Euler allocations in the presence of non-linear reinsurance: comment on Major (2018)*

Silvana M. Pesenti†[1], Andreas Tsanakas[1], and Pietro Millossovich[1,2]

[1]Cass Business School, City, University of London
[2]DEAMS, University of Trieste

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Abstract

Major (2018) discusses Euler/Aumann-Shapley allocations for non-linear portfolios. He argues convincingly that many (re)insurance portfolios, while non-linear, are nevertheless positively homogeneous, owing to the way that deductibles and limits are typically set. For such non-linear but homogeneous portfolio structures, he proceeds with defining and studying a particular type of capital allocation. In this comment, we build on Major's (2018) insights but take a slightly different direction, to consider Euler capital allocations for distortion risk measures applied to homogeneous portfolios. Thus, the important problem of capital allocation in portfolios with non-linear reinsurance is solved.

Keywords Distortion risk measures, capital allocation, Euler allocation, Aumann-Shapley, reinsurance, aggregation.

1 Preliminaries

We use notation slightly different to Major (2018), which is better suited to the exposition of the ideas in this note. Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $\mathcal{X}$ and (for a positive integer n) $\mathcal{X}^n$ be, respectively, the sets of random variables and n-dimensional random vectors on that space, which are bounded from below. Positive outcomes of random variables in $\mathcal{X}$ represent financial losses. For any $Y \in \mathcal{X}$, denote its distribution by F_Y, its (left-)quantile function by F_Y^{-1}, and by U_Y a uniform random variable on $(0,1)$ comonotonic to Y, such that $Y = F_Y^{-1}(U_Y)$ almost surely. A *distortion risk measure*

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†Corresponding author. Email: Silvana.Pesenti@cass.city.ac.uk

$\rho_\zeta : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ can be defined as (Wang et al., 1997; Acerbi, 2002)

$$\rho_\zeta(Y) := \int_0^1 F_Y^{-1}(u)\zeta(u)du = \mathbb{E}(Y\zeta(U_Y)),$$

where ζ is a density on $(0,1)$. The risk measure ρ_ζ is positively homogeneous, that is, $\rho_\zeta(\beta Y) = \beta\rho_\zeta(Y)$ for any $Y \in \mathcal{X},\ \beta \geq 0$.

Consider a linear portfolio $Y^{\boldsymbol{w}} = \sum_{j=1}^n w_j X_j$, where $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}_+^n$ and $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$ are vectors of exposures and losses respectively, for the n lines of business that an insurance portfolio is made of. Let the capital requirement for $Y^{\boldsymbol{w}}$ be calculated as $\rho_\zeta(Y^{\boldsymbol{w}})$ for a distortion risk measure ρ_ζ. The *Euler capital allocation* for the portfolio $Y := Y^{\boldsymbol{1}}$ with unit exposures is given by the functional (Tasche, 2004):

$$\boldsymbol{d}(\boldsymbol{X}) : \mathcal{X}^n \to \mathbb{R}^n, \quad d_i(\boldsymbol{X}) := \frac{\partial}{\partial w_i}\rho_\zeta(Y^{\boldsymbol{w}})\bigg|_{\boldsymbol{w}=\boldsymbol{1}}.$$

By the positive homogeneity of ρ_ζ and Euler's theorem for homogeneous functions, we have that $\sum_{j=1}^n d_j(\boldsymbol{X}) = \rho_\zeta(Y)$. In particular, subject to differentiability, we have (Tsanakas, 2004)

$$d_i(\boldsymbol{X}) = \mathbb{E}(X_i\zeta(U_Y)), \quad i = 1, \ldots, n.$$

2 Major's contribution

Insurance portfolios are often non-linear, typically due to the presence of non-proportional reinsurance contracts. This makes Euler allocations as discussed above not obviously applicable, particularly when reinsurance contracts cover more than one line of business; equivalently when reinsurance recoveries cannot be easily attributed to individual lines of business. A non-linear portfolio can be formalised by an operator $\mathcal{F} : \mathcal{X}^n \to \mathcal{X}$. Assume that, for the purposes of the capital allocation exercise, the random vector $\boldsymbol{X}$ is fixed so that the portfolio loss is $\mathcal{F}(\boldsymbol{w} * \boldsymbol{X})$, where '$*$' stands for the Hadamard (elementwise) vector product. One can then represent the portfolio structure via a function $h : \mathbb{R}^n \times \mathcal{X}^n \to \mathbb{R}$, such that $h(\boldsymbol{w}, \boldsymbol{X}) := \mathcal{F}(\boldsymbol{w} * \boldsymbol{X})$, where the possible dependence of h on the distribution of $\boldsymbol{X}$ is suppressed. We denote the portfolio with unit weights as $Y = \mathcal{F}(\boldsymbol{X}) \equiv \mathcal{F}(\boldsymbol{1} * \boldsymbol{X})$. Let $h_i(\boldsymbol{z}) = \frac{\partial h(\boldsymbol{w},\boldsymbol{z})}{\partial w_i}\Big|_{\boldsymbol{w}=\boldsymbol{1}}$. If h is positively homogeneous in the first argument, that is $h(\beta\boldsymbol{w}, \boldsymbol{z}) = \beta h(\boldsymbol{w}, \boldsymbol{z})$ for any $\beta \geq 0$ and $\boldsymbol{z}$, $\boldsymbol{w} \in \mathbb{R}^n$, then the following decompositions hold:

$$h(\boldsymbol{1}, \boldsymbol{z}) = \sum_{j=1}^n h_i(\boldsymbol{z}) \implies \mathcal{F}(\boldsymbol{X}) = \sum_{j=1}^n h_i(\boldsymbol{X}).$$

Major (2018) argues convincingly that, when the non-linearity of $\mathcal{F}$ arises from reinsurance, the portfolio may still be considered positively homogeneous, as in practice reinsurance deductibles and limits are set (typically implicitly) as positively homogeneous functionals of the loss variables. For example, he considers the reinsurance portfolio:

$$\mathcal{F}(\boldsymbol{X}) = \min\left\{\left(X_1 + X_2 - F^{-1}_{X_1+X_2}(p)\right)_+ \ , \ F^{-1}_{X_1+X_2}(q) - F^{-1}_{X_1+X_2}(p)\right\}, \tag{1}$$
$$h(\boldsymbol{w}, \boldsymbol{z}) = \min\left\{\left(w_1 z_1 + w_2 z_2 - F^{-1}_{w_1X_1+w_2X_2}(p)\right)_+ \ , \ F^{-1}_{w_1X_1+w_2X_2}(q) - F^{-1}_{w_1X_1+w_2X_2}(p)\right\}$$

for $0 < p < q < 1$ and $F_{X_1+X_2}$ is the distribution of $X_1 + X_2$. It is straightforward to check that the function h is positively homogeneous in $\boldsymbol{w}$ and remains so if the percentiles are replaced by e.g. multiples of means or standard deviations.

Major (2018) proceeds by considering the positively homogeneous (in the loss variable $\boldsymbol{X}$) functional

$$\psi_\zeta(\boldsymbol{X}, \mathcal{F}) := \mathbb{E}\left(\mathcal{F}(\boldsymbol{X})\zeta\left(U_{\sum_{j=1}^n X_j}\right)\right).$$

This functional can be understood as an expectation of the portfolio loss subject to a probability distortion derived from the 'underlying' or 'gross of reinsurance' loss $\sum_{j=1}^n X_j$, which operates as a benchmark with respect to which the risk of any non-linear portfolio $\mathcal{F}(\boldsymbol{X})$ is evaluated.

The resulting capital allocation is defined via the partial derivatives of $\psi_\zeta(\boldsymbol{w} * \boldsymbol{X}, \mathcal{F})$, which are shown to be equal to (Major, 2018, Th. 3),

$$c_i^{\mathcal{F}}(\boldsymbol{X}) := \frac{\partial}{\partial w_i}\psi_\zeta(\boldsymbol{w} * \boldsymbol{X}, \mathcal{F})\bigg|_{\boldsymbol{w}=\boldsymbol{1}} = \mathbb{E}\left(h_i(\boldsymbol{X})\zeta\left(U_{\sum_{j=1}^n X_j}\right)\right) + E_2. \tag{2}$$

The term E_2 is quite involved and vanishes for example if $\mathcal{F}$ is a function of $\sum_{j=1}^n X_j$ alone (Major, 2018, Th. 5).

3 Euler allocations for reinsurance portfolios

The allocation proposed by Major (2018) makes the implicit assumption that portfolio risk is evaluated with respect to $\sum_{j=1}^n X_j$, which can be interpreted as an insurance portfolio gross of reinsurance. However, actual economic capital is calculated by the risk measure of the non-linear (e.g. net of reinsurance) portfolio, $\mathcal{F}(\boldsymbol{X})$. In other words, in many capital allocation applications, the amount that needs to be allocated is $\rho_\zeta(\mathcal{F}(\boldsymbol{X}))$ rather than $\psi_\zeta(\boldsymbol{X}, \mathcal{F})$. Fortunately, building on Major's (2018) insights and previous work on risk measure sensitivity (Hong, 2009; Hong and Liu, 2009; Tsanakas and Millossovich, 2016), such a capital allocation is easily obtained.

Assume that, as before, $Y = \mathcal{F}(\boldsymbol{X})$, $h(\boldsymbol{w}, \boldsymbol{X}) = \mathcal{F}(\boldsymbol{w} * \boldsymbol{X})$, and h is homogeneous in $\boldsymbol{w}$. Then $\rho_\zeta(\mathcal{F}(\boldsymbol{w} * \boldsymbol{X}))$ is also homogeneous in $\boldsymbol{w}$. Consequently, we can define the Euler allocation for a non-linear portfolio $\mathcal{F}$:

$$\boldsymbol{d}^{\mathcal{F}}(\boldsymbol{X}) : \mathcal{X}^n \to \mathbb{R}^n, \quad d_i^{\mathcal{F}}(\boldsymbol{X}) := \frac{\partial}{\partial w_i}\rho_\zeta(\mathcal{F}(\boldsymbol{w} * \boldsymbol{X}))\bigg|_{\boldsymbol{w}=\boldsymbol{1}},$$

where it holds that $\sum_{j=1}^{n} d_j^{\mathcal{F}}(\boldsymbol{X}) = \rho_\zeta(\mathcal{F}(\boldsymbol{X})) = \rho_\zeta(Y)$.

The explicit calculation of the allocation $d^{\mathcal{F}}(\boldsymbol{X})$ for distortion risk measures follows from Hong (2009); Hong and Liu (2009) or, alternatively, Tsanakas and Millossovich (2016). Subject to differentiability conditions stated in those papers, we have the allocation

$$d_i^{\mathcal{F}}(\boldsymbol{X}) = \mathbb{E}(h_i(\boldsymbol{X})\zeta(U_Y)), \quad i = 1, \ldots, n, \tag{3}$$

which obviously subsumes the linear case. This allocation satisfies a version of the well studied core property (Tsanakas, 2004; Kalkbrener, 2005), that is, if ζ is non-decreasing or equivalently if ρ_ζ is subadditive, we have that

$$d_i^{\mathcal{F}}(\boldsymbol{X}) \leq \rho_\zeta(h_i(\boldsymbol{X})),$$

often interpreted as a requirement that the allocation does not produce incentives for portfolio fragmentation. The following two examples illustrate how the capital allocation $\boldsymbol{d}^{\mathcal{F}}$ differs from that of Major (2018).

Example. First consider the portfolio structure given in (1). As Major (2018) notes, by positive homogeneity we can write $\mathcal{F}(\boldsymbol{X}) = \sum_{j=1}^{2} h_i(\boldsymbol{X})$, where

$$\begin{aligned} h_i(\boldsymbol{X}) = {} & \boldsymbol{I}_{\{X_1+X_2 \in [F^{-1}_{X_1+X_2}(p), F^{-1}_{X_1+X_2}(q)]\}} \Big(X_i - \mathbb{E}\big(X_i | X_1 + X_2 = F^{-1}_{X_1+X_2}(p)\big)\Big) \\ & + \boldsymbol{I}_{\{X_1+X_2 > F^{-1}_{X_1+X_2}(q)\}} \Big(\mathbb{E}\big(X_i | X_1 + X_2 = F^{-1}_{X_1+X_2}(q)\big) - \mathbb{E}\big(X_i | X_1 + X_2 = F^{-1}_{X_1+X_2}(p)\big)\Big). \end{aligned}$$

The above calculation utilises quantile derivatives, see Tasche (2004). Notice that, since the portfolio defined in (1) is a non-decreasing function of $X_1 + X_2$, the E_2 term in (2) vanishes. Moreover, the random variables $\mathcal{F}(\boldsymbol{X})$ and $X_1 + X_2$ are comonotonic. Hence, we can choose $U_{\mathcal{F}(\boldsymbol{X})} = U_{X_1+X_2}$ almost surely and therefore $\rho_\zeta(\mathcal{F}(\boldsymbol{X})) = \psi_\zeta(\boldsymbol{X}, \mathcal{F})$. This implies that the Euler allocation we propose coincides with Major's allocation. Indeed

$$d_i^{\mathcal{F}}(\boldsymbol{X}) = \mathbb{E}(h_i(\boldsymbol{X})\zeta(U_Y)) = \mathbb{E}\left(h_i(\boldsymbol{X})\zeta\left(U_{X_1+X_2}\right)\right) = c_i^{\mathcal{F}}(\boldsymbol{X})$$

by comparing equations (2) and (3).

Thus, in the case when the portfolio $\mathcal{F}(\boldsymbol{X})$ is comonotonic to $\sum_{i=1}^{n} X_i$, the Euler allocation and the allocation proposed by Major are equivalent. In general however, the two allocations differ, even if the E_2 term in (2) vanishes, for example if the portfolio $\mathcal{F}(\boldsymbol{X})$ is a function of $\sum_{i=1}^{n} X_i$ that is not non-decreasing. The probability distortions derived with reference to $\mathcal{F}(\boldsymbol{X})$ (approach taken in this note) and $\sum_{i=1}^{n} X_i$ (approach taken by Major) are in general different. This is demonstrated in the following example.

Example. Consider now a different portfolio structure, where for some $\lambda \geq 1,\ p \in (0,1)$, we have

$$\mathcal{F}(\boldsymbol{X}) = \min\Big\{ \big(X_1 - \lambda\mathbb{E}(X_1)\big)_+ + \big(X_2 - \lambda\mathbb{E}(X_2)\big)_+,\ F^{-1}_{X_1+X_2}(p) - \lambda\mathbb{E}(X_1 + X_2) \Big\}.$$

Table 1: Comparison of risk measures ψ_ζ, ρ_ζ and respective allocations $\boldsymbol{c}^{\mathcal{F}}, \boldsymbol{d}^{\mathcal{F}}$, with standard errors for a simulated sample of size 10^6.

	$\lambda = 1$	$\lambda = 1.8$
$\psi_\zeta(\boldsymbol{X}, \mathcal{F})$	3.902 (0.004)	0.563 (0.004)
$\dfrac{\boldsymbol{c}^{\mathcal{F}}(\boldsymbol{X})}{\psi_\zeta(\boldsymbol{X}, \mathcal{F})}$	36.4%, 63.6% (0.1%, 0.1%)	62.7%, 37.3% (0.6%, 0.6%)
$\rho_\zeta(\mathcal{F}(\boldsymbol{X}))$	3.956 (0.004)	0.691 (0.005)
$\dfrac{\boldsymbol{d}^{\mathcal{F}}(\boldsymbol{X})}{\rho_\zeta(\mathcal{F}(\boldsymbol{X}))}$	36.9%, 63.1% (0.1%, 0.1%)	54.2%, 45.8% (0.6%, 0.6%)

In this case, it is seen that $\mathcal{F}(\boldsymbol{X})$ is not comonotonic with $X_1 + X_2$ and thus $E_2 \neq 0$. Hence, $\rho_\zeta(\mathcal{F}(\boldsymbol{X})) \neq \psi_\zeta(\boldsymbol{X}, \mathcal{F})$ and the Euler allocation $d_i^{\mathcal{F}}(\boldsymbol{X})$ does not coincide with the allocation $c_i^{\mathcal{F}}(\boldsymbol{X})$ of Major.

We demonstrate this by a numerical example. First note that for the given portfolio,

$$\begin{aligned} h(\boldsymbol{w}, \boldsymbol{z}) = \min\{&\big(w_1 z_1 - \lambda\mathbb{E}(w_1 X_1)\big)_+ + \big(w_2 z_2 - \lambda\mathbb{E}(w_2 X_2)\big)_+, \\ &F^{-1}_{w_1 X_1 + w_2 X_2}(p) - \lambda\mathbb{E}(w_1 X_1 + w_2 X_2)\}, \\ h_i(\boldsymbol{X}) = \boldsymbol{I}_A \boldsymbol{I}_{\{X_i > \lambda\mathbb{E}(X_i)\}}&\big(X_i - \lambda\mathbb{E}(X_i)\big) + \boldsymbol{I}_{A^c}\left(\mathbb{E}\big(X_i | X_1 + X_2 = F^{-1}_{X_1+X_2}(p)\big) - \lambda\mathbb{E}(X_i)\right), \end{aligned}$$

where $A = \left\{\big(X_1 - \lambda\mathbb{E}(X_1)\big)_+ + \big(X_2 - \lambda\mathbb{E}(X_2)\big)_+ \leq F^{-1}_{X_1+X_2}(p) - \lambda\mathbb{E}(X_1 + X_2)\right\}$.

Let $X_1 \sim \Gamma(4,1)$, $X_2 \sim \Gamma(8,1)$ be independent, such that X_1 has a lower standard deviation, but higher skewness coefficient, than X_2. Same as Major (2018), we consider a distortion risk measure with $\zeta(u) = \frac{1}{2}(1-u)^{-1/2}$, $0 < u < 1$. For the portfolio parameters, we fix $p = 0.999$ and let $\lambda \in \{1, 1.8\}$.

In Table 1, values for the risk measures $\psi_\zeta(\boldsymbol{X}, \mathcal{F})$ and $\rho_\zeta(\mathcal{F}(\boldsymbol{X}))$ are reported, as well as the corresponding Euler capital allocations $\boldsymbol{c}^{\mathcal{F}}$ and $\boldsymbol{d}^{\mathcal{F}}$, normalised to add up to 1. The results were derived from 500 sets of simulated samples, each of size 10^6. On each of the 500 samples, the risk measures and capital allocations were calculated. The reported values are the average risk measures and allocations across the 500 samples. In addition, we report estimated standard errors (pertaining to a sample size of 10^6), calculated as standard deviations of the risk measure and allocation estimates across the 500 samples.

As λ increases in value from 1 to 1.8, dependence between $X_1 + X_2$ and $\mathcal{F}(\boldsymbol{X})$ weakens, such that the two random variables attain extreme values for different states. This implies that the differences between $\rho_\zeta(\mathcal{F}(\boldsymbol{X}))$ and $\psi_\zeta(\boldsymbol{X}, \mathcal{F})$, as well as the respective allocations, become more pronounced, as can be seen in the table. In particular, the relative allocations are nearly identical for $\lambda = 1$, with X_2 being allocated almost twice the amount of capital than X_1. For $\lambda = 1.8$, emphasis is placed on the tails of the variables X_1, X_2, as is apparent from the form of $\mathcal{F}$. As a result, for both allocations, the picture is reversed, with X_1 allocated a larger percentage of the risk; this may be explained by the higher skewness of X_1. This change in allocations appears to be more pronounced in the allocation $\boldsymbol{c}^{\mathcal{F}}$ compared to $\boldsymbol{d}^{\mathcal{F}}$.

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