Asymmetric Pricing and Replenishment Controls for Substitutable Products

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We study settings in which a firm offering substitutable products may face restrictions in its ability to either replenish or adjust the prices of some of its products, resulting in asymmetries in the pricing and replenishment controls available for each product. Specifically, we first consider a firm selling two substitutable products, a *seasonal* and a *regular* product, that differ in how their inventories are managed over a finite selling horizon. The seasonal product has an initial inventory with no further replenishment opportunities and is dynamically priced throughout the selling horizon, whereas the regular product has a static price but can be replenished periodically subject to a limited capacity. We characterize the firm's optimal replenishment decision for the regular product as well as the dynamic pricing and initial quantity selection decisions for the seasonal product. Through the insights gained by the optimal policy structure, we also develop a simple-to-implement and effective heuristic policy. In addition, we investigate profit implications of markdown policies and study how potential differences in quality perceptions between the products impact the optimal policy. Lastly, we consider further types of asymmetries resulting in pricing with partial replenishment or replenishment with partial pricing and provide insights on the value of additional pricing and replenishment flexibilities. Our study helps broaden our understanding of joint pricing and replenishment decisions for substitutable products under circumstances where these decisions may not all be available for all products.

Key words: dynamic pricing; revenue management; substitutable products; inventory control

1. Introduction

Our goal in this paper is to study the implications of asymmetries in the pricing and replenishment controls of a firm offering substitutable products. Various circumstances may restrict a firm's ability to replenish or adjust the price of some of its products. As one example, consider a retailer that carries comparable items produced by different manufacturers. Often, the retailer may have different arrangements with manufacturers in terms of the retailer's flexibility to adjust the price of their product. While one manufacturer may allow pricing flexibility to a retailer, another that provides a substitutable product may require the retailer to adhere to its strict pricing policy.

According to the Federal Trade Commission (FTC) Guidelines following a Supreme Court ruling in 2007, "*if a manufacturer, on its own, adopts a policy regarding a desired level of prices, the law allows the manufacturer to deal only with retailers who agree to that policy.*" Thus, a retailer may lack any pricing control on a particular item they carry among an assortment of substitutable products. In addition, a retailer may also lack access to further replenishment opportunities for some of their products, due to, for example, incorporating a one-off variant such as a limited edition, special collection, or seasonal item to their regular product portfolio. Asymmetric pricing and replenishment controls may also arise due to simultaneous availability of products in different stages of their life cycles. For example, a gradually phased-out product that temporarily coexists with a newer version may face restrictions in further replenishment while possibly allowing more flexibility in its pricing.

As such settings demonstrate, it is important to broaden our understanding of optimal joint pricing and replenishment decisions for substitutable products under circumstances where these decisions may not all be available for all products. To do so, we begin by considering a specific setting in which a firm offering two substitutable products has the ability to adjust the price of only one of their products and has replenishment opportunities only for the other. Though we focus on this particular completely asymmetric setting in the main discussions that follow, we also later briefly extend our analysis to consider further asymmetries where both products allow price adjustments but only one can be replenished, or where both can be replenished but only one allows price adjustments.

To aid our exposition for the main setting, we use the terms *regular* and *seasonal* in order to distinguish between the two different types of products. In particular, we refer to the product that has a fixed price and that can be replenished periodically as the *regular* product. The *seasonal* product, on the other hand, has an initial perishable inventory but its price can be dynamically set in each period. Note that our use of the terms regular and seasonal are not in a restrictive literal sense but rather in a broader theme to capture and label the operational differences in how the respective product's inventory is managed by the firm over the selling horizon. It is evident that, due to the substitutability between the products, the firm's price choice for the seasonal product also impacts the inventory of, and thus the replenishment decisions for the regular product, and vice versa. One of our main goals in this paper is to identify the role this asymmetry in the pricing and replenishment opportunities plays on the firm's optimal decisions.

An additional interesting behavior of this setting is that the two types of products are not guaranteed to be simultaneously available throughout the horizon. By definition, once the inventory for the seasonal product is depleted, the firm will consider any spill over demand and resort to only managing its regular product inventory. Taking into account all these interactions, we are also interested in identifying how the firm should select its initial quantity for the seasonal product.

To investigate these questions, we formulate a multi-period stochastic dynamic program. At the beginning of each period, the firm first observes its current inventory level for the regular and seasonal products. If the seasonal item is no longer available, the firm only decides on the replenishment quantity for the regular product. If both items are available, the firm simultaneously sets a replenishment quantity for the regular product and a price for the seasonal product that will influence the current period demand for both the seasonal and the regular product. If applicable, at the beginning of the selling horizon, the firm also decides on an initial stocking quantity for the seasonal product to cover demand across the horizon.

Our first contribution in this paper is the characterization of the structure of the optimal pricing and replenishment policies under this asymmetric control setting. We find that the firm's replenishment decision is governed by a modified state-dependent base-stock policy and that the optimal price set for the seasonal product decreases with its own inventory but is increasing in, and partially decoupled from, the regular product inventory. Through a numerical study, we show that the optimal price for the seasonal product for a given inventory position does not necessarily possess monotonicity with respect to the time remaining in the selling horizon. Further, we use the insights gained from the structural results on the optimal policy to develop a simple-to-implement and efficient heuristic policy. Through our analysis and numerical tests, we also investigate how potential quality differences between the products influence the optimal policy and how some markdown policies in this setting impact the firm's profit potential. In addition, we provide further insights on joint pricing and replenishment control through extensions that incorporate other types of asymmetries resulting in pricing with partial replenishment and replenishment with partial pricing. We show that when the firm can also change the price of the regular product, its response to excess inventory for the regular product is no longer an increase in the price of the seasonal product but a decrease. On the other hand, if the firm can gain a replenishment opportunity for the seasonal product, its price is no longer always decreasing in its own inventory but becomes independent for certain starting inventory positions. Lastly, we consider the additional value a firm can receive by implementing dynamic pricing for the regular product or by allowing further replenishment opportunities for the seasonal product and find that allowing replenishment for the seasonal product can generally provide a greater improvement in the firm's profit compared to incorporating dynamic pricing for the regular product.

The rest of the paper is organized as follows. In Section 2, we review the related literature. We introduce the model and problem formulation in Section 3, and provide a characterization of the optimal policy structure in Section 4. In Section 5, we first numerically further investigate the optimal policy by providing sensitivity results with respect to various problem parameters and the remaining time in the selling horizon. We then develop and test a heuristic policy based on the insights gained from the structure of the optimal policy. In Section 6, we explore how potential quality differences impact the optimal policy and provide insights on the profit implications of markdown policies. Finally, we provide brief extensions to further types of asymmetries in pricing and replenishment controls in Section 7 before concluding in Section 8.

2. Related Literature

The setting we consider in this paper is closely related to two main streams of work within the dynamic pricing and inventory control literature, namely, dynamic pricing without replenishment (occasionally also referred to as dynamic pricing of perishable products) and dynamic pricing with replenishment. Whereas the former considers pricing decisions for a fixed initial inventory sold over a finite selling horizon with no further replenishment opportunities, the latter lets the firm jointly set prices and replenishment quantities. For a comprehensive review of the literature on dynamic pricing, we refer the reader to Elmaghraby and Keskinocak (2002), Bitran and Caldentey (2003), Chan et al. (2004), and more recently to Chen and Simchi-Levi (2012), and Chen and Chen (2015).

We first highlight closely related works on dynamic pricing without replenishment. Gallego and van Ryzin (1994) study the optimal pricing decisions for a single product over a finite selling horizon in a continuous time setting. When the demand curve is stationary over the horizon (i.e., for any given price, the expected demand is identical at any point in time), they show that the optimal price decreases with inventory. They further find that, for any particular inventory level, the optimal price is increasing in the remaining time to sell (i.e., decreases as the end of the horizon approaches). Bitran and Mondeschein (1997) study a closely related problem to that in Gallego and van Ryzin (1994) while considering heterogeneous valuations among consumers in the form of reservation prices. They show that, when reservation prices are time stationary, the optimal price behavior is similar to that in Gallego and van Ryzin (1994). Zhao and Zheng (2000) later extend the findings of Bitran and Mondeschein (1997) and show that optimal price remains decreasing in inventory and increasing in the remaining selling time as long as consumers' willingness to pay a premium for the product does not increase over time.

In addition to the above mentioned works that consider a single perishable product, there has also been considerable interest in studying the pricing decisions for perishable products in a multiple product setting. In a follow up study, Gallego and van Ryzin (1997) consider dynamic pricing decisions for multiple products where each product requires a particular selection of resources. Studying the firm's pricing decisions that take into account the remaining inventory for each resource, they develop heuristics based on the deterministic formulation of the problem and show that they perform close to optimal. Later, Bertsimas and de Boer (2005) propose heuristics based on a decomposition approach for a similar setting but in a periodic review context. Dong et al. (2009) study dynamic pricing of perishable products by adopting the multinomial logit model and show that optimal prices are obtained by applying identical marginal revenues for each product. They also numerically show that while the price of each product decreases in its own inventory, it may increase or decrease with the inventory of other products and with the remaining time until the end of the horizon. Akcay et al. (2010) study optimal pricing policies in a similar setting by considering instances where products are either vertically or horizontally differentiated. While their findings for the setting where products are horizontally differentiated are similar to those in Dong et al. (2009), they find that when products are vertically differentiated, optimal prices possess time monotonicity and the prices depend on the higher quality products only through their aggregate inventory rather than individually.

As we had classified earlier, the main common theme across these studies is that the firm only sets dynamic pricing decisions for fixed initial inventories and there are no replenishments for any of the products. In addition, it is also important to note that previous work in this area has mainly considered continuous time models. As the solutions derived from such models cannot be readily implemented in retail settings that we are considering, structural results for periodic review models are also of particular interest. Thus, with respect to this body of work, the main contribution of our study is that it provides a full characterization of the dynamic pricing decisions for a perishable product in a periodic review context and with the presence of a replenishable substitutable product.

A second stream in the literature considers joint pricing and replenishment decisions. Whitin (1955) is among the first to study joint pricing and inventory decisions for a single period problem. Thowsen (1975), and later Federgruen and Heching (1999) study a joint pricing and inventory control problem for a single product in a multi-period setting and show that, when backlogs are allowed, the optimal pricing and replenishment decisions are characterized by a list-price, base-stock policy. In brief, the list-price, base-stock policy leads the firm to raise the inventory of the product to a base-stock level and charge a list price. If the product is already overstocked, then no production takes place and a discount is applied. Further, they find that the price of the product is decreasing with inventory. Chen and Simchi-Levi (2004) further extend this work by studying the effect of fixed ordering costs on a firm's pricing and replenishment decisions. More recently, Feng et al. (2014) generalize conditions under which a base-stock list-price policy is optimal in the single product setting, including instances in which a particular demand-price relationship results in a non-monotonic price with respect to inventory. In addition, there is also recent work on joint pricing and inventory control for items that deteriorate from one period to the next. Sainathan (2013) considers the pricing and ordering decisions in such a setting where a product has limited shelf life and deteriorates in the next period. He finds that the optimal price for the new product may increase or decrease with the inventory for the old product and that the benefit of offering the old product is higher when replenishment is costlier and when the quality level of the new product is higher. Chen et al. (2014) incorporates multiple stages of deterioration and show that the ordering decisions are more sensitive to fresher inventory and that the pricing decisions are more sensitive to older inventory.

As a follow up to the single product settings, Zhu and Thonemann (2009) consider the pricing and replenishment decisions for two price-substituable products utilizing a linear, additive demand-price relationship and show that the optimal policy is again characterized by regions of list prices and price discounts, this time, defined by state-dependent base-stock levels. Specifically, they show that the base-stock level for a product decreases with the inventory of the other product. When the cross-price sensitivities are identical, they also show that the price of a product is decreasing with respect to both inventory levels. Song and Xue (2007) study more general demand models for substitutable products and provides algorithms to compute the optimal policies. Ceryan et al. (2013) incorporates capacity restrictions on production quantities and study how the availability of a flexible replenishment capacity, which can be allocated to either of the substitutable products, impacts dynamic pricing decisions. In a subsequent work, Ceryan et al. (2017) study the influence of product upgrades in a setting where products are vertically differentiated and show that offering upgrades helps the firm preserve the price separation between vertically differentiated products in a dynamic pricing context.

One common characteristic of these works on joint pricing and replenishment is the symmetry across products regarding the firm's available tools to manage their inventories. Regarding our contributions with respect to the joint pricing and replenishment literature, the main differentiating feature of our work is its consideration of asymmetric, partial pricing and replenishment decisions which raise additional complications as the inability to replenish may prevent all products to remain available throughout the horizon. It is also important to note that this partial pricing and replenishment setting is not a special case of the joint pricing and replenishment problems studied earlier. For instance, a main result from this literature (as we have previously mentioned) is that the optimal price of a product is decreasing with the inventory level of either product when cross price sensitivities are identical. On the other hand, when the firm is limited to control only one of the prices rather than both, we will reveal that the price is instead increasing with the inventory level of the other product under a similar demand-price relationship. In addition, from a technical standpoint, we also show that when the firm is able to price only one of the products, the value function does not satisfy the diagonal dominance property previously shown to hold for the joint pricing and replenishment problem but instead requires the preservation of a modified variant of this property. Finally, in addition to providing full structural results, we also contribute to the existing body of work by developing a simple-to-implement and efficient heuristic policy for this partial pricing and replenishment problem.

3. Problem Formulation

We consider a firm that sells two substitutable products that differ in how their inventories are managed by the firm over a finite selling horizon of length T. One of the products, which we will refer to as the '*regular*' product, can be replenished periodically and the firm charges a fixed price, p_r, for this product throughout the selling horizon. The other product, referred to as the '*seasonal*' product, is stocked only once at the beginning of the selling horizon and its price, p_s^t, can be dynamically set in each period. (We provide an extension to our analysis that also incorporates dynamic pricing decisions for the regular product in Section 7.) For example, the seasonal product may be a limited edition clothing article that is offered alongside a regularly replenished, classic article that is generally not included in price promotions. We note that we use the terms regular and seasonal, with the corresponding subscripts r and s, not in a restrictive literal sense but as shorthand labels to distinguish the products according to their operational distinctions.

We represent the price-demand relationship between the products through a linear, additive, stochastic demand model as commonly adopted in the related literature. (As a side note, such linear price-demand relationships arise, for example, when a representative consumer maximizes a quadratic utility function as described in Dixit (1979) and in Singh and Vines (1984).) We temporarily let $\hat{D}_r^t(p_r, p_s^t, \epsilon_r^t) = \hat{a}_{r,0}^t - \hat{a}_r p_r + b_r p_s^t + \epsilon_r^t$ and $\hat{D}_s^t(p_r, p_s^t, \epsilon_s^t) = \hat{a}_{s,0}^t + \hat{a}_s p_r - b_s p_s^t + \epsilon_s^t$ denote the current period demand for the regular and seasonal product, respectively. As the firm adopts a constant price policy for the regular product, and in order to aid expositional clarity, we define $a_r^t := \hat{a}_{r,0}^t - \hat{a}_r p_r$ and $a_s^t := \hat{a}_{s,0}^t + \hat{a}_s p_r$. Thus, we express the current period demand for the regular and seasonal products as $D_r^t(p_s^t, \epsilon_r^t) = a_r^t + b_r p_s^t + \epsilon_r^t$ and $D_s^t(p_s^t, \epsilon_s^t) = a_s^t - b_s p_s^t + \epsilon_s^t$, respectively. In this framework, the terms a_r^t and a_s^t can be considered as the overall demand intercepts for the regular and seasonal products taking into account the influence of the price of the regular product. In addition, b_r and b_s denote individual and cross-price sensitivity coefficients with respect to the seasonal price. We allow demand intercepts a_r^t and a_s^t to vary with time, which enables us to capture effects such as a stronger interest in a fashion item earlier in the season rather than towards the end of the horizon. For tractability however, we require b_s and b_r to be time invariant, which is reasonable as one might expect that changes in consumers' price sensitivities are more gradual. We assume $b_s > b_r > 0$, reflecting the substitutable nature of the products and that a change in the price of the seasonal product, p_s^t, affects its own demand more strongly than the demand for the other product. The terms ϵ_r^t and ϵ_s^t refer to zero-mean, continuously distributed, independent random variables with nonnegative support on product demands. For future reference, we refer to the expected demand for the products by $d_r^t(p_s^t)$ and $d_s^t(p_s^t)$, i.e., $d_r^t(p_s^t) = a_r^t + b_r p_s^t$ and $d_s^t(p_s^t) = a_s^t - b_s p_s^t$. (Note: We also briefly extend our analysis to incorporate dynamic pricing decisions for the regular product in Section 7 and refer back to the initial two-price demand-price model in the modified formulation.)

At the beginning of each period, the firm first reviews the current inventory levels x_r^t and x_s^t for the regular and seasonal products, respectively. For periods in which there is available seasonal product inventory, i.e., when $x_s^t > 0$, the firm decides on a price, p_s^t, for the seasonal item along with a replenishment level, y_r^t, for the regular item that is constrained by a replenishment capacity K. We denote the per unit replenishment cost for the regular product by c_r. Owing to recurring replenishment opportunities only for the regular product and no further replenishments for the seasonal product, we let the firm backorder any excess demand only for the regular product. If the firm faces a shortage for the seasonal product, we allow it to satisfy this excess demand at its first occurrence via an external resource at a certain cost. In such a case, the seasonal product is no longer available until the end of the horizon and the problem for the subsequent period(s) reduces to a (multi-period) single item, capacitated inventory control problem with a corresponding demand adjustment for the regular product. Specifically, let $\tilde{p}_s^t = a_s^t / b_s$, i.e., $\tilde{p}_s^t$ is the *'null price'* for the seasonal product in period t that would eliminate all demand for that product. We let $\tilde{d}_r^t$, where $\tilde{d}_r^t = a_r^t + b_r \tilde{p}_s^t$, denote the corresponding expected regular product demand when the firm no longer sells the seasonal product.

To facilitate our analysis, we apply a change of variables and set the decision variables as d_s^t, the mean demand selection for the seasonal product, and z_r^t, the expected inventory level for the regular product after replenishment and depletion by demand, i.e., $z_r^t = y_r^t - d_r^t$. In other words, z_r^t corresponds to the target safety stock level for the regular product. For any decision pair (d_s^t, z_r^t), we can immediately obtain the corresponding price set for the seasonal product through $p_s^t(d_s^t) = (a_s^t - d_s^t)/b_s$, the resulting expected demand for the regular product as $d_r^t(d_s^t) = a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t)$, and the replenishment level for the regular product as $y_r^t(d_s^t, z_r^t) = z_r^t + a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t)$. Consequently, for initial states where the firm has available seasonal inventory (i.e., $x_s^t > 0$), we indicate the feasible values for the decision variables with $\mathcal{F}(x_r^t) := \{(d_s^t, z_r^t) \,|\, d_s^t \geq 0,\, x_r^t \leq z_r^t + a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t) \leq x_r^t + K\}$. Similarly, for states with fully depleted seasonal inventory (i.e., $x_s^t = 0$), we indicate the feasible values for the single decision variable (corresponding to selecting the safety-stock for the regular product) with $\tilde{\mathcal{F}}(x_r^t) := \{(z_r^t) \,|\, x_r^t \leq z_r^t + \tilde{d}_r^t \leq x_r^t + K\}$.

Following the change of variables, we can express the expected revenue within a period with available seasonal inventory as $R^t(d_s^t) = p_s^t(d_s^t)\, d_s^t + p_r\, d_r^t(d_s^t)$ where $p_s^t(d_s^t)$ and $d_r^t(d_s^t)$ are the seasonal price and the expected regular demand as defined earlier. Explicitly stated, $R^t(d_s^t) = p_r\left(a_r^t + \frac{b_r}{b_s} a_s^t\right) + \left(\frac{a_s^t - p_r b_r}{b_s}\right) d_s^t - \frac{{d_s^t}^2}{b_s}$. If the firm's seasonal inventory is depleted, a modified expected revenue expression for which revenue is only generated by the regular product can be stated as $\tilde{R}^t = p_r \tilde{d}_r^t$. Further, letting $H_r(z_r^t)$ and $H_s(x_s^t, d_s^t)$ correspond to expected holding and shortage costs for the regular and seasonal product, respectively, we have $H_r(z_r^t) = \mathrm{E}_{\epsilon_r^t}\left[h_r^+\,(z_r^t - \epsilon_r^t)^+ + h_r^-\,(z_r^t - \epsilon_r^t)^-\right]$ where h_r^+ and h_r^- refer to unit holding and backorder costs for the regular product, and where $(z_r^t - \epsilon_r^t)^+ := \max(0, (\mathrm{z_r^t} - \mathrm{\epsilon_r^t}))$, and $(z_r^t - \epsilon_r^t)^- := \max(0, (\mathrm{\epsilon_r^t} - \mathrm{z_r^t}))$. Similarly, $H_s(x_s^t, d_s^t) = \mathrm{E}_{\epsilon_s^t}\left[h_s^+\,(x_s^t - d_s^t - \epsilon_s^t)^+ + h_s^-\,(\epsilon_s^t - x_s^t + d_s^t)^-\right]$, where h_s^+ is the unit holding cost for the seasonal product and h_s^- is the unit cost to satisfy any excess demand by external means.

We formulate the pricing and replenishment decisions through a multi-period stochastic dynamic programming model by letting $V^t(x_r^t, x_s^t)$ denote the expected discounted profit-to-go function under the optimal policy starting at state (x_r^t, x_s^t) in period t of an horizon consisting of T periods. For states corresponding to depleted seasonal inventory, i.e., when $x_s^t = 0$, we also define $\tilde{V}^t(\cdot) := V^t(x_r^t, 0)$ where $\tilde{V}^t(\cdot)$ denotes the optimal value function starting period t with a regular product inventory level of x_r^t. In other words, $\tilde{V}^t(\cdot)$ corresponds to a reduced, single product, capacitated, (multi-period) inventory control problem. Lastly, we use $\mathbb{1}_{\{\cdot\}}$ to denote the indicator function regarding whether the firm ends a period with positive seasonal inventory. Below, we present the problem formulation:

$$V^t(x_r^t, x_s^t) = \begin{cases} \max_{d_s^t, z_r^t \in \mathcal{F}(x_r^t)} R(d_s^t) - c_r\,(z_r^t + d_r^t(d_s^t) - x_r^t) - H_r(z_r^t) - H_s(x_s^t, d_s^t) \\ \qquad\qquad + \beta\, \mathrm{E}_{\epsilon_r^t, \epsilon_s^t}\Big[V^{t+1}\big(z_r^t - \epsilon_r^t, x_s^t - d_s^t - \epsilon_s^t\big)\mathbb{1}_{\{x_s^t - d_s^t - \epsilon_s^t > 0\}} \\ \qquad\qquad\qquad + \tilde{V}^{t+1}\big(z_r^t - \epsilon_r^t\big)\mathbb{1}_{\{x_s^t - d_s^t - \epsilon_s^t \leq 0\}}\Big], & \text{if } x_s^t > 0 \\ \tilde{V}^t\big(x_r^t\big), & \text{if } x_s^t = 0 \end{cases} \tag{1}$$

where,

$$\tilde{V}^t(x_r^t) = \max_{z_r^t \in \tilde{\mathcal{F}}(x_r^t)} \tilde{R} - c_r \cdot (z_r^t + \tilde{d}_r^t - x_r^t) - H_r(z_r^t) + \beta \,\mathrm{E}_{\epsilon_r^t}\left[\tilde{V}^{t+1}\big(z_r^t - \epsilon_r^t\big)\right] \tag{2}$$

with $V^T(x_r, x_s) = \tilde{V}^T(x_r) = -c_r x_r^-$.

In (1), the expression corresponding to $x_s^t > 0$ is the optimal value starting in state (x_r^t, x_s^t) in period t with positive seasonal inventory. The first term in this expression, $R(d_s^t)$, is the expected revenue for the current period as defined earlier. The term $c_r^t \left(z_r^t + d_r^t(d_s^t) - x_r^t\right)$ accounts for the replenishment cost for the regular product, and the terms $H_r(z_r^t)$ and $H_s(x_s^t, d_s^t)$ refer to the expected holding and shortage costs as stated earlier. The discounted expected value-to-go consists of two parts. The first term in the expression $\mathrm{E}_{\epsilon_r^t, \epsilon_s^t}\left[\,\cdot\,\right]$ is the value-to-go function for the next period if the current period ends with positive seasonal product inventory whereas the second term is the value-to-go function for the next period otherwise. As we described earlier, once the seasonal inventory is depleted, the firm's problem essentially reduces to solving (2), i.e., identifying a safety stock level for the regular product over the remaining horizon that maximizes its expected profit subject to capacity limitations.

Lastly, we also would like to address problem instances where the firm makes an initial quantity selection for the seasonal product. In such a setting, at the beginning of the planning horizon, i.e., when $t = 0$, the firm selects an initial stocking quantity, Q_s, for the seasonal product along with a price, p_s^0, to charge during the initial period. The firm also decides on a reorder quantity for the regular product by determining a replenishment level, y_r^0. As the regular product may preexist in the market before the initial order for the seasonal product is made, we incorporate the possible presence or shortage of any previous regular product inventory rolling over into the start of the horizon. Thus, the firm considers the regular product's existing inventory position while determining its initial stocking quantity and price for the seasonal product as well as the reorder quantity for the regular product. For a concise representation, we can examine the firm's profit for any particular starting inventory position (x_r^0, Q_s) at $t = 0$ following the optimal pricing and replenishment decisions as described above, and state the initial quantity problem as choosing the seasonal product quantity Q_s that maximizes the overall profit. Letting $V_I^t(x_r^t)$ denote the optimal value of starting the horizon with an existing regular product inventory position x_r^t, we can write the initial quantity selection problem as:

$$V_I(x_r^0) = \max_{Q_s \geq 0} \quad V^0\big(x_r^0, Q_s\big) - c_s Q_s \tag{3}$$

where c_s is the unit purchase cost for the seasonal product.

4. Characterization of the Optimal Policy

Our main results in this section are the characterizations of the structure of the optimal pricing and replenishment decisions. We then extend our analysis by identifying the structure of the initial quantity decisions for the seasonal product to also address settings where such an initial order may be applicable.

4.1. Optimal Replenishment Policy for the Regular Product

At the beginning of each period, the firm first observes its inventory position for the regular and seasonal products. If it has available seasonal inventory, it simultaneously decides on the price it will apply for the seasonal product and the replenishment level for the regular product. If the seasonal product is no longer available, the firm's decision reduces to selecting a replenishment level for the regular product. As we will discuss subsequently, we find that an underlying state space segmentation for the optimal replenishment policy also prescribes the optimal pricing policy. Therefore, we first present the optimal replenishment decisions for the regular product.

THEOREM 1. (Optimal Replenishment Policy) *Optimal replenishment for the regular product follows a modified base-stock policy defined by a state-dependent base-stock level $\bar{x}_r^t(x_s^t)$.*

(*a*) *For any starting inventory pair (x_r^t, x_s^t), it is optimal to replenish $\min\left((\bar{x}_r^t(x_s^t) - x_r^t)^+, K\right)$ units of the regular product.*

(*b*) *The state-dependent base-stock level $\bar{x}_r^t(x_s^t)$ is strictly decreasing in x_s^t.*

Proof: The proof of Theorem 1 and all subsequent results are provided in the online appendix.

As stated in Theorem 1 (a) and depicted in the subsequent Figure 1 (a), the optimal replenishment for the regular product in each period is defined by a modified state-dependent base-stock level, denoted by $\bar{x}_r^t(x_s^t)$, that is a function of the seasonal product inventory. The firm replenishes the regular product as much as its capacity permits to reach this desired base-stock level. That is, if the initial inventory position for the regular product is such that it requires replenishment and is below $\bar{x}_r^t(x_s^t) - K$, then the firm will use its capacity, K, to its full extent and replenish K units of the product. When the initial inventory position for the regular product falls in the interval $[\bar{x}_r^t(x_s^t) - K, \bar{x}_r^t(x_s^t)]$, the firm is not constrained by the available capacity and is therefore able to replenish the regular product to reach its desired base-stock level $\bar{x}_r^t(x_s^t)$. If the regular product is already overstocked, i.e., if $x_r^t > \bar{x}_r^t(x_s^t)$, then no replenishment for the regular product takes place. Finally, if the firm no longer has any availability for the seasonal product, its modified base-stock level is given by $\bar{x}_r^t(0)$ for the subsequent periods.

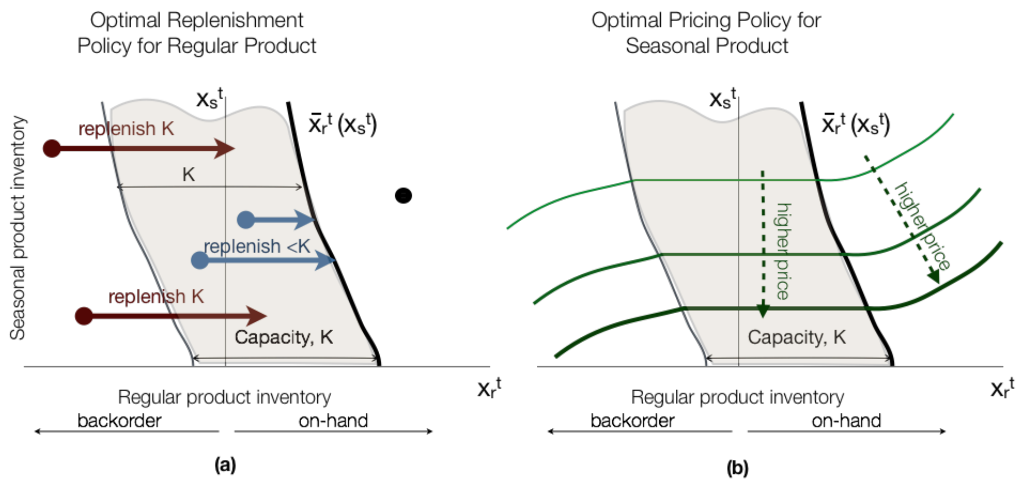


Figure 1 **Structure of the (a) optimal replenishment policy for the regular product, and (b) the optimal pricing policy for the seasonal product.**

Part (b) of Theorem 1 indicates that the desired base-stock level, $\bar{x}_r^t(x_s^t)$, is a strictly decreasing function of the seasonal product inventory. In other words, a higher seasonal product inventory results in the firm lowering its base-stock level for the regular product. Similarly, lower seasonal product inventory prompts the firm to increase its base-stock level. This result is intuitive due to the partially-substitutable nature of the products. We defer the specification of the precise mechanism behind this monotonicity result until after we present the optimal pricing policy for the seasonal product.

4.2. Optimal Pricing Policy for the Seasonal Product

We next describe the optimal pricing policy for the seasonal product. We note that the pricing decision for the seasonal product is only applicable when the firm has available seasonal inventory at the beginning of the period, i.e., when $x_s^t > 0$. First, we recall that the previously stated replenishment policy segmented the state space into three regions based on whether the product is overstocked, or, if it requires replenishment, whether the firm's capacity is sufficient or is limiting to reach the desired base-stock level. We find that these regions give rise to different behaviors for the firm's optimal pricing strategy as stated in the following result.

Theorem 2. (Optimal Pricing Policy) *The optimal price for the seasonal product, $p_s^{t*}(x_r^t, x_s^t)$, is strictly decreasing in x_s^t and weakly increasing in x_r^t. Specifically, for any starting inventory pair (x_r^t, x_s^t),*

- *If $\bar{x}_r^t(x_s^t) - K \le x_r^t \le \bar{x}_r^t(x_s^t)$, the price for the seasonal product is independent of the inventory level of the regular product.*

- *Otherwise, the price for the seasonal product is strictly increasing with the inventory level of the regular product.*

As described in Theorem 2, we find that the price of the seasonal product is strictly decreasing with its own inventory. This result is in line with the general finding in the pricing of perishable inventory literature that, for a given point in time, the optimal price for a perishable product decreases with its inventory. Our result shows that this monotonicity property extends to a setting when the price change on a perishable product impacts not only its own demand but also the demand for a statically priced and replenishable substitutable product. This price monotonicity also leads to the monotonic behavior of the optimal replenishment policy as reported in Theorem 1 (b) earlier. Specifically, as the price of the seasonal product decreases with its own inventory, when the availability of the seasonal item is higher, the firm effectively increases the expected demand for the seasonal item and decreases its expected demand for the regular item. This lowered expected demand for the regular product in turn also decreases the firm's desired base-stock level for this product.

Next, we find that the optimal price for the seasonal product is partially decoupled from the inventory level of the regular product. Specifically, when the regular product requires replenishment and its capacity is adequate to bring its inventory level to its base-stock target (as defined earlier in Theorem 1), the optimal price for the seasonal product is independent of the starting inventory level of the regular product. That is, as long as the regular product is not overstocked and its capacity for the regular product is not limiting its replenishment, the firm's optimal price for the seasonal product only considers the seasonal product inventory. If on the other hand, the regular product is understocked such that its capacity is not sufficient to bring its inventory up to the target base-stock level, then the price of the seasonal product depends on the starting inventory level of the regular product. Particularly, the firm decreases its price for the seasonal product when the regular product inventory is lower. This stems from the fact that such a price decrease helps alleviate the impact of replenishment limitations by encouraging more customers to purchase the seasonal product rather than the regular product. Similarly, when the regular product is overstocked, the firm's optimal price selection for the seasonal product also depends on the inventory level of the regular product. Specifically, the firm increases the price of the seasonal product to shift more demand to the regular item in an effort to decrease the excess inventory for the regular product. It is important to note that a main result from the joint pricing and replenishment literature is that, when cross price sensitivities are identical, the optimal price of a product is decreasing with the inventory level of either product (see for example Zhu and Thonemann (2009) and Ceryan et al. (2013)). On the other hand, we find that in the setting we consider where the firm is limited to control only one of the prices rather than both, the optimal price is instead increasing with the inventory level of the other product under a similar demand-price relationship. We also would like to briefly note that the optimal price for the seasonal product does not possess monotonicity with respect to time remaining in the selling horizon. We expand on the price behavior with respect to time through numerical examples in Section 5.

4.3. Initial Quantity Selection for the Seasonal Product

Having described the optimal pricing and replenishment decisions across the horizon, we now would like to explore the firm's initial stocking problem for the seasonal product, formally defined earlier in (3). In practice, one might interpret the initial quantity selection problem differently in various contexts. For example, when a product is phased out and is replaced by a new product, the initial order selection can be thought of as the size of the last batch of the phased-out product to be produced before the product is retired. As another example from the apparel retailing setting, the initial quantity selection may also be viewed as the size of a one-off order for a limited edition seasonal item that will be offered alongside a recurring classic item. Our next result describes how the firm should select its initial order quantity.

THEOREM 3. (Initial Quantity Selection) *The optimal initial quantity for the seasonal product, $Q_s^*(x_r^0)$, is weakly decreasing with x_r^0. For any existing regular inventory, x_r^0, at the beginning of the horizon, $Q_s^*(x_r^0)$ is independent of x_r^0 if $\bar{x}_r^0 - K \le x_r^0 \le \bar{x}_r^0$, and strictly decreases with x_r^0 otherwise.*

As described in Theorem 3, the optimal initial order quantity Q_s^* for the seasonal product exhibits two distinct behaviors depending on the existing inventory level of the regular product at the beginning of the horizon. If the regular product is either overstocked or has insufficient capacity, then the firm's choice of the initial quantity for the seasonal product strictly decreases with the regular inventory. Consider for example the instance where the regular product is understocked at the beginning of the horizon with insufficient capacity. The firm responds to this unfavorable inventory level at the beginning of the horizon by selecting a higher initial stock for the seasonal item. In particular, the firm's initial order for the seasonal product will be higher for a larger amount of shortage of the regular product. This increase in the initial order quantity is attributable to the firm's pricing policy. Specifically, when the firm faces shortages for the regular item, it decreases the price of the seasonal product, as discussed in Theorem 2, in order to suppress the demand for the regular item, which in turn, induces a higher demand for the seasonal item. Thus, in anticipation of an increased seasonal demand, the firm also increases the initial order size for the seasonal product. When the regular product is overstocked, a similar but reverse dynamic is at play. In order to reduce regular product's excess inventory, the firm prompts more customers to choose the regular product by increasing the price of the seasonal product. The following demand reduction for the seasonal product leads the firm to select a lower initial order quantity. When, at the beginning of the horizon, the regular item is neither overstocked nor understocked such that its capacity is not limiting its replenishment, we find that the initial order quantity for the seasonal product is independent of the regular product inventory. In other words, the firm's optimal quantity choice for the seasonal product decouples from the regular product inventory as long as the regular product is understocked with no capacity limitations, similar to the decoupling behavior in the pricing policy as described in Theorem 2.

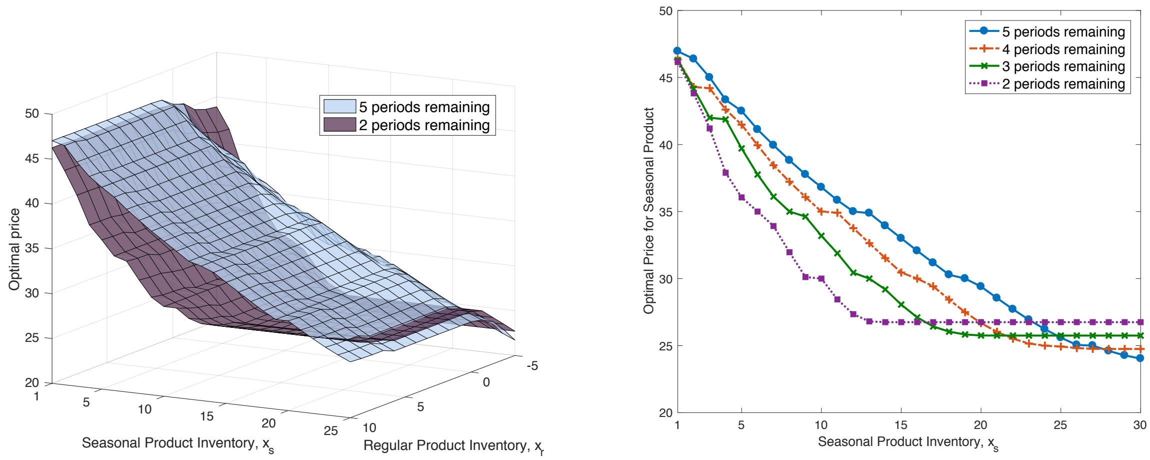
5. Numerical Study

In this section, we first demonstrate the non-monotonic behavior of the optimal price with respect to the remaining time in the selling horizon and briefly comment on the sensitivity of the optimal policy with respect to various problem parameters. We then develop an easy-to-implement and effective heuristic policy based on insights gained from the characterization of the optimal policy.

5.1. Non-Monotonic Behavior of Optimal Prices with respect to Remaining Time

To visualize the price behavior, we consider an example problem instance where the mean demand for the regular and the seasonal products are given by $d_r^t(p_s^t) = 7 - 0.2(25) + 0.1p_s^t = 2 + 0.1p_s^t$ and $d_s^t(p_s^t) = 7.5 + 0.1(25) - 0.2p_s^t = 10 - 0.2p_s^t$, respectively, where the price for the regular product is set as $p_r = 25$ (the value in the parentheses) and the base demand intercept values, 7 and 7.5, suggest a slightly stronger base market potential for the seasonal product. We set the replenishment cost for the regular product at $c_r = 10$, the holding costs for the regular and seasonal products as $h_r^+ = h_s^+ = 2$, and the shortage costs for the regular and seasonal products as $h_r^- = 20$, $h_s^- = 50$, respectively, where the relatively high value for the seasonal shortage penalty reflects the firm's need to resolve and meet excess demand through an external source, which may also likely be greater than any price it would receive from selling it. We let the demand uncertainties for the regular and seasonal products be independently and uniformly distributed within the range $[-2, 2]$, set a selling horizon of five periods with a discount factor of $\beta = 1$, and assume a fixed replenishment capacity of $K = 8$ units per period for the regular product.

Figure 2 (a) simultaneously displays the optimal pricing policies across the state space when there are either two or five periods remaining until the end of the selling horizon. Figure 2 (a) shows that, apart from instances with a very high inventory level for the seasonal product (e.g.,



(a) Optimal Price for (i) 2 periods and (ii) 5 periods remaining in the selling horizon

(b) Optimal Price for $x_r = 0$ and across various periods remaining in the selling horizon

Figure 2 Non-monotonic behavior of optimal prices across various inventory positions and time periods

approximately 23 units for this example), the firm generally chooses to apply a lower price at any such particular inventory position when the remaining selling horizon is shorter (e.g., two periods rather than five periods). This can be expected as the firm wishes to increase the demand for the seasonal product when there are fewer periods left to sell this product (similar to the behavior in Gallego and van Ryzin, 1994). An exception to this behavior occurs when the firm's inventory for the seasonal product is very high (e.g., 25 units) that it is very unlikely to sell all seasonal inventory by the end of the horizon. In this case, the firm applies a lower price earlier in the horizon (i.e., when there are 5 periods to go) in order to avoid recurring inventory costs associated with a unit that otherwise might not be eventually sold. We observe that a second such exception also occurs when the firm simultaneously has very low seasonal and regular inventory as depicted in Figure 2 (a). In this case, despite the desire to charge a high price in both periods due to limited seasonal inventory, the firm selects a relatively lower price for the seasonal product when there are more periods remaining in order to suppress regular demand to avoid multiple periods of backorders that may arise due to limited replenishment capacity.

Figure 2 (b) provides additional expositional clarity by illustrating a particular cross section of the pricing policy corresponding to a regular inventory position of zero units across multiple successive periods. The reported prices indicate the optimal prices corresponding to a particular seasonal inventory level and time in the selling horizon. We extend our earlier discussion based on two different periods to multiple periods following a similar logic. For this cross section of the pricing policy, we observe that for any particular seasonal inventory level that is less than 16 units,

	Price (seasonal product)	Replenishment Level (regular product)
Demand intercept, seasonal (a_s)	↑	↑
Demand intercept, regular (a_r)	↓	↑
Individual price sensitivity (b_s)	↓	↓
Cross price sensitivity (b_r)	↑ ↓	↑
Price, regular (p_r)	↑	↓
Replenishment cost, regular (c_r)	↓	↓
Replenishment capacity (K)	↑	↑ ↓
Holding cost, seasonal (h_s^+)	↓	↓
Shortage cost, seasonal (h_s^-)	↑	↑
Holding cost, regular (h_r^+)	↑	↓
Shortage cost, regular (h_r^-)	↓	↑

Table 1 Sensitivity of the optimal policy to various problem parameters

the firm selects a lower price when it has fewer periods to sell the seasonal product. Conversely, if the firm has a large amount of inventory for the seasonal product, e.g., 30 units, it selects a lower price when it has more periods to sell the product (in order to avoid recurring inventory costs as we have discussed previously). Consequently, for seasonal inventory levels in between, the pricing behavior may exhibit temporal reversals. For example, Figure 2 (b) demonstrates that for a starting seasonal inventory level of 20 units, the firm first decreases the price as it moves from five periods-to-go to three periods-to-go, but then increases the price when only two periods remain.

5.2. Sensitivity of the Optimal Policy

Next, we would like to briefly comment on the sensitivity of the optimal policy. Table 1 summarizes numerical results on how the optimal price for the seasonal product and the replenishment level for the regular product change as various problem parameters change. The symbols ↓ and ↑ in the table stand for weakly decreasing and weakly increasing, respectively. When both appear simultaneously, the corresponding optimal decision may increase or decrease. As indicated in the table, when the demand intercept for the seasonal product (a_s) increases (due to an increase in the base market potential only and assuming a constant regular product price), the firm responds to this strengthening of demand by increasing the price of the seasonal product. This price increase also results in an increase in the expected demand for the regular product, thus the replenishment level for this product increases as well. If, instead, the demand for the regular product strengthens through an increase in (a_r) (again due to an increase in the base market potential only), the firm again raises the replenishment level for this product. However, it also simultaneously decreases the price of the seasonal product to attract some of this increased demand to the seasonal product. Next, as the demand for the seasonal product becomes more sensitive to its own price (i.e., an increase in b_s), the firm reduces the price to counter the weakened demand, which in turns reduces the demand for the regular product and thus its replenishment level. When the demand for the regular product becomes more sensitive to the price of the seasonal product (i.e., an increase in b_r), the firm selects a higher replenishment level for the regular product but may increase or decrease the seasonal price depending on its inventory position. Specifically, if the firm has high regular inventory, it raises the price of the seasonal product. This allows it to capture more of the regular demand as well as to avoid additional holding costs for excess regular inventory. On the other hand, if the firm has low inventory for the regular product, selecting a lower price helps the firm reduce backorder costs. An increase in the price of the regular product (p_r) strengthens the demand for the seasonal product and weakens the demand for the regular product to which the firm responds by increasing the seasonal price but lowering the regular replenishment. If the replenishment capacity (K) increases, the firm does not need to lower its seasonal price (and consequently regular demand) as much to prevent excessive backorders, thus the price for the seasonal product increases. As the firm becomes less sensitive to backorders, it affords to keep a lower safety stock, thus decreasing the desired base-stock levels for the regular product. When the firm is facing excessive backorders that requires full use of the capacity though, a larger capacity allows the firm increase its replenishment level. The sensitivity results for the regular replenishment, holding and shortage costs are intuitive.

5.3. A Heuristic Policy

Determining the optimal policy is rather computationally expensive as it requires the solution of a dynamic program with continuous decision variables through a value iteration algorithm using successive discretizations and value function approximations. Therefore, and especially from a practical standpoint, it is worthwhile to investigate easier-to-implement heuristic policies. Hence, our next focus is on developing one such heuristics utilizing the insights we gained through our earlier characterization of the optimal policy structure.

We derive the heuristic policy in three steps. In the first stage, we utilize the partial decoupling property between the price of the seasonal product and the inventory of the regular product in order to obtain an initial price for the seasonal product. In the second stage, we adjust this initial price upwards if the firm faces any scarceness of inventory for the seasonal product. Finally, in the last stage, we adjust the price downwards if the firm faces restrictions for the replenishment for the regular product due to capacity limitations. The direction of both of these adjustments also follow the monotonicity of the optimal policy. (We do not make adjustments for excess regular inventory as any reasonable excess is transient since it will eventually be drawn below the desired base-stock level.) For expositional clarity, we first consider stationary instances and suppress time indices when applicable. We comment on modifications for nonstationary settings subsequently. We expand on the derivation of the heuristic policy below.

Step 1: We note that the unit ordering cost for the seasonal product can be viewed as sunk cost and that a sale of a seasonal product will bring in the product's revenue as well as savings on holding costs that would have been incurred until the end of the selling horizon. Therefore, we derive an initial price $\hat{p}^t$ for period t by solving the following current period revenue maximization problem: $\max_{\hat{p}^t}\ (\hat{p}^t + \sum_t \beta^{(T-t)} h_s^+)(a_s - b_s\hat{p}^t) + (p_r - c_r)(a_r + b_r\hat{p}^t)$. The first term in parenthesis represents the per unit net contribution (sales price plus the holding cost savings until the end of the horizon) of a seasonal product sale. The second term in parenthesis denotes the expected seasonal demand at price point $\hat{p}^t$. Similarly, the remaining terms collectively refer to the net revenue obtained by the corresponding regular product sales. Solving for $\hat{p}^t$ and capping the price by the null price $\tilde{p}$, we get $\hat{p}^t = \min\left(\frac{a_s+(p_r-c_r)b_r}{2b_s} - \frac{h_s^+}{2}\sum_t \beta^{(T-t)}\,,\, \tilde{p}\right)$.

Step 2: Let $\hat{x}_s^t = (T-t)\, d_s(\hat{p}^t)$ denote the total expected demand for the seasonal product until the end of the horizon at price point $\hat{p}^t$. If $x_s^t < \hat{x}_s^t$, i.e., the current inventory for the seasonal product is not adequate to meet demand throughout the horizon, we increase the price to allocate the current inventory x_s^t evenly across periods. Let ${p'}^t$ denote this updated price. We have ${p'}^t = \min\left(\max\left(\hat{p}^t\,,\, \frac{a_s - x_s^t/(T-t)}{b_s}\right)\,,\, \tilde{p}\right)$, where the inner maximum adjusts the price upwards if inventory is scarce and the outer minimum limits the upward adjustment by the null price.

Step 3: Let $d_r({p'}^t)$ denote the expected demand for the regular product at price point ${p'}^t$. In addition, let ${y'}^t$ denote the corresponding desired replenishment level for the regular product where ${y'}^t$ satisfies the critical fractile solution $F_{\epsilon_r}({y'}^t - d_r({p'}^t)) = h_r^-/(h_r^+ + h_r^-)$ with F_{ϵ_r} representing the cumulative distribution function for the additive uncertainty term for the regular product demand. Further, let $C({y'}^t - d_r)$ denote the expected holding and shortage costs for the regular product when the firm orders up to ${y'}^t$ and faces an expected demand d_r. If $K \geq {y'}^t - x_r^t$, then the firm's capacity is sufficient and thus no further price adjustment is necessary and we set the price as $p^t = {p'}^t$ and the replenishment level as $y^t = x_r^t + \left({y'}^t - x_r^t\right)^+$. If, on the other hand, $K < {y'}^t - x_r^t$, then the firm replenishes K units using all available capacity and we decrease the seasonal price to suppress some of the demand for the regular product. Specifically, we find p^t by solving $\max_{p^t}\ (p^t + \sum_t \beta^{(T-t)} h_s^+)(a_s - b_s p^t) + (p_r - c_r)(a_r + b_r p^t) - C(x_r^t + K - d_r(p^t))$ such that $p^t \leq {p'}^t$ and $d_r({p'}^t) - d_r(p^t) \leq {y'}^t - x_r^t - K$, where the second inequality limits units suppressed by the initial deficiency.

Next, we test the performance of the heuristic policy. To do so, we treat the previous problem instance given in Section 5.1 as the base case and systematically decrease and increase the value

Case #	p_r	c_r	h_r^+	h_s^+	h_r^-	h_s^-	a_r	a_s	b_r	b_s	Lower Initial Inv. ($Q_s = 15$) Optimal Profit	Heuristic Profit	% diff.	Higher Initial Inv. ($Q_s = 30$) Optimal Profit	Heuristic Profit	% diff.
0	25	10	2	2	20	50	2	10	0.1	0.2	827.4	820.4	0.8	796.3	795.7	0.1
1	15	10	2	2	20	50	2	10	0.1	0.2	549.8	542.7	1.3	571.2	567.9	0.6
2	50	10	2	2	20	50	2	10	0.1	0.2	1527.4	1520.2	0.5	1407.8	1407.8	0.0
3	25	5	2	2	20	50	2	10	0.1	0.2	966.6	959.3	0.8	912.7	912.4	0.0
4	25	15	2	2	20	50	2	10	0.1	0.2	688.4	681.6	1.0	682.6	681.0	0.2
5	25	10	1	2	20	50	2	10	0.1	0.2	836.1	829.1	0.8	805.0	804.3	0.1
6	25	10	5	2	20	50	2	10	0.1	0.2	805.6	798.7	0.9	774.5	773.8	0.1
7	25	10	2	1	20	50	2	10	0.1	0.2	857.1	851.4	0.7	874.7	874.5	0.0
8	25	10	2	5	20	50	2	10	0.1	0.2	747.9	727.6	2.7	587.4	583.1	0.7
9	25	10	2	2	10	50	2	10	0.1	0.2	829.1	822.0	0.9	797.8	797.2	0.1
10	25	10	2	2	40	50	2	10	0.1	0.2	826.4	819.4	0.8	795.5	794.8	0.1
11	25	10	2	2	20	75	2	10	0.1	0.2	824.8	811.4	1.6	796.3	795.1	0.2
12	25	10	2	2	20	100	2	10	0.1	0.2	823.7	802.5	2.6	796.2	794.6	0.2
13	25	10	2	2	20	50	1	10	0.1	0.2	752.6	745.6	0.9	721.3	720.7	0.1
14	25	10	2	2	20	50	4	10	0.1	0.2	952.3	936.7	1.6	944.6	943.5	0.1
15	25	10	2	2	20	50	2	8	0.1	0.2	611.6	604.5	1.2	504.4	504.4	0.0
16	25	10	2	2	20	50	2	12	0.1	0.2	1046.9	1039.0	0.8	1129.0	1123.8	0.5
17	25	10	2	2	20	50	2	10	0.05	0.2	694.2	687.3	1.0	701.7	699.3	0.3
18	25	10	2	2	20	50	2	10	0.15	0.2	945.8	931.5	1.5	896.4	896.2	0.0
19	25	10	2	2	20	50	2	10	0.1	0.15	1078.2	1071.3	0.6	1069.3	1068.9	0.0
20	25	10	2	2	20	50	2	10	0.1	0.25	675.9	666.9	1.3	633.5	632.2	0.2

Table 2 **Performance of the heuristic policy for lower and higher initial inventory levels for the seasonal product and across various parameters.**

for each parameter. For this five period problem, when the firm has ample initial inventory for the seasonal product, the firm's optimal price choice results in mean demands of approximately 4.4 and 5.2 units per period for the regular and seasonal products, respectively, i.e., its pricing implies that it expects to sell approximately 26 units of the seasonal product throughout the horizon. For each problem instance, we test the performance of the heuristic policy based on two different initial inventory levels for the seasonal product, a lower value of $Q_s = 15$, which allows an average availability of only three units per period and a higher value of $Q_s = 30$ which allows an average availability of six units per period. We set the initial inventory for the (replenishable) regular product to zero. Table 2 reports the optimal and the heuristic profit corresponding to these two starting inventory levels across all problem instances. To summarize, we first note that the average difference between the optimal and the heuristic policy across all cases is 1.2% and 0.2%, respectively, for the lower and higher initial seasonal inventory levels. We find that the heuristic policy has performed well compared to the optimal policy across all instances. We also observe that the heuristics performed better for higher initial seasonal inventory compared to lower initial seasonal inventory across each problem instance. This is expected since higher initial seasonal inventory levels require fewer price adjustments. (As a side note, for a very low initial seasonal inventory level of five units, which allows on average only one unit to be sold per period, the average difference across all instances is 2.2%.)

Before we conclude, we would like to comment on extensions to instances with nonstationary parameters. The steps we described earlier can easily be extended to handle nonstationary parameters. For example, if the seasonal product demand weakens towards the end of the horizon, the total expected demand in Step 2 and the allocations in case there is scarce inventory can be computed by accounting for the expected demand at each period accordingly. As an example, consider a problem instance in which the demand intercept for the seasonal product decreases by one unit per period, i.e., from ten to six over the five period horizon. For this instance, the difference between the heuristics and the optimal policy is 1.9% (optimal: 639.7 vs. heuristic: 627.4) and 0.0% (optimal: 526.9 vs. heuristic: 526.9) for the lower and higher seasonal inventory levels, respectively.

6. Extensions

6.1. Impact of Quality Differences on the Optimal Policy

Next, we would like to comment on how potential quality differences between products impact the optimal policy. Below, we first present a choice model assuming that the seasonal product is perceived to be of higher quality, e.g., a preferred limited edition product. We also recognize that in some settings the regular product instead may be deemed of higher quality, e.g. a new product replacing a retiring ('seasonal') product. We will comment on this latter case with the reverse ranking subsequently.

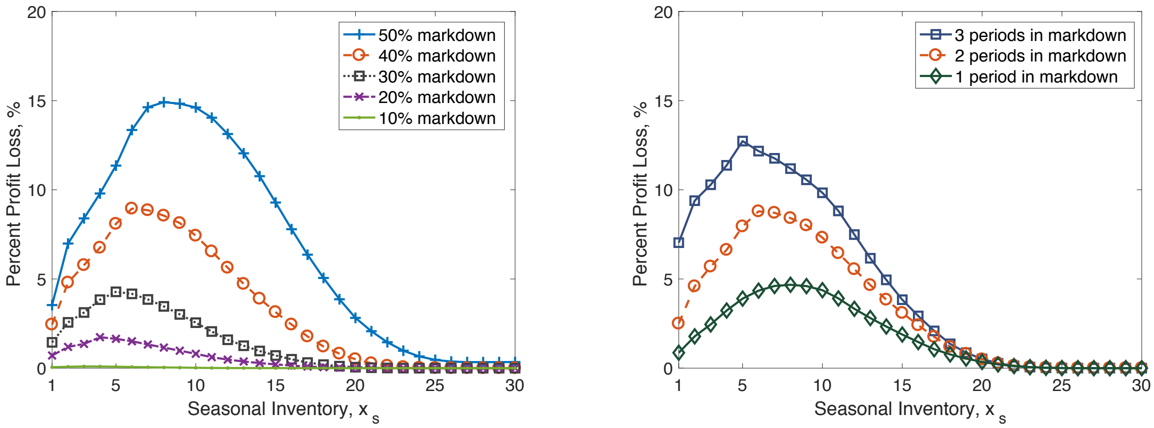
Following Mussa and Rosen (1978), Mantin et al. (2014), and Ceryan et al. (2017), consider a choice model in which a customer with valuation v receives a net utility of $q_i v - p_i$ when choosing product type i, for $i = \{r, s\}$, where q_i and p_i denote the quality level and price of product type i, respectively. Since we currently assume that the seasonal product is of higher quality, we have $q_s > q_r$. Hence, a consumer with valuation $v > (p_s - p_r)/(q_s - q_r)$ will prefer the seasonal product over the regular product. We assume v is uniformly distributed with density δ on $[0, \bar{v}]$ where $\delta\bar{v}$ is the total market size. Further, suppose that there exists other products in the market with quality and price levels $(\underline{p}, \underline{q})$ and $(\bar{p}, \bar{q})$ such that $\underline{p} < p_i < \bar{p}$ and $\underline{q} < q_i < \bar{q}$. Then, assuming that all products can coexist in the market, the choice model results in the following expected demand expressions for the products: $d_s(p_s, p_r) = \delta\left[\frac{\bar{p}}{(\bar{q}-q_s)} - \left(\frac{1}{\bar{q}-q_s} + \frac{1}{q_s-q_r}\right)p_s + \frac{1}{(q_s-q_r)}p_r\right]$ and $d_r(p_s, p_r) = \delta\left[\frac{\underline{p}}{(q_r-\underline{q})} + \frac{1}{(q_s-q_r)}p_s - \left(\frac{1}{q_s-q_r} + \frac{1}{q_r-\underline{q}}\right)p_r\right]$. (Note: it can be verified that for both products to be viable, the firm's price choice for the higher quality seasonal product exceeds the price of the regular product, and that as the quality difference between the products increases, the demand for each product becomes less sensitive to the price of the other product.) First, note that defining $a_s := \delta\left(\frac{\bar{p}}{(\bar{q}-q_s)} + \frac{p_r}{(q_s-q_r)}\right)$, $a_r := \delta\left(\frac{\underline{p}}{(q_r-\underline{q})} - \left(\frac{1}{q_s-q_r} + \frac{1}{q_r-\underline{q}}\right)p_r\right)$, $b_s := \delta\left(\frac{1}{\bar{q}-q_s} + \frac{1}{q_s-q_r}\right)$, and $b_r :$ $\delta\left(\frac{1}{(q_s-q_r)}\right)$ results in our original formulation, thus the optimal policy structure provided in Section 4 continues to hold.

Next, we would like to briefly comment on the sensitivity of the optimal policy with respect to the quality of the products. For settings where the seasonal product is of higher quality, our numerical tests indicate that, across all inventory states, both the optimal price for the seasonal product and the replenishment level for the regular product increases as the quality of the seasonal product increases. For example, consider a problem instance in which $q_r = 1$, $\bar{v} = 50$, $\delta = 0.1$, $\underline{q} = 0$, $\underline{p} = 0$, $\bar{q}$ and $\bar{p}$ are relatively large with $\bar{p}/\bar{q} = \bar{v}$, $p_r = 10$, $c_r = 5$, $h_s^+ = h_r^+ = 2$, $h_s^- = 50$, $h_r^- = 20$, $K = 8$, $T = 5$, and $\beta = 1$. As an example, for the initial inventory position $x_s = 10$ and $x_r = 0$ with five periods to go, we find that quality levels $q_s = \{1.50, 1.75, 2.00\}$ correspond to optimal seasonal prices $p_s = \{22.3, 30.7, 38.0\}$ and regular replenishment levels $y_r = \{3.1, 3.3, 3.4\}$, respectively. The underlying intuition is as follows. A higher quality for the seasonal product results in a stronger demand for the seasonal product and the firm responds to this strengthening of demand by increasing its price. Overall, this increased seasonal price also results in a stronger demand for the regular product, thus requiring a higher replenishment level. If on the other hand, the quality of the regular product is higher (and using accordingly updated demand-price relationships), we observe a different behavior. We find that increases in the quality of the regular product results in the firm to lower its price for the seasonal product. However, even though this price reduction acts to weaken the demand for the regular product, its impact does not outweigh the demand strengthening effect due to increased quality. Overall, the firm still experiences a higher demand for its regular product and thus increases its replenishment.

6.2. Price Markdowns

We also would like to briefly remark on the profit implications of markdown policies prevalent in retail settings. Here, we will limit our attention to gain preliminary insights into how the timing and depth of markdowns impact the firm's profit potential. To aid our exposition, we refer to the problem instance described earlier in Section 5.1.

First, we consider a setting in which the firm applies a markdown policy during the second half of a four period selling horizon (i.e., in the last two periods) by setting a price that reflects at least a certain percentage off from a maximum allowable price. That is, as opposed to having the previous period's price forming an upper bound on the current period price, we let the upper bound be pre-determined reflecting a certain percentage off the maximum price (for this problem instance, the



(a) Profit Loss (%) vs depth of markdowns

(b) Profit Loss (%) vs timing of markdowns

Figure 3 Percent Profit Loss across various (a) depths of markdowns, and (b) timing of markdowns

maximum price is 50). Figure 3 (a) displays the firm's percent profit loss across different markdown levels as a function of the available seasonal inventory while assuming zero initial inventory for the regular product. We compute the percent profit loss by comparing the optimal profit obtained with no pre-set markdowns to the optimal profit obtained taking into account the pre-set markdown limits. We observe that when the firm has a sufficiently large amount of seasonal inventory, the profit loss from markdowns is small. For example, for a starting seasonal inventory level of 25 units, the profit loss incurred by a firm applying markdown levels ranging from 10% to 50% is limited to less than 0.5%. This behavior is expected as a high inventory level causes the firm to select lower prices even if its price choice was not limited by a specific markdown policy. At lower inventory levels however, forcing a markdown may result in large profit losses. For example, the firm's profit loss can be up to 15% for a markdown of 50% when its inventory for the seasonal product is around 10 units. For very low inventory levels, the impact of a markdown policy in the second half of the horizon again has a relatively smaller impact as in these cases the firm is more likely to sell out before markdown periods begin, thus reducing the relevancy of a particular markdown policy. Figure 3 (b) shows how the firm's profit potential is influenced by the timing of markdowns. Specifically, it considers applying a 40% markdown as early as starting at three periods-to go until the end of the horizon to as late as only applying the markdown in the last period. We again observe a similar behavior (following similar reasoning as discussed above) that the timing of markdowns has limited consequences when the firm has higher inventory levels, the impact of the timing is greatest for moderately low inventory levels, and the impact is again relatively lower for very low inventory levels.

7. Further Asymmetries between Pricing and Replenishment

Finally, we would like to briefly extend our discussion to settings where (i) both products can be dynamically priced but only one allows replenishment, which we refer to as 'pricing with partial replenishment', and (ii) only one product can be dynamically priced but both allow replenishments, referred to as 'replenishment with partial pricing'.

7.1. Pricing with Partial Replenishment

In this section, we consider a setting where the firm can exercise full pricing control on all products but is only able to replenish one of the products. In other words, we extend our base model to also incorporate pricing decisions for the regular product. We retain much of the same notation but now treat the expected demand for the regular product, d_r^t, as an additional decision variable, which, together with d_s^t, specifies the optimal prices for both products, p_r^t and p_s^t, in period t through the (two price) demand-price model as described initially in the problem formulation section. As generally assumed in the related literature and to aid exposition, we assume identical cross price sensitivity coefficients within this section. We redefine the expected revenue functions as $R(d_r^t, d_s^t) := p_r^t(d_r^t, d_s^t) \cdot d_r^t + p_s^t(d_r^t, d_s^t) \cdot d_s^t$ and $\tilde{R}(d_r^t) := p_r^t(d_r^t, 0) \cdot d_r^t$, and the feasible sets as $\mathcal{F}(x_r^t) := \{(d_r^t, d_s^t, z_r^t) \,|\, x_r^t \le z_r^t + d_r^t \le x_r^t + K\}$ and $\tilde{\mathcal{F}}(x_r^t) := \{(d_r^t, z_r^t) \,|\, x_r^t \le z_r^t + d_r^t \le x_r^t + K\}$. With the terminal value defined as previously, the modified problem formulation is as follows:

$$V^t(x_r^t, x_s^t) = \begin{cases} \max_{d_r^t, d_s^t, z_r^t \in \mathcal{F}(x_r^t)} R(d_r^t, d_s^t) - c_r \cdot (z_r^t + d_r^t - x_r^t) - H_r(z_r^t) - H_s(x_s^t, d_s^t) \\ \qquad + \beta \, \mathrm{E}_{\epsilon_r^t, \epsilon_s^t} \Big[V^{t+1}\big(z_r^t - \epsilon_r^t, x_s^t - d_s^t - \epsilon_s^t\big) \mathbb{1}_{\{x_s^t - d_s^t - \epsilon_s^t > 0\}} \\ \qquad\qquad + \tilde{V}^{t+1}\big(z_r^t - \epsilon_r^t\big) \mathbb{1}_{\{x_s^t - d_s^t - \epsilon_s^t \le 0\}} \Big], & \text{if } x_s^t > 0 \\ \tilde{V}^t\big(x_r^t\big), & \text{if } x_s^t = 0 \end{cases} \tag{4}$$

where,

$$\tilde{V}^t(x_r^t) = \max_{d_r^t, z_r^t \in \tilde{\mathcal{F}}(x_r^t)} \tilde{R}(d_r^t) - c_r \cdot (z_r^t + d_r^t - x_r^t) - H_r(z_r^t) + \beta \, \mathrm{E}_{\epsilon_r^t} \big[\tilde{V}^{t+1}\big(z_r^t - \epsilon_r^t\big) \big] \tag{5}$$

In this modified formulation, depleted seasonal inventory reduces the subsequent periods to the single item joint pricing and replenishment problem given in (5), which is essentially the setting studied by Federgruen and Heching (1999). For periods with seasonal product availability, we provide the main results regarding the structure of the optimal policy below.

Theorem 4. *(a) Optimal replenishment for the regular product follows the state-dependent base-stock policy structure described in Theorem 1, i.e., it is optimal to replenish* $\min\big((\bar{x}_r^t(x_s^t) -$

$x_r^t)^+, K\big)$ units of the regular product where $\bar{x}_r^t(x_s^t)$ is strictly decreasing in x_s^t.

(b) It is optimal to apply a list price for the regular product if $\bar{x}_r^t(x_s^t) - K < x_r^t < \bar{x}_r^t(x_s^t)$, charge a surplus if $x_r^t < \bar{x}_r^t(x_s^t) - K$, and give a discount if $x_r^t > \bar{x}_r^t(x_s^t)$. The optimal price for the regular product is independent of both the regular and seasonal inventory levels if $\bar{x}_r^t(x_s^t) - K < x_r^t < \bar{x}_r^t(x_s^t)$ and decreases with either inventory level otherwise.

(c) The optimal seasonal price is independent of the regular product inventory level and decreases in its own inventory level if $\bar{x}_r^t(x_s^t) - K < x_r^t < \bar{x}_r^t(x_s^t)$. The seasonal price decreases in either inventory level otherwise.

We would like to highlight a few important points regarding the optimal policy structure described in Theorem 4. First, when the firm can both price and replenish the regular product, a list-price, base-stock type policy (as described in the related literature, e.g., Federgruen and Heching, 1999) remains optimal in the presence of a non-replenishable, dynamically priced, substitutable product. Second, for starting inventory levels for which the firm's capacity is sufficient to reach the desired base-stock level for the regular product, we find that the optimal price for the seasonal product remains decreasing in its own inventory and independent of the regular product's inventory (as in Theorem 2), while the optimal price for the regular product is independent of both inventory levels. Third, when the regular product has excess inventory, being able to lower the price for the regular product now also prompts the firm to decrease the price of the seasonal product as well. Previously, under the same demand price model but without the ability to alter and lower the price of the regular product, the firm's response to further excesses in regular product inventory was to increase the price of the seasonal product.

7.2. Replenishment with Partial Pricing

We also would like to consider a setting in which the firm is able to replenish both products but has pricing control for only one of the products. For continuity of notation, we continue to refer to the dynamically priced product as the seasonal product although it now also allows replenishment. We modify our original formulation by introducing a new variable z_s^t that denotes the target safety stock level for the seasonal product. We let K_r and K_s denote the replenishment capacity for the regular and the seasonal product, respectively. Consequently, the set of feasible values for decision variables can now be stated as $\mathcal{F}(x_r^t, x_s^t) := \{(d_s^t, z_r^t, z_s^t) \,|\, x_r^t \le z_r^t + a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t) \le x_r^t + K_r,\ x_s^t \le z_s^t + d_s^t \le x_s^t + K_s\}$. We let the terminal value function be defined as $V^T(x_r, x_s) = -c_r x_r^- - c_s x_s^-$. In addition, similar to the previously defined $H_r(z_r^t)$, we let $H_s(z_s^t)$ correspond to the expected holding and shortage costs for the seasonal product and now allow backorders for the seasonal product as well. The modified formulation and the corresponding structural results are as follows:

$$V^t(x_r^t, x_s^t) = \max_{d_r^t, z_r^t, z_s^t, \in \mathcal{F}(x_r^t, x_s^t)} R(d_s^t) - c_r\left(z_r^t + d_r^t(d_s^t) - x_r^t\right) - c_s\left(z_s^t + d_s^t - x_s^t\right) - H_r(z_r^t) - H_s(z_s^t)$$
$$+\beta\, \mathrm{E}_{\epsilon_r^t, \epsilon_s^t}\left[V^{t+1}\left(z_r^t - \epsilon_r^t, z_s^t - \epsilon_s^t\right)\right] \quad (6)$$

Theorem 5. *(a) It is optimal to replenish* $\min\left((\bar{x}_r^t(x_s^t) - x_r^t)^+, K_r\right)$ *units of the regular product and* $\min\left((\bar{x}_s^t(x_r^t) - x_s^t)^+, K_s\right)$ *units of the seasonal product where* $\bar{x}_r^t(x_s^t)$ *is strictly decreasing in* x_s^t *and* $\bar{x}_s^t(x_r^t)$ *is strictly decreasing in* x_r^t*.*
(b) It is optimal to charge a list price for the seasonal product if $\bar{x}_s^t(x_r^t) - K_s < x_s^t < \bar{x}_s^t(x_r^t)$ *and* $\bar{x}_r^t(x_s^t) - K_r < x_r^t < \bar{x}_r^t(x_s^t)$*. The seasonal price is independent of its own inventory if* $\bar{x}_s^t(x_r^t) - K_s < x_s^t < \bar{x}_s^t(x_r^t)$ *and is independent of the regular product inventory if* $\bar{x}_r^t(x_s^t) - K_r < x_r^t < \bar{x}_r^t(x_s^t)$*. Otherwise, the seasonal price is decreasing in its own inventory and increasing in the regular product inventory.*

Theorem 5 indicates that optimal replenishment for both products follow modified base-stock policies as described earlier. It is important to note that a base stock, list price policy for the seasonal product is only optimal if the regular product also requires replenishment and its capacity is sufficient. Otherwise, a list price is no longer optimal for the seasonal product and its optimal price is increasing with the regular product inventory. Further, allowing replenishment for the seasonal product generates a region of starting inventory positions in which its price is independent of its own inventory (when it is replenished with sufficient capacity), whereas in the original setting with no seasonal replenishment, the optimal price for the seasonal product was always decreasing in its own inventory.

7.3. Value of Pricing and Replenishment Flexibilities

Lastly, we would like to briefly comment on the additional value a firm can receive by implementing dynamic pricing for the regular product or by allowing further replenishment opportunities for the seasonal product. To do so, we numerically compute the optimal policies corresponding to formulations (4) and (5) in Section 7.1, and (6) in Section 7.2 and compare the results with the original setting. (To provide a fair comparison, when a setting assumes a static price for the regular product, we let this price be set equal to the list-price that would have been obtained through dynamic pricing. For all remaining parameters, we adopt the values as described in Section 5.) For each of the three settings, we consider a four-period problem and compute the optimal profit corresponding to 256 initial inventory positions where $x_s = \{0, 1, 2, ..., 15\}$ and $x_r = \{-5, ..., 10\}$.

We find that the original setting results in an average profit of 757.7 across the range of the initial inventory positions. If the firm also incorporates dynamic pricing decisions for the regular product, its average profit across the range of the initial inventory positions increases to 758.1. On the other hand, allowing replenishments for the seasonal product results in an average profit of 788.5 (i.e., a gain of 4.1% compared to the original setting). Overall, we find that allowing replenishment for the seasonal product can generally provide a greater improvement in the firm's profit compared to incorporating dynamic pricing for the regular product. This can be expected as a replenishable product requires narrower price adjustments (e.g., a list-price is applied within a certain region of inventory positions) and thus the benefit gained from dynamic pricing becomes more limited. We also find that the value of pricing flexibility for the regular product is higher when that product either faces capacity restrictions or excess inventories and that there is virtually no gain if the inventory position is intermediate. Regarding replenishment flexibility, we find that the value of a replenishment opportunity for the seasonal product is most pronounced when the seasonal inventory level is low and that the value of replenishment flexibility decreases with seasonal inventory.

8. Conclusions

Our focus in this paper has been asymmetric pricing and replenishment controls for a firm offering substitutable products. Such asymmetries arise when a firm faces restrictions in its ability to either replenish or adjust the prices of some of its products. We mainly consider a setting in which the firm may dynamically adjust the price of a perishable seasonal item that has no further supply, but is required to maintain a stable price for another partially substitutable item that can be regularly replenished over a finite selling horizon. We characterize the firm's optimal dynamic pricing and initial ordering decisions for the seasonal perishable product, and the replenishment decisions for the regular product. We find that the price of the seasonal product decreases in its own inventory while it is partially decoupled from the regular product inventory, and show that the optimal price does not necessarily possess time monotonicity for a given inventory position. Through a numerical study, we demonstrate the drivers behind the temporal price reversal behavior. Further, we utilize the insights gained through the optimal policy characterization to develop a simple and effective heuristic policy. We also provide additional insights regarding how potential differences in quality perceptions between the products impact the optimal policy and the profit implications of markdown policies. Together with extensions that also consider further types of asymmetries resulting in pricing with partial replenishment or replenishment with partial pricing, we believe the insights provided in the study along with the well-performing heuristic policy would be of practical importance to firms operating under circumstances where dynamic pricing and replenishment decisions may not all be available for all products within an assortment of substitutable products.

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Online Appendix

Asymmetric Pricing and Replenishment Controls for Substitutable Products

Oben Ceryan

Notation: Below, we provide a list of key notation used in the text and in the Appendix.

(x_r^t, x_s^t)	:	inventory position at the beginning of period t for the regular (x_r^t) and the seasonal product (x_s^t)
p_r	:	price of the regular product
p_s^t	:	price of the seasonal product
d_r^t	:	mean demand for the regular product
d_s^t	:	mean demand for the seasonal product (decision variable)
y_r^t	:	replenishment level for the regular product
z_r^t	:	target safety stock level for the regular product (decision variable, $z_r^t = y_r^t - d_r^t$)
K	:	replenishment capacity per period for the regular product
$V^t(x_r^t, x_r^t)$	:	optimal value function starting at state (x_r^t, x_s^t) at the beginning of period t
$\tilde{V}^t(x_r^t)$	:	optimal value function starting at state (x_r^t) at the beginning of period t with depleted seasonal inventory

Proofs of Theorems:

In order to characterize the optimal policy structure, we first state the following Inductional Assumption:

Inductional Assumption:

1. The value function $V^{t+1}(x_r, x_s)$ possesses the following properties:

   (a) $V^{t+1}(x_r, x_s)$ is jointly concave,

   (b) $V^{t+1}(x_r, x_s)$ is submodular and satisfies the following modified diagonal dominance conditions: $\frac{\partial^2 V^{t+1}(x_r,x_s)}{\partial x_r^2} \leq \left(\frac{b_s}{b_r}\right) \frac{\partial^2 V^{t+1}(x_r,x_s)}{\partial x_r \partial x_s} \leq 0$ and $\frac{\partial^2 V^{t+1}(x_r,x_s)}{\partial x_s^2} \leq \left(\frac{b_r}{b_s}\right) \frac{\partial^2 V^{t+1}(x_r,x_s)}{\partial x_r \partial x_s} \leq 0$,

   (c) $\left.\frac{\partial V^{t+1}(x_r,x_s)}{\partial x_s}\right|_{x_s=0} \leq h_s^-$.

2. The value function $\tilde{V}^{t+1}(x_r)$ is concave.

After showing the optimal policy structure for period t, we will show that the above properties for the value functions survive the dynamic programming recursion and extend to the previous period as well. In the following, we assume twice continuous differentiability and, for expositional clarity, we introduce and let $V_1^t(x_r, x_s)$ and $V_2^t(x_r, x_s)$ to represent $\frac{\partial V^t(x_r,x_s)}{\partial x_r}$ and $\frac{\partial V^t(x_r,x_s)}{\partial x_s}$, respectively. For the second partials, we let $V_{11}^t(x_r, x_s)$ to denote $\frac{\partial^2 V^t(x_r,x_s)}{\partial x_r^2}$, and $V_{12}^t(x_r, x_s)$ to denote $\frac{\partial^2 V^t(x_r,x_s)}{\partial x_r x_s}$. We define $V_{21}^t(x_r, x_s)$ and $V_{22}^t(x_r, x_s)$, and $\tilde{V}_{11}^t(x_r)$ similarly.

Proof of Theorem 1:

We start by introducing and defining $J^t(x_s^t, d_s^t, z_r^t)$ for $x_s^t > 0$ such that:

$$\begin{aligned} J^t(x_s^t, d_s^t, z_r^t) = \; & R(d_s^t) \; - c_r^t\left(z_r^t + d_r^t(d_s^t)\right) - H_r(z_r^t) - H_s(x_s^t, d_s^t) \\ & + \beta \, \mathrm{E}_{\epsilon_r^t, \epsilon_s^t}\left[V^{t+1}\left(z_r^t - \epsilon_r^t, x_s^t - d_s^t - \epsilon_s^t\right)\mathbb{1}_{\{x_s^t - d_s^t - \epsilon_s^t > 0\}} + \tilde{V}^{t+1}\left(z_r^t - \epsilon_r^t\right)\mathbb{1}_{\{x_s^t - d_s^t - \epsilon_s^t \leq 0\}}\right] \end{aligned} \tag{7}$$

Then, we can write (1) as

$$V^t(x_r^t, x_s^t) = \begin{cases} c_r^t x_r^t + \max_{d_s^t, z_r^t \in \mathcal{F}(x_r^t)} J^t(x_s^t, d_s^t, z_r^t) & \text{if } x_s^t > 0 \\ \tilde{V}^t(x_r^t), & \text{if } x_s^t = 0 \end{cases} \tag{8}$$

where,

$$\tilde{V}^t(x_r^t) = \max_{z_r^t \in \tilde{\mathcal{F}}(x_r^t)} \tilde{R} - c_r^t \cdot (z_r^t + \tilde{d}_r^t - x_r^t) - H_r(z_r^t) + \beta\, \mathrm{E}_{\epsilon_r^t}\left[\tilde{V}^{t+1}\left(z_r^t - \epsilon_r^t\right)\right] \tag{9}$$

We first show that $J^t(x_s^t, d_s^t, z_r^t)$ is strictly concave in d_s^t and z_r^t. Note that, we have $\frac{\partial^2 J^t(\cdot)}{\partial {z_r^t}^2} < 0$ and $\frac{\partial^2 J^t(\cdot)}{\partial {d_s^t}^2} < 0$ due to the Induction Assumption. Similarly, the Induction Assumption also results in the determinant of the Hessian matrix to be strictly positive, therefore we have $J^t(x_s^t, d_s^t, z_r^t)$ strictly concave in d_s^t and z_r^t. Hence, there exists is a unique pair (d_s^t, z_r^t) that maximizes the profit function. Next, we would like to explore how the optimal solution depends on the initial state. To do so, we introduce the Lagrangian variables $\lambda_1^t \geq 0$, $\lambda_2^t \geq 0$, and $\mu^t \geq 0$ associated with the constraints $x_r^t \leq z_r^t + a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t)$, $z_r^t + a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t) \leq x_r^t + K$, and $d_s^t \geq 0$, respectively as indicated by $\mathcal{F}(x_r^t)$ in (1). Recall that the expression $z_r^t + a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t)$ corresponds to the inventory of the regular product after replenishment. Notice that, for any $K > 0$, the constraints $x_r^t \leq z_r^t + a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t)$ and $z_r^t + a_r^t + \frac{b_r}{b_s}(a_s^t - d_s^t) \leq x_r^t + K$ cannot be simultaneously active, thus λ_1^t and λ_2^t cannot simultaneously be nonzero. Hence, if we introduce a non-restricted variable λ^t defined as $\lambda^t = \lambda_1^t - \lambda_2^t$, any value of λ^t will uniquely determine the values of λ_1^t and λ_2^t. For example, $\lambda^t > 0$ indicates that $\lambda_1^t > 0$ and $\lambda_2^t = 0$. Similarly, $\lambda^t < 0$ indicates that $\lambda_1^t = 0$ and $\lambda_2^t > 0$. Finally, $\lambda^t = 0$ implies that $\lambda_1^t = \lambda_2^t = 0$. The first order conditions are as follows:

$$\frac{\partial J^t(\cdot)}{\partial z_r^t} + \lambda^t = 0 \tag{10}$$

$$\frac{\partial J^t(\cdot)}{\partial d_s^t} - \frac{b_r}{b_s}\lambda^t + \mu^t = 0 \tag{11}$$

We distinguish six cases based on whether λ^t is positive, zero, or negative, and whether μ^t is positive or zero.

Consider first the case for which $\lambda^t = 0$ and $\mu^t = 0$. Then, optimal d_s^{t*} and z_r^{t*} simultaneously solves $\frac{\partial J^t(x_s^t, d_s^{t*}, z_r^{t*})}{\partial z_r^t} = 0$ and $\frac{\partial J^t(x_s^t, d_s^{t*}, z_r^{t*})}{\partial d_s^t} = 0$. Next, we identify how d_s^{t*} and z_r^{t*} changes with respect to x_r^t and x_s^t. For this case, we immediately have $\frac{\partial d_s^{t*}}{\partial x_r^t} = 0$ and $\frac{\partial z_r^{t*}}{\partial x_r^t} = 0$. Consequently, both the optimal seasonal price p_s^{t*} and the replenishment level for the regular product y_r^{t*} are independent of the initial regular product inventory. Next, differentiating the first order conditions (10) and (11) with respect to x_s^t leads to the price of the seasonal product and the replenishment level for the regular product to be strictly decreasing in x_s^t.

Next, consider the case for which $\lambda^t < 0$ and $\mu^t = 0$. Since for this case, the capacity is binding, we have $y_r^{t*} = x_r^t + K$. To find how the optimal price changes with respect to x_r^t, we differentiate the first order conditions along with the constraint $y_r^{t*} = x_r^t + K$ and solve three equalities for the three variables $\frac{\partial z_r^{t*}}{\partial x_r^t}$, $\frac{\partial d_s^{t*}}{\partial x_r^t}$, and $\frac{\partial \lambda^{t*}}{\partial x_r^t}$. The remaining analysis is similar to the previous case and we omit the details for brevity. For this case, we find $\frac{\partial z_r^{t*}}{\partial x_r^t} > 0$ and $\frac{\partial d_s^{t*}}{\partial x_r^t} < 0$. Therefore, we also have $\frac{\partial p_s t*}{\partial x_s^t} > 0$, indicating that the price of the seasonal product increases with the regular inventory. In addition, we also find $\frac{\partial \lambda^{t*}}{\partial x_r^t} > 0$. Similarly, by differentiating the first order conditions with respect to x_s^t, we get $\frac{\partial z_r^{t*}}{\partial x_s^t} > 0$, $\frac{\partial d_s^{t*}}{\partial x_s^t} > 0$, and $\frac{\partial \lambda^{t*}}{\partial x_s^t} > 0$. The case for which $\lambda^t > 0$ and $\mu^t = 0$, implies $y_r^{t*} = x_r^t$ and the monotonicities of the decision variables are identical to the case with $\lambda^t > 0$ and $\mu^t = 0$.

Now consider the case for which $\lambda^t = 0$ and $\mu^t > 0$. This indicates that $d_s^{t*} = 0$. Differentiating the first order conditions and solving for the monotonicities of the decision variables in a similar fashion leads to $\frac{\partial z_r^{t*}}{\partial x_r^t} = 0$, $\frac{\partial z_r^{t*}}{\partial x_s^t} \leq 0$, $\frac{\partial \mu^{t*}}{\partial x_r^t} = 0$, $\frac{\partial \mu^{t*}}{\partial x_s^t} > 0$. Since, $d_s^{t*} = 0$, we have $\frac{\partial p_s^{t*}}{\partial x_r^t} = \frac{\partial p_s^{t*}}{\partial x_s^t} = 0$. In addition, we consequently have $\frac{\partial y_r^{t*}}{\partial x_r^t} = 0$ and $\frac{\partial y_r^{t*}}{\partial x_s^t} \leq 0$.

Finally, we define $\bar{x}_r^t(x_s^t)$ to indicate the desired replenishment level for the regular product when initial inventory or capacity is not binding, i.e. $\bar{x}_r^t(x_s^t) = y_r^{t*}$. As shown, y_r^{t*}, and thus, $\bar{x}_r^t(x_s^t)$ is decreasing in x_s^t. When the initial inventory level for the regular product is below $\bar{x}_r^t(x_s^t)$, the firm replenishes up to $\bar{x}_r^t(x_s^t)$. If the initial inventory level for the regular product is less than $\bar{x}_r^t(x_s^t) - K$, the firm uses the replenishment capacity to its full extent and brings the inventory level to $x_r^t + K$. Otherwise, if the regular inventory is already above $\bar{x}_r^t(x_s^t)$, then no replenishment takes place. When the seasonal inventory is depleted, the firm's optimal replenishment is of a base-stock policy, a classical inventory result, where, following our notation, we denote the base stock level with $\bar{x}_r^t(0)$.

Proof of Theorem 2:

Theorem 2 follows immediately from the results obtained within the proof of Theorem 1. Specifically, when $\lambda^t = 0$ and $\mu^t = 0$, we had $\frac{\partial p_s t*}{\partial x_r^t} = 0$ and $\frac{\partial p_s t*}{\partial x_s^t} < 0$. That is, in this region the price of the seasonal product is independent of the regular product's inventory and is strictly decreasing with its own inventory. When $\mu^t > 0$, the demand is set to zero with the corresponding null price. When $\lambda^t < 0$ and $\mu^t = 0$, i.e., when capacity limits firm's replenishment, we had $\frac{\partial p_s t*}{\partial x_r^t} > 0$ and $\frac{\partial p_s t*}{\partial x_s^t} < 0$. In other words, in this region, the price of the seasonal product is strictly increasing in the regular product inventory and remains strictly decreasing in its own inventory. The results for the region with $\lambda^t > 0$ and $\mu^t = 0$ is similar, and also with $\frac{\partial p_s t*}{\partial x_r^t} > 0$ and $\frac{\partial p_s t*}{\partial x_s^t} < 0$. That is, when the firm has excess stock for the regular product, its price for the seasonal product increases.

Completing the Induction:

We now verify that the Inductional Assumptions survive the dynamic programming recursions.

Lemma 1. *: The properties on the value functions as outlined in the Inductional Assumption hold for period t.*

Proof: We start by showing the proof for Part 1 (ii), i.e., showing that $V^t(x_r, x_s)$ is submodular and satisfies the following modified diagonal dominance conditions:
$\frac{\partial^2 V^t(x_r,x_s)}{\partial x_r^2} \le \left(\frac{b_s}{b_r}\right) \frac{\partial^2 V^t(x_r,x_s)}{\partial x_r \partial x_s} \le 0$ and $\frac{\partial^2 V^t(x_r,x_s)}{\partial x_s^2} \le \left(\frac{b_r}{b_s}\right) \frac{\partial^2 V^t(x_r,x_s)}{\partial x_r \partial x_s} \le 0$. Through the Envelope Theorem, we have $V_{11}^t := \frac{\partial^2 V^t(x_r,x_s)}{\partial x_r^2} = -\frac{\partial \lambda^{t*}}{\partial x_r}$, $V_{12}^t := \frac{\partial^2 V^t(x_r,x_s)}{\partial x_r \partial x_s} = -\frac{\partial \lambda^{t*}}{\partial x_s}$. We can also similarly write:

$$V_{21}^t := \frac{\partial^2 V^t(x_r,x_s)}{\partial x_s \partial x_r} = (h_s^+ + h_s^-) f_s^t(x_s^t - d_s^t)\left(\frac{\partial d_s t*}{\partial x_r^t}\right) + \beta \iint_{S_1} \left(V_{21}^{t+1}\left(\frac{\partial z_r t*}{\partial x_r^t}\right) - V_{22}^{t+1}\left(\frac{\partial d_s t*}{\partial x_r^t}\right)\right) f_r^t(\xi_r) f_s^t(\xi_s) d(\xi_r) d(\xi_s)$$

$$\begin{aligned} V_{22}^t := \frac{\partial^2 V^t(x_r,x_s)}{\partial x_s^2} = & \left(-(h_s^+ + h_s^-) f_s^t(x_s^t - d_s^t) + \beta \int_{\underline{\epsilon}_r}^{\bar{\epsilon}_r} V_2^{t+1}(.,0) f_s^t(x_s^t - d_s^t) f_r^t(\xi_r) d(\xi_r)\right)\left(1 - \frac{\partial d_s t*}{\partial x_s^t}\right) \\ & + \beta \iint_{S_1} \left(V_{21}^{t+1}\left(\frac{\partial z_r t*}{\partial x_s^t}\right) - V_{22}^{t+1}\left(1 - \frac{\partial d_s t*}{\partial x_s^t}\right)\right) f_r^t(\xi_r) f_s^t(\xi_s) d(\xi_r) d(\xi_s) \end{aligned}$$

where $S_1 := \{(\xi_r, \xi_s) : \underline{\epsilon}_r \le \xi_r^t \le \bar{\epsilon}_r$ and $\underline{\epsilon}_s \le \xi_s^t \le x_s^t - d_s^{t*}\}$. When $\lambda^t = 0$ and $\mu^t = 0$, we have $\frac{\partial^2 V^t(x_r,x_s)}{\partial x_r^2} = 0$, $\frac{\partial^2 V^t(x_r,x_s)}{\partial x_r \partial x_s} = \frac{\partial^2 V^t(x_r,x_s)}{\partial x_s \partial x_r} 0$, and $\frac{\partial^2 V^t(x_r,x_s)}{\partial x_s^2} < 0$. Therefore, submodularity and modified diagonal dominance holds. When $\lambda^t > 0$ and $\mu^t = 0$, substituting in the earlier derived expressions for $\frac{\partial z_r t*}{\partial x_r^t}$, $\frac{\partial z_r t*}{\partial x_s^t}$, $\frac{\partial d_s t*}{\partial x_r^t}$, $\frac{\partial d_s t*}{\partial x_s^t}$, and $\frac{\partial \lambda^{t*}}{\partial x_r^t}$, $\frac{\lambda^{t*}}{\partial x_s^t}$ establishes $V_{11}^t \le \left(\frac{b_s}{b_r}\right) V_{12}^t \le 0$, and that $V_{22}^t \le \left(\frac{b_r}{b_s}\right) V_{12}^t \le 0$, verifying submodularity and the modified diagonal dominance. The cases for $\lambda^t > 0$ and $\mu^t = 0$, and when $\mu^t > 0$ are similar and omitted for brevity.

Part 1 (i) stating that $V^t(x_r, x_s)$ is jointly concave follows immediately through nonpositive diagonal elements, diagonal dominance, and submodularity.

For Part 1 (iii), we have

$$\begin{aligned} \left.\frac{\partial V^t(x_r,x_s)}{\partial x_s}\right|_{x_s=0} & = -h_s^+ F_s^t(0) + h_s^- - h_s^- F_s^t(0) + \beta \int_{\underline{\epsilon}_r}^{\bar{\epsilon}_r} \int_{\underline{\epsilon}_s}^{0} V_2^{t+1}(.,0) f_s^t(0) f_r^t(\xi_r) d(\xi_s) d(\xi_r) \\ & = -h_s^+ F_s^t(0) + h_s^- + \beta \int_{\underline{\epsilon}_r}^{\bar{\epsilon}_r} \int_{\underline{\epsilon}_s}^{0} (-h_s^- + V_2^{t+1}(.,0)) f_s^t(0) f_r^t(\xi_r) d(\xi_s) d(\xi_r) < h_s^- \end{aligned}$$

where the inequality follows from Part 1 (iii) of the Inductional Assumption.