The pseudo McMillan degree of implicit transfer functions of RLC networks

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Abstract We study the structure of a given RLC network without sources. Since the McMillan degree of the implicit network transfer function is not a suitable measure for the complexity of the network, we introduce the pseudo McMillan degree to overcome these shortcomings. Using modified nodal analysis models, which are linked directly to the natural network topology, we show that the pseudo McMillan degree equals the sum of the number of capacitors and inductors minus the number of fundamental loops of capacitors and fundamental cutsets of inductors; this is the number of independent dynamic elements in the network. Exploiting this representation we derive a minimal realization of the given RLC network, that is one where the number of involved (independent) differential equations equals the pseudo McMillan degree.

Keywords RLC networks · modified nodal analysis · McMillan degree · minimal realization

1 Introduction

In the present paper we consider complexity measures for linear time-invariant RLC networks with finitely many elements and without sources. We investigate models of RLC networks which arise from modified nodal analysis (MNA),

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see [14] and the survey [27], and can thus be described by a linear differential-algebraic equation of the form

$$E\dot{x}(t) = Ax(t), \tag{1}$$

where

$$sE - A = \begin{bmatrix} sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top & s\mathcal{L} \end{bmatrix} \in \mathbb{R}[s]^{n\times n}, \tag{2}$$

$$x = \begin{pmatrix} \eta \\ i_{\mathcal{L}} \end{pmatrix}, \tag{3}$$

and

$$\left.\begin{aligned} &\mathcal{C} \in \mathbb{R}^{n_{\mathcal{C}}\times n_{\mathcal{C}}}, \mathcal{G} \in \mathbb{R}^{n_{\mathcal{G}}\times n_{\mathcal{G}}}, \mathcal{L} \in \mathbb{R}^{n_{\mathcal{L}}\times n_{\mathcal{L}}}, \\ &A_{\mathcal{C}} \in \mathbb{R}^{n_e\times n_{\mathcal{C}}}, A_{\mathcal{R}} \in \mathbb{R}^{n_e\times n_{\mathcal{G}}}, A_{\mathcal{L}} \in \mathbb{R}^{n_e\times n_{\mathcal{L}}}, \\ &n = n_e + n_{\mathcal{L}}. \end{aligned}\right\} \tag{4}$$

Here $\mathbb{R}[s]$ denotes the ring of polynomials with coefficients in the set of real numbers $\mathbb{R}$. $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the constitutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials[1] and $i_{\mathcal{L}}(t)$ is the vector of currents through inductances. By $n_{\mathcal{C}}, n_{\mathcal{G}}, n_{\mathcal{L}}$ we denote the number of capacitances, resistances and inductances in the network, resp., and n_e+1 is the number of nodes in the network graph. The matrix pencil $sE - A$ is regular, i.e., $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$, provided that the network is connected and passive, cf. [4,22]. Then the implicit transfer function associated with (1) exists and is given by $G(s) = (sE - A)^{-1}$.

The complexity analysis of RLC networks is related to the problem of network redesign, see [19,20], i.e., the desire to change the natural dynamics of the network by modification of its elements and/or topology. In order to identify appropriate changes it is necessary to have a measure for the complexity of the network. It is commonly believed that the McMillan degree of the implicit transfer function $G(s)$ is such a measure. According to [12,29,31], the McMillan degree defines, roughly speaking, the minimum number of dynamic elements which are necessary to fully describe the network. We will show that this is not completely accurate.

A classical result of circuit theory, see e.g. [26, p. 322] and [25], says that the McMillan degree of a positive real implicit transfer function $G(s)$ equals the minimal number of reactive elements (i.e., capacitors and inductors) in any passive synthesis of $G(s)$. A synthesis procedure which achieves this minimal number for any (scalar) positive real function was provided in the seminal work by Brune [7]. However, a drawback of this procedure is that it requires the use of transformers, which is undesirable since there are some issues with their physical realization, cf. [15, p. 3]. The synthesis procedure developed by Bott and Duffin [6] resolves this drawback and does not need any transformers, however the number of reactive elements in the synthesized circuit is larger

[1] The node potential η_i expresses the voltage between the ith node in the network graph and the ground node.

than the McMillan degree in general, while improvements are available, see also [15–17,30].

The present paper does not deal with the synthesis problem, but with the structural properties of an implicit operator (represented by $G(s)$) describing the dynamics of a given and fix RLC network which we aim to analyze. We stress that, in particular, there is no overlap with the works [15–17,30] which are concerned with network synthesis. Also note that the latter refers to realization issues of transfer functions of single-input, single-output (SISO) systems; in our approach we consider properties of the implicit internal transfer function that has no inputs and no outputs. Apart from this fundamental difference between implicit internal models (discussed here) and the realization of SISO systems discussed in network synthesis (e.g., [15–17,30]), there are some common issues such as the McMillan degree, but this is where the similarity stops.

While it follows from the above mentioned synthesis results that the McMillan degree of the implicit transfer function of a given RLC network is a lower bound for the number of its reactive elements, a characterization of the McMillan degree in terms of the network topology is not available in the literature. In fact, linking the McMillan degree to the dynamic elements of the network is not an easy task, as it turns out that for some circuits the McMillan degree is smaller than the number of dynamic elements. In these circuits the dynamic elements are not independent in a certain sense. In order to rigorously define what "independent" means, we use the concepts of fundamental loops and cutsets. We further introduce the new concept of "pseudo McMillan degree" and show that it equals the sum of the number of capacitors and inductors minus the number of fundamental loops of capacitors and fundamental cutsets of inductors, i.e., the number of independent dynamic elements in the network. We stress that the McMillan degree and the pseudo McMillan degree are not equal in general, but show some relations between them. Note that this significantly improves earlier results obtained in [24].

Let us also mention that early results [5,8] define the order of complexity of an electrical network as the number of natural frequencies and derive some formulas in terms of the network topology. However, this measure is not equal to the number of independent dynamic elements in the network in general.

The implicit description that forms the core of the paper provides the natural setup for redesign of the network. Network redesign problems are often considered in the context of impedance and admittance models (see [32]) as discussed in [2,21,22,24]. The impedance-admittance operator is the implicit transfer function $G(s) = (sE - A)^{-1}$, which is a rational matrix function and not a scalar transfer function. The operator $G(s)$ describes the network, which has no inputs and no outputs, and by selecting different sets of inputs and outputs we can generate different transfer functions. From this viewpoint, $G(s)$ acts as a generator of all possible transfer functions associated with the network. We stress again that we consider the characterization of the (pseudo) McMillan degree of $G(s)$ and not the synthesis of scalar transfer functions. This problem has not been treated before except for [24] which provides an

exterior algebra characterization of the McMillan degree. In contrast to this, in the present paper we consider models arising from modified nodal analysis, which are linked directly to the natural network topology. This allows to derive the representation of the (pseudo) McMillan degree in terms of the network topology.

As a second main result, we exploit the representation of the (pseudo) McMillan degree to derive a minimal realization of a given RLC network, that is an equation of the form (1) where the number of involved (independent) differential equations equals the pseudo McMillan degree. To avoid any confusion, we stress that in the present paper a minimal realization does *not* mean a synthesized network with a minimal number of reactive elements. We illustrate our results by means of two examples.

2 Graph theoretical preliminaries

In this section we introduce the graph theoretical concepts (cf. for instance [10]) on which the modified nodal analysis is based. We further introduce the notions of fundamental loops and cutsets and characterize their number in terms of the incidence matrix of the network graph.

Definition 1 A *graph* is a triple $\mathcal{G} = (V, E, \varphi)$ consisting of a *node set* V and a *branch set* E together with an *incidence map*

$$\varphi : E \to V \times V, \quad e \mapsto \varphi(e) = (\varphi_1(e), \varphi_2(e)) \,,$$

where $\varphi_1(e) \neq \varphi_2(e)$ for all $e \in E$, i.e., the graph does not contain self-loops. If $\varphi(e) = (v_1, v_2)$, we call e to be *directed from* v_1 *to* v_2; v_1 is called the *initial node* and v_2 the *terminal node* of e.

Let $V' \subseteq V$ and let E' be a set of branches satisfying

$$E' \subseteq E|_{V'} := \{\ e \in E \ \mid \varphi_1(e) \in V' \text{ and } \varphi_2(e) \in V' \ \} \,.$$

Further let $\varphi|_{E'}$ be the restriction of φ to E'. Then the triple $\mathcal{K} := (V', E', \varphi|_{E'})$ is called a *subgraph of* $\mathcal{G}$. If $V' = V$, then $\mathcal{K}$ is called a *spanning subgraph*. A *proper subgraph* is one with $E \neq E'$.

For each branch e, define an additional branch $-e$ being directed from the terminal to the initial node of e, that is $\varphi(-e) = (\varphi_2(e), \varphi_1(e))$ for $e \in E$. Now define the set $\tilde{E} = \{\ e \mid e \in E \text{ or } -e \in E \ \}$. A tuple $w = (w_1, \ldots, w_r) \in \tilde{E}^r$, where for $i = 1, \ldots, r-1$,

$$v_0 := \varphi_1(v_1), \quad v_i := \varphi_2(w_i) = \varphi_1(w_{i+1})$$

is called *path from* v_0 *to* v_r; w is called *elementary path*, if $v_1, \ldots, v_r$ are distinct. A *loop* is an elementary path with $v_0 = v_r$. Two nodes v, v' are called *connected*, if there exists a path from v to v'. The graph itself is called connected, if any two nodes are connected. A subgraph $\mathcal{K} = (V', E', \varphi|_{E'})$ is called a *component of connectivity*, if it is connected and $\mathcal{K}^c := (V \setminus V', E \setminus E', \varphi|_{E \setminus E'})$ is a subgraph.

A *tree* is a minimally connected graph, i.e., it is connected without having any connected proper spanning subgraph. A spanning subgraph of a connected graph $\mathcal{G}$, which is a tree, is called a *tree in* $\mathcal{G}$. If $\mathcal{G}$ is not connected, with k components of connectivity, and $\mathcal{T}_i$ is a tree in any such component for $i = 1, \ldots, k$, then $\mathcal{T} = \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_k$ is called a *forest in* $\mathcal{G}$.

A spanning subgraph $\mathcal{K} = (V, E', \varphi|_{E'})$ is called a *cutset* of $\mathcal{G} = (V, E, \varphi)$, if its branch set is non-empty, $\mathcal{G} - \mathcal{K} := (V, E \setminus E', \varphi|_{E \setminus E'})$ is a disconnected subgraph and $\mathcal{G} - \mathcal{K}'$ is a connected subgraph for any proper spanning subgraph $\mathcal{K}'$ of $\mathcal{K}$.

In this work we consider only *finite graphs*, i.e., graphs with finite node set and finite branch set.

Definition 2 Let $\mathcal{G}$ be a graph, $\mathcal{K}, \mathcal{L}$ be spanning subgraphs of $\mathcal{G}$, and ℓ be a path of $\mathcal{G}$.

(i) $\mathcal{L}$ is called a $\mathcal{K}$-*cutset*, if $\mathcal{L}$ is a subgraph of $\mathcal{K}$ and a cutset of $\mathcal{G}$.
(ii) ℓ is called a $\mathcal{K}$-*loop*, if ℓ is a loop of $\mathcal{K}$.

A graph can have many $\mathcal{K}$-loops and $\mathcal{K}$-cutsets, resp., but not all of them are independent. In the following we introduce the crucial notions of fundamental $\mathcal{K}$-loops and $\mathcal{K}$-cutsets, which generalize the notions of fundamental loops and cutsets given e.g. in [1].

Definition 3 Let $\mathcal{G}$ be a graph and $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$. Further let $\mathcal{T}_1$ be a forest in $\mathcal{K}$ and $\mathcal{T}_2$ be a forest in $\mathcal{G} - \mathcal{K}$. Then

(i) every branch in $\mathcal{K} - \mathcal{T}_1$ closes a unique loop in $\mathcal{K}$ that consists of that branch and branches from $\mathcal{T}_1$ only. These loops are called *fundamental* $\mathcal{K}$-*loops of* $\mathcal{G}$.
(ii) $\mathcal{T}_2$ can be completed to a tree $\mathcal{T}_3$ in $\mathcal{G}$ by adding branches from $\mathcal{K}$ (if necessary). Every branch in $\mathcal{T}_3 - \mathcal{T}_2$ defines a unique cutset of $\mathcal{G}$ that consists of that branch and branches which are common to $\mathcal{G} - \mathcal{T}_3$ and $\mathcal{K}$ only. These cutsets are called *fundamental* $\mathcal{K}$-*cutsets of* $\mathcal{G}$.

Similar to [1] we may show that any $\mathcal{K}$-loop/$\mathcal{K}$-cutset can be expressed in terms of fundamental $\mathcal{K}$-loops/$\mathcal{K}$-cutsets, for any fix choice of trees/forests $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ as in Definition 3. Therefore, in particular, the number of fundamental $\mathcal{K}$-loops/$\mathcal{K}$-cutsets in a graph $\mathcal{G}$ is independent of the choice of the trees/forests and we may define, using the notation from Definition 3,

$$\begin{aligned}\mathrm{FL}_{\mathcal{K}} &:= |\{\ \ell \mid \ell \text{ is a fundamental } \mathcal{K}\text{-loop of } \mathcal{G} \text{ corresponding to } \mathcal{T}_1\ \}| \\ &= |\{\ e \mid e \text{ is a branch of } \mathcal{K} - \mathcal{T}_1\ \}|, \\ \mathrm{FC}_{\mathcal{K}} &:= |\{\ c \mid c \text{ is a fundamental } \mathcal{K}\text{-cutset of } \mathcal{G} \text{ corresponding to } \mathcal{T}_2 \text{ and } \mathcal{T}_3\ \}| \\ &= |\{\ e \mid e \text{ is a branch of } \mathcal{T}_3 - \mathcal{T}_2\ \}|.\end{aligned}$$

In the following we introduce the notion of an incidence matrix, which is helpful in describing the topology of RLC networks. In particular, we derive formulas for $\mathrm{FL}_{\mathcal{K}}$ and $\mathrm{FC}_{\mathcal{K}}$ using incidence matrices.

Definition 4 Let a graph $\mathcal{G} = (V, E, \varphi)$ with l branches $E = \{e_1, \dots, e_l\}$ and k nodes $V = \{v_1, \dots, v_k\}$ be given. Then the *all-node incidence matrix* of $\mathcal{G}$ is given by $A_0 = (a_{ij}) \in \mathbb{R}^{k \times l}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \varphi_1(e_j) = v_i, \\ -1, & \text{if } \varphi_2(e_j) = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since the rows of A_0 sum up to the zero row vector, one may delete an arbitrary row of A_0 to obtain a matrix A having the same rank as A_0. We call A an *incidence matrix* of $\mathcal{G}$. Usually, the chosen row corresponds to the ground node from V.

A spanning subgraph $\mathcal{K}$ of the graph $\mathcal{G}$ has an incidence matrix $A_\mathcal{K}$ which is constructed by deleting columns of the incidence matrix A of $\mathcal{G}$ corresponding to the branches of the complementary spanning subgraph $\mathcal{G} - \mathcal{K}$. By a suitable reordering of the branches, the incidence matrix reads

$$A = \begin{bmatrix} A_\mathcal{K} & A_{\mathcal{G}-\mathcal{K}} \end{bmatrix}. \tag{5}$$

In the following result we derive the number of fundamental $\mathcal{K}$-loops/$\mathcal{K}$-cutsets in terms of the incidence matrices $A_\mathcal{K}$, $A_{\mathcal{G}-\mathcal{K}}$; this improves the result in [28, Lem. 2.1 & Lem. 2.3].

Theorem 1 *Let $\mathcal{G}$ be a connected graph with incidence matrix $A \in \mathbb{R}^{(k-1)\times l}$. Further, let $\mathcal{K}$ be a spanning subgraph and assume that the branches of $\mathcal{G}$ are sorted in a way that* (5) *is satisfied. Then the following holds true:*

(i) $\mathrm{FL}_\mathcal{K} = \dim \ker A_\mathcal{K}$,

(ii) $\mathrm{FC}_\mathcal{K} = \dim \ker A_{\mathcal{G}-\mathcal{K}}^\top$.

Proof Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be trees/forests as in Definition 3.
We show (i): Let m denote the number of branches of $\mathcal{K}$, n the number of its node and p the number of its components of connectivity. Note that $A_\mathcal{K} \in \mathbb{R}^{(k-1)\times m}$. Since $\mathcal{T}_1$ is a forest in $\mathcal{K}$, it has $n - p$ branches, hence

$$\mathrm{FL}_\mathcal{K} = m - (n - p).$$

By [27, Thm. 4.3] we have that $\operatorname{rk} A_\mathcal{K} = n - p$, and hence

$$\mathrm{FL}_\mathcal{K} = m - \operatorname{rk} A_\mathcal{K} = \dim \ker A_\mathcal{K}.$$

We show (ii): Since $\mathcal{T}_3$ is a tree in $\mathcal{G}$, it has $k - 1$ branches. Therefore, we have

$$\mathrm{FC}_\mathcal{K} = k - 1 - r,$$

where r is the number of branches in $\mathcal{T}_2$. Since $\mathcal{T}_2$ is a forest in $\mathcal{G} - \mathcal{K}$, it has $k - q$ branches, where q is the number of components of connectivity of $\mathcal{G} - \mathcal{K}$, thus

$$\mathrm{FC}_\mathcal{K} = q - 1.$$

By [27, Thm. 4.3] we have that $\operatorname{rk} A_{\mathcal{G}-\mathcal{K}} = k-q$, where $A_{\mathcal{G}-\mathcal{K}} \in \mathbb{R}^{(k-1)\times(l-m)}$, and hence

$$q = k - \operatorname{rk} A_{\mathcal{G}-\mathcal{K}}^\top = k - \big(k-1-\dim\ker A_{\mathcal{G}-\mathcal{K}}^\top\big) = \dim\ker A_{\mathcal{G}-\mathcal{K}}^\top + 1,$$

which completes the proof of the theorem. □

3 Network equations

It is well-known [9,14] that the graph underlying an electrical network can be described by an incidence matrix $\mathbf{A} \in \mathbb{R}^{(k-1)\times l}$, which can be decomposed into submatrices

$$\mathbf{A} = \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} \end{bmatrix}$$

for the quantities in (4), where $n_e = k-1$ and $l = n_{\mathcal{C}} + n_{\mathcal{G}} + n_{\mathcal{L}}$. Each submatrix is the incidence matrix of a specific subgraph of the network graph. $A_{\mathcal{C}}$ is the incidence matrix of the subgraph consisting of all network nodes and all branches corresponding to capacitors. Similarly, $A_{\mathcal{R}}$ and $A_{\mathcal{L}}$ are the incidence matrices corresponding to the resistor and inductor subgraphs, resp. Then, using the standard MNA modeling procedure [14], see also the survey [27], which is just a clever arrangement of Kirchhoff's laws together with the characteristic equations of the devices, results in a differential-algebraic equation (1) with (2)–(4). $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the constitutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials and $i_{\mathcal{L}}(t)$ is the vector of currents through inductances.

Definition 5 For a given RLC network, any differential-algebraic equation (1) satisfying (2)–(4), which arises from the MNA modeling procedure [14], is said to be an MNA model of the network.

It is a reasonable assumption that an electrical network is connected (see also [22]); otherwise, since the components of connectivity do not physically interact, one might consider them separately. Furthermore, in the present paper we consider networks with *passive* devices. These assumptions lead to the following assumptions on an MNA model (2)–(4) of the network (cf. [27]).

(A1) $\operatorname{rk}\begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} \end{bmatrix} = n_e$,

(A2) $\mathcal{C} = \mathcal{C}^\top > 0, \mathcal{L} = \mathcal{L}^\top > 0, \mathcal{G} + \mathcal{G}^\top > 0$.

It is shown in [4, Cor. 4.5] that under the conditions (A1) and (A2), the pencil $sE - A$ in (2) is regular.

In the following we will use expressions like $\mathcal{C}$-loop for a loop in the circuit graph whose branch set consists only of branches corresponding to capacitors. Likewise, a $\mathcal{L}$-cutset is a cutset in the circuit graph whose branch set consists only of branches corresponding to inductors. Using the concepts of fundamental loops and cutsets from Section 2 it is reasonable to say that the number of independent capacitors equals the total number of capacitors minus

the number of fundamental loops of capacitors; likewise, the number of independent inductors equals the total number of inductors minus the number of fundamental cutsets of inductors. This justifies the following definition.

Definition 6 Consider a MNA model (1) with (2)–(4) of a RLC network. We call $n_{\mathcal{C}} - \mathrm{FL}_{\mathcal{C}}$ the *number of independent capacitors* of the network and $n_{\mathcal{L}} - \mathrm{FC}_{\mathcal{L}}$ the *number of independent inductors* of the network.

4 The McMillan and pseudo McMillan degrees

In this section we investigate the McMillan degree of implicit network transfer functions and, because of several shortcomings, we introduce the new concept of pseudo McMillan degree. For both concepts we derive a formula in terms of the system matrices as well as a topological interpretation.

The McMillan degree of a proper[2] rational matrix $G(s) \in \mathbb{R}(s)^{n\times n}$, where $\mathbb{R}(s)$ is the quotient field of $\mathbb{R}[s]$, is the total number of its poles, and can be defined via its *Smith-McMillan form*

$$U(s)^{-1}G(s)V(s)^{-1} = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, \ldots, 0\right) \in \mathbb{R}(s)^{n\times n}, \quad (6)$$

where $U(s), V(s) \in \mathbb{R}[s]^{n\times n}$ are unimodular (i.e. invertible over $\mathbb{R}[s]^{n\times n}$), $\operatorname{rk} G(s) = r$, $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$, $\psi_{i+1}(s) \mid \psi_i(s)$ for all $i = 1, \ldots, r-1$. If $G(s)$ is not proper, then it can be decomposed into a proper rational matrix and a polynomial matrix, and the McMillan degree can be defined following [18, p. 466].

Definition 7 Consider $G(s) \in \mathbb{R}(s)^{n\times n}$. If $G(s)$ is proper with Smith-McMillan form (6), then we call

$$\delta_M G(s) := \deg \prod_{i=1}^{r} \psi_i(s)$$

the *McMillan degree* of $G(s)$. Otherwise, if $G(s) = G_p(s) + P(s)$ with proper $G_p(s) \in \mathbb{R}(s)^{n\times n}$ and polynomial $P(s) \in \mathbb{R}[s]^{n\times n}$, then we call

$$\delta_M G(s) := \delta_M G_p(s) + \delta_M P\big(s^{-1}\big)$$

the *McMillan degree* of $G(s)$.

The McMillan degree has several important properties, for instance that $\delta_M G(s)^{-1} = \delta_M G(s)$ for any invertible $G(s) \in \mathbb{R}(s)^{n\times n}$, and these properties may also serve to uniquely define the McMillan degree, see e.g. [18,25]. For the implicit transfer function $G(s) = (sE - A)^{-1}$ of a system (1) with regular matrix pencil $sE - A$, it is a consequence of the Weierstraß canonical form (see [11]) that

$$\delta_M G(s) = \operatorname{rk} E. \quad (7)$$

[2] $G(s) \in \mathbb{R}(s)^{n\times n}$ is *proper*, if $\lim_{s\to\infty} G(s) \in \mathbb{R}^{n\times n}$ exists.

The following example shows that the McMillan degree neither equals the number of dynamic elements, nor the number of independent dynamic elements of a given RLC network in general.

Example 1 Consider the RLC network depicted in Figure 1. According to the

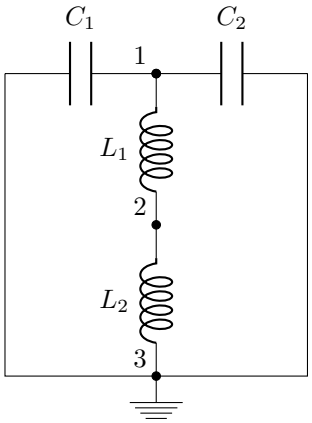

Fig. 1: RLC network

numbering of the nodes, the element-related incidence matrices are as follows:

$$A_{\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}, \quad A_{\mathcal{L}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix},$$

and

$$\mathcal{C} = \operatorname{diag}(C_1, C_2), \quad \mathcal{L} = \operatorname{diag}(L_1, L_2).$$

An essential step is now to observe that one of the three node potentials can be chosen freely. Therefore, we may, for instance, choose the potential at node 3 to be zero, which is equivalent to choosing this node as the ground node as in Figure 1. As a result, the corresponding node potential is not relevant in the modified nodal model and we may delete the corresponding row (here it is the last row) in the incidence matrices, that is

$$A_{\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{\mathcal{L}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Therefore, the matrix pencil (2) corresponding to the MNA model is

$$sE - A = \begin{bmatrix} s(C_1 + C_2) & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & sL_1 & 0 \\ 0 & -1 & 0 & sL_2 \end{bmatrix}$$

and the McMillan degree of $G(s) = (sE - A)^{-1}$ is $\delta_M G(s) = \operatorname{rk} E = 3$. However, the number of dynamic elements is $n_{\mathcal{C}} + n_{\mathcal{L}} = 4$ and, furthermore, we observe that the circuit contains one fundamental $\mathcal{C}$-loop and one fundamental $\mathcal{L}$-cutset, so the number of independent dynamic elements is $n_{\mathcal{C}} + n_{\mathcal{L}} - \mathrm{FL}_{\mathcal{C}} - \mathrm{FC}_{\mathcal{L}} = 2$.

Finally, let us compare the above findings to the "order of complexity" introduced in [5,8] as the number of natural frequencies in the network. According to [8], this order of complexity is given, in terms of the network topology, as

$$\sigma = n_{\mathcal{L}} + (n_e + 1) + 1 - S_{\mathcal{C}} - S_{\mathcal{CR}},$$

where $S_{\mathcal{C}}$, $S_{\mathcal{CR}}$ denote the number of components of connectivity of the subnetwork formed by the capacitors only and by the capacitors and the resistors only, respectively. For the network in Figure 1 it can be seen that $\sigma = 4$, which neither equals $\delta_M G(s)$, nor the number of independent dynamic elements.

Therefore, both the order of complexity from [5,8] and the McMillan degree are not suitable as a measure for the complexity of the dynamic part of a given network. To appropriately deal with the independent dynamic elements of a network using an algebraic concept associated with the implicit transfer function $G(s) = (sE - A)^{-1}$ we introduce the new concept of pseudo McMillan degree, which is simply defined as the total number of poles in the Smith-McMillan form, even in the case of a non-proper rational matrix.

Definition 8 Consider $G(s) \in \mathbb{R}(s)^{n\times n}$ with Smith-McMillan form (6), then we call

$$\delta_{PM} G(s) := \deg \prod_{i=1}^{r} \psi_i(s)$$

the *pseudo McMillan degree* of $G(s)$.

Obviously, $\delta_{PM} G(s) = \delta_M G(s)$ for any proper rational matrix. On the other hand, $\delta_{PM} G(s)^{-1} \neq \delta_{PM} G(s)$ in general, for an arbitrary invertible $G(s) \in \mathbb{R}(s)^{n\times n}$, so this nice property of the McMillan degree is not retained. For the implicit transfer function $G(s) = (sE - A)^{-1}$ of a system (1) with regular matrix pencil $sE - A$ we find that, again using the Weierstraß canonical form,

$$\delta_{PM} G(s) = \deg \det(sE - A). \tag{8}$$

Therefore, we obtain the relation

$$\delta_{PM} G(s) \leq \delta_M G(s), \quad G(s) = (sE - A)^{-1}.$$

We are now in a position to derive the first main result of the present paper.

Theorem 2 *Consider a MNA model* (1) *with* (2)–(4) *of a RLC network. Then, for* $G(s) = (sE - A)^{-1}$*, we have*

$$\delta_M G(s) = n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}},$$
$$\delta_{PM} G(s) = n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}} - \dim \ker [A_{\mathcal{R}}, A_{\mathcal{C}}]^{\top}.$$

Proof It is straightforward to compute that, using (7),

$$\delta_M G(s) = \operatorname{rk} E = \operatorname{rk} \begin{bmatrix} A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^\top & 0 \\ 0 & \mathcal{L} \end{bmatrix} = n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}}.$$

To show the second equation, choose matrices V, W with full column rank such that

$$\operatorname{im} V = \operatorname{im}[A_{\mathcal{R}}, A_{\mathcal{C}}], \qquad \operatorname{im} W = \ker [A_{\mathcal{R}}, A_{\mathcal{C}}]^\top = \big(\operatorname{im} V\big)^\perp,$$

and let $m := \operatorname{rk} W$. With

$$T := \begin{bmatrix} V & W & 0 \\ 0 & 0 & I_{n_{\mathcal{L}}} \end{bmatrix} \in \mathbf{GL}_{n_e + n_{\mathcal{L}}},$$

where $\mathbf{GL}_n$ denotes the set of invertible matrices from $\mathbb{R}^{n \times n}$, we obtain

$$T^\top (sE - A) T = \begin{bmatrix} V^\top \big(sA_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^\top + A_{\mathcal{R}} \mathcal{G} A_{\mathcal{R}}^\top\big) V & 0 & V^\top A_{\mathcal{L}} \\ 0 & 0 & W^\top A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top V & -A_{\mathcal{L}}^\top W & s\mathcal{L} \end{bmatrix}. \tag{9}$$

Then, the Schur complement formula (see e.g. [13, Lem. A.1.17]), yields

$$\begin{aligned} \det(sE - A) &= \det \big(V^\top \big(sA_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^\top + A_{\mathcal{R}} \mathcal{G} A_{\mathcal{R}}^\top\big) V\big) \\ &\quad \cdot \det \begin{bmatrix} 0 & W^\top A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top W & s\mathcal{L} + A_{\mathcal{L}}^\top V \big(V^\top \big(sA_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^\top + A_{\mathcal{R}} \mathcal{G} A_{\mathcal{R}}^\top\big) V\big)^{-1} V^\top A_{\mathcal{L}} \end{bmatrix}. \end{aligned}$$

Let

$$P(s) := s\mathcal{L} + A_{\mathcal{L}}^\top V \big(V^\top \big(sA_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^\top + A_{\mathcal{R}} \mathcal{G} A_{\mathcal{R}}^\top\big) V\big)^{-1} V^\top A_{\mathcal{L}},$$

then, again using the Schur complement formula,

$$\begin{aligned} &\det(sE - A) \\ &\quad = \det \big(V^\top \big(sA_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^\top + A_{\mathcal{R}} \mathcal{G} A_{\mathcal{R}}^\top\big) V\big) \det P(s) \det W^\top A_{\mathcal{L}} P(s)^{-1} A_{\mathcal{L}}^\top W. \end{aligned}$$

We show that $A_{\mathcal{L}}^\top W$ has full column rank: Let $x \in \mathbb{R}^m$ be such that $A_{\mathcal{L}}^\top W x = 0$, then

$$Wx \in \ker A_{\mathcal{L}}^\top \cap \operatorname{im} W = \ker A_{\mathcal{L}}^\top \cap \ker [A_{\mathcal{R}}, A_{\mathcal{C}}]^\top = \ker [A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{L}}]^\top \overset{(A1)}{=} \{0\},$$

and the full column rank of W implies $x = 0$. Write $P(s) = s\mathcal{L} + G_p(s)$, where $G_p(s)$ is proper. Then

$$P(s)^{-1} = (s\mathcal{L})^{-1} \big(I + (s\mathcal{L})^{-1} G_p(s)\big)^{-1} = \sum_{k=0}^{\infty} (-1)^k s^{-k-1} \mathcal{L}^{-k-1} G_p(s)^k,$$

and

$$W^\top A_{\mathcal{L}} P(s)^{-1} A_{\mathcal{L}}^\top W = s^{-1} W^\top A_{\mathcal{L}} \mathcal{L}^{-1} A_{\mathcal{L}}^\top W + s^{-1} G_{sp}(s),$$

where $G_{sp}(s)$ is strictly proper. Since $W^\top A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top W \in \mathbf{GL}_m$, the highest power of s appearing in $\det W^\top A_\mathcal{L} P(s)^{-1} A_\mathcal{L}^\top W$ is s^{-m}. Furthermore, the highest power of s appearing in $\det P(s)$ is $s^{n_\mathcal{L}}$. By (8) and the above observations we obtain

$$\delta_{PM} G(s) = \deg\det\left(V^\top\left(sA_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top + A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top\right)V\right) + n_\mathcal{L} - m.$$

We consider the matrix pencil $s\tilde{E} - \tilde{A} := V^\top\left(sA_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top + A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top\right)V$. We show that $\ker\tilde{E} \cap \ker(\tilde{A} + \tilde{A}^\top) = \{0\}$: Let $x \in \ker\tilde{E} \cap \ker(\tilde{A} + \tilde{A}^\top)$, then

$$x^\top V^\top A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top Vx = 0 \quad \text{and} \quad x^\top V^\top A_\mathcal{R}(\mathcal{G} + \mathcal{G}^\top)A_\mathcal{R}^\top Vx = 0,$$

which implies, using (A2), that $A_\mathcal{C}^\top Vx = 0$ and $A_\mathcal{R}^\top Vx = 0$. Therefore,

$$Vx \in \ker[A_\mathcal{R}, A_\mathcal{C}]^\top \cap \operatorname{im} V = \ker[A_\mathcal{R}, A_\mathcal{C}]^\top \cap \operatorname{im}[A_\mathcal{R}, A_\mathcal{C}] = \{0\},$$

and full column rank of V implies $x = 0$. Invoking that $\ker\tilde{A} \subseteq \ker(\tilde{A} + \tilde{A}^\top)$ by (A2), it now follows from [4, Cor 2.6 & Lem. 2.6] that $s\tilde{E} - \tilde{A}$ is regular. We show that its index (see e.g. [23, Def. 2.9] for a definition) is at most one. Seeking a contradiction, assume that it is larger than one. Then [3, Prop. 2.10] implies that there exist $x, y \in \mathbb{R}^q \setminus \{0\}$, where $q = \operatorname{rk} V$, such that $\tilde{E}y = \tilde{A}x$ and $\tilde{E}x = 0$. Therefore,

$$x^\top(\tilde{A} + \tilde{A}^\top)x = x^\top\tilde{E}y + y^\top\tilde{E}x = 0,$$

hence $(\tilde{A} + \tilde{A}^\top)x = 0$ which gives $x \in \ker\tilde{E} \cap \ker(\tilde{A} + \tilde{A}^\top) = \{0\}$, a contradiction. Since the index of $s\tilde{E} - \tilde{A}$ is at most one we find that (see e.g. [23])

$$\deg\det(s\tilde{E} - \tilde{A}) = \operatorname{rk}\tilde{E}.$$

Furthermore,

$$\operatorname{rk}\tilde{E} = \operatorname{rk} V^\top A_\mathcal{C} = \operatorname{rk}\begin{bmatrix} V^\top A_\mathcal{C} \\ 0 \end{bmatrix} = \operatorname{rk}\begin{bmatrix} V^\top A_\mathcal{C} \\ W^\top A_\mathcal{C} \end{bmatrix} = \operatorname{rk}\begin{bmatrix} V^\top \\ W^\top \end{bmatrix} A_\mathcal{C} = \operatorname{rk} A_\mathcal{C},$$

and so we finally obtain

$$\delta_{PM} G(s) = \operatorname{rk} A_\mathcal{C} + n_\mathcal{L} - m = \operatorname{rk} A_\mathcal{C} + n_\mathcal{L} - \dim\ker[A_\mathcal{R}, A_\mathcal{C}]^\top.$$

□

We present an interpretation of Theorem 2 in terms of the network topology. By Theorem 1, $\dim\ker A_\mathcal{C}$ equals the number of fundamental $\mathcal{C}$-loops and $\dim\ker[A_\mathcal{R}, A_\mathcal{C}]^\top$ equals the number of fundamental $\mathcal{L}$-cutsets in the network, thus the following is an immediate consequence of Theorems 1 and 2.

Corollary 1 *Using the notation from Theorem 2 we have that*

$$\delta_M G(s) = n_\mathcal{C} + n_\mathcal{L} - \mathrm{FL}_\mathcal{C},$$
$$\delta_{PM} G(s) = n_\mathcal{C} + n_\mathcal{L} - \mathrm{FL}_\mathcal{C} - \mathrm{FC}_\mathcal{L}.$$

As a consequence of Corollary 1, we see that $\delta_{PM} G(s)$ equals the number of independent capacitors and inductors in the network, whereas the difference $\delta_M G(s) - \delta_{PM} G(s)$ equals the number $\mathrm{FC}_\mathcal{L}$ of fundamental $\mathcal{L}$-cutsets.

5 Minimal realization

In this section we derive a minimal realization of a given RLC network in the following sense.

Definition 9 A system of the form (1) is called a *minimal realization* of a RLC network, if its number of (independent) differential equations equals the pseudo McMillan degree of the implicit transfer function $G(s)$ of the network, i.e., $\operatorname{rk} E = \delta_{PM} G(s)$, and there is a one-to-one correspondence to the solutions of an MNA model of the network.

In order to obtain a minimal realization we start with an MNA model (1) satisfying (2)–(4) of the RLC network and its transformation in (9), using the notation from the proof of Theorem 2. Now let Y be a matrix with full column rank such that

$$\operatorname{im} Y = \ker W^\top A_{\mathcal{L}} = \left(\operatorname{im} A_{\mathcal{L}}^\top W\right)^\perp,$$

and, recalling that $A_{\mathcal{L}}^\top W$ has full column rank,

$$S := \begin{bmatrix} I_{n_e} & 0 & 0 \\ 0 & A_{\mathcal{L}}^\top W & Y \end{bmatrix} \in \mathbf{GL}_{n_e+n_{\mathcal{L}}}.$$

Then

$$\begin{aligned} &S^\top T^\top (sE - A) T S \\ &= \begin{bmatrix} V^\top \left(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\right)V & 0 & V^\top A_{\mathcal{L}} A_{\mathcal{L}}^\top W & V^\top A_{\mathcal{L}} Y \\ 0 & 0 & W^\top A_{\mathcal{L}} A_{\mathcal{L}}^\top W & 0 \\ -W^\top A_{\mathcal{L}} A_{\mathcal{L}}^\top V & -W^\top A_{\mathcal{L}} A_{\mathcal{L}}^\top W & sW^\top A_{\mathcal{L}}\mathcal{L}A_{\mathcal{L}}^\top W & sW^\top A_{\mathcal{L}}\mathcal{L}Y \\ -Y^\top A_{\mathcal{L}}^\top V & 0 & sY^\top \mathcal{L} A_{\mathcal{L}}^\top W & sY^\top \mathcal{L} Y \end{bmatrix}. \end{aligned}$$

Obviously, $W^\top A_{\mathcal{L}} A_{\mathcal{L}}^\top W \in \mathbf{GL}_m$ and hence there is a one-to-one correspondence between the solutions of the MNA model (1) and the solutions of the system

$$\begin{aligned} V^\top A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top V\, \dot{x}_1(t) &= -V^\top A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top V\, x_1(t) - V^\top A_{\mathcal{L}} Y\, x_4(t), \\ W^\top A_{\mathcal{L}}\mathcal{L}Y\, \dot{x}_4(t) &= W^\top A_{\mathcal{L}} A_{\mathcal{L}}^\top V\, x_1(t) + W^\top A_{\mathcal{L}} A_{\mathcal{L}}^\top W\, x_2(t), \\ Y^\top \mathcal{L} Y\, \dot{x}_4(t) &= Y^\top A_{\mathcal{L}}^\top V\, x_1(t). \end{aligned}$$

Again using that $W^\top A_{\mathcal{L}} A_{\mathcal{L}}^\top W \in \mathbf{GL}_m$, the second equation can be solved for x_2 and we obtain a one-to-one correspondence to the solutions of the system

$$\tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t), \qquad \tilde{x}(t) = \begin{pmatrix} x_1(t) \\ x_4(t) \end{pmatrix}, \tag{10}$$

with

$$s\tilde{E} - \tilde{A} = \begin{bmatrix} V^\top \left(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\right)V & V^\top A_{\mathcal{L}} Y \\ -Y^\top A_{\mathcal{L}}^\top V & sY^\top \mathcal{L} Y \end{bmatrix}.$$

The variables x_1 in (10) may be interpreted as those corresponding to independent capacitors in the network and x_4 as those corresponding to independent inductors.

Theorem 3 *Consider a MNA model* (1) *with* (2)–(4) *of a RLC network. Then the system* (10) *is a minimal realization of that network. In particular, for the matrices in* (10) *we find*

$$\operatorname{rk} A_{\mathcal{C}}^{\top} V = n_{\mathcal{C}} - \mathrm{FL}_{\mathcal{C}} \quad \textit{and} \quad \operatorname{rk} Y = n_{\mathcal{L}} - \mathrm{FC}_{\mathcal{L}}.$$

Proof It is obvious that there is a one-to-one correspondence between the solutions of (1) and (10). Furthermore,

$$\operatorname{rk} \tilde{E} = \operatorname{rk} V^{\top} A_{\mathcal{C}} + \operatorname{rk} Y^{\top} \mathcal{L} Y = \operatorname{rk} A_{\mathcal{C}} + \operatorname{rk} Y,$$

where we have used that $\operatorname{rk} V^{\top} A_{\mathcal{C}} = \operatorname{rk} A_{\mathcal{C}}$ as shown in proof of Theorem 2. We may further calculate that

$$\operatorname{rk} Y = \dim \ker W^{\top} A_{\mathcal{L}} = n_{\mathcal{L}} - \operatorname{rk} W^{\top} A_{\mathcal{L}} = n_{\mathcal{L}} - \operatorname{rk} A_{\mathcal{L}}^{\top} W = n_{\mathcal{L}} - m,$$

since $A_{\mathcal{L}}^{\top} W$ has full column rank $m = \operatorname{rk} W$. Therefore,

$$\operatorname{rk} Y = n_{\mathcal{L}} - \operatorname{rk} W = n_{\mathcal{L}} - \dim \ker \left[A_{\mathcal{R}}, A_{\mathcal{C}}\right]^{\top}$$

and hence

$$\operatorname{rk} \tilde{E} = \operatorname{rk} A_{\mathcal{C}} + n_{\mathcal{L}} - \dim \ker \left[A_{\mathcal{R}}, A_{\mathcal{C}}\right]^{\top} = \delta_{PM} G(s)$$

for $G(s) = (sE - A)^{-1}$ by Theorem 2. The last statement is a consequence of Corollary 1. □

Since there is a one-to-one correspondence between the solutions of (1) and (10) it follows that $\det(s\tilde{E} - \tilde{A}) = \det(sE - A)$ and hence

$$\deg \det(s\tilde{E} - \tilde{A}) = \delta_{PM}(sE - A)^{-1} = \operatorname{rk} \tilde{E},$$

thus the index of the matrix pencil $s\tilde{E} - \tilde{A}$ is at most one. Therefore, the procedure which leads from (1) to (10) may also be viewed as an *index reduction method* (cf. [23]), because the possibly higher index of $sE - A$ in (1) is reduced to at most one in (10). Note that this procedure is simpler than those presented in [1], however we do not consider any sources in the network here.

6 Examples

We illustrate our obtained results by means of two examples.

6.1 Example 1 revisited

Consider again the RLC network depicted in Figure 1 from Example 1. We have observed that for $G(s) = (sE - A)^{-1}$ we have $\delta_M G(s) = 3$. Since

$$\det(sE - A) = s^2(C_1 + C_2)(L_1 + L_2) + 1,$$

the pseudo McMillan degree is given by $\delta_{PM} G(s) = 2$. From Theorem 2 we obtain the same values:

$$n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}} = 2+1 = 3 \quad \text{and} \quad n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}} - \dim \ker [A_{\mathcal{R}}, A_{\mathcal{C}}]^\top = 2+1-1 = 2.$$

Since $\operatorname{rk} E = 3 \neq 2 = \delta_{PM} G(s)$, the MNA model is not a minimal realization. We see that $sE - A$ is already in the form (9), so it remains to choose

$$Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \operatorname{im} Y = \ker W^\top A_{\mathcal{L}} = \ker [-1, 1].$$

Then, with

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbf{GL}_4$$

we obtain

$$S^\top (sE - A) S = \begin{bmatrix} s(C_1 + C_2) & 0 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 2 & sL_1 + sL_2 & -sL_1 + sL_2 \\ 1 & 0 & -sL_1 + sL_2 & sL_1 + sL_2 \end{bmatrix},$$

and hence a minimal realization is given by (10) with

$$s\tilde{E} - \tilde{A} = \begin{bmatrix} s(C_1 + C_2) & 1 \\ 1 & s(L_1 + L_2) \end{bmatrix}.$$

6.2 Example 2

Consider the RLC network depicted in Figure 2.

After deleting the row corresponding to the ground node the incidence matrices read

$$A_{\mathcal{R}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}, \quad A_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_{\mathcal{L}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix},$$

$$\mathcal{G} = \operatorname{diag}(R_1^{-1}, R_2^{-1}, R_3^{-1}, R_4^{-1}), \quad \mathcal{C} = [C_1], \quad \mathcal{L} = \operatorname{diag}(L_1, L_2).$$

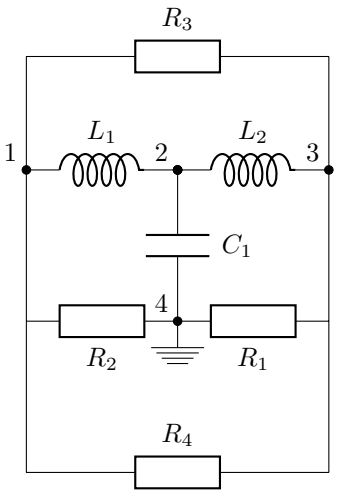

Fig. 2: RLC network

Therefore, the matrix pencil (2) corresponding to the MNA model is

$$sE - A = \begin{bmatrix} sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{R}A_{\mathcal{R}}^\top & A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top & s\mathcal{L} \end{bmatrix}$$
$$= \begin{bmatrix} R_1^{-1}+R_3^{-1}+R_4^{-1} & 0 & -R_3^{-1}-R_4^{-1} & 1 & 0 \\ 0 & sC_1 & 0 & -1 & 1 \\ -R_3^{-1}-R_4^{-1} & 0 & R_2^{-1}+R_3^{-1}+R_4^{-1} & 0 & -1 \\ -1 & 1 & 0 & sL_1 & 0 \\ 0 & -1 & 1 & 0 & sL_2 \end{bmatrix}$$

and the McMillan degree of $G(s) = (sE-A)^{-1}$ is $\delta_M G(s) = \operatorname{rk} E = 3$. Further, we calculate

$$\det(sE-A) = s^3\, C_1L_1L_2(R_1^{-1}R_2^{-1}+R_1^{-1}R_3^{-1}+R_1^{-1}R_4^{-1}+R_2^{-1}R_3^{-1}+R_2^{-1}R_4^{-1}) + s^2(\ldots) + s(\ldots) + R_1^{-1} + R_2^{-1},$$

and hence the pseudo McMillan degree of $G(s)$ is $\delta_{PM} G(s) = 3$. These are the same values as we obtain from Theorem 2:

$$n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}} = 2+1 = 3 \quad \text{and} \quad n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}} - \dim\ker\,[A_{\mathcal{R}}, A_{\mathcal{C}}]^\top = 2+1-0 = 3,$$

and we observe that the network neither contains $\mathcal{C}$-loops nor $\mathcal{L}$-cutsets. Since $\operatorname{rk} E = 3$, the MNA model itself is already a minimal realization.

7 Conclusion

In the present paper we have argued that the McMillan degree is not a suitable measure for the complexity of the dynamic part of an RLC network, because

it neither equals the number of dynamic elements nor the number of independent dynamic elements. To resolve this drawback we have introduced the new concept of pseudo McMillan degree and shown that the pseudo McMillan degree of an implicit network transfer function equals the number of independent dynamic elements. The latter is given by the sum of the number of capacitors and inductors minus the number of fundamental loops of capacitors and fundamental cutsets of inductors. A minimal realization of the RLC network is then derived, where the number of involved (independent) differential equations equals the pseudo McMillan degree.

The starting point for our analysis has been the modified nodal analysis model, which preserves the natural graph topology of the network, but in general leads to an implicit non-minimal representation. The results presented here provide an extension to the results derived in [24] based on the impedance-admittance network description, which provides an appropriate framework for network re-engineering. The corresponding integral-differential rational description also leads to a state space description that is in general non-minimal [22], but that preserves the natural nodal/loop topologies of the network. Extending the results on the (pseudo) McMillan degree obtained in the present paper to this alternative description is a topic of future research.

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