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DECOMPOSITION NUMBERS FOR THE CYCLOTOMIC BRAUER ALGEBRAS IN CHARACTERISTIC ZERO

C. BOWMAN, A. G. COX, AND M. DE VISSCHER

Abstract. We study the representation theory of the cyclotomic Brauer algebra via truncation to idempotent subalgebras which are isomorphic to a product of walled and classical Brauer algebras. In particular, we determine the block structure and decomposition numbers in characteristic zero.

Introduction

The symmetric and general linear groups satisfy a double centraliser property over tensor space. This relationship is known as Schur–Weyl duality and allows one to pass information between the representation theories of these algebras. The Brauer algebra is an enlargement of the symmetric group algebra and is in Schur-Weyl duality with the orthogonal (or symplectic) group.

The cyclotomic Brauer algebra B_n^m is a corresponding enlargement of the complex reflection group algebra H_n^m of type $G(m,1,n)$. This was introduced by [HO01] as a specialisation of the cyclotomic BMW algebra, and has been studied by various authors (see for example [GH09, RX07, RY04, Yu07]).

The algebra H_n^m is Morita equivalent to a direct sum of products of symmetric group algebras. One might ask if this equivalence extends to the cyclotomic Brauer algebra. Although there is no direct equivalence, we will see that the underlying combinatorics of B_n^m is that of a product of classical Brauer and walled Brauer algebras.

Our main result is that certain co-saturated idempotent subalgebras of B_n^m are isomorphic to a product of classical Brauer and walled Brauer algebras. Over a field of characteristic zero, this induces isomorphisms between all higher extension groups $\mathrm{Ext}^i(\mathcal{F}(\Delta), -)$. Hence we obtain the decomposition numbers and block structure of the cyclotomic Brauer algebra in characteristic zero from the corresponding results for the Brauer and walled Brauer algebras [Mar, CD11].

We exhibit a tower of recollement structure [CMPX06] for B_n^m, and discuss certain signed induction and restriction functors associated with this. We expect that this structure will also be a useful tool in the positive characteristic case.

Diagrams for the cyclotomic Brauer algebra come with an orientation due to the relationship with the cyclotomic BMW algebra. However, one can define a similar algebra without orientation, which we shall call the unoriented cyclotomic Brauer algebra. In an Appendix we show that our results can be easily modified for this algebra, to reduce its study to a product now just of Brauer algebras. The advantage of this unoriented version is that analogues can be defined associated to general complex reflection groups of type $G(m,p,n)$; we will consider the representation theory of such algebras in a subsequent paper.

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1. Cyclotomic Brauer algebras

In this section we define the cyclotomic Brauer algebra, $B_n^m = B_n^m(\delta)$ over an algebraically closed field k of characteristic $p \geq 0$. We assume throughout the paper that m is invertible in k and we fix a primitive m-th root of unity ξ.

1.1. Definitions

Given $m, n \in \mathbb{N}$ and $\delta = (\delta_0, \ldots, \delta_{m-1}) \in k^m$, the *cyclotomic Brauer algebra* $B_n^m(\delta)$ is a finite dimensional associative k-algebra spanned by certain Brauer diagrams. An (m,n)-*diagram* consists of a *frame* with n distinguished points on the northern and southern boundaries, which we call *nodes*. We number the northern nodes from left to right by $1 \ldots n$ and the southern nodes similarly by $\bar{1}, \ldots, \bar{n}$. Each node is joined to precisely one other by a strand; strands connecting the northern and southern edge will be called *through strands* and the remainder *arcs*. There may also be closed loops inside the frame, those diagrams without closed loops are called *reduced* diagrams.

Each strand is endowed with an orientation and labelled by an element of the cyclic group $\mathbb{Z}/m\mathbb{Z}$. We may reverse the orientation by relabelling the strand with the inverse element in $\mathbb{Z}/m\mathbb{Z}$. We identify diagrams in which the strands connect the same pairs of nodes and (after being identically oriented) have the same labels.

As a vector space, B_n^m is the k-span of all reduced (m,n)-diagrams. Figure 1 gives an example of two such elements in $B_6^3(\delta)$.

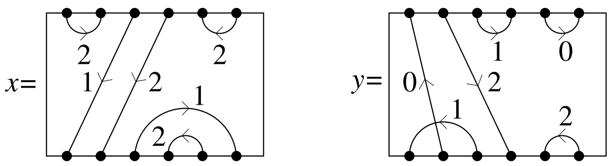

Figure 1. Two elements in $B_6^3(\delta)$.

We define the product $x \cdot y$ of two reduced (m,n)-diagrams x and y using the concatenation of x above y, where we identify the southern nodes of x with the northern nodes of y. More precisely, we first choose compatible orientations of the strands of x and y. Then we concatenate the diagrams and add the labels on each strand of the new diagram to obtain another (m,n)-diagram.

Any closed loop in this (m,n)-diagram can be oriented such that as the strand passes through the leftmost central node in the loop it points downwards. If this oriented loop is labelled by $i \in \mathbb{Z}/m\mathbb{Z}$ then the diagram is set equal to δ_i times the same diagram with the loop removed.

Example 1.1.1. Consider the product $x \cdot y$ of the elements in Figure 1. After concatenation we obtain the element in Figure 2. Reading from left to right in the diagram we have that $1 - 0 \equiv 1$, $2 + 2 \equiv 1$, and $1 - 2 - 1 + 0 \equiv 1$, (mod 3) and therefore we obtain the reduced diagram in Figure 3 by removing the closed loop labelled by 1, and multiply by δ_1.

From now on, we will omit the label on any strand labelled by $0 \in \mathbb{Z}/m\mathbb{Z}$.

We will need to speak of certain elements of the algebra with great frequency. Let $t_{i,j}$ (for $1 \leq i, j \leq n$) be the diagram with only 0 labels and having through strands from i to $\bar{j}$, j to $\bar{i}$, and l to $\bar{l}$ for all $l \neq i, j$. Let t_i^r (for $1 \leq i \leq n$ and $0 \leq r \leq m-1$) be the diagram with through strands from l to $\bar{l}$ for all l, with the through strand from i labelled by r and all other labels being 0. These elements are illustrated in Figure 4. We let $e_{i,j}$ (for $1 \leq i, j \leq n$) be the diagram with only 0 labels and having arcs from i to j and $\bar{i}$ to $\bar{j}$, and through strands from l to $\bar{l}$ for all $l \neq i, j$. This element is illustrated in Figure 5.

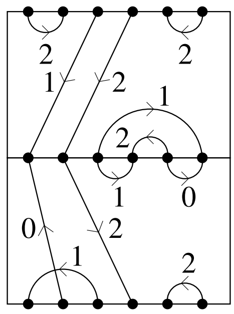

FIGURE 2. The unoriented and unreduced product of x and y.

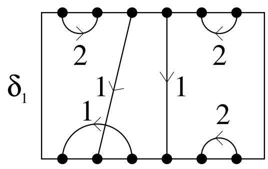

FIGURE 3. The reduced product of x and y.

The elements $t_{i,i+1}$ (with $i \leq n-1$) and t_1^1 are generators of the group algebra

$$H_n^m = k((\mathbb{Z}/m\mathbb{Z}) \wr \Sigma_n)$$

as a subalgebra of $B_n^m(\delta)$. Note that $B_1^m(\delta) \cong k(\mathbb{Z}/m\mathbb{Z})$; for convenience, we set $B_0^m(\delta) = k$. It is easy to see that the cyclotomic Brauer algebra is generated by the elements $t_{i,i+1}, t_1^1$, and $e_{1,2}$ (for $i \leq n-1$).

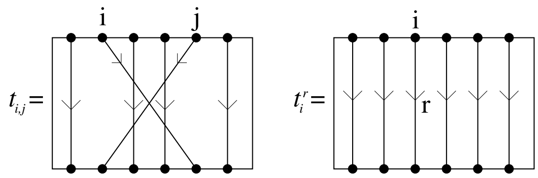

FIGURE 4. The elements $t_{i,j}$ and t_i^r.

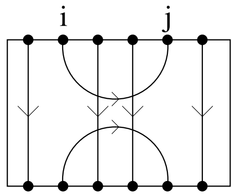

FIGURE 5. The element $e_{i,j}$.

We have defined the cyclotomic Brauer algebra in terms of $\delta = (\delta_0, \dots, \delta_{m-1}) \in k^m$. However, we shall find that it is signed polynomials in these parameters which govern the representation theory of the algebra.

Definition 1.1.2. For each $0 \leq r \leq m-1$ we define the rth signed cyclotomic parameter to be

$$\overline{\delta}_r = \frac{1}{m} \sum_{i=0}^{m-1} \xi^{ir} \delta_i.$$

Note that $\overline{\delta}_r$ and $\overline{\delta}_{m-r}$ are swapped by the map $\xi \leftrightarrow \xi^{-1}$.

Remark 1.1.3. Cyclotomic Brauer algebras were originally defined by Häring-Oldenburg [HO01]. Our definition can easily be seen to be equivalent to that of Rui and his collaborators (see [RY04] and [RX07]). The version considered by Goodman and Hauschild Mosley [GH09] and Yu [Yu07] is the specialisation of this algebra obtained by setting $\delta_r = \delta_{m-r}$.

Remark 1.1.4. In [RY04] and [RX07] semi-simplicity conditions are given for B_n^m in terms of the signed parameters. We note that there is a mistake in the statement of [RY04, Theorem 8.6] which runs through both of these papers. This is a simple misreading of the (correct) circulant matrices calculated in the proof of the theorem.

Their vanishing conditions are given in terms of $\overline{\delta}_r - m\epsilon_{(r,0)}$ where $\epsilon_{(r,0)}$ is the Kronecker function. Correct versions of these statements can be deduced by substituting this by $\overline{\delta}_r - m\epsilon_{(r,m-r)}$. Compare [CDDM08, Theorem 6.2] and [CDM09, Proposition 4.2] to see how the $-m\epsilon_{(r,m-r)}$ relates to the semi-simplicity of the Brauer algebra versus the walled Brauer algebra.

Remark 1.1.5. We have that $B_n^2(\delta)$ is a subalgebra of the recently defined Brauer algebra of type C_n (see [CLY]). This can be seen by 'unfolding' the diagrams (as outlined in [MGP07, Section 4.3]) and using [Bow12, Theorem 3.6].

1.2. Classical Brauer and walled Brauer algebras

The classical Brauer algebra $B(n, \delta)$ ($\delta \in k$) is given by the particular case $B_n^1(\delta)$ with $\delta_0 = \delta$. Note that the orientation of the strands in Brauer diagrams plays no role in this case and so can be ignored.

The walled Brauer algebra $WB(r, s, \delta)$ is the subalgebra of $B(r+s, \delta)$ spanned by the so-called walled Brauer diagrams. Explicitly, we place a vertical wall in the $(r+s)$-Brauer diagrams after the first r northern (resp. southern) nodes and we require that arcs must cross the wall and through strands cannot cross the wall.

2. Representations of H_n^m

In this section we review the construction of the Specht modules for the group algebra H_n^m of the complex reflection group $(\mathbb{Z}/m\mathbb{Z}) \wr \Sigma_n$.

2.1. Compositions and partitions

An *m-composition* of n is an m-tuple of non-negative integers $\omega = (\omega_0, \dots, \omega_{m-1})$ such that $\sum_{i=0}^{m-1} \omega_i = n$. A partition is a finite decreasing sequence of non-negative integers. An *m-partition* of n is an m-tuple of partitions $\lambda = (\lambda^0, \dots, \lambda^{m-1})$ such that $\sum_{i=0}^{m-1} |\lambda^i| = n$ (where $|\lambda^i|$ denotes the sum of the parts of the partition λ^i). Given an m-partition λ we associate the m-composition

$$|\lambda| = (|\lambda^0|, |\lambda^1|, \dots, |\lambda^{m-1}|).$$

For an m-composition of n, ω, we define another m-composition $[\omega]$ by

$$[\omega] = ([\omega_0], [\omega_1], \dots, [\omega_{m-1}] = n)$$

where $[\omega_r] = \sum_{i=0}^{r} \omega_i$ for $0 \leq r \leq m-1$. For an m-partition λ we define $[\lambda] = [|\lambda|]$.

The *Young diagram* of an m-partition is simply the m-tuple of Young diagrams of each partition. We do not distinguish between the m-partition λ and its Young diagram. For an m-partition λ, define the set $\mathrm{rem}(\lambda)$ (resp.$\mathrm{add}(\lambda)$) of all *removable boxes* (respectively *addable boxes*) to be those which can be removed from (respectively added to) λ such that the result is the Young diagram of an m-partition. We can refine this by insisting that a removable (respectively addable) box has sign ξ^r if it can be removed (respectively added) to λ^r, for $0 \leq r \leq m-1$. We denote these sets by ξ^r-$\mathrm{rem}(\lambda)$ and ξ^r-$\mathrm{add}(\lambda)$ respectively.

2.2. Idempotents

We define some idempotents in H_n^m which play a very important role in this paper. Note that $k(\mathbb{Z}/m\mathbb{Z} \times \ldots \times \mathbb{Z}/m\mathbb{Z})$ occurs naturally as the subalgebra of H_n^m spanned by all diagrams where node i is connected to node $\bar{i}$ for all $1 \leq i \leq n$. As m is invertible in k we have that $k(\mathbb{Z}/m\mathbb{Z})$ is semisimple and decomposes into a sum of 1-dimensional modules given by ξ^r $(0 \leq r \leq m-1)$. We denote by T_i^r the idempotent in the copy of $k(\mathbb{Z}/m\mathbb{Z})$ on the i-th strand corresponding to ξ^r. This idempotent is given as follows.

Definition 2.2.1. For each $1 \leq i \leq n$ and each $0 \leq r \leq m-1$, define the idempotent

$$T_i^r = \frac{1}{m} \sum_{0 \leq q \leq m-1} \xi^{qr} t_i^q.$$

Now we will consider certain products of these idempotents. Let ω be an m-composition of n. We have

$$0 \leq [\omega_0] \leq [\omega_1] \leq [\omega_2] \leq \ldots \leq [\omega_{m-1}] = n.$$

So for each $1 \leq i \leq n$ there is a unique $0 \leq r \leq m-1$ with

$$[\omega_{r-1}] < i \leq [\omega_r]$$

(where we set $[\omega_{-1}] = 0$). In this case we write $i \in [\omega_r]$. Now we define the (orthogonal) idempotent π_ω as follows.

Definition 2.2.2. Let ω be an m-composition of n. Then we define

$$\pi_\omega = \prod_{r=0}^{m-1} \prod_{i \in [\omega_r]} T_i^r.$$

The element π_ω is a linear combination of diagrams, but can be viewed as putting the element T^0 on each of the first ω_0 strands of the identity diagram, then the element T^1 on each of the next ω_1 strands,..., and finally T^{m-1} on each of the last ω_{m-1} strands.

2.3. Specht modules of H_n^m

For an m-composition $\omega = (\omega_0, \omega_1, \ldots, \omega_{m-1})$ of n we define the Young subgroup Σ_ω of Σ_n by

$$\Sigma_\omega = \Sigma_{\omega_0} \times \Sigma_{\omega_1} \times \ldots \times \Sigma_{\omega_{m-1}}$$

and the corresponding Young subalgebra H_ω^m of H_n^m by

$$H_\omega^m = k((\mathbb{Z}/m\mathbb{Z}) \wr \Sigma_\omega).$$

Definition 2.3.1. Let λ, μ be m-partitions of n. We say that λ dominates μ and write $\mu \trianglelefteq_n \lambda$ if

$$[\lambda^{j-1}] + \sum_{i=1}^{k} \lambda_i^j \geq [\mu^{j-1}] + \sum_{i=1}^{k} \mu_i^j$$

for all $0 \leq j \leq m-1$ and $k \geq 0$ (where we set $[\lambda^{-1}] = [\mu^{-1}] = 0$).

Given any $k\Sigma_n$-module M and any $r \in \mathbb{Z}/m\mathbb{Z}$ we define the H_n^m-module $M^{(r)}$ by setting $M^{(r)} \downarrow_{\Sigma_n} = M$ and each t_i $(1 \leq i \leq n)$ acts on $M^{(r)}$ by scalar multiplication by ξ^r. In particular, if λ is a partition of n and we denote by $S(\lambda)$ the corresponding Specht module for $k\Sigma_n$ then we have an H_n^m-module $S(\lambda)^{(r)}$ for each $0 \leq r \leq m-1$. This module is the Specht H_n^m-module labelled by $(\emptyset, \ldots, \emptyset, \lambda, \emptyset, \ldots, \emptyset)$ where λ is in the r-th position. More generally we have the following result.

Proposition 2.3.2 (Section 5 of [GL96])**.** *The algebra H_n^m is cellular with respect to the dominance order $\trianglelefteq_n$ on the set of m-partitions of n. For a given m-partition λ of n, the cell module $\mathbf{S}(\lambda)$ is given by*

$$\mathbf{S}(\lambda) \cong (S(\lambda^0)^{(0)} \otimes \ldots \otimes S(\lambda^{m-1})^{(m-1)})\uparrow_{H^m_{|\lambda|}}^{H^m_n}.$$

We call $\mathbf{S}(\lambda)$ the Specht module for H_n^m labelled by λ.

It is well known (see for example [DM02]) that the algebra H_n^m is Morita equivalent to the direct sum of group algebras of Young subgroups of Σ_n. These arise as idempotent subalgebras of H_n^m. Indeed, the idempotent subalgebra $\pi_\omega H_n^m \pi_\omega$ is isomorphic to $k\Sigma_\omega$. Now, using the description of cell modules given in Proposition 2.3.2, we see that $\pi_\omega \mathbf{S}(\lambda) = 0$ unless $\omega = |\lambda|$. Now suppose $|\lambda| = \omega$ and let h be a diagram in $H_n^m \setminus H^m_{|\lambda|}$. So h has a strand between i and $\bar{j}$ say with $i \in [\omega_r]$ and $j \in [\omega_s]$ for some $s \neq r$. Then we have $\pi_\omega h = \pi_\omega T_i^r h = \pi_\omega h T_j^r$. As $s \neq r$ we have $\pi_\omega h \otimes x = 0$ for any $x \in S(\lambda^0)^{(0)} \otimes \ldots \otimes S(\lambda^{m-1})^{(m-1)}$. Thus we have

$$\pi_\omega \mathbf{S}(\lambda) \cong \begin{cases} S(\lambda^0) \otimes S(\lambda^1) \otimes \ldots \otimes S(\lambda^{m-1}) & \text{if } |\lambda| = \omega \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.1}$$

3. Cell modules for B_n^m

In this section we show that B_n^m is an iterated inflation (in the sense of [KX01]), and so is a cellular algebra. We recall the construction of the cell modules and study the restriction and induction rules for these. When the algebras are quasi-hereditary we obtain a tower of recollement (in the sense of [CMPX06]).

3.1. Iterated inflation and cell modules

Definition 3.1.1. Suppose that $n, l \in \mathbb{N}$ with $l \leq \lfloor n/2 \rfloor$. An (n,l)-dangle is a partition of $\{1, \ldots, n\}$ into l two-element subsets (called arcs) and $n - 2l$ one-element subsets (called free nodes). An (m,n,l)-dangle is an (n,l)-dangle to which an integer $r \in \mathbb{Z}/m\mathbb{Z}$ has been assigned to every subset of size 2.

We can represent an (n,l)-dangle d by a set of n nodes labelled by the set $\{1, \ldots, n\}$, where there is an arc (denoted v_{ij}) joining i to j if $\{i,j\} \in d$, and there is a vertical line starting from i if $\{i\} \in d$. An (m,n,l)-dangle can be represented graphically by first labelling each arc of the underlying (n,l)-dangle and then giving it the following orientation: we let all one element sets have a downward orientation and all two element sets have a right orientation. An example of an $(m,7,3)$-dangle for $m \geq 3$ is given in Figure 6.

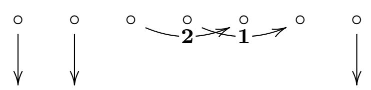

Figure 6. An $(m,7,3)$ dangle.

We let $V(m,n,l)$ denote the vector space spanned by all (m,n,l)-dangles.

Let $\Lambda(m,n)$ denote the set of m-partitions of $n - 2l$ for all $l \leq \lfloor n/2 \rfloor$. We can extend the dominance ordering given in Definition 2.3.1 to this set as follows.

Definition 3.1.2. Let $\lambda, \mu \in \Lambda(m,n)$. We set $\lambda \trianglerighteq \mu$ if and only if either $\sum_{i=0}^{m-1} |\lambda^i| > \sum_{i=0}^{m-1} |\mu^i|$ or $\sum_{i=0}^{m-1} |\lambda^i| = \sum_{i=0}^{m-1} |\mu^i| = n - 2l$ (for some l) and $\mu \trianglelefteq_{n-2l} \lambda$.

Each (m, n)-diagram in B_n^m with $n-2l$ through strands can be decomposed as two (m, n, l)-dangles, giving the top and bottom of the diagram, and an element of $\mathbb{Z}/m\mathbb{Z} \wr \Sigma_{n-2l}$ giving the through strands. Using this decomposition we get the following result.

Theorem 3.1.3. *The cyclotomic Brauer algebra $B_n^m(\delta)$ is an iterated inflation with inflation decomposition*

$$B_n^m(\delta) = \bigoplus_{l=0}^{\lfloor n/2 \rfloor} V(m,n,l) \otimes V(m,n,l) \otimes H_{n-2l}^m.$$

Therefore B_n^m is cellular with respect to the dominance ordering $\trianglerighteq$ on $\Lambda(m, n)$, and the anti-involution $*$ given by reflection of a diagram through its horizontal axis.

Proof. This follows by standard arguments, see for example [KX01]. $\square$

For any H_n^m-module M, we define an action of the algebra B_n^m on the vector space $V(m,n,l)\otimes M$. For any (m, n)-diagram X, any (m, n, l)-dangle d and any element $x \in M$ we define $X(d \otimes x)$ as follows. Place the diagram X above the (m, n, l)-dangle d. Choose a compatible orientation of the strands and then concatenate to give an $(m, n, l+t)$-dangle Xd and an element $\sigma \in H_n^m$ acting on the free $n - 2(l + t)$ nodes. If $t > 0$ then we set $X(d \otimes x) = 0$, and otherwise, we define $X(d \otimes x) = (Xd) \otimes \sigma x$.

Corollary 3.1.4. *We have that the cell modules for $B_n^m(\delta)$ are of the form*

$$\Delta_n(\lambda) = V(m,n,l) \otimes \mathbf{S}(\lambda)$$

where $\mathbf{S}(\lambda)$ is the Specht module for H_{n-2l}^m defined in Section 2.3.

3.2. Tower of algebras, restriction and induction

Let $n \geq 2$. Suppose first that $\delta \neq 0 \in k^m$ and fix a $\delta_r \neq 0$ for some $0 \leq r \leq m-1$. We then define the idempotent $e_{n-2} = \frac{1}{\delta_r} t_{n-1}^r e_{n-1,n}$ as illustrated in Figure 7. Note that it is a scalar multiple of a diagram with $n-2$ through strands. If $\delta = 0$ and $n \geq 3$ then we define e_{n-2} to be the idempotent $e_{n-1,n}e_{n-2,n-1}$, as illustrated in Figure 8.

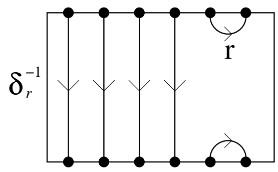

FIGURE 7. The idempotent e_{n-2} when $\delta_r \neq 0$.

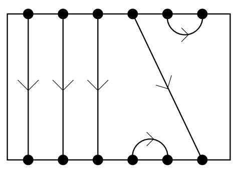

FIGURE 8. The idempotent e_{n-2} when $\delta = 0$ and $n \geq 3$.

It is easy to see that

$$e_{n-2} B_n^m e_{n-2} \cong B_{n-2}^m \tag{3.2.1}$$

and
$$B_n^m/B_n^m e_{n-2} B_n^m \cong H_n^m \tag{3.2.2}$$
just as for the Brauer algebra [CDM09, Lemma 2.1]. In particular, any H_n^m-module can naturally be viewed as a B_n^m-module.

Via (3.2.1) we define an exact localisation functor through the idempotent e_{n-2}.
$$\begin{aligned} F_n \ : \ B_n^m\text{-mod} &\longrightarrow B_{n-2}^m\text{-mod} \\ M &\longmapsto e_{n-2}M \end{aligned}$$
and a right exact globalisation functor
$$\begin{aligned} G_n \ : \ B_n^m\text{-mod} &\longrightarrow B_{n+2}^m\text{-mod} \\ M &\longmapsto B_{n+2}^m e_n \otimes_{B_n^m} M. \end{aligned}$$
Note that $F_{n+2}G_n(M) \cong M$ for all $M \in B_n^m$-mod, and hence G_n is a full embedding. It is easy to check that for any $\lambda \in \Lambda(m,n)$ we have
$$F_n(\Delta_n(\lambda)) \cong \begin{cases} \Delta_{n-2}(\lambda) & \text{if } \lambda \in \Lambda(m, n-2) \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.3}$$
We have, for any H_n^m-module, that
$$B_{n+2}^m e_n \otimes_{B_n^m} M \ \cong \ V(m, n+2, 1) \otimes_k M$$
as B_{n+2}^m-modules where the action on the LHS is given by left multiplication in B_{n+2}^m and the action on the RHS is given in Section 3.1 (see [HHKP10, Proposition 4.1]). In particular, if λ is an m-partition of $n-2l$, we have
$$\Delta_n(\lambda) = G_{n-2}G_{n-4}\dots G_{n-2l}\mathbf{S}(\lambda)$$
and hence
$$G_n\Delta_n(\lambda) = \Delta_{n+2}(\lambda). \tag{3.2.4}$$

Lemma 3.2.1. *For each $n \geq 1$, the algebra B_n^m can be identified as a subalgebra of B_{n+1}^m via the homomorphism which takes an (m,n)-diagram X in B_n^m to the $(m,n+1)$-diagram in B_{n+1}^m obtained by adding two vertices $n+1$ and $\overline{n+1}$ with a strand between them labelled by zero.*

Lemma 3.2.1 implies that we can consider the usual restriction and induction functors. We refine these functors as a direct sum of signed versions. This refinement is given by inducing via
$$B_{n-1}^m \subset B_{n-1}^m \otimes B_1^m \subset B_n^m,$$
where $B_1^m \cong k(\mathbb{Z}/m\mathbb{Z})$ and corresponds to the rightmost string in the diagrams. Note that for any B_n^m-module M we have that $T_n^r M$ is naturally a B_{n-1}^m-module. In fact, it is given by the summand of $M{\downarrow_{B_{n-1}^m \otimes B_1^m}}$ on which B_1^m acts by ξ^r. So we can define the following signed induction and restriction functors.
$$\begin{aligned} \xi^r\text{-res}_n \ &: B_n^m\text{-mod} \longrightarrow B_{n-1}^m\text{-mod} \\ M &\longmapsto T_n^r M{\downarrow_{B_{n-1}^m}} \end{aligned}$$
and
$$\begin{aligned} \xi^r\text{-ind}_n \ &: B_n^m\text{-mod} \longrightarrow B_{n+1}^m\text{-mod} \\ M &\longmapsto \mathrm{ind}_{B_n^m \otimes B_1^m}^{B_{n+1}^m}(M \boxtimes kT_{n+1}^r). \end{aligned}$$

We can relate these functors to globalisation and localisation, as in [CMPX06, (A4)].

Lemma 3.2.2. *(i) For all $n \geq 2$ we have that*

$$B_n^m e_{n-2} \cong B_{n-1}^m$$

as a left B_{n-1}^m, right B_{n-2}^m-bimodule.
(ii) For all B_n^m-modules M we have

$$\xi^r\text{-res}_{n+2}(G_n(M)) \cong \xi^{m-r}\text{-ind}_n(M).$$

Proof. (i) Every diagram in $B_n^m e_{n-2}$ has an edge between $\overline{n-1}$ and $\bar{n}$. Define a map from $B_n^m e_{n-2}$ to B_{n-1}^m by sending a diagram X to the diagram with $2(n-1)$ vertices obtained from X by removing the line connecting $\overline{n-1}$ and $\bar{n}$ and the line from n (labelled by r), and pairing the vertex $\overline{n-1}$ to the vertex originally paired with n in X (labelling this line with r and preserving the orientation). It is easy to check that this gives an isomorphism.
(ii) We have to show that

$$T_{n+2}^r B_{n+2}^m e_n \otimes_{B_n^m} M \;\cong\; B_{n+1}^m \otimes_{B_n^m \otimes B_1^m} (M \boxtimes kT_{n+1}^{m-r}).$$

The left hand side is spanned by all elements obtained from diagrams in $B_{n+2}^m e_n$ by attaching the idempotent T^r to node $n+2$. Following the map given in (i) gives the required isomorphism. $\square$

Remark 3.2.3. Note that this construction will generalise to any tower of recollement (as in [CMPX06]) where $A_{n-1} \otimes A_1 \subset A_n$, and A_1 admits a (non-trivial) direct sum decomposition.

Given a family of modules M_i we will write $\biguplus_i M_i$ to denote some module with a filtration whose quotients are exactly the M_i, each with multiplicity one. This is not uniquely defined as a module, but the existence of a module with such a filtration will be sufficient for our purposes.

Proposition 3.2.4. *(i) For $\lambda \in \Lambda(m,n)$ we have short exact sequences*

$$0 \longrightarrow \biguplus_{\square \in \xi^{m-r}\text{-rem}(\lambda)} \Delta_{n+1}(\lambda - \square) \longrightarrow \xi^r\text{-ind}_n\, \Delta_n(\lambda) \longrightarrow \biguplus_{\square \in \xi^{r}\text{-add}(\lambda)} \Delta_{n+1}(\lambda + \square) \longrightarrow 0$$

and

$$0 \longrightarrow \biguplus_{\square \in \xi^{r}\text{-rem}(\lambda)} \Delta_{n-1}(\lambda - \square) \longrightarrow \xi^r\text{-res}_n\, \Delta_n(\lambda) \longrightarrow \biguplus_{\square \in \xi^{m-r}\text{-add}(\lambda)} \Delta_{n-1}(\lambda + \square) \longrightarrow 0.$$

(ii) In each of the filtered modules which arise in (i), the filtration can be chosen so that partitions labelling successive quotients are ordered by dominance (see Definition 3.1.2), with the top quotient maximal among these. When H_n^m is semisimple the $\biguplus$ all become direct sums.

Proof. We prove the result for the functor ξ^r-res$_n$. The result for ξ^r-ind$_n$ then follows immediately from Proposition 3.2.2(ii) and (3.2.4).

We let W be the subspace of $\xi^r\text{-res}_n\, \Delta_n(\lambda)$ spanned by all elements of the form $T_n^r d \otimes x$ with $d \in V(m,n,l), x \in \mathbf{S}(\lambda)$ such that the node n is free in d. It is clear that this subspace is a $B_{n-1}^m(\delta)$-submodule. We shall prove that $W = \biguplus_{\square \in \xi^r\text{-rem}(\lambda)} \Delta_{n-1}(\lambda - \square)$.

By the restriction rules for cyclotomic Hecke algebras (see [Mat09] for details), it will be enough to show that

$$W \cong V(m, n-1, l) \otimes \xi^r\text{-res}_{n-2l}\, \mathbf{S}(\lambda)$$

where $\xi^r\text{-res}_{n-2l}\, \mathbf{S}(\lambda) = T_{n-2l}^r \mathbf{S}(\lambda)$ viewed as a H_{n-2l-1}^m-module. The map sending $T_n^r d \otimes x$ to $\phi(d) \otimes T_{n-2l}^r x$ where $\phi(d)$ is the $(m, n-1, l)$-dangle obtained from d by removing node n is clearly an isomorphism.

We will now show that

$$U = \xi^r\text{-res}_n\, \Delta(\lambda)/W \cong V(m, n-1, l-1) \otimes (\mathrm{ind}_{H_{n-2l}^m \otimes H_1^m}^{H_{n+1-2l}^m}(\mathbf{S}(\lambda) \boxtimes kT^{m-r}))$$

which gives the required result using [Mat09]. Let d be an (m,n,l)-dangle which has an arc from node n to some other node. Number the free vertices of d and the node connected to n in order from left to right with the integers $1,\dots n+1-2l$. Say that the node connected to n is numbered with i. Define $\psi(d)$ to be the $(m,n-1,l-1)$-dangle obtained from d by removing the arc $\{i,n\}$ and deleting the node n (so that i becomes a free node). And define the permutation $\sigma_i = (i, n-2l+1, n-2l, n-2l-1, \dots i+1) \in \Sigma_{n-2l+1}$. This element is obtained by pulling down node n in d (as in the proof of Proposition 3.2.2(i)), giving the permutation σ_i of the free vertices $\{1,2,\dots,n-2l+1\}$. An example is given in Figures 9 and 10. In Figure 9 we have a $(m,13,3)$-dangle with vertices labelled with the numbers 1 to 8 (here $i=5$) and we see in Figure 10 that by pulling down the last vertex we obtain an $(m,12,2)$-dangle (ignoring the crossings of the through-strands) and the permutation $\sigma_5 = (5,8,7,6)$ given on the through-strands.

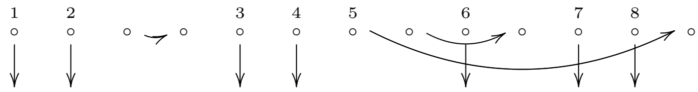

FIGURE 9. An $(m,13,3)$-dangle with labelled vertices.

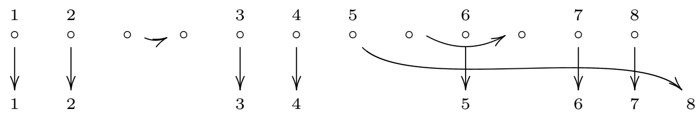

FIGURE 10. Pulling down the last vertex in Figure 9 to obtain a $(m,12,2)$-dangle and the permutation σ_5.

Now the map sending $T_n^r d \otimes x$ to $\psi(d) \otimes (\sigma_i \otimes_{H^m_{n-2l} \otimes H^m_1} (x \otimes T^{m-r}_{n-2l+1}))$ gives the required isomorphism. □

We have seen that the induction and restriction functors for B_n^m decompose into signed versions. We saw in Lemma 3.2.2 and Proposition 3.2.4 that when $r \neq m-r$, the ξ^r- and ξ^{m-r}-functors 'pair-off' in a manner reminiscent of those for the walled Brauer algebra (see [CDDM08, Theorem 3.3]). In the case that $r = m-r$ we saw that the ξ^r-functors behave like those of the classical Brauer algebra (see [DWH99, Theorem 4.1] and [CDM09, Proposition 2.7]). We will make this connection explicit in Section 5.

3.3. Quasi-heredity

A cellular algebra is quasi-hereditary if and only if it has the same number of cell modules and simple modules, up to isomorphism. Using this and standard arguments for iterated inflations [KX98], we deduce the following.

Theorem 3.3.1. *Let k be a field of characteristic $p \geq 0$, $m,n \in \mathbb{N}$, and $\delta \in k^m$. If n is even suppose $\delta \neq 0 \in k^m$. The algebra $B_n^m(\delta)$ is quasi-hereditary if and only if $p > n$ and p does not divide m, or $p = 0$.*

Assumption 3.3.2. *From now on, we will assume that $\delta \neq 0$ if n is even and that k satisfies the conditions in Theorem 3.3.1, and so $B_n^m(\delta)$ is quasi-hereditary.*

The cell modules $\Delta_n(\lambda)$ are then the *standard modules* for this quasi-hereditary algebras, and we will call them so. Each standard module $\Delta_n(\lambda)$ has simple head $L_n(\lambda)$ and the set

$$\{L_n(\lambda) : \lambda \in \Lambda(m,n)\}$$

form a complete set of non-isomorphic simple B_n^m-modules. We denote by $P_n(\lambda)$ the projective cover of $L_n(\lambda)$.

Note that in this case the first and last term in the exact sequences given in Proposition 3.2.4 (i) are just direct sums of standard modules. In fact the dominance order $\trianglerighteq_{n-2l}$ plays no role here and the order for the quasi-hereditary structure can simply be taken to be the size of the multi-partitions. In particular we have that $[\Delta_n(\lambda) : L_n(\mu)] \neq 0$ implies $\sum_{i=0}^{m-1} |\lambda^i| \leq \sum_{i=0}^{m-1} |\mu^i|$. We will see in the next section (see Definition 4.1.4 and Corollary 4.2.2) how this ordering can be refined.

The results in this Section have shown

Theorem 3.3.3. *Under Assumption 3.3.2 the algebras $B_n^m(\delta)$ form a tower of recollement.*

4. The cyclotomic poset and combinatorics of B_n^m

4.1. The cyclotomic poset

Recall that $\Lambda(m,n)$ denotes the set of m-partitions of $n-2l$ for all $l \leq \lfloor n/2 \rfloor$. We let $\Lambda|m,n|$ denote the set of m-compositions of $n-2l$ for all $l \leq \lfloor n/2 \rfloor$. There is a many-to-one map $\Lambda(m,n) \to \Lambda|m,n|$ given by

$$(\lambda^0, \ldots, \lambda^{m-1}) \mapsto (|\lambda^0|, \ldots, |\lambda^{m-1}|).$$

For example the map $\Lambda(3,9) \to \Lambda|3,9|$ maps $((1^2),(2,1),(2,2))$ to $(2,3,4)$.

For a given $m,n \in \mathbb{N}$ we define a partial ordering $\preceq$ on $\Lambda|m,n|$ as follows.

Definition 4.1.1. For m-compositions $\omega = (\omega_0, \omega_1, \ldots, \omega_{m-1})$ and $\omega' = (\omega'_0, \omega'_1, \ldots, \omega'_{m-1}) \in \Lambda|m,n|$, we say $\omega \preceq \omega'$ if and only if

(i) $\omega_r \leq \omega'_r$ for all $0 \leq r \leq m-1$
(ii) $\omega_r - \omega'_r = \omega_{m-r} - \omega'_{m-r}$ for $r \neq 0, m/2$
(iii) $\omega_r - \omega'_r \in 2\mathbb{Z}$ for $r = 0, m/2$.

Any irreducible component in the Hasse diagram of this poset has a unique minimal element.

Let $\omega \preceq \omega'$, with $a_r = \omega_r - \omega'_r$ for $r \neq m-r$ and $a_r = (\omega_r - \omega'_r)/2$ for $r = 0, m/2$ (note that $a_r = a_{m-r}$ by assumption). We write $\omega \preceq_{\sum a_r \xi^r} \omega'$ where the sum is over all $0 \leq r \leq \lfloor m/2 \rfloor$.

Example 4.1.2. The diagram in Figure 11 is an irreducible component of the Hasse diagram of the poset $(\Lambda|3,6|, \preceq)$. We have annotated the edges with the relevant signs. We see that $(0,0,0) \preceq_{1+\xi} (2,1,1)$.

$$\begin{tikzcd}
(0,3,3) \arrow[dr, dash, "\xi"] & & (2,2,2) \arrow[dl, dash, "1"'] \arrow[dr, dash, "\xi"] & & (4,1,1) \arrow[dl, dash, "1"'] \arrow[dr, dash, "\xi"] & & (6,0,0) \arrow[dl, dash, "1"'] \\
& (0,2,2) \arrow[dr, dash, "\xi"] & & (2,1,1) \arrow[dl, dash, "1"'] \arrow[dr, dash, "\xi"] & & (4,0,0) \arrow[dl, dash, "1"'] & \\
& & (0,1,1) \arrow[dr, dash, "\xi"] & & (2,0,0) \arrow[dl, dash, "1"'] & & \\
& & & (0,0,0) & & &
\end{tikzcd}$$

Figure 11. An irreducible component of the Hasse poset on $\Lambda|3,6|$.

Proposition 4.1.3. *If $\lambda, \mu \in \Lambda(m,n)$ are such that $[\Delta_n(\mu) : L_n(\lambda)] \neq 0$, then we must have that $|\mu| \preceq |\lambda|$.*

Proof. We prove this by induction on n. If $n=0$ then $B_0^m = k$, therefore there is only one simple module $\Delta_0(\emptyset) = L_0(\emptyset)$ and there is nothing to prove.

Let $n \geq 1$ and suppose that $[\Delta_n(\mu) : L_n(\lambda)] \neq 0$, that is we have a non-zero homomorphism $\Delta_n(\lambda) \to \Delta_n(\mu)/N$ for some submodule N of $\Delta_n(\mu)$. By localisation, we may assume that λ is an m-partition of n, so that $L(\lambda) = \Delta(\lambda)$, and that μ is an m-partition of $n-2l$ for some $l \leq \lfloor n/2 \rfloor$.

As $n \geq 1$, λ has at least one removable box, ϵ, say in the rth part of the m-partition λ. Then by Proposition 3.2.4 (and noting that, under our assumption, each term is a direct sum) we have that

$$\xi^r\text{-}\operatorname{ind}_{n-1}\Delta_{n-1}(\lambda-\epsilon) \twoheadrightarrow \Delta_n(\lambda).$$

Therefore we have

$$\xi^r\text{-}\operatorname{ind}_{n-1}\Delta_{n-1}(\lambda-\epsilon) \to \Delta_n(\mu)/N.$$

By adjointness of $\xi^r\text{-}\operatorname{ind}_{n-1}$ and $\xi^r\text{-}\operatorname{res}_n$ we have

$$\operatorname{Hom}_{B_n^m}(\xi^r\text{-}\operatorname{ind}_{n-1}\Delta_{n-1}(\lambda-\epsilon), \Delta_n(\mu)/N) \cong \operatorname{Hom}_{B_{n-1}^m}(\Delta_{n-1}(\lambda-\epsilon), \xi^r\text{-}\operatorname{res}_n\Delta_n(\mu)/N)$$

By Proposition 3.2.4 we can conclude that either:

$$[\Delta_{n-1}(\mu-\epsilon') : L_{n-1}(\lambda-\epsilon)] \neq 0$$

and $\epsilon' \in \xi^r\text{-}\operatorname{rem}(\mu)$ or

$$[\Delta_{n-1}(\mu+\epsilon'') : L_{n-1}(\lambda-\epsilon)] \neq 0.$$

and $\epsilon'' \in \xi^{m-r}\text{-}\operatorname{add}(\mu)$.

In the first case we have by our inductive assumption that

$$|\mu-\epsilon'| \preceq_{(\sum_i a_i\xi^i)} |\lambda-\epsilon|$$

for some $a_i \geq 0$. We have that $\epsilon' \in \xi^r\text{-}\operatorname{rem}(\mu)$ and $\epsilon \in \xi^r\text{-}\operatorname{rem}(\mu)$ and so

$$|\mu| \preceq_{(\sum_i a_i\xi^i)} |\lambda|$$

as required. In the second case we have by our inductive assumption that

$$|\mu+\epsilon''| \preceq_{(\sum_i a_i\xi^i)} |\lambda-\epsilon|$$

for some $a_i \geq 0$. We have that $\epsilon'' \in \xi^{m-r}\text{-}\operatorname{add}(\mu)$ and $\epsilon \in \xi^r\text{-}\operatorname{rem}(\mu)$ and so

$$|\mu| \preceq_{(\sum_i b_i\xi^i)} |\lambda|$$

where $a_i = b_i$ for all $i \neq r$, and $a_r = b_r - 1$. □

Definition 4.1.4. We define a partial order, $\leq$, on $\Lambda(m,n)$ by taking $\lambda \leq \mu$ if $\lambda^i \subseteq \mu^i$ for all $0 \leq i \leq m-1$ and $|\lambda| \preceq |\mu|$.

4.2. The restriction of standard modules to H_n^m

Here we calculate the multiplicities

$$[\Delta_n(\lambda)\downarrow_{H_n^m} : \mathbf{S}(\mu)]$$

for $\lambda \vdash n-2l$ and $\mu \vdash n$. The case $l=1$ was already done in [RX07, Proposition 2.5], where they remark (see [RX07, Remark 2.6]) that the general case given in [RY04, Section 4.4] is incorrect.

Recall that we have $\Delta_n(\lambda) = V(m,n,l) \otimes \mathbf{S}(\lambda)$. From the explicit action of B_n^m given in Section 3.1, it is easy to see that we have

$$\begin{aligned} \Delta_n(\lambda)\downarrow_{H_n^m} &= (V(m,n,l)\otimes\mathbf{S}(\lambda))\downarrow_{H_n^m} \\ &\cong H_n^m \otimes_{H_{2l}^m \otimes H_{n-2l}^m} (V(m,2l,l)\otimes\mathbf{S}(\lambda)). \end{aligned}$$

So the first step is to understand the structure of $V(m,2l,l)\downarrow_{H_{2l}^m}$.

Each $(m, 2l, l)$-dangle has l arcs denoted by (i_p, j_p) (for $p = 1, \ldots, l$) where i_p (resp. j_p) is the left (resp. right) vertex of the arc. Note that for any arc (i_p, j_p) in v and any $r \in \mathbb{Z}/m\mathbb{Z}$ we have

$$T_{i_p}^r v = T_{j_p}^{m-r} v. \tag{4.2.1}$$

It follows that as a $(\mathbb{Z}/m\mathbb{Z})^{2l}$-module, $V(m, 2l, l)$ decomposes as

$$V(m,2l,l)\downarrow_{(\mathbb{Z}/m\mathbb{Z})^{2l}} = \bigoplus_{\substack{v \\ (r_1,\ldots,r_l)}} k(T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_l}^{r_l} v)$$

where the sum is over all $(m, 2l, l)$-dangles v with all arcs labelled by 0 and over all l-tuples $(r_1, r_2, \ldots, r_l) \in (\mathbb{Z}/m\mathbb{Z})^l$. The generators of Σ_{2l} act as follows: For each $p \neq q \in \{1, 2, \ldots l\}$ we have

$$t_{i_p,j_p} T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_p}^{r_p} \ldots T_{i_l}^{r_l} v = T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_p}^{m-r_p} \ldots T_{i_l}^{r_l} v, \tag{4.2.2}$$

$$t_{i_p,i_q} T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_p}^{r_p} \ldots T_{i_q}^{r_q} \ldots T_{i_l}^{r_l} v = T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_p}^{r_q} \ldots T_{i_q}^{r_p} \ldots T_{i_l}^{r_l} (t_{i_p,i_q} v). \tag{4.2.3}$$

For $(r_1, r_2, \ldots, r_l) \in (\mathbb{Z}/m\mathbb{Z})^l$ define the weight $wt(r_1, r_2, \ldots, r_l)$ to be $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_{\lfloor m/2 \rfloor})$ where

$$\varphi_i = |\{p : r_p = i \,\text{or}\, m - i\}|.$$

It follows from (4.2.2) and (4.2.3) that

$$V(m, 2l, l)\downarrow_{H_{2l}^m} = \oplus_\varphi V(m, 2l, l)^\varphi \tag{4.2.4}$$

where

$$V(m,2l,l)^\varphi = \bigoplus_{\substack{v \\ wt(r_1,\ldots,r_l)=\varphi}} k(T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_p}^{r_p} \ldots T_{i_l}^{r_l} v)$$

and the sum is over all $(m, 2l, l)$-dangles v with all arcs labelled by 0. Now, it follows again from (4.2.2) and (4.2.3) that $V(m, 2l, l)^\varphi$ is a cyclic H_{2l}^m-module. We now construct an explicit generator for this module. Let v_φ be the $(m, 2l, l)$-dangle with all arcs labelled by 0 and with set of arcs given by

$$\cup_{0 \le i \le \lfloor m/2 \rfloor} \mathrm{Arc}(i)$$

where we have

$$\mathrm{Arc}(0) = \{(1, 2), (3, 4), \ldots, (2\varphi_0 - 1, 2\varphi_0)\}$$

and for $0 < i < m/2$

$$\begin{aligned} \mathrm{Arc}(i) = \{ & (2(\varphi_0 + \ldots + \varphi_{i-1}) + 1, 2(\varphi_0 + \ldots \varphi_i)), \\ & (2(\varphi_0 + \ldots + \varphi_{i-1}) + 2, 2(\varphi_0 + \ldots \varphi_i) - 1), \ldots \\ & (2(\varphi_0 + \ldots + \varphi_{i-1}) + \phi_i, 2(\varphi_0 + \ldots \varphi_{i-1}) + \varphi_i + 1)\}, \end{aligned}$$

and if $i = m/2$ we have

$$\mathrm{Arc}(m/2) = \{(2(\varphi_0 + \ldots + \varphi_{m/2-1}) + 1, 2(\varphi_0 + \ldots + \varphi_{m/2-1}) + 2), \ldots, (2l - 1, 2l)\}.$$

The $(m, 2l, l)$-dangle v_φ is depicted in Figure 12.

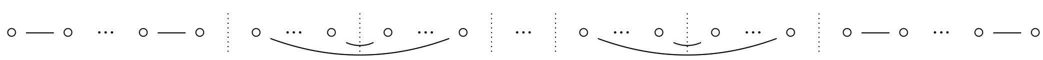

FIGURE 12. The $(m, 2l, l)$-dangle v_φ.

Now we define

$$T^\varphi = \prod_{0 \le i \le \lfloor m/2 \rfloor} \prod_{(i_p, j_p) \in \mathrm{Arc}(i)} T_{i_p}^i.$$

It follows from (4.2.2) and (4.2.3) that $T^\varphi v_\varphi$ is a generator for the H^m_{2l}-module $V(m,2l,l)^\varphi$. The stabiliser of $k(T^\varphi v_\varphi)$ is given by

$$\mathrm{Stab}(k(T^\varphi v_\varphi)) = k((\mathbb{Z}/m\mathbb{Z})\wr(\Sigma_2\wr\Sigma_{\varphi_0}))\otimes\left(\bigotimes_{0<i<m/2} k((\mathbb{Z}/m\mathbb{Z})\wr\Sigma_{\varphi_i})\right)\otimes k((\mathbb{Z}/m\mathbb{Z})\wr(\Sigma_2\wr\Sigma_{\varphi_{m/2}}))$$

where we ignore the last term if m is odd, and the group Σ_{φ_i} is viewed as a subgroup of $\Sigma_{\varphi_i}\times\Sigma_{\varphi_i}$ via the diagonal embedding. As a module for its stabiliser, we have

$$k(T^\varphi v_\varphi) = (k_{\Sigma_2\wr\Sigma_{\varphi_0}})^{(0)}\otimes\left(\bigotimes_{0<i<m/2}((k\otimes k)_{\Sigma_{\varphi_i}})^{(i)\otimes(m-i)}\right)\otimes(k_{\Sigma_2\wr\Sigma_{\varphi_{m/2}}})^{(m/2)}.$$

Thus we have

$$V(m,2l,l)^\varphi\cong[(k_{\Sigma_2\wr\Sigma_{\varphi_0}})^{(0)}\otimes\left(\bigotimes_{0<i<m/2}((k\otimes k)_{\Sigma_{\varphi_i}})^{(i)\otimes(m-i)}\right)\otimes(k_{\Sigma_2\wr\Sigma_{\varphi_{m/2}}})^{(m/2)})]\uparrow^{H^m_{2l}}_{\mathrm{Stab}(k(T^\varphi v_\varphi))}. \tag{4.2.5}$$

We can now prove the main result of this section.

Theorem 4.2.1. *Let $\lambda,\mu\in\Lambda(m,n)$. If $\lambda\not\trianglelefteq\mu$, then $[\Delta_n(\lambda)\downarrow_{H^m_n}:\mathbf{S}(\mu)]=0$. Otherwise, we have that $|\lambda|\preceq_{\sum a_r\xi^r}|\mu|$, and*

$$[\Delta_n(\lambda)\downarrow_{H^m_n}:\mathbf{S}(\mu)] = \prod_{i\neq 0,m/2}\left(\sum_{\tau\vdash a_i} c^{\mu^i}_{\lambda^i,\tau}c^{\mu^{m-i}}_{\lambda^{m-i},\tau}\right)\prod_{j=0,m/2}\left(\sum_{\substack{\eta\vdash 2a_j\\ \eta\ even}} c^{\mu^j}_{\lambda^j,\eta}\right)$$

Proof. Using the decomposition of $V(m,2l,l)$ given in (4.2.4) and (4.2.5) and the construction of the Specht modules given in Section 2.2 we have

$$\begin{aligned}
\Delta_n(\lambda)\downarrow_{H^m_n} &\cong (V(m,2l,l)\otimes\mathbf{S}(\lambda))\uparrow^{H^m_n}_{H^m_{2l}\otimes H^m_n}\\
&\cong \oplus_\varphi(V(m,2l,l)^\varphi\otimes\mathbf{S}(\lambda))\uparrow^{H^m_n}_{H^m_{2l}\otimes H^m_n}\\
&\cong \oplus_\varphi[(k_{\Sigma_2\wr\Sigma_{\varphi_0}})^{(0)}\otimes\left(\bigotimes_{0<i<m/2}((k\otimes k)_{\Sigma_{\varphi_i}})^{(i)\otimes(m-i)}\right)\otimes(k_{\Sigma_2\wr\Sigma_{\varphi_{m/2}}})^{(m/2)}\\
&\qquad\otimes\bigotimes_{1\leq i\leq m}S(\lambda^i)^{(i)}]\uparrow^{H^m_n}_{\mathrm{Stab}(T^\varphi v_\varphi)\otimes H^m_{|\lambda|}}.
\end{aligned}$$

Rearranging according to the action of $(\mathbb{Z}/m\mathbb{Z})^n$ and using transitivity of induction we get

$$\begin{aligned}
\Delta_n(\lambda)\downarrow_{H^m_n} \cong\ & \oplus_\varphi[((k\uparrow^{\Sigma_{2\varphi_0}}_{\Sigma_2\wr\Sigma_{\varphi_0}}\otimes S(\lambda^0))\uparrow^{\Sigma_{2\varphi_0+|\lambda^0|}}_{\Sigma_{2\varphi_0}\times\Sigma_{|\lambda^0|}})^{(0)}\\
&\otimes\left(\bigotimes_{0<i<m/2}(((k\otimes k)\uparrow^{\Sigma_{\varphi_i}\times\Sigma_{\varphi_i}}_{\Sigma_{\varphi_i}}\otimes S(\lambda^i)\otimes S(\lambda^{m-i})\uparrow^{\Sigma_{\varphi_i+|\lambda^i|}\times\Sigma_{\varphi_i+|\lambda^{m-i}|}}_{\Sigma_{\varphi_i}\times\Sigma_{\varphi_i}\times\Sigma_{|\lambda^i|}\times\Sigma_{|\lambda^{m-i}|}})^{(i)\otimes(m-i)}\right)\\
&\otimes((k\uparrow^{\Sigma_{2\varphi_{m/2}}}_{\Sigma_2\wr\Sigma_{\varphi_{m/2}}}\otimes S(\lambda^{m/2}))\uparrow^{\Sigma_{2\varphi_{m/2}+|\lambda^{m/2}|}}_{\Sigma_{2\varphi_{m/2}}\times\Sigma_{|\lambda^{m/2}|}})^{(m/2)}]\uparrow^{H^m_n}_{H^m_{\varphi+|\lambda|}},
\end{aligned}$$

where

$$H^m_{\varphi+|\lambda|} = H^m_{2\varphi_0+|\lambda^0|}\otimes\left(\bigotimes_{0<i<m/2}(H^m_{\varphi_i+|\lambda^i|}\otimes H^m_{\varphi_i+|\lambda^{m-i}|})\right)\otimes H^m_{2\varphi_{m/2}+|\lambda^{m/2}|}.$$

Note that

$$(k\uparrow^{\Sigma_{2\varphi_0}}_{\Sigma_2\wr\Sigma_{\varphi_0}} \otimes S(\lambda^0))\uparrow^{\Sigma_{2\varphi_0+|\lambda^0|}}_{\Sigma_{2\varphi_0}\times\Sigma_{|\lambda^0|}}$$

is exactly the restriction to $\Sigma_{2\varphi_0+|\lambda^0|}$ of the standard module labelled by λ^0 for the classical Brauer algebra $B(2\varphi_0+|\lambda^0|,\delta')$ (any parameter δ'). And similarly for the last term. Note also that

$$((k\otimes k)\uparrow^{\Sigma_{\varphi_i}\times\Sigma_{\varphi_i}}_{\Sigma_{\varphi_i}} \otimes S(\lambda^i)\otimes S(\lambda^{m-i}))\uparrow^{\Sigma_{\varphi_i+|\lambda^i|}\times\Sigma_{\varphi_i+|\lambda^{m-i}|}}_{\Sigma_{\varphi_i}\times\Sigma_{\varphi_i}\times\Sigma_{|\lambda^i|}\times\Sigma_{|\lambda^{m-i}|}}$$

is exactly the restriction to $\Sigma_{\varphi_i+|\lambda^i|}\times\Sigma_{\varphi_i+|\lambda^{m-i}|}$ of the standard module labelled by $(\lambda^i,\lambda^{m-i})$ for the walled Brauer algebra $WB(\varphi_i+|\lambda^i|,\varphi_i+|\lambda^{m-i}|,\delta')$ (any parameter δ'). The result now follows from [HW90, Theorem 4.1] and [Hal96, Corollary 7.24] (by replacing φ_i by a_i in the statement). $\square$

Corollary 4.2.2. *Let $\lambda,\mu\in\Lambda(m,n)$. If $[\Delta_n(\lambda):L_n(\mu)]\neq 0$ then $\lambda\leq\mu$.*

5. Truncation to idempotent subalgebras

In this section we show that maximal co-saturated idempotent subalgebras of B^m_n are isomorphic to a tensor product of classical and walled Brauer algebras. Hence we determine the space of homomorphisms between standard modules and the decomposition numbers for B^m_n.

5.1. Co-saturated sets

For $\omega\in\Lambda|m,n|$ an m-composition of n, we define $(\preceq\omega)\subseteq\Lambda|m,n|$ to be the subset of all m-compositions less than or equal to ω with respect to $\preceq$. We define Λ_ω to be the pre-image of $(\preceq\omega)$ in $\Lambda(m,n)$, that is the set of all m-partitions λ with $|\lambda|\preceq\omega$.

Example 5.1.1. The diagram in Figure 13 is the Hasse diagram of the poset $(\preceq(4,1,1))\subset\Lambda|3,6|$. We have again annotated the edges with the signed partial ordering.

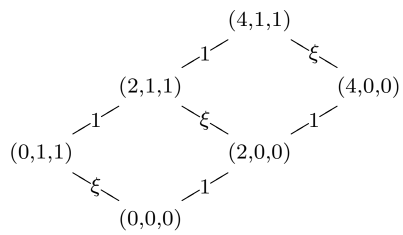

Figure 13. A sub-poset of the Hasse poset in Figure 11.

We have chosen to work with the partial order $\leq$ on $\Lambda(m,n)$ as it is a refinement of the natural partial ordering given by inclusion on the set of multipartitions. Note, however, that $B^m_n(\delta)$ is quasi-hereditary with respect to the opposite partial order $\leq_{\mathrm{opp}}$ on $\Lambda(m,n)$, and we have that $\Lambda_\omega\subseteq(\Lambda(m,n),\leq_{\mathrm{opp}})$ is a co-saturated subset. So we can apply the results from [Don98, Appendix] on idempotent subalgebras corresponding to co-saturated subsets for quasi-hereditary algebras. The first thing we need is an idempotent corresponding to Λ_ω.

5.2. Idempotents and standard modules

In this section we consider the effect of applying the idempotents π_ω defined in Section 2.2 to standard modules.

Recall that an (m,n,l)-dangle v can be described as a set of l disjoint pairs $(i_p<j_p)\in\{1,\dots,n\}^2$, called arcs, where each arc is labelled by an element of $\mathbb{Z}/m\mathbb{Z}$. We say that v belongs to ω if every

arc (i_p, j_p) (for $p = 1, \ldots, l$) satisfies $i_p \in [\omega_{r_p}]$ and $j_p \in [\omega_{m-r_p}]$ for some $0 \leq r_p \leq \lfloor m/2 \rfloor$ (where we set $[\omega_m] = [\omega_0]$). In this case we define $\omega \setminus v$ by

$$\omega \setminus v = \omega - \sum_{p=1}^{l} (\epsilon_{r_p} + \epsilon_{m-r_p}) \in \Lambda|m, n|$$

where $\epsilon_{r_p} = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in position r_p (and similarly for ϵ_{m-r_p}).

Proposition 5.2.1. *Let v be an (m, n, l)-tangle, $\lambda \in \Lambda(m, n)$ and $x \in \mathbf{S}(\lambda)$. Then we have*

$$\pi_\omega(v \otimes x) = \begin{cases} T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_l}^{r_l} v \otimes \pi_{\omega \setminus v} x & \textit{if } v \textit{ belongs to } \omega \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, we have that $\pi_\omega \Delta_n(\lambda) \neq 0$ if and only if $\lambda \in \Lambda_\omega$.

Proof. Note that for each arc (i_p, j_p) of v we have

$$T_{i_p}^{r} v = T_{j_p}^{m-r} v.$$

Now as $\{T_i^r \,:\, 0 \leq r \leq m-1\}$ form a set of orthogonal idempotents we have

$$T_{i_p}^{r} T_{j_p}^{s} v = T_{i_p}^{r} T_{i_p}^{m-s} v = \begin{cases} T_{i_p}^{r} v & \text{if } s = m - r \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have that $\pi_\omega(v \otimes x) = 0$ unless v belongs to ω. Now it is easy to see that π_ω acts on the free vertices of v by $\pi_{\omega \setminus v}$. So if v belongs to ω we get

$$\pi_\omega(v \otimes x) = T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_l}^{r_l} v \otimes \pi_{\omega \setminus v} x$$

as required. Now using equation (2.3.1) we have that

$$\pi_{\omega \setminus v} \mathbf{S}(\lambda) \cong \begin{cases} S(\lambda^0) \otimes S(\lambda^1) \otimes \ldots \otimes S(\lambda^{m-1}) & \text{if } |\lambda| = \omega \setminus v \\ 0 & \text{otherwise.} \end{cases}$$

Finally note that $\lambda \in \Lambda_\omega$ if and only if $|\lambda| = \omega \setminus v$ for some v belonging to ω. This proves the last part of the proposition. $\square$

5.3. Truncation functors

We now consider the truncation functor defined by the idempotent π_ω. From now on we shall denote $\pi_\omega B_n^m \pi_\omega$ by B_ω^m. The truncation functor is defined by

$$\begin{aligned} f_\omega : & B_n^m\text{-mod} \to B_\omega^m\text{-mod} \\ & M \longmapsto \pi_\omega M. \end{aligned}$$

Using the last part of Proposition 5.2.1 and [Don98, A3.11] we have the following result.

Proposition 5.3.1. *(i) A complete set of non-isomorphic simple B_ω^m-modules is given by*

$$\{f_\omega L_n(\lambda) \,:\, \lambda \in \Lambda_\omega\}.$$

(ii) A complete set of non-isomorphic indecomposable projective B_ω^m-modules is given by

$$\{f_\omega P_n(\lambda) \,:\, \lambda \in \Lambda_\omega\}.$$

(iii) The algebra B_ω^m is a quasi-hereditary algebra with respect to the partial order $\leq_{\text{opp}}$ on Λ_ω. Its standard modules are given by $f_\omega \Delta_n(\lambda)$ for all $\lambda \in \Lambda_\omega$.

For $M \in B_n^m$-mod we write $M \in \mathcal{F}_\omega(\Delta)$ to indicate that M has a filtration with subquotients belonging to $\{\Delta_n(\lambda) \,:\, \lambda \in \Lambda_\omega\}$.

Proposition 5.3.2 (A3.13 [Don98]). *Let $X, Y \in B_n^m$-mod with $X \in \mathcal{F}_\omega(\Delta)$. For all $i \geq 0$ we have*

$$\mathrm{Ext}^i_{B_n^m}(X, Y) \cong \mathrm{Ext}^i_{B_\omega^m}(f_\omega X, f_\omega Y).$$

As $P_n(\lambda) \in \mathcal{F}_\omega(\Delta)$ for all $\lambda \in \Lambda_\omega$ and $[\Delta_n(\mu) : L_n(\lambda)] = \dim \mathrm{Hom}(P_n(\lambda), \Delta_n(\mu))$ we have the following corollary.

Corollary 5.3.3. *For all $\lambda, \mu \in \Lambda_\omega$ we have*

$$[\Delta_n(\mu) : L_n(\lambda)] = [f_\omega \Delta_n(\mu) : f_\omega L_n(\lambda)].$$

5.4. The idempotent subalgebras B_ω^m

We now wish to understand the structure of these idempotent subalgebras. Therefore we start by considering the image, in B_ω^m, of the generators of the cyclotomic Brauer algebra B_n^m.

Lemma 5.4.1. *Let $1 \leq i, j \leq n$ and let ω be an m-composition of n. Then we have*
(i) $\pi_\omega t_i^k \pi_\omega = \xi^{-kr}\pi_\omega$ if $i \in [\omega_r]$ for some $0 \leq r \leq m-1$.
(ii) $\pi_\omega t_{i,j} \pi_\omega \neq 0$ if and only if $i, j \in [\omega_r]$ for some $0 \leq r \leq m-1$.
(iii) $\pi_\omega e_{i,j} \pi_\omega \neq 0$ if and only if $i \in [\omega_r]$ and $j \in [\omega_{m-r}]$ for some $0 \leq r \leq m-1$.

Proof. This follows from the definition of π_ω, equation (4.2.1) and the fact that the T_i^r's for $0 \leq r \leq m-1$ (and fixed i) are orthogonal idempotents. □

We now state the main result of this section.

Theorem 5.4.2. *Let ω be an m-composition of n. The algebra B_ω^m is isomorphic to a product of Brauer and walled Brauer algebras with parameters $\overline{\delta}_r$ for $0 \leq r \leq \lfloor m/2 \rfloor$. More specifically*

$$B_\omega^m \cong B(\omega_0, \overline{\delta}_0) \otimes \bigotimes_{r=1}^{\lfloor m/2 \rfloor} WB(\omega_r, \omega_{m-r}, \overline{\delta}_r)$$

if m is odd, and

$$B_\omega^m \cong B(\omega_0, \overline{\delta}_0) \otimes \left(\bigotimes_{r=1}^{(m/2)-1} WB(\omega_r, \omega_{m-r}, \overline{\delta}_r) \right) \otimes B(\omega_{m/2}, \overline{\delta}_{m/2})$$

if m is even.

Remark 5.4.3. In our definition of multiplication for B_n^m we chose one of two possible orientations of the closed loops. Had we favoured the alternative orientation, the above proposition would be stated in terms of the conjugate parameters $\overline{\delta}_r$ such that $m/2 \leq r \leq m-1$. This makes no difference to the representation theory as we obtain non-semisimple specialisations only when these parameters are integral — in which case $\overline{\delta}_r = \overline{\delta}_{m-r}$.

Proof. We will assume that m is even in the proof. The case m odd is obtained by ignoring all the terms corresponding to $m/2$.

We view the tensor product of Brauer and walled Brauer algebras as a diagram algebra spanned by certain Brauer diagrams with n northern and n southern nodes. More precisely, as vector spaces, we embed $B(\omega_0, \overline{\delta}_0) \otimes \left(\bigotimes_{0<r<m/2} WB(\omega_r, \omega_{m-r}, \overline{\delta}_r)\right) \otimes B(\omega_{m/2}, \overline{\delta}_{m/2})$ into $B(n)$, the vector space spanned by Brauer diagrams on $2n$ nodes, by partitioning the n northern and n southern nodes according to ω, that is we draw a wall after the first ω_0 nodes, then another wall after the next ω_1 nodes, etc. We embed the diagrams in $B(\omega_0, \overline{\delta_0})$ using the first ω_0 northern and southern nodes. For $0 < r < m/2$ we embed the diagrams in $B(\omega_r, \omega_{m-r}, \overline{\delta_r})$ using all nodes $i, \overline{i} \in [\omega_r]$ or $[\omega_{m-r}]$. Finally, we embed $B(\omega_{m/2}, \overline{\delta}_{m/2})$ using all nodes $i, \overline{i} \in [\omega_{m/2}]$. An example of such a diagram is given in Figure 14.

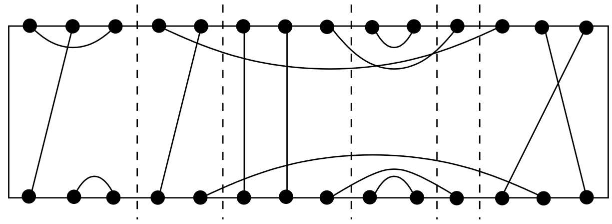

FIGURE 14. An example of the embedding of

$$B(3,\overline{\delta}_0)\otimes WB(2,3,\overline{\delta}_1)\otimes WB(3,1,\overline{\delta}_2)\otimes B(2,\overline{\delta}_3)$$

into $B(14)$ corresponding to the 6-composition ω of 14 given by $\omega=(3,2,3,2,1,3)$.

Now the multiplication is given by concatenation. Note that each closed loop obtained by concatenation only contains nodes $i\in[\omega_r]$ or $[\omega_{m-r}]$ for some $0\le r\le\lfloor m/2\rfloor$; we then remove this closed loop and multiply by the scalar $\overline{\delta_r}$.

We denote by $\sigma_{i,j}$, resp. $u_{i,j}$, the unoriented version of $t_{i,j}$, resp. $e_{i,j}$. So the algebra $B(\omega_0,\overline{\delta}_0)\otimes\left(\bigotimes_{0<r<m/2}WB(\omega_r,\omega_{m-r},\overline{\delta}_r)\right)\otimes B(\omega_{m/2},\overline{\delta}_{m/2})$ is generated by the $\sigma_{i,j}$ for all $i<j\in[\omega_r]$ with $0\le r\le m-1$, and the $u_{i,j}$ for all $i<j$ with $i\in[\omega_r]$ and $j\in[\omega_{m-r}]$ for some $0\le r\le\lfloor m/2\rfloor$ (where we set $[\omega_m]=[\omega_0]$). The relations satisfied by these generators can be found in [BW89, Section 5] and [N07, Theorem 4.1]. Now define the map

$$\phi: B(\omega_0,\overline{\delta}_0)\otimes\left(\bigotimes_{0<r<m/2}WB(\omega_r,\omega_{m-r},\overline{\delta}_r)\right)\otimes B(\omega_{m/2},\overline{\delta}_{m/2})\;\to\;B^m_\omega$$

on generators by setting $\phi(1)=\pi_\omega$, $\phi(\sigma_{i,j})=\pi_\omega t_{i,j}\pi_\omega$ for all $i<j\in[\omega_r]$ for some $0\le r\le m-1$, and $\phi(u_{i,j})=\pi_\omega e_{i,j}\pi_\omega$ for all $i<j$ with $i\in[\omega_r]$ and $j\in[\omega_{m-r}]$ for some $0\le r\le\lfloor m/2\rfloor$ (where we set $[\omega_m]=[\omega_0]$). It is clear from the description in terms of diagrams that ϕ gives a bijection and that all the relations involving only the $\sigma_{i,j}$'s, or the $\sigma_{i,j}$'s and the $u_{i,j}$'s are satisfied. It remains to show that for $i\in[\omega_r]$ and $j\in[\omega_{m-r}]$ we have $(\pi_\omega e_{i,j}\pi_\omega)^2=\overline{\delta_r}(\pi_\omega e_{i,j}\pi_\omega)$. Now we have

$$\begin{aligned}(\pi_\omega e_{i,j}\pi_\omega)^2 &= \pi_\omega e_{i,j}\pi_\omega e_{i,j}\pi_\omega\\ &= \pi_\omega e_{i,j}T_i^rT_j^{m-r}e_{i,j}\pi_\omega\\ &= \pi_\omega e_{i,j}(T_i^r)^2e_{i,j}\pi_\omega\\ &= \pi_\omega e_{i,j}T_i^re_{i,j}\pi_\omega\\ &= \frac{1}{m}\sum_{a=0}^{m-1}\xi^{ar}\pi_\omega e_{i,j}t_i^ae_{i,j}\pi_\omega=\frac{1}{m}\sum_{a=0}^{m-1}\xi^{ar}\delta_a\pi_\omega e_{i,j}\pi_\omega=\overline{\delta_r}(\pi_\omega e_{i,j}\pi_\omega).\end{aligned}$$

□

5.5. Homomorphisms and decomposition numbers

Recall that the standard modules for the classical Brauer algebra $B(n,\delta)$ are indexed by partitions λ of $n-2l$ for $0\le l\le\lfloor n/2\rfloor$. For each partition λ of $n-2l$, the standard $B(n,\delta)$-module $\Delta_{B(n)}(\lambda)$ can be constructed by inflating the Specht module $S(\lambda)$ along $V(1,n,l)$, and it has simple head $L_{B(n)}(\lambda)$. In particular, when $\lambda\vdash n$ we have $\Delta_{B(n)}(\lambda)=S(\lambda)$. The standard modules for the walled Brauer algebra $WB(r,s,\delta)$ are indexed by bi-partitions (λ,μ) of $(r-l,s-l)$ for $0\le l\le\min\{r,s\}$. For each bi-partition (λ,μ), the standard $WB(r,s,\delta)$-module $\Delta_{WB(r,s)}(\lambda,\mu)$ can be constructed similarly by inflating the tensor product of Specht modules $S(\lambda)\otimes S(\mu)$ along the corresponding subspace of dangles, and it has simple head $L_{WB(r,s)}(\lambda,\mu)$. In particular, when $\lambda \vdash r$ and $\mu \vdash s$ we have $\Delta_{WB(r,s)}(\lambda,\mu) = S(\lambda) \otimes S(\mu)$.

Proposition 5.5.1. *For $\lambda \in \Lambda_\omega$ the module $\Delta_n^\omega(\lambda) = f_\omega \Delta_n(\lambda)$ is isomorphic to*

$$\Delta_{B(\omega_0)}(\lambda^0) \otimes \bigotimes_{1\le r\le \lfloor m/2\rfloor} \Delta_{WB(\omega_r,\omega_{m-r})}(\lambda^r,\lambda^{m-r}) \otimes \Delta_{B(\omega_{m/2})}(\lambda^{m/2})$$

(under the isomorphism given in Theorem 5.4.2) where we ignore the last term when m is odd.

Proof. Let $N = \Delta_{B(\omega_0)}(\lambda^0) \otimes \bigotimes_{1\le r\le \lfloor m/2\rfloor} \Delta_{WB(\omega_r,\omega_{m-r})}(\lambda^r,\lambda^{m-r}) \otimes \Delta_{B(\omega_{m/2})}(\lambda^{m/2})$. By Proposition 5.3.1(iii), we know that $f_\omega\Delta_n(\lambda)$ is a standard module. So we only need to show that it is labelled by the same multi-partition as N. Now N is characterised by the fact that when we localise it to

$$B(|\lambda^0|) \otimes \left(\bigotimes_{1\le r\le \lfloor m/2\rfloor} WB(|\lambda^r|,|\lambda^{m-r}|) \right) \otimes B(|\lambda^{m/2}|)$$

we get a module isomorphic to

$$S(\lambda^0) \otimes \left(\bigotimes_{1\le r\le \lfloor m/2\rfloor} (S(\lambda^r)\otimes S(\lambda^{m-r})) \right) \otimes S(\lambda^{m/2}).$$

But it is clear that $f_\omega\Delta_n(\lambda)$ satisfies this condition using Proposition 5.2.1 and Section 2.3. □

Corollary 5.5.2. *Let $\lambda,\mu \in \Lambda(m,n)$ and define $\omega = |\lambda|$. Then (i) $\mathrm{Hom}_{B_n^m}(\Delta_n(\lambda),\Delta_n(\mu))$ is isomorphic to*

$$\mathrm{Hom}_{B(\omega_0,\overline{\delta_0})}(\Delta_B(\lambda^0),\Delta_B(\mu^0)) \otimes \bigotimes_{0<r<m/2} \mathrm{Hom}_{WB(\omega_r,\omega_{m-r},\overline{\delta_r})}(\Delta_{WB}(\lambda^r,\lambda^{m-r}),\Delta_{WB}(\mu^r,\mu^{m-r}))$$
$$\otimes \mathrm{Hom}_{B(\omega_{m/2},\overline{\delta_{m/2}})}(\Delta_B(\lambda^{m/2}),\Delta_B(\mu^{m/2})).$$

(ii) The decomposition numbers $[\Delta_n(\mu) : L_n(\lambda)]$ factorise as

$$[\Delta_B(\mu^0):L_B(\lambda^0)] \times \prod_{0<r<m/2} [\Delta_{WB}(\mu^r,\mu^{m-r}) : L_{WB}(\lambda^r,\lambda^{m-r})] \times [\Delta_B(\mu^{m/2}):L_B(\lambda^{m/2})].$$

(We ignore the last term in (i) and (ii) when m is odd).

Proof. By localisation (3.2.3), we can always assume that λ is an m-partition of n. Now, we have seen in Corollary 4.2.2 that a necessary condition for a non-zero homomorphism (or decomposition number) is that $\lambda \ge \mu$. Thus we have $\lambda,\mu \in \Lambda_\omega$ where $\omega = |\lambda|$. We then obtain the results using Propositions 5.5.1 and 5.3.2 and Corollary 5.3.3. □

Remark 5.5.3. Let ω be an m-composition of n and let $\lambda,\mu \in \Lambda_\omega$. Then, using Proposition 5.3.2, we have more generally that

$$\mathrm{Ext}^i_{B_n^m}(\Delta_n(\lambda),\Delta_n(\mu)) \cong \mathrm{Ext}^i_{B_\omega^m}(\Delta_n^\omega(\lambda),\Delta_n^\omega(\mu))$$

for all $i \ge 0$.

Remark 5.5.4. (i) With this factorisation of the decomposition numbers at hand, one can easily deduce the block structure of B_n^m (in terms of that of the walled and classical Brauer algebras).

(ii) The decomposition numbers for the Brauer and walled Brauer algebras in characteristic zero are known by [Mar] and [CD11], and so we have determined the decomposition numbers for the cyclotomic Brauer algebra in characteristic zero.

6. Appendix: The unoriented cyclotomic Brauer algebra

There is another version of the cyclotomic Brauer algebra, which we will denote by $\tilde{B}_n^m(\delta)$, spanned by unoriented reduced (m,n)-diagrams. As a vector space, it coincides with B_n^m but the multiplication is simply given by concatenation, addition (in $\mathbb{Z}/m\mathbb{Z}$) of the labels on each strands, and replacing any closed loop labelled by r with scalar multiplication by δ_r.

All the arguments in this paper apply to the unoriented cyclotomic Brauer algebra as well, and it turns out that the corresponding idempotent subalgebras are isomorphic to a tensor product of classical Brauer algebras in this case. Hence this gives a factorisation of the decomposition numbers of $\tilde{B}_n^m$ as a product of decomposition numbers for the classical Brauer algebras. We will now briefly sketch the modifications required.

The algebra $\tilde{B}_n^m$ is still an iterated inflation of the algebras H_n^m but along the spaces of unoriented dangles. All the results in Section 3 hold as before if we replace $m-r$ by r in Lemma 3.2.2(ii) and Proposition 3.2.4. In Section 4, note that in equation (4.2.1) and equation (4.2.2) we have to replace $m-r$ with r again. Now following the argument in Section 4.2 we obtain

$$[\Delta_n(\lambda)\downarrow_{H_n^m}: \mathbf{S}(\mu)] = \prod_{0\le j\le m-1} \sum_{\substack{\eta\vdash 2a_j \\ \eta \text{ even}}} c^{\mu^j}_{\lambda^j,\eta}.$$

We modify the partial ordering $\preceq$ and $\le$ accordingly. For $\omega,\omega' \in \Lambda|m,n|$ we set $\omega \preceq \omega'$ if and only if $\omega_r - \omega'_r \ge 0$ and $\omega_r - \omega'_r$ is even for all $0 \le r \le m-1$. We then define $\le$ on $\Lambda(m,n)$ by setting $\lambda \le \mu$ if and only if $|\lambda| \preceq |\mu|$ and $\lambda^r \subseteq \mu^r$ for all $0 \le r \le m-1$. We then have that Corollary 4.2.2 holds with respect to this new partial order. In Section 5 we can define the co-saturated subset Λ_ω as before (using the new partial order). Replacing $m-r$ with r throughout the arguments we obtain that

$$\tilde{B}_\omega^m \cong \bigotimes_{0\le r\le m-1} B(\omega_r, \overline{\delta_r}),$$

and hence we get the required factorisation of homomorphisms between standard modules and of decomposition numbers.

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