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Dade's ordinary conjecture implies the Alperin–McKay conjecture

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Abstract. We show that Dade's ordinary conjecture implies the Alperin–McKay conjecture. We remark that some of the methods can be used to identify a canonical height zero character in a nilpotent block.

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Dade proved in [4] that his projective conjecture [4, 15.5] implies the Alperin–McKay conjecture. Navarro showed in [11, Theorem 9.27] that the group version of Dade's ordinary conjecture implies the McKay conjecture. We show here that Dade's ordinary conjecture [3, 6.3] implies the Alperin–McKay conjecture. Let p be a prime number.

Theorem 1. *If Dade's ordinary conjecture holds for all p-blocks of finite groups, then the Alperin–McKay conjecture holds for all p-blocks of finite groups.*

The proof combines arguments from Sambale [17] and formal properties of chains of subgroups in fusion systems from [7]. Let $(K, \mathcal{O}, k)$ be a p-modular system. We assume that k is algebraically closed, and let $\bar{K}$ be an algebraic closure of K. By a character of a finite group, we will mean a $\bar{K}$-valued character. For a finite group G and a block B of $\mathcal{O}G$, let $\mathrm{Irr}(B)$ denote the set of irreducible characters of G in the block B, and let $\mathrm{Irr}_0(B)$ denote the set of irreducible height zero characters of G in B. For a central p-subgroup Z of G and a character η of Z, let $\mathrm{Irr}_0(B|\eta)$ denote the subset of $\mathrm{Irr}_0(B)$ consisting of

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those height zero characters which cover the character η. The following lemma is implicit in [17].

Lemma 2. *Let P be a finite p-group, let $\mathcal{F}$ be a saturated fusion system on P, and let $Z \leq Z(\mathcal{F})$. Suppose that η is a linear character of P. There exists a linear character $\hat{\eta}$ of P such that $\hat{\eta}|_Z = \eta|_Z$ and $\mathfrak{foc}(\mathcal{F}) \leq \mathrm{Ker}(\hat{\eta})$.*

Proof. First consider the case that $\eta|_Z$ is faithful. Then $Z \cap [P, P] = 1$. Hence by [5, Lemma 4.3], $\mathfrak{foc}(\mathcal{F}) \cap Z = 1$. The result is now immediate. Now suppose $Z_0 = \mathrm{Ker}(\eta|_Z)$ and let $\bar{\mathcal{F}} = \mathcal{F}/Z_0$. By the previous argument, applied to P/Z_0 and $\bar{\mathcal{F}}$, there exists a character $\hat{\eta}$ of P/Z_0 such that $\hat{\eta}|_{Z/Z_0} = \eta|_{Z/Z_0}$ and $\mathfrak{foc}(\bar{\mathcal{F}}) \leq \mathrm{Ker}(\hat{\eta})$. Denote also by $\hat{\eta}$ the inflation of $\hat{\eta}$ to P. Then $\hat{\eta}$ has the required properties since $\mathfrak{foc}(\bar{\mathcal{F}}) = \mathfrak{foc}(\mathcal{F})Z_0/Z_0$. $\square$

The following result is a special case of a result due to Murai; we include a proof for convenience.

Lemma 3 (cf. [9, Theorem 4.4]). *Let G be a finite group, B be a block of $\mathcal{O}G$, and P a defect group of B. Let Z be a central p-subgroup of G and let η be an irreducible character of Z such that $\mathrm{Irr}_0(B|\eta) \neq \emptyset$. Then η extends to P.*

Proof. By replacing K by a suitable finite extension, we may assume that K is a splitting field for all subgroups of G. Let $i \in B^P$ be a source idempotent of B and let V be a KG-module affording an element of $\mathrm{Irr}_0(B|\eta)$. Then $n := \dim_K(iV)$ is prime to p (see [13]). Since i commutes with P, iV is a KP-module via $x \cdot iv = ixv$, where $x \in P, v \in V$. Let $\rho : P \to \mathrm{GL}_n(K)$ be a corresponding representation and let $\delta : P \to K^\times$ be the determinantal character of ρ. Then $\delta|_Z = \eta^n$. The result follows since n is prime to p. $\square$

Lemma 4. *Let G be a finite group, let B be a block of $\mathcal{O}G$ with a defect group P, and let Z be a central p-subgroup of G. Then $|\mathrm{Irr}_0(B)|$ equals the product of $|\mathrm{Irr}_0(B|1_Z)|$ with the number of distinct linear characters η of Z which extend to P.*

Proof. Let $\mathcal{F} = \mathcal{F}_{(P,e_P)}(G, B)$ be the fusion system of B with respect to a maximal B-Brauer pair (P, e_P), and let η be a linear character of Z which extends to P. Since $Z \leq Z(\mathcal{F})$, by Lemma 2 there exists a linear character $\hat{\eta}$ of P such that $\hat{\eta}|_Z = \eta$ and $\mathfrak{foc}(\mathcal{F}) \leq \mathrm{Ker}(\hat{\eta})$. By the properties of the Broué-Puig $*$-construction [1,16] the map $\chi \mapsto \hat{\eta} * \chi$ is a bijection between $\mathrm{Irr}_0(B|1_Z)$ and $\mathrm{Irr}_0(B|\eta)$. The result follows by Lemma 3. $\square$

Slightly strengthening the terminology in [10], we say that a pair (G, B) consisting of a finite group G and a block B of $\mathcal{O}G$ is a *minimal counterexample to the Alperin–McKay conjecture* if B is a counterexample to the Alperin–McKay conjecture and if G is such that first $|G : Z(G)|$ is smallest possible and then $|G|$ is smallest possible.

Proposition 5. *Let (G, B) be a minimal counterexample to the Alperin–McKay conjecture. Then $O_p(G) = 1$.*

Proof. By a result of Murai [10], we have that $Z := O_p(G)$ is central in G. Let P be a defect group of B and let C be the block of $\mathcal{O}N_G(P)$ in Brauer correspondence with B. By Lemma 4, $|\operatorname{Irr}_0(B)| = |\operatorname{Irr}_0(C)|$ if and only if $|\operatorname{Irr}_0(\bar{B})| = |\operatorname{Irr}_0(\bar{C})|$ where $\bar{B}$ (resp. $\bar{C}$) is the block of $\mathcal{O}G/Z$ (resp. $\mathcal{O}N_G(P)/Z$) dominated by B (resp. C). The result follows since $N_{G/Z}(P/Z) = N_G(P)/Z$ and $\bar{B}$ and $\bar{C}$ are in Brauer correspondence. □

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\mathcal{C}$ be a full subcategory of $\mathcal{F}$ which is upwardly closed; that is, if Q, R are subgroups of P such that Q belongs to $\mathcal{C}$ and if $\operatorname{Hom}_{\mathcal{F}}(Q, R)$ is nonempty, then also R belongs to $\mathcal{C}$. Drawing upon notation and facts from [7, §5], $S_{\lhd}(\mathcal{C})$ is the category having as objects nonempty chains $\sigma = Q_0 < Q_1 < \cdots < Q_m$ of subgroups Q_i of P belonging to $\mathcal{C}$ such that $m \geq 0$ and Q_i is normal in Q_m, for $0 \leq i \leq m$. Morphisms in $S_{\lhd}(\mathcal{C})$ are given by certain 'obvious' commutative diagrams of morphisms in $\mathcal{F}$; see [7, 2.1, 4.1] for details. With this notation, the *length* of a chain σ in $S_{\lhd}(\mathcal{C})$ is the integer $|\sigma| = m$. The chain σ is called *fully normalised* if Q_0 is fully $\mathcal{F}$-normalised and if either $m = 0$ or the chain $\sigma_{\geq 1} = Q_1 < Q_2 < \cdots < Q_m$ is fully $N_{\mathcal{F}}(Q_0)$-normalised. Every chain in $S_{\lhd}(\mathcal{C})$ is isomorphic (in the category $S_{\lhd}(\mathcal{C})$) to a fully normalised chain. There is an involution n on the set of fully normalised chains which fixes the chain of length zero P and which sends any other fully normalised chain σ to a fully normalised chain $n(\sigma)$ of length $|\sigma| \pm 1$. This involution is defined as follows. If $\sigma = P$, then set $n(\sigma) = \sigma$. If $\sigma = Q_0 < Q_1 < \cdots < Q_m$ is a fully normalised chain different from P such that $Q_m = N_P(\sigma)$, then define σ by removing the last term Q_m; if $Q_m < N_P(\sigma)$, then define σ by adding $N_P(\sigma)$ as last term to the chain. Then $n(\sigma)$ is fully normalised, and $n(n(\sigma)) = \sigma$. Denote by $[S_{\lhd}(\mathcal{C})]$ the partially ordered set of isomorphism classes of chains in $S_{\lhd}(\mathcal{C})$, and for each chain σ by $[\sigma]$ its isomorphism class. We have a partition

$$[S_{\lhd}(\mathcal{C})] = \{[P]\} \cup \mathcal{B} \cup n(\mathcal{B}),$$

where $\mathcal{B}$ is the set of isomorphism classes of fully normalised chains σ satisfying $|n(\sigma)| = |\sigma| + 1$. The following Lemma is a very special case of a functor cohomological statement [7, Theorem 5.11].

Lemma 6. *With the notation above, let $f : [S_{\lhd}(\mathcal{C})] \to \mathbb{Z}$ be a function on the set of isomorphism classes of chains in $S_{\lhd}(\mathcal{C})$ satisfying $f([\sigma]) = f([n(\sigma)])$ for any fully normalised chain σ in $S_{\lhd}(\mathcal{C})$. Then*

$$\sum_{[\sigma] \in [S_{\lhd}(\mathcal{C})]} (-1)^{|\sigma|} f([\sigma]) = f([P]).$$

Proof. The hypothesis on f implies that the contributions from chains in $\mathcal{B}$ cancel those from chains in $n(\mathcal{B})$, whence the result. □

Proposition 7. *Let G be a finite group such that $O_p(G) = 1$, and let B be a block of $\mathcal{O}G$ with nontrivial defect groups. Suppose that Dade's ordinary conjecture holds for B and that the Alperin–McKay conjecture holds for any block of any proper subgroup of G. Then the Alperin–McKay conjecture holds for the block B.*

Proof. Let (P, e) be a maximal B-Brauer pair, and denote by $\mathcal{F}$ the associated fusion system on P. For d a positive integer, denote by $\mathbf{k}_d(G, B)$ the number of ordinary irreducible characters in B of defect d. If $p^d = |P|$, then $\mathbf{k}_d(G, B)$ is the number of height zero characters, and if $p^d > |P|$, then $\mathbf{k}_d(G, B) = 0$.

Let $\mathcal{C}$ be the full subcategory of $\mathcal{F}$ consisting of all nontrivial subgroups of P. We briefly describe the standard translation process between chains in a fusion system of a block and the associated chains of Brauer pairs. The map sending a chain $\sigma = Q_0 < Q_1 < \cdots < Q_m$ in $S_{\lhd}(\mathcal{C})$ to the unique chain of nontrivial B-Brauer pairs $\tau = (Q_0, e_0) < (Q_1, e_1) < \cdots < (Q_m, e_m)$ contained in (P, e) induces a bijection between isomorphism classes of chains in $S_{\lhd}(\mathcal{C})$ and the set of G-conjugacy classes of normal chains of nontrivial B-Brauer pairs (cf. [7, 2.5]). If σ is fully normalised, then the corresponding chain of Brauer pairs $\tau = (Q_0, e_0) < (Q_1, e_1) < \cdots < (Q_m, e_m)$ has the property that $e_\tau = e_m$ remains a block of $N_G(\tau)$, and by [7, 5.14], $N_P(\sigma) = N_P(\tau)$ is a defect group of e_τ as a block of $N_G(\tau)$. Denote by $n(\tau)$ the chain of Brauer pairs corresponding to $n(\sigma)$.

Let $d > 0$ such that $p^d = |P|$. Define a function f on $S_{\lhd}(\mathcal{C})$ by setting

$$f([\sigma]) = \mathbf{k}_d(N_G(\tau), e_\tau)$$

for any fully normalised chain σ and corresponding chain τ of Brauer pairs. If $N_P(\sigma)$ is a proper subgroup of P, then $f([\sigma]) = 0$, and if $N_P(\sigma) = P$, then $f([\sigma])$ is the number of height zero characters of the block e_τ of $N_G(\tau)$. Dade's ordinary conjecture for B, reformulated here in terms of chains of Brauer pairs, asserts that $\mathbf{k}_d(G, B)$ is equal to the alternating sum

$$\sum_{[\sigma] \in S_{\lhd}(\mathcal{C})} (-1)^{|\sigma|} f([\sigma]).$$

The passage between formulations in terms of normalisers of chains of Brauer pairs rather than normalisers of chains of p-subgroups is well known; see e.g. [6, 4.5], [15].

If $|n(\sigma)| = |\sigma| + 1$, then, setting $H = N_G(\tau)$, we have $N_G(n(\tau)) = N_H(N_P(\tau), e_{n(\tau)})$; that is, $(N_P(\tau), e_{n(\tau)})$ is a maximal (H, e_τ)-Brauer pair. By the assumptions, the Alperin–McKay conjecture holds for the block e_τ of H. This translates to the equality $f([\sigma]) = f([n(\sigma)])$. That is, the function f satisfies the hypotheses of Lemma 6. Thus the above alternating sum is equal to $f([P])$, which by definition is $\mathbf{k}_d(N_G(P, e), e)$, and thus the Alperin–McKay conjecture holds for B. □

Theorem 1 follows now immediately from combining Propositions 5 and 7.

Remark 8. By work of Dade [2] and Okuyama and Wajima [12], the Alperin–McKay conjecture holds for blocks of finite p-solvable groups. G. R. Robinson pointed out that Proposition 5 yields another short proof of this fact.

Remark 9. Let G be a finite group, B a block algebra of $\mathcal{O}G$, (P, e_P) a maximal (G, B)-Brauer pair with associated fusion system $\mathcal{F}$ on P, and let Z be a central p-subgroup of G. Let η be a linear character of Z, and suppose that η extends to a linear character $\hat{\eta}$ of P satisfying $\mathfrak{foc}(\mathcal{F}) \leq \mathrm{Ker}(\hat{\eta})$. The proof of

Lemma 4 is based on the fact that the $*$-construction $\chi \mapsto \hat{\eta} * \chi$ yields a bijection $\mathrm{Irr}(B|1_Z) \to \mathrm{Irr}(B|\eta)$. There is some slightly more structural background to this. For $\chi \in \mathrm{Irr}(B)$, denote by $e(\chi)$ the corresponding central primitive idempotent in $K \otimes_{\mathcal{O}} B$. Set

$$e_1 = \sum_{\chi \in \mathrm{Irr}_0(B|1_Z)} e(\chi), \quad e_\eta = \sum_{\chi \in \mathrm{Irr}_0(B|\eta)} e(\chi).$$

Identify B with its image in $K \otimes_{\mathcal{O}} B$. Multiplying B by the central idempotents e_1 and e_η in $K \otimes_{\mathcal{O}} B$ yields the two $\mathcal{O}$-free $\mathcal{O}$-algebra quotients Be_1 and Be_η of B. By [8, Theorem 1.1], there is an $\mathcal{O}$-algebra automorphism α of B which induces the identity on $k \otimes_{\mathcal{O}} B$ and which acts on $\mathrm{Irr}(B)$ as the map $\chi \to \hat{\eta} * \chi$. Thus the extension of α to $K \otimes_{\mathcal{O}} B$ sends e_1 to e_η, and hence induces an $\mathcal{O}$-algebra isomorphism

$$Be_1 \cong Be_\eta.$$

We conclude this note with an observation regarding canonical height zero characters in nilpotent blocks, based in part on some of the above methods.

Let G be a finite group, B a block algebra of $\mathcal{O}G$, P a defect group of B, and $i \in B^P$ a source idempotent of B. Denote by $\mathcal{F}$ the fusion system of B on P determined by the choice of i. Suppose that K is a splitting field for all subgroups of G. For V a finitely generated $\mathcal{O}$-free B-module, denote by

$$\Delta_{V,P,i} : P \to \mathcal{O}^\times$$

the map sending $u \in P$ to the determinant of the $\mathcal{O}$-linear automorphism of iV induced by the action of u on V (this makes sense since all elements in P commute with i). By standard properties of determinants, this map depends only on the $(B^P)^\times$-conjugacy class of i and the isomorphism class of the $K \otimes_{\mathcal{O}} B$-module $K \otimes_{\mathcal{O}} V$. Thus if V affords a character $\chi \in \mathrm{Irr}(B)$, we write $\Delta_{\chi,P,i}$ instead of $\Delta_{V,P,i}$.

Proposition 10. *With the notation above, let $\chi \in \mathrm{Irr}(B)$ and $\eta \in \mathrm{Irr}(P/\mathfrak{foc}(P))$. Regard η as a linear character of P. We have*

$$\Delta_{\eta * \chi,P,i} = \eta^{\chi(i)} \Delta_{\chi,P,i}.$$

Proof. The statement makes sense as the value of χ on an idempotent is a positive integer. Let V be an $\mathcal{O}$-free $\mathcal{O}G$-module affording χ. By [8, Theorem 1.1] there exists an $\mathcal{O}$-algebra automorphism α of B such that the module V^α (obtained from twisting V by α) affords $\eta * \chi$ and such that $\alpha(ui) = \eta(u)ui$ for all $u \in P$. Since in particular $\alpha(i) = i$, it follows that

$$\Delta_{V^\alpha,P,i}(u) = \Delta_{V,P,i}(\eta(u)u)$$

for all $u \in P$. The result follows as $\mathrm{rank}_{\mathcal{O}}(iV) = \chi(i)$. □

Denote by $\mathrm{Irr}'(B)$ the set of all $\chi \in \mathrm{Irr}(B)$ such that $\Delta_{\chi,P,i}$ is the trivial map (sending all elements in P to 1). Set $\mathrm{Irr}'_0(B) = \mathrm{Irr}'(B) \cap \mathrm{Irr}_0(B)$. The maximal local pointed groups on B are G-conjugate. Thus if P' is any other defect group of B and $i' \in B^{P'}$ a source idempotent, then there exist $g \in G$ and $c \in (B^{P'})^\times$ such that $P' = gPg^{-1}$ and $i' = cgig^{-1}c^{-1}$. Therefore the map

$\Delta_{V,P,i}$ is trivial if and only if the map $\Delta_{V,P',i'}$ is trivial, and hence the sets $\mathrm{Irr}'(B)$ and $\mathrm{Irr}'_0(B)$ are independent of the choice of P and i. The following is immediate.

Proposition 11. *The sets* $\mathrm{Irr}'(B)$ *and* $\mathrm{Irr}'_0(B)$ *are invariant under any automorphism of* G *which stabilises* B.

The next result shows that if B is nilpotent, then $\mathrm{Irr}'_0(B)$ consists of a single element.

Proposition 12. *Suppose that* B *is nilpotent. Then* $|\,\mathrm{Irr}'_0(B)| = 1$. *Moreover, if* p *is odd, then the unique element of* $\mathrm{Irr}'_0(B)$ *is the unique* p*-rational height zero character in* B.

Proof. Let $\chi \in \mathrm{Irr}_0(B)$. Since i is a source idempotent of B, $\chi(i)$ is prime to p. Hence if η, ζ are linear characters of P, then $\eta^{\chi(i)} = \zeta^{\chi(i)}$ implies that $\eta = \zeta$. Since B is nilpotent, we have that $\mathfrak{foc}(\mathcal{F}) = [P, P]$ and $|\,\mathrm{Irr}_0(B)| = |P : [P, P]|$. Thus, by Proposition 10, the map $\chi \mapsto \Delta_{\chi,P,i}$ is a bijection from $\mathrm{Irr}_0(B)$ to $\mathrm{Irr}(P/[P, P])$. This proves the first assertion.

Suppose that p is odd. Let χ_0 be the unique p-rational character in $\mathrm{Irr}_0(B)$. Let $W(k)$ be the ring of Witt vectors in $\mathcal{O}$. By the structure theory of nilpotent blocks (see [14]), there exists a $W(k)G$-module V affording χ_0. Since the source idempotent i can be chosen to be in $W(k)G$, we have that $\Delta_{\chi,P,i}$ takes values in $W(k)$. Since p is odd, it follows that the trivial character of P is the unique linear character of P which takes values in $W(k)$. □

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