(\xi)e^{-x} & x > \xi . \end{cases}$$

Since $G(x,\xi) = G(\xi,x)$,

$$G(x,\xi) = \begin{cases} Ae^{(x-\xi)}, & x < \xi \\ Ae^{-(x-\xi)} & x > \xi . \end{cases}$$

Using the continuity properties of the Green's function

$$G\Big|_{x=\xi+0} = G\Big|_{x=\xi-0} \quad \text{and} \quad \frac{\partial G}{\partial x}\Big|_{x=\xi+0} - \frac{\partial G}{\partial x}\Big|_{x=\xi-0} = -1$$

one can obtain $A = -\frac{1}{2}$. Hence

$$G(x,\xi) = -\frac{1}{2}e^{-|x-\xi|} \qquad (4.5.3)$$

which is the Green's function of the differential operator $(1 - \frac{d^2}{dx^2})$.

Then (4.5.2) can be inverted to

$$u_t = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\xi|}\frac{\partial}{\partial\xi}\{u(\xi,t) + \frac{1}{2}u^2(\xi,t)\}d\xi$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\xi)}\{u(\xi,t) + \frac{1}{2}u^2(\xi,t)\}d\xi \;.$$

Integrating the last equation once with respect to t and using the initial condition $u(x,0) = g(x)$ we have

$$u = g(x) + \int_0^t\int_{-\infty}^{\infty} k(x-\xi)\{u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau)\}d\xi d\tau \qquad (4.5.4a)$$

where $k(x-\xi) = \frac{1}{2}\,\mathrm{sgn}(x-\xi)e^{-|x-\xi|}$.

Hence the original initial value problem (4.5.1) is formally equivalent to an integral equation and we can now re-write this in the form:

$$u = Au = Bu + g(x) \qquad (4.5.4b)$$

where $Bu = \int_0^t\int_{-\infty}^{\infty} k(x-\xi)\{u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau)\}d\xi d\tau$.

Having obtained the integral form of the general solution of (4.5.1) in the form (4.5.4) we turn to study the well-posedness of the problem.

4.5.1 Existence

The existence proof is done in two steps. The existence of the solution is first obtained locally and then extended globally.

The treatment, given below, closely follows that of Benjamin et al [5].

Local existence

The proof of existence of a solution to (4.5.1) is reduced to existence of a solution to (4.5.4). For this purpose let $\mathcal{C}_{t_o}$ be the space of all continuous and bounded functions defined on $R \times [0,t_o]$ with norm defined by $||.|| = \sup_{\substack{x \in R \\ 0 < t \leq t_o}} |.|$. One can show that $\mathcal{C}_{t_o}$ is complete normed linear space (Banach space). Consider now the integral equation (4.5.4) where the integral operator A acts on $\mathcal{C}_{t_o}$, i.e., $A: \mathcal{C}_{t_o} \to \mathcal{C}_{t_o}$. Then for any $v_1, v_2 \in \mathcal{C}_{t_o}$ and any $x \in R$, $t \in [0,t_o]$,

$$|Av_1 - Av_2| = |(g+Bv_1) - (g+Bv_2)|$$

$$= |Bv_1 - Bv_2|$$

$$= |\int_0^t \int_{-\infty}^{\infty} k(x-\xi)\{(v_1-v_2) + \frac{1}{2}(v_1^2-v_2^2)\}d\xi d\tau|$$

$$\leq \int_0^t \int_{-\infty}^{\infty} |k(-\xi)||v_1 - v_2||1 + \frac{1}{2}(v_1+v_2)|d\xi d\tau$$

$$\leq \sup_{\substack{x\in R \\ t\in[0,t_o]}} |v_1 - v_2|\{1 + \frac{1}{2} \sup_{\substack{x\in R \\ t\in[0,t_o]}} |v_1 - v_2|$$

$$\int_0^t \int_{-\infty}^{\infty} |k(x-\xi)d\xi d\tau\}$$

$$\leq ||v_1 - v_2||_{\mathcal{C}_{t_o}}\{1 + \frac{1}{2}||v_1||_{\mathcal{C}_{t_o}} + \frac{1}{2}||v_2||_{\mathcal{C}_{t_o}}\}t$$

$$(\text{Since } \int_{-\infty}^{\infty} \|k(x-\xi)\| d\xi = 1). \qquad (4.5.5)$$

Taking the supremum of both sides for $x,t \in R \times [0,t_o]$ and using the definition of the norm in the space $\mathcal{C}_{t_o}$, we have

$$||Av_1-Av_2||_{\mathcal{C}_{t_o}} \leq (1 + \tfrac{1}{2}\ ||v_1||_{\mathcal{C}_{t_o}} + \tfrac{1}{2}\ ||v_2||_{\mathcal{C}_{t_o}})||v_1-v_2||_{\mathcal{C}_{t_o}}\ t_o \qquad (4.5.6)$$

which implies that A is continuous mapping of $\mathcal{C}_{t_o}$ into itself. Furthermore, it satisfies a Lipschitz condition on the ball $||v|| \leq R$ with Lipschitz constant λ such that

$$t_o(1+R) \leq \lambda < 1. \qquad (4.5.7)$$

Since, using the condition (4.5.7) in (4.5.6),

$$||Av_1-Av_2|| \leq t_o(1+R)\ ||v_1-v_2||_{\mathcal{C}_{t_o}} \leq \lambda\ ||v_1-v_2||_{\mathcal{C}_{t_o}}. \qquad (4.5.8)$$

Choosing $v_2 = 0$ and $v_1 = v$ in (4.5.8), then

$$||Av||_{\mathcal{C}_{t_o}} \leq \lambda\ ||v||_{\mathcal{C}_{t_o}}, \qquad 0 < \lambda < 1,$$

i.e., A is Lipschitz. Hence A is a contractive mapping for values of t_o, where $t_o(1+R) < \lambda < 1$.

Similarly

$$||Bv||_{\mathcal{C}_{t_o}} \leq \lambda\ ||v||_{\mathcal{C}_{t_o}}. \qquad (4.5.9)$$

But using (4.5.4) and (4.5.9),

$$||Av||_{\mathcal{C}_{t_o}} \leq \sup_{x\in R} |g(x)| + ||Bv||_{\mathcal{C}_{t_o}}$$

$$\leq \sup_{x\in R} |g(x)| + \lambda\ ||v||_{\mathcal{C}_{t_o}}.$$

Now for the mapping A to be contractive over the ball $||v|| = R$ for any t_o, we must have

$$\sup_{x \in R} |g(x)| \leqslant (1-\lambda)R . \qquad (4.5.10)$$

It can be easily checked that (4.5.7) and (4.5.10) can be satisfied simultaneously by choosing $\lambda = \frac{1}{2}$, $R = 2M$ and $\sup_{x \in R} |g(x)| < M$. Firstly (4.5.10) is satisfied, and (4.5.7) is satisfied for any value t_o such that $t_o \leqslant \frac{1}{2+4M}$.

Hence A is contractive over the ball $||v||_{\mathcal{C}_{t_o}} \leqslant R$. Thus according to the fixed point theorems for Banach spaces, the integral equation (4.5.4) has a unique solution which is continuous and bounded for all t such that $0 \leqslant t \leqslant t_o$. This proves the following lemma:

<u>Lemma 4.5.1</u>

Let $g(x)$ be a continuous function such that $\sup_{x \in R} |g(x)| \leqslant M < \infty$. Then there exists a $t_o(M) > 0$ such that the integral equation (4.5.4) has a solution satisfying $u(x,0) = g(x)$ which is bounded and continuous for $x \in R$ and $0 \leqslant t \leqslant t_o$. ☐

Next we show that the solution guaranteed by lemma (4.5.1) has sufficient regularity to be a classical solution of (4.5.1).

<u>Lemma 4.5.2</u>

If $g \in C^2(R)$. Then any solution of (4.5.4) which is an element of $\mathcal{C}_T$ (for a given $T > 0$) is also an element of $\mathcal{C}_T^{2,\infty}$. ☐

Proof

Let u be a solution of (4.5.4) guaranteed by lemma 4.5.1, i.e.,

$$u = Au = g(x) + \int_0^t [\int_{-\infty}^{\infty} k(x-\xi)\{u+\tfrac{1}{2}u^2\}d\xi]d\tau.$$

Since $u \in \mathcal{C}_{t_o}$, then u is bounded and uniformly continuous on $\mathbb{R} \times [0,t_o]$. Hence $u = Au$ is continuously differentiable function i.e., u_t exists and is given by

$$u_t = (Au)_t = \int_{-\infty}^{\infty} K(x-\xi)\{u+\tfrac{1}{2}u^2\}d\xi.$$

Furthermore u_t is continuous and bounded on both x and t on $\mathbb{R} \times [0,t_o]$. Hence u_{tt} exists and is given by

$$u_{tt}(x,t) = \int_{-\infty}^{\infty} K(x-\xi)\{u_t(\xi,t) + u(\xi,t)\, u_t(\xi,t)\}d\xi.$$

Hence, by induction, the k th derivatives with respect to t exists and is given by

$$\frac{\partial^{k}u}{\partial t^k} = \int_{-\infty}^{\infty} K(x-\xi)\, \frac{\partial^{k-1}}{\partial t^{k-1}}\, \{u+\tfrac{1}{2}u^2\}d\xi, \quad K = 2,3, \ldots .$$

To obtain the x-derivative, the range of integration is divided at $\xi = x$, i.e.

$$u = g(x) + \tfrac{1}{2}\int_0^t \int_{-\infty}^{x} e^{\xi-x}\{u+\tfrac{1}{2}u^2\}d\xi d\tau - \tfrac{1}{2}\int_0^t \int_{x}^{\infty} e^{x-\xi}\{u+\tfrac{1}{2}u^2\}d\xi d\tau. \tag{4.5.11}$$

Since $u(x,t)$ is a solution of (4.5.4) in $\mathcal{C}_T$ then $u(x,0) = g(x)$ is continuous and bounded, then u_x exists, being given by

$$u_x(x,t) = g'(x) + \int_0^t (u+\tfrac{1}{2}u^2)d\tau - \tfrac{1}{2}\int_0^t \int_{-\infty}^{x} (u+\tfrac{1}{2}u^2)e^{\xi-x}d\xi d\tau$$

$$- \tfrac{1}{2}\int_0^t \int_x^{\infty} e^{x-\xi}(u+\tfrac{1}{2}u^2)d\xi d\tau$$

$$= g'(x) + \int_0^t (u+\tfrac{1}{2}u^2)d\tau - \int_0^t \int_{-\infty}^{\infty} \tfrac{1}{2}e^{|x-\xi|}(u+\tfrac{1}{2}u^2)d\xi d\tau.$$

This shows that u_x is continuous and bounded, then the first integral is a continuously differentiable function of x. Since $g \in C^2(R)$ we can differentiate again and obtain

$$u_{xx}(x,t) = g''(x) + \int_0^t (u_x+uu_x)d\tau + \tfrac{1}{2}\int_0^t \int_{-\infty}^{x} e^{\xi-x}(u+\tfrac{1}{2}u^2)d\xi d\tau$$

$$- \tfrac{1}{2}\int_0^t (u+\tfrac{1}{2}u^2)d\tau - \tfrac{1}{2}\int_0^t \int_{-\infty}^{x} e^{x-\xi}(u+\tfrac{1}{2}u^2)d\xi d\tau$$

$$= g'' + \int_0^t (u_x+uu_x)d\tau + \int_0^t \int_{-\infty}^{\infty} K(x-\xi)(u+\tfrac{1}{2}u^2)d\xi d\tau$$

$$= g''(x) - g(x) + u(x,t) + \int_0^t (u_x + uu_x)d\tau\ .$$

(using (4.5.4))

Thus u_{xx} exists and is continuous and bounded. Clearly u has the regularity of g in its x-derivatives and so $u \in \mathcal{C}_T^{2,\infty}$.
(For the definition of the space $\mathcal{C}_T^{i,j}$, see appendix D).

Combining lemma 4.5.1 and lemma 4.5.2, the proof of the following theorem is obtained.

Theorem 4.6

Let $g(x) \in C^2(R)$ and be bounded, then there exists a $t_o > 0$ such that the initial value problem (4.5.1) has a local classical solution for any $t, 0 \leq t \leq t_o$. □

Global existence

Having obtained the local existence of solution of the regularized long wave equation (4.5.1) we turn to extend this solution for larger t. For this purpose the interval $[0,t_o]$ is replaced by the interval $[t_o,t_1]$ to see whether $u(x,t_o)$, guaranteed by theorem 4.6 can provide the same set of properties that when assumed for $u(x,0)$ enabled the existence of solution of (4.5.1) to be proved for $0 \leq t \leq t_o$. If this is the case then the solution of (4.5.1) exists in the interval $[t_o,t_1]$ and this argument can be repeated any number of times which leads to the existence of the solution globally. For doing this we introduce the following lemma which proof is provided by Benjamin et al [5].

Lemma 4.5.3

(i) If $\{w_n\}$ is a sequence of functions in $\mathcal{E}_{t_o}$ and if $w_n(x,t)$ is asymptotically null for all n, then so is $\lim_{n\to\infty} w_n$.

(ii) If $w(x,t)$ is continuous and asymptotically null, then so are $\int_{-\infty}^{\infty} e^{|x-\xi|} w(\xi)d\xi$ and $\lim_{n\to\infty} \int_{-\infty}^{\infty} K(x-\xi)w_n(\xi)d\xi$.

(iii) If $u(x,t)$ is a solution of the integral equation (4.5.4) guaranteed by lemma (4.5.1) and if $g, g', \ldots, g^{(K)}$ are continuous and asymptotically then $\partial_x^P \partial_t^m u$ is aysmptotically null for all $m \geq 0$, $0 \leq P \leq K$. □

From now on the arguments depend on the assumption that we are dealing with solutions which satisfy $u, u_x, u_{xt} \to 0$ as $|x| \to \infty$ (i.e. asymptotically null). For such solutions, the above lemma leads to the following theorem which assures the global existence.

Theorem 4.7

Let $g(x)$ satisfy

$\int_{-\infty}^{\infty} (g^2+g'^2)dx = E_o < \infty$ and $g \in C^2(\mathbb{R})$, then the partial differential equation (4.5.1) has a solution $u \in \mathcal{C}_{\infty}^{2,\infty}$ which satisfies $u(x,0) = g(x)$. □

Proof

Let $u(x,t)$ be a solution of (4.5.4) assured by lemma (4.5.1), then u is a classical solution of (4.5.1) [by theorem 4.6], i.e., u satisfies

$$u_t + u_x + uu_x - u_{xxt} = 0$$

pointwise on $\mathbb{R} \times [0,t_o]$. Multiplying the last equation by u and integrating with respect to x between $x = -L$ and $x = +L$, we have

$$\frac{d}{dt} \int_{-L}^{L} \tfrac{1}{2}(u^2+u_x^2)dx + [\tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 - uu_x]_{-L}^{L} = 0.$$

Integrating with respect to t and using $u(x,0) = g(x)$, we have

$$\int_{-L}^{L} \tfrac{1}{2}(u^2+u_x^2)dx - \tfrac{1}{2}\int_{-L}^{L} (g^2+g'^2)dx = -\int_{0}^{t} [\tfrac{1}{2}u^2+\tfrac{1}{3}u^3-uu_x]_{-L}^{L} d\tau .$$

Since $\int_{-L}^{L} (g^2+g'^2)dx$ remains bounded as $L \to \infty$ and the integrand in the right hand side is uniformly bounded as $L \to \infty$, then as $L \to \infty$, $\int_{-L}^{L} (u^2+u_x^2)dx$ must be bounded.

Using lemma (4.5.3) (iii), the integrand in the right hand side vanishes as $L \to \infty$. Hence

$$E(u) = \int_{-\infty}^{\infty} (u^2+u_x^2)dx = \int_{-\infty}^{\infty} (g^2+g'^2)dx < \infty \qquad (4.5.12)$$

through the interval $[0,t_o]$.

Thus $u(x,t_o)$ provides the same set of properties that when assumed for $g(x)$ enabled the existence of solution to be proved for $0 \leq t \leq t_o$. Hence the theorem is proved. ▨

4.5.2 Uniqueness

Theorem 4.8

The solution of the initial value problem

$$u_t + u_x + uu_x - u_{xxt} = 0, \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = g(x) \qquad (4.5.1)$$

guaranteed by theorem (4.7) is unique. □

Proof

Let u and v be two solutions of the initial value problem (4.5.1) and $w = v-u$, then w satisfies the initial value problem

$$w_t + w_x + \tfrac{1}{2}[w(u+v)]_x - w_{xxt} = 0, \qquad w(x,0) = 0. \qquad (4.5.13)$$

Multiplying (4.5.13) by w, i.e.,

$$ww_t + ww_x + \tfrac{1}{2}w[w(u+v)]_x - ww_{xxt} = 0. \qquad (4.5.14)$$

Integrating (4.5.14) with respect to x between $x = -R$, $x = R$

$$0 = \int_{-R}^{R} \tfrac{1}{2}(w^2+w_x^2)_t dx + [\tfrac{1}{2}w_x^2 - ww_{xt}]_{-R}^{R} + \tfrac{1}{2}\int_{-R}^{R} w[w(u+v)]_x dx$$

$$= \tfrac{1}{2}\int_{-R}^{R} \frac{d}{dt}(w^2+w_x^2)dx + \tfrac{1}{2}[w_x^2 - ww_{xt}]_{-R}^{R} + \tfrac{1}{2}[w^2(u+v)]_{-R}^{R}$$

$$- \int_{-R}^{R} (v+u)ww_x dx .$$

Since u, u_x, u_{xt}, v, v_x and v_{xt} vanish as $|x| \to \infty$, then so are w, w_x and w_{xt}. Hence as $R \to \infty$

$$\tfrac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(w^2+w_x^2)dx = \tfrac{1}{2}\int_{-\infty}^{\infty} ww_x(u+v)dx$$

$$\leq \tfrac{1}{2}\sup_{x\in R}|u+v| \int_{-\infty}^{\infty} ww_x dx$$

$$= \tfrac{1}{2}c(t)\int_{-\infty}^{\infty}\tfrac{1}{2}(w^2+w_x^2)dx$$

where $c(t) = \sup_{x\in R}|u+v|$.

Integrating with respect to t between 0 and t, we have

$$\int_{-\infty}^{\infty}(w^2+w_x^2)dx \leq \int_{-\infty}^{\infty}[w^2(x,0) + w_x^2(x,0)]dx \;\; \exp\{\tfrac{1}{2}\int_0^t c(\tau)d\tau\}$$

$$= 0 \;\; (\text{since } w(x,0) = 0) \qquad (4.5.15)$$

i.e., $\int_{-\infty}^{\infty}(w^2+w_x^2)dx = 0 \to w \equiv 0.$

Hence $u \equiv v$ for all $t > 0$.

Thus the solution of (4.5.1) is unique. ■

Note

The choice of $g \in C^2(R)$ in the above analysis was replaced by Bona & Smith [9] by the assumption $g \in H^2(R)$ (see the appendix D for the definition of H^k) and was considered in the last chapter. Hence theorems 4.7 and 4.8 can be slightly changed to the following:

Theorem 4.9

Let $g \in H^m$, $m \geq 2$, then there exists a unique solution to the initial value problem (4.5.1) lies in $\mathcal{H}_T^m$ for all finite $T > 0$. □

4.5.3 The dependence of solution on the initial data

Let u and v be two solutions of (4.5.1) such that $v(x,0) = g_1(x)$ and $u(x,0) = g_2(x)$. Then $w = u-v$ satisfies the initial value problem

$$w_t + w_x + \tfrac{1}{2}[w(u+v)]_x - w_{xxt} = 0, \quad w(x,0) = g_1-g_2 = \Delta g. \tag{4.5.16}$$

Using similar calculations as in theorem 4.8, we have

$$\int_{-\infty}^{\infty} (w^2+w_x^2)dx \leq ||\Delta g||_{H^1} \exp\left(\tfrac{1}{2}\int_0^t c(\tau)d\tau\right) \tag{4.5.17}$$

where $c(t) = \sup_{x \in R} |v(x,t) + u(x,t)|$.

Let, now, $||\Delta g||_{H^1} < \delta$, then

$$c(t) = \sup_{x\in R} |v(x,t) + u(x,t)|$$

$$\leq ||v||_{H^1} + ||u||_{H^1} \qquad \text{[by the properties of } H^1]$$

$$= ||g_1||_{H^1} + ||g_2||_{H^1} \qquad \text{[by (4.5.12].}$$

Thus,

$$\max_{0\leq t\leq T} c(t) \leq ||g_1||_{H^1} + ||g_2||_{H^1}$$

$$= 2||g_1||_{H^1} + (||g_2||_{H^1} - ||g_1||_{H^1}) = 2||g_1||_{H^1} + \delta. \tag{4.5.18}$$

The relations (4.5.17) and (4.5.18) are combining together and give

$$\sup_{0\leq t\leq T} \int_{-\infty}^{\infty} (w^2+w_x^2)dx \leq \delta \exp\{||g_1||_{H^1} + \frac{\delta}{2}\}T, \quad \text{i.e.}$$

$$||w||_{\mathcal{H}^1_T} \leq \delta \exp\{||g_1||_{H^1} + \frac{\delta}{2}\}T$$

(see appendix D, for the definition of the space $\mathcal{H}^1_T$).

Hence u and v are close to each other provided that g_1 and g_2 are. Thus the following theorem is proved

<u>Theorem 4.10</u>

The solutions of (4.5.1) depend continuously on the initial data. □

The analysis in the above three subsections imply that (4.5.1) is well- posed.

4.6 Conclusion

In this chapter we reviewed the mathematical properties of the RLW equation. We found that the equation has a stable solitary wave solution, perhaps the only exact asymptotically null solution which is known for this equation. Thus, unlike the KdV, there is no information whether the N-soliton solution exists or not. It is believed that the RLW does not have N-soliton solutions because the equation has only three conserved functionals and hence cannot be linearized by the inverse method. Although the numerical results carried out by Abdullov et al and Bona et al show that there is inelastic interaction between the solitary waves of the RLW, as far as we know no analytic proof for the non existence of N-soliton solutions is yet known.

The analysis above shows that the RLW theory is not complete.

CHAPTER FIVE

CLASSIFICATION AND REDUCTION OF THE GENERAL CLASS OF EQUATIONS

In this chapter we complete the classification derived in chapter 2 on the existence of solitary waves of the general class of equations. By defining solutions to be equivalent if they are connected by a nonsingular linear transformation we show that the general class can be reduced to four equivalence classes, considerably simplifying the problem of proving the well-posedness of the equations. We also show that the equivalence transformations preserve solitary waves and conservation laws. Finally, we consider the question of the existence of multisoliton solutions in the KdV equivalence class.

5.1 General Classification

The existence of solitary wave solutions of the general class of equations,

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0 \tag{5.1.1}$$

was established in chapter 2, and a general classification in terms of the speed of the solitary waves was initiated for the cases $a_6 = 0$. This section is devoted to completing this classification.

For this context let $a_6 \neq 0$. Hence from section 2.1, the solitary wave solution has the form

$$u_s(x,t) = \frac{3}{\alpha}\,\mathrm{sech}^2\{\frac{1}{2\sqrt{\beta}}[x-(1+c)t]\} \tag{5.1.2}$$

where,

$$\alpha c = a_1 - a_2(1+c), \quad \beta c = a_3 - a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3 \qquad (5.1.3)$$

The necessary condition for the existence of the solitary wave solutions is β should be positive. The cubic equation (5.1.3) for βc has three roots. The types of these roots are obtained according to the properties of the discriminant determinant Δ, where,

$$\Delta = \frac{1}{108a_6^2}\{4a_3a_5^3 - 18a_3a_4a_5a_6 + 27a_3^2a_6^2 + 4a_4^3a_6 - a_4^2a_5^2\}. \qquad (5.1.4)$$

Remark:

If $\Delta = 0$, $a_5^2 = 3a_4a_6$ and $2a_5^3 = 9a_4a_5a_6 - 27a_3a_6^2$, then βc has three real and equal roots. If $\Delta < 0$, then βc has two complex conjugate roots and one real root. If $\Delta > 0$, then βc has three real and distinct roots. And if $\Delta = 0$, then βc has two real and equal roots and one simple real root.

Thus four cases arise in the classification below. This classification is presented for $c > 0$. The results for $c < 0$ follow from this by reversing the direction of the c-axis and interchanging the interpretation of its corresponding figures.

(1) $\Delta = 0$, $a_5^2 = 3a_4a_6$ and $2a_5^3 = 9a_4a_5a_6 - 27a_3a_6^2$

In this case βc has two possible graphs

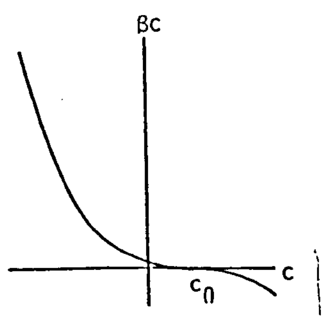


Fig.(1.a)

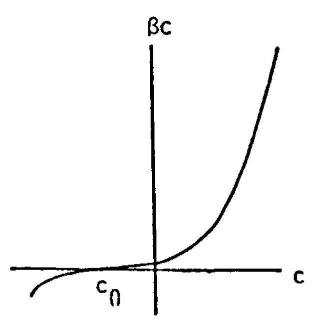


Fig.(1.b)

The cubic for figure (1.a) has real root c_0 of order three. This figure corresponds to the condition $a_6 > 0$. Hence, if $c_0 > 0$, the solitary wave would exist inside the interval $(0, c_0)$ and the speed is bounded above, whilst if $c_0 < 0$, the solitary wave does not exist.

The cubic for figure (1.b) has a real root c_0 of order three, subject to the condition $a_6 < 0$. Thus
$c_0 > 0 \rightarrow c \in (c_0, \infty)$ for which the solitary wave would exist and
$c_0 < 0 \rightarrow c \in (0, \infty)$ for which the solitary wave would exist.

(2) $\Delta < 0$

In this case βc has two possible graphs

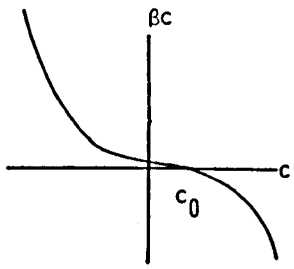


Fig.(2.a)

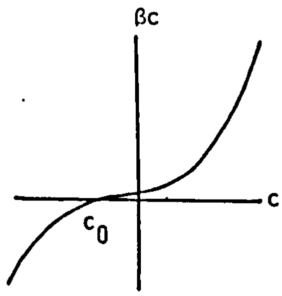


Fig.(2.b)

The cubic for figures (2.a) and (2.b) have one real root and two complex conjugate roots. This case reduces to the cases (1.a) and (1.b), discussed above,.

(3) $\Delta > 0$

In this case βc has two possible graphs, having three real distinct roots

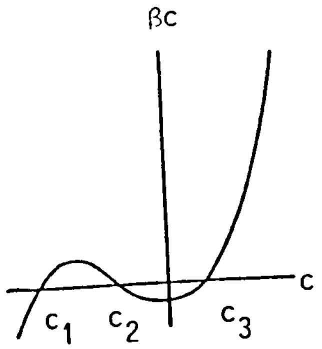


Fig.(3.a)

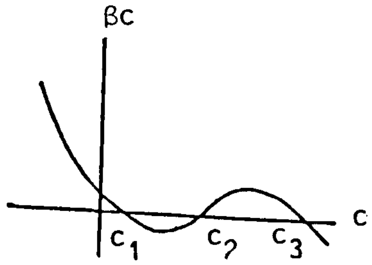


Fig.(3.b)

For this case, let c_1, c_2 and c_3 be the roots of the cubic βc and without loss of generality let $c_1 < c_2 < c_3$. Between any two consecutive roots the graph attains its maximum or minimum values (i.e. $(\beta c)_{max}$ or $(\beta c)_{min}$ respectively) at the values

$$\frac{a_5 \pm \sqrt{a_5^2 - 3a_4 a_6}}{3a_6} .$$

The cubic for figure (3.a) corresponds to the condition $c_{min} > c_{max}$, where $\beta c_{min} = (\beta c)_{min}$ and $\beta c_{max} = (\beta c)_{max}$.

Hence the following subcases arise.

(i) $\underline{c_3 > c_2 > c_1 > 0}$.

There would be no solitary waves in the interval $[c_2, c_3] \cup [0, c_1]$ otherwise the solitary wave exists

(ii) $\underline{c_1 < c_2 < c_3 < 0}$.

The solitary wave exists and the speed is unbounded above, i.e., $c \in (0, \infty)$

(iii) $\underline{c_1 < c_2 < 0 < c_3}$.

This case implies that $c \in (c_3,\infty)$ for which the solitary wave would exist

(iv) $c_1 < 0 < c_2 < c_3$.

The solitary wave would exist only, for those values c, $c \in (0,c_2) \cup (c_3,\infty)$.

The cubic βc for figure (3.b) corresponds to the condition $c_{max} > c_{min}$ and the following subcases arise

(i) $0 < c_1 < c_2 < c_3$.

The solitary wave would exist for the values c, $c \in (0,c_1) \cup (c_2,c_3)$

(ii) $c_1 < c_2 < c_3 < 0$.

For this case there would be no solitary waves

(iii) $c_1 < c_2 < 0 < c_3$.

The solitary wave would exist only for the values c, $c \in (0,c_3)$

(iv) $c_1 < 0 < c_2 < c_3$.

The solitary wave exists for c, $c \in (c_2,c_3)$.

(4) $\Delta = 0$

In this case, the cubic βc has only one multiple real root c_1 and one simple root c_2. This multiple root is a turning point for the graph of βc. Hence four possible graphs.

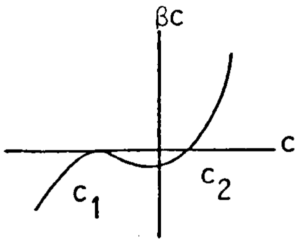


Fig.(4.a)

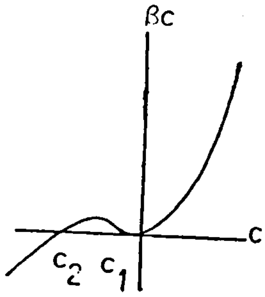


Fig.(4.b)

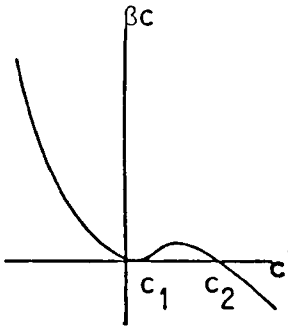


Fig.(4.c)

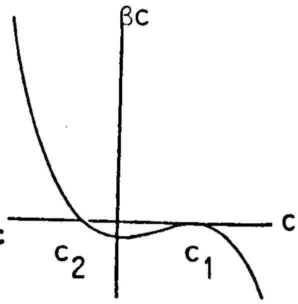


Fig.(4.d)

The cubic βc for the graph (4.a) correspond to the condition, $c_1 < c_2$ and $c_{max} < c_{min}$. Hence, if $c_1 < 0 < c_2$ or $0 < c_1 < c_2$, the solitary wave would exist only for the values of c, $c \in (c_2, \infty)$. If $c_1 < c_2 < 0$, then the solitary wave exists for the values of c, $c \in (0 , \infty)$.

The cubic βc for figure (4.b) corresponds to the conditions $c_2 < c_1$ and $c_{max} < c_{min}$. Hence, if $c_2 < 0 < c_1$, the solitary waves would exist for the values c, $c \in (0, c_1) \cup (c_1, \infty)$. If $0 < c_2 < c_1$, the solitary wave exists for c, $c \in (c_2, c_1) \cup (c_1, \infty)$. Whilst, if $c_2 < c_1 < 0$, the solitary wave exists for the values of c, $c \in (0, \infty)$.

The cubic βc for figure (4.c) corresponds to the conditions $c_1 < c_2$ and $c_{min} < c_{max}$. Hence, if $0 < c_1 < c_2$, the solitary wave exist for c, $c \in (0,c_1) \cup (c_1,c_2)$. If $c_1 < 0 < c_2$, then the solitary wave would exist for c, $c \in (0,c_2)$. Otherwise the solitary wave does not exist.

For figure (4.d), i.e. $c_2 < c_1$ and $c_{min} < c_{max}$, the solitary wave would exist only for the values c, $c \in (0,c_2)$ if $0 < c_2 < c_1$. Otherwise the solitary wave does not exist.

5.2 The class $W \cap S$

In the parameter space, let $W = \{(a_1, a_2, \ldots, a_6) \in R^6\}$, i.e. W represents the general class of equations

$$u_t + u_x + a_1 u u_x + a_2 u u_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0.$$

Then $W \cap S$ is defined to be the class of all elements of W which possess a solitary wave solution

$$u_s = \frac{3}{\alpha} \operatorname{sech}^2 \frac{1}{2\sqrt{\beta}} [x-(1+c)t],$$

$\alpha c = a_1 - a_2(1+c)$ and $\beta c = a_3 - a_4(1+c) + a_5(1+c)^2 - a_6(1+c)^3$.
The analysis, introduced in chapter 2 together with section 5.1 indicates that inside the class $W \cap S$ all the equations are restricted to those which keep the solutions real. Furthermore from the definition of the solitary wave solutions, u_s, the following points are noted:

(1) If c is kept fixed, the amplitude $(\frac{3}{\alpha})$ is a function of a_1 and a_2. Hence positive or negative amplitudes are possible by varying a_1 or a_2 or both. Furthermore, these coefficients are the dominant terms in producing the sech^2 profile.

(2) The width of the solitary wave $(4\pi\sqrt{\beta})$ is a function of the dispersion terms only, i.e. of the parameters a_3, a_4, a_5 and a_6.

(3) The width and the amplitude of the solitary wave are coupled in terms of c.

(4) In the parameter space W, $W \cap S$ can be regarded as a subspace of the topological space R^6.

It follows from (4) that it makes sense to discuss any of the topological properties for $W \cap S$ in the parameter space.

Definition (5.2.1)

(i) A topological space E is connected if it cannot be represented as a union of two disjoint open sets. Otherwise it is disconnected.

(ii) A maximal connected subset of a topological space, i.e. a connected subset which is not properly contained in any larger connected subset, is called a component of the space.

Theorem 5.1

(1) $W \cap S$ is disconnected, i.e. not connected.

(2) $W \cap S$ has four possible components. □

Proof

Let W_R and W_C be two subsets of $W \cap S$, where

$W_R = \{e \in W \cap S:$ βc has only real roots and

$W_C = \{e \in W \cap S:$ βc contains complex roots$\}$.

Then W_R and W_C satisfies

(1) $W_R \cap W_C = \Phi$ (the empty set) (2) $W_R \cup W_C = W \cap S$.

Hence $W \cap S$ has a proper separated partition which proves (1).

(2) In the proof of (1) W_R can be separated into W_{R_1}, W_{R_2} and W_{R_3}, where, βc has three real equal roots in W_{R_1}, has two equal roots and one simple in W_{R_2}, and has three distinct real roots in W_{R_3}. Thus $W \cap S = W_{R_1} \cup W_{R_2} \cup W_{R_3} \cup W_C$ and $W_C \cap W_{R_1} \cap W_{R_2} \cap W_{R_3} = \Phi$. Each one of the four classes is clearly connected and cannot be contained in any larger connected set. Thus each one is a component of $W \cap S$. ■

Definition 5.2.2

If $e_1, e_2 \in W \cap S$ and $e_1 = (a_1, a_2, \ldots, a_6)$, $e_2 = (b_1, b_2, \ldots, b_6)$ in the parameter space then the segment e_1e_2 is the set of points $(a_1 + s(b_1-a_1),\ a_2 + s(b_2-a_2),\ \ldots,\ a_6 + s(b_6-a_6))$ and $s \in [0,1]$.

Example 5.2.1

$a_1 = a_3 = 1$, $a_2 = a_4 = a_5 = a_6 = 0$ (KdV) and $b_1 = -b_4 = 1$, $b_5 = b_3 = b_6 = 0$ (RLW).

Then $e_1e_2 = (1,\ 0,\ (1-s),\ -s,\ 0,\ 0)$ which represents the equation

$$u_t + u_x + uu_x + (1-s)u_{xxx} - su_{xxt} = 0 \quad \text{(Regularized KdV).}$$

Since inside $W \cap S$, a solitary wave solution exists along any segment, joining any two elements of $W \cap S$, then a subset N of all segments, joining the KdV with all other elements of $W \cap S$ is now connected. Hence, constructing sequences $(s_i)_{i \in [0,1]}$ such that a solitary wave solution exists for each value s_i is now possible.

5.3 Reduction to equivalence classes

The general class (5.1.1) splits, with respect to Cauchy problem, into three distinctive subclasses:

(i) The class $W_6(a_6 \neq 0)$ for which (5.1.1) is third order in t and three bits of data, u, u_t and u_{tt}, are given at $t = 0$.

(ii) The class $W_5(a_6 = 0$ and $a_5 \neq 0)$, then (5.1.1) is second order in t and both u and u_t have to be specified.

(iii) The class $W_{43}(a_6 = 0 = a_5)$, then (5.1.1) is first order in t and only u must be given at $t = 0$.

To discuss the well-posedness of the general class, which is done in the next section, it is convenient to reduce this class to a number of disjoint subclasses such that the properties of the general class are characterized by these subclasses. The proof of this reduction will be established separately for the classes, which we call W_6, W_5 and W_{43}.

5.3.1 The reduction theorem of W_6

The equation of this class is

$$u_t + u_x + a_1uu_x + a_2uu_t + a_3u_{xxx} + a_4u_{xxt} + a_5u_{xtt} + a_6u_{ttt} = 0. \tag{5.3.1}$$

Consider the nonsingular linear transformation

$$x \to x' = \frac{x}{1-n} - \frac{n}{1-n}t, \quad t \to t' = \frac{x}{1-m} - \frac{m}{1-m}t, \quad u(x',t) \to v(x,t'), \tag{5.3.2}$$

and $n;\ m \neq 1$. Under the transformation (5.3.2), (5.3.1) reduces to

$$v_{t'} + v_{x'} + b_1vv_{x'} + b_2vv_{t'} + b_3v_{x'x'x'} + b_4v_{x'x't'} + b_5v_{x't't'} + b_6v_{t't't'} = 0 \tag{5.3.3a}$$

where,

$$b_1 = \frac{a_1 - na_2}{1 - n}, \quad b_2 = \frac{a_1 - ma_2}{1 - m}, \quad b_3 = \frac{a_3 - a_4n + a_5n^2 - a_6n^3}{(1 - n)^3},$$

$$b_4 = \frac{3a_3 - (m + 2n)a_4 + n(n + 2m)a_5 - 3mn^2a_6}{(1 - n)^2(1 - m)},$$

$$b_5 = \frac{3a_3 - (2m + n)a_4 + m(2n + m)a_5 - 3nm^2a_6}{(1 - n)(1 - m)^2}, \quad \text{and}$$

$$b_6 = \frac{a_3 - a_4m + a_5m^2 - a_6m^3}{(1-m)^3}. \tag{5.3.3b}$$

Define a relation ρ on W_6 such that, if $e_1, e_2 \in W_6$, then $e_1 \rho e_2$ if and only if they have the same type of roots of the cubic equation

$$a_3 - a_4\lambda + a_5\lambda^2 - a_6\lambda^3 = 0 . \qquad (5.3.4)$$

Now, ρ is clearly an equivalence relation. Thus it partitions W_6 into equivalence classes. Since (5.3.4) has only four types of roots λ_1, λ_2 and λ_3, i.e.,

(1) $\lambda_1 = \lambda_2 = \lambda_3$ and λ_i is real, $i = 1, 2$ and 3 .

(2) $\lambda_1 = \lambda_2 \neq \lambda_3$ and λ_i is real, $i = 1, 2$ and 3 .

(3) $\lambda_1 \neq \lambda_2 \neq \lambda_3$ and λ_i is real, $i = 1, 2$ and 3 .

(4) $\lambda_1 = \bar{\lambda}_2$, λ_3 , λ_3 is real but λ_1 and λ_2 are complex conjugate.

Then W_6 is partitioned into four equivalence classes, each of them being characterized by one type of the roots of (5.3.4). Furthermore, this partition depends only on the dispersive coefficients a_3, a_4, a_5 and a_6.

<u>Theorem 5.2</u>

The class W_6 can be reduced to the following equivalence classes

(i) The subclass $(c_1, c_2, c_3, 0, 0, 0)$, i.e.

$$v_t + v_x + c_1 v v_x + c_2 v v_t + c_3 v_{xxx} = 0 \quad \text{(KdV class)}$$

(ii) The subclass $(d_1, d_2, 0, d_4, 0, 0)$, i.e.

$$v_t + v_x + d_1 v v_x + d_2 v v_t + d_4 v_{xtt} = 0 \quad \text{(RLW class)}$$

(iii) The subclass $(\gamma_1, \gamma_2, 0, \gamma_4, \gamma_5, 0)$, i.e.

$$v_t + v_x + \gamma_1 vv_x + \gamma_2 vv_t + \gamma_4 v_{xxt} + \gamma_5 v_{xtt} = 0 \quad (W_{54} \text{ class})$$

(iv) The subclass $(\delta_1, \delta_2, \delta_3, 0, \delta_5, 0)$, i.e.

$$v_t + v_x + \delta_1 vv_x + \delta_2 vv_t + \delta_3 v_{xxx} + \delta_5 v_{xtt} = 0. \quad (W_{53} \text{ class}).\square$$

To prove this theorem, the following lemma is introduced

Lemma 5.3.1

If the cubic equation (5.3.4) for a given equation

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0,$$

has a unit root λ (i.e. $\lambda = 1$), then there exists a nonsingular linear transformation which transforms this equation to another equation

$$w_t + w_x + c_1 ww_x + c_2 ww_t + c_3 w_{xxx} + c_4 w_{xxt} + c_5 w_{xtt} + c_6 w_{ttt} = 0$$

for which the cubic (5.3.4) for c_i does not possess the unit root. $\square$

Thus from now on we suppose that: $\lambda \neq 1$, a root of (5.3.4).

Proof of theorem 5.2

The above analysis proves that W_6 splits into four equivalence classes, characterized by the different kinds of the roots of (5.3.4).

(i) $\lambda_1 = \lambda_2 = \lambda_3$ are all real. Choose m and n in (5.3.2) as follows: $m = \lambda_1$ and $n = 0$. Hence $m \neq n$ such that the transformation (5.3.2) is nonsingular. Since m is a multiple root of order three of (5.3.4), then

$$a_5^2 = 3a_4a_6 \quad \text{and} \quad 2a_5^3 - 9a_4a_5a_6 + 27a_3a_6^2 = 0. \qquad (5.3.5)$$

Hence combining the two conditions (5.3.5) yields

$$a_4a_5 = 9a_3a_6. \qquad (5.3.6)$$

Using this result we calculate the coefficients b_i in (5.3.3), then

$b_6 = 0$ is clear since m is a root of (5.3.4)

$$b_5 = \frac{3a_3 - (2m + n)a_4 + m(2n + m)a_5 - 3nm^2a_6}{(1 - n)(1 - m)^2}$$

$$= \frac{3(a_3 - a_4\lambda_1 + a_5\lambda_1^2 - a_6\lambda_1^3) - \lambda_1(-a_4 + 2a_5\lambda_1 - 3a_6\lambda_1^2)}{(1 - \lambda_1)^2}$$

$= 0$ [since λ_1 is a root of order three].

Similarly $b_4 = 0$,

$$b_3 = a_3, \quad b_2 = \frac{3a_1a_6 - a_2a_5}{3a_6 - a_5} \quad \text{and} \quad b_1 = a_1.$$

i.e. (5.3.1) reduces to

$$v_t + v_x + c_1vv_x + c_2vv_t + c_3v_{xxx} = 0 \quad \text{(KdV class)} \qquad (5.3.7)$$

where, $c_i = b_i \quad i = 1, 2$ and 3 which proves (i)

(ii) $\underline{\lambda_1 = \lambda_2 \neq \lambda_3}$ are all real.

Choose $m = \lambda_1$ and $n = \lambda_3$ in (5.3.2). The transformation remains nonsingular, and since m is a root of order two of (5.3.4), then similar to the above case $b_6 = b_5 = 0$. Furthermore since n is a root of order one, then $b_3 = 0$ but $b_4 \neq 0$ and (5.3.1) reduces to the class

$$v_t + v_x + d_1 vv_x + d_2 vv_t + d_4 v_{xxt} = 0 \quad \text{(RLW class)} \qquad (5.3.8)$$

where d_1, d_2 and d_3 are in terms of a_i $(i = 1, 2, \ldots, 6)$

(iii) $\underline{\lambda_1 \neq \lambda_2 \neq \lambda_3}$ are all real.

Choose $m = \lambda_i$ and $n = \lambda_j$, $(i \neq j)$. Both m and n are simple roots of (5.3.4), then $b_3 = b_6 = 0$ but $b_4 \neq 0$ and $b_5 \neq 0$ in (5.3.3b). Thus W_6 reduces to

$$v_t + v_x + \gamma_1 vv_x + \gamma_2 vv_t + \gamma_4 v_{xxt} + \gamma_5 v_{xtt} = 0 \qquad (5.3.9)$$

(v) $\underline{\lambda_1 = \bar{\lambda}_2 , \lambda_3}$.

Choose $m = \lambda_3$ and n such that $b_4 = 0$ in (5.3.3b). Hence $b_6 = 0 = b_4$, $b_3 \neq 0$. Thus W_6 reduces to

$$v_t + v_x + \delta_1 vv_x + \delta_2 vv_t + \delta_3 v_{xxx} + \delta_5 v_{xtt} = 0 \quad (W_{53} \text{ class}) \qquad (5.3.10)$$

where δ_i are in terms of a_i, $i = 1, 2, \ldots, 6$. ■

<u>Corollary (5.3.1)</u>

If $\frac{a_1}{a_2}$ is a root of (5.3.4), then the class W_6 reduces to the simpler classes

(i) $v_t + v_x + c_1 vv_x + c_3 v_{xxx} = 0$

(ii) $v_t + v_x + d_1 vv_x + d_4 v_{xxt} = 0$

(iii) $v_t + v_x + \gamma_1 vv_x + \gamma_4 v_{xxt} + \gamma_5 v_{xtt} = 0$

(iv) $v_t + v_x + \delta_1 vv_x + \delta_3 v_{xxx} + \delta_5 v_{xtt} = 0$

where c_i, d_i, γ_i and δ_i are in terms of $a_1, a_2, \ldots, a_6$. □

Note:

The only nonlinear term in the reduced classes, is vv_x.

Proof

In the above theorem if $\frac{a_1}{a_2}$ is a root of (5.3.4) we choose m equal to this root, then from (5.3.3b) $b_2 = 0$. This completes the reduction of W_6. ∎

5.3.2 The reduction theorem of W_5

The equation of this class is

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} = 0. \tag{5.3.11}$$

Consider the nonsingular linear transformation

$$x \to x' = \frac{x}{1-k} - \frac{k}{1-k}t, \quad t \to t' = \frac{x}{1-p} - \frac{p}{1-p}t, \quad u(x,t) \to v(x',t') \tag{5.3.12}$$

Under the transformation (5.3.12), (5.3.11) transforms to

$$v_{t'} + v_{x'} + b_1 vv_{x'} + b_2 vv_{t'} + b_3 v_{x'x'x'} + b_4 v_{x'x't'} + b_5 v_{x't't'} + b_6 v_{t't't'} = 0 \tag{5.3.13a}$$

where

$$b_1 = \frac{a_1 - ka_2}{1 - k}, \quad b_2 = \frac{a_1 - pa_2}{1 - p}, \quad b_3 = \frac{a_3 - a_4 k + a_5 k^2}{(1 - k)^3},$$

$$b_4 = \frac{3a_3 - (p + 2k)a_4 + (k^2 + 2kp)a_5}{(1 - k)^2(1 - p)} \tag{5.3.13b}$$

$$b_5 = \frac{3a_3 - (2p + k)a_4 + (2kp + p^2)a_5}{(1 - k)(1 - p)^2}, \quad b_6 = \frac{a_3 - a_4 p + a_5 p^2}{(1 - p)^3},$$

and p can be chosen such that $b_6 \neq 0$ and the transformation (5.3.12) remains nonsingluar. Hence equation (5.3.13) forms an element of W_6 and the reduction to the equivalence classes can be done by following the analysis in the above section. Since the equivalence relation can be defined in terms of the roots of the cubic equation

$$b_3 - b_4\lambda + b_5\lambda^2 + b_6\lambda^3 = 0 \tag{5.3.14}$$

i.e. using (5.3.13b) in the cubic (5.3.14), one can find the roots of (5.3.14) in the form

$$\lambda_1 = -\frac{1-p}{1-k}, \quad \lambda_2 = -\frac{(1-p)}{2(1-k)} \frac{[2a_3-(p+k)a_4+2kpa_5]+(p-k)\sqrt{a_4^2-4a_3a_5}}{a_3 - a_4p + a_5p^2}$$

and (5.3.15)

$$\lambda_3 = -\frac{(1-p)}{2(1-k)} \frac{[2a_3 - (p+k)a_4 + 2kpa_5] - (p-k)\sqrt{a_4^2 - 4a_3a_5}}{a_3 - a_4p + a_5p^2} .$$

Then

(i) $a_4^2 = 4a_3a_5 \rightarrow \lambda_2 = \lambda_3$ and if in addition $p = \frac{a_4}{2a_5}$, then $\lambda_1 = \lambda_2 = \lambda_3$ and the class W_5 reduces to the KdV class

(ii) $a_4^2 = 4a_3a_5 \rightarrow \lambda_1 \neq \lambda_2 = \lambda_3$ all real. Hence W_5 reduces the RLW class

(iii) $a_4^2 > 4a_3a_5 \rightarrow \lambda_1 \neq \lambda_2 \neq \lambda_3$ and all real. Hence W_5 reduces to the class W_{54}

(iv) $a_4 < 4a_3a_5$, then $\lambda_1, \lambda_2 = \bar{\lambda}_3$ and W_5 reduces to the class W_{53}.

This proves the reduction theorem of the class W_5.

5.3.3 The reduction of the class W_{43}

This class has the form

$$u_t + u_x + a_1uu_x + a_2uu_t + a_3u_{xxx} + a_4u_{xxt} = 0 \ .$$

This class contains individually the KdV and the RLW classes as follows

(i) If $a_4 = 0$, then W_{43} reduces to

$$u_t + u_x + a_1uu_x + a_2uu_t + a_3u_{xxx} = 0 \qquad \text{(KdV class)}$$

(ii) If $a_3 = 0$, then W_{43} reduces to

$$u_t + u_x + a_1uu_x + a_2uu_t + a_4u_{xxt} \qquad \text{(RLW class)}$$

The analysis in the sections 5.3.1, 5.3.2 and 5.3.3 completes the reduction theorem of the general class.

5.4 Well-posedness

Consider the initial value problem which corresponds to the general class

$$u_t + u_x + a_1uu_x + a_2uu_t + a_3u_{xxx} + a_4u_{xxt} + a_5u_{xtt} + a_6u_{ttt} = 0$$

where u, u_t and u_{tt} are given on an arbitrary space curve $x = x(s)$, $t = t(s)$ and s is a parameter. Then this initial value problem is said to be well-posed if it has a unique solution which depends continuously on the initial data.

As shall be proved in the next chapter, the initial value problem corresponding to the general class can be reduced to a semilinear system of first order partial differential equations

$$U_t + AU_x + B = 0 \qquad (5.4.1)$$

where A, B are matrices, U is a column matrix and A does not depend on U. Then the method of characteristics is the natural procedure to assure the well-posedness. This procedure comes to a stop if A is a singular or if the initial curve which supports the data is a characteristic curve of the system (5.4.1).

To avoid these obstacles the reduction, introduced in section 5.3, is used to advantage. In this reduction we have concentrated on the equivalence of solutions. However this implies also an equivalence of the initial data. For example, for the class W_6, u, u_t and u_{tt} are given on any initial curve ($t = 0$ say). Using the reduction theorem of W_6, these data reduce to v, v_t and v_{tt} on the skew curve. Since the reductions in all cases reduce the number of t-derivatives by at least one, this means that at least one bit of data becomes redundant and raises the question about the specification of the data. This will be discussed in detail in the next chapter when we consider the question of well-posedness for the reduced equation.

Thus, using this reduction, the original initial value problem reduces to four disjoint initial value problems corresponding to the classes KdV, RLW, W_{54} and W_{53}. Here disjoint means that the solution for one does not imply the solution of other. Under this reduction the solution does not in fact lose any regularity since the reduction is via a transformation which is nonsingular. Thus the theory of well-posedness of the KdV and RLW, introduced in chapters 2 and 3 are used to advantage.

In the next chapter the well-posedness of the general class shall be studied in detail via the above analysis.

5.5 Conservation laws

In chapter 7 a number of conservation laws for the general class will be established together with a detailed discussion of how many conservation laws exist for the general class. The idea of conservation laws was initiated in chapter 3 where the proof for the existence of an infinite number of conservation laws of the KdV equation is found. This idea was revived in chapter 4 and it was shown that the RLW equation has only three conservation laws. Since the KdV and the RLW lie in two disjoint classes (KdV and RLW classes respectively), then the question which naturally arise is whether the reduction introduced in section 5.3 preserves the existence of such conservation laws or not. The answer is clearly positive by the definition of the conservation law and shall be illustrated by the first conservation law.

Example 5.5.1 (the first conservation law)

By resolving the general class (5.1.1) in x and t derivatives, the first conservation law of the general class has the form

$$\frac{\partial}{\partial t}[u + \frac{a_2}{2} u^2 + a_5 u_{xt} + a_6 u_{tt}] + \frac{\partial}{\partial x}[u + \frac{a_1}{2} u^2 + a_3 u_{xx} + a_4 u_{xt}] = 0. \tag{5.5.1}$$

Consider the transformation (5.3.2), i.e.

$$x' = \frac{x}{1-n} - \frac{n}{1-n}t, \quad t' = \frac{x}{1-m} - \frac{m}{1-m}t, \quad \text{and}$$

$$u(x,t) = v(x',t'). \quad \text{Hence}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1-n}\frac{\partial v}{dx'} + \frac{1}{1-m}\frac{\partial v}{dt'} \quad \text{and ,}$$

$$\frac{\partial u}{\partial t} = \frac{-n}{1-n}\frac{\partial v}{\partial x'} - \frac{m}{1-m}\frac{\partial v}{\partial t'} . \tag{5.5.2}$$

Thus (5.5.1) reduces to

$$(\frac{1}{1-n}\frac{\partial}{\partial x'} + \frac{1}{1-m}\frac{\partial}{\partial t'})\{v + \frac{a_1}{2}v^2 + a_3[\frac{1}{(1-n)^2}v_{x'x'}$$

$$+ \frac{2}{(1-n)(1-m)}v_{x't'} + \frac{1}{(1-m)^2}v_{t't'}] + a_4[\frac{-n}{(1-n)^2}v_{x'x'}$$

$$- \frac{(n+m)}{(1-n)(1-m)}v_{x't'} - \frac{m}{(1-m)}v_{t't'}]\}$$

$$- (\frac{n}{1-n}\frac{\partial}{\partial x'} + \frac{m}{1-m}\frac{\partial}{\partial t'})\{v + \frac{a_2}{2}v^2 + a_5[\frac{-n}{(1-n)^2}v_{x'x'}$$

$$- \frac{(n+m)}{(1-n)(1-m)}v_{x't'} - \frac{m}{(1-m)^2}v_{t't'}]$$

$$+ a_6[\frac{n^2}{(1-n)^2}v_{x'x'} + \frac{2mn}{(1-n)(1-m)}v_{x't'} + \frac{m^2}{(1-m)^2}v_{t't'}]\}$$

$= 0$, i.e.

$$\frac{\partial}{\partial t'}\{[v + \frac{1}{2}(\frac{a_1 - a_2 m}{1-m})v^2 + [\frac{a_3 - a_4 n + a_5 nm - a_6 mn^2}{(1-n)^2(1-m)}]v_{x'x'}$$

$$+ [\frac{2a_3 - a_4(n+m) + a_5 m(n+m) - 2a_6 m^2 n}{(1-n)(1-m)^2}]v_{x't'}$$

$$+ [\frac{a_3 - a_4 m + a_5 m^2 - a_6 m^3}{(1-m)^3}]v_{t't'}\}$$

$$+ \frac{\partial}{\partial x'}\{v + \frac{1}{2}(\frac{a_1 - a_2 n}{1-n})v^2 + [\frac{a_3 - a_4 n + a_5 n^2 - a_6 n^3}{(1-n)^3}]v_{x'x'}$$

$$+ [\frac{2a_3 - a_4(n+m) + a_5 n(n+m) - 2a_6 n^2 m}{(1-n)^2(1-m)}]v_{x't'}$$

$$+ [\frac{a_3 - a_4 m + a_5 m^2 - a_6 m^3}{(1-m)^2(1-n)}]v_{t't'}\} = 0 \qquad (5.5.3)$$

(5.5.3) can be re-written in the form

$$\frac{\partial}{\partial t'}\{v + \frac{1}{2}(\frac{a_1 - a_2 m}{1 - m})v^2 + [\frac{a_3 - a_4 m + a_5 m^2 - a_6 m^3}{(1 - m)^3}]v_{t't'}\}$$

$$+ \frac{\partial}{\partial x}\{v + \frac{1}{2}\frac{a_1 - a_2 n}{1 - n}v^2 + (\frac{a_3 - a_4 n + a_5 n^2 - a_6 n^3}{(1 - n)^3})v_{x'x'}$$

$$+ [\frac{3a_3 - a_4(2n + m) + a_5(n + 2m) - 3a_6 mn^2}{(1 - n)^2(1 - m)}]v_{x't'}$$

$$+ [\frac{3a_3 - a_4(n + 2m) + a_5 m(2m + n) - 3a_6 m^2 n}{(1 - n)(1 - m)^2}]v_{t't'}\} = 0 ,$$

(5.5.4a)

i.e.,

$$\frac{\partial}{\partial t'}\{v + \frac{1}{2}c_2 v^2 + c_6 v_{t't'}\} + \frac{\partial}{\partial x'}\{v + \frac{1}{2}c_1 v^2 + c_3 v_{x'x'}$$

$$+ c_4 v_{x't'} + c_5 v_{t't'}\} = 0$$

where,

$$c_1 = \frac{a_1 - a_2 n}{1 - n}, \quad c_2 = \frac{a_1 - a_2 m}{1 - m}, \quad c_3 = \frac{a_3 - a_4 n + a_5 n^2 - a_6 n^3}{(1 - n)^3},$$

$$c_4 = \frac{3a_3 - a_4(2n + m) + a_5 n(n + 2m) - 3a_6 mn^2}{(1 - n)^2(1 - m)},$$

$$c_5 = \frac{3a_3 - a_4(n + 2m) + a_5 m(m + 2n) - 3a_6 nm^2}{(1 - n)(1 - m)^2} \quad \text{and}$$

$$c_6 = \frac{a_3 - a_4 m + a_5 m^2 - a_6 m^3}{(1 - m)^3}. \qquad (5.5.4b)$$

Comparing (5.5.4) and (5.3.3) proves that the reduction preserves the existence of conservation laws.

This result produces a convenient procedure to study the conservation laws of the general class by shifting this study to the four disjoint classes instead. This shall be done in chapter 7.

5.6 Preservation of the solitary waves

In this section we prove that the reduction, introduced in section (5.3) preserves the existence of solitary wave solutions.

Theorem 5.3

If $u_s = \frac{3}{\alpha}\,\text{sech}^2\, \frac{1}{2\sqrt{\beta}}[x - (1 + c)t]$ is a solitary wave solution of the equation

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0 \tag{5.6.1}$$

then

$v_s = \frac{3}{\alpha'}\,\text{sech}^2\, \frac{1}{2\sqrt{\beta'}}[x' - (1 + c')t']$ is a solitary wave solution of the equation

$$v_{t'} + v_{x'} + b_1 vv_{x'} + b_2 vv_{t'} + b_3 v_{x'x't'} + b_4 v_{x'x'x'}$$
$$+ b_5 v_{x't't'} + b_6 v_{t't't'} = 0 \tag{5.6.2}$$

where $b_1, b_2, \ldots, b_6$ are defined by (5.3.3b),

$\alpha c = a_1 - a_2(1 + c)$, $Bc = a_3 - a_4(1 + c) + a_5(1 + c)^2$

$- a_6(1 + c)^3$, $\alpha' c' = b_1 - b_2(1 + c')$,

$B'c' = b_3 - b_4(1 + c') + b_5(1 + c')^2 - b_6(1 + c')^3$,

and $(1 + c)$, $(1 + c')$ are the speeds of the solitary waves in the respective coordinate systems. □

Proof

The definition of the rest frame, introduced in chapter 2, implies, $\frac{dx}{dt} = 1 + c$ and $\frac{dx'}{dt'} = 1 + c'$, i.e.

$$x = (1 + c)t \quad \text{and} \quad x' = (1 + c'). \tag{5.6.3}$$

From section 5.3, the equation (5.6.1) reduces to (5.6.2) via the nonsingular transformation

$$x' = \frac{x}{1-n} - \frac{n}{1-n}t\ , \quad t' = \frac{x}{1-m} - \frac{m}{1-m}t \quad \text{and}$$

$$u(x,t) = v(x',t').$$

Then, using (5.6.3)

$$\frac{x}{1-n} - \frac{n}{1-n}t = (1+c')\left(\frac{x}{1-m} - \frac{m}{1-m}t\right). \qquad (5.6.4)$$

Substituting the first relation of (5.6.3) in (5.6.4), then

$$\left[\frac{1+c}{1-n} - \frac{n}{1-n}\right]t = (1+c')\left[\frac{1+c}{1-m} - \frac{m}{1-m}\right]t.$$

Hence $(1+c')$ has the form

$$(1+c') = \left(\frac{1-m}{1-n}\right)\left[\frac{(1+c)-n}{(1+c)-m}\right]. \qquad (5.6.5)$$

Thus

$$\alpha'c' = b_1 - b_2(1+c')$$

$$= \frac{a_1 - na_2}{1-n} - \frac{a_1 - ma_2}{1-m}\left(\frac{1-m}{1-n}\right)\left(\frac{1+c-n}{1+c-m}\right)$$

$$= \frac{(a_1 - na_2)(1+c-m) - (a_1 - ma_2)(1+c-n)}{(1-n)(1+c-m)}$$

$$= \frac{(n-m)a_1 - (n-m)a_2(1+c)}{(1-n)(1+c-m)}$$

$$= \frac{(n-m)[a_1 - a_2(1+c)]}{(1-n)(1+c-m)} = \frac{(n-m)}{(1-n)(1+c-m)}\ \alpha c\ .$$

(5.6.6)

But (5.6.5) implies $\quad c' = \dfrac{(n-m)c}{(1-n)(1+c-m)}$ (5.6.7)

Then (5.6.6) yields $\alpha = \alpha'$

i.e. the amplitude is invariant under the reduction.

Similarly, it can be shown, after tedious calculations, that

$$\beta' c' = \left[\frac{n - m}{(1 - n)(1 + c - m)}\right]\beta \ . \tag{5.6.8}$$

Hence

$$\beta' = \left[\frac{n - m}{(1 - n)(1 + c - m)}\right]^2 \beta \tag{5.6.9}$$

i.e., if $\beta > 0$, then $\beta' > 0$

which proves the theorem. ∎

Example 5.6.1

Let $a_1 = 2$, $a_2 = 1$, $a_3 = 8$, $a_4 = 12$, $a_5 = 6$ and $a_6 = 1$

i.e. $u_t + u_x + 2uu_x + uu_t + 8u_{xxx} + 12u_{xxt} + 6u_{xtt} + u_{ttt} = 0.$

(5.6.10)

Consider the transformation

$$x' = x, \quad t' = 2t - x, \quad u(x,t) = v(x',t'). \tag{5.6.11}$$

Then (5.6.10) reduces under this transformation to

$$v_{t'} + v_{x'} + 2vv_{x'} + 8v_{x'x'x'} = 0. \tag{5.6.12}$$

$\beta c = 8 - 12(1 + c) + 6(1 + c)^2 - (1 + c)^3 = (1 - c)^3$ and

$$\alpha c = 1 - c \tag{5.6.13}$$

i.e., the solitary wave solution of (5.6.10) has form

$$u_s = \frac{3c}{1 - c} \operatorname{sech}^2 \left\{\frac{1}{2}\sqrt{\frac{c}{(1 - c)^3}}[x - (1 + c)t]\right\}, \tag{5.6.14}$$

Similarly, for the equation (5.6.12)

$$\beta'c' = 8 \quad \text{and} \quad \alpha'c' = 2, \tag{5.6.15}$$

Hence (5.615) implies

$$8 = \beta'c' = \beta' . \frac{2c}{1 - c} \rightarrow \beta' = 4\left(\frac{1 - c}{c}\right) = 4\left(\frac{1}{1 - c}\right)^2\left(\frac{1 - c}{c}\right)^3$$

[by substituting $n = 0$, $m = 2$ in theorem 5.3], and $\alpha' = \alpha$.

Hence the solitary wave solution of (5.6.12) has the form

$$v_s = \frac{3c'}{2} \operatorname{sech}^2 \frac{1}{2}\sqrt{\frac{c'}{8}}\,[x' - (1 + c')t']. \tag{5.6.16}$$

5.7 Preservation of the N soliton solution

We have seen that the above reduction to the four classes preserves solitary waves. However, we know that the solitary wave of the KdV is also a soliton, i.e. that there are exact N-soliton solutions for any $N \in Z$. The question therefore arise as to whether the N-soliton solutions are preserved under the reduction. The definition of the N-soliton solution for a given equation implies that the solution decomposes asymptotically to N solitary waves of the equation. Since the reduction preserves a solitary wave, then under the reduction, the N-soliton solution transforms to a solution for the reduced equation and the solitary waves transform to soliatry waves for the reduced equation. Hence the reduced solution is N-soliton solution for the new equation. We look first to the following example of an equation from the simple KdV class, defined in corollary 5.3.1, to show that the N-soliton solutions can be obtained by a technique, comes from the KdV equation.

Example 5.7.1

$$u_t + u_x - 6uu_x + 6uu_t + u_{xxx} - 3u_{xxt} + 3u_{xtt} - u_{ttt} = 0. \quad (5.7.1)$$

Consider the transformation

$$u = v^2 + v_x - v_t \; . \quad (5.7.2)$$

Substituting (5.7.2) in (5.7.1) leads to the fact that if v evolves according to the equation

$$v_t + v_x - 6v^2v_x + 6v^2v_t + v_{xxx} - 3v_{xxt} + 3v_{xtt} - v_{ttt} = 0 \quad (5.7.3)$$

then u evolves according to (5.7.1). If u is known, (5.7.2) can be linearized by choosing

$$v = \frac{1}{\psi}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\psi \; . \quad (5.7.4)$$

Hence, (5.7.2) has the form

$$\psi u = \psi_{xx} - 2\psi_{xt} + \psi_{tt} \quad (5.7.5)$$

which is parabolic equation in ψ and without loss of generality u can be shifted by a constant λ. Hence (5.7.5) reduces to the form:

$$\left[\left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial t} + \frac{\partial^2}{\partial t^2}\right) - (u - \lambda)\right]\psi = 0. \quad (5.7.6)$$

To solve (5.7.6) it is convenient to use the nonsingular linear transformation

$$x \to x' = x, \quad t \to t' = \frac{x}{2} + \frac{t}{2}, \quad \psi(x,t) \to \phi(x',t') \text{ and}$$

$$u(x,t) = w(x',t') \quad (5.7.7)$$

to transform (5.7.6) to the equation

$$\left[\frac{\partial^2}{\partial x^2} - (\lambda - w)\right]\phi = 0 \qquad (5.7.8)$$

where the substitution (5.7.7) in (5.7.1) implies that w evolves according to the KdV equation

$$w_{t'} + w_{x'} - 6ww_{x'} + w_{x'x'x'} = 0. \qquad (5.7.9)$$

Hence (5.7.8) is a Schrödinger equation with potential w, energy level λ and wave function ϕ. Then by using the inverse scattering method, introduced in chapter 3, the N soliton solution of (5.7.9) has the form

$$w(x',t') = -2\frac{d^2}{dx'^2}\ln f,$$

$$f = \begin{vmatrix} 1+\frac{c_1^2}{2k_1}e^{2k_1x'} & \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)x'} & \cdots & \frac{c_1c_N}{k_1+k_N}e^{(k_1+k)x'} \\ \vdots & \vdots & & \vdots \\ \frac{c_Nc_1}{k_N+k_1}e^{(k+k_1)x'} & \frac{c_Nc_2}{k_N+k_2}e^{(k_1+k_2)x'} & \cdots & 1+\frac{c_N^2}{2k_N}e^{2k_{N+1}} \end{vmatrix}$$

Hence by using the inverse of the transformation (5.7.7), the N-soliton solution of (5.7.1) has the form

$$u(x,t) = -2\left[\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x\partial t} + \frac{\partial^2}{\partial t^2}\right]\ln F, \qquad (5.7.10)$$

$F = T^{-1}f$, where T^{-1} is the inverse of the transformation (5.7.7). In fact (5.7.10) is N soliton solution of (5.7.1), this fact can be shown by studying the behaviour of the solution as $|t| \to \infty$, where one can show that the solution splits into a number of solitary waves of (5.7.1) in a similar sense as for the KdV equation (see Wadati and Toda [4]).

The above example shows that the reduction preserves the N-soliton solution [in the simple KdV class]. Furthermore the procedure, introduced for obtaining the N-soliton solution of the KdV equation can be extended to obtain the N-soliton solution of all the elements of the simple KdV class.

Outside the simple KdV class, we shall see that the above technique comes to a stop and the transformation which couples any element with its modified form only exists in the simple KdV class.

<u>Theorem 5.4</u>

If v evolves according to the modified general class

$$v_t + v_x - 6a_1v^2v_x - 6a_2v^2v_t + a_3v_{xxx} + a_4v_{xxt} + a_5v_{xtt}$$
$$+ a_6v_{ttt} = 0 \tag{5.7.11}$$

then

$$u = v^2 + \alpha v_x + \beta v_t \tag{5.7.12}$$

evolves according to the general class

$$u_t + u_x - 6a_1uu_x - 6a_2uu_t + a_3u_{xxx} + a_4u_{xxt} + a_5u_{xtt}$$
$$+ a_6u_{ttt} = 0 \tag{5.7.13}$$

if and only if the cubic equation

$$a_3 - a_4\lambda + a_5\lambda^2 - a_6\lambda^3 = 0 \tag{5.7.14}$$

has three equal real roots. □ [See appendix (B) for the proof].

The condition of the theorem together with the reduction analysis in section (5.3) implies that the equation (5.7.11) must be inside the simple KdV class. If this is the case, then (5.7.12) can be linearized by making use of the transformation

$$u = \frac{1}{\psi} \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial t} \right) \psi \, . \qquad (5.7.15)$$

Then the exact solution is obtained in a sense similar to that as in example (5.7.1).

CHAPTER SIX

THE WELL-POSEDNESS OF THE GENERAL CLASS OF EQUATIONS

In this chapter the well-posedness of the general class

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0 \tag{6.1}$$

is studied. We prove, first of all, that the general class can be reduced for certain data to a semi-linear system of first order partial differential equations. We find the characteristics of this sytem and show that it is equivalent to a system of ordinary differential equations in which differentiation is along characteristic direction. These equations can be intergrated to give the solution of the system provided that the data is not specified on a characteristic. This method of solution is called the method of characteristics. Thus, its availability for the general class (6.1) depends upon the data not being specified on a characteristic. This leads us to divide the general class into two subclasses according as to whether the method of characteristics can be used. We call the subclass in which the method of characteristics is applicable the nonsingular class and the remaining the singular class.

We establish well-posedness for the nonsingular class by applying the well-known theorems on uniqueness, existence and continuous dependence on the initial conditions for semi-linear systems. As regards the singular class, we divide it further according to the multiplicity of the (essential) characteristic roots. In the case of a triple root we show that it corresponds to the general KdV, for a double root to the RLW, for a distinct roots to W_{54} and for a pair of conjugate roots to W_{53} . (These classes were defined in chapter 5). Thus the KdV represents one of the subsets of equations in the general class for which the method of characteristics fails.

Now, clearly, well-posedness of some parts of singular class follows from the results established for the KdV and RLW in chapters 3 and 4 respectively. However, these do not deal with the inclusion of a uu_t term, but are confined to the so-called "simple" KdV and RLW classes. As part of our own contribution we extend these results to certain equations in the singular class which include the uu_t term. Finally, some applications are provided to show that the KdV and the RLW equations are well-posed for any skew data.

6.1 Reduction to a semi-linear system of first order partial differential equations

Consider the initial value problem which corresponds to the general class of equations (6.1). Let the initial curve which supports the data be non-characteristic, as shall be defined in the next section, and without loss of generality let this curve be the usual one $t = 0$, i.e.,

$$u_t + u_x + a_1uu_x + a_2uu_t + a_3u_{xxx} + a_4u_{xxt} + a_5u_{xtt} + a_6u_{ttt} = 0 \tag{6.1.1a}$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad u_{tt}(x,0) = h(x). \tag{6.1.1b}$$

We introduce now the following:

Lemma 6.1.1

The initial value problem for the general class of equations

(6.1.1) with non-characteristic initial data may be reduced to a non-characteristic initial value problem for a first order system of partial differential equations. □

<u>Proof</u>

Re-writing (6.1.1a) in the form:

$$F(u, p, q, r, s, \tau, v, w, \mu, \nu) = a_6 v + a_5 w + a_4 \mu + a_3 \nu$$
$$+ (a_1 q + a_2 p)u + q + p = 0, \qquad (6.1.2)$$

where,

$$p = u_t, \quad q = u_x, \quad \tau = u_{xx}, \quad s = u_{xt}, \quad r = u_{tt}, \quad \nu = u_{xxx},$$
$$\mu = u_{xxt}, \quad w = u_{xtt}, \quad \text{and} \quad v = u_{ttt}. \qquad (6.1.3)$$

Subject to the initial conditions (6.1.1b), and differentiating (6.1.2) with respect to t yields

$$\frac{\partial F}{\partial t} = F_u u_t + F_p p_t + F_q q_t + F_r r_t + F_s s_t + F_\tau \tau_t + F_v v_t + F_w w_t$$
$$+ F_\mu \mu_t + F_\nu \nu_t$$
$$= (a_1 q + a_2 p)u_t + (1 + a_2 u)p_t + (1 + a_1 u)q_t + a_6 v_t + a_5 w_t$$
$$+ a_4 \mu_t + a_3 \nu_t = 0 \qquad (6.1.4)$$

and,

$$u_t = p, \quad p_t = r, \quad q_t = s, \quad r_t = v, \quad s_t = r_x = w, \quad \tau_t = s_x,$$
$$w_t = v_x, \quad \mu_t = w_x \quad \text{and} \quad \nu_t = \mu_x \quad \text{(from (6.1.3))}. \qquad (6.1.5)$$

Inserting (6.1.5) into (6.1.4) we have

$$(a_1q + a_2p)p + (1 + a_2u)r + (1 + a_1u)s + a_6v_t + a_5v_x + a_4w_x + a_3\mu_x = 0. \qquad (6.1.6)$$

Thus (6.1.5) and (6.1.6) can be combined to form the following system:

$$\begin{array}{lll} \multicolumn{3}{l}{(a_1q + a_2p)p + (1 + a_2u)r + (1 + a_1u)s + a_6v_t + a_5v_x + a_4w_x + a_3\mu_x = 0,} \\ u_t = p\,, & p_t = r\,, & q_t = s. \\ r_t = v & s_t = r_x = w & \tau_t = s_x \\ w_t = v_x & \mu_t = w_x & \nu_t = \mu_x \end{array} \qquad (6.1.7)$$

which is a system of first order partial differential equations in the dependent variables u, p, q, r, s, τ, v, w, μ and ν.

The initial conditions may be obtained from equations (6.1.1b), and amount to the specification of u, p, q, r, s, τ, v, w, μ and ν. However, v is not known explicity, but since the initial conditions are assumed specified on a non-characteristic curve, then v may always be determined. Thus the initial conditions on $t = 0$ become:

$$\begin{array}{lll} u(x,0) = f(x) & p(x,0) = g(x) & q(x,0) = f'(x) \\ r(x,0) = h(x) & s(x,0) = g'(x) & \tau(x,0) = f''(x) \\ w(x,0) = h'(x) & \mu(x,0) = g''(x) & \nu(x,0) = f'''(x) \end{array} \qquad (6.1.8)$$

$$v(x,0) = G(f(x),\ g(x),\ f'(x),\ h(x),\ g'(x),\ f''(x),\ h'(x),\ g''(x),\ f'''(x))$$

The system (6.1.7) can be expressed in the matrix form

$$U_t + AU_x + C = 0, \qquad U(x,0) = H(x) \tag{6.1.9}$$

where U, A and C are

$$U^T = [u\ p\ q\ r\ s\ \tau\ v\ w\ \mu\ \nu], \tag{6.1.10a}$$

$$C^T = [\ -p\ -r\ -s\ -v\ 0\ 0\ j\ 0\ 0\ 0] \tag{6.1.10b}$$

with $j = \frac{1}{a_6}\ [(a_1q + a_2p)p + (1 + a_2u)r + (4a_1u)s], \quad a_6 \neq 0$

and

$$A = \left[\begin{array}{cccc|cc|cccc} & & & & 0 & 0 & & & & \\ & & & & 0 & 0 & & & & \\ & O(4\times4) & & & 0 & 0 & & O(4\times4) & & \\ & & & & 0 & 0 & & & & \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & 0 & 0 & \frac{a_5}{a_6} & \frac{a_4}{a_6} & \frac{a_3}{a_6} & 0 \\ & O(4\times4) & & & 0 & 0 & -1 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & -1 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right], \tag{6.1.10c}$$

with O(n×n) the n×n zero matrix.

This completes the proof of the lemma. ∎

Definition 6.1.1

The system of equations (6.1.9) is called quasi-linear if A and C depend on x, t and U. If A is independent of U, the system is called semi-linear. If C is also a linear function of U, the system is called linear.

Using definition 6.1.1 and lemma 6.1.1 gives the following theorem:

Theorem 6.1

The initial value problem (6.1.1) for the general class of equations with non-characteristic data can be reduced to a non-characteristic initial value problem for a first order semi-linear system of partial differential equations. □

Remarks

(1) When $a_6 = 0$ in the above reduction then A becomes singular. In this case, as was shown in section 5.2, one can find a nonsingular linear transformation which takes the original equation to one with $a_6 \neq 0$. However, the non-characteristic curve which supports the data of the original equation may be transformed to a characteristic curve for the new equation, i.e., the new data becomes characteristic. This gives a contradiction with the assumptions of theorem 6.1. The way out of this contradiction together with a classification of the problem in terms of the singularity of A will be discussed in section 6.5. Thus, in the present section and up to section 6.5 we assume that $a_6 \neq 0$.

(2) The equivalence of the solutions corresponding to the new system of equations and the original equation can be proved by

noting that if u is a solution of (6.1.2) and (6.1.1b), then the vector U will be a solution of the new system (6.1.7) with initial conditions (6.1.8). Conversely, suppose that U is a solution of (6.1.7) with initial conditions (6.1.8). Then, if these quantities do not simultaneously satisfy both the original equation and the new system, there will be defined the non-zero quantities

$$\alpha_t = p - u_t, \quad \alpha_x = q - u_x, \quad \alpha_{tt} = u_{tt} - r, \quad \alpha_{xt} = u_{xt} - s,$$

$$\alpha_{xx} = u_{xx} - \tau, \quad \alpha_{ttt} = u_{ttt} - v, \quad \alpha_{xtt} = v_{xtt} - w,$$

$$\alpha_{xxt} = u_{xxt} - \mu, \quad \text{and} \quad \alpha_{xxx} = u_{xxx} - \nu.$$

Now, from the second equation of (6.1.7) we have $\alpha_t = u_t - u_t = 0$, and from the definition of α_{tt}

$$\alpha_{tt} = \frac{\partial p}{\partial t} - u_{tt} = \frac{\partial}{\partial t}u_t - u_{tt} = 0, \quad \text{so that} \quad \alpha_{tt} = 0.$$

Using the initial condition at $t = 0$, we have

$$\alpha_{xt} = \frac{\partial}{\partial t} u_x - s = \frac{\partial}{\partial x}p - s = g'(x) - g'(x) \rightarrow \alpha_{xt}|_{t=0} = 0.$$

To establish that α_{xt} is identically zero, we form the relation

$$\frac{\partial}{\partial t} \alpha_{xt} = \frac{\partial}{\partial t} u_{xt} - \frac{\partial s}{\partial t}$$

$$= \frac{\partial}{\partial x} r - \frac{\partial}{\partial x} u_{tt} = - \frac{\partial}{\partial x}\alpha_{tt} = 0 \rightarrow \alpha_{xt} \equiv 0.$$

Similar arguments establish that the other differences are identically zero. Thus the quantities u, p, q, r, s, τ, v, w, μ and ν satisfy also the differential equation (6.1.2), which proves the equivalence between the solutions.

6.2 Characteristics of the system

Definition 6.2.1

A characteristic of the sytem (6.1.9) is a curve along which the values of U, combined with the equations (6.1.9) are insufficient to determine the derivatives of U normal to this curve.

The problem of determining the derivative of U normal to our data is easily resolved by considering the effect on system (6.1.9) of a change of coordinates,

$$t \to t \quad \text{and} \quad x \to \phi(x,t) = \text{constant} \qquad (6.2.1)$$

[t is left unchanged since the discussion is for evolution equations].

Then the system (6.1.9) reduces under (6.2.1) to

$$\left(\frac{\partial U}{\partial t} + \frac{\partial \phi}{\partial t}\frac{\partial U}{\partial \phi}\right) + A\frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial x} + C(U) = 0, \quad \text{i.e.,}$$

$$\left(I\frac{\partial \phi}{\partial t} + A\frac{\partial \phi}{\partial x}\right)\frac{\partial U}{\partial \phi} + \frac{\partial U}{\partial t} + C(U) = 0 \qquad (6.2.2)$$

where $\frac{\partial U}{\partial \phi}$ is the normal derivative of U to $t = 0$. This normal derivative is determined if

$$\det\left[I\frac{\partial \phi}{\partial t} + A\frac{\partial \phi}{\partial x}\right] \neq 0. \qquad (6.2.3)$$

Combining this result with definition 6.2.1, then the characteristics of the system (6.1.9) are given by the equation

$$\det\left[I\frac{\partial \phi}{\partial t} + A\frac{\partial \phi}{\partial x}\right] = 0. \qquad (6.2.4)$$

Putting $\lambda = -\frac{\partial \phi}{\partial t} / \frac{\partial \phi}{\partial x} = \frac{dx}{dt}$, then (6.2.4) can be written as

$$\det(A - \lambda I) = 0. \qquad (6.2.5)$$

Equation (6.2.5) is called the characteristic equation of the system (6.1.9) where λ is now an eigenvalue of the matrix A.

The above analysis leads to the following:

Theorem 6.2

The characteristics of the system (6.1.9) which corresponds to the general class of equations (6.1.1) are given by the roots of the equation

$$\lambda^7[a_3 - a_4\lambda + a_5\lambda - a_6\lambda^3] = 0 \qquad (6.2.6)$$

where $\lambda = \frac{dx}{dt}$. □

Proof

By using the expression of A from (6.1.10c) and expanding $\det(A-\lambda I) = 0$ then, obviously (6.2.6) follows and the theorem is proved. ■

We conclude this section by defining the hyperbolicity of a general system of which our case is a specific example.

Definition 6.2.2

(1) If all the roots of equation (6.2.5) are real and distinct the system of equations (6.1.9) is called totally hyperbolic.

(2) If some of the roots of (6.2.5) are complex, the system is called ultra-hyperbolic.

(3) If all the roots of (6.2.5) are complex, the system (6.1.9) is elliptic.

(4) The system is hyperbolic if (6.2.5) has at least one real root.

Note that the method of reduction is not unique and can introduce redundant eigenvalues into the characteristic equation. These can be disregarded since they do not lead to any inconsistency or loss of generality. The number of genuine eigenvalues necessary to solve a given equation is equal to the order of the differential equation. For example in the second order differential equation $u_{tt} + cu_{xx} + u = 0$, if $u_1 = u_t$ and $u_2 = u_x$ the equation reduces to the system

$$\begin{bmatrix} u_1 \\ u_2 \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & c & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u \end{bmatrix}_x + \begin{bmatrix} u \\ 0 \\ -(u_1 + u_2) \end{bmatrix} = 0$$

for which the genuine eigenvalues are $\lambda = \pm\sqrt{-c}$ and the redundant eigenvalue is $\lambda = 1$. Thus, if we reject $\lambda = 1$ then the equation is elliptic if $c > 0$ and totally hyperbolic if $c < 0$.

We now turn to exploit the results of theorem 6.2 to reduce the sysem (6.1.9) into a simple form for discussion and this is done in the next section.

6.3 Normal form of the first order system

In the previous section we demonstrated that our system (6.1.9) is of hyperbolic type and proved that its characteristics are given by the eigenvalues of the eigenvalue problem

$$AX = \lambda X. \tag{6.3.1}$$

It is now convenient to transform the system (6.1.9) to a simple form in which the differentiation should be in one direction only, i.e., directed along a characteristic of the system. This new system is called the normal (canonical) form of (6.1.9).

For doing this let the eigenvectors corresponding to the eigenvalues λ_i of A span E^{10} and let T be the matrix in which each column is one of those eigenvectors. Then T is nonsingular. Suppose that

$$U = TV. \tag{6.3.2}$$

Inserting this transformation into (6.1.9), then

$$(TV)_t + A(TV)_x + C = 0, \qquad TV(x,0) = H(x). \tag{6.3.3}$$

Hence,

$$TV_t + T_tV + AT_xV + ATV_x + C = 0. \tag{6.3.4}$$

Multiplying both sides of (6.3.4) by the inverse of T, i.e., T^{-1},

$$V_t + T^{-1}ATV_x + \tilde{C} = 0, \tag{6.3.5}$$

$$\tilde{C} = T^{-1}C + T^{-1}AT_xV + T^{-1}T_tV . \tag{6.3.6}$$

Since A is a matrix of constant coefficients, (6.1.10c) then the eigenvalues of A do not depend on x, t and U consequently T does not depend on x, t and U and this implies that $T_t = 0 = T_x$.

But since $T^{-1}AT = D$ is diagonal, (6.3.5) can be written as

$$V_t + DV_x + \tilde{C} = 0 \quad , D = \mathrm{diag}(\lambda_1, \ldots, \lambda_{10}). \tag{6.3.7a}$$

with the initial condition

$$V(x,0) = T^{-1}U(x,0) = \Psi(x). \qquad (6.3.7b)$$

Finally, equations (6.3.7a,b) can be written in terms of components and the ith component, which corresponds to the ith characteristic, has the form:

$$v_t^i + \lambda^i v_x^i + \tilde{c}^i = 0, \qquad v^i(x,0) = \psi^i(x). \qquad (6.3.8)$$

From the theory of a single first order partial differential equation, it follows that on the characteristic traces for the equation, the equation reduces to an ordinary differential equation. Hence, $v_t^i + \lambda^i v_x^i$ is a directional derivative in the direction λ^i. Thus, every equation in the form (6.3.7) contains a differentiation in one direction only which is the characteristic direction. The form (6.3.7) is called the normal form of the system (6.1.9).

Remarks

(1) The reduction of the original system to its corresponding normal form (6.3.7) is viable even if some of the eigenvalues are multiple [14].

(2) The case when some of the roots are complex is left to the end of this chapter.

6.4 The method of characteristics

The characteristics of the system (6.1.9) were obtained in the last section. The basic rationale underlying the use of the characteristics is that by an appropriate choice of coordinates the original system (6.1.9) can be replaced by a system expressed in characteristic coordinates (normal form). The method of characteristics is expressible briefly as; firstly, solving the equation (6.2.5) to locate the characteristic curves and secondly, integrating the equation (6.3.7) as ordinary differential equations along the characteristics. Hence the solution of the original system can be constructed. This is illustrated by the following simple example.

<u>Example 6.4.1</u> (the wave equation)

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x). \tag{6.4.1}$$

To find the solution of this equation by using the method of characteristics, we firstly reduce it to a system of first order quasilinear partial differential equations. Thus, let

$$F(u,\ p,\ q,\ r\ s,\ \tau) \equiv u_{tt} - u_{xx} = \tau - r = 0, \tag{6.4.2}$$

where

$$u_t = q, \quad u_x = p, \quad u_{xx} = r, \quad u_{xt} = s \quad \text{and} \quad u_{tt} = \tau. \tag{6.4.3}$$

Differentiating (6.4.2) with respect to t and using (6.4.3) gives

$$\tau_t - r_t = 0 \tag{6.4.4}$$

where,

$$u_t = q, \quad p_t = s, \quad q_t = \tau, \quad r_t = s_x \quad \text{and} \quad s_t = \tau_x. \tag{6.4.5}$$

Combining (6.4.4) and (6.4.5), then the original equation (6.4.1) reduces to the system

$$\tau_t = s_x, \qquad u_t = q, \qquad p_t = s,$$
$$q_t = \tau, \qquad r_t = s_x, \qquad s_t = \tau_x. \tag{6.4.6}$$

with the initial data

$$u(x,0) = f(x) \qquad p(x,0) = f'(x) \qquad q(x,0) = g(x)$$
$$r(x,0) = f''(x) \qquad s(x,0) = g'(x), \tag{6.4.7}$$
$$\tau(x,0) = G(f(x),\ g(x),\ f'(x),\ f''(x),\ g'(x)).$$

The system (6.4.6) and (6.4.7) can be written in the matrix form

$$U_t = AU_x + BU, \qquad U(x,0) = H(x), \tag{6.4.8}$$

where,

$$A = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & 0(4\times4) & & 0 & 0 \\ & & & & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right], \qquad B = \left[\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline & & & & 0 & 1 \\ & & 0(4\times4) & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \end{array}\right]$$

(6.4.9)

and $U^T = [u\ p\ q\ r\ s\ \tau]$.

The characteristic equation for (6.4.8), i.e., $\det(A - \lambda I) = 0$ is

$$\begin{bmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 0 & 1 & -\lambda \end{bmatrix} = \lambda^4(\lambda^2-1) = 0. \tag{6.4.10}$$

Thus, the characteristic roots are

$$\lambda = 0,\ 0,\ 0,\ 0,\ 1 \ \text{ and } \ -1 \tag{6.4.11}$$

indicating that the equation is hyperbolic. (In fact it is totally hyperbolic since the roots $\lambda = 0$ are redundant). The eigenvectors corresponding these eigenvalues (6.4.11) are the solutions of the equation

$$AX = \lambda X \tag{6.4.12}$$

and are as follows:

$\lambda = 0$; $e_1 = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T$, $e_2 = [0 \ \ 1 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T$,

$e_3 = [0 \ \ 0 \ \ 1 \ \ 0 \ \ 0 \ \ 0]^T$, $e_4 = [0 \ \ 0 \ \ 0 \ \ 1 \ \ 0 \ \ 0]^T$

$\lambda = -1$; $e_5 = [0 \ \ 0 \ \ 0 \ \ -1 \ \ 1 \ \ -1]^T$ and

$\lambda = 1$; $e_6 = [0 \ \ 0 \ \ 0 \ \ 1 \ \ 1 \ \ 1]^T$.

Let T be the matrix whose columns are e_i, $i = 1, 2, \ldots$ and 6, i.e.,

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{6.4.13}$$

Then the inverse of this matrix exists and has the form

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad (6.4.14)$$

Then, clearly, we have

$$T^{-1}AT = \text{diag}\,(0,\ 0,\ 0,\ 0,\ -1,\ 1), \qquad (6.4.15a)$$

$$T^{-1}BT = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (6.4.15b)$$

Let, now, $U = TV$, then the system (6.4.8) reduces to

$$(TV)_t = A(TV)_x + BTV. \qquad (6.4.16)$$

Since T does not depend on both x and t, then (6.4.16) implies that

$$V_t = (T^{-1}AT)V_x + (T^{-1}BT)V. \qquad (6.4.17)$$

Using (6.4.15), equation (6.4.17) can be written in the form:

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -v_5 \\ v_6 \end{bmatrix}_x + \begin{bmatrix} v_3 \\ v_5+v_6 \\ -v_5+v_6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.4.18}
$$

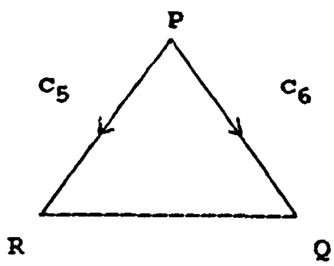
with the initial data $V(x,0) = T^{-1}U(x,0)$, i.e.,

$$
V(x,0) = [f \quad f' \quad g \quad 0 \quad \tfrac{1}{2}(g'-f") \quad \tfrac{1}{2}(g'+f")]^T. \tag{6.4.19}
$$

Consider now the last two components of the system (6.4.18) and (6.4.19). The equations of the characteristics are $\frac{dx}{dt} = -1$ and and $\frac{dx}{dt} = 1$ respectively. Then,

$$
(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})v_5 = 0 = v_5(P) = v_5(R),
$$

$$
(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})v_6 = 0 \rightarrow v_6(P) = v_6(Q).
$$



(Fig. 6.1)

Now the third component gives

$$
\frac{\partial v_3}{\partial t} = -v_5 + v_6
$$

$$
= -\tfrac{1}{2}\,[g'(x-t) - f"(x-t)] + \tfrac{1}{2}\,[g'(x+t) + f"(x+t)].
$$

Thus, integrating with respect to t gives

$$V_3(P) = -\frac{1}{2}\,[-g(x-t) + f'(x-t)] + \frac{1}{2}\,[g(x+t) + f'(x+t)] + P(x). \tag{6.4.20}$$

Computing at $t = 0$ gives

$$g = \frac{1}{2}\,g - \frac{1}{2}\,f' + \frac{1}{2}\,g + \frac{1}{2}\,f' + P(x) - P(x) = 0.$$

Thus,

$$V_3(P) = -\frac{1}{2}\,[-g(x-t) + f'(x-t)] + \frac{1}{2}\,[g(x+t) + f'(x+t)]. \tag{6.4.21}$$

Next, the first component yields

$$\frac{\partial V_1}{\partial t} = V_3 = \frac{1}{2}\,[\,g(x-t) - f'(x-t)] + \frac{1}{2}\,[g(x+t) + f'(x+t)].$$

Thus

$$V_1 = \frac{1}{2}\int_0^t g(x-\bar{t})d\bar{t} + \frac{1}{2}\int_0^t g(x+\bar{t})d\bar{t}$$

$$+ \frac{1}{2}[f(x-t) + f(x+t)] + q(x). \tag{6.4.22}$$

The first integral is determined by putting $x-\bar{t} = \xi$, then

$$\frac{1}{2}\int_0^t g(x-\bar{t})d\bar{t} = -\frac{1}{2}\int_x^{x-t} g(\xi)d\xi = \frac{1}{2}\int_{x-t}^{x} g(\xi)d\xi.$$

Similarly:

$$\frac{1}{2}\int_0^t g(x+\bar{t})d\bar{t} = \frac{1}{2}\int_x^{x-t} g(\xi)d\xi.$$

Inserting these two integrals into (6.4.22) we have

$$V_1 = \frac{1}{2}\int_{x-t}^{x+t} g(\xi)d\xi \quad + \quad \frac{1}{2}[f(x+t) + f(x-t)] + q(x). \qquad (6.4.23)$$

Since $V_1(x,0) = u(x,0) = f(x)$, (6.4.23) implies that

$f = 0 + \frac{1}{2}(f+f) + q(x) \quad \rightarrow \quad q(x) = 0,$ and

$$V_1|_p = u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)d\xi. \qquad (6.4.24)$$

Which is the D'Alembert formula .

6.5 Well-posedness classification

The analysis so far has concentrated on elements of the general class with $a_6 \neq 0$ and non-characteristic data. (The reduction to a first order system introduced in section 6.1 fails for the case $a_6 = 0$ or if the data are characteristic). It was pointed out at the end of section 6.2, that the case $a_6 = 0$ can be avoided by using a nonsingular linear transformation to transform this equation to one with $a_6 \neq 0$. Thus the reduction makes sense again as long as the transformed data is non-characteristic (with respect to the transformed equation). However, if the transformed data remain or become characteristic, then the underlying reduction fails completely without any visible avoidance. Hence the method of characteristics cannot be used to solve the original initial value problem.

To study the well-posedness of the general class of equations we classify it into two subclasses in terms of the above reduction. In one of them, well-posedness will be investigated by using the method of characteristics, whilst in the other well-posedness will be studied by means of the reduction to equivalence classes introduced in section 5.2.

Definition 6.5.1 (the nonsingular class)

The nonsingular class is the subclass of the general class of equations W, whose elements satisfy one or other of the following conditions:

(1) $a_6 \neq 0$ and the data are non-characteristic.

(2) if $a_6 = 0$ there exists a nonsingular linear transformation which transforms the initial value problem to one satisfying (1).

Note that condition (1) of this definition can be replaced by:

(1') A is nonsingular, where A is the matrix given by the expression (6.1.10c), and the data are non-characteristic.

Definition 6.5.2 (the singular class)

The complement of the nonsingular class in W is called the singular class, i.e., the subclass of W whose elements satisfy one or other of the following conditions:

(1) $a_6 \neq 0$ and the data is on a characteristic curve,

(2) $a_6 = 0$ and there is no linear transformation which transforms a_6 to a non-zero value and leaves the data on a non-characteristic curve.

Thus, any element of W clearly belongs to either the nonsingular or the singular class.

Remark:

Since we are only interested in those elements of W which have solitary waves, i.e., $W \cap S$, then the above definitions are restricted to the class $W \cap S$.

These definitions lead to the following theorem for our general class of equations:

Theorem 6.3

Let $e = (a_1, a_2, \ldots, a_6) \in W \cap S$.

(1) If $a_6 \neq 0$, then e belongs to the nonsingular class if and only if the initial curve supporting the data is non-characteristic, i.e. $\lambda = \frac{dx}{dt}$ and $a_3 - a_4\lambda + a_5\lambda^2 - a_6\lambda^3 \neq 0$.

(2) If $a_6 = 0$, then e belongs to the nonsingular class if the initial curve supporting the data is neither the usual curve (i.e., $t = 0$) nor the curve given by

$$\{2a_3 - a_4 m \pm \sqrt{a_4^2 - 4a_3a_5}\}t = \{a_4 - 2a_5 m \pm \sqrt{a_4^2 - 4a_3a_5}\}x$$

for every m such that $a_3 - a_4 m + a_5 m^2 \neq 0$. □

Proof

Combining definition 6.5.1 and the characteristic equation (6.2.5) then A is nonsingular and the initial curve supporting the data is non-characteristic. Thus (1) follows.

To prove (2), let $a_6 = 0$, i.e, $e \in W \cap S$ is defined by

$$u_t + u_x + a_1 u u_x + a_2 u u_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} = 0 \qquad (6.5.1)$$

with $u(x,t)$ and $u_t(x,t)$ given on any line $t = \alpha x$.

Subjecting (6.5.1) to the nonsingular linear transformation

$$x \to \frac{x}{1-k} - \frac{k}{1-k}\,t, \quad t \to \frac{x}{1-m} - \frac{m}{1-m}\,t, \quad u(x,t) \to v(x,t), \qquad (6.5.2)$$

then, as in section 5.2, (6.5.1) reduces to

$$v_t + v_x + b_1 v v_x + b_2 v v_t + b_3 v_{xxx} + b_4 v_{xxt} + b_5 v_{xtt} + b_6 v_{ttt} = 0 \qquad (6.5.3)$$

where

$$b_1 = \frac{a_1 - k a_2}{1-k}, \qquad b_2 = \frac{a_1 - m a_2}{1-m}, \qquad b_3 = \frac{a_3 - a_4 k + a_5 k^2}{(1-k)^3},$$

$$b_4 = \frac{3a_3 - (m+2k)a_4 + (k^2+2km)a_5}{(1-k)^2(1-m)},$$

$$b_5 = \frac{3a_3 - (2m+k)a_4 + (2km+m^2)a_5}{(1-k)(1-m)^2}, \quad \text{and} \quad b_6 = \frac{a_3 - a_4 m + a_5 m^2}{(1-m)^3}. \qquad (6.5.4)$$

Hence $b_6 \neq 0$ if $a_3 - a_4 m + a_5 m^2 \neq 0$.

On the other hand the initial line $t = \alpha x$ for (6.5.1) transforms to the initial line

$$t = \left(\frac{m-1}{k-1}\right)\left(\frac{\alpha k-1}{\alpha m-1}\right) \qquad (6.5.5)$$

and the two bits of data u, u_t for (6.5.1) reduce to two bits of data v and v_t for (6.5.3) on the line defined by (6.5.5).

Now, the characteristic equation of (6.5.3) is

$$b_3 - b_4\lambda + b_5\lambda^2 - b_6\lambda^3 = 0, \qquad \lambda = \frac{dx}{dt}, \qquad (6.5.6)$$

and the roots are:

$$\lambda_1 = \frac{1-m}{1-k}, \quad \lambda_2 = \frac{1}{2}\left(\frac{1-m}{1-k}\right)\left\{\frac{2a_3-(m+k)a_4+(m-k)\sqrt{a_4^2-4a_3a_5}}{a_3-a_4m+a_5m^2}\right\}$$

$$\text{and} \quad \lambda_3 = \frac{1}{2}\left(\frac{1-m}{1-k}\right)\left\{\frac{2a_3-(m+k)a_4-(m-k)\sqrt{a_4^2-4a_3a_5}}{a_3-a_4m+a_5m^2}\right\}. \tag{6.5.7}$$

Thus the characteristic lines of (6.5.3) are

$$x = \left(\frac{m-1}{k-1}\right)t, \quad x = \frac{1}{2}\left(\frac{m-1}{k-1}\right)\left\{\frac{2a_3-(m+k)a_4\pm(m-k)\sqrt{a_4^2-4a_3a_5}}{a_3-a_4m+a_5m^2}\right\}t.$$

By using the inverse of the transformation (6.5.2), these characteristic lines correspond to the lines

$$t = 0 \quad \text{and} \quad t = \left\{\frac{a_4-2a_5m\pm\sqrt{a_4^2-4a_3a_5}}{2a_3-a_4\pm\sqrt{a_4^2-4a_3a_5}}\right\}x$$

respectively. Thus (2) is proved. ■

Example 6.5.1 (KdV with skew data)

Consider the initial value problem:

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{6.5.8}$$

$$u(x,t) = g(x) \quad \text{on the line} \quad t = kx \; (k \neq 0) \tag{6.5.9}$$

under the nonsingular linear transformation

$$x \to x, \quad t \to \frac{t}{1-k} - \frac{k}{1-k}x, \quad \text{and} \quad u(x,t) = v(x,t) \tag{6.5.10}$$

(6.5.8) and (6.5.9) become

$$v_t + v_x + vv_x - \frac{k}{1-k}vv_t + v_{xxx} - \frac{3k}{1-k}v_{xxt} + \frac{3k^2}{(1-k)^2}v_{xtt}$$

$$- \frac{k^3}{(1-k)^3}v_{ttt} = 0, \tag{6.5.11}$$

$$v(x,0) = g(x). \tag{6.5.12}$$

The characteristic equation of (6.5.11) is

$$1 + \frac{3k}{1-k}\lambda + \frac{3k^2}{(1-k)^2}\lambda^2 + \frac{k^3}{(1-k)^3}\lambda^3 = 0, \qquad \lambda = \frac{dx}{dt}. \tag{6.5.13}$$

Thus $\lambda_1 = \lambda_2 = \lambda_3 = \frac{k-1}{k}$, i.e. (6.5.11) has only one characteristic curve which is given by

$$t = \frac{k}{k-1}x . \tag{6.5.14}$$

This line clearly corresponds to the line $t = 0$ in the original coordinates. Thus two cases arise

(1) if $k = 0$, then (6.5.8) belongs to the singular class.

(2) if $k \neq 0$, then (6.5.8) belongs to the nonsingular class.

We shall use this result later to prove well-posedness for the KdV equation not only for the usual data but also for skew data.

<u>Example 6.5.2</u> (BBM with skew data)

$$u_t + u_x + uu_x - u_{xxt} = 0 , \tag{6.5.15}$$

$$u(x,t) = g(x) \text{ on the line } t = kx, \ (k \neq 0). \tag{6.5.16}$$

Under the transformation (6.5.10), equations (6.5.15) and (6.5.16) become

$$v_t + v_x + vv_x - \frac{k}{1-k}vv_t - \frac{k}{1-k}v_{xxt} + \frac{2k}{(1-k)^2}v_{xtt} - \frac{k^2}{(1-k)^3}v_{ttt} = 0 \tag{6.5.17}$$

$$v(x,0) = g(x).$$

The characteristic equation of (6.5.17) has the roots

$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = \frac{k-1}{k}$.

Thus, (6.5.17) has two characteristic lines given by

$$x = \text{constant} \quad \text{and} \quad x = \frac{k-1}{k} t.$$

Hence we have the following:

(1) if $k = 0$, i.e., the data is usual, then the BBM belongs to the singular class.

(2) if $k \neq 0$ then, the BBM belongs to the nonsingular class.

Example 6.5.3 (Joseph Egri model)

$$u_t + u_x + uu_x + u_{xtt} = 0 \qquad (6.5.18)$$

$u(x,t)$ and $u_t(x,t)$ given on the line $t = kx$.

Proceeding as in the above examples one can show that:

(1) if $k = 0$, then the J.E. Model belongs to the singular class.

(2) if $k \neq 0$, then the J.E. Model belong to the nonsingular class.

Having classified the general class of equations into the two subclasses, namely, the singular and the nonsingular classes, we now look at the well-posedness of these classes.

6.6 Well-posedness of the nonsingular class

This section is devoted to the proof of the well-posedness of the nonsingular class defined by Definition 6.5.1. For this purpose we first establish integral formulae for the solution of the nonsingular class.

6.6.1 The integral formulae of the nonsingular class

It was proved in section 6.1, that the nonsingular class can be reduced to the semi-linear system of first order partial differential equations

$$U_t + AU_x + C = 0, \quad U(x,0) = H(x)$$

and it was shown that the latter system reduces to the normal form

$$V_t + DV_x + \tilde{C} = 0, \quad V(x,0) = \Psi(x)$$

where $D = \text{diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_{10})$ and $\tilde{C}$ is defined by (6.3.6).

Thus the ith component is

$$v_t^i + \lambda^i v_x^i = \tilde{\tilde{C}}^i, \quad v^i(x,0) = \psi^i(x), \quad i = 1, 2, \ldots \quad (6.6.1)$$

where $\tilde{\tilde{C}} = -\tilde{C}$. Along characteristics, the equations (6.6.1) are ordinary differential equations, since the differentiation is now in one direction only. This is the clue to establishing the integral formula.

Definition 6.6.1 (Domain of determinacy)

Consider the linear or semi-linear system $U_t + AU_x + C = 0$. The domain of determinacy for this system is defined to be the set of all points $p(x,t)$ which can be connected to the initial interval by characteristic trajectories.

Now, if $p(x,t)$ is any point in the domain of determinacy of the system (6.6.1), then integrating along the characteristic PQ_i we have

$$\overset{i}{v}(p) = \overset{i}{v}(Q^i) + \int_{Q_i}^{p} \overset{\approx}{C}(\overset{i}{v})d\eta \qquad (6.6.2)$$

where Q_i are those points on the initial intervals, connected to p by the ith characteristic, $i = 1, 2, \ldots$. Equation (6.6.2) gives

$$v^i(p) = \Psi^i(Q_i) + \int_{Q_i}^{p} \overset{\approx}{C}(\overset{i}{v})d\eta \qquad (6.6.3)$$

which is integral fomulae of the underlying system.

6.6.2 Uniqueness

To prove the uniqueness of the solution of the system

$$U_t + AU_x + C = 0, \qquad U(x,0) = H(x)$$

where A and C are given by (6.1.10c,b) it is important to note that it can be re-written in the form

$$U_t + AU_x + BU = 0, \qquad U(x,0) = H(x). \qquad (6.6.4)$$

Since by the expression of C it can be shown that $C = BU$, where U is given by (6.1.10a) and

$$B = \left[\begin{array}{ccc|cc|ccccc}
 & & & 0 & 0 & & & & & \\
 & O(3\times3) & & -1 & 0 & & & O(5\times3) & & \\
 & & & 0 & -1 & & & & & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{a_1q+a_2p}{a_6} & 0 & \frac{1+a_2u}{a_6} & \frac{1+a_1u}{a_6} & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & 0 & 0 & & & & & \\
 & O(3\times3) & & 0 & 0 & & & O(5\times3) & & \\
 & & & 0 & 0 & & & & &
\end{array}\right]. \qquad (6.6.5)$$

Lemma 6.6.1 [3]

If $W(x,t)$ is a solution of the linear system

$$W_t + AW_x + BW = 0, \qquad W(x,0) = 0 \tag{6.6.6}$$

where A is symmetric, then $W = 0$. □

The proof of this lemma is left to appendix C.

Using the result of the above lemma, then the uniqueness of the solution of the original system (6.6.4) can be proved.

Theorem 6.4

If U is a solution of the semi-linear system (6.6.4) then U is unique. □

Proof

As was demonstrated in section 6.3, the semi-linear system (6.6.4) can be reduced, by a nonsingular linear transformation to the normal form

$$V_t + DV_x + \bar{C}(V) = 0, \qquad V(x,0) = \Psi(x) \tag{6.6.7}$$

where $U = TV$, $D = \text{diag.}(\lambda_1, \ldots, \lambda_{10})$, $\bar{C} = T^{-1}BT$ and λ_i are the eigenvalues of the matrix A. Hence to prove the uniqueness of the system (6.6.4) it suffices, without loss of generality, to prove that the solution of (6.6.7) is unique.

Let V_1 and V_2 be two solutions of (6.6.7) and $W = V_1 - V_2$, then W satisfies

$$W_t + DW_x + \bar{C}(V_1) - \bar{C}(V_2) = 0, \quad W(x,0) = 0. \tag{6.6.8}$$

Using the mean-value theorem, we have

$$\bar{C}(V_1) - \bar{C}(V_2) = K(V_1, V_2)(V_1 - V_2)$$

$$= K(V_1, V_2)W.$$

Then (6.6.8) reduces to

$$W_t + DW_x + K(V_1, V_2)W = 0, \quad W(x,0) = 0. \tag{6.6.9}$$

The latter system is a linear system with D diagonal and K does not depend on W. Now, since $W(x,0) = 0$ then by using lemma 6.6.1.

$$W(x,t) = 0, \quad \text{i.e.,} \quad V_1 \equiv V_2.$$

Consequently the solution of the system (6.6.4) is unique. ∎

Example 6.5.4

$$u_t + u_x + uu_x - u_{xxt} + u_{ttt} = 0, \tag{6.6.10a}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{and} \quad u_{tt}(x,0) = h(x). \tag{6.6.10b}$$

The initial value problem (6.6.10a) and (6.6.10b) clearly belongs to the nonsingular class. To prove that the solution of this problem is unique we reduce the problem into a system of first order partial differential equations.

Using the procedures, introduced in section 6.1, (6.6.10) reduces to the semi-linear system

$$U_t + AU_x + C = 0, \quad U(x,0) = G(x), \tag{6.6.11}$$

where,

$$A = \left[\begin{array}{c|c|c} 0(4\times4) & \begin{matrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{matrix} & 0(4\times4) \\ \hline \begin{matrix}0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\end{matrix} & \begin{matrix}0 & 0\\ -1 & 0\end{matrix} & \begin{matrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{matrix} \\ \hline 0(4\times4) & \begin{matrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\end{matrix} \end{array}\right]. \qquad (6.6.12a)$$

$$U^T = [u\ p\ q\ r\ s\ \tau\ v\ w\ \mu\ \nu], \qquad (6.6.12b)$$

with u, p, q, r, s, τ, v, w, μ and ν as in (6.1.3) and

$$C^T = [-p\ \ -r\ \ -s\ \ -v\ \ 0\ \ 0\ \ j\ \ 0\ \ 0\ \ 0]. \qquad (6.6.12c)$$

with $j = qp + r + (1+u)s$.

Similarly, the initial data (6.6.10b) reduces to

$$\psi^T = [f(x)\ \ g(x)\ \ f'(x)\ \ h(x)\ \ g'(x)\ \ f''(x)\ \ h'(x)\ \ g''(x)\ \ f'''(x)\ G] \qquad (6.6.13)$$

with $G = G(f, g, f', h, g', f'', h', g'', f''')$.

The eigenvalues of A are

$\lambda = 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1$ and the corresponding eigenvectors are as follows:

$\lambda = 0$; e_i are the usual basis vectors for R^8 e.g.,

$$e_i = [1,\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T, \qquad i = 1, \ldots, 8$$

$$\lambda = 1;\quad e_9 = [0\ 0\ 0\ 0\ 0\ 0\ 1\ {-1}\ 1\ {-1}]^T$$

$$\lambda = -1;\quad e_{10} = [0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1]^T.$$

Now, denote by T to the matrix whose columns are e_i, $i = 1, 2, \ldots, 10$, i.e.,

$$T = \left[\begin{array}{c|ccccc} \text{diag}(1,\ 1,\ 1,\ 1,\ 1) & & & 0(5\times5) & & \\ \hline & 1 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 1 & 1 \\ 0(5\times5) & 0 & 0 & 1 & -1 & 1 \\ & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 0 & -1 & 1 \end{array}\right]. \qquad (6.6.14)$$

Then,

$$T^{-1} = \left[\begin{array}{c|ccccc} \text{diag}(1,\ 1,\ 1,\ 1,\ 1) & & & 0(5\times5) & & \\ \hline & 1 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & -1 & 0 \\ 0(5\times5) & 0 & 0 & 1 & 0 & -1 \\ & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]. \qquad (6.6.15)$$

Now, if we subject the system (6.6.11) to the nonsingular transformation $U = TV$ where T is given by (6.6.14), the system reduces to its characteristic form, i.e.,

$$V_t + DV_x + \tilde{C} = 0, \qquad V(x,0) = \Phi(x), \tag{6.6.16}$$

where $D = \mathrm{diag}(0, 0, 0, 0, 0, 0, 0, 0, 1, -1)$ and

$$\tilde{C} = [-p \quad -r \quad -s \quad -v \quad 0 \quad 0 \quad (qp + r + (1+u)s) \quad 0 \quad 0 \quad 0]^T. \tag{6.6.17}$$

Hence, to prove the uniqueness of the original system (6.6.11) it suffices to prove the uniqueness of (6.6.16). Thus, let V_1 and V_2 be two solutions of (6.6.16) and $W = V_1 - V_2$. Then W satisfies the initial value problem:

$$W_t + DW_x + \tilde{C}(V_1) - \tilde{C}(V_2) = 0, \qquad W(x,0) = 0. \tag{6.6.18}$$

By using the definition of $\tilde{C}$ from (6.6.17) and the relations:

$$q_1p_1 - q_2p_2 = (q_1 - q_2)p_1 + (p_1 - p_2)q_2 = qp_1 + pq_2 \quad \text{and}$$

$$u_1s_1 - u_2s_2 = (u_1 - u_2)s_1 + (s_1 - s_2)u_2 = us_1 + su_2,$$

where $p_1 = v_{1t}$, $p_2 = v_{2t}$, $p = w_t$ ... etc, then

$$\tilde{C}(V_1) - \tilde{C}(V_2) = [-p \quad -r \quad -s \quad -v \quad 0 \quad 0 \quad k \quad 0 \quad 0 \quad 0]^T$$

with $k = q_1p_1 - q_2p_2 + r_1 - r_2 + u_1s_1 - u_2s_2$

$$= qp_1 + pq_2 + r + us_1 + su_2.$$

Hence $\tilde{C}(V_1) - \tilde{C}(V_2) = BW$, where

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & q_2 & p_1 & 1 & u_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e., the system (6.6.18) reduces to

$$W_t + DW_x + B(V_1, V_2)W = 0, \qquad W(x,0) = 0.$$

Since this system is linear in W and the matrix D is symmetric and $W(x,0) = 0$, the hypotheses of lemma (6.6.1) are satisfied and it follows that the system (6.6.18) has at most one solution. Consequently, the original equation (6.6.10) has at most one solution also.

6.6.3 Existence

The existence theory for hyperbolic systems of quasi-linear partial differential equations

$$U_t + AU_x + C = 0, \qquad U(x,0) = H(x)$$

is an old problem and has been studied by many people [4], [5], [11], ... For the analytic problem, i.e. when A and C are analytic in x and t and H is analytic in x, then the solution exists and depends continously on the data in the small (i.e., for suitably narrow neighbourhood of $x = 0$, $t = 0$) by the Cauchy-Kowalewsky theorem. This result was extended by Lax [13] who considered the quasi-linear system i.e. the system in which linearity and semi-linearity of the systems are special cases. By using a priori estimates of the solutions and an iterative scheme, Lax was able to show firstly that for analytic data the solution exists not only in the small but it can be continued analytically until it reaches the boundary of the domain of analyticity. Secondly, by approximating a non-analytic problem by a sequence of analytic problems and using the above results, the solution of a non-analytic initial value problem which is now a generalized solution is shown to exist. Lax proved that if all the matrices A, C and T (where T is the matrix of eigen-vectors of A) have continuous first derivatives and the first derivative of $H(x)$ is almost everywhere continuous, the first derivatives of the generalized solution are continuous at all regular points of the system, i.e., points that do not lie on characteristics through points of discontinuity of the initial data.

We now turn to prove existence for the semi-linear system (6.6.4) which has the normal form

$$V_t + DV_x + \vec{C} = 0, \qquad V(x,0) = \Psi(x) \qquad (6.3.7)$$

where $D = \text{diag}\,(\lambda_1, \lambda_2, \ldots)$.

To prove existence for (6.3.7) the following lemma is needed:

Lemma 6.6.2

The system of differential equations (6.3.7) can be replaced equivalently by a system of nonlinear integral equations. □

Proof

Let $D_k = \frac{\partial}{\partial t} + \lambda_k \frac{\partial}{\partial x}$ in the kth component of (6.3.7) then D_k can be regarded as differentiation along the characteristic C_k. Thus, by similar arguments as were used to derive the integral formulae (6.6.3), the system (6.3.7) corresponds to the nonlinear integral equations

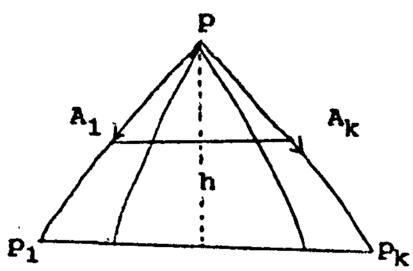
$$v = Lv, \tag{6.6.19a}$$

$$Lv^k(\xi, \tau) = \psi^k(x_k) + \int_0^{\tau} \tilde{c}^k(x_k, \eta, v)d\eta \tag{6.6.19b}$$

which proves the lemma. ■

Before introducing the theorem which guarantees the existence of the solution of (6.3.7) we define the region in which the existence proof is valid.

Let H be a closed domain in the x, t space in which all the characteristics C_i followed from a point p in H backwards in t meet a given section J of the initial data line $t = 0$ in the points p_i, as in figure 6.2.



(Fig. 6.2)

Let S be the set of all functions v with domain H having continuous derivatives and equal to $\Psi(x)$ on $t = 0$. Finally, we defined the norm of the elements of S to be the largest value of the functions attained in the closed domain H. However, if we chose $||\Psi(x)|| = N$ and restrict admissible functions in S by choosing $||v|| \leq 2N$, then there exists a common upper bound $\mu > 0$ such that [3]:

$$||\tilde{C}_v^k|| < \mu, \quad ||\tilde{C}_v^k|| < \mu, \quad ||\tilde{C}_x^k|| < \mu \quad \text{and} \quad ||\tilde{C}_t^k|| < \mu, \qquad (6.6.20)$$

where $\tilde{C}_v^k$ is the functional gradiant of $\tilde{C}^k$ with respect to v.

Note that $\tilde{C}_x^k = \tilde{C}_t^k = 0$ for the system (6.6.4).

Now, we introduce the following theorem:

<u>Theorem 6.5</u>

Let $\Psi(x)$, $\tilde{C}$ have continuous first derivatives, then the system

$$V_t + DV_x + \tilde{C} = 0, \qquad V(x,0) = \Psi(x) \qquad (6.3.7)$$

possesses a solution which has the same differentiability as $\Psi(x)$. □

<u>Proof</u>

If we choose h sufficiently small, then (6.6.19) implies that $||v^k|| \leq ||\Psi(x)|| + \mu h = N + \mu h \leq 2N$.

The system (6.6.19) lends itself immediately to a process of solution by iteration and for a suitably narrow strip H_h the desired fixed element will be constructed as the uniform limit, as $n \to \infty$ of $v_{n+1} = Lv_n$, starting with $v_o(x,t) = \Psi(x)$. For doing this we prove that the operator L in (6.6.19) is contracting in the supremum norm.

Let V_1 and V_2 be two elements in S, then (6.6.19) yields

$$LV_1 - LV_2 = \int_0^{\tau} [\tilde{C}(x,\tau,V_1) - \tilde{C}(x,\tau,V_2)d\tau \qquad (6.6.21)$$

$$= \int_0^{\tau} \tilde{C}_V(x,\tau,\bar{V})(V_1-V_2)d\tau \text{ (using the mean value theorem)}$$

where $\bar{V}$ is the intermediate value. Thus

$$||LV_1 - LV_2|| \leq \mu h||V_1-V_2||. \qquad (6.6.22)$$

If h is small enough such that $\mu h < \theta < 1$, then L is a contraction operator in the supremum norm.

Similarly, if $Z_n = V_{n+1} - V_n$, then

$$||Z_n|| < \theta||Z_{n-1}||, \quad 0 < \theta < 1 \qquad (6.6.23)$$

i.e. $Z_n \to 0$ as $n \to \infty$ uniformly in the strip H_h. Thus the sequence $\{V_n\}$ converges uniformly to a continuous function V in S and clearly has the initial value $\Psi(x)$. Hence, by using the fixed point theorem V is a solution of the integral equations (6.6.19). Furthermore, V solves the sytem (6.3.7) in the normal form since the directional differential operator on the integral in (6.6.19) produces the integrand.

We must still show that the solution $V(x,t)$ has continuous first derivatives with respect to x and t. To prove this it is enough to show that V has, at all points, continuous first derivatives in the characteristic direction and with respect to x, since the t-derivatives follows from the known directional (characteristic) derivatives.

Now, the existence and continuity of V in the characteristic direction follows directly from the system (6.6.19) and from the continuity of the solution obtained. To prove the existence and continuity of the derivatives $\frac{\partial V}{\partial x}$ we observe, first of all, that the assumed continuous differentiability of $\Psi(x)$ and $\tilde{C}$ implies that all the approximations constructed in proving the existence of a solution, have continuous derivative with respect to x.

Differentiating the $(n+1)$ st approximation,

$$V_{n+1}(\xi,\tau) = \Psi(x(0,\tau,\xi)) + \int_0^{\tau} \tilde{C}(x,t,V_n)d\eta$$

with respect to ξ. Thus

$$\frac{\partial V_{n+1}}{\partial \xi} = \Psi'(x(0,\tau,\xi)\,\frac{\partial \xi}{\partial x} + \int_0^{\tau}(\frac{\partial \tilde{C}}{\partial V_n}\,\frac{\partial V_n}{\partial x}\,\frac{\partial x}{\partial \xi} + \frac{\partial \tilde{C}}{\partial x}\,\frac{\partial x}{\partial \xi})d\eta$$

$$= \Psi'\frac{\partial \xi}{\partial x} + \int_0^{\tau} x_{\xi}\,[\tilde{C}_V V_x + \tilde{C}_x]d\eta. \qquad (6.6.24)$$

Similar to the assumption made about the system (6.6.19) we can prove the uniform convergence of the sequence $\{\frac{\partial V_n}{\partial x}\}$ (x instead of ξ), $n = 1, 2, \ldots,$ by using the same method which we used to prove the convergence of $\{V_n\}$. This give us $\lim_{n\to\infty} \frac{\partial V_n}{\partial x} = \frac{\partial V}{\partial x}$, which suffices to prove the existence of the solution of the characteristic system (6.3.7) <u>locally</u>. To show that the solution exists <u>globally</u>, i.e. in a larger region, we use the line $t = h$ as new initial line and solve the problem by the same procedures, as above, in the strip $h < t < 2h$. We continue stepwise in this way which implies the existence of the solution in an arbitrary large t so long as the assumption of the continuity and boundedness remains satisfied. ■

The existence of the original system

$$U_t + AU_x + C = 0, \qquad U(x,0) = G(x)$$

is, then, obtained from the above theorem since this system reduces equivalently to the system in the last theorem as in section 6.3.

6.6.4 Continuous dependence of the solution on the initial data

Theorem 6.6.

Let $U(x,t)$ and $W(x,t)$ be two solution of (6.3.7), such that $U(x,0) = \Psi(x)$, $W(x,0) = \Phi(x)$ and $||\Phi - \Psi|| < \delta$. Then $||w - u|| < \epsilon$ and $\epsilon \to o$ as $\delta \to o$ (where $||.||$ is the supremum norm defined in the previous section). □

Proof

Let $\Phi(x) - \Psi(x) = \alpha(x)$, where $||\alpha(x)|| < \delta$, and $U(x,t) - W(x,t) = Z(x,t)$. Then, as in theorem 6.5 Z satisfies the integral equation

$$Z(x,t) = \delta(x) + \int_0^{T} \tilde{C}_V(x,\eta,V)(U-W)d\eta$$

$$= \delta(x) + \int_0^{T} \tilde{C}_V(x,\eta,V)Z(x,\eta)d\eta \qquad (6.6.25)$$

(where V is intermediate value).

Let $\max_{x,t\in S} |Z(x,t)| = \epsilon$, then by estimates analogous to that used in the existence proof

$$\epsilon < \delta + \epsilon T \mu , \quad (||\tilde{C}|| < \mu). \qquad (6.6.26)$$

Replacing Z in the integral equation (6.6.25) by the right hand side of (6.6.26) and repeating the procedure, we obtain

$$\epsilon < \delta(1 + \mu\tau) + \epsilon \frac{\mu^2\tau^2}{2}.$$

Repeating this aperation n times we have

$$\epsilon < \delta[1 + \mu\tau + \frac{\mu^2\tau^2}{2!} + \dots + \frac{\mu^{n-1}\tau^{n-1}}{(n-1)!}] + \epsilon \frac{\mu^n\tau^n}{n!}.$$

Now, as $n \to \infty$, we get

$$\epsilon < \delta\, e^{\mu t} .$$

Thus if t is bounded, then $\delta \to o$ implies $\epsilon \to o$ which proves the theorem. ▩

6.7 Well-posedness of the singular class

The singular class was defined in section 6.5 as the complement of the nonsingular class in $W \cap S$ in the sense of the capability of the method of characteristics to ensure the well-posedness of the problem. That is, in the singular class, the technique used in the previous section is no longer applicable.

To study the well-posedness of the singular class we recall the reduction to equivalence classes obtained in section 5.2. This reduction not only reduces the singular class to the four equivalence classes, KdV, RLW, W_{54} and W_{53}, but it also reduces the prescribed data from being characteristic to the usual data as in the following theorem:

Theorem 6.7

Consider the initial value problem corresponding to the general class of equations

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0 \tag{6.7.1}$$

where the initial data u, u_t and u_{tt} are given on a characteristic line $x = mt$, $m \neq 0$. Then, this problem reduces to the four equivalence classes KdV, RLW, W_{54}, and W_{53} classes, i.e.,

$$\begin{aligned}
&v_t + v_x + c_1 vv_x + c_2 vv_t + c_3 v_{xxx} = 0 , \\
&v_t + v_x + d_1 vv_x + d_2 vv_t + d_4 v_{xxt} = 0 , \\
&v_t + v_x + \gamma_1 vv_x + \gamma_2 vv_t + \gamma_4 v_{xxt} + \gamma_5 v_{xtt} = 0 , \\
&v_t + v_x + \delta_1 vv_x + \delta_2 vv_t + \delta_3 v_{xxx} + \delta_5 v_{xtt} = 0 ,
\end{aligned} \tag{6.7.2}$$

respectively, and the corresponding characteristic data u, u_t and u_{tt} reduce to v, v_t and v_{tt} on $t = 0$. □

Proof

Since $x = mt$ is a characteristic of (6.7.1), then m satisfies the equation

$$a_3 - a_4\lambda + a_5\lambda^2 - a_6\lambda^3 = 0.$$

Now, consider the nonsingualr linear transformation

$$x \to \bar{x} = \frac{x}{1-n} - \frac{nt}{1-n} , \quad t \to \bar{t} = \frac{x}{1-m} - \frac{mt}{1-m} \text{ and}$$

$$u(x,t) \to v(\bar{x},\bar{t}) = u(x,t).$$

The reduction of equation (6.7.1) to the equations of the four equivalence classes KdV, RLW, W_{54}, and W_{53} is ensured by the reduction theorem 5.2. Clearly the characteristic line $x = mt$ transforms to the line $t = 0$ and the data on this line are obtained from the following:

$$u(x,\frac{x}{m}) = v(\bar{x},0) ,$$

$$u_t(x,\frac{x}{m}) = \frac{-n}{1-n} v_{\bar{x}} (\bar{x},0) - \frac{m}{1-m} v_{\xi}(\bar{x},0), \qquad (6.7.3)$$

$$u_{tt}(x,\frac{x}{m}) = \frac{n^2}{(1-n)^2} v_{\bar{x}\bar{x}} + \frac{2mn}{(1-n)(1-m)} v_{\bar{x}\xi} (\bar{x},0) + \frac{m^2}{(1-m)^2} v_{\xi\xi} (\bar{x},0).$$

This completes the proof. ■

Remarks

(1) In the proof of the above theorem if $m = \frac{a_1}{a_2}$, then according to the Corollary 5.3.1 the general class reduces to the simple four equivalence classes, i.e., with the disappearance of the uu_t term.

(2) It is seen from the result of the above theorem that the reduction reduces the t-derivatives by at least one so that at least one bit of data becomes redundant. If the transformed data of each of the four classes are consistent, then well-posedness of the singular subset of the general class is guaranteed if we can prove that the set of equation (6.7.2) are well-posed for data on $t = 0$.

Thus, to investigate the well-posedness of the singular class it suffices to study the well-posedness of each of the four classes in (6.7.2).

6.7.1 The RLW class

The initial value problem corresponding to the RLW equation

$$v_t + v_x + vv_x - v_{xxt} = 0, \quad v(x,0) = g(x) \qquad (6.7.4)$$

has been studied in chapter 4 where its well-posedness is ensured by theorems 4.7 to 4.10.

Now, the RLW equation generates a subset of the singular class, i.e., the simple RLW class and the data $v(x,0)$ generates one bit of data for this subset. But according to the Cauchy problem two more are needed for this subset corresponding to $v_t(x,0)$ and $v_{tt}(x,0)$. If v, v_t and v_{tt} are consistent (i.e. can be generated from the solution of (6.7.4)) then the RLW being well-posed, leads to the well-posedness of this subset.

The analysis above was restricted to the simple RLW subclass i.e., no uu_t term. A similar analysis can be done if uu_t is present. Thus consider the initial value problem:

$$u_t + u_x + uu_x + \epsilon uu_t - u_{xxt} = 0, \quad u(x,0) = f(x). \qquad (6.7.5)$$

To study the existence of a solution of (6.7.5) it is convenient to establish the integral formula of the solution.

Integral formula of solution of (6.7.5)

Re-writing (6.7.5) in the form

$$(1 - \frac{\partial^2}{\partial x^2})u_t = -(u_x + uu_x + \epsilon uu_t). \qquad (6.7.6)$$

The left hand side is an ordinary differential equation in u_t.

Thus, by using the Green's function of the differential operator $(1 - \frac{\partial^2}{\partial x^2})$, which was establish in chapter 4, (6.7.6) reduces to

$$u_t = \int_{-\infty}^{\infty}[\tfrac{1}{2} e^{-|x-\xi|}(u + \frac{u^2}{2})_\xi + (\epsilon(\frac{u^2}{2})_t)]d\xi.$$

Integrating once with respect to t between 0 and t, then

$$u = g(x) + \int_0^t \int_{-\infty}^{\infty} k(x-\xi)[u + \frac{u^2}{2}]d\xi d\tau - \epsilon \int_{-\infty}^{\infty} k(x-\xi)\frac{u^2}{2}d\xi$$

$$+ \epsilon \int_{-\infty}^{\infty} k(x-\xi)\frac{g^2}{2}(\xi)d\xi$$

$$= g(x) + \epsilon \int_{-\infty}^{\infty} k(x-\xi)\frac{g^2}{2}(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi)[u(\xi,\tau)$$

$$+ \tfrac{1}{2}u^2(\xi,\tau)]d\xi d\tau - \epsilon \int_{-\infty}^{\infty} k(x-\xi)\frac{u^2}{2}(\xi,t)d\xi,$$

$$= \phi(x) + \int_0^t \int_{-\infty}^{\infty} k(x-\xi)[u + \tfrac{1}{2}u^2]d\xi d\tau - \epsilon \int_{-\infty}^{\infty} k(x-\xi)\frac{u^2}{2}(\xi,t)d\xi, \quad (6.7.7a)$$

where

$$k(Z) = \tfrac{1}{2}\,\mathrm{sgn}(Z)e^{-|Z|} \quad \text{and} \quad \phi(x) = g(x) + \epsilon \int_{-\infty}^{\infty} k(x-\xi)\frac{g^2}{2}(\xi)d\xi,$$

i.e.,

$$u = AU = \phi(x) + BU \quad (6.7.7b)$$

which is the integral formula of (6.7.5) where

$$BU = \int_0^t \int_{-\infty}^{\infty} k(x-\xi)[u(\xi,\tau) + \tfrac{1}{2}u^2(\xi,\tau)]d\xi d\tau - \epsilon\int_{-\infty}^{\infty} k(x-\xi)\frac{u^2}{2}(\xi,t)d\xi.$$

We now turn to prove the existence of a solution of (6.7.7).

Existence of solution of (6.7.7)

Lemma 6.7.1

Let $g(x)$ be a continuous function such that

$$\sup_{x\in R} |g(x) + g^2(x)| < b < \infty$$

then there exists a $t_o(b) > 0$ and an $\epsilon(b)$ such that the integral equation (6.7.7) has a solution $u(x,t)$, satisfying $u(x,0) = g(x)$ which is bounded and continuous for $x \in R$, $t \in [0, t_o]$. □

Proof

Let ℓ_T be the space of all continuous and bounded functions with norm defined by

$$||u||_{\ell_t} = \sup_{\substack{-\infty<x<\infty \\ 0\leq t\leq T}} |u(x,t)| .$$

Suppose u_1 and u_2 are two elements of ℓ_T such that $||u||_{\ell_T} < r$, then

$$|AU_1 - AU_2| \leq |BU_1 - BU_2|$$

$$\leq \sup_{x,t}|u_1 - u_2|[1 + \tfrac{1}{2}|u_1 + u_2|] \int_0^t\int_{-\infty}^{\infty}|k(x-\xi)d\xi d\tau$$

$$+ |\epsilon| \sup_{x,t}|u_1 - u_2||u_1 + u_2| \int_{-\infty}^{\infty}|k(x-\xi)d\xi$$

$$\leq ||u_1 -u_2||_{\ell_t}[1 + \tfrac{1}{2}||u_1 + u_2||_{\ell_t}]t$$

$$+ |\epsilon|||u_1 - u_2||_{\ell_t}||u_1 + u_2||_{\ell_t}.$$

Taking the supremum of both sides in the strip $R \times [0,t_o]$ we have,

$$||Au_1 - Au_2||_{\ell_{t_o}} \leq ||u_1 - u_2||_{\ell_{t_o}}[1 + \tfrac{1}{2}||u_1||_{\ell_{t_o}} + \tfrac{1}{2}||u_2||_{\ell_{t_o}}]t_o \qquad (6.7.8)$$

$$+ |\epsilon|\,||u_1 - u_2||_{\ell_{t_o}}[||u_1||_{\ell_{t_o}} + ||u_2||_{\ell_{t_o}}].$$

Thus,

$$||Au_1 - Au_2||_{\ell_{t_o}} \leq [(1+r)t_o + 2r|\epsilon|]\,||u_1 - u_2||_{\ell_{t_o}}$$

which implies that A is continuous mapping of the space ℓ_{t_o} into itself. Morever, the ball $||u||_{\ell_{t_o}} < r$ satisfies a Lipschitz condition with Lipschitz constant $\theta < 1$ if

$$(1+r)t_o + 2r|\epsilon| < \theta < 1. \qquad (6.7.9)$$

Also in the above calculation if $u_2 = 0$ and $u_1 = u$, then

$$||Bu||_{\ell_{t_o}} \leq \theta\,||u||_{\ell_{t_o}}.$$

Morever, the ball is mapped into itself if

$$b < (1-\theta)r.$$

Thus A is contractive operator. Hence according to the fixed point theorem on Banach spaces, A has a fixed point u in the ball $||u||_{\ell_{t_o}} \leq r$ which is a solution of (6.7.7). ∎

Note that the inequality (6.7.9) restricts the amplitude of solutions for which existence is guaranteed. Specifically we have that a necessary condition is that

$$r < \frac{1}{2}\epsilon .$$

Now, using the same proof as was given to prove lemma 4.5.2, the following theorem is proved.

<u>Theorem 6.8</u>

If $g \in C^2(R)$. Then any solution of (6.7.7) which is an element of $\mathcal{C}_T$ (for a given $T > 0$) is also an element of $\mathcal{C}_T^{2,\infty}$. □

Hence the result from theorem 6.8 implies that the solution of the integral equation (6.7.7) has sufficient regularity to be a classical solution of the initial value problem (6.7.5) in the infinite strip $R \times [0,t_0]$.

The above theorem means that in the subset of the singular class corresponding to the RLW (with the presence of uu_t term) a solution exists at least locally.

6.7.2 <u>The KdV class</u>

In the simple KdV class (i.e., no uu_t term)

$$v_t + v_x + vv_x + v_{xxx} = 0, \quad v(x,0) = g(x) \qquad (6.7.10)$$

by adding the term $-\epsilon v_{xxt}$ and using the method of regularization we have shown in chapter 3 that the initial value problem (6.7.10) is well-posed.

Now, this result is used to generate well-posedness of the corresponding subset of the general class under certain data as follows:

Theorem 6.9

Consider the general singular subclass

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0 \tag{6.7.11a}$$

where a_1/a_2 is a triple root of the cubic equation

$$a_3 - a_4 m + a_5 m^2 - a_6 m^3 = 0 \tag{6.7.11b}$$

and u, u_t, u_{tt} are prescribed on the characteristic line

$$x = \frac{a_1}{a_2} t$$

if the corresponding three data v, v_t, v_{tt} are consistent then the problem is well-posed. □

Proof

Using corollary (5.3.1) the equation (6.7.11a) reduces to the KdV equation and morever, the characteristic data u, u_t, u_{tt} reduces to three bits of data v, v_t, v_{tt} on $t = 0$. Hence, if $v(x,0), v_t(x,0)$ $v_{tt}(x,0)$ are consistent then since the KdV is well-posed it follows that the theorem is proved. ■

If the triple root of (6.7.11b) is not $\frac{a_1}{a_2}$ then (6.7.11a) reduces to the KdV in the general sense, it includes vv_t term. Hence, the well-posedness of this problem is needed. Unfortunately the method of regularization cannot be applied here and the reason is that there are not enough conservation laws to estimate the bounds of the corresponding regularized problem.

6.7.3 The Class W_{54}

In this section we carry out an existence proof for solutions of this class. This proof is unfortunately not very strong since it does not lead to existence of global solutions in both x and t. We are still trying to sort out the difficulties. So, until then we introduce the proof below:

Consider the specific initial value problem (since the general form is clearly treated similarly)

$$u_t + u_x + uu_x + uu_t + u_{xxt} + u_{xtt} = 0, \quad -\infty < a < x < b < \infty, \quad t > 0$$

$$u(x,0) = g(x), \quad u_t(x,0) = h(x). \tag{6.7.12}$$

Equations (6.7.12) correspond to the system

$$u_t + u_x = v(x,t), \qquad u(x,0) = g(x) \tag{6.7.13a}$$

$$v_{xt} + (1+u)v(x,t) = 0, \quad v(x,0) = h(x) + g'(x). \tag{6.7.13b}$$

Equations (6.7.13a) can be solved to give an expression for u in terms of v. For this purpose subjecting (6.7.13a) to the transformation:

$$x \to \xi = x - t, \qquad t \to \eta = t, \qquad u(x,t) \to u(\xi,\eta) \text{ and}$$

$$v(x,t) \to v(\xi,\eta) \tag{6.7.14}$$

we have

$$u_\eta = v(\xi,\eta) \qquad u(\xi,0) = g(\xi). \tag{6.7.15}$$

Since the transformation (6.7.14) is a nonsingular linear transformation then, to find the solution of (6.7.13a) and prove its existence it suffices to prove that the solution of (6.7.15) exsits. To do this we integrate (6.7.15) with respect to η and obtain:

$$u(\xi,\eta) = g(\xi) + \int_0^{\eta} v(\xi,\eta)d\tau. \qquad (6.7.16)$$

Hence

$$|u| \leq |g| + \sup_{0\leq\eta\leq T} |v(\xi,\eta)|\eta, \qquad 0 < \eta < T. \qquad (6.7.17)$$

Taking the supremum of both sides with respect to ξ and η and using the nonsingularity of (6.7.14), we have

$$||u||_{\ell_T} \leq T||v||_{\ell_T} + M \qquad (6.7.18)$$

where ℓ_T is the function space of all continuous and bounded functions on $a \leq x \leq b$, $t \geq 0$, defined in section 6.7.1 and $\sup_{\xi} |g(\xi)| \leq M$. Hence, the following lemma is proved:

Lemma 6.7.2

If $g(\xi)$ is bounded and $v(\xi,\eta)$ exists and belongs to the space ℓ_T then $u(\xi,\eta)$ exists and is bounded. □

Since the transformation (6.7.14) is nonsingualar then, the solution of (6.7.13a) now exists under the same assumptions of the above lemma.

We turn now to prove that v, the solution of (6.7.13b) exists. For doing this we shall use the relation (6.7.18) since the nonsingularity of the transformation (6.7.14) provides similar relation for $u(x,t)$. Thus, integrating (6.7.13b) with respect to x and t, $a \leq x \leq b$, $0 \leq t \leq T$, we have

$$v(x,t) = v(x,0) - \int_0^t \int_a^x (1+u)v(\xi,\tau)d\xi d\tau \tag{6.7.19a}$$

$$= Av = g'(x) + h(x) + Bv \tag{6.7.19b}$$

with $Bv = - \int_0^t \int_a^x (1+u)v(\xi,\tau)d\xi d\tau.$

<u>Lemma 6.7.3</u>

If $\sup_{a\leq x\leq b} |g'(x)| \leq L$ and $\sup_{a\leq x\leq b} |h(x)| \leq N$, there exists a T depending on L and N such that (6.7.19) has a solution satisfying $v(x,0) = g'(x) + h(x)$. □

<u>Proof</u>

Let $v_1, v_2 \in \mathcal{C}_T$ such that $||v_i|| \leq R$, $i = 1, 2$, then,

$|Av_1 - Av_2| = |Bv_1 - Bv_2|$

$$\leq \left| \int_0^t\int_a^x \{(1+u_1)v_1 - (1+u_2)v_2\}d\xi d\tau\right|$$

$$= \left| \int_0^t\int_a^x \{(v_1-v_2) + \tfrac{1}{2} [(u_1+u_2)(v_1-v_2) + (u_1-u_2)(v_1+v_2)]\}d\xi d\tau\right|$$

$$\leq \left| \int_0^t\int_a^x (v_1-v_2)[1 + \tfrac{1}{2}(u_1+u_2)]d\xi d\tau\right| + \left| \int_0^t\int_a^x \tfrac{1}{2}(u_1-u_2)(v_1+v_2)d\xi d\tau\right|$$

$$\leq \sup_{x,t} \{|v_1-v_2|\ |1 + \tfrac{1}{2}(u_1+u_2)|\ (b-a)t$$

$$+ \sup_{x,t} \tfrac{1}{2}\ |u_1-u_2|\ |v_1+v_2|\ (b-a)t$$

$$\leq ||v_1-v_2||_{\mathcal{C}_T} \{1 + \tfrac{1}{2}\ ||u_1+u_2||_{\mathcal{C}_T}\}\ (b-a)t$$

$$+ \tfrac{1}{2}\ ||u_1-u_2||_{\mathcal{C}_T}\ ||v_1+v_2||_{\mathcal{C}_T}\ (b-a)t. \tag{6.7.20}$$

But using (6.7.18) we have

$$||u_1+u_2||_{\mathcal{C}_T} \leq T\,||v_1+v_2||_{\mathcal{C}_T} + 2M, \tag{6.7.21a}$$

$$||u_1-u_2||_{\mathcal{C}_T} \leq T\,||v_1-v_2||_{\mathcal{C}_T}. \tag{6.7.21b}$$

Inserting the relations (6.7.21) into (6.7.20) we have

$$|Av_1-Av_2| \leq ||v_1-v_2||_{\mathcal{C}_T}\{1 + \tfrac{1}{2}(T\,||v_1+v_2||_{\mathcal{C}_T} + 2M)(b-a)t$$

$$+ \tfrac{1}{2}T\,||v_1-v_2||_{\mathcal{C}_T}\,||v_1+v_2||_{\mathcal{C}_T}(b-a)t$$

$$\leq \{1 + 2\,T\,R + M\}(b-a)t\;||v_1-v_2||_{\mathcal{C}_T}.$$

Taking the supremum of both sides with respect to x and t,

$$||Av_1-Av_2||_{\mathcal{C}_T} \leq \{1 + 2\,T\,R + M\}(b-a)T\;||v_1-v_2||_{\mathcal{C}_T}$$

$$\leq \theta||v_1-v_2||_{\mathcal{C}_T} \tag{6.7.22}$$

with $\{1 + 2\,T\,R + M\}(b-a)T < \theta < 1.$ (6.7.23)

Hence, A is a continuous operator. Also in the above calculations if $v_2 \equiv 0$ and $v_1 = v$ then

$$||Bv||_{\mathcal{C}_T} \leq ||v||_{\mathcal{C}_T}. \tag{6.7.24}$$

Now, if $\sup_{a\leq x\leq b}|g'(x)| \leq L$ and $\sup_{a\leq x\leq b} h'(x) \leq N$ it is seen that the ball $||v||_{\mathcal{C}_T} \leq R$ is mapped into itself if in addition to (6.7.24)

$$L + N \leq (1-\theta)R. \tag{6.7.25}$$

Thus, (6.7.23) and (6.7.24) imply that A is contractive over the ball in $\mathcal{C}_T$. Hence, using the fixed point theorem for Banach spaces v(x,t), being the fixed point of A satisfies the integral equation (6.7.19) and the lemma is proved. ▣

Combining the two lemmas 6.7.2 and 6.7.3 implies that if the initial data g(x) and h(x) are continuous and bounded then the original equation (6.7.12) has a bounded solution over the rectangle $[0,T] \times [a,b]$. This gives existence proof for solution of the class W_{54}.

6.8 Applications

We turn now to provide some applications in order to make the theory of well-posedness more understandable.

6.8.1 The KdV with skew data

Consider the initial value problem

$$u_t + u_x + uu_x + u_{xxx} = 0, \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x,kx) = g(x). \tag{6.8.1}$$

Two cases arise

(1) $k = 0$:

The initial value problem (6.8.1) is the same problem studied in chapter 3. Hence, for $g \in H^k$, $k \geq 3$ there exists a unique solution u(x,t) of (6.8.1) depending continuously on the data.

(2) $k \neq 0$:

Using the transformation

$$x \to x, \quad t \to \frac{t}{1-k} - \frac{kx}{1-k}, \quad \text{and} \quad u(x,t) \to v(x,t) \qquad (6.8.2)$$

then (6.8.1) reduces to

$$v_t + v_x + vv_x - \frac{k}{1-k} vv_t + v_{xxx} - \frac{3k}{(1-k)} v_{xxt} + \frac{3k^2}{(1-k)^2} v_{xtt}$$

$$- \frac{k^3}{(1-k)^3} v_{ttt} = 0 \, , \qquad (6.8.3a)$$

$$v(x,0) = g(x). \qquad (6.8.3b)$$

Now, (6.8.3) is an element in the nonsingular class. Let $v_t(x,0)$ and $v_{tt}(x,0)$ be given, i.e.,

$$v_t(x,0) = f(x), \qquad (6.8.3c)$$

$$v_{tt}(x,0) = h(x). \qquad (6.8.3d)$$

Applying the well-posedness theory of the nonsingular class to the problem (6.8.3), then, if $g(x)$, $f(x)$ and $h(x)$ are continuous and have continuous first derivatives, (6.8.3) is well-posed. Now, inserting the inverse of the transformation (6.8.2) in (6.8.3) we have

$$u_t + u_x + uu_x + u_{xxx} = 0 \, , \qquad (6.8.4a)$$

$$u(x,kx) = g(x) \, , \qquad (6.8.4b)$$

$$u_t(x,kx) = (1-k)f(x) \, , \qquad (6.8.4c)$$

$$u_{tt}(x,kx) = (1-k)^2 h(x). \qquad (6.8.4d)$$

Hence, (6.8.4) is well-posed provided that the data (6.8.4b, c,d) are at least C^1. But since from equation (6.8.4) differentiability with respect to t decreases the differentiability with respect to x by three, then prescribing the data on the skew curve restricts the well-posedness of the KdV equation into a smaller function space than for the usual data, precisely, the space $\mathcal{H}_T^k$, $k \geq 9$ which is a subspace of $\mathcal{H}_T^k$, $k \geq 3$.

6.8.2 The BBM with skew data

$$u_t + u_x + uu_x - u_{xxt} = 0. \qquad (6.8.5a)$$

$$u(x,kx) = g(x). \qquad (6.8.5b)$$

(1) $k = 0$:

The initial value problem (6.8.5) is the same problem which was studied in chapter 4, i.e. the well-posedness is already ensured.

(2) $k \neq 0$:

By the same nonsingular linear transformation (6.8.2), (6.8.5) reduces to:

$$v_t + v_x + vv_x - \frac{k}{1-k} vv_t - \frac{1}{1-k} v_{xxt} + \frac{2k}{(1-k)^2} v_{xtt}$$

$$- \frac{k^2}{(1-k)^3} v_{ttt} = 0 \qquad (6.8.6a)$$

$$v(x,0) = g(x) \qquad (6.8.6b)$$

which is an element of the nonsingular class. Hence similar to the above example this initial value problem is solved for two more arbitrary data $v_t(x,0)$ and $v_{tt}(x,0)$ to ensure the well-posedness. Then, using the inverse of the transformation (6.8.2), leads to well-posedness of (6.8.5) with two more bits of data $u_t(x,kx)$ and $u_{tt}(x,kx)$ which have to be in.

6.9 Conclusion

In this chapter, the well-posedness of the general class was investigated. For this investigation it was convenient to reduce the general class to a system of first order partial differential equations. It is found that if $a_6 \neq 0$ and the data are noncharacteristic, then the general class being reduced to a semi-linear system of first order partial differential equations, can be transformed to a system of ordinary differential equations on its characteristics. The proof of this fact was carried out for the case where all the characteristics are real. This proof can be done if some of these characteristics are complex by reducing the system to two systems of real characteristics and the reduction to systems of ordinary differential equations is clearly obtained again. This result leads to a classification of the problem into two main classes namely nonsingular and singular classes. For the nonsingular class the method of characteristics is applied to obtain well-posedness.

The failure of this method on the singular class is due to: (1) the data are characteristic (2) the singularity of A. This singular class consists of the four equivalence classes KdV, RLW, W_{54}, and W_{53} classes, introduced in the previous chapter, with usual data. This reduction turns the problem from being a six parameter problem, with characteristic data, into one with usual data and one less t-derivative. Furthermore, the reduction is via nonsingular linear transformations and hence preserves well-posedness where it is known. Consequently the theory of well-posedness of the KdV and the RLW, introduced in chapters 3 and 4 respectively, are used to advantage to establish well-posedness in the corresponding simple classes. Finally, our proof of existence for an element of W_{54} can be used to imply existence for the corresponding subset of the singular class. We have also established two results on the well-posedness of the KdV and the RLW on skew data.

CHAPTER SEVEN

CONSERVATION LAWS

An important property of the general class

$$u_t + u_x + a_1uu_x + a_2uu_t + a_3u_{xxx} + a_4u_{xxt} + a_5u_{xtt}$$

$$+ a_6u_{ttt} = 0 \quad (a_i \in R, \quad i = 1, 2, \ldots, 6)$$

is the possible existence of a number of independent conservation laws. This property plays a significant part in both mathematical and physical interests. The conservation law associated with a given equation was defined in chapter 3 wherever the conserved form for this equation is expressed by an equation of the form $\frac{\partial}{\partial t} T + \frac{\partial}{\partial x} X = 0$, where T, the conserved density, and $-X$, the flux are polynomials of x, t, u and the various derivatives of u. This conservation law is used as an indicator of whether the equation has an N-soliton solution or not. Thus, it is a mathematical property. Furthermore it is a physical property, since it is used for deriving a priori estimates and to obtain integrals of motion. For example, if the flux X is zero as $|x| \to \infty$, then $\int_{-\infty}^{\infty} T dx = \text{constant}$.

The idea of conservation laws was first introduced in chapter 3 where the proof of the existence of an infinite number of conservation laws of the KdV equation was given. In chapter 4 this idea was revived and it was shown that the RLW equation has only three conservation laws. These two equations lie in two disjoint subclasses of the general class, as was demonstrated in chapter 5. This demonstration was given via a reduction which preserves the conservation law property.

Thus, the general class includes other elements which have the same conservation properties as the KdV and the RLW equations.

In this chapter, a number of conservation laws of the general class are derived, followed by a careful examination of how many of such conservation laws can be found, and consequently how much from the general class are of the same character as the KdV equation.

7.1 Derivation of some possible conservation laws

We consider the general class of equations

$$u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt}$$

$$+ a_6 u_{ttt} = 0, \quad a_i \in R, \quad i = 1, 2, \ldots, 6 \qquad (7.1.1)$$

and write its nth conservation law in the form

$$\frac{\partial T_n}{\partial t} + \frac{\partial X_n}{\partial x} = 0. \qquad (7.1.2)$$

The derivations that follow are done by means of elementary operations and are stated in two theorems. However, first we note the following lemma

Lemma 7.1.1 [3]

Two conservation laws are said to be dependent if there exists constants c_1, c_2 such that

$$c_1 T_1 + c_2 T_2 = P_x \text{ (for some P)}$$

where T_i(is 1, 2) are the conserved densities. □

Note that if $T=P_x$, $X = -P_t$ (for some P) the conservation law is trivially satisfied.

Theorem 7.1

The first two conservation laws of (7.1.1) have densities and fluxes given by,

$$T_1 = u + a_2 \frac{u^2}{2} + a_5 u_{xt} + a_6 u_{tt} ,$$

$$X_1 = u + a_1 \frac{u^2}{2} + a_3 u_{xx} + a_4 u_{xt} ,$$

$$T_2 = \frac{u^2}{2} + a_2 \frac{u^3}{3} - a_4 \frac{u_x^2}{2} + a_5 uu_{xt} + a_6 (uu_{tt} - \frac{u_t^2}{2}), \text{ and}$$

$$X_2 = \frac{u^2}{2} + a_1 \frac{u^3}{3} + a_3 (uu_{xx} - \frac{u_x^2}{2}) + a_4 uu_{xt} - a_5 \frac{u_t^2}{2} . \square$$

Proof

Equation (7.1.1) can be re-written in the form

$$\frac{\partial}{\partial t}[u + a_2 \frac{u^2}{2} + a_5 u_{xt} + a_6 u_{tt}] + \frac{\partial}{\partial x}[u + a_1 \frac{u^2}{2} + a_3 u_{xx} + a_4 u_{xt}] = 0. \tag{7.1.3}$$

Hence T_1 and X_1 follow by (7.1.2).

Multiplying (7.1.1) by u, we have

$$uu_t + uu_x + a_1 u^2 u_x + a_2 u^2 u_t + a_3 uu_{xxx} + a_4 uu_{xxt} + a_5 uu_{xtt} + a_6 uu_{ttt} = 0. \tag{7.1.4}$$

Now, using the relations:

$$uu_{xxx} = \frac{\partial}{\partial x} (uu_{xx} - \frac{u_x^2}{2}), \quad uu_{xxt} = \frac{\partial}{\partial x} uu_{xt} - \frac{\partial}{\partial t} \frac{u_x^2}{2} , \tag{7.1.5}$$

$$uu_{xtt} = \frac{\partial}{\partial t} uu_{xt} - \frac{\partial}{\partial x} \frac{u_t^2}{2} , \quad \text{and} \quad uu_{ttt} = \frac{\partial}{\partial t} (uu_{tt} - \frac{u_t^2}{2})$$

means that (7.1.4) can be re-written in the conserved form (7.1.2) with

$$T_2 = \frac{u^2}{2} + a_2 \frac{u^3}{3} - a_4 \frac{u_x^2}{2} + a_5 u u_{xt} + a_6(u u_{tt} - \frac{u_t^2}{2}) \quad \text{and}$$

$$X_2 = \frac{u^2}{2} + a_1 \frac{u^3}{3} + a_3(u u_{xx} - \frac{u_x^2}{2}) + a_4 u u_{xt} - a_5 \frac{u_t^2}{2} .$$

Thus the theorem is proved. ▨

In the above theorem two conservation laws of (7.1.1) were derived and they are clearly independent where no one can be reduced to the other under possible integrations. We turn, now, to study the possibility of deriving the third conservation law.

<u>Theorem 7.2</u>

If the coefficients a_i satisfy the condition

$$a_3 - a_4(\frac{a_1}{a_2}) + a_5(\frac{a_1}{a_2})^2 - a_6(\frac{a_1}{a_2})^3 = 0. \tag{7.1.6}$$

Then this subclass has a third conservation law with

$$T_3 = \frac{1}{3} u^3 + a_2 \frac{u^4}{4} + a_4 u^2 u_{xx} + a_6 u^2 u_{tt} - (2K a_2 a_1^2 u + \frac{a_3}{a_1} + a_1 K) u_x^2$$

$$+ \frac{a_6}{a_2} u_t^2 + (2a_1 a_2 K u + \frac{2a_6}{a_2}) u_x u_t + (\frac{a_3 a_4}{a_1} + a_1 a_2 a_3 k) u_{xx}^2$$

$$+ (\frac{a_3 a_6}{a_1} - \frac{a_4 a_6}{a_2} - a_1 a_2 a_5 K) u_{xt}^2 + \frac{a_6^2}{a_2} u_{tt}^2 + (\frac{2a_3 a_5}{a_1} - \frac{2a_3 a_6}{a_2}) u_{xx} u_{xt}$$

$$+ \frac{2a_3 a_6}{a_1} u_{xx} u_{tt} - 2a_1 a_2 a_6 K u_{xt} u_{tt} ,$$

and

$$X_3 = \frac{1}{3}u^3 + a_1\frac{u^4}{4} + a_3u^2u_{xx} + a_5u^2u_{tt} + \frac{a_3}{a_1}u_x^2$$

$$- (2a_1a_2^2Ku + \frac{a_6}{a_2} + a_1a_2K)u_t^2 + (2Ka_1^2a_2u + \frac{2a_3}{a_1})u_xu_t + \frac{a_3^2}{a_1}u_{xx}^2$$

$$+ (\frac{a_3a_6}{a_2} - \frac{a_3a_5}{a_1} - a_1a_2a_4K)u_{xt}^2 + (\frac{a_5a_6}{a_2} + a_1a_2a_6K)u_{tt}^2$$

$$- 2a_1a_2a_3Ku_{xx}u_{xt} + \frac{2a_3a_6}{a_2}u_{xx}u_{tt} + (\frac{2a_4a_6}{a_2} - \frac{2a_3a_6}{a_1})u_{xt}u_{tt} ,$$

where

$$K = \frac{a_2a_3 - a_1a_4}{a_1^3a_2} = \frac{a_1a_6 - a_2a_5}{a_1a_2^3} . \qquad \square$$

Proof

To prove this theorem, we assume, without loss of generality, that $a_1, a_2 \neq 0$ (since if either a_1 or a_2 vanish there always exists a nonsingular linear transformation to put the equation in the form (7.1.1), i.e., with both nonlinear terms present).

Now, inserting the substitution

$$x \to \frac{1}{a_1}x \quad \text{and} \quad t \to \frac{1}{a_2}t \tag{7.1.7}$$

into (7.1.1), we have

$$\frac{1}{a_2}u_t + \frac{1}{a_1}u_x + uu_x + uu_t + \frac{a_3}{a_1^3}u_{xxx} + \frac{a_4}{a_1^2a_2}u_{xxt} + \frac{a_5}{a_1a_2^2}u_{xtt}$$

$$+ \frac{a_6}{a_2^3}u_{ttt} = 0 . \tag{7.1.8}$$

Multiplying (7.1.8) by u^2, the resulting equation can be re-written in the form:

$$\frac{\partial}{\partial t}\left[\frac{1}{a_2}\frac{u^3}{3} + \frac{u^4}{4} + \frac{a_4}{a_1^2 a_2} u^2 u_{xx} + \frac{a_6}{a_2^3} u^2 u_{tt}\right] + \frac{\partial}{\partial x}\left[\frac{1}{a_1}\frac{u^3}{3} + \frac{u^4}{4} + \frac{a_3}{a_1^3} u^2 u_{xx}\right.$$

$$\left. + \frac{a_5}{a_1 a_2^2} u^2 u_{tt}\right] - \frac{2a_3}{a_1^3} uu_x u_{xx} - \frac{2a_4}{a_1^2 a_2} uu_t u_{xx} - \frac{2a_5}{a_1 a_2^2} uu_x u_{tt}$$

$$- \frac{2a_6}{a_2^3} uu_t u_{tt} = 0 ,$$

i.e.,

$$\frac{\partial}{\partial t}\left[\frac{1}{a_2}\frac{u^3}{3} + \frac{u^4}{4} + \frac{a_4}{a_1^2 a_2} u^2 u_{xx} + \frac{a_6}{a_2^3} u^2 u_{tt}\right] + \frac{\partial}{\partial x}\left[\frac{1}{a_1}\frac{u^3}{3} + \frac{u^4}{4} + \frac{a_3}{a_1^3} u^2 u_{xx}\right.$$

$$\left. + \frac{a_5}{a_1 a_2^2} u^2 u_{tt}\right] - \frac{2a_3}{a_1^3}\left[uu_x u_{xx} + uu_t u_{xx}\right] + \left(\frac{2a_3}{a_1^3} - \frac{2a_4}{a_1^2 a_2}\right) uu_t u_{xx}$$

$$- \frac{2a_6}{a_2^3}\left[uu_x u_{tt} + uu_t u_{tt}\right] + \left(\frac{2a_6}{a_2^3} - \frac{2a_5}{a_1 a_2^2}\right) uu_x u_{tt} = 0. \qquad (7.1.9)$$

Using the relations:

$$uu_t u_{xx} = uu_x u_{xt} + \frac{\partial}{\partial x}(uu_x u_t) - \frac{\partial}{\partial t} uu_x^2 ,$$

$$(7.1.10)$$

$$uu_x u_{tt} = uu_t u_{xt} + \frac{\partial}{\partial t}(uu_x u_t) - \frac{\partial}{\partial x} uu_t^2 ,$$

reduces (7.1.9) to the form:

$$\frac{\partial}{\partial t}\left[\frac{1}{a_2}\frac{u^3}{3} + \frac{u^4}{4} + \frac{a_4}{a_1^2 a_2} u^2 u_{xx} + \frac{a_6}{a_2^3} u^2 u_{tt}\right]$$

$$+ \frac{\partial}{\partial x}\left[\frac{1}{a_1}\frac{u^3}{3} + \frac{u^4}{4} + \frac{a_3}{a_1^3} u^2 u_{xx} + \frac{a_5}{a_1 a_2^2} u^2 u_{tt}\right] - \frac{2a_3}{a_1^3}(uu_x + uu_t)u_{xx}$$

$$- \frac{2a_6}{a_2^3}(uu_x + uu_t)u_{tt} + \frac{2(a_2 a_3 - a_1 a_4)}{a_1^3 a_2} uu_x u_{xt}$$

$$+ \frac{2(a_1 a_6 - a_2 a_5)}{a_1 a_2^3} uu_t u_{xt} + \frac{\partial}{\partial t}\left[\frac{-2(a_2 a_3 - a_1 a_4)}{a_1^3 a_2} uu_x^2 + \frac{2(a_1 a_6 - a_2 a_5)}{a_1 a_2^3} uu_x u_t\right]$$

$$+ \frac{\partial}{\partial x} [\frac{2(a_2a_3 - a_1a_4)}{a_1^3a_2} uu_xu_t - \frac{2(a_1a_6 - a_2a_5)}{a_1a_2^3} uu_t^2] = 0. \quad (7.1.11)$$

Hence, (7.1.11) has the form:

$$\frac{\partial}{\partial t} T_3' + \frac{\partial}{\partial x} X_3' - \frac{2a_3}{a_1^3} (uu_x + uu_t)u_{xx} - \frac{2a_6}{a_2^3} (uu_x + uu_t)u_{tt}$$

$$+ \frac{2(a_2a_3 - a_1a_4)}{a_1^3a_2} uu_xu_{xt} + \frac{2(a_1a_6 - a_2a_5)}{a_1a_2^3} uu_tu_{xt} = 0 \quad (7.1.12a)$$

where,

$$T_3' = \frac{1}{a_2} \frac{u^3}{3} + \frac{u^4}{4} + \frac{a_4}{a_1^2a_2} u^2u_{xx} + \frac{a_6}{a_2^3} u^2u_{tt} - \frac{2(a_2a_3 - a_1a_4)}{a_1^3a_2} uu_x^2$$

$$+ \frac{2(a_1a_6 - a_2a_5)}{a_1a_2^2} uu_xu_t, \quad (7.1.12b)$$

$$X_3' = \frac{1}{a_1} \frac{u^3}{3} + \frac{u^4}{4} + \frac{a_3}{a_1^3} u^2u_{xx} + \frac{a_5}{a_1a_2^2} u^2u_{tt} + \frac{2(a_2a_3 - a_1a_4)}{a_1^3a_2} uu_xu_t$$

$$- \frac{2(a_1a_6 - a_2a_5)}{a_1a_2^3} uu_t^2. \quad (7.1.12c)$$

But the condition (7.1.6) of the theorem implies that

$$\frac{a_1a_6 - a_2a_5}{a_1a_2^3} = \frac{a_2a_3 - a_1a_4}{a_1^3a_2} = K. \quad (7.1.13)$$

Substituting (7.1.13) in (7.1.12), we have

$$\frac{\partial}{\partial t}[T_3' - 2Kuu_x^2 + 2Kuu_xu_t] + \frac{\partial}{\partial x}[X_3' + 2Kuu_xu_t - 2Kuu_t^2]$$

$$- \frac{2a_3}{a_1^3}(uu_x + uu_t)u_{xx} - \frac{2a_6}{a_2^3}(uu_x + uu_t)u_{tt} + 2K(uu_x + uu_t)u_{xt} = 0, \quad (7.1.14)$$

Now, multiplying (7.1.8) by $\frac{2a_3}{a_1^3} u_{xx}$, the resulting equation can be re-written in the form:

$$\frac{\partial}{\partial t}\left[\frac{a_3}{a_1^3 a_2} u_x^2 + \frac{a_3 a_4}{a_1^5 a_2} u_{xx}^2 + \frac{2a_3 a_5}{a_1^4 a_2^2} u_{xx}u_{xt} + \frac{a_3 a_6}{a_1^3 a_2^3} u_{xt}^2 + \frac{2a_3 a_6}{a_1^3 a_2^3} u_{xx}u_{tt}\right]$$

$$+ \frac{\partial}{\partial x}\left[\frac{2a_3}{a_1^3 a_2} u_x u_t + \frac{a_3}{a_1^4} u_x^2 + \frac{a_3^2}{a_1^6} u_{xx}^2 - \frac{a_3 a_5}{a_1^4 a_2^2} u_{xt}^2 - \frac{2a_3 a_6}{a_1^3 a_2^3} u_{xt}u_{tt}\right.$$

$$\left. + \frac{2a_3}{a_2^3} (uu_x + uu_t)u_{xx}\right] = 0. \qquad (7.1.15)$$

Multiplying (7.1.8) by $\frac{2a_6}{a_2^3} u_{tt}$, the resulting equation has the form:

$$\frac{\partial}{\partial t}\left[\frac{a_6}{a_2^4} u_t^2 + \frac{2a_6}{a_1 a_2^3} u_x u_t - \frac{2a_3 a_6}{a_1^3 a_2^3} u_{xx}u_{xt} - \frac{a_4 a_6}{a_1^2 a_2^4} u_{xt}^2 + \frac{a_6^2}{a_2^6} u_{tt}^2\right]$$

$$+ \frac{\partial}{\partial x}\left[\frac{-a_6}{a_1 a_2^3} u_t^2 + \frac{2a_3 a_6}{a_1^3 a_2^3} u_{xx}u_{tt} + \frac{a_3 a_6}{a_1^3 a_2^3} u_{xt}^2 + \frac{2a_4 a_6}{a_1^2 a_2^4} u_{xt}u_{tt} + \frac{a_5 a_6}{a_1 a_2^5} u_{tt}^2\right]$$

$$+ \frac{2a_6}{a_2^3} (uu_x + uu_t)u_{tt} = 0. \qquad (7.1.16)$$

Multiplying, finally, (7.1.8) by $-2Ku_{xt}$, yields:

$$\frac{\partial}{\partial t}\left[\frac{-K}{a_1} u_x^2 + \frac{a_3}{a_1^3} Ku_{xx}^2 - \frac{a_5 K}{a_2^2 a_1} u_{xt}^2 - \frac{2a_6}{a_2^3} Ku_{xt}u_{tt}\right]$$

$$+ \frac{\partial}{\partial x}\left[\frac{-K}{a_2} u_t^2 - \frac{2a_3 K}{a_1^3} u_{xt}u_{xx} - \frac{a_4 K}{a_1^2 a_2} u_{xt}^2 + \frac{a_6 K}{a_2^3} u_{tt}^2\right]$$

$$- 2K(uu_x + uu_t)u_{xt} = 0. \qquad (7.1.17)$$

Adding the four equations (7.1.14) to (7.1.17), we have

$$\frac{\partial}{\partial t}[T_3' - (\frac{a_3}{a_1^3a_2} + \frac{K}{a_1})u_x^2 + \frac{a_6}{a_2^4}u_t^2 + \frac{2a_6}{a_1a_2^3}u_xu_t + (\frac{a_3a_4}{a_1^5a_2} + \frac{a_3K}{a_1^3})u_{xx}^2$$

$$+ (\frac{a_3a_6}{a_1^3a_2^3} - \frac{a_4a_6}{a_1^4a_2^4} - \frac{a_5K}{a_1a_2^2})u_{xt}^2 + \frac{a_6^2}{a_2^6}u_{tt}^2 + (\frac{2a_3a_5}{a_1^4a_2^2} - \frac{2a_3a_6}{a_1^3a_2^3})u_{xx}u_{xt}$$

$$+ \frac{2a_3a_6}{a_1^3a_2^3}u_{xx}u_{tt} - \frac{2a_6K}{a_2^3}u_{xt}u_{tt}]$$

$$+ \frac{\partial}{\partial x}[X_3' - (\frac{a_6}{a_1a_2^3} + \frac{K}{a_2})u_t^2 + \frac{2a_3}{a_1^3a_2}u_xu_t + \frac{a_3^2}{a_1^6}u_{xx}^2$$

$$+ (\frac{a_3a_6}{a_1^3a_2^3} - \frac{a_3a_5}{a_1^4a_2^2} - \frac{a_4K}{a_1^2a_2})u_{xt}^2 + (\frac{a_5a_6}{a_1a_2^5} + \frac{a_6K}{a_2^3})u_{tt}^2 - \frac{2a_3K}{a_1^3}u_{xx}u_{xt}$$

$$+ \frac{2a_3a_6}{a_1^3a_2^3}u_{xx}u_{tt} + (\frac{2a_4a_6}{a_1^2a_2^4} - \frac{2a_3a_6}{a_1^3a_2^3})u_{xt}u_{tt} = 0. \qquad (7.1.18)$$

Thus equation (7.1.18) represents the third conservation law of the equation (7.1.8). Hence to establish the third conservation law of the original equation (7.1.1), we use the inverse of (7.1.2), i.e.

$$x \to a_1x \quad \text{and} \quad t \to a_2t \qquad (7.1.19)$$

and the expression T_3' and X_3', i.e., (7.1.12b) and (7.1.12c) (respectively).

Then (7.1.1) has a third conservation law with:

$$T_3 = \frac{1}{3}u^3 + a_2\frac{u^4}{4} + a_4u^2u_{xx} + a_6u^2u_{tt} - (2Ka_2a_1^2u + \frac{a_3}{a_1} + a_1K)u_x^2$$

$$+ \frac{a_6}{a_2}u_t^2 + (2a_1a_2Ku + \frac{2a_6}{a_2})u_xu_t + (\frac{a_3a_4}{a_1} + a_1a_2a_3K)u_{xx}^2$$

$$+ (\frac{a_3a_6}{a_1} - \frac{a_4a_6}{a_2} - a_1a_2a_5K)u_{xt}^2 + \frac{a_6^2}{a_2}u_{tt}^2 + (\frac{2a_3a_5}{a_1} - \frac{2a_3a_6}{a_2})u_{xx}u_{xt}$$

$$+ \frac{2a_3a_6}{a_1}u_{xx}u_{tt} - 2a_1a_2a_6Ku_{xt}u_{tt},$$

and

$$X_3 = \frac{1}{3}u^3 + a_1\frac{u^4}{4} + a_3u^2u_{xx} + a_5u^2u_{tt} + \frac{a_3}{a_1}u_x^2$$

$$- (2a_1a_2^2Ku + \frac{a_6}{a_2} + a_1a_2K)u_t^2 + (2Ka_1^2a_2u + \frac{2a_3}{a_1})u_xu_t + \frac{a_3^2}{a_1}u_{xx}^2$$

$$+ (\frac{a_3a_6}{a_2} - \frac{a_3a_5}{a_1} - a_1a_2a_4K)u_{xt}^2 + (\frac{a_5a_6}{a_2} + a_1a_2a_6K)u_{tt}^2$$

$$- 2a_1a_2a_3Ku_{xx}u_{xt} + \frac{2a_3a_6}{a_2}u_{xx}u_{tt} + (\frac{2a_4a_6}{a_2} - \frac{2a_3a_6}{a_1})u_{xt}u_{tt}$$

which proves the theorem. ∎

Note that the condition (7.1.6) of theorem 7.2 couples all the coefficients of the general class (7.1.1). We call it the coupling coefficients condition. In the next section we shall use this condition to classify the problem.

7.2 Classification of the problem using the coupling coefficients condition

Combining the definition of the characteristic equation of the general class (7.1.1), defined in chapter 6, with the coupling coefficients conditions (7.1.6) implies, clearly, that for all the coefficients $a_i (i=1, 2, \ldots, 6)$ of the general class, the coupling coefficients condition is satisfied if and only if $\frac{a_1}{a_2}$ is a root of the characteristic equation

$$a_3 - a_4\lambda + a_5\lambda^2 - a_6\lambda^3 = 0. \qquad (7.2.1)$$

This leads to a classification of the problem, in terms of the order of the roots of the cubic equation (7.2.1) as follows:

(1) If $\frac{a_1}{a_2}$ is a root of (7.2.1) of order three, then the corollary (5.3.1), introduced in chapter 5 implies that the corresponding subset of the general class reduces to equivalence class of the KdV equation (simple KdV class) and hence has an infinite number of conservation laws, as was proved in chapter 3.

(2) If $\frac{a_1}{a_2}$ is a root of (7.2.1) of order two, then the corresponding subset is reduced to the equivalence class of the RLW equation (simple RLW class) consequently, it has only three conservation laws, as was shown in chapter 4.

(3) If $\frac{a_1}{a_2}$ is a simple root of (7.2.1), then corollary (5.3.1) of the reduction theorem 5.2 implies that two subcases arise:

(i) If all the roots of (7.2.1) are real and simple, the corresponding subset reduces to the simple W_{54} class, i.e.,

$$u_t + u_x + b_1uu_x + b_4u_{xxt} + b_5u_{xtt} = 0. \qquad (7.2.2)$$

In the next section it will be proved that (7.2.2) has only three conservation laws.

(ii) If (7.2.1) has two complex conjugate roots, then this subset reduces to the simple W_{53} class, i.e.,

$$u_t + u_x + b_1 u u_x + b_3 u_{xxx} + b_5 u_{xtt} = 0. \qquad (7.2.3)$$

We shall show that this also has three conservation laws.

(4) If $\frac{a_1}{a_2}$ is not a root of (7.2.1) then the above analysis has only given us two conservation laws for this subset of equations. We shall discuss this case in the last section.

Having classified the general class of equations into four equivalence classes where the informations about the existence of conservation laws of the first two classes (simple KdV and simple RLW classes) are known, we turn now, to examine the other two classes by introducing a general formalism for proving the existence of conservation laws. This will be exploited later for solving the problem of the specific equations, simple W_{54} and simple W_{53} classes.

7.3 General formalism for proving existence of conservation laws

Let N be the space of all points with coordinates x_i, u^j, $i = 1, 2, \ldots, n$ and $j=1, 2, \ldots, m$ and M be the space of all points x_i, $i=1, 2, \ldots, n$. Define the projection operator Π such that $\Pi : N \to M$; i.e., $\Pi(x_i, u^j) = x_i$, $i=1, 2, \ldots, n$, $j=1, 2, \ldots, m$.

Let N_k be the space of all points with coordinates x_i, u^j and the various derivatives of u^j with respect to x_i. Thus, if we denote by R to the kth order differential equation with independent variables x_i and dependent variables u^j, then $R \subset N_k$.

Now, using the above notions we introduce the following definitions:

Definition (7.3.1) (infinite prolongation)

Let $R \subset N_k$ be the kth-order differential equation, i.e., R can be defined as $F_1 = F_2 = \ldots F_r = 0$ where $F_i (i=1, 2, \ldots, r)$ are smooth functions on N_k. We define the infinite prolongation $R_\infty \subset N_\infty$ as the subspace obtained by equating F and its various total derivatives to zero.

Example 7.3.1

$$u_{xxt} = u_t - uu_x - u_{xtt}. \qquad (7.3.1)$$

For this example $N = R^3$ with coordinates (x,t,u) and $M = R^2$ with coordinates (x,t). Hence equation (7.3.1) is a subspace of N_3 and F can be taken as

$$F = u_t - uu_x - u_{xxt} - u_{xtt}. \qquad (7.3.2)$$

Thus to obtain the infinite prolongation R_∞ of (7.3.1), we equate F and its various total derivatives to zero, i.e., by taking into account all the differential consequences of (7.3.1)

$$u_{xxxt} = u_{xt} - uu_{xx} - u_x^2 - u_{xxtt},$$

$$u_{xxtt} = u_{tt} - uu_{xt} - u_x u_t - u_{xttt} \ldots\ldots,$$

we arrive to the infinite prolongation $R_\infty \subset N_\infty$. This prolongation admits, in fact, a global coordinate system, e.g.

$$x,\ t,\ u_k,\ v_k,\ w_k \text{ and } \mu_k \qquad (7.3.3)$$

where

$u_k(k \geq 0) = u_{xx\ldots x}$(k-times x), $v_x(k \geq 1) = u_{tt\ldots t}$(k-times t),

$w_k(k \geq 2) = u_{xtt\ldots t}$(k-times t) and $\mu_k(k \geq 3) = u_{txx\ldots x}$((k-1)-times x), Example 7.3.2

$$u_{xxx} = u_t - uu_x - u_{xtt} = 0. \qquad (7.3.4)$$

In this example N and M are chosen as in the above example and

$$F = u_t - uu_x - u_{xxx} - u_{xtt} = 0.$$

Hence the infinite prolongation of (7.3.4) are obtained as in the above example and it admits the global coordinate system (7.3.3).

Defintion 7.3.2 (The algebra)
The subset A of the space C^∞ is called the algebra of smooth functions if, whenever f, g $\in$ A and α any real number, f+g, fg, $\alpha f \in$ A.

Let A denote the algebra of smooth functions on R_∞ and B the algebra of smooth functions on N_∞. Then

$$A = B/I$$

where I is the ideal of functions vanishing on R_∞. Then in the coordinate system (7.3.3) the total derivatives with respect to x and t on the algebra A are written as

$$D_x = \frac{\partial}{\partial x} + \sum_{k\geq 0} u_{k+1}\frac{\partial}{\partial u_k} + \sum_{k\geq 1} w_{k+1}\frac{\partial}{\partial v_k} + \sum_{k\geq 2}\frac{\partial w_k}{\partial x}\frac{\partial}{\partial w_k} + \sum_{k\geq 3}\mu_{k+1}\frac{\partial}{\partial \mu_k} \qquad (7.3.5a)$$

$$D_t = \frac{\partial}{\partial t} + \sum_{k\geq 2}\mu_{k+1}\frac{\partial}{\partial u_k} + \sum_{k\geq 1} v_{k+1}\frac{\partial}{\partial v_k} + \sum_{k\geq 2} w_{k+1}\frac{\partial}{\partial w_k} + \sum_{k\geq 3}\frac{\partial \mu_k}{\partial t}\frac{\partial}{\partial \mu k} + v_1\frac{\partial}{\partial u}$$

$$+ w_2\frac{\partial}{\partial u_1}\,. \qquad (7.3.5b)$$

Remark

From the definition of D_t, it can be clearly shown that if $\text{Ker } D_t = C^{\infty}(t)$ (the set of all functions of t), then the linear dependence of conserved densities implies linear dependence of corresponding conservation laws.

Definition 7.3.3

(1) The universal operator ℓ_F of a given equation $R \subset N_k$ is defined by the matrix

$$(\ell_F)_{ij} = \sum_{\xi} \frac{\partial F_i}{\partial u^j_{\xi}} D_{\xi} \qquad (7.3.6)$$

for the multi-indices ξ, where F_i are the components of R, as in the above notions, and D_{ξ} is the total derivative operator such that $D_{x_1 x_2 \ldots x_n} = D_{x_1} D_{x_2} \ldots D_{x_n}$.

(2) The conjugate operator ℓ_F^* of ℓ_F is derived from (7.3.6) by the transposition and taking the conjugate of each scalar element of the matrix where

$(D_i D_j)^* = D_j^* D_i^*$, $D_i^* = -D_i$ and $g^* = g$ for the function coefficient.

Example (7.3.3)

The universal operator for the equation in example (7.3.1) is

$$\ell_F = D_x^2 D_t + D_t^2 D_x - D_t + u D_x + u_x$$

and the conjugate operator ℓ_F^* has the form

$$\ell_F^* = -D_x^2 D_t - D_t^2 D_x + D_t - u D_x.$$

Similarly the equation in example 7.3.2 has

$$\ell_F = D_x^3 + D_t^2 D_x - D_t + uD_x + u_x$$

and

$$\ell_F^* = -D_x^3 - D_t^2 D_x + D_t - uD_x$$

where u_x which appears in ℓ_F is the operator multiplication by u_x and the conjugate of this multiplication operator vanish in ℓ_F^*.

In the following, and without any confusion, we denote by ℓ_F^* to the restriction of the conjugate operator ℓ_F^* to the infinite prolongation R_∞ of R.

Using the above definition of the conjugate operator ℓ_F^* the relation between the space of conservation laws and ℓ_F^* was given in [1], [4] and is summarized as follows:

<u>Theorem 7.3</u> [4]

The space of conservation laws can be injected into Ker ℓ_F^*. □

The above theorem implies that the dimension of the space of conservation laws is not greater than the dimension of Ker ℓ_F^*. Hence to prove the existence of conservation laws it suffices to calculate the dimension of Ker ℓ_F^*. This will be used in the next section to prove the existence of conservation laws of the simple W_{54} class.

7.4 Conservation laws of the simple W_{54} class

This section is devoted to the study of conservation laws of the simple W_{54} class (7.2.2). For this context we begin by the equation

$$u_{xxt} = u_t - uu_x - u_{xxt} \qquad (7.4.1)$$

which is an element of the class (7.2.2) and prove that the dimension of its space of conservation laws cannot be greater than three. This proof will be adapted later for the original equation (7.2.2).

Using the notions in section 7.3, the infinite prolongation R_∞ of (7.4.1) was obtained in example 7.3.1. Morever it was pointed out that it admits the global coordinate system (7.3.3).

If, now, A, B stands for the algebras of smooth functions on R_∞ and N_∞ (respectively), corresponding to (7.4.1), then

$$A = B/I \qquad (7.4.2)$$

where I is, now, differentially generated by the function

$$F = u_{xxt} + u_{xtt} - u_t + uu_x. \qquad (7.4.3)$$

Thus, in the coordinate system (7.3.3) the total derivatives D_x and D_t are obtained from (7.3.5), i.e., by using the equation (7.4.1) we have

$$\frac{dw_k}{dx} = v_{k-1} - w_{k+1} - \sum_{i=0}^{k-2} \binom{k-2}{i} v_i w_{k-1-i}, \text{ and}$$

$$\frac{d\mu_k}{dt} = \mu_{k-1} - \mu_{k+1} - \sum_{i=0}^{k-2} \binom{k-2}{i} u_i u_{k-i-1}.$$

Substituting $\frac{dw_k}{dx}$ and $\frac{d\mu_k}{dt}$ into (7.3.5) yield

$$D_x = \frac{\partial}{\partial x} + \sum_{k\geq 0} u_{k+1}\frac{\partial}{\partial u_k} + \sum_{k\geq 1} w_{k+1}\frac{\partial}{\partial v_k}$$

$$+ \sum_{k\geq 2}[v_{k-1} - w_{k+1} - \sum_{i=0}^{k-2}\binom{k-2}{i} v_i w_{k-1-i}]\frac{\partial}{\partial w_k} + \sum_{k\geq 3}\mu_{k+1}\frac{\partial}{\partial \mu_k}$$

(7.4.4)

and

$$D_t = \frac{\partial}{\partial t} + \sum_{k\geq 2}\mu_{k+1}\frac{\partial}{\partial u_k} + \sum_{k\geq 1} v_{k+1}\frac{\partial}{\partial v_k} + \sum_{k\geq 2} w_{k+1}\frac{\partial}{\partial w_k}$$

$$+ \sum_{k\geq 3}[\mu_{k-1} - \mu_{k+1} - \sum_{i=0}^{k-2}\binom{k-2}{i} u_i u_{k-i-1}]\frac{\partial}{\partial \mu_k} + v_1\frac{\partial}{\partial u} + w_2\frac{\partial}{\partial u_1}.$$

(7.4.5)

The above definition of the total differential operators on A leads to the following:

Lemma 7.4.1

Ker $D_x = C^\infty(t)$ (the set of all C^∞ functions of t). □

Proof

Let $D_x g = 0$ (7.4.6)

where $g \in A_n$, A_n, being the subalgebra of functions x, t, u_k, v_k, w_k and μ_k with $k \leq n$ which satisfies

$$A = \bigcup_{n=0}^{\infty} A_n.$$

Thus coefficient of u_{n+1} in (7.4.6), i.e., $(\frac{\partial g}{\partial u_n})$ must vanish, i.e., g does not depend on u_n. Similarly the coefficients of u_n, i.e., $\frac{\partial g}{\partial u_{n-1}}$ must vanish, yielding $\frac{\partial g}{\partial u_{n-1}} = 0$. Thus we come to

$$\frac{\partial q}{\partial u_1} = 0. \quad \text{Similarly} \quad \frac{\partial q}{d\mu_n} = \frac{\partial q}{\partial \mu_{n-1}} = \ldots\ldots = \frac{\partial q}{\partial \mu_3} = 0. \tag{7.4.7}$$

Since the coefficients of w_{n+1} in (7.4.6) vanish, then

$$\frac{\partial q}{\partial v_n} - \frac{\partial q}{\partial w_n} = 0. \tag{7.4.8}$$

Now, equation (7.4.1) can be re-written in the equivalent form

$$u_{xtt} = u_t - uu_x - u_{xxt}. \tag{7.4.9}$$

Then,

$$\frac{\partial w_2}{\partial x} = u_{xxt} = \mu_3,$$

$$\frac{\partial w_3}{\partial x} = u_{xxtt} = u_{xt} - uu_{xx} - u_x^2 - u_{xxxt}$$

$$= w_2 - uu_2 - u_1^2 - \mu_4,$$

$$\frac{\partial w_4}{\partial x} = w_3 - u\mu_3 - v_1u_2 - 2u_1w_2 - \mu_3 + uu_3 + 3u_1u_2 + \mu_5,$$

$$\vdots$$

$$\frac{dw_k}{dx} = w_{k-1} + (-1)^{k+1}\,\mu_{k+1} - \tau, \quad \tau \text{ does not depend on both}$$

w_{k-1} and μ_{k+1}.

Thus, the total x-derivative can be re-written equivalently in the form:

$$D_x = \frac{\partial}{\partial x} + \sum_{k\geq 0} u_{k+1}\frac{\partial}{\partial u_k} + \sum_{k\geq 2} w_{k+1}\frac{\partial}{\partial v_k} + (\mu_3\frac{\partial}{\partial w_2} + [w_2 - \mu_4 - uu_1 - u_1^2]\frac{\partial}{\partial w_3}$$

$$+ [w_3 - u\mu_3 - v_1u_2 - 2u_1w_2 - \mu_3 + uu_3 + 3u_1u_2 + \mu_5]\frac{\partial}{\partial w_4} + \ldots.$$

$$+ [w_{k-1} + (-1)^k\mu_{k+1} - \tau]\frac{\partial}{\partial w_k} + \sum_{k\geq 3}\mu_{k+1}\frac{\partial}{\partial \mu_k} \qquad (7.4.10)$$

where τ does not depend on μ_{k+1}.

Then, coefficient μ_{n+1} in (7.4.6) must vanish, i.e.

$$(-1)^n\frac{\partial g}{\partial w_n} + \frac{\partial g}{\partial \mu_n} = 0. \qquad (7.4.11)$$

Solving (7.4.8) and (7.4.11) gives

$$\frac{\partial g}{\partial w_n} = 0 \text{ and } \frac{\partial g}{\partial v_n} = 0 \quad (\text{since } \frac{\partial g}{\partial \mu_n} = 0).$$

Hence $g \in A_{n-1}$, and by induction, we come to $g = g(x, t, u)$.

Thus $D_x g = 0 \rightarrow \frac{\partial g}{\partial u}u_1 + \frac{\partial g}{\partial x} = 0 \rightarrow \frac{\partial g}{\partial u} = 0$ and $\frac{\partial g}{\partial x} = 0$,

i.e. $g = g(t) \in C^\infty(t)$, which proves the lemma. ■

Similar to the proof of lemma 7.4.1 one can prove the following

<u>Lemma 7.4.2</u>

Ker $D_t = C^\infty(x)$ (the set of all C^∞ functions of x). □

<u>Lemma 7.4.3</u>

Ker $(D_x + D_t) \subseteq C^\infty(x, t)$.

<u>Proof</u>

Let $(D_x + D_t)g = 0, \quad g \in A_n$ (7.4.12)

using (7.4.4) and (7.4.5), then

$$0 = (D_x + D_t)g = (\frac{\partial}{\partial x} + \frac{\partial}{\partial t})g + \sum_{k\geq 3} (\mu_{k+1} + u_{k+1})\frac{\partial g}{\partial u_k}$$

$$+ \sum_{k\geq 1} (w_{k+1} + v_{k+1})\frac{\partial g}{\partial v_k} + \sum_{k\geq 2} [v_{k-1} - \sum_{i=0}^{k-2} \binom{k-2}{i} v_i w_{k-1-i}]\frac{\partial g}{\partial w_k}$$

$$+ \sum_{k\geq 3} [\mu_{k-1} - \sum_{i=0}^{k-2} [\mu_{k-1} - \sum_{i=0}^{k-2} \binom{k-2}{i} u_i u_{k-i-1}]\frac{\partial g}{\partial \mu_k} + (u_2 + w_2)\frac{\partial g}{\partial w_k}$$

$$+ (u_1 + v_1)\frac{\partial g}{\partial u} . \qquad (7.4.13)$$

Thus, the coefficients of u_{n+1}, u_n, v_{n+1} and v_n in (7.4.12) must vanish, i.e.

$$\frac{\partial g}{\partial u_n} = \frac{\partial g}{\partial u_{n-1}} = \frac{\partial g}{\partial v_n} = \frac{\partial g}{\partial v_{n-1}} = 0 \text{ (respectively)} \qquad (7.4.14)$$

and coefficient $\mu_{n-1} = 0$, implies

$$\frac{\partial g}{\partial u_{n-2}} + \frac{\partial g}{\partial \mu_n} = 0. \qquad (7.4.15)$$

Similarly, coefficient $u_{n-1} = 0$, implies

$$\frac{\partial g}{\partial u_{n-2}} - u \frac{\partial g}{\partial \mu_n} = 0. \qquad (7.4.16)$$

Hence, combining (7.4.15) and (7.4.16) implies $\frac{\partial g}{\partial \mu_n} = 0$. By the same procedure, since the coefficients of v_{n-1} and w_{n-1} must vanish, then, combining the two coefficients, implies $(1+u) \frac{\partial}{\partial w_n} = 0$, thus $\frac{\partial}{\partial w_n} = 0$. Consequently $g \in A_{n-1}$. By the induction, we come to $g \in A_1$, i.e. $g = g(x, t, u, u_1)$ and,

$$(D_x + D_t)g = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} + (u_1 + v_1) \frac{\partial g}{\partial u} + (u_2 + w_2)\frac{\partial g}{\partial u_1}$$

$$+ (w_2 + v_2)\frac{\partial g}{\partial v_1} = 0. \qquad (7.4.17)$$

In the equation (7.4.17)

coefficient $u_2 = 0 \to \frac{\partial q}{\partial u_1} = 0$, coefficient $v_2 = 0 \to \frac{\partial q}{\partial v_1} = 0$,

thus $\frac{\partial q}{\partial u}$ = coefficient $(u_1 + v_1) = 0$, i.e.

$$\frac{\partial q}{\partial x} + \frac{\partial q}{\partial t} = 0 \to g = g(x - t) \in C^{\infty}(x, t)$$

and lemma 7.4.3 is proved. ■

Lemma 7.4.4

If $g \in A = \bigcup_n A_n$ is linear over $C^{\infty}(x, t)$ in u_i, $i \geq 0$ and $g \in \mathrm{Im} D_t$, then $g = 0$. □

Proof

Suppose $g = D_t f$ for some $f \in A_n$. Then, in a sense similar to that used to prove lemma 7.4.1, f does not depend on $v_1, v_2, \ldots, v_n; w_2, w_3, \ldots, w_n$ and $\mu_3, \mu_4, \ldots, \mu_n$, then

$$D_t f = \frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial u} + w_2 \frac{\partial f}{\partial u_1} + \sum_{i \geq 2} \mu_{i+1} \frac{\partial f}{\partial u_i} = g = \sum_i \alpha_i u_i. \qquad (7.4.18)$$

Now, equating the coefficients of μ_{i+1}, w_2 and v_1 in both sides, then f does not depend on $u_i (i \geq 0)$.

Hence $\alpha_i = 0$, i.e., $g = 0$. ■

The above four lemmas determine the properties of the total differential operators over the algebra A, which are required to prove the main result of this section.

We come now to prove the existence of conservation laws of the equation (7.4.1). The result of theorem 7.3 and the analysis, introduced in section 7.3, reduce the problem of finding the conservation laws of (7.4.1) to that of calculating the number of the elements in Ker ℓ_F^*,

$$\ell_F^* = -D_x^2 D_t - D_x D_t^2 + D_t - uD_x \tag{7.4.19}$$

where D_x and D_t are given by (7.4.4) and (7.4.5) respectively. So we shall prove that if $\phi \in A_n$ and $\ell_F^*(\phi) = 0$, then

$$\phi = \alpha(v_2 + w_2 + \frac{u^2}{2}) + \gamma u + \theta \tag{7.4.20}$$

where α, γ, θ are constants and $n \leq 2$. For this purpose we introduce:

Lemma 7.4.5

Let $\phi \in A_2$ and $\ell_F^*(\phi) = 0$, then ϕ does not depend on both u_2 and μ_3. Furthermore ϕ is linear in u_1 and v_1. □

Proof

Since $\phi \in A_2$, then by the definition of A_2, ϕ does not depend on μ_3. Hence D_x and D_t have the form

$$D_x = \frac{\partial}{\partial x} + u_3 \frac{\partial}{\partial u_2} + w_3 \frac{\partial}{\partial v_2} + [v_1 - uu_1 - w_3]\frac{\partial}{\partial w_2} + w_2 \frac{\partial}{\partial v_1}$$

$$+ u_2 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u}, \tag{7.4.21}$$

$$D_t = \frac{\partial}{\partial t} + [v_1 - uu_1 - w_3]\frac{\partial}{\partial u_2} + v_3 \frac{\partial}{\partial v_2} + w_3 \frac{\partial}{\partial w_2} + v_2 \frac{\partial}{\partial v_1}$$

$$+ v_1 \frac{\partial}{\partial u}. \tag{7.4.22}$$

Using the relations

$$\frac{dw_2}{dx} = v_1 - uu_1 - w_3, \quad \frac{dw_3}{dx} = v_2 - uw_2 - u_1 v_1 - w_4$$

$$\frac{dw_4}{dx} = v_3 - uw_3 - 2v_1w_2 - u_1v_2 - w_5 \text{ and } \frac{dw_5}{dx} = v_4 - uw_4 - 3v_1w_3$$

$$- 3v_2w_2 - u_1v_3 - w_6.$$

Then by equating the coefficients of u_4 and v_4 in $\ell_F^*(\phi)$ to zero, we have

$$D_t(\frac{\partial\phi}{\partial u_2}) = 0 \quad \text{and} \quad D_x(\frac{\partial\phi}{\partial v_2}) = 0. \qquad (7.4.23)$$

Hence, using lemmas 7.4.1 and 7.4.2, then

$$\phi = A(x)u_2 + \alpha(t)v_2 + \psi \qquad (7.4.24)$$

where $\psi \in A_2$ and does not depend on both u_2 and v_2. Similarly coefficient w_4 in $\ell_F^*(\phi)$ must vanish, yields

$$(D_x + D_t)\frac{\partial}{\partial w_2} + B(x,t) = 0 \qquad (7.4.25)$$

where $B(x,t)$ is in terms of $A(x)$ and $\alpha(t)$. Hence by using lemma 7.4.3, ϕ is linear in w_2, i.e.

$$\phi = A(x)u_2 + \alpha(t)v_2 + \beta(x,t)w_2 + \psi_1, \; \psi_1 \in A_1. \qquad (7.4.26)$$

Then (7.4.21) and (7.4.22) reduce (respectively) into

$$D_x\phi = Au_3 + A_xu_2 + \alpha w_3 + B[v_1 - uu_1 - w_3] + B_xw_2$$

$$+ \{\frac{\partial\psi_1}{\partial x} + u_2\frac{\partial\psi_1}{\partial u_1} + w_2\frac{\partial\psi_1}{\partial v_1} + u_1\frac{\partial\psi_1}{\partial u}\}, \qquad (7.4.27)$$

$$D_t\phi = A[v_1 - uu_1 - w_3] + \alpha v_3 + \alpha_tv_2 + Bw_3 + B_tw_2$$

$$+ \{\frac{\partial\psi_1}{\partial t} + w_2\frac{\partial\psi_1}{\partial u_1} + v_2\frac{\partial\psi_1}{\partial v_1} + v_1\frac{\partial\psi_1}{\partial u}\}. \qquad (7.4.28)$$

Hence,

$$D_x^2 D_t = A[v_1 - uu_1 - 2w_3 - uu_3 - 3u_1u_2 + uv_1 - u^2u_1 - 2uw_3 + 2u_1w_2$$

$$+ u_2v_1 + v_3 - 2v_1w_2 - u_1v_2 - w_5] + 2A_x [w_2 - uu_2 - u_1^2 - v_2$$

$$+ uw_2 + u_1v_1 + w_4] + A_{xx} [v_1 - uu_1 - w_3] + \alpha[v_3 - uw_3$$

$$- 2v_1w_2 - u_1v_2 - w_5] + \alpha_t[v_2 - uw_2 - u_1v_1 - w_4] + B[w_3 - uv_1$$

$$+ u^2u_1 + 2uw_3 - 2u_1w_2 - u_2v_1 - v_3 + uw_3 + 2v_1w_2 + u_1v_2 + w_5]$$

$$+ 2B_x[v_2 - uw_2 - u_1v_1 - w_4] + B_{xx}w_3 + B_t[w_2 - uu_2 - u_1^2 - v_2$$

$$+ uw_2 + u_1v_1 + w_4] + 2B_{xt}[v_1 - uu_1 - w_3] + B_{xxt}w_2$$

$$+ \frac{\partial^3\psi_1}{\partial x^2\partial t} + u_2 \frac{\partial^3\psi_1}{\partial u_1\partial x\partial t} + w_2 \frac{\partial^3\psi_1}{\partial v_1\partial x\partial t} + u_1 \frac{\partial^3\psi_1}{\partial u\partial x\partial t} + u_2D_x \frac{\partial^2\psi_1}{\partial u_1\partial t}$$

$$+ u_3 \frac{\partial^2\psi_1}{\partial u_1\partial t} + w_2D_x \frac{\partial^2\psi_1}{\partial v_1\partial t} + [v_1 - uu_1 - w_3] \frac{\partial^2\psi_1}{\partial v_1\partial t} + u_1D_x \frac{\partial^2\psi_1}{\partial u\partial t}$$

$$+ u_2 \frac{\partial^2\psi_1}{\partial u\partial t} + w_2D_x^2 \frac{\partial\psi_1}{\partial u_1} + 2[v_1 - uu_1 - w_3]D_x \frac{\partial\psi_1}{\partial u_1}$$

$$+ [w_2 - uu_2 - u_1^2 - v_2 + uw_2 + u_1v_1 + w_4] \frac{\partial\psi_1}{\partial u_1} + v_2D_x^2 \frac{\partial\psi_1}{\partial v_1}$$

$$+ 2w_3D_x \frac{\partial\psi_1}{\partial v_1} + [v_2 - uw_2 - u_1v_1 - w_4] \frac{\partial\psi_1}{\partial v_1} + v_1D_x^2 \frac{\partial\psi_1}{\partial u}$$

$$+ 2w_2D_x \frac{\partial\psi_1}{\partial u} + [v_1 - uu_1 - w_3] \frac{\partial\psi_1}{\partial u} ,$$

$$
\begin{aligned}
D_t^2D_x = {} & A[w_3 - uv_1 + u^2u_1 + 2uw_3 - u_2v_1 - 2u_1w_2 - v_3 + 2v_1w_2 \\
& + u_1v_2 + w_5] + A_x[v_2 - uw_2 - u_1v_1 - w_4] + \alpha w_5 + 2\alpha_t w_4 \\
& + \alpha_{tt}w_3 + B[v_3 - uw_3 - 2v_1w_2 - u_1v_2 - w_5] + 2B_t[v_2 - uw_2 \\
& - u_1v_1 - w_4] + B_xw_4 + 2B_{xt}w_3 + B_{tt}[v_1 - uu_1 - w_3] \\
& + B_{xtt}w_3 + (\frac{\partial^2}{\partial x\partial t^2} + w_2 \frac{\partial^3}{\partial u_1\partial x\partial t} + v_2 \frac{\partial^3}{\partial v_1\partial x\partial t} + v_1 \frac{\partial^3}{\partial u\partial x\partial t} \\
& + u_2D_t \frac{\partial^2}{\partial u_1\partial t} + [v_1 - uu_1 - w_3] \frac{\partial^2}{\partial u_1\partial t} + w_2D_t \frac{\partial^2}{\partial v_1\partial t} \\
& + u_1D_t \frac{\partial^2}{\partial u\partial t} + w_2 \frac{\partial^2}{\partial u\partial t} + w_2D_tD_x \frac{\partial}{\partial u_1} + w_3D_x \frac{\partial}{\partial u_1} \\
& + [v_1 - uu_1 - w_3]D_t \frac{\partial}{\partial u_1} + [v_2 - uw_2 - u_1v_1 - w_4] \frac{\partial}{\partial u_1} \\
& + v_2D_tD_x \frac{\partial}{\partial v_1} + v_3D_x \frac{\partial}{\partial v_1} + w_3D_t \frac{\partial}{\partial v_1} + w_4 \frac{\partial}{\partial v_1} + v_1D_tD_x \frac{\partial}{\partial u} \\
& + v_2D_x \frac{\partial}{\partial u} + w_2D_x \frac{\partial}{\partial u} + w_3 \frac{\partial}{\partial u})\psi_1.
\end{aligned}
$$

Since $\ell_F^*(\phi) = 0$, then by equating the coefficients of v_3, u_3 and v_2 in both sides to zero, then

$$
D_x \frac{\partial\psi_1}{\partial v_1} = 0, \quad D_t(\frac{\partial\psi_1}{\partial u_1}) = 0 \quad \text{and} \quad D_x(\frac{\partial\psi_1}{\partial u} - \alpha u) = B_x + B_t + A_x. \tag{7.4.29}
$$

Thus ϕ is linear in both v_1 and u_1 (by the lemmas 7.4.2 and 7.4.1). The last relation in (7.4.29) implies

$$D_x(\frac{\partial\psi_1}{\partial u} - \alpha u + C(x,t)) = 0 \qquad (7.4.30)$$

where $D_x C = B_x + B_t + A_x$. Then, by lemma 7.4.1

$$\frac{\partial\psi_1}{\partial u} - \alpha u + C(x,t) \in C^\infty(t), \text{ i.e.}$$

$$\frac{\partial\psi_1}{\partial u} = \alpha u + \gamma(x,t). \quad \text{Thus} \quad \psi_1 = \alpha\frac{u^2}{2} + \gamma u \quad \text{and}$$

$$\phi = A(x)u_2 + \alpha(t)v_2 + B(x,t)w_2 + \frac{\alpha}{2}u^2 + \gamma u + \delta(t)v_1$$

$$+ du_1 + \theta(x,t)$$

where γ, δ, d and $\theta \in C^\infty(x,t)$, $A \in C^\infty(x)$ and α, $\delta \in C^\infty(t)$. Furthermore, from the definition of ϕ, coefficient u_1u_2 in $\ell_F^*(\phi) = 0$ must vanish, then $A = 0$ i.e. ϕ does not depend on u_2. Hence

$$\phi = \alpha(t)v_2 + \beta(x,t)w_2 + \frac{\alpha(t)}{2}u^2 + \gamma(x,t)u + \delta(t)v_1$$

$$+ d(x,t)u_1 + \theta(x,t). \blacksquare \qquad (7.4.31)$$

<u>Lemma 7.4.6</u>

Let ϕ has the expression (7.4.31) and $\ell_F^*(\phi) = 0$, then all the coefficients α, β, γ, δ, d and θ are constants. □

<u>Proof</u>

Since $\phi \in A_2$ is in the form (7.4.31), the expression of D_x and D_t reduces to the forms

$$D_x\phi = \alpha w_3 + \beta[v_1 - uu_1 - w_3] + \beta_x w_2 + \alpha uu_1 + \gamma u_1 + \gamma_x u + \delta w_2 + du_2$$

$$+ d_x u_1 + \theta_x , \qquad (7.4.32)$$

$$D_t\Phi = \alpha v_3 + \alpha_t v_2 + \beta w_3 + \beta_t w_2 + \alpha u v_1 + \frac{\alpha_t}{2} u^2 + \gamma v_1 + \gamma_t u + \delta v_2$$

$$+ \delta_t v_1 + d w_2 + d_t u_1 + \theta_t \ . \qquad (7.4.33)$$

Thus, $D_x^2 D_t$ and $D_t^2 D_x$ take the forms:

$$\begin{aligned}
D_x^2 D_t = {} & (\alpha-\beta)[v_3 - u w_3 - 2v_1 w_2 - u_1 v_2 - w_5] - \beta_x w_4 \\
& + (\alpha_t - \beta_t + \beta_x + \delta - d)[v_2 - u w_2 - u_1 v_1 - w_4] \\
& + (\beta_{xx} - \beta_{xt} - d_x) w_3 + (\beta_{xt} + \gamma + \delta_t + d_x)[v_1 - u u_1 - w_3] \\
& + (\beta_{xxt} + \gamma_x + d_{xx}) w_2 + (d + \beta_t)[w_2 - u u_2 - u_1^2] \\
& + (d_x + \beta_{xt})[v_1 - u u_1] + (\alpha-\beta)[u v_1 - u^2 u_1 - u w_3 + 2u_1 w_2 \\
& + u_2 v_1] - \beta_x [u w_2 + u_1 v_1] + (\gamma_t + d_{xt}) u_2 + (\gamma_{xt} + d_{xxt}) u_1 \\
& + \beta w_3 + \beta_x v_2 + \gamma_x w_2 + \gamma_{xx} v_1 + \gamma_{xt} u_1 + \gamma_{xxt} u + d_t u_3 \\
& + d_{xt} u_2 + \alpha_t [u u_2 + u_1^2] + \theta_{xxt} \ ,
\end{aligned}$$

$$\begin{aligned}
D_t^2 D_x = D_t D_x D_t = {} & (\alpha-\beta) w_5 + (\alpha_t - \beta_t) w_4 + (\alpha_t - \beta_t + \beta_x + \delta - d) w_4 \\
& + (\alpha_{tt} - \beta_{tt} + \beta_{xt} + \delta_t - d_t) w_3 + (\beta_{xt} + \gamma + \delta_t + d_x) w_3 \\
& + (\beta_{xtt} + \gamma_t + \delta_{tt} + d_{xt}) w_2 + (d + \beta_t)[v_2 - u w_2 - u_1 v_1] \\
& + (d_t + \beta_{tt})[v_1 - u u_1] + (\alpha-\beta)[u w_3 + 2v_1 w_2 + u_1 v_2] \\
& + (\alpha_t - \beta_t)[u w_2 + u_1 v_1] + (\gamma_t + d_{xt}) w_2 + (\gamma_{tt} + d_{xtt}) u_1 \\
& + \beta v_3 + \beta_t v_2 + \gamma_x v_2 + 2\gamma_{xt} v_1 + \gamma_{xtt} u + d_t [v_1 - u u_1 - w_3] \\
& + d_{tt} u_2 + \alpha_t [u w_2 + u_1 v_1] + \alpha_{tt} u u_1 + \theta_{xtt} .
\end{aligned}$$

Now, since $\ell_F^*(\phi) = 0$, i.e.,

$$(D_x^2D_t + D_t^2D_x - D_t + uD_x)\phi = 0.$$

Thus, equating the coefficients of u_2v_1, u_1^2, u_1v_1, v_2, u_2 and u^2 in $\ell_F^*(\phi)$ to zero we have, respectively, the following relations.

$$\alpha - \beta = 0, \quad -(d + \beta_t) + \alpha_t = 0, \quad \delta + \alpha_t = 0,$$

$$\beta_t + \gamma_x = 0 \text{ and } \frac{\alpha_t}{2} + \gamma_x = 0, \; (\alpha_t - d - \beta_t)u + (\alpha-\beta)v_1$$

$$+ \gamma_t + d_{tt} = 0. \qquad (7.4.34)$$

Solving these relations together, we have

$$\alpha = \beta \in \mathbf{R}, \qquad d = 0 = \delta \qquad \text{and} \qquad \gamma \in \mathbf{R}.$$

Then $\ell_F^*(\phi) = \ell_F^*(\theta)$. Hence $\theta_x = \theta_t = 0$, i.e., $\theta \in \mathbf{R}$. Thus

$$\phi = \alpha(v_2 + w_2 + \frac{u^2}{2}) + \gamma u + \theta, \quad \alpha, \quad \text{and} \quad \theta \in \mathbf{R}. \; ▣$$

Now, the two lemmas 7.4.5 and 7.4.6 lead to the following:

<u>Theorem 7.4</u>

If $\phi \in A_n$, $n \leq 2$ and $\ell_F^*(\phi) = 0$, then

$$\phi = \alpha(v_2 + w_2 + \frac{u^2}{2}) + \; u + \theta, \; \alpha, \quad \text{and} \quad \theta \in \mathbf{R}. \; \square$$

In the above we have shown that ℓ_F^* has a nontrivial kernel lying in A_2 and since this kernel is three dimensional this corresponds to the three conservation laws that we derived earlier. We now show that there are no more conservation laws by proving that the kernel of ℓ_F^* lies entirely in A_2. For this purpose we introduce

Lemma 7.4.7

Let ϕ be nontrivially in A_n, $n > 2$, (i.e. contains terms which are in A_n and not in A_{n-1}) and $\ell_F^*(\phi) = 0$, then

(i) ϕ is linear in its highest terms, i.e., u_n, w_n, v_n and μ_n

(ii) ϕ does not depend on μ_n, w_n, μ_{n-1} and w_{n-1}. □

Proof

By using the definitions of D_x and D_t, (7.4.4) and (7.4.5), respectively, and the assumption $\ell_F^*(\phi) = 0$, then equating the coefficients of v_{n+2}, u_{n+2}, w_{n+2} and μ_{n+2} in $\ell_F^*(\phi)$ to zero, the following relations are obtained

$$D_x\left(\frac{\partial\phi}{\partial v_n}\right) = 0, \quad D_t\left(\frac{\partial\phi}{\partial w_n}\right) = 0, \quad (D_x + D_t)\frac{\partial\phi}{\partial w_n} = \theta_1 \quad \text{and}$$

$$(D_x + D_t)\frac{\partial\phi}{\partial \mu_n} = \theta_2 \,, \qquad (7.4.35)$$

where θ_1 and θ_2 are in terms of $\frac{\partial\phi}{\partial u_n}$ and $\frac{\partial\phi}{\partial v_n}$ (see appendix C).

The original equation (7.4.1) can be regarded as a coupling relation between μ_n and w_n ($n \geq 2$), i.e.,

$$\mu_3 + w_3 = v_1 - uu_1 \,,$$

$$\mu_4 - w_4 = w_2 - uu_2 - u_1^2 - v_2 + uw_2 + u_1v_1 + w_4 \,,$$

$$\ldots, \quad \mu_n + (-1)^{n+1}w_n = \psi_1 \,, \quad \psi_1 \in A_{n-2} \qquad (7.4.36)$$

i.e. ψ_1 does not depend on μ_n, w_n, μ_{n-1} and w_{n-1}.

Then the last two identities of (7.4.35) imply

$$(D_x + D_t)\,[\frac{\partial\Phi}{\partial w_n} + (-1)^n \frac{\partial\Phi}{\partial\mu_n}] = \theta(x,t) \qquad (7.4.37)$$

where θ is in terms of θ_1 and θ_2. Thus by using the lemmas 7.4.1, 7.4.2 and 7.4.3, we obtain

$$\Phi = \alpha(x)u_n + \beta(t)v_n + \Upsilon(x,t)[w_n + (-1)^n\mu_n] + \psi, \quad \psi \in A_{n-1}$$

which completes the proof. ■

Lemma 7.4.8

Let Φ be as in the above lemma, then Φ does not depend on both u_n and v_n. □

Proof

Since $\ell_F^*(\Phi) = 0$, then using the expression of Φ

i.e.,

$\Phi = \alpha(x)u_n + \beta(t)v_n + \psi_1$, $\psi_1 \in A_{n-1}$. Hence, equating the coefficient of v_{n+1} to zero yields

$$D_x(\frac{\partial\psi_1}{\partial v_{n-1}}) = \beta'(x,t) \qquad (7.4.38)$$

where B' is in terms of α, β and consequently Φ is linear in v_{n-1} by lemma 7.4.1.

Similarly

coefficient $u_{n+1} = 0 \rightarrow D_t \dfrac{\partial\psi_1}{\partial u_{n-1}} = \beta''(x,t) + \alpha(x)u_1$. (7.4.39)

Thus by the definition of D_t and lemma 7.4.4, $\alpha = 0$ i.e. ϕ does not depend on u_n and is linear in u_{n-1}. By using the coupling relation (7.4.36) and in a sense similar to that used in lemma 7.4.7

coefficient $w_{n+1} = 0 \rightarrow (D_x + D_t)\frac{\partial}{\partial w_n} = j(x,t)$ and coefficient

$\mu_{n+1} = 0 \rightarrow (D_x + D_t)\frac{\partial \Psi_1}{\partial \mu_{n-}} = k(x,t)$

where j and k are in terms of $\frac{\partial \Psi_1}{\partial v_{n-1}}$, $\frac{\partial \Psi_1}{\partial u_{n-1}}$. This implies that ϕ does not depend on μ_{n-1}, w_{n-1}, μ_{n-2} and w_{n-2} (as in the above lemma). But coefficient v_n in $\ell_F^*(\phi)$ must vanish, then

$$D_x \frac{\partial}{\partial v_{n-2}} - u_1 \frac{\partial}{\partial v_n} = c(x,t) \qquad (7.4.40)$$

coefficient $w_n = 0$, then

$$D_t \frac{\partial \Psi}{\partial v_{n-2}} - rv_1 \frac{\partial \Psi}{\partial v_n} + 2D_x \frac{\partial \Psi}{\partial v_{n-2}} = P(x,t), \quad r \in R \qquad (7.4.41)$$

[where c and P are in terms of the derivatives with respect to w_i, μ_i, u_j, v_j, $i = n$, $n-1$ and $n-2$ and $j=n$ and $n-1$].

Then (7.4.40) and (7.4.41) implies

$$D_t(\frac{\partial \Psi}{\partial v_{n-2}} - r\beta u) = 2\beta u_1 + (P - 2c). \qquad (7.4.42)$$

Thus, using the definition of D_t and lemma (7.4.4), equation (7.4.42) implies $\beta = 0$, i.e.,

ϕ does not depend on $v_n \rightarrow \phi \in A_{n-1}$ gives a contradiction and the lemma is proved. ■

The two lemma 7.4.7 and 7.4.8 lead to the following:

Theorem 7.5

Let $\phi \in A_n$ (nontrivially) and $\ell_F^*(\phi) = 0$, then $n \leq 2$. □

Hence the two theorems 7.4 and 7.5 demonstrate that the dimension of ker ℓ_F^* is three and is generated by 1, u and $\frac{u^2}{2} + u_{xt} + u_{tt}$. Then the space of conservation laws of (7.4.1) is three dimensional, by theorem 7.3. Thus equation (7.4.1) has only three conservation laws, having the form

$$\frac{\partial}{\partial t}[u + u_{xt}] + \frac{\partial}{\partial x}[-\frac{u^2}{2} u_{xt}] = 0 ,$$

$$\frac{\partial}{\partial t}[\frac{u^2}{2} - uu_{xt} + \frac{u_x^2}{2}] + \frac{\partial}{\partial x}[\frac{u^3}{3} - uu_{xt} - \frac{u_t^2}{2}] = 0 ,$$

$$\frac{\partial}{\partial t}[\frac{u^3}{3} + u_t^2] + \frac{\partial}{\partial x}[u_t^2 - \frac{u^4}{4} - u^2(u_{xt} + u_{tt}) - (u_{xt} + u_{tt})^2] = 0$$

where the first is obtained by re-writing (7.4.1) in a conserved form, the second is obtained by multiplying (7.4.1) by u and re-writing the resulting equation in a conserved form, and the third is given by multiplying (7.4.1) by u^2 and re-writing the resulting equation in a conserved form.

Now, if we replace u_n, μ_n and w_n by $-b_1u_n$, $-b_4\mu_n$ and $-b_5w_n$ (respectively) for all $n \geq 1$ in all the above calculations we come to the proof of the following,

Theorem 7.6

The equation

$$u_t + u_x + b_1uu_x + b_4u_{xxt} + b_5u_{xtt} = 0$$

has only three independent conservation laws which is the main result of this section.

7.5 Conservation laws of The Simple W_{53}

In a sense similar as to that used in section 7.4, the equation of the class W_{53} has only three conservation laws as in the following,

Theorem 7.7.

The equation

$$u_t + u_x + b_1 u u_x + b_3 u_{xxx} + b_5 u_{xtt} = 0 \qquad (7.5.1)$$

has only three conservation laws and have the forms:

$$\frac{\partial}{\partial t}[u + b_5 u_{xt}] + \frac{\partial}{\partial x}[u + \frac{b_1}{2} u^2 + b_3 u_{xx}] = 0, \qquad (7.5.2)$$

$$\frac{\partial}{\partial t}[\frac{u^2}{2} + b_5 u u_{xt}] + \frac{\partial}{\partial x}[\frac{u^2}{2} + \frac{b_1}{3} u^3 + b_3 u u_{xx} - b_3 \frac{u_x^2}{2} - b_5 \frac{u_t^2}{2}] = 0,$$

$$(7.5.3)$$

$$\frac{\partial}{\partial t}[\frac{u^3}{3} - \frac{b_3}{b_1} u_x^2 + \frac{b_5}{b_1} u_t^2 + \frac{2b_5}{b_1} u_x u_t] + \frac{\partial}{\partial x}[\frac{u^3}{3} + b_1 \frac{u^4}{4} + b_3 u^2 u_{xx}$$

$$+ b_5 u^2 u_{tt} + \frac{2b_3}{b_1} + \frac{2b_3}{b_1}(u_x u_t + \frac{u_x^2}{2}) + \frac{b_3^2}{b_1} u_{xx}^2 - \frac{b_5}{b_1} u_t^2 + \frac{2b_5 b_3}{b_1} u_{xx} u_{tt}$$

$$+ \frac{b_5^2}{b_1} u_{tt}^2] = 0. \qquad (7.5.4)$$

where (7.5.2) is obtained by re-writing (7.5.1) in a conserved form, (7.5.3) is given by multiplying (7.5.1) by u and re-writing the resulting equation in a conserved form, and (7.5.4) is obtained by multiplying (7.5.1) by $u^2 + \frac{2b_3}{b_1} u_{xx} + \frac{2b_5}{b_1} u_{tt}$ and re-writing the resulting equation in a conserved form.

7.6 The complement of the simple classes

This section is devoted to the study of the case in which $\frac{a_1}{a_2}$ is not a root of the coupling coefficients condition

$$a_3 - a_4(\frac{a_1}{a_2}) + a_5(\frac{a_1}{a_2})^2 - a_6(\frac{a_1}{a_2})^3 = 0.$$

For this case it is thought that the corresponding subset has only two conservation laws, which were derived earlier. We shall not go through the proof of this results because of the laborious calculations which arise when we attempt to calculate the dimension of $\ker \ell_F^*$, where ℓ_F is the corresponding universal operator on the algebra A, defined in section 7.3. We merely illustrate this case by the following example to provide an indicator to our belief.

Example 7.6.1

$$u_{xxt} = u_t - uu_x - uu_t + u_{xtt}. \qquad (7.6.1)$$

In this example $a_1|a_2 = 1$, $\alpha_4 = -a_5 = -1$, thus the coupling coefficient condition breaks down.

We shall show that $\ker \ell_F^*$ is two dimensional and is generated by 1 and u, where

$$\ell_F^* = -D_x^2 D_t + D_t^2 D_x + D_t - uD_x - uD_t \qquad (7.6.2)$$

and D_x, D_t are the total differential operators with respect to x and t (respectively), defined as in section 7.3. These two differential operators has similar properties, as were given in section 7.4.

Now to calculate the dimension of $\ker \ell_F^*$, let $\phi \in A_2$ (non-trivially) and

$$\ell_F^*(\phi) = 0. \qquad (7.6.3)$$

On the algebra A_2, the total differential operator D_x and D_t have the forms:

$$D_x = \frac{\partial}{\partial x} + u_3 \frac{\partial}{\partial u_2} + w_3 \frac{\partial}{\partial v_2} + [v_1 - uu_1 - uv_1 + w_3]\frac{\partial}{\partial w_2}$$

$$+ u_2 \frac{\partial}{\partial u_1} + w_2 \frac{\partial}{\partial v_1} + u_1 \frac{\partial}{\partial u}, \qquad (7.6.4)$$

$$D_t = \frac{\partial}{\partial t} + [v_1 - uu_1 - uv_1 + w_3]\frac{\partial}{\partial u_2} + v_3 \frac{\partial}{\partial v_2} + w_3 \frac{\partial}{\partial w_2}$$

$$+ w_2 \frac{\partial}{\partial u_1} + v_2 \frac{\partial}{\partial v_1} + v_1 \frac{\partial}{\partial u}. \qquad (7.6.5)$$

Now, equating the coefficients of u_4, v_4, w_4, u_3 and v_3 in $\ell_F^*(\phi)$ to zero, we have

$$D_x(\frac{\partial}{\partial u_2}) = 0, \quad D_t(\frac{\partial}{\partial v_2}) = 0, \quad (D_x + D_t)\frac{\partial}{\partial w_2} = c(x,t), \quad D_x(\frac{\partial}{\partial u_1}) = 0,$$

and $D_t(\frac{\partial}{\partial v_1}) = 0$ (respectively). (7.6.6)

Hence, using the properties of D_x and D_t, we have

$$\phi = A(x)u_2 + \alpha(t)v_2 + \beta(x,t)w_2 + \gamma(x)u_1 + \delta(t)v_1$$

$$+ \psi(x,t,u). \qquad (7.6.7)$$

Inserting the expression of ϕ in (7.6.3) and equating the coefficients of u_1u_2, v_1v_2 and w_3 in $\ell_F^*(\phi)$ to zero we have:

$$3A = 0, \quad -2(A + B + \alpha) + 3(A + B) = 0, \quad \text{and}$$

$$-\alpha u + A_{xx} + B_{xx} - A_x - B_x - \alpha_t + \delta = 0 \quad \text{(respectively)}. \tag{7.6.8}$$

The first two identities in (7.6.8) imply,

$A = 0$, $B = 2\alpha$ i.e. $B = B(t)$: substituting the values of A and B in the third identity of (7.6.8) implies

$$-\alpha(t)u(x,t) - \alpha_t(t) + \delta(t) = 0$$

i.e. $\alpha = 0$. Thus $B = 0$

which gives a contradiction with the choice of ϕ.

If now $\phi \in A_1$, i.e. $\phi = \phi(x,t,u)$ and $\ell_F^*(\phi) = 0$ then it is easy to show that ϕ is linear in u, by equating the coefficients of u_2 in $\ell_F^*(\phi)$ to zero, i.e.

$$D_t \frac{\partial}{\partial u} = 0.$$

Thus

$$\phi = A(x)u + \theta(x,t).$$

Hence,

$$D_x\phi = Au_1 + A_xu + \theta_x ,$$

$$D_t\phi = Av_1 + \theta_t ,$$

$$D_x^2D_t\phi = A[v_1 - uu_1 - uv_1 + w_3] + 2A_xw_2 + A_{xx}v_1 + \beta_{xxt} ,$$

and

$$D_t^2 D_x \phi = Aw_3 + A_x v_2 + \theta_{xtt} .$$

Since $\ell_F^*(\phi) = 0$, then equating the coefficient of w_2 in $\ell_F^*(\phi)$ to zero implies $A_x = 0$, i.e. $A \in R$.

Thus, $\ell_F^*(\phi) = \ell_F^*(\theta) \rightarrow \theta \in R$

i.e. ker ℓ_F^* is two dimensional and is generated by: 1, u and consequently, from the relation between the ker ℓ_F^* and the space of conservation law, equation (7.6.1) has only two conservation laws.

7.7 Conclusion

In this chapter, the conservation laws property was discussed. First, two such conservation laws were established via elementary operations. These operations were used to derive a third conservation law of the problem. All the equations of the general class, satisfying the coupling coefficients condition, were proved that have a third conservation law. This condition together with the reduction theorem, introduced in chapter 5, classify the problem into four equivalence classes, KdV, RLW, W_{54} and W_{53} classes in the simple sense, i.e., with the disappearence of uu_t term. The informations about conservation laws of the first two classes is already known from chapters 3 and 4, i.e., infinite number of conservation laws exist for the KdV, whilst only three conservation laws of the RLW exists. The conservation laws of the simple W_{54} class is, then, studied. We prove that this class has only three conservation laws. We, next, showed that the simple W_{53} class has only three conservation laws too.

Finally we turned to the equations which do not satisfy the coupling coefficient condition. Here the two nonlinear terms are present. For this case it is thought that it has only two conservation laws. This was illustrated by one example. If this is the case, then we come to the main conclusion that the only equations which have infinite number of conservation laws lie in the simple KdV class. Whilst all the equations which have the RLW feature satisfy the coupling coefficients condition.

CHAPTER EIGHT

CONCLUDING REMARKS

In this thesis we have studied a general class of semi-linear third order partial differential equations with quadratic nonlinearity. The original equation for this class of models is the Korteweg-de Vries equation which was first derived as an approximation of the Euler equations of hydrodynamics. However, its appearance in many other physical systems meant that a more general method of derivation was required. Broer suggested that one could construct the equation as a structural perturbation of the basic linear equation $u_t + u_x = 0$ by adding on the third order dispersive term u_{xxx} and the quadratic nonlinearity $(\frac{1}{2} u^2)_x$. The main idea behind this construction is that the properties of the KdV can be understood in terms of the interaction between nonlinearity and dispersion. This assumption was then used by other workers to construct alternative models to the KdV on the basis that these models had dispersion relations which use more in accord with physical behaviour. Now, as models for describing experimental behaviours these alternatives may have been adequate, but from the mathematical point of view the question remained as to whether the hypothesis that the properties of the equation are due to the interaction between nonlinearity and dispersion was itself a correct one. This question was studied in detail by Abbas for this set of equations and he found that the hypothesis was not a viable one either for the KdV or any of its alternatives.

Since the hypothesis on dispersion is no longer valid, the discussion shifts to a consideration of the properties held by this class of equations. All these equations share the mathematical property of being third order with quadratic nonlinearities and they all have stable solitary waves. However, it was found that the KdV has a number of exceptional properties which are not known to be shared by any of the alternatives. This raises questions as to whether the KdV is a unique equation of its type and if so, whether we can develop criteria for understanding this uniqueness. This is the motivation for the work in this thesis. Now, this study clearly requires a much more mathematical approach than was adopted by Abbas and in this thesis we have provided foundations for such an approach by establishing certain necessary conditions which the class should satisfy in order for the questions to be meaningful. Our main effort was concentrated on introducing a nontrivial classification scheme, establishing the well-posedness of the class and proving the existence of conservation laws and our contribution in these areas are described below:

The general class was first reduced to a set of equations all of which possessed linearity stable solitary wave solutions. This eliminated equations which a priori are not similar to the KdV. This was done on the basis that the properties we wish to consider, e.g., the existence of solitons are generally felt to depend upon the equations having solitary wave solutions. We then completed the classification of solitary waves begun by Abbas and classified the equations themselves in terms of the number of time derivatives. This classification was based on the equivalence relation that two equations are equivalent if there is a nonsingular linear transformation which maps one to the other. This reduced the set of equations to three subclasses, i.e., those which essentially contained three time derivatives, those which essentially contained two time derivatives and those which had just one time derivative. The subclasses containing one and two time derivatives were further subdivided into two classes each accroding to the presence of the term u_{xxx} or u_{xxt}. In particular we noted that the set of equations with one time derivative divided naturally into a generalized KdV class and a generalized RLW class. The other two classes we called W_{53} and W_{54}.

Well-posedness was then discussed in terms of the equivalence classes defined above. For equations with three time-derivatives the method of characteristics was used to establish well-posedness for non-charateristic data. This was done by first reducing the equations to a first order system of partial differential equations in the standard manner. By going to characteristic coordinates this system becomes a system of ordinary differential equations and hence well-posedness followed by integration. The proof was carried out for the case where all the characteristics are real but it can be easily extended to the case when some of the roots are complex (ultra-hyperbolic) by reducing the corresponding equations to pairs of hyperbolic systems of first order partial differential equations with real characteristics.

For those equivalence classes which has less than three time derivatives the method of characteristics fails and we established well-posedness for certain subclasses as follows: The known theorems on the well-posedness of the KdV and RLW, which were discussed in chapter 3 and 4, enabled us to deduce well-posedness for those equations which are equivalent to the KdV and RLW.

These are what we called the simple classes of KdV and RLW equations, i.e., they did not contain uu_t term. Using a modification of the method used to prove existence for the RLW equation, we then proved existence for the general RLW class. However, we noted that not every equation has unique solutions. In the case of general KdV class we have no results since the only method we find was a modification of that used for KdV and this fails since there were not enough conserved functionals to obtain the necessary a priori estimates. Finally, we presented a restricted existence proof for those equations with two time derivatives and a u_{xxt} term. It is interesting to note that the analysis of well-posedness discussed above can be extended to corresponding modified equations where the quadratic nonlinearity is replaced by a cubic nonlinearity, i.e. $(u^3)_x$ and $(u^3)_t$. We also studied how the well-posedness of the KdV and RLW could be extended to initial data given on lines other than $t = 0$, i.e., to skew data.

Finally, we looked at the question of the existence of conservation laws. Since in considering local conservation laws we are not concerned with initial data, the a_6 term was eliminated by a suitable transformation and the equations reduced to the four equivalence classes KdV, RLW, W_{54} and W_{53}. Using elementary operations we established the existence of two conservation laws for the whole class. We then derived a necessary condition for the existence of a third conservation law. This condition was a condition on the coefficients of the equation, which we called the coupling coefficient condition specifically, we found that if a_1/a_2 is a root of the cubic equation

$$a_3 - a_4\lambda + a_5\lambda^2 - a_6\lambda^3 = 0,$$

then the corresponding equations have at least three conservation laws. Applying the condition eliminated the uu_t term and hence reduced the equations to the simple versions of the four classes mentioned above. According to the nature of the roots of the cubic we then obtained the following results. If a_1/a_2 is a triple root then we have the simple KdV class and, using the theorem for the KdV given in chapter 3, we deduced that this class has an infinite number of conservation laws. If a_1/a_2 is a double root then we have the simple RLW class and, using the theorem for the RLW given in chapter 4, we deduced that this class has only three conservation laws. The other two cases, i.e., W_{54} and W_{53} occur when a_1/a_2 is one of three distinct roots and when a_1/a_2 is the only real root respectively. In the case of W_{54}, we proved, using methods similar to those used in the proof of the RLW conservation laws theorem, that this class has only three conservation laws. For W_{53}, we indicated that this proof can be extended to show that this class also has only three conservation laws. Thus, if the coupling coefficient condition is satisfied, we have a complete classification of the numbers of conservation laws, i.e., the simple KdV class has an infinite number and the three other classes have only three. If a_1/a_2 is not a root of the cubic then, as we have mentioned above, all the equations in this subclass have at least two conservation laws. We conjecture that these are the only conservation laws, but we did not carry out the proof since the calculation becomes extremely laborious because of the appearance of the uu_t term in the equation.

We come now to the question of future research. This thesis has laid the basis for a strict mathematical investigation as to why the KdV is unique in the class considered. In this study we have not achieved all our objectives and the areas which need to be completed are the extension of well-posedness to the general KdV and W_{53} classes, and the proof of the number of conservation laws in the non-simple sector. We feel that these can be attained and hence the set of equations considered will form a well-defined neighbourhood of the KdV. The next stage will be to develop sequencing procedures to examine the behaviour of the properties of the KdV as a limiting point of this neighbourhood. The most interesting question will be why soliton solutions suddenly occur in this limit and where they come from. We are now working on these aspects as well as trying to complete the objectives set out in this thesis.

APPENDIX A

In this appendix we present the proof of lemmas 3.7.2 and 3.7.3 which were necessary for the proof of the existence theorem 3.7 of the KdV equation (3.7.1).

A.1 Proof of lemma 3.7.2

(1) Multiplying the RKdV (3.7.9) by u, we have

$$uu_t + u^2u_x + uu_{xxx} - \epsilon uu_{xxt}. \tag{A1}$$

If $u(x,t)$ vanishes together with all its x-derivatives as $|x| \to \infty$, then integrating (A1) with respect to x, $-\infty < x < \infty$, yields

$$\frac{d}{dt}\int_{-\infty}^{\infty}(u^2+\epsilon u_x^2)dx = 0. \tag{A2}$$

Thus (A2) implies

$$\int_{-\infty}^{\infty}(u^2+\epsilon u_x^2)dx = \int_{-\infty}^{\infty}(g^2+\epsilon g'^2)dx, \quad (g'=\frac{dg}{dx}). \tag{A3}$$

Now,

$$||u||^2_{L^2(R)} = \int_{-\infty}^{\infty}u^2dx \leq \int_{-\infty}^{\infty}(u^2+\epsilon u_x^2)dx, \quad (0<\epsilon\leq 1)$$

$$= \int_{-\infty}^{\infty}(g^2+\epsilon g'^2)dx$$

$$\leq \int_{-\infty}^{\infty}(g^2+g'^2)dx = ||g||_{H^1}$$

which proves (1).

(2) Multiplying (3.7.9) by u^2, gives

$$u^2u_t + u^3u_x + u^2u_{xxx} - \epsilon u^2u_{xxt} = 0 \qquad (A4)$$

which can be arranged in the form

$$\frac{\partial}{\partial t}\left[\frac{u^3}{3}\right] + \frac{\partial}{\partial t}\left[\frac{u^4}{4} + u^2u_{xx} - \epsilon u^2u_{xt}\right]$$

$$+ 2\epsilon uu_x(\epsilon u_{xt} - u_{xx}) = 0. \qquad (A5)$$

Using (3.7.9) and (A5), we have

$$\frac{\partial}{\partial t}\left[\frac{u^3}{3} - u_x^2\right] + \frac{\partial}{\partial x}\left[\frac{u^4}{4} + u^2u_{xx} - \epsilon u^2u_{xt}\right.$$

$$\left. + 2u_xu_t - \epsilon u_t^2 + (\epsilon u_{xt} - u_{xx})^2\right] = 0. \qquad (A6)$$

Integrating the last equation with respect to x, $-\infty < x < \infty$, where u, u_x and u_t vanish together with all their x-derivatives as $|x| \to \infty$, we have

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left(\frac{u^3}{3} - u_x^2\right)dx = 0 \qquad (A7)$$

which implies that

$$\int_{-\infty}^{\infty}\left(\frac{u^3}{3} - u_x^2\right)dx = \int_{-\infty}^{\infty}\left(\frac{g^3}{3} - g'^2\right)dx. \qquad (A8)$$

Now,

$$||u||_{H^1} = \int_{-\infty}^{\infty}(u^2 + u_x^2)dx$$

$$\leq \int_{-\infty}^{\infty}(u^2 + \frac{1}{3}u^3 + g'^2 - \frac{1}{3}g^3)dx \quad \text{(from A8)}$$

$$\leq \int_{-\infty}^{\infty} u^2 dx + \frac{1}{3} \sup_{x\in R} |u(x,t)| \int_{-\infty}^{\infty} u^2 dx + ||g||^2_{H^1}$$

$$- \frac{1}{3} ||g||^3_{H^1}$$

$$\leq ||g||^2_{H^1} + \frac{1}{3} ||u||_{H^1} ||g'||^2_{H^1} + ||g||^2_{H^1}$$

$$- \frac{1}{3} ||g||^3_{H^1}$$

[since $\sup_{x\in R} |u| \leq ||u||_{H^1}$].

Thus

$$||u||^2_{H^1} - \frac{1}{3} ||u||_{H^1} ||g||^2_{H^1} \leq 2||g||^2_{H^1} + \frac{1}{3} ||g||^3_{H^1} ,$$

i.e.,

$$[||u||_{H^1} - \frac{1}{6} ||g||^2_{H^1}]^2 - \frac{1}{36} ||g||^4_{H^1}$$

$$\leq 2||g||^2_{H^1} - \frac{1}{3} ||g||^3_{H^1} .$$

Hence,

$$||u||_{H^1} \leq \frac{1}{6} ||g||^2_{H^1} + ||g||_{H^1} [2+ \frac{1}{3}||g||_{H^1} + \frac{1}{36}||g||^2_{H^1}]^{\frac{1}{2}}$$

$$= \xi (||g||_{H^1}) , \qquad \text{(A9)}$$

$\xi(0) = 0$ and ξ is clearly monotone increasing function, i.e. (2) is proved.

(3) By multiplying (3.7.9) by,

$$u^3 + 3u_x^2 + 6uu_{xx} + \frac{18}{5} u_{xxx} ,$$

the proof of (3) is obtained in a sense similar to (2).

(4) This part is proved by induction. Since from (3) u is bounded in $\mathcal{H}_T^2$ with bound depending only on T, ϵ_0 and $||g||_{H^3}$. Suppose, now, that u is bounded in $\mathcal{H}_T^{k-1}$ with a bound depending only on T, ϵ_0 and $||g||_{H^k}$ and independent of $\epsilon \in (0, \epsilon_0]$. Then to prove that u is bounded in $\mathcal{H}_T^k$ with a bound depending only on T, ϵ_0, $||g||_{H^k}$ and $\epsilon^{\frac{1}{2}}||g||_{H^k}$, multiply the RKdV (3.7.9) by $u_{(2k)}$, i.e.,

$$u_{(2k)}u_t + u_{(2k)}uu_x + u_{(2k)}u_{xxx} - \epsilon u_{(2k)}u_{xxt} = 0 \qquad \text{(A10)}$$

(where $u_{(s)} = \frac{\partial^s u}{\partial x^s}$).

Let, $I_1 = \int_{-\infty}^{\infty} u_{(2k)}u_t dx$, $\qquad I_2 = \int_{-\infty}^{\infty} u_{(2k)}uu_x dx$,

$$I_3 = \int_{-\infty}^{\infty} u_{(2k)}u_{xxx}dx \text{ and } I_4 = \int_{-\infty}^{\infty} u_{(2k)}u_{xxt}dx. \qquad \text{(A11)}$$

Then by integrating (A10) with respect to x and using (A11) we have

$$I_1 + I_2 + I_3 - \epsilon I_4 = 0. \qquad \text{(A12)}$$

We calculate, now, the integrals I_i, (i = 1, 2, 3, and 4], using the assumptions on u and their derivatives. Then

$$I_1 = \int_{-\infty}^{\infty} u_{(2k)} u_t dx = [u_{(2k-1)} u_t]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_{(2k-1)} u_{(1),t}\, dx$$

$$= \int_{-\infty}^{\infty} u_{(2k-1)} u_{(1),t}\, dx \quad (u_{(k)} \to 0 \text{ as } |x| \to \infty)$$

$$= (-1)^2 \int_{-\infty}^{\infty} u_{(2k-2)} u_{(2),t}\, dx$$

(in the second integration).

Then, after k integrations

$$I_1 = (-1)^k \int_{-\infty}^{\infty} u_{(k)} u_{(k),t}\, dx = (-1)^k \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} u^2_{(k)} dx. \tag{A13}$$

Similarly, after k integrations

$$I_2 = (-1)^k \frac{1}{2} \int_{-\infty}^{\infty} u_{(k)} u^2_{(k+1)} dx \;, \quad I_3 = 0 \;, \text{ and}$$

$$I_4 = (-1)^k \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} u^2_{(k+1)} dx \;. \tag{A14}$$

Thus, after k integrations (A10) reduces to

$$\frac{d}{dt} \int_{-\infty}^{\infty} [u^2_{(k)} + \epsilon u^2_{(k+1)}] dx = - \int_{-\infty}^{\infty} u_{(k)} (u^2)_{(k+1)} dx. \tag{A15}$$

Using Leibintz's rule to expand the term $(u^2)_{(k+1)}$, (A15) reduces to

$$\frac{d}{dt} \int_{-\infty}^{\infty} [u^2_{(k)} + \epsilon u^2_{(k+1)}] dx = - \int_{-\infty}^{\infty} u_{(k)} \sum_{r=0}^{r=k+1} \binom{k+1}{r} u_{(r)} u_{(k+1-r)} dx$$

$$= - \int_{-\infty}^{\infty} \{ \binom{k+1}{0} uu_{(k+1)}u_{(k)} + \binom{k+1}{1} u_1 u^2_{(k)} + uu_{(k+1)}u_{(k)}$$

$$+ \binom{k+1}{1} u_{(1)}u^2_{(k)} + u_{(k)} \sum_{r=2}^{k-2} \binom{k+1}{r} u_{(r)}u_{(k+1-r)} + u^2_{(k-1)}u_{(k)}\} dx$$

$$= - \int_{-\infty}^{\infty} [b_1 uu_{(k+1)}u_{(k)} + b_2 uu^2_{(k)} + u_{(k)} \sum_{r=2}^{k-2} b_r u_{(k+1-r)}u_{(r)}$$

$$+ u^2_{(k-1)}u_k] dx$$

where b_i are constants $\in R$ and the last term only occurs when $k = 3$ and must vanish under the integration with respect to x. Hence

$$\frac{d}{dt} \int_{-\infty}^{\infty} [u^2_{(k)} + \epsilon\, u^2_{(k+1)}] dx = - \int_{-\infty}^{\infty} b_1 u \frac{d}{dx} \frac{u^2_{(k)}}{2} dx$$

$$+ \int_{-\infty}^{\infty} b_2 u_1 u^2_{(k)} dx + \int_{-\infty}^{\infty} u_{(k)} \sum_{r=2}^{k-2} b_r u_{(k+1-r)}u_{(r)} dx. \quad \text{(A16)}$$

Since u is bounded in $\mathcal{H}^{k-1}_T$ (by the assumption) independently of ϵ, $\epsilon \in (0, \epsilon_0)$, it follows that

$$||u_{(j)}|| \leq C, \text{ for } j < k-1 \text{ and } ||u_j||_\infty \leq C \text{ for } j \leq k-2$$

where $C = C(T, \epsilon_0, ||g||_{H^k})$. Then (A16) reduces to

$$\frac{d}{dt}\int_{-\infty}^{\infty}(u^2_{(k)} + \epsilon u^2_{(k+1)})dx \leq C\int_{-\infty}^{\infty} u^2_{(k)} dx + \sum_{r=2}^{k-2} b_r ||u_{(k)}|| \, ||u_{(r)}||_\infty ||u_{(k+1-r)}||$$

$$\leq C \int_{-\infty}^{\infty} u^2_{(k)} dx + C \left[\int_{-\infty}^{\infty} u^2_{(k)} dx\right]^{\frac{1}{2}}$$

$$\leq C\left[\int_{-\infty}^{\infty} u^2_{(k)}\,dx + 1\right] \leq C\left[\int_{-\infty}^{\infty} u^2_{(k)} + \epsilon u^2_{(k+1)}\right]dx$$

i.e.,
$$\frac{dE_k}{dt} \leq C\,(E_k+1) \tag{A17}$$

where $E_k(t) = \int_{-\infty}^{\infty} [u^2_{(k)} + \epsilon\, u^2_{(k+1)}]dx.$

By using Gronwall's inequality (A17) implies

$$E_k(t) = E_k(0)e^{ct} + e^{ct} - 1$$

i.e., $E_k(t)$ is bounded with bounds depending on $||g||_{H^k}$ and $\epsilon^{\frac{1}{2}}\,||g||_{H^{k+1}}$ (since $E_k(0) = ||g||_{H^k} + \epsilon^{\frac{1}{2}}\,||g||_{H^{k+1}}$).

Hence, (4) is proved.

(5) To prove (5), re-write (3.7.9) in the form

$$(1 - \epsilon\frac{d^2}{dx^2})ut = -uu_x - u_{xxx}\,. \tag{A18}$$

Hence, the Green's function subject to $u(\pm\infty,t) = 0$ for the operator $(1 - \epsilon\frac{d^2}{dx^2})$ is

$$G(x,\epsilon) = \begin{cases} \frac{1}{2\sqrt{\epsilon}}\exp\frac{(x-\xi)}{\sqrt{\epsilon}} & x < \xi \\ \frac{1}{2\sqrt{\epsilon}}\exp-\frac{(x-\xi)}{\sqrt{\epsilon}} & x > \xi \end{cases}.$$

Then (A18) reduces to the integral equation

$$u_t(x,t) = -\int_{-\infty}^{\infty} K_\epsilon(x-\xi)\,[uu_x + u_{xxx}]d\xi$$

$$= - K_\epsilon * (uu_x + u_{xxx}) \qquad \text{(A19)}$$

where $K_\epsilon(Z) = \text{Sgn}\, Ze^{-|Z|\sqrt{\epsilon}}$ and $*$ is the convolution.

If $(uu_x + u_{xxx}) \in \mathcal{H}^k_T$, then by using Hausdorff-young inequality $K_\epsilon * (uu_x + u_{xxx})$ is bounded in $\mathcal{H}^k_T$ independently of ϵ, $\epsilon \in (0,\epsilon_0)$, i.e., u_t is bounded in $\mathcal{H}^k_T$, independently of ϵ, and by the properties of $\mathcal{H}^k_T$, $\frac{\partial^m}{dx^m} u_t$ is bounded in $\mathcal{H}_T$. Furthermore u is bounded $\mathcal{H}^{k,1}_T$ for each k, independently of ϵ, $\epsilon \in (0,\epsilon_0]$. Thus by the induction procedure the proof of (5) is obtained. ∎

A2 Proof of lemma 3.7.3

(1) By using the transformed variables

$$\epsilon^{\frac{1}{3}j}\, ||g_\epsilon||^2_{H^{k+j}} = \epsilon^{\frac{1}{3}j}\int_{-\infty}^{\infty} [1+\lambda^2+ \ldots+\lambda^{2(k+j)}]\,|\hat{g}_\epsilon(\lambda)|^2 d\lambda$$

$$= \epsilon^{\frac{1}{3}j}\int_{-\infty}^{\infty}\left[\frac{1+\lambda^2+\ldots+\lambda^{2(k+j)}}{1+\lambda^2+\ldots+\lambda^{2k}}\right][1+\lambda^2+\ldots+\lambda^{2k}]\,|\hat{g}_\epsilon(\lambda)|^2 d\lambda$$

$$= \epsilon^{\frac{1}{3}j}\int_{-\infty}^{\infty}\left[\frac{1+\lambda^2+\ldots+\lambda^{2(k+j)}}{1+\lambda^2+\ldots+\lambda^{2k}}\,\phi^2(\epsilon^{\frac{1}{6}}\lambda)\right][1+\lambda^2+\ldots+\lambda^{2k}]\,|\hat{g}(\lambda)|^2 d\lambda$$

[Since $\hat{g}_\epsilon(k) = \phi(\epsilon^{\frac{1}{6}}k)\hat{g}(k)$].

Hence

$$\epsilon^{\frac{1}{3}j}||g_\epsilon||^2_{H^{k+j}} \le \sup_{\lambda\in R} [\epsilon^{\frac{1}{3}j} \frac{1+\lambda^2+...+\lambda^{2(k+j)}}{1+\lambda^2+...+\lambda^{2k}} \phi^2(\epsilon^{\frac{1}{6}}\lambda)]||g||^2_{H^k} .$$

If $M = \epsilon^{\frac{1}{6}}\lambda$, $\alpha = \epsilon^{\frac{1}{3}}$, then $\lambda = \frac{M}{\sqrt{\alpha}}$.

Then

$$\epsilon^{\frac{1}{3}j}||g_\epsilon||^2_{H^{k+j}} \le \sup_{\lambda\in R} \left|\epsilon^{\frac{1}{3}j} \frac{1+\left(\frac{M}{\sqrt{\alpha}}\right)^2+\left(\frac{M}{\sqrt{\alpha}}\right)^4+...+\left(\frac{M}{\sqrt{\alpha}}\right)^{2(k+j)}}{1+\left(\frac{M}{\sqrt{\alpha}}\right)^2+\left(\frac{M}{\sqrt{\alpha}}\right)^4+...+\left(\frac{M}{\sqrt{\alpha}}\right)^{2k}} \phi^2(M)||g||^2_{H^k}\right|$$

$$= ||g||^2_{H^k} \sup_{M\in R} \alpha^j \frac{1+...+\left(\frac{M^2}{\sqrt{\alpha}}\right)^{k+j}}{1+...+\left(\frac{M^2}{\sqrt{\alpha}}\right)^{k}} \phi^2(M)$$

$$\le ||g||_{H^k} \sup_{M\in R} \frac{\alpha^{k+j}+...+M^{2(k+j)}}{\alpha^k +...+M^{2k}}.$$

Estimating separately the ranges $|M| < 1$ and $|M| \ge 1$, then

$$\epsilon^{\frac{1}{3}j}||g_\epsilon||^2_{H^{k+j}} \le ||g||^2_{H^k} (k+j) \sup_{M\in R} \{1+M^{2j}\phi^2(M)\} \le C\ ||g||^2_{H^k} .$$

Hence $||g_\epsilon||_{H^{k+j}} \le \epsilon^{-\frac{1}{6}j} C\ ||g||_{H^k}$ (1) holds uniformly for each bounded subset of H^k, where C_1 does not depend on g and ϵ.

(2) The proof of (2) is similar to the proof of (1).

$$(3) \qquad ||g - g_\epsilon||^2_{H^k} = \int_{-\infty}^{\infty}(1+\lambda^2+\ldots+\lambda^{2k})||\hat{g} - \hat{g}_\epsilon||d\lambda$$

$$= \int_{-\infty}^{\infty}(1+\lambda^2+\ldots+\lambda^{2k})|\hat{g} - \hat{\Phi}\,(\epsilon^{\frac{1}{6}}\lambda)\hat{g}(\lambda)|^2\, d\lambda$$

$$= \int_{-\infty}^{\infty}(1-\hat{\Phi}(\epsilon^{\frac{1}{6}}\lambda))^2\,(1+\lambda^2+\ldots+\lambda^{2k})||\hat{g}(\lambda)||^2\, d\lambda$$

$$= \int_{-\infty}^{\infty}\psi^2(\epsilon^{\frac{1}{6}}\lambda)\,[1+\lambda^2+\ldots+\lambda^{2k}]||\hat{g}(\lambda)||^2\, d\lambda$$

(A20)

where $\psi = 1 - \Phi$ and as $\epsilon \to 0$ the integrand term vanishes almost everywhere. Using the Lebesgue dominated convergence theorem

$$||g - g_\epsilon||_{H^k} \to 0 \quad \text{as} \quad \epsilon \to 0 . \qquad \text{(A21)}$$

Since for all n

$$||g_{n\epsilon}-g_\epsilon||^2_{H^k} = ||(g_n-g)_\epsilon||^2 = \int_{-\infty}^{\infty}\psi^2(\epsilon^{\frac{1}{6}}\lambda)(1+\lambda^2+\ldots+\lambda^{2k})|\hat{g}_n(\lambda)-\hat{g}(\lambda)|^2 d\lambda$$

$$\leq \int_{-\infty}^{\infty}(1+\lambda^2+\ldots+\lambda^{2k})|\hat{g}_n(\lambda)-\hat{g}(\lambda)|^2 d\lambda = ||g_n-g||^2_{H^k} .$$

But

$$||g_{n\epsilon}-g_n||_{H^k} \leq ||g_{n\epsilon}-g_\epsilon||_{H^k} + ||g_\epsilon-g||_{H^k} + ||g-g_n||_{H^k}$$

$$\leq ||g_n-g||_{H^k} + ||g_\epsilon-g||_{H^k} + ||g-g_n||_{H^k}.$$

(A22)

Hence if $g_n \to g$ in H^k and δ be given, then by choosing N so large we choose ϵ_o so small such that $\|g_{m\epsilon} - g_m\|_{H^k} \leq \frac{1}{3}\delta$, $1 \leq m \leq N$ and $\|g_\epsilon - g\| < \frac{1}{3}\delta$, $\epsilon \in (0, \epsilon_o]$.

(A23)

Hence (A22) leads to

$$\|g_{n\epsilon} - g_n\|_{H^k} \leq \frac{1}{3}\delta + \frac{1}{3}\delta + \frac{1}{3}\delta = \delta \quad \text{for all} \quad n > N$$

i.e. $g_{n\epsilon} \to g_n$ uniformly. Since sequentially compacteness equivalent to the compacteness, (3) is proved.

(4) Since $g_\epsilon \in H^\infty$ (by the definition) then by lemma (3.7.2) u_ϵ has an upper bound depending only on T, ϵ_o, $\|g_\epsilon\|_{H^k}$ and $\epsilon^{\frac{1}{2}}\|g_\epsilon\|_{H^{k+1}}$ but since

$$\|g_\epsilon\|_{H^k} \leq \|g\|_{H^k} \text{ and } \epsilon^{\frac{1}{2}}\|g_\epsilon\|_{H^{k+1}} \leq \epsilon^{\frac{1}{2}}\epsilon^{-\frac{1}{6}}C\|g\|_{H^k} = C\epsilon^{\frac{1}{3}}\|g\|_{H^k}$$

(where $\epsilon^{\frac{1}{2}j}\|g_\epsilon\|_{H^{k+1}} \leq C\|g\|_{H^k}$).

Then $\|u_\epsilon\|_{H^k}$ has an upper bound depending only on T, ϵ_o and $\|g\|_{H^k}$. Similarly,

$\|g_\epsilon\|_{H^{k+m}} \leq \|g\|_{H^{k+m}}$, $k+m>3 \to \epsilon^{m/6}u_\epsilon$ is bounded in $\mathcal{H}_T^{k+m}$ independently of sufficient small ϵ for each finite $T > 0$, $m > 1$ which proves (4).

(5) Re-write the RKdV equation 3.7.9 in the form

$$(1 - \epsilon\frac{\partial^2}{\partial x^2})\frac{\partial u_\epsilon}{\partial t} = -[\tfrac{1}{2}u_\epsilon^2 + \frac{\partial^2}{\partial x^2}u_\epsilon]_x . \qquad \text{(A24)}$$

Then using the Green function of the operator $(1 - \epsilon \frac{\partial^2}{\partial x^2})$, then (A24) implies

$$||\partial_t u_\epsilon||_{H^{k-3}} \le ||u_\epsilon||_{H^{k-3}} \; ||\frac{\partial}{\partial x} u_\epsilon||_{H^{k-3}} + ||\frac{\partial^3}{\partial x^3} u_\epsilon||_{H^{k-3}}. \qquad (A25)$$

Then $\frac{\partial}{\partial t} u_\epsilon$ is bounded in $\mathcal{H}^{k-3}_T$ independently of sufficiently small ϵ for all finite $T > 0$.

Similarly

$$\epsilon^{\frac{1}{6}m}||\frac{\partial}{\partial t} u_\epsilon||_{H^{k+m-3}} \le ||u_\epsilon||_{H^{k+m-3}}||\frac{\partial}{\partial x} u_\epsilon||_{H^{k+m-3}} + ||\frac{\partial}{\partial x^3}u_\epsilon||_{H^{k+m-3}}$$

$$\le \epsilon^{\frac{1}{6}m}(||u_\epsilon||_{H^{k+m-3}} \; ||u_\epsilon||_{H^{k+m-2}} + ||u_\epsilon||_{H^{k+m}}) < C$$

where C does not depend on sufficiently small ϵ (from 4) which proves (5). ∎

APPENDIX B

Proof of theorem 5.4

Using 5.7.12 in (5.7.13) then

$$u_t + u_x + a_1uu_x + a_2uu_t + a_3u_{xxx} + a_4u_{xxt} + a_5u_{xtt} + a_6u_{ttt}$$

$$= [2vv_t + \alpha v_{xt} + \beta v_{tt}] + [2vv_x + \alpha v_{xx} + \beta v_{xt}]$$

$$- 6a_1[v^2 + \alpha v_x + \beta v_t]\ [2vv_x + \alpha v_{xx} + \beta v_{xt}]$$

$$- 6a_2[v^2 + \alpha v_x + \beta v_t][2vv_t + \alpha v_{xt}$$

$$+ \beta v_{tt}] + a_3[2vv_{xxx} + 6v_xv_{xx} + \beta v_{xxxt} + \alpha v_{xxxx}]$$

$$+ 4[2vv_{xxt} + 2v_tv_{xx} + 4v_xv_{xt} + \alpha v_{xxxt} + \beta v_{xxtt}]$$

$$+ a_5[2vv_{xtt} + 2v_xv_{tt} + 4v_tv_{xt} + \alpha v_{xxtt} + \beta v_{xttt}]$$

$$+ a_6[2vv_{ttt} + 6v_tv_{tt} + \alpha v_{xttt} + \beta v_{tttt}]$$

$$= (2v + \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t})[v_t + v_x - 6a_1v^2v_x - 6a_2v^2v_t + a_3v_{xxx}$$

$$+ a_4v_{xxt} + a_5v_{xtt} + a_6v_{ttt}] + 6(a_3 - \alpha^2a_1)v_xv_{xx}$$

$$+ 2(2a_4 - 3\alpha\beta a_1 - 3\alpha^2a_2)v_xv_{xt} + 2(a_4 - 3\alpha\beta a_1)v_tv_{xx}$$

$$+ 2(2a_5 - 3\beta^2a_1 - 3\alpha\beta a_2)v_tv_{xt} + 2(a_5 - 3\alpha\beta a_2)v_xv_{tt}$$

$$+ 6(a_6 - \beta^2a_2)v_tv_{tt}. \qquad (B.1)$$

Let v evolves according to (5.7.11), then (B.1) reduces to

$$u_t + u_x - 6a_1uu_x - 6a_2uu_t + a_3u_{xxx} + a_4u_{xxt} + a_5u_{xtt}$$

$$+ a_6u_{ttt} = 6(a_3 - \alpha^2a_1)v_xv_{xx} + 2(2a_4 - 3\alpha\beta a_1 - 3\alpha^2a_2)v_xv_{xt}$$

$$+ 2(a_4 - 3\alpha\beta a_1)v_t v_{xx} + 2(2a_5 - 3\beta^2 a_1 - 3\alpha\beta a_2)v_t v_{xt}$$

$$+ 2(a_5 - 3\alpha\beta a_2)v_x v_{tt} + 6(a_6 - \beta^2 a_2)v_t v_{tt} \qquad (B.2)$$

(B.2) vanishes if and only if the relations

$$(a_3 - \alpha^2 a_1) = 0, \quad (2a_4 - 3\alpha\beta a_1 - 3\alpha^2 a_2) = 0,$$

$$a_4 - 3\alpha\beta a_1 = 0, \quad 2a_5 - 3\beta^2 a_1 - 3\alpha\beta a_2 = 0, \quad a_5 - 3\alpha\beta a_2 = 0$$

and

$$a_6 - \beta^2 a_2 = 0 \qquad (B.3)$$

satify simultaneously. Hence

$$a_3 = \alpha^2 a_1, \quad a_4 = 3\alpha\beta a_1, \quad a_5 = 3\alpha\beta a_2 \text{ and } a_6 = \beta^2 a_2$$

then the second relation in (B.3) implies, $a_1 = \frac{\alpha}{\beta} a_2$.

Hence the cubic equation (5.7.14) implies

$$0 = \frac{\alpha^3}{\beta} a_1 - 3\alpha^2\lambda + 3\alpha\beta\lambda^2 - \beta^3\lambda^3.$$

Thus

$$\alpha^3 - 3\alpha^2\beta\lambda + 3\alpha\beta^2\lambda^2 - \beta^3\lambda^3 = 0. \qquad (B.4)$$

Equation (B.4) clearly has three real equal roots λ_o,

$$\lambda_o = \frac{\alpha}{\beta} \qquad (B.5)$$

which is the necessary condition of the KdV class.

APPENDIX C

Proof of lemma 6.6.1

Without loss of generality, we assume for the proof that A is a symmetric matrix and recall that linear hyperbolic system can be transformed into a symmetric form (e.g. normal form as in section 6.3). Let $p(\xi, \tau)$ be an arbitrary point in the domain of determinacy with coordinates ξ, τ and draw the characteristics $c_1, c_2, \ldots$ and c_k backward to meet the line $t = 0$ at $p_1, p_2, \ldots, p_k$.

Using the Green's identity [5]

$$(W, AW)_x = (W_x, AW) + (W, A_xW) + (W, AW_x). \qquad \text{(C1)}$$

Then, since A is symmetric, then $(W, AW_x) = (AW, W_x)$.

However (C1) reduces to

$$2(W, AW_x) = (W, AW)_x - (W, A_xW) \qquad \text{(C2)}$$

[where (.,.) stands for the inner product].

Now, taking the inner product of the equation in (6.6.6) with the vector W, then

$$(W, W_t) + (W, AW_x) + (W, BW) = 0. \qquad \text{(C3)}$$

Inserting (C2) in (C3), we have

$$\frac{1}{2}(W, W)_t + \frac{1}{2}(W, AW)_x - \frac{1}{2}(W, A_xW) + (W, BW) = 0. \qquad \text{(C4)}$$

Introducing the transformation

$$W = e^{\gamma t}V. \qquad \text{(C5)}$$

Then (6.6.6) reduces to

$$\gamma e^{\gamma t}V + e^{\gamma t}V_t + Ae^{\gamma t}V_x + Be^{\gamma t}V = 0, \quad \text{i.e.}$$

$$V_t + AV_x + (B + \gamma I)V = 0, \quad \text{or for simplicity}$$

$$V_t + AV_x + B_1V = 0 \tag{C6}$$

where $B_1 = B + \gamma I$.

Then, by taking the inner product of (C6) with V and in a sense similar to that used to derive (C4)

$$\frac{1}{2}(V,V)_t + \frac{1}{2}(V,AV)_x - \frac{1}{2}(V,A_xV) + (V,B_1V) = 0. \tag{C7}$$

Hence V can be replaced by W in (C7) and yields
$\frac{1}{2}(W,W)_t + \frac{1}{2}(W,AW)_x = (W, (-B - \frac{1}{2}A_x)W)$, i.e.

$$\frac{1}{2}(W,W)_t + \frac{1}{2}(W,AW)_x = (W,B_2W), \tag{C8}$$

where $B_2 = -B - \frac{1}{2}A_x = -B - \gamma I - \frac{1}{2}A_x$. (C9)

If now γ is sufficiently large then B_2 is negative definite,

i.e. $(W, B_2W) \leq 0$. (C10)

Thus integrating (C8) over the trapezoid Γ_d which has the boundary $P_1P_kA_kA_1$, where A_k and A_1 are two points on the characteristic c_k and c_1 (respectively). Then

$$0 \geq \frac{1}{2}\iint_{\Gamma_d} [(W,W)_t + (W,AW)_x]dxdt. \tag{C11}$$

Now, if x_n and t_n be the components of the outward normal unit vector, then by using Green formula , (C11) implies

$$0 \geq \frac{1}{2} \iint_{\Gamma_d} [(W,W)_t + (W,AW)_x] dxdt = \frac{1}{2} \int_{\Gamma_d} [(W,W)t_n + (W,AW)x_n] ds$$

$$= \frac{1}{2} \int_{A_1A_k+P_kP_1} (W,W)dx + \frac{1}{2} \int_{C_1+C_k} x_n(W,[A + \frac{t_n}{x_n} I]W)ds \qquad (C12)$$

If $E(d) = \frac{1}{2} \int_{A_1}^{A_k} (W,W)dx$, then (C12) gives

$$E(d) - E(0) = -\frac{1}{2} \int_{C_1+C_k} x_n(W,[A + \frac{t_n}{x_n} I])ds. \qquad (C13)$$

The right hand side of (C13) is in fact nonpositive. (For the proof of this fact it is refered to [3]).

Thus $E(d) \leq E(0)$.

By $E(0) = 0$, from the assumptions of the lemma, then $E(d) = 0$; therefore $W = 0$. ∎

APPENDIX D

In this appendix, some calculations, which were necessary for proving theorem 7.5 are presented.

$$D_x = \frac{\partial}{\partial x} + \sum_{k\geq 0} u_{k+1}\frac{\partial}{\partial u_k} + \sum_{k\geq 1} w_{k+1}\frac{\partial}{\partial v_k} + \sum_{k\geq 2}[v_{k-1} - w_{k+1}$$

$$- \sum_{i=0}^{k-2}\binom{k-2}{i} v_i w_{k-i-1}]\frac{\partial}{\partial w_k} + \sum_{k\geq 3}\mu_{k+1}\frac{\partial}{\partial \mu_k}. \qquad (D1)$$

$$D_t = \frac{\partial}{\partial t} + \sum_{k\geq 2}\mu_{k+1}\frac{\partial}{\partial u_k} + \sum_{k\geq 1} v_{k+1}\frac{\partial}{\partial v_k} + \sum_{k\geq 2} w_{k+1}\frac{\partial}{\partial w_k}$$

$$+ \sum_{k\geq 3}[\mu_{k-1} - \mu_{k+1} - \sum_{i=0}^{k-2}\binom{k-2}{i} u_i u_{k-i-1}]\frac{\partial}{\partial \mu_k} + v_1\frac{\partial}{\partial u} + w_2\frac{\partial}{\partial u_1}.$$

$$(D2)$$

Using (D1) and (D2), we have

$$D_xD_t = (\frac{\partial^2}{\partial x\partial t} + \sum_{k\geq 0} u_{k+1}\frac{\partial^2}{\partial u_k\partial t} + \sum_{k\geq 1} w_{k+1}\frac{\partial^2}{\partial v_k\partial t} + \sum_{k\geq 2}[v_{k-1} - w_{k+1}$$

$$- \sum_{i=0}^{k-2}\binom{k-2}{i} v_i w_{k-i-1}]\frac{\partial^2}{\partial w_k\partial t} + \sum_{k\geq 2}\mu_{k+1}\frac{\partial^2}{\partial \mu_k\partial t})$$

$$+ \sum_{k\geq 3}[\mu_{k+1}D_x\frac{\partial}{\partial u_k} + \mu_{k-2}\frac{\partial}{\partial u_k}] + \sum_{k\geq 1}[v_{k+1}D_x\frac{\partial}{\partial v_k} + w_{k+2}\frac{\partial}{\partial v_k}]$$

$$+ \sum_{k+2}\{w_{k+1}D_x\frac{\partial}{\partial w_k} + [v_k - w_{k+2} - \sum_{i=0}^{k-1}\binom{k-1}{i} v_i w_{k-i}]\frac{\partial}{\partial w_k}\}$$

$$+ \sum_{k\geq 3}[\mu_{k-1} - \mu_{k+1} - \sum_{i=0}^{k-2}\binom{k-2}{i} u_i u_{k-i-1}]D_x\frac{\partial}{\partial \mu_k}$$

$$+ \sum_{k\geq 3} [\mu_k - \mu_{k+2}] - \sum_{i=0}^{k-2} \binom{k-2}{i} (u_i u_{k-i} + u_{i+1} u_{k-i-1}] \frac{\partial}{\partial \mu_k}$$

$$+ v_1 D_x \frac{\partial}{\partial u} + w_2 \frac{\partial}{\partial u} + w_2 D_x \frac{\partial}{\partial u_1} + [v_1 - uu_1 - w_3] \frac{\partial}{\partial u_1}. \qquad (D3)$$

Thus

$$D_x^2 D_t = \frac{\partial^3}{\partial x^2 \partial t} + \sum_{k\geq 0} u_{k+1} \frac{\partial^3}{\partial u_k \partial x \partial t} + \sum_{k\geq 1} w_{k+1} \frac{\partial^3}{\partial v_k \partial x \partial t}$$

$$+ \sum_{k\geq 2} [v_{k-1} - w_{k+1} - \sum_{i=0}^{k-2} \binom{k-2}{i} v_i w_{k-i-1}] \frac{\partial^3}{\partial w_k \partial x \partial t}$$

$$+ \sum_{k\geq 3} \mu_{k+1} \frac{\partial^3}{\partial \mu_k \partial x \partial t} + \sum_{k\geq 0} [u_{k+1} D_x \frac{\partial^2}{\partial u_k \partial t} + u_{k+2} \frac{\partial^2}{\partial u_k \partial t}\}$$

$$+ \sum_{k\geq 1} [w_{k+1} D_x \frac{\partial^2}{\partial v_k \partial t} + \{v_k - w_{k+2} - \sum_{i=0}^{k-1} \binom{k-1}{i} v_i w_{k-i})] \frac{\partial^2}{\partial v_k \partial t}$$

$$+ \sum_{k\geq 2} [v_{k-1} - w_{k+1} - \sum_{i=0}^{k-2} \binom{k-2}{i} v_i w_{k-i-1}] D_x \frac{\partial^2}{\partial w_k \partial t}$$

$$+ \sum_{k\geq 2} [w_k - v_k - w_{k+2} + \sum_{i=0}^{k-1} \binom{k-1}{i} v_i w_{k-i}$$

$$- \sum_{i=0}^{k-2} \binom{k-2}{i} (v_i w_{k-i} + w_{i+1} w_{k-i-1}] \frac{\partial^2}{\partial w_k \partial t}$$

$$+ \sum_{k\geq 2} [\mu_{k+1} D_x \frac{\partial^2}{\partial \mu_k \partial t} + \mu_{k+2} \frac{\partial^2}{\partial \mu_k \partial t}\})$$

$$+ \sum_{k\geq 0} [\mu_{k+2} D_x^2 \frac{\partial}{\partial u_k} + 2\mu_{k+2} D_x \frac{\partial}{\partial u_k} + \mu_{k+3} \frac{\partial}{\partial u_k}]$$

$$+ \sum_{k\geq 1} [v_{k+1} D_x^2 \frac{\partial}{\partial v_k} + 2w_{k+2} D_x \frac{\partial}{\partial v_k}$$

$$+ (v_{k+1} - w_{k+3} - \sum_{i=0}^{k} \binom{k}{i} v_i w_{k+i+1}) \frac{\partial}{\partial v_k}]$$

$$+ \sum_{k\geq 2} \{w_{k+1} D_x^2 \frac{\partial}{\partial w_k} + 2\ [v_k - w_{k+2} - \sum_{i=0}^{k-1} \binom{k-1}{i} v_i w_{k-i}] D_x \frac{\partial}{\partial w_k}\}$$

$$+ \sum_{k\geq 2} \{w_{k+1} - v_{k+1} + w_{k+3} - \sum_{0}^{k} \binom{k}{i} v_i w_{k-i}$$

$$- \sum_{0}^{k-1} \binom{k-1}{i} v_i [v_{k-1-i} - w_{k-i+1} - \sum_{0}^{k-i-2} \binom{k-i-2}{j} v_j w_{k-2j-i}]$$

$$+ w_{i+1} w_{k-i}\} \frac{\partial}{\partial w_k} + \sum_{k\geq 3} [\mu_{k-1} - \mu_{k+1} - \sum_{0}^{k-2} \binom{k-2}{i} u_i u_{k-i-1}] D_x^2 \frac{\partial}{\partial \mu_k}$$

$$+ 2 \sum_{k\geq 3} [\mu_k - \mu_{k+2} - \sum_{0}^{k-2} \binom{k-2}{i} (u_i u_{k-i} + u_{i+1} u_{k-i-1})] D_x \frac{\partial}{\partial \mu_k}$$

$$+ \sum_{k\geq 3} \{\mu_{k+1} - \mu_{k+3} + \sum_{0}^{k-2} \binom{k-2}{i} [u_i u_{k-i-1} + 2u_{i+1} u_{k-i}$$

$$+ u_{i+2} u_{k-i-1}]\} \frac{\partial}{\partial u_k} + v_1 D_x^2 \frac{\partial}{\partial u} + 2w_2 D_x \frac{\partial}{\partial u} + [v_1 - uu_1 - w_3] \frac{\partial}{\partial u}$$

$$+ w_2 D_x^2 \frac{\partial}{\partial u_1} + 2[v_1 - uu_1 - w_3] D_x \frac{\partial}{\partial u_1}$$

$$+ [w_2 - uu_2 - u_1^2 - v_2 + uw_2 + u_1 v_1 + w_4] \frac{\partial}{\partial u_1}. \qquad \text{(D4)}$$

Similarly

$$D_t^2 D_x = \frac{\partial^3}{\partial t \partial x \partial t} + \sum_{k\geq 0} \mu_{k+1} \frac{\partial^3}{\partial u_k \partial x \partial t} + \sum_{k\geq 1} v_{k+1} \frac{\partial^3}{\partial v_k \partial x \partial t}$$

$$+ \sum_{k\geq 2} w_{k+1} \frac{\partial^3}{\partial w_k \partial x \partial t} + \sum_{k\geq 3} [\mu_{k-1} - \mu_{k+1}$$

$$- \sum_{0}^{k-2} \binom{k-2}{i} u_i u_{k-i-1}] \frac{\partial^3}{\partial \mu_k \partial x \partial t} + w_2 \frac{\partial^3}{\partial u_1 \partial x \partial t}$$

$$+ v_1 \frac{\partial^3}{\partial u \partial x \partial t} + \sum_{k\geq 0} [u_{k+1} D_t \frac{\partial^2}{\partial u_k \partial t} + \mu_{k+2} \frac{\partial^2}{\partial u_k \partial t}]$$

$$+ \sum_{k\geq 1} [w_{k+1} D_t \frac{\partial^2}{\partial v_k \partial t} + w_{k+2} \frac{\partial^2}{\partial v_k \partial t}] + \sum_{k\geq 2} \{[v_{k-1} - w_{k+1}$$

$$- \sum_{0}^{k-2} \binom{k-2}{i} v_i w_{k-i-1}] D_t \frac{\partial^2}{\partial w_k \partial t} + [v_k - w_{k+2}$$

$$- \sum_{0}^{k-2} \binom{k-2}{i} (v_i w_{k-i} + v_{i+1} w_{k-i-1})] \frac{\partial^2}{\partial w_k \partial t} \}$$

$$+ \sum_{k\geq 3} \mu_{k+1} D_t \frac{\partial^2}{\partial \mu_k \partial t} + \sum_{k\geq 3} [\mu_k - \mu_{k+2}$$

$$- \sum_{0}^{k-1} \binom{k-1}{i} u_i u_{k-i}] \frac{\partial^2}{\partial \mu_k \partial t} + \sum_{k\geq 3} \{\mu_{k+1} D_t D_x \frac{\partial}{\partial u_k} + [\mu_k - \mu_{k+1}$$

$$- \sum_{0}^{k-1} \binom{k-1}{i} u_i u_{k-i}] D_x \frac{\partial}{\partial u_k} + \mu_{k+2} D_t \frac{\partial}{\partial u_k}$$

$$+ [\mu_{k+1} - \mu_{k+3} - \sum_{0}^{k} \binom{k}{i} u_i u_{k-i+1}] \frac{\partial}{\partial u_k} \}$$

$$+ \sum_{k\geq 1} \{v_{k+1} D_t D_x \frac{\partial}{\partial v_k} + v_{k+2} D_x \frac{\partial}{\partial v_k} + w_{k+2} D_t \frac{\partial}{\partial v_k} + w_{k+3} \frac{\partial}{\partial v_k} \}$$

$$+ \sum_{k\geq 2} \{w_{k+1} D_t D_x \frac{\partial}{\partial w_k} + w_{k+2} D_x \frac{\partial}{\partial w_k}$$

$$+ [v_k - w_{k+2} - \sum_0^{k-1}\binom{k-1}{i} v_i w_{k-i}] D_t \frac{\partial}{\partial w_k} + [v_{k+1} - w_{k+3}$$

$$- \sum_0^{k-1}\binom{k-1}{i}(v_i w_{k-i+1} + v_{i+1} w_{k-i}]\frac{\partial}{\partial w_k}\} + \sum_{k\geq 3}\{[\mu_{k-1} - \mu_{k+1}$$

$$- \sum_0^{k-2}\binom{k-2}{i} u_i u_{k-i-1}] D_t D_x \frac{\partial}{\partial \mu_k} + [\mu_{k-2} - \mu_k$$

$$- \sum_0^{k-3}\binom{k-3}{i} u_i u_{k-i-2} - \mu_k + \mu_{k+2} + \sum_0^{k-1}\binom{k-1}{i} u_i u_{k-i}$$

$$- \sum_0^{k-2}\binom{k-2}{i}(u_i \mu_{k-i} + \mu_{i+1} u_{k-i-1}] D_x \frac{\partial}{\partial \mu_k}\} + \sum_{k\geq 3}\{\mu_k - \mu_{k+2}$$

$$- \sum_0^{k-2}\binom{k-2}{i} u_i u_{k-i} + u_{i+1} u_{k-i+1}) D_t \frac{\partial}{\partial \mu_k} + [\mu_{k-1} - \mu_{k+1}$$

$$- \sum_0^{k-2}\binom{k-2}{i} u_i u_{k-i-1} - \mu_{k+1} + \mu_{k+3} + \sum_0^{k}\binom{k}{i} u_i u_{k-i+1}$$

$$- \sum_0^{k-2}\binom{k-2}{i}(u_i \mu_{k-i+1} + \mu_{i+1} u_{k-i} + u_{i+1} \mu_{k-i+2}$$

$$+ \mu_{i+2} u_{k-i+1})]\} \frac{\partial}{\partial \mu_k} + v_1 D_t D_x \frac{\partial}{\partial u} + v_2 D_x \frac{\partial}{\partial u} + w_2 D_t \frac{\partial}{\partial u}$$

$$+ w_3 \frac{\partial}{\partial u} + w_2 D_t D_x \frac{\partial}{\partial u} + w_3 D_x \frac{\partial}{\partial u} + [v_1 - u u_1 - w_3] D_t \frac{\partial}{\partial u_1}$$

$$+ [v_2 - u w_2 - u_1 v_1 - w_4]\frac{\partial}{\partial u_1}. \qquad \text{(D5)}$$

$$D_x^2 = \frac{\partial^2}{\partial x^2} + \sum_{k\geq 0} u_{k+1} \frac{\partial^2}{\partial u_k \partial x} + \sum_{k\geq 1} w_{k+1} \frac{\partial^2}{\partial v_k \partial x} + \sum_{k\geq 2} [v_{k-1} - w_{k+1}$$

$$- \sum_0^{k-2} \binom{k-2}{i} v_i w_{k-i-1}] \frac{\partial}{\partial w_k \partial x} + \sum_{k\geq 3} \mu_{k+1} \frac{\partial^2}{\partial \mu_k \partial x} + \sum_{k\geq 0} u_{k+1} D_x \frac{\partial}{\partial u_k}$$

$$+ \sum_{k\geq 0} u_{k+2} \frac{\partial}{\partial u_k} + \sum_{k\geq 1} w_{k+1} D_x \frac{\partial}{\partial v_k} + \sum_{k\geq 3} [v_k - w_{k+2}$$

$$- \sum_0^{k-1} \binom{k-1}{i} v_i w_{k-i}] \frac{\partial}{\partial v_k} + \sum_{k\geq 2} [v_{k-1} - w_{k+1}$$

$$- \sum_0^{k-2} \binom{k-2}{i} v_i w_{k-i-1}] D_x \frac{\partial}{\partial w_k} + \sum_{k\geq 2} \{w_k - v_k + w_{k+2}$$

$$+ \sum_0^{k-1} \binom{k-1}{i} v_i w_{k-i} - \sum_0^{k-2} \binom{k-2}{i} [w_{i+1} w_{k-i-1} + v_i (v_{k-i-2} - w_{k-i}$$

$$- \sum_0^{k-i-3} \binom{k-i-3}{j} v_j w_{k-2j-2})]\} \frac{\partial}{\partial w_k} + \sum \mu_{k+1} D_x \frac{\partial}{\partial \mu_k}$$

$$+ \sum \mu_{k+2} \frac{\partial}{\partial \mu_k}. \qquad \text{(D6)}$$

$$D_t D_x = \sum_{k\geq 3} [\mu_{k-1} - \mu_{k+1} - \sum_0^{k-2} \binom{k-2}{i} u_i u_{k-i-1}] \frac{\partial^2}{\partial \mu_k \partial t} + \frac{\partial^2}{\partial t \partial x}$$

$$+ \sum_{k\geq 2} \mu_{k+1} \frac{\partial^2}{\partial u_k \partial t} + \sum_{k\geq 1} v_{k+1} \frac{\partial^2}{\partial v_k \partial t} + \sum_{k\geq 2} w_{k+1} \frac{\partial^2}{\partial w_k \partial t}$$

$$+ v_1 \frac{\partial^2}{\partial u \partial t} + w_2 \frac{\partial^2}{\partial u_1 \partial t} + \sum_{k\geq 0} u_{k+1} D_t \frac{\partial}{\partial u_k} + \sum_{k\geq 0} \mu_{k+2} \frac{\partial}{\partial u_k}$$

$$+ \sum_{k\geq 0} w_{k+1} D_t \frac{\partial}{\partial v_k} + \sum_{k\geq 1} w_{k+2} \frac{\partial}{\partial v_k}$$

$$+ \sum_{k\geq 2} [v_{k-1} - w_{k+1} - \sum_{0}^{k-2} \binom{k-2}{i} v_i w_{k-i-1}] D_t \frac{\partial}{\partial w_k} + \sum_{k\geq 2} [v_k - w_{k+2}$$

$$- \sum_{0}^{k-2} \binom{k-2}{i} (v_i w_{k-i} + v_{i+1} w_{k-i-1}) \frac{\partial}{\partial w_k} + \sum_{k\geq 3} \mu_{k+1} D_t \frac{\partial}{\partial \mu_k}$$

$$+ \sum_{k\geq 3} [\mu_k - \mu_{k+2} - \sum_{0}^{k-1} \binom{k-1}{i} u_i u_{k-i}] \frac{\partial}{\partial \mu_k}. \qquad \text{(D7)}$$

Using the equations (D1), (D2), ..., (D6) and

$$\ell_P^*(\phi) = (D_x^2 D_t + D_t^2 D_x - D_t + uD_x)\phi = 0.$$

Then

coefficient $u_{n+2} = D_t \frac{\partial \phi}{\partial u_n} = 0 \rightarrow \phi$ is linear in u_n,

coefficient $v_{n+2} = D_x \frac{\partial \phi}{\partial v_n} = 0 \rightarrow \phi$ is linear in v_n,

coefficient $w_{n+2} = -(D_x + D_t) \frac{\partial \phi}{\partial w_n} + 2D_x \frac{\partial \phi}{\partial v_n} + D_t \frac{\partial \phi}{\partial v_n} = 0 \rightarrow$

$$-(D_x + D_t) \frac{\partial \phi}{\partial w_n} + D_t \frac{\partial \phi}{\partial v_n} = 0 \rightarrow \text{ is linear in } w_n$$

[since ϕ is in v_n and $\ker(D_x + D_t) = C^\infty(x,t)$].

Similarly

coefficient $\mu_{n+2} = -(D_x + D_t) \frac{\partial}{\partial \mu_n} + 2D \frac{\partial}{\partial u_n} + 2D_t \frac{\partial}{\partial u_n} = 0 \rightarrow \phi$ is linear in μ_n [since ϕ is linear in both u_n and v_n].

If we use now the coupling relation (7.3.36), i.e.

$$\mu_{n+2} = (-1)^{n+2} w_{n+2} + \Psi_1 \qquad (D8)$$

where Ψ_1 does not depend on w_{n+2}, μ_{n+2}, w_{n+1} and μ_{n+1}, then replacing μ_n by w_n from D8, in all the above calculations, then

coefficient $w_{n+2} = 0$ →

$$(D_x + D_t)[\frac{\partial}{\partial w_n} - (-1)^{n+1} \frac{\partial}{\partial \mu_n}] = \Upsilon(x,t) \qquad (D9)$$

where $\Upsilon(x,t)$ is now in terms of the derivative of Φ over u_n and v_n.

The solution of D9 has the form $f(w_n + (-1)^{n+1}\mu_n)$. By using D8 again, this solution reduces to $f(\Psi_2)$, Ψ_2 does not depend on w_n, μ_n, w_{n-1} and μ_{n-1}.

Then

coefficient $v_{n+1} = 0$ → $D_x \frac{\partial \Phi}{\partial v_{n-1}} = \jmath$, $\jmath$ is constant, depending only on the derivatives of Φ over v_n, and

coefficient $u_{n+1} = 0$ → $D_t \frac{\partial \Psi}{\partial u_{n-1}} = \beta''(x,t) + \alpha(x)u_1$, where $\alpha(x)$ is the derivative of Φ over u_n, i.e. Φ is linear in v_{n-1} and does not depend on u_n [since by lemma 7.3.4, $\alpha(x) = 0$]. Now, the coefficients w_n, can be calculated and equated to zero, in a similar sense to prove that Φ does not depend on v_n, either.

APPENDIX E

Glossary of spaces

(1) $L^2(R) = \{f(x) \mid \int_{-\infty}^{\infty} |f|^2 dx < \infty\}, \quad ||f||^2_{L^2(R)} = \int_{-\infty}^{\infty} |f(x)|^2 dx.$

(2) $H^k(R) = \{g(x) \in L^2(R) \mid \frac{d^i g}{dx^i} \in L^2(R) \quad \text{for} \quad 1 \leq i \leq k\},$

$$||g(x)||_{H^k(R)} = \sum_{i=0}^{k} \int_{-\infty}^{\infty} |\frac{d^i g}{dx^i}| dx = \sum_{i=0}^{k} ||\frac{d^i g}{dx^i}||_{L^2(R)}.$$

(3) $\mathcal{H}_T^k = C(0,\ T;\ H^k) = \{u(x,t) : R \times [0,T] \to R$ for each

$t \in [0,T],\ u(\cdot,t) \in H^k$ and $u : [0,T] \to H^k$ is continuous

and bounded$\}$, $\quad ||u||_{\mathcal{H}_T^k} = \sup_{0 \leq t \leq T} ||u(\cdot,t)||_{H^k}$.

(4) $\mathcal{H}_T^{k,m} = \{u(x,t) \in {}^{k}_{T} \mid \partial_t^i u \in {}^{k}_{T},\quad 0 \leq i \leq m\},$

$$||u||_{k,m} = \sup_{0 \leq t \leq T} \ \sup_{0 \leq i \leq m} ||\partial_t^i u(x,t)||_{H^k}.$$

(5) $\mathcal{X}_{s,T} = \mathcal{H}_T^s \cap \mathcal{H}_T^{s-3,1} \cap \mathcal{H}_T^{s-6,2} \cap \quad \cdots$

$= \{u(x,t) \in {}^{s}_{T} \mid \partial_t^i u \in {}^{s-3j}$ for j such that

$s-3j \geq 0\}.$

(6) $\mathcal{C}_T = [C(R \times [0,T])] = \{v(x,t) \mid v$ is continuous and uniformly

bounded on $R \times [0,T]\}$,

$$||v||_{\mathcal{C}_T} = \sup_{\substack{x \in R \\ 0 \leq t \leq T}} |v(x,t)|.$$

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