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Approximate comparison of functions computed by distance automata

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Abstract Distance automata are automata weighted over the semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$ (the tropical semiring). Such automata compute functions from words to $\mathbb{N} \cup \{\infty\}$. It is known from Krob that the problems of deciding '$f \leqslant g$' or '$f = g$' for f and g computed by distance automata is an undecidable problem. The main contribution of this paper is to show that an approximation of this problem is decidable.

We present an algorithm which, given $\varepsilon > 0$ and two functions f, g computed by distance automata, answers "yes" if $f \leqslant (1-\varepsilon)g$, "no" if $f \not\leqslant g$, and may answer "yes" or "no" in all other cases.

The core argument behind this quasi-decision procedure is an algorithm which is able to provide an approximated finite presentation of the closure under products of sets of matrices over the tropical semiring.

Lastly, our theorem of affine domination gives better bounds on the precision of known decision procedures for cost automata, when restricted to distance automata.

Keywords Distance automata, asymptotic behaviour, weighted automata, tropical semiring, comparison

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Contents

1 Introduction

One way to see language theory, and in particular the theory of regular languages, is as a toolbox of constructions and decision procedures allowing high level handling of languages. These high level operations can then be used as black-boxes in various decision procedures, such as in verification.

Since the early times of automata theory, the need for the effective handling of functions rather than sets (as languages) was already apparent. Schützenberger proposed already in the sixties models of finite state machines used for computing functions. These are now known as weighted automata [9] and are the subject of much attention from the research community. In general, weighted automata are non-deterministic automata, with transitions carrying weights over some semiring $(S, \oplus, \otimes)$. The value computed by such an automaton over a given word is then the sum (under $\oplus$) over every run over this word of the product (under $\otimes$) of the weights of the transitions along the run.

Several instances of this model are very relevant for modelling the behaviour of systems, and henceforth attract much attention. This is in particular the case of probabilistic automata (over the semiring $(\mathbb{R}^+, +, \times)$ with some additional stochastic assumption enforcing weights to remain in $[0, 1]$), and distance automata which are automata weighted over the semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$. In such an automaton, each transition is labelled with a non-negative integer (usually 0 or 1), and the weight of a word is the minimum over all possible paths going from an initial state to a final state, of the sum of the weights over the transitions. The value is infinite if there is no accepting run. These automata naturally capture some optimisation problems since computing the value of an input word amounts to find a path of minimal weight.

The subject of this paper is to develop algorithmic tools for distance automata, and more precisely for comparing distance automata. We know from the beginning that exact comparison is beyond reach.

Theorem 1 (Krob [6]) *The problem of determining, given two functions f, g computed by distance automata, whether $f = g$ is undecidable. The problem $f \leqslant g$ is also undecidable, even if g is deterministic.*

Moreover, the problem of determining, given two functions f, g computed by distance automata, whether there exists $a \in \mathbb{N}$ such that $f \leqslant ag + a$ is also undecidable.

Despite this, some positive results exist but for a comparison relation less precise than inequality, namely *domination.* Given two functions $f, g : \mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$, we say that *$f$ is dominated* by g (and we note $f \preccurlyeq g$) if there is a function $\alpha : \mathbb{N} \to \mathbb{N}$, extended with $\alpha(\infty) = \infty$, such that

$$f \leqslant \alpha \circ g \ .$$

Moreover, if α is a polynomial, we say that *f is polynomially dominated by g.* The following theorem shows some good properties of the domination relation.

Theorem 2 ([2] extending results and techniques from [4, 8, 11, 5, 1]) *The domination of functions computed by distance automata is decidable. Furthermore, for such functions, domination is equivalent to polynomial domination* [1].

The motivation of this work is to improve Theorem 2 and to answer the following question:

> Are there some "approximations" of the comparison of functions computed by distance automata that are finer than domination but still decidable?

We answer positively this question in two ways. Firstly, we show:

Theorem 3 (affine domination) *Given two functions f and g computed by distance automata, if f is dominated by g then f is affinely dominated by g, i.e., $f \leqslant \alpha \circ g$ for some polynomial α of degree* 1.

A consequence of this theorem is that the decision procedure provided by Theorem 2 in fact decides affine domination, which is finer than polynomial domination[2].

Our second, and main contribution is an even more accurate decision-like procedure. We say that an algorithm, given two functions f and g and some real $\varepsilon > 0$, *ε-approximates inequality* if:

- if $f \leqslant (1-\varepsilon)g$, the output is "yes",
- if $f \not\leqslant g$, the output is "no",
- otherwise the output can be either "yes" or "no".

Hence, if such an algorithm answers "yes", one has a guaranty that $f \leqslant g$. Conversely if f is ε-inferior to g (meaning $f \leqslant (1-\varepsilon)g$), one is sure that the algorithm answers "yes". Our second and main result reads as follows:

Theorem 4 (approximate comparison) *There is an algorithm which ε-approximates the inequality of functions computed by distance automata.*

This result is a consequence of the core theorem (Theorem 6) stating that it is possible, given a set of matrices X in the tropical semiring, to approximate (in a suitable way) the set

$$\left\{ \frac{1}{k}(M_1 \otimes \cdots \otimes M_k) \ : \ M_1, \ldots, M_k \in X \right\}$$

where $\otimes$ denotes the product of matrices. More precisely, the core theorem states that it is possible to approximate the upper envelope of the set of pairs

$$\{(M_1 \otimes \cdots \otimes M_k, k) \ : \ M_1, \ldots, M_k \in X\}$$

for a suitable notion of approximation. This core theorem, Theorem 6, requires several definitions to be introduced beforehand.

[1] Technically, polynomial domination is not stated in [2], but can be derived directly from the proofs which explicitly compute the function α using operations preserving polynomials.

[2] Theorem 2 holds for more general classes of automata, cost automata, for which affine domination does not hold. Affine domination is specific to distance automata.

Organization of the paper.

In Section 2, we present some classical definitions and we introduce distance automata. We also prove theorem of affine domination (Theorem 3). In Section 3, we formally state our core theorem (Theorem 6), and we apply it for answering our original motivation, and show the decidability of the approximate comparison between distance automata. Finally, Section 4 is devoted to the proof of the core theorem and Section 5 concludes the paper.

2 Comparing distance automata

In this section, we consider the problem of comparing the functions computed by distance automata. In particular, we establish Theorem 3, and we reduce Theorem 4 to our core theorem, Theorem 6.

We start by introducing the basic definitions and describing distance automata, and their relationship with matrices over the tropical semiring.

2.1 Standard definitions

A *semigroup* $(S,\cdot)$ is a set S equipped with an associative, binary *product* operation "$\cdot$". If the product has furthermore a neutral element, it is called a *monoid.* The monoid is called *commutative* when $\cdot$ is commutative. An *idempotent* in a monoid is an element e such that $e\cdot e=e$. Given a subset A of a semigroup, $\langle A\rangle$ denotes the closure of A under product, *i.e.,* the least sub-semigroup that contains A. Given two subsets X,Y of a semigroup, $X\cdot Y$ denotes the set $\{a\cdot b\ :\ a\in X,\ b\in Y\}$.

A *semiring* is a set S equipped with two binary operations $\oplus$ and $\otimes$ such that $(S,\oplus)$ is a commutative monoid with neutral element 0, $(S,\otimes)$ is a monoid of neutral element 1, 0 is absorbing for $\otimes$ (i.e., $x\otimes 0=0\otimes x=0$) and $\otimes$ distributes over $\oplus$. We will consider three semirings: $(\mathbb{R}^+\cup\{\infty\},\min,+)$, denoted $\overline{\mathbb{R}^+}$, its restriction to $\mathbb{N}\cup\{\infty\}$, denoted $\overline{\mathbb{N}}$, and its restriction to $\{0,\infty\}$ denoted $\mathbb{B}$. The semiring $\mathbb{B}$ is called the *Boolean semiring*, since if we identify 0 with "true" and ∞ with "false", then $\oplus$ is the disjunction and $\otimes$ the conjunction. Remark that in the three cases, the "0" is ∞, and the "1" is 0.

Let S be one of the above semirings. The set of matrices with m rows and n columns over S is denoted $\mathcal{M}_{m,n}(S)$. For $M\in\mathcal{M}_{m,n}(S)$, we denote by $\phi(M)$ the matrix over $\mathbb{B}$ in which all entries of M different from ∞ are replaced by 0. We define the multiplication $A\otimes B$ of two matrices A,B (provided the number n of columns of A equals the number of rows of B) as usual by:

$$(A\otimes B)_{i,j}=\bigoplus_{0<k\leqslant n}(A_{i,k}\otimes B_{k,j})=\min_{0<k\leqslant n}(A_{i,k}+B_{k,j})\ .$$

For a positive integer k, we also use the notation $M^k=\underbrace{M\otimes\cdots\otimes M}_{k\text{ times}}$.

For $\lambda \in S$, we denote by λA the matrix such that $(\lambda A)_{i,j} = \lambda A_{i,j}$ for all i, j, with the convention that $\lambda\infty = \infty$ (the standard product is used here, not the one of the semiring). Note in particular that $\phi(M) = 0M$. We also denote by $B + \lambda$ the matrix such that $(B + \lambda)_{i,j} = B_{i,j} + \lambda$ for all i, j. Finally, we write $A \leqslant B$ if $A_{i,j} \leqslant B_{i,j}$ for all i, j where the ordering is the natural ordering on numbers.

We cite here basic facts used in the proofs.

Lemma 1 *Let $k, \ell, m \in \mathbb{N} \cup \{\infty\}$, $a \in \mathbb{N}$ and $M, N, M', N' \in \mathcal{M}_{n,n}(\mathbb{N})$.*

- *If $k \leqslant \ell$ and $M \leqslant N$ then $kM \leqslant \ell N$.*
- *If $M \leqslant N$ and $M' \leqslant N'$ then $M \otimes M' \leqslant N \otimes N'$.*
- $\ell M \otimes \ell N = \ell(M \otimes N)$
- *If the greatest finite entry of M is a then $(k + \ell)M \leqslant kM + \ell a$.*
- $(\ell M + k) \otimes (mN + k) = (\ell M \otimes mN) + 2k$

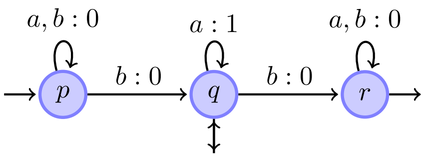
2.2 Distance automata

An *alphabet* is a finite set. The set of *words* over an alphabet $\mathbb{A}$ is denoted $\mathbb{A}^*$. A *distance automaton* is a tuple $(\mathbb{A}, Q, I, F, T)$, where Q is a finite set of *states* (that we can assume to be $\{1, \ldots, n\}$) where I (resp. T) is a row-vector (resp. column-vector) indexed by Q, and F is a morphism from words to $\mathcal{M}_{n,n}(\overline{\mathbb{N}})$. The function f computed by a distance automaton $(\mathbb{A}, Q, I, F, T)$ over an input word u is:

$$\begin{aligned} f : \mathbb{A}^* &\to \overline{\mathbb{N}} \\ u &\mapsto I \otimes F(u) \otimes T \ . \end{aligned}$$

We assume from now on that the initial and final vectors I, T of distance automata only range over $\{0, \infty\}$. The theorems are equally true without this assumption, but this simplifies slightly the proof. In practice the theorems without this restriction can be obtained by simple reductions to this case.

We have defined so far distance automata in terms of matrices. One can see this object in a more "automaton" form as follows. There is a transition labelled $a : x$ from state p to state q if $x < \infty$ and $x = F(a)_{p,q}$. A state p is *initial* if $I_{1,p} = 0$. It is *final* if $T_{i,1} = 0$. An example of distance automaton is as follows:



One can redefine the function computed by a distance automaton as follows. A *run* of an automaton over a word $a_1 \ldots a_k$ is a sequence $p_0, \ldots, p_k$ of states. The *weight of a run* is the sum of the weights of its transitions, *i.e.*,

$F(a_1)_{p_0,p_1}+\cdots+F(a_k)_{p_{k-1},p_k}$. Remark that if there is some non-existing transition in this sequence, say from p_{i-1} to p_i, this means that $F(a_i)_{p_{i-1},p_i}=\infty$, and as a consequence the run has an infinite weight. A run is *accepting* if p_0 is initial and p_k is final. One defines the function accepted by the automaton as:

$$\begin{aligned} f : \mathbb{A}^* &\to \overline{\mathbb{N}} \\ u &\mapsto \inf\{\mathrm{weight}(\rho) \ : \ \rho \text{ accepting run over } u\} \ . \end{aligned}$$

This definition is equivalent to the matrix version presented above.

For instance, the function computed by the above distance automaton associates to each word $u = a^{n_0}ba^{n_1}\dots ba^{n_k}$ the value $\min(n_0,\dots,n_k)$.

2.3 Superior limits

In this section, we present Theorem 5. This result is a refinement of known proofs concerning distance automata and will prove useful for further reductions.

In order to define the superior limit of a set of matrices, a topology is required. We consider the topology of one point compactification of the naturals, where the basic open sets are the singletons $\{n\}$ and the intervals $[n,\infty)$ for $n \in \mathbb{N}$.

Given $X \subseteq \mathcal{M}_{n,n}(\overline{\mathbb{N}})$, a matrix N belongs to the *superior limit* of X if:

- N is the limit of some sequence of matrices from X,
- there exists no $M \in X$ such that $M > N$.

Let us call $\limsup(X)$ the set of matrices in the superior limit of S. Following from Higman's theorem, we can observe that $\limsup(X)$ is finite.

Theorem 5 (consequence of [4, 8]) *There is a PSPACE algorithm which, given a monoid morphism F from $\mathbb{A}^*$ to $\mathcal{M}_{n,n}(\overline{\mathbb{N}})$ and a regular language $L \subseteq \mathbb{A}^*$, enumerates* $\limsup(F(L))$.

This is an adaptation of Leung's proof of decidability of limitedness for distance automata [8] (it subsumes this result). We are not aware of any similar statement in the literature, though it can be deduced from previous works.

2.4 A first reduction: the theorem of affine domination

Our goal in this section is to establish the theorem of affine domination (Theorem 3). Notations introduced here will be reused in Section 3.2.

Let us fix two distance automata over the same alphabet $\mathbb{A}$. The first one, $\mathcal{A}_f = (\mathbb{A}, Q_f, F, I_f, T_f)$ calculates a function f. The second one, $\mathcal{A}_g = (\mathbb{A}, Q_g, G, I_g, T_g)$ calculates a function g.

Define $R_{p,0,q} \subseteq \mathbb{A}^*$ to be the set of words over which there is a run of $\mathcal{A}_g$ of weight 0 from state p to state q. Let ℓ be a non-null weight occurring in some transition of $\mathcal{A}_g$, and p, q be states in Q_g. Define $R_{p,\ell,q} \subseteq \mathbb{A}^*$ to contain the words over which there is a run of $\mathcal{A}_g$ from state p to state q which uses one transition of weight ℓ, and otherwise only transitions of weight 0.

Proof (Proof of theorem 3) Define K to be the largest number that occurs in at least one of the matrices of the union, over states p, q and weights of transitions ℓ, of the sets $\limsup(F(R_{p,\ell,q}))$ (such a number exists since by Theorem 5 it is the maximum of finitely many numbers). Given a matrix M, call an *m-expansion* of M a matrix $M' \geqslant M$ such that for all i, j, $M_{i,j} > K$ implies $M'_{i,j} \geqslant m$. We first show a claim concerning expansions.

Claim. For all $M \in F(R_{p,\ell,q})$ and all m there exists an m-expansion $M' \in F(R_{p,\ell,q})$ of M.
Indeed, by definition of the superior limit, there is some $L \in \limsup(F(R_{p,\ell,q}))$ such that $L \geqslant M$. Furthermore, by choice of K, whenever $M_{i,j} > K$, $L_{i,j} = \infty$. Finally, again by definition of the superior limit, L is the limit of a sequence of matrices in $F(R_{p,\ell,q})$. Hence, for all m, there exists a matrix M' in this sequence which is sufficiently close to L that it is an m-expansion of M. This proves the claim.

Let us turn now to the core of the proof. Our goal is to prove that if f is dominated by g, (*i.e.,* there exists $\alpha : \mathbb{N} \to \mathbb{N}$ extended with $\alpha(\infty) = \infty$ such that $f \leqslant \alpha \circ g$), then $f \leqslant K(1 + g)$. The proof is by contraposition. Thus, assume $f \not\leqslant K(1 + g)$. This means $f(u) > Kg(u) + K$ for some word u. We will prove that f is not dominated by g.

The first case is $g(u) = 0$. This means that $u \in R_{p,0,q}$ with p initial and q final. Using the above claim, one can chose for all m a word $v^{(m)} \in R_{p,0,q}$ such that $F(v^{(m)})$ is an m-expansion of $F(u)$. Since $f(u) > K$ then that for every initial state r and final state s of $\mathcal{A}_f$, $F(u)_{r,s} > K$. This means that for all such r, s, $F(v^{(m)})_{r,s} \geqslant m$. It follows that $f(v^{(m)}) \geqslant m$. Hence, over the sequence $(v^{(m)})_m$, g is bounded and f tends to infinity. This forbids the existence of a function α such that $f \leqslant \alpha \circ g$ so f is not dominated by g.

Assuming $g(u) \neq 0$, the argument is similar. Remark first that $g(u)$ is finite since $f(u) > Kg(u) + K$. This means one can find $p_0, \ldots, p_k$ with p_0 initial, p_k final, and such that:

$$u = u_1 \ldots u_k, \quad u_1 \in R_{p_0,\ell_1,p_1}, \ldots, u_k \in R_{p_{k-1},\ell_k,p_k} \ ,$$

where $\ell_1, \ldots, \ell_k$ are all non-null and of sum $g(u)$. By the above claim, for all $i = 1 \ldots k$, and all m, one can select $v_i^{(m)}$ in R_{p_{i-1},ℓ_i,p_i} such that $F(v_i^{(m)})$ is an m-expansion of $F(u_i)$. Consider now the word $v^{(m)} = v_1^{(m)} \ldots v_k^{(m)}$. Clearly $g(v^{(m)}) = g(u)$. For the sake of contradiction, assume now that $f(v^{(m)}) < m$ for some m. This means that there exists $q_0, \ldots, q_k$ such that q_0 is initial, q_k is final, and $F(v_i^{(m)})_{q_{i-1},q_i} < m$ for all $i = 1 \ldots k$. Since $F(v_i^{(m)})$ is an m-expansion of $F(u_i)$, this implies $F(u_i)_{q_{i-1},q_i} \leqslant K$. It follows that $f(u) \leqslant Kk \leqslant Kg(u)$. A contradiction. Hence $f(v^{(m)}) \geqslant m$. Thus, g is bounded by $(v^{(m)})_m$ while f is not. As a consequence, f is not dominated by g.

3 Description of the core theorem

In this section, we define sufficient material for stating our core theorem (Theorem 6). Then we apply it to the comparison of distance automata. Its proof is the subject of Section 4.

3.1 Weighted matrices, approximation, finitely presented sets, and the core theorem

In this section we state our core approximation result, Theorem 6. This theorem states that given a set of weighted matrices, it is possible to compute a finite presentation of its closure under product up to some approximation. Hence we have to introduce weighted matrices, the approximation, and what are finite presentations before disclosing the statement. This requires some specific definitions that we now present. Fix a positive integer n, and all matrices implicitly belong to $\mathcal{M}_{n,n}(\overline{\mathbb{R}^+})$.

A *weighted matrix* is an ordered pair (M,ℓ) where $M \in \mathcal{M}_{n,n}(\overline{\mathbb{R}^+})$ and $\ell \in \mathbb{N}$ is non-null. The positive integer ℓ is called the *weight* of the weighted matrix. The set of weighted matrices is denoted by $\mathcal{W}_{n,n}$. Weighted matrices have a semigroup structure $(\mathcal{W}_{n,n},\otimes)$, where $(M,\ell)\otimes(M',\ell')$ stands for $(M\otimes M',\ell+\ell')$. Given subsets X,Y of $\mathcal{W}_{n,n}$, one denotes by $X\otimes Y$ the set $\{M\otimes N \ : \ M\in X,\ N\in Y\}$, and by $\langle X\rangle$ the closure under $\otimes$ of X. With this terminology, our goal is, to approximate $\langle X\rangle$ for a given finite set of matrices X.

We describe now the notion of approximation that we use. Given some $\varepsilon>0$ and two weighted matrices (M,ℓ) and (M',ℓ'), we write

$$(M,\ell)\preccurlyeq_\varepsilon (M',\ell') \qquad \text{if} \qquad \ell\geqslant\ell',\quad \phi(M)=\phi(M') \text{ and } M\leqslant M'+\varepsilon\ell\ .$$

Remark 1 Note that $(M,\ell)\preccurlyeq_\varepsilon(M',\ell')$ implies $\frac{1}{\ell}M\leqslant\frac{1}{\ell'}M'+\varepsilon$, and this is the intention behind this definition, i.e., being able to consider weighted matrices up to a multiplicative error of ε. In fact $(M,\ell)\preccurlyeq_\varepsilon(M',\ell')$ is a more restrictive definition than simply $\frac{1}{\ell}M\leqslant\frac{1}{\ell'}M'+\varepsilon$. This is necessary since we want this notion to be robust with respect to the operations used later on in the proof. This robustness is made explicit in Lemma 2 below.

This definition extends to sets of weighted matrices as follows. Given two sets X,X' of weighted matrices, we write $X\preccurlyeq_\varepsilon X'$ if for every $(M,\ell)\in X$, there exists $(M',\ell')\in X'$ such that $(M,\ell)\preccurlyeq_\varepsilon(M',\ell')$. We also define $X\approx_\varepsilon X'$ to hold if both $X\preccurlyeq_\varepsilon X'$ and $X'\preccurlyeq_\varepsilon X$ (and we say that X is ε-equivalent to X').

The following lemma establishes some simple, yet essential, properties of the $\preccurlyeq_\varepsilon$ relations (as a consequence, the same properties hold for $\approx_\varepsilon$).

Lemma 2 *Given $X,X',Y,Y',Z\subseteq\mathcal{W}_{n,n}$ and $\varepsilon,\eta>0$,*

1. *if $X\preccurlyeq_\varepsilon Y$ and $Y\preccurlyeq_\eta Z$ then $X\preccurlyeq_{\varepsilon+\eta} Z$,*
2. *if $X\preccurlyeq_\varepsilon X'$ and $Y\preccurlyeq_\varepsilon Y'$ then $X\otimes Y\preccurlyeq_\varepsilon X'\otimes Y'$,*

3. if $X \preccurlyeq_\varepsilon X'$ then $\langle X\rangle \preccurlyeq_\varepsilon \langle X'\rangle$.

Proof 1. If $(M,\ell) \preccurlyeq_\varepsilon (M',\ell') \preccurlyeq_\eta (M'',\ell'')$, then one gets $\ell \geqslant \ell' \geqslant \ell''$, $\phi(M) = \phi(M') = \phi(M'')$ and $M \leqslant M' + \varepsilon\ell \leqslant M'' + \eta\ell' + \varepsilon\ell \leqslant M'' + (\varepsilon+\eta)\ell$. This easily extends to sets of weighted matrices.
2. Assume that both $(M,\ell) \preccurlyeq_\varepsilon (M',\ell')$ and $(N,t) \preccurlyeq_\varepsilon (N',t')$. Then, $\ell + t \geqslant \ell' + t'$, $\phi(M \otimes N) = \phi(M' \otimes N')$ and $M \otimes N \leqslant (M' + \varepsilon\ell) \otimes (N' + \varepsilon t) \leqslant M' \otimes N' + \varepsilon(\ell + t)$. This naturally extends to sets of weighted matrices.
3. By induction, applying the second item.

□

The last ingredient required is to describe how to represent (infinite) sets of weighted matrices. Call a set of weighted matrices $W \subseteq \mathcal{W}_{n,n}$ *finitely presented* if it is a finite union of singleton sets, and of sets of the form $\{(kM,k) :\ k \geqslant \ell, k \in \mathbb{N}\}$ where $M \in \mathcal{M}_{n,n}(\overline{\mathbb{R}^+})$ and ℓ is a positive integer. Our algorithm manipulates finitely presented sets of weighted matrices. For $a \in \mathbb{N}$, let us note $\mathcal{W}^a_{n,n} \subseteq \mathcal{W}_{n,n}$ the set of weighted matrices (M,ℓ) such that every finite coefficient of M is smaller than $a\ell$. Note that for each finitely presented set P, there is an $a \in \mathbb{N}$ such that $P \subseteq \mathcal{W}^a_{n,n}$.

The core technical contribution of this paper can now be stated, as follows.

Theorem 6 (core theorem) *Given a finitely presented set $X \subseteq \mathcal{W}_{n,n}$ and a real $\varepsilon > 0$, there exists effectively a finitely presented set* $\texttt{closure}(\varepsilon, X) \subseteq \mathcal{W}_{n,n}$ *such that:*

$$\texttt{closure}(\varepsilon, X) \approx_\varepsilon \langle X\rangle\ .$$

The proof of this result will be the subject of Section 4. The application of this theorem to the comparison of distance automata is presented in Section 3.2. The two sections are independent.

3.2 The reduction construction

The goal of this section is to prove the approximate comparison theorem (Theorem 4).

We reuse definitions and notations of automata $\mathcal{A}_f$ and $\mathcal{A}_g$ given in the Section 2.4. In particular, we use the sets $R_{p,\ell,q}$ again.

Our goal is to construct a finite set of weighted matrices X that captures the relationship between f and g. The key ideas behind this reduction are the following. Each matrix (M,ℓ) in X corresponds to a set of runs of g, that start in a given state p and end in a given state q, and use exactly one transition of non-null weight ℓ. The corresponding matrix M is in charge of (a) simulating the behaviour of F over some word corresponding to such a run (there may be infinitely many such runs, but only the finitely many matrices of the superior limit need to be considered), and (b) keeping information concerning the first and last state of the run of $\mathcal{A}_g$ for being able to check that pieces of the run of $\mathcal{A}_g$ are correctly concatenated.

One also needs to define the part of the matrix in charge of controlling the validity of the run of $\mathcal{A}_g$. The construction behind Lemma 3 below is the one of a deterministic automaton, that reads words over the alphabet Q_g^2, and accepts a word $(p_1,q_1)\dots(p_k,q_k)$ if, either p_1 is not initial, or q_k is not final, or if $q_{i-1}\neq p_i$ for some i. One can verify that this language is accepted by a deterministic and complete automaton with states $Q_g \uplus \{i,\perp\}$. The unique initial state is i, and, when reading the word $(p_1,q_1)\dots(p_k,q_k)$, the automaton reaches state $\perp$ if p_1 is not initial or $q_{i-1}\neq p_i$ for some i, otherwise it reaches state q_k. The final states are the one not in T_g plus $\perp$ plus possibly i if there are no states that are both initial and final in g. Translated in matrix form, this yields Lemma 3.

Lemma 3 *There are $(|Q_g|+2,|Q_g|+2)$-matrices $(C^{p,q})_{p,q\in Q_g}$ over $\mathbb{B}$ and vectors I_C and T_C such that for all $p_1,q_1,\dots,p_k,q_k\in Q_g$,*

$$I_C\otimes C^{p_1,q_1}\otimes\cdots\otimes C^{p_k,q_k}\otimes T_C = \begin{cases}\infty & \text{if } p_1\in I_g,\ q_1=p_2,\dots,q_{k-1}=p_k \text{ and } q_k\in T_g,\\ 0 & \text{otherwise.}\end{cases}$$

Proof This is implemented in matrix form as follows. For each p,q where $p,q\in Q_g$, set the matrix $C^{p,q}$ that has indices in $Q_g\cup\{i,\perp\}$, to be such that:

$$(C^{p,q})_{p',q'} = \begin{cases} 0 & \text{if } p'=i,\ p\in I_g \text{ and } q'=q,\\ 0 & \text{if } p'=i,\ p\notin I_g \text{ and } q'=\perp,\\ 0 & \text{if } p'=p \text{ and } q'=q,\\ 0 & \text{if } p'\neq i \text{ and } p'\neq p \text{ and } q'=\perp,\\ \infty & \text{otherwise.}\end{cases}$$

Define furthermore I_C be the vector with all entries ∞ but i which is 0, and let T_C be the vector with all entries equal to 0 except T_g and i if there is a state both initial and final in $\mathcal{A}_g$.

We can now construct the set X as follows:

$$X = \left\{\left(\begin{pmatrix} M & \infty\\ \infty & C^{p,q}\end{pmatrix},\ell\right)\ :\ M\in\limsup(F(R_{p,\ell,q}))\right\}$$

and the vectors

$$I = (I_f\ I_C) \qquad\text{and}\qquad T=\begin{pmatrix}T_f\\ T_C\end{pmatrix}\ .$$

The following lemma shows the validity of the construction, and more particularly how it relates the comparison of distance automata to the computation of the closure of a set of weighted matrices.

Lemma 4 *For all $\beta>0$, $f\leqslant\beta g$ if and only if for all $(W,\ell)\in\langle X\rangle$, $I\otimes W\otimes T\leqslant\beta\ell$.*

Proof Assume first $f \not\leqslant \beta g$, which means $f(u) > \beta g(u)$ for some u. Then clearly, $g(u)$ is finite and hence, there is an accepting run ρ of g over u. This means that one can find $p_0, \ldots, p_k$ with p_0 initial, p_k final, such that:

$$u \in R_{p_0,\ell_1,p_1} R_{p_1,\ell_2,p_2} \ldots R_{p_{k-1},\ell_k,p_k} \ ,$$

where $\ell_1, \ldots, \ell_k$ are all non-null and of sum $\ell = g(u)$. For all $i = 1 \ldots k$, set M_i to be some matrix in $\limsup(F(R_{p_{i-1},\ell_i,p_i}))$ such that $F(u_i) \leqslant M_i$. Let also C_i be C^{p_{i-1},p_i}. Clearly, the weighted matrix

$$(W_i, \ell_i) \qquad \text{with} \quad W_i = \begin{pmatrix} M_i & \infty \\ \infty & C_i \end{pmatrix}$$

belongs to X. Hence (W, ℓ) belongs to $\langle X \rangle$, where $W = W_1 \otimes \cdots \otimes W_k$. We then have $I \otimes W \otimes T = \min(x_f, x_C)$ with

$$x_f = I_f \otimes M_1 \otimes \cdots \otimes M_k \otimes T_f \qquad \text{and} \qquad x_C = I_C \otimes C_1 \otimes \cdots \otimes C_k \otimes T_C \ .$$

By choice of the M_i's, $x_f \geqslant I_f \otimes F(u) \otimes T_f = f(u)$. Furthermore, by Lemma 3, $x_C = \infty$. It follows that $I \otimes W \otimes T \geqslant f(u) > \beta g(u) = \beta\ell$.

Assume now that $f \leqslant \beta g$. Consider some $(W, \ell) \in \langle X \rangle$, it is obtained as $(W, \ell) = (W_1, \ell_1) \otimes \cdots \otimes (W_k, \ell_k)$ with $(W_i, \ell_i) \in X$ for all i. By definition of X, each of the W_i's can be written, for some $p_i, q_i \in Q_g$, as

$$W_i = \begin{pmatrix} M_i & \infty \\ \infty & C^{p_i,q_i} \end{pmatrix} \qquad \text{with} \quad M_i \in \limsup F(R_{p_i,\ell_i,q_i}).$$

Once more, one has $I \otimes W \otimes T = \min(x_f, x_C)$ with

$$x_f = I_f \otimes M_1 \otimes \cdots \otimes M_k \otimes T_f \qquad \text{and} \qquad x_C = I_C \otimes C_1 \otimes \cdots \otimes C_k \otimes T_C \ .$$

Remark first that if $x_C = 0$, clearly, $I \otimes W \otimes T = 0 \leqslant \beta\ell$. Hence, let us assume that $x_C = \infty$. This means by Lemma 3 that p_1 is initial, q_k is final, and $p_i = q_{i-1}$ for all $i = 2 \ldots k$. One needs to prove $x_f \leqslant \beta\ell$.

Assume for the sake of contradiction that $x_f > \beta\ell$. By continuity of the product, and using the definition of the superior limit, there exist words $u_1, \ldots, u_k$ such that for all $i = 1 \ldots k$, $u_i \in R_{p_i,\ell_i,q_i}$, and $I_f \otimes F(u_1) \otimes \cdots \otimes F(u_k) \otimes T_f > \beta\ell$. Furthermore, by definition of the sets R_{p_i,ℓ_i,q_i}, the fact that p_1 is initial, that q_k is final, and that $q_{i-1} = p_i$ for all $i = 2 \ldots k$, it follows that $g(u_1 \ldots u_k) = \ell$. It follows that $f(u_1 \ldots u_k) > \beta g(u_1 \ldots u_k)$. A contradiction.

We are now ready to establish the main theorem of the paper, that we recall first.

Theorem 4 (approximate comparison) *There is an algorithm which ε-approximates the inequality of functions computed by distance automata.*

Proof (Proof of Theorem 4) Let us consider two functions f and g computed by distance automata and some $\varepsilon > 0$. The algorithm works as follows. It computes the set X of weighted matrices as defined earlier in Section 3.2 as well as the corresponding vectors I, T. Using Theorem 6, it computes a finitely presented set Y of weighted matrices such that $Y \approx_{\frac{\varepsilon}{2}} \langle X \rangle$. Then it tests the existence in Y of a weighted matrix (M, ℓ) such that $I \otimes \frac{1}{\ell} M \otimes T > 1 - \frac{\varepsilon}{2}$. This is easy to do for finitely presented sets. If such a weighted matrix exists, the algorithm answers "no". It answers "yes" otherwise. Let us show the correctness of this approach.

- Assume $f \leqslant (1-\varepsilon)g$, and that, for the sake of contradiction, the algorithm answers "no". This means that $I \otimes \frac{1}{\ell} M \otimes T > 1 - \frac{\varepsilon}{2}$ for some weighted matrix $(M, \ell) \in Y$. Furthermore, there exists $(M', \ell') \in \langle X \rangle$ such that $(M, \ell) \preccurlyeq_{\frac{\varepsilon}{2}} (M', \ell')$. This implies $\frac{1}{\ell} M \leqslant \frac{1}{\ell'} M' + \frac{\varepsilon}{2}$. It follows that $I \otimes M' \otimes T > (1-\varepsilon)\ell'$. This contradicts Lemma 4.
- Assume $f \not\leqslant g$, then by Lemma 4, there exists a matrix $M \in \langle X \rangle$ such that $I \otimes \frac{1}{\ell} M \otimes T > 1$. Furthermore, there exists $M' \in Y$ such that $(M, \ell) \preccurlyeq_{\frac{\varepsilon}{2}} (M', \ell')$. This implies $\frac{1}{\ell} M \leqslant \frac{1}{\ell'} M' + \frac{\varepsilon}{2}$, and hence $I \otimes \frac{1}{\ell'} M' \otimes T > 1 - \frac{\varepsilon}{2}$. The algorithm answers "no".

4 Proof of the core theorem

In this section we prove our core theorem, Theorem 6. It is the combination of several arguments. The first one is the use of the factorisation forest theorem of Simon, and is the subject of Section 4.1.

4.1 The main induction: the factorisation forest theorem of Simon

The factorisation forest theorem of Simon [10] is a powerful combinatorial tool for understanding the structure of semigroups. We will not describe the original statement of this theorem, in terms of trees of factorisations, but rather a direct consequence of it which is central in our proof.

Theorem 7 (equivalent to the factorisation forest theorem [10][3]) *Given a semigroup morphism ϕ from a semigroup $(S, \otimes)$ (possibly infinite) to a finite semigroup $(T, \cdot)$, and a set $X \subseteq S$, let $X_0 = X$ and for all $k \geqslant 0$ define*

$$X_{k+1} = X_k \cup X_k \otimes X_k \cup \bigcup_{e \cdot e = e \in T} \langle X_k \cap \phi^{-1}(e) \rangle.$$

Then $X_N = \langle X \rangle$ for $N = 3|T| - 1$.

[3] Modern proofs of this theorem can be found in [7,3], in particular with the exact bound of $N = 3|T| - 1$ (Simon's original proof only provides $N = 9|T|$).

This theorem teaches us that to compute the closure under product in the semigroup S, it is sufficient to know how to compute (a) the union of sets, (b) the product of sets, (c) the intersection of a set with the inverse image of an idempotent under ϕ, and (d) the closure under product of sets of elements that all have the same idempotent image under ϕ. Of course, this theorem is interesting when the semigroup T and the mapping ϕ are cleverly chosen.

In our case, we are going to use the above proposition with $(S, \otimes) = (\mathcal{W}_{n,n}, \otimes)$, $(T, \cdot) = (\mathcal{M}_{n,n}(\mathbb{B}), \otimes)$, and ϕ the morphism which maps (M, ℓ) to $\phi(M)$. Our algorithm will compute, given a finitely presented set of weighted matrices X, an approximation of $\langle X \rangle$ following the same inductive construction as in the factorisation forest theorem. This is justified by the two following lemmas, that are proved in Sections 4.2 and 4.3. respectively

Lemma 5 *For all $\varepsilon > 0$ and all finitely presented sets $X, Y \subseteq \mathcal{W}_{n,n}$ there exists effectively a finitely presented set* $\texttt{product}(\varepsilon, X, Y) \subseteq \mathcal{W}_{n,n}$ *such that*

$$\texttt{product}(\varepsilon, X, Y) \approx_\varepsilon X \otimes Y \ .$$

Let X be a set of weighted matrices. We set

$$\phi(X) = \{\phi(M) \mid (M, \ell) \in X\} \ .$$

Lemma 6 *For all $\varepsilon > 0$ and for every finitely presented sets $X \subseteq \mathcal{W}_{n,n}$ such that $\phi(X) = \{E\}$ for some idempotent E, there exists effectively a finitely presented set* $\texttt{idempotent}(\varepsilon, X) \subseteq \mathcal{W}_{n,n}$ *such that*

$$\texttt{idempotent}(\varepsilon, X) \approx_\varepsilon \langle X \rangle \ .$$

Assuming that Lemmas 5 and 6 hold, it is easy to provide an algorithm for Theorem 6, i.e., an algorithm which, given a finitely presented set $X \subseteq \mathcal{W}_{n,n}$ computes a finitely presented set $\texttt{closure}(\varepsilon, X) \subseteq \mathcal{W}_{n,n}$ such that $\texttt{closure}(X) \approx_\varepsilon \langle X \rangle$. This algorithm mimics the induction involved in the statement of the factorisation forest theorem, Theorem 7.

Consider $S = \mathcal{W}_{n,n}$, $T = \mathcal{M}_{n,n}(\mathbb{B})$ and the morphism ϕ. Let X be a finitely presented set, and let $X_0, X_1, \ldots$ be defined as in Theorem 7.

Lemma 7 *Let $\varepsilon > 0$. For $k \geqslant 0$ one can compute a finitely presented set Y_k such that $Y_k \approx_\varepsilon X_k$.*

Proof By induction on k, using Lemmas 5 and 6.

For $k = 0$, it follows from the definitions that $X_0 = X = Y_0$.

Let $k \geqslant 0$, by the induction hypothesis one can compute a finitely presented set Y_k such that $Y_k \approx_{\frac{\varepsilon}{2}} X_k$. Set:

$$Y_{k+1} = Y_k \cup \texttt{product}(\frac{\varepsilon}{2}, Y_k, Y_k) \cup \bigcup_{\substack{e \otimes e = e \\ e \in \mathcal{M}_{n,n}(\mathbb{B})}} \texttt{idempotent}(\frac{\varepsilon}{2}, Y_k \cap \phi^{-1}(e))$$

First, Y_{k+1} is a presented set by definition and by Lemma 5 and Lemma 6. By Lemma 5 and Lemma 2

$$\texttt{product}(\frac{\varepsilon}{2}, Y_k, Y_k) \approx_{\frac{\varepsilon}{2}} Y_k \otimes Y_k \approx_{\frac{\varepsilon}{2}} X_k \otimes X_k \ .$$

Finally, by Lemma 2, $\texttt{product}(\frac{\varepsilon}{2}, Y_k, Y_k) \approx_\varepsilon X_k \otimes X_k$. Similarly, by Lemma 6, for all idempotent e, $\texttt{idempotent}(\frac{\varepsilon}{2}, Y_k \cap \phi^{-1}(e)) \approx_\varepsilon \langle X_k \cap \phi^{-1}(e)\rangle$. Thus $Y_{k+1} \approx_\varepsilon X_{k+1}$.

Finally, it is sufficient to apply Lemma 7 to $k = N = 3|T| - 1$. By Theorem 7, we obtain a presented set Y_N satisfying $Y_N \approx_\varepsilon < X >$.

Hence, what remains to be done is to establish Lemmas 5 and 6.

4.2 Approximate products of finitely presented sets

In this part, we give a proof of Lemma 5 which describes how to approximate the products of two finitely presented sets of weighted matrices. We also provide an extension of it to products of any bounded length, namely Lemma 10.

We first establish Lemma 8 which states that it is possible to control the effect of slight changes of length in the choices of weighted matrices in finitely presented sets.

Lemma 8 *For all $\varepsilon > 0$, all reals $a \geqslant 0$ and all positive integers p, there exists $\eta > 0$ such that for all positive integers $\ell_1, \ell_2, \ldots, \ell_p$, $\ell'_1, \ell'_2, \ldots, \ell'_p$, ℓ with*

$$\ell_i \leqslant \ell'_i + \eta\ell \qquad \textit{for all } i = 1 \ldots p,$$

and all matrices $M_1, M_2, \ldots, M_p \in \mathcal{M}_{n,n}(\overline{\mathbb{R}^+})$ with entries no greater than a,

$$(\ell_1 M_1 \otimes \ell_2 M_2 \otimes \cdots \otimes \ell_p M_p, \ell) \preccurlyeq_\varepsilon (\ell'_1 M_1 \otimes \ell'_2 M_2 \otimes \cdots \otimes \ell'_p M_p, \ell) \ .$$

Proof Set $\eta = \frac{\varepsilon}{pa}$. We have:

$$\begin{aligned} \ell_1 M_1 \otimes \cdots \otimes \ell_p M_p &\leqslant (\ell'_1 + \eta\ell) M_1 \otimes \cdots \otimes (\ell'_p + \eta\ell) M_p \\ &\leqslant (\ell'_1 M_1 + a\eta\ell) \otimes \cdots \otimes (\ell'_p M_p + a\eta\ell) \\ &\leqslant \ell'_1 M_1 \otimes \cdots \otimes \ell'_p M_p + (pa\eta)\ell \\ &\leqslant \ell'_1 M_1 \otimes \cdots \otimes \ell'_p M_p + \varepsilon\ell \ . \end{aligned}$$

Moreover, the Boolean projections of the both hand-sides of the comparison are the same. Hence $(\ell_1 M_1 \otimes \ell_2 M_2 \otimes \cdots \otimes \ell_p M_p, \ell) \preccurlyeq_\varepsilon (\ell'_1 M_1 \otimes \ell'_2 M_2 \otimes \cdots \otimes \ell'_p M_p, \ell)$. □

Remark that in the above statement, η depends on p. This means that the result is only useful for products of a bounded number of matrices.

Lemma 9 *For all positive integers x, p and all reals $\eta > 0$, there is a positive integer k such that, for all reals $\lambda_1, \ldots, \lambda_p \in [0,1]$ satisfying*

$$\sum_{i=1}^{p} \lambda_i = 1 \ ,$$

and all $\ell \geqslant k$, there are integers $y_1, \ldots, y_p \geqslant x$ such that

$$\sum_{i=1}^{p} y_i = \ell \quad \textit{and} \quad |y_i - \lambda_i \ell| \leqslant \eta\ell \ .$$

Proof Let us fix k to be any positive integer such that

$$\eta k > px \qquad \text{and} \quad k \geqslant p(p+1)x \ .$$

Let now $\ell \geqslant k$ and $\lambda_1, \lambda_2, \ldots, \lambda_p \in [0,1]$ such that $\sum_{i=1}^{p} \lambda_i = 1$.

Before constructing the y_i's, let us define the z_i's as follows:

$$z_i = \max(\lceil \lambda_i \ell \rceil, x) \qquad \text{for all } i = 1, \ldots, p.$$

Note first that in all cases $z_i \geqslant \lambda_i \ell$. Furthermore, $z_i < \lambda_i \ell + x$ and since $x < \frac{\eta k}{p} \leqslant \eta k \leqslant \eta \ell$, we obtain $|z_i - \lambda_i \ell| \leqslant \eta \ell$ for all $i = 1, \ldots, p$. Note also that

$$\ell = \sum_{i=1}^{p} \lambda_i \ell \leqslant \sum_{i=1}^{p} z_i \leqslant \sum_{i=1}^{p} (\lambda_i \ell + x) = \ell + px \ . \tag{1}$$

Hence, the z_i's would be a perfect choice for the y_i's, but for the fact that their sum may be a bit larger than ℓ, at most by px. We will correct this by modifying the largest of the z_i's.

Let m be the index maximizing λ_m. Let $y_i = z_i$ for all $i \neq m$, and $y_m = \ell - \sum_{i \neq m} z_i$. By definition $\sum_{i=1\ldots p} z_i = \ell$. According to (1), $y_m \in [z_m - px, z_m]$. Let us prove that the conclusion of the lemme holds.

Since $\lambda_m \geqslant \frac{1}{p}$ then $z_m \geqslant \lambda_m \ell \geqslant \lambda_m k \geqslant \frac{k}{p} \geqslant (p+1)x$, hence $y_m \geqslant z_m - px \geqslant x$. Furthermore, since $\lambda_m \ell - \eta \ell \leqslant z_m - px \leqslant y_m$, we have $|y_m - \lambda_m \ell| \leqslant \eta \ell$. □

We can now complete the proof of Lemma 5.

Lemma 5 *For all $\varepsilon > 0$ and all finitely presented sets $X, Y \subseteq \mathcal{W}_{n,n}$ there exists effectively a finitely presented set* $\texttt{product}(\varepsilon, X, Y) \subseteq \mathcal{W}_{n,n}$ *such that*

$$\texttt{product}(\varepsilon, X, Y) \approx_\varepsilon X \otimes Y \ .$$

Proof Let X, Y be finitely presented sets of weighted matrices, and $\varepsilon > 0$. Since finitely presented sets of weighted matrices are closed under union and thanks to distributivity of union over $\otimes$, it is sufficient to establish the result for the atomic blocks of the finite presentation. Namely, it is sufficient to consider the case $X = \{(M, \ell)\}$ or $X = \{(xM, x) \mid x \geqslant \ell\}$ together with $Y = \{(N, k)\}$ or $Y = \{(yN, y) \mid y \geqslant k\}$. This results in four possibilities, among which only three remain up to symmetry: (a) $X = \{(M, \ell)\}$ and $Y = \{(N, k)\}$, (b) $X = \{(M, \ell)\}$ and $Y = \{(yN, y) \mid y \geqslant k\}$, and finally (c) $X = \{(xM, x) \mid x \geqslant \ell\}$ and $Y = \{(yN, y) \mid y \geqslant k\}$.

- Case $X = \{(M, \ell)\}$ and $Y = \{(N, k)\}$, then we can set
$$\texttt{product}(\varepsilon, X, Y) = \{(M \otimes N, \ell + k)\} \qquad (= X \otimes Y) \ .$$
- Case $X = \{(M, \ell)\}$ and $Y = \{(yN, y) \mid y \geqslant k\}$, then we set a to be the greatest coefficient of $\frac{1}{\ell}M$ and N. Let us apply now Lemma 8 with parameters ε, a and $p = 2$, and obtain some $\eta > 0$. Set z to be an integer such that $\eta z \geqslant \ell$. Then set $Z = Z_1 \cup Z_2$ where
$$\begin{aligned} Z_1 &= \bigcup_{k \leqslant y < z} \{(M \otimes yN, \ell + y)\} \\ \text{and} \qquad Z_2 &= \{(y(\phi(M) \otimes N), y) \mid y \geqslant \ell + z\} \ . \end{aligned}$$
Note that this set Z is finitely presented (in particular because Z_1 is finite). We now prove that $X \otimes Y \approx_\varepsilon Z$. The idea is that Z_1 captures exactly the products of matrices in X and Y of length smaller than $z + \ell$ while Z_2 gives an approximation of longer products.
First direction: $X \otimes Y \preccurlyeq_\varepsilon Z$**.** Consider a weighted matrix in $X \otimes Y$. It is of the form $(M, \ell) \otimes (yN, y)$ for some $y \geqslant k$. If $k \leqslant y < z$, then $(M, \ell) \otimes (yN, y) \in Z_1$.
Otherwise $\eta y \geqslant \eta z \geqslant \ell \geqslant 1$. We obtain:
$$\begin{aligned} (M, \ell) \otimes (yN, y) &= (1M \otimes yN, \ell + y) \\ &\preccurlyeq_\varepsilon (0M \otimes (\ell + y)N, \ell + y) \\ &\qquad\qquad \text{(by Lemma 8 with } 1 \leqslant \eta(\ell + y)) \\ &= ((\ell + y)(\phi(M) \otimes N), \ell + y) \\ &\in Z_2 \ . \end{aligned}$$
Overall $X \otimes Y \preccurlyeq_\varepsilon Z$.
Opposite direction: $Z \preccurlyeq_\varepsilon X \otimes Y$**.** We prove that $Z \preccurlyeq_\varepsilon X \otimes Y$. Since $Z_1 \subseteq X \otimes Y$ it is sufficient to prove $Z_2 \preccurlyeq_\varepsilon X \otimes Y$. Let us consider a weighted matrix in Z_2, it is of the form $(\phi(M) \otimes yN, y)$ for some $y \geqslant \ell + z$. We have $\eta y \geqslant \eta z \geqslant \ell$ and by Lemma 8,
$$\begin{aligned} (\phi(M) \otimes yN, y) &= (0M) \otimes yN, y) \\ &\preccurlyeq_\varepsilon (1M \otimes (y - \ell)N, y) && \text{(by Lemma 8)} \\ &= (M, \ell) \otimes ((y - \ell)N, y) \\ &\in X \otimes Y \ . \end{aligned}$$
Overall $Z \preccurlyeq_\varepsilon X \otimes Y$.
- Case $X = \{(xM, x) \mid x \geqslant \ell\}$ and $Y = \{(yN, y) \mid y \geqslant k\}$. Let a be the greatest coefficient of M and N. Let us apply Lemma 8 with parameters ε, a, and $p = 2$, and obtain a corresponding η. We now use Lemma 9 with parameter $x = \max(k, \ell)$, $p = 2$ and η, and obtain an integer z as a result. Define now $Z = Z_1 \cup Z_2$ where,
$$Z_1 = \{(xM \otimes yN, x + y) \mid \ell \leqslant x < z, \ k \leqslant y < z\}$$

and

$$Z_2 = \bigcup_{\lambda \in ([0,1] \cap \eta \mathbb{N})} \{(t(\lambda M \otimes (1-\lambda)N), t) \mid t \geqslant z\} \ .$$

The set Z_1 is finite, and merely lists all weighted matrices of weight less than z in $X \otimes Y$. The set Z_2 takes barycenters of M and N, and produces corresponding weighted matrices for all possible weights greater or equal to z. To make Z_2 finitely presented, instead of taking all barycentres $\lambda M \otimes (1-\lambda)N)$ for $\lambda \in [0,1]$, we discretize λ by having it ranging in $[0,1] \cap \eta\mathbb{N}$. We note first that such a set Z is finitely presented. Let us prove now that $X \otimes Y \approx_\varepsilon Z$. There are two directions.

First direction: $X \otimes Y \preccurlyeq_\varepsilon Z$**.** Let us consider a weighted matrix in $X \otimes Y$. It is of the form $(xM, x) \otimes (yN, y)$ for some $x \geqslant \ell$ and some $y \geqslant k$. If $x < z$ and $y < z$, then $(xM, x) \otimes (yN, y) \in Z_1$ by definition.
Otherwise, $x \geqslant z$ or $y \geqslant z$. The weighted matrix can then be rewritten as

$$(xM, x) \otimes (yN, y) = \left((x+y)\left(\frac{x}{x+y}M \otimes \frac{y}{x+y}N\right), x+y\right) \ .$$

Futhermore, $x + y \geqslant z$. Let us now choose $\lambda \in ([0,1] \cap \eta\mathbb{N})$ such that $\left|\frac{x}{x+y} - \lambda\right| \leqslant \eta$. We also immediately have $\left|\frac{y}{x+y} - (1-\lambda)\right| \leqslant \eta$. Hence by Lemma 8,

$$\begin{aligned}(xM, x) \otimes (yN, y) &\preccurlyeq_\varepsilon ((x+y)(\lambda M \otimes (1-\lambda)N), x+y) \\ &\in Z_2 \ .\end{aligned}$$

Overall $X \otimes Y \preccurlyeq_\varepsilon Z$.

Second direction: $Z \preccurlyeq_\varepsilon X \otimes Y$**.** Conversely, let us first note that $Z_1 \subseteq X \otimes Y$. Hence, what remains to be proved is $Z_2 \preccurlyeq_\varepsilon X \otimes Y$. Let us consider a weighted matrix in Z_2. It is of the form $(t(\lambda M \otimes (1-\lambda)N), t)$ with $t \geqslant z$ and $\lambda \in [0,1]$. By Lemma 9, there are $x \geqslant \max(k, \ell)$ and $y \geqslant \max(k, \ell)$ such that $x + y = t$, $|x - \lambda t| \leqslant \eta t$ and $|y - (1-\lambda)t| \leqslant \eta t$. By Lemma 8, we get:

$$\begin{aligned}(t(\lambda M \otimes (1-\lambda)N), t) &\preccurlyeq_\varepsilon \left((x+y)\left(\frac{x}{x+y}M \otimes \frac{y}{x+y}N\right), x+y\right) \\ &\in X \otimes Y \ .\end{aligned}$$

Overall $Z \preccurlyeq_\varepsilon X \otimes Y$.

□

We have just proved Lemma 5 that gives an approximation of the product of two finitely presented sets of weighted matrices. This lemma will be used also in the proof of the more difficult Lemma 6. We will in fact use a slight generalisation of the result to a product of a bounded number of weighted matrices.

Lemma 10 (generalisation of Lemma 5) *For all $\varepsilon > 0$, and all finitely presented sets $X_1, \ldots, X_p \subseteq \mathcal{W}_{n,n}$, there is a computable and finitely presented set Z such that:*

$$Z \approx_\varepsilon X_1 \otimes \cdots \otimes X_p \ .$$

Proof This is true for $p = 2$ (Lemma 5). Suppose this is true for an integer $p \geqslant 2$, then $X_1 \otimes \cdots \otimes X_{p+1} \approx_{\frac{\varepsilon}{2}} \texttt{product}(\frac{\varepsilon}{2}, X_1, X_2) \otimes \cdots \otimes X_{p+1}$ by Lemma 2 and Lemma 5. Then by induction hypothesis, there is a computable and finitely presented set Z such that $\texttt{product}(\frac{\varepsilon}{2}, X_1, X_2) \otimes \cdots \otimes X_{p+1} \approx_{\frac{\varepsilon}{2}} Z$. Finally by Lemma 2, we obtain $X_1 \otimes \cdots \otimes X_{p+1} \approx_\varepsilon Z$. □

4.3 Approximate closure under products of finitely presented sets having the same idempotent projection

We shall now prove Lemma 6, which is the most difficult part in the proof of the core theorem. Let us fix an idempotent $E \in \mathcal{M}_{n,n}(\mathbb{B})$ and some finitely presented set of weighted matrices $X \subseteq \phi^{-1}(E)$. Our goal is to construct, given some $\varepsilon > 0$ a finitely presented set $\texttt{idempotent}(\varepsilon, X)$ such that $\texttt{idempotent}(\varepsilon, X) \approx_\varepsilon \langle X \rangle$. In the rest of this section, all weighted matrices belong to $\phi^{-1}(E)$.

The proof is divided in four parts. We first describe the general structure of the proof in Section 4.3.1, stating the key intermediate lemmas, and using them for establishing Lemma 6. The subsequent sections, namely Sections 4.3.1, 4.3.2 and 4.3.4, are then devoted to the proofs of these intermediate lemmas.

4.3.1 Description of the key intermediate lemmas, and the proof of Lemma 6

Our goal is to approximate $\langle X \rangle$ for $X \subseteq \phi^{-1}(E)$. The fact that all matrices are sent to the same idempotent is a big help in the sense that the structure of matrices is now fixed. Nevertheless, in a product the coefficients of the entries of the matrices may vary a lot. To overcome this problem, we introduce the central notion of uniform (weighted) matrices.

A matrix M such that $\phi(M) = E$ is *uniform* if

$$E \otimes M \otimes E = M.$$

A weighted matrix is *uniform* if its matrix part is uniform. Note that for all M such that $\phi(M) = E$, $E \otimes M \otimes E$ is a uniform matrix.

We will see below several properties of uniform matrices. What is interesting for us is that (a) the closure of a presentable set of uniform weighted matrices is approximable (Lemma 11 below). Another important point is that (b) we are able to define a notion of 'small product', and that it is possible to approximate the set of small uniform products of weighted matrices from X. Finally, an extraction argument (c) states that in any sufficiently long product it is possible to extract products of uniform small products. The combination of these three points yields the proof of Lemma 6.

In this section describing the general structure of the proof we make all the points (a), (b) and (c) precise, and then conclude the proof of Lemma 6. We postpone to the following subsection the precise proofs involved in points (a), (b) and (c), that happen to use fairly distinct arguments.

The point (a) above is the easiest to state.

Lemma 11 *For all $\varepsilon > 0$ and all finitely presented sets of uniform matrices $X \subseteq \phi^{-1}(E)$, there exists effectively a finitely presented set Z such that*

$$Z \approx_\varepsilon \langle X \rangle \ .$$

The proof of this statement is the subject of Section 4.3.3.

For describing point (b) we provide the notion of a 'small product'. The results concerning small products are developed in Section 4.3.2. Such products are parameterized by some $\eta > 0$ and some integer p. Essentially, a small product is a product in which in the total weight ℓ, a weight at least equal to $(1-\eta)\ell$ has been contributed by a small number of weighted matrices, namely at most p of them.

Definition 1 Let p be some positive integer and $\eta > 0$. Define $\langle X\rangle_{p,\eta}$ to be the set of weighted matrices

$$(M, \ell) = (M_1, \ell_1) \otimes \cdots \otimes (M_k, \ell_k)$$

where each (M_i, ℓ_i) belongs to X, and there exists $1 \leqslant i_1 < \cdots < i_s \leqslant k$ with $s \leqslant p$ such that

$$\sum_{j=1}^{s} \ell_{i_j} \geqslant (1-\eta)\ell \ .$$

The idea behind $\langle X\rangle_{p,\eta}$ is that it is an under-approximation of $\langle X\rangle$, that it contains all products of weighted matrices from X up to length p, and that even better, it is robust under the insertion of (possibly many) matrices of small weights everywhere. The following lemma states that small products can be effectively approximated (note the precise alternation of quantifiers, that is necessary for the rest of the proof to go through).

Lemma 12 *For all $\varepsilon > 0$ and all $a > 0$, there exists effectively $\eta > 0$ such that for all finitely presented $X \subseteq \mathcal{W}^a_{n,n} \cap \phi^{-1}(E)$ and all $p \geqslant 1$, there exists a finitely presented set Y such that*

$$\langle X\rangle_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X\rangle \ .$$

The proof of Lemma 12 is developed in Section 4.3.2.

We now combine the notion of small products with uniformity. For this, we define $\langle X\rangle^u_{p,\eta}$ exactly as $\langle X\rangle_{p,\eta}$, except that i_1 cannot be 1 and i_s cannot be k, which means that the first and last matrices of the product have to have a small weight.

Definition 2 Let p be some positive integer and $\eta > 0$. Define $\langle X\rangle^u_{p,\eta}$ to be the set of weighted matrices

$$(M, \ell) = (M_1, \ell_1) \otimes \cdots \otimes (M_k, \ell_k)$$

where each (M_i, ℓ_i) belongs to X, and there exists $1 < i_1 < \cdots < i_s < k$ with $s \leqslant p$ such that

$$\sum_{j=1}^{s} \ell_{i_j} \geqslant (1-\eta)\ell \ .$$

It so happens that matrices in $\langle X\rangle^u_{p,\eta}$ are almost uniform in the sense that they are ε-close to a uniform matrix since they are products (of weighted matrices sent to the same idempotent) such that the first and last term account for a sufficiently small percentage of the weight. The way we use this remark is by adapting Lemma 12.

Lemma 13 *For all $\varepsilon > 0$ and all $a > 0$, there exists effectively $\eta > 0$ such that for all finitely presented $X \subseteq \mathcal{W}^a_{n,n} \cap \phi^{-1}(E)$ and all $p \geqslant 1$, there exists a finitely presented set Y such that*

$$\langle X\rangle^u_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X\rangle \ .$$

Furthermore, Y only contains uniform weighted matrices.

At this point in the proof, we know how to approximate small products, their uniform variants, and we know how to approximate the closure of presentable sets of uniform weighted matrices. The key missing ingredient is to prove that, by combining these results together, we capture all possible products constructed upon some set of matrices (included in $\phi^{-1}(E)$). This is the subject of the following extraction lemma.

Lemma 14 *For all $X \subseteq \phi^{-1}(E)$ and all $\eta > 0$ there is an integer p such that:*

$$\langle X\rangle = \langle X\rangle_{p,\eta} \cup \langle X\rangle_{p,\eta} \otimes \langle\langle X\rangle^u_{p,\eta}\rangle \otimes \langle X\rangle_{p,\eta} \ .$$

The proof of this result is the subject of Section 4.3.4.

The combination of the above lemmas yields a direct proof of Lemma 6, that we recall now.

Lemma 6 *For all $\varepsilon > 0$ and for every finitely presented sets $X \subseteq \mathcal{W}_{n,n}$ such that $\phi(X) = \{E\}$ for some idempotent E, there exists effectively a finitely presented set* $\texttt{idempotent}(\varepsilon, X) \subseteq \mathcal{W}_{n,n}$ *such that*

$$\texttt{idempotent}(\varepsilon, X) \approx_\varepsilon \langle X\rangle \ .$$

Proof Let $\varepsilon > 0$. By Lemmas 12 and 13 we obtain some $\eta > 0$ (we take the minimum of the values of η produced by these two lemmas). Then, using Lemma 14 (with this value of η), we obtain an integer p such that

$$\langle X\rangle = \langle X\rangle_{p,\eta} \cup \langle X\rangle_{p,\eta} \otimes \langle\langle X\rangle^u_{p,\eta}\rangle \otimes \langle X\rangle_{p,\eta}\ .$$

Relying then on Lemmas 12 and 13 (with this value of p), there are effectively finitely presented sets T and V (V consisting of uniform matrices) such that

$$\langle X\rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{4}} T \preccurlyeq_{\frac{\varepsilon}{4}} \langle X\rangle \qquad \text{and} \qquad \langle X\rangle^u_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{4}} V \preccurlyeq_{\frac{\varepsilon}{4}} \langle X\rangle\ .$$

Then, using Lemma 2, we obtain

$$\langle X\rangle \preccurlyeq_{\frac{\varepsilon}{4}} T\cup T\otimes\langle V\rangle\otimes T \preccurlyeq_{\frac{\varepsilon}{4}} \langle X\rangle \cup \langle X\rangle \otimes \langle\langle X\rangle\rangle \otimes \langle X\rangle = \langle X\rangle\ ,$$

and hence $\langle X\rangle \approx_{\frac{\varepsilon}{4}} T\cup T\otimes\langle V\rangle\otimes T$.

Moreover, since all the weighted matrices in V are uniform, by Lemma 11, there is effectively a finitely presented set Y, such that $\langle V\rangle \approx_{\frac{\varepsilon}{4}} Y$. Now, using Lemma 2, we get

$$\langle X\rangle \approx_{\frac{\varepsilon}{2}} T\cup T\otimes Y\otimes T\ .$$

Finally, using Lemma 10 and the closure of finitely presented sets under union, there exists effectively a finitely presented set Z such that $T\cup T\otimes Y\otimes T \approx_{\frac{\varepsilon}{2}} Z$. We conclude using once more Lemma 2 that $\langle X\rangle \approx_\varepsilon Z$. □

4.3.2 Approximating small products: proofs of Lemmas 12 and 13

Let us recall Lemma 12 then prove it.

Lemma 12 *For all $\varepsilon > 0$ and all $a > 0$, there exists effectively $\eta > 0$ such that for all finitely presented $X \subseteq \mathcal{W}^a_{n,n} \cap \phi^{-1}(E)$ and all $p \geqslant 1$, there exists a finitely presented set Y such that*

$$\langle X\rangle_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X\rangle\ .$$

Proof Given $\varepsilon > 0$ and $a \geqslant 0$, let $\eta = \frac{\varepsilon}{2a}$.

Now, given a finitely presented set X and an integer p, define Z to be the results of products of at least one, and at most $2p+1$ weighted matrices from X, i.e.,

$$Z = \bigcup_{1\leqslant r\leqslant 2p+1} X^r \qquad\qquad \text{(where } X^r \text{ denotes } \overbrace{X\otimes\cdots\otimes X}^{r \text{ times}})$$

Two things are immediately clear from this definition. The first one is that $Z \subseteq \langle X\rangle$. The second is that, according to Lemma 10, there exists effectively a finitely presented set Y such that $Y \approx_{\frac{\varepsilon}{2}} Z$. We prove below that this Y fulfills the conclusion of the lemma. For this it is sufficient to prove that $\langle X\rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Z$.

Indeed, then we would have $\langle X\rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Z \preccurlyeq_{\frac{\varepsilon}{2}} Y \preccurlyeq_{\frac{\varepsilon}{2}} Z \subseteq \langle X\rangle$, which by Lemma 2 implies the expected $\langle X\rangle_{p,\eta} \preccurlyeq_{\varepsilon} Y \preccurlyeq_{\varepsilon} \langle X\rangle$.

Hence, let us prove now that $\langle X\rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Z$. Let us consider a matrix (M,ℓ) from $\langle X\rangle_{p,\eta}$. Our goal is to turn it into a 'resemblant' matrix from Z. Let us consider the product that has produced the matrix

$$(M,\ell) = (M_1,\ell_1) \otimes \cdots \otimes (M_k,\ell_k)\ ,$$

where $(M_1,\ell_1),\ldots,(M_k,\ell_k)$ belong to X, and there are $1 \leqslant i_1 < \cdots < i_s \leqslant k$ with $1 \leqslant s \leqslant p$ such that $\ell_{i_1} + \cdots + \ell_{i_s} \geqslant (1-\eta)\ell$. For convenience, let us define $i_0 = 0$ and $i_{s+1} = k+1$. The idea is to factorize this product as follows:

$$(M,\ell) = (N_0,n_0) \otimes (M_{i_1},\ell_{i_1}) \otimes (N_1,n_1) \otimes \cdots \otimes (M_{i_s},\ell_{i_s}) \otimes (N_s,n_s)$$

where for all $j = 0\ldots s$,

$$(N_j,n_j) = (M_{i_{j-1}+1},\ell_{i_{j-1}+1}) \otimes \cdots \otimes (M_{i_j-1},\ell_{i_j-1})\ .$$

Note that the definition of (N_j,n_j) may involve an empty product. In this case, set $(N,n_j) = 1$ where 1 is a neutral element added to weighted matrices[4]. Let us define now (N'_j,n'_j) to be 1 if $(N_j,n_j) = 1$, and to be (S,m) otherwise, where (S,m) is a matrix of minimal weight in X. Clearly now the matrix

$$(M',\ell') = (N'_0,n'_0) \otimes (M_{i_1},\ell_{i_1}) \otimes (N'_1,n'_1) \otimes \cdots \otimes (M_{i_s},\ell_{i_s}) \otimes (N'_s,n'_s)$$

belongs to Z. Let us show that $(M,\ell) \preccurlyeq_{\frac{\varepsilon}{2}} (M',\ell')$.

For this, let us note that for all $j = 0\ldots s$ such that $N_j \neq 1$, all the finite coefficients of N_j are at most equal to an_j (they can't be ∞ because N_j and N'_j are both sent by ϕ to E). Hence $N_j \leqslant N'_j + an_j$. Since furthermore $\sum_{j=1}^{s} n_j \leqslant \eta\ell$, it follows that $M \leqslant M' + a\eta\ell = M' + \frac{\ell}{2}$. On the weight side, since for all $j = 0\ldots s$, $n'_j \leqslant n_j$, we clearly have $\ell' \leqslant \ell$. Overall, $\langle X\rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Z$ as announced. □

The following lemma shows that there is a finite number of maximal uniform matrices smaller than a given matrix. We use it for proving Corollary 1.

Lemma 15 *Given a matrix M, there exist a finite set of matrices U such that*

- *all matrices in U are uniform,*
- *$N \leqslant M$ for all $N \in U$,*
- *for all uniform matrices $K \leqslant M$, there exists $N \in U$ such that $K \leqslant N$.*

Proof Let M be a matrix, and S be the set of values in its entries. Construct now the set of matrices

$$U = \{N \mid \phi(N) = E,\ E \otimes N \otimes E = N,\ N \leqslant M,\ \text{all entries in } N \text{ belong to } S \cup \{\infty\}\}\ .$$

[4] It corresponds formally to the virtual weighted matrix $(I_n, 0)$ (virtual since weight 0 is not allowed).

Of course, since all the entries of the matrices have to belong to the finite set S, U is finite. It is also easily effectively computable. The fact that all matrices N in U are uniform and that $N \leqslant M$ is from the definition. What remains to be proved is the last point.

Thus, consider a uniform matrix K such that $K \leqslant M$. We have to turn it into a matrix N that uses only entries from S. For this, we consider the map

$$\begin{aligned} f : \overline{\mathbb{R}^+} &\to S \cup \{\infty\} \\ x &\mapsto \inf\{y \in S \mid y \geqslant x\} \ . \end{aligned}$$

Let N be $f(K)$ (f is applied component-wise to all the entries of the matrix). It is clear that all the entries of N belong to $S \cup \{\infty\}$ by construction (a). Since f preserves the order, we have $N = f(K) \leqslant f(M) = M$ (b). Since f preserves the order, and multiplication with E only involves the computation of minima (that are preserved under f),

$$E \otimes N \otimes E = E \otimes f(K) \otimes E = f(E \otimes K \otimes E) = f(K) = N \ . \qquad \text{(c)}$$

Finally, since $K \leqslant N \leqslant M$, $E = \phi(K) \leqslant \phi(N) \leqslant \phi(M) = E$. Hence $\phi(N) = E$ (d). Overall, by (a+b+c+d), $N \in U$. Finally, by definition of f, $K \leqslant f(K) = N$. □

If we apply Lemma 15 to all the matrices involved in the definition of a finitely presented set, we immediately get the following corollary.

Corollary 1 *Given a finitely presented set Y, there exists effectively a finitely presented set Z such that*

- *all weighted matrices in Z are uniform,*
- *for all weighted matrices $(K, \ell) \in Z$, there exists $(M, \ell) \in Y$ with $K \leqslant M$,*
- *for all uniform weighted matrices (K, ℓ) such that $(M, \ell) \in Y$ for some $K \leqslant M$, there exists $(N, \ell) \in Z$ such that $K \leqslant N$.*

We are now ready to prove Lemma 13, which we restate for the sake of completeness.

Lemma 13 *For all $\varepsilon > 0$ and all $a > 0$, there exists effectively $\eta > 0$ such that for all finitely presented $X \subseteq \mathcal{W}^a_{n,n} \cap \phi^{-1}(E)$ and all $p \geqslant 1$, there exists a finitely presented set Y such that*

$$\langle X \rangle^u_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X \rangle \ .$$

Furthermore, Y only contains uniform weighted matrices.

Proof The idea is to use Lemma 12 for obtaining a Y that is a solution. We then use Corollary 1 in order to transform it into a finitely presented set of uniform matrices.

We first claim ($\star$) that for η such that $a\eta \leqslant \frac{\varepsilon}{2}$, all p and all weighted matrices $(M, \ell) \in \langle X \rangle^u_{p,\eta}$, then $(M, \ell) \preccurlyeq_{\frac{\varepsilon}{2}} (E \otimes M \otimes E, \ell)$ (in fact $\approx_{\frac{\varepsilon}{2}}$ holds, but we do not need the other direction). Indeed, the weighted matrix (M,ℓ) can be decomposed as

$$(M,\ell) = (M_1,\ell_1) \otimes (M_2,\ell_2) \otimes (M_3,\ell_3) \ ,$$

where $(M_1,\ell_1),(M_3,\ell_3) \in X$, $(M_2,\ell_2) \in \langle X\rangle_{p,\eta}$ and $\ell_1+\ell_3 \leqslant \eta\ell$. We have

$$\begin{aligned} M = M_1 \otimes M_2 \otimes M_3 &\leqslant (E\otimes M_1)\otimes M_2 \otimes (M_3 \otimes E) + a(\ell_1+\ell_3) \\ &\leqslant E\otimes M\otimes E + a\eta\ell \leqslant E\otimes M\otimes E + \frac{\varepsilon}{2}\ell \ . \end{aligned}$$

Claim $(\star)$ is established.

Let us prove now the lemma itself. Let $\varepsilon > 0$ and $a > 0$ be fixed. Using Lemma 12 with parameter $\frac{\varepsilon}{2}$ and the same a, we obtain some $\eta > 0$. Now given a finitely presentable set X, we know that there exists a finitely presented set Y such that $\langle X\rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Y \preccurlyeq_{\frac{\varepsilon}{2}} \langle X\rangle$.

Let us now apply Corollary 1 to Y, and obtain a set Z. Let us show it fulfil the conclusions of the lemma. For this, consider some $(K,k) \in \langle X\rangle^u_{p,\eta}$, let us show that $(K,k) \preccurlyeq_\varepsilon Z$. According to Claim $(\star)$, $(K,k) \approx_{\frac{\varepsilon}{2}} (K',k)$ where $K' = E\otimes K\otimes E$ is uniform. By construction of Y, $(K',k) \preccurlyeq_{\frac{\varepsilon}{2}} (M,\ell)$ for some $(M,\ell) \in Y$. This means that[5]

$$K' - \frac{\varepsilon}{2}k \leqslant M \ .$$

Hence, according to Corollary 1, there exists some $(N,\ell) \in Z$ with $K' - \frac{\varepsilon}{2}k \leqslant N$. Hence $(K,k) \preccurlyeq_{\frac{\varepsilon}{2}} (K',k) \preccurlyeq_{\frac{\varepsilon}{2}} (N,\ell) \in Z$. Overall, we have

$$\langle X\rangle^u_{p,\eta} \preccurlyeq_\varepsilon Z \preccurlyeq_0 Y \preccurlyeq_\varepsilon \langle X\rangle \ ,$$

where the $\preccurlyeq_0$ comes from Corollary 1. □

4.3.3 The closure of finitely presented sets of uniform matrices: proof of Lemma 11

The goal of this section is to prove that the closure under product of a finitely presented set of uniform matrices is effectively approximable. The key in this proof is to understand the structure of uniform matrices. In fact, we will even disclose a stronger notion than uniformity: normalized idempotency. This section starts by describing, analysing and giving results concerning these important notions, and end with the proof of Lemma 11 itself. In this section, all matrices are supposed to be sent by ϕ to the same idempotent matrix E.

Lemma 16 *Uniform matrices are closed under product.*

[5] Technically, here, some entries of the matrices may become negative. Since the arguments in Corollary 1 only involve the order, this does not make any difference.

Proof Let M and M' be uniform matrices of idempotent projection E, then:

$$\begin{aligned} E\otimes M\otimes M'\otimes E &= E\otimes(E\otimes M\otimes E)\otimes(E\otimes M'\otimes E)\otimes E && \text{(uniformity)}\\ &= (E\otimes M\otimes E)\otimes(E\otimes M'\otimes E) && \text{(idempotency of } E\text{)}\\ &= M\otimes M'. \end{aligned}$$

□

Given a uniform matrix M of idempotent projection E, let i and j be two indices. Let us define the relation $\to$ between indices by $i\to j$ if $E_{i,j}=0$. The relation $i\leftrightarrow j$ holds if both $i\to j$ and $j\to i$, or if $i=j$. Pairs (i,j) such that $i\to j$ but $i\not\leftarrow j$ are called *transient*. These definitions depend upon E, but since the idempotent E is fixed in this section, no confusion should arise and it will be omitted.

Lemma 17 *The relation $\to$ is a pre-order and $\leftrightarrow$ an equivalence relation.*

Proof Assume $i\to j$ and $j\to k$, i.e., $E_{i,j}=0$ and $E_{j,k}=0$. Since $E=E\otimes E$, we have $E_{i,k}=\min_\ell(E_{i,\ell}+E_{\ell,l})=0$ (choosing $\ell=j$). This means that $i\to k$. Thus $\to$ is transitive. Reflexivity is obvious from the definition. Hence $\to$ is a pre-order and $\leftrightarrow$ an equivalence relation. □

When $i\leftrightarrow j$ then i and j play exactly the same role anywhere in any product of uniform matrices. This is formalized by the following lemma and its corollary.

Lemma 18 *Given a uniform matrix M, and indices $i\to i'$ and $j'\to j$,*

$$M_{i,j}\leqslant M_{i',j'}\ .$$

Proof We claim first that $M_{i,j}\leqslant M_{i,j'}$. Indeed, either $j=j'$ and the claim obviously holds, or $E_{j',j}=0$. Using the definition of a uniform matrix, we have $M=M\otimes E$. This means in particular that $M_{i,j}\leqslant M_{i,j'}+E_{j',j}=M_{i,j}$.

Symmetrically, $M_{i,j'}\leqslant M_{i',j'}$, and hence $M_{i,j}\leqslant M_{i,j'}\leqslant M_{i',j'}$. □

Corollary 2 *Whenever $i\leftrightarrow i'$ and $j\leftrightarrow j'$, then $M_{i,j}=M_{i,',j'}$.*

Our next lemma teaches us that for coefficients on the diagonal, the products of uniform matrices are very easy to understand.

Lemma 19 *Given $M_1,\ldots,M_m$ uniform matrices,*

$$(M_1\otimes\ldots\otimes M_m)_{i,i}=\sum_{k=1}^m (M_k)_{i,i}\ .$$

Proof The first inequality, $(M_1\otimes\ldots\otimes M_m)_{i,i}\leqslant\sum_{k=1}^m(M_k)_{i,i}$ holds for any matrices, and simply comes from the fact that the term $\sum_{k=1}^m(M_k)_{i,i}$ appears in the minimum defining $(M_1\otimes\ldots\otimes M_m)_{i,i}$.

Conversely, $(M_1\otimes\ldots\otimes M_m)_{i,i}$ is defined as a finite minimum which is reached, say by the term $v=\sum_{k=1}^m(M_k)_{i_{k-1},i_k}$ for some sequence of indices

$i = i_0, i_1, \ldots, i_{m-1}, i_m = i$ belonging to $1 \ldots n$. If $v = \infty$, then obviously $\sum_{k=1}^{m}(M_k)_{i,i} \leqslant v$. Otherwise, this means that $(M_k)_{i_{k-1},i_k} < \infty$ for all $k = 1 \ldots m$, which means $i_{k-1} \to i_k$. Since furthermore $i = i_0 \leftrightarrow i_k = i$, using the transitivity of $\to$, all the indices $i_0, \ldots, i_k$ are $\leftrightarrow$-equivalent to i. Thus by Corollary 2 we have $v = \sum_{k=1}^{m}(M_k)_{i_{k-1},i_k} = \sum_{k=1}^{m}(M_k)_{i,i}$. □

Let us recall that our goal is to compute the closure under product of a set of uniform matrices. Among uniform matrices, some will play a particularly important roles: the matrices for which the iteration is straightforward to compute. A uniform matrix M is called *normalized idempotent* if

$$M \otimes M = 2M \ .$$

These matrices are not strictly speaking idempotents, but they are if one accepts a renormalizing multiplying coefficient. The following lemma presents the key properties of normalized idempotent matrices.

Lemma 20 *For a uniform matrix, the following statements are equivalent:*

1. *M is normalized idempotent,*
2. *for all indices g, h such that $g \to h$, there exists i such that $g \to i$, $i \to h$ and $M_{g,h} = M_{i,i}$,*
3. *$(aM) \otimes (bM) = (a+b)M$ for all non-negative reals a, b, and*
4. *$M^r = rM$ for all positive integers r.*

Proof **From 1 to 2.** Assume that M is a normalized idempotent matrix. Let g, h be such that $g \to h$.

We claim first that for k a power of 2, $M^k = kM$. Indeed, this is true for $k = 1$, and by induction, for all k power of 2, $M^{2k} = M^k \otimes M^k = (kM) \otimes (kM) = k(M \otimes M) = k(2M) = (2k)M$. The claim is established.

Let us consider now k some power of 2 larger than n. We have $(M^k)_{g,h} = kM_{g,h}$ by the above claim. Since $(M^k)_{g,h}$ is computed as a minimum, there exists a witness sequence $g = i_0, i_1, \ldots, i_k = h$ such that $M_{i_0,i_1} + \cdots + M_{i_{k-1},i_k} = kM_{g,h}$. According to Lemma 18, $M_{i_{j-1},i_j} \leqslant M_{g,h}$ for all $j = 1 \ldots k$. Assume that $M_{i_{j-1},i_j} < M_{g,h}$ for some j among $1 \ldots k$, then we would have $M_{i_0,i_1} + \cdots + M_{i_{k-1},i_k} < kM_{g,h}$, a contradiction. Hence $M_{i_0,i_1} = \cdots = M_{i_{k-1},i_k} = M_{g,h}$. Applying now the pigeonhole principle and the fact that $\to$ is transitive, we have $i_{j-1} \leftrightarrow i_j$ for some i. The index i_j is then a witness of the second item as we have $g \to i_j$, $i_j \to h$ and $M_{i_j,i_j} = M_{i_{j-1},i_j} = M_{g,h}$ (using Corollary 2).

From 2 to 3. Let g, h be indices such that $g \to h$. We have $M_{g,h} = M_{i,i}$ for some i such that $g \to i$, $i \to h$ and $M_{g,h} = M_{i,i}$. Note that by Lemma 18, $M_{g,h} \leqslant M_{g,i} \leqslant M_{i,i}$, and thus $M_{g,h} = M_{g,i}$, and similarly $M_{g,h} = M_{i,h}$. We can now compute $(a+b)M_{g,h} = aM_{g,h} + bM_{g,h} = aM_{g,i} + bM_{i,h} \geqslant ((aM) \otimes (bM))_{g,h}$ (using the definition of $\otimes$). Conversely, by definition of $\otimes$, there exists some i such that $((aM) \otimes (bM))_{g,h} = (aM)_{g,i} + (bM)_{i,h}$. We have $((aM) \otimes (bM))_{g,h} = aM_{g,i} + bM_{i,h} \geqslant aM_{g,h} + bM_{g,h} = (a+b)M_{g,h}$ (using again Lemma 18).

From 3 to 4. Let us assume item 3. The proof is by induction on r. For $r = 1$, we indeed have $M^r = rM$. Now, $M^{r+1} = M^r \otimes M = (rM) \otimes M = (r+1)M$ (using the induction hypothesis and item 3).

From 4 to 1. It is sufficient to take $r = 2$. □

Now that we understand the structure of normalized idempotent matrices, we shall see that given a uniform matrix M, the series $\frac{1}{r}M^r$ converges to some normalized idempotent matrix, denoted $\bar{M}$.

Lemma 21 *Given a uniform matrix M, there exists effectively a unique normalized idempotent matrix that coincides with M over its diagonal. It is denoted $\bar{M}$. It furthermore satisfies:*

1. *for all uniform matrices M, $M \leqslant \bar{M}$,*
2. *for all uniform matrices M, N with $M_{i,i} \leqslant N_{i,i}$ for $i = 1 \ldots m$, $\bar{M} \leqslant \bar{N}$,*
3. *for all $\varepsilon > 0$, and all $a > 0$, there exists an integer r such that for all uniform matrices M of finite entries not exceeding a, $r\bar{M} \leqslant M^r + \varepsilon r$.*

Proof Let us define $\bar{M}$ for all indices g, h by

$$\bar{M}_{g,h} = \min\{M_{i,i} \mid g \to i,\ i \to h\}\ .$$

Let us show that $\bar{M}$ is uniform. For this, let us compute $(E \otimes \bar{M} \otimes E)_{g,h}$:

$$\begin{aligned}
(E \otimes \bar{M} \otimes E)_{g,h} &= \inf\{\bar{M}_{g',h'} \mid g \to g',\ h' \to h\} && \text{(definitions of } \otimes \text{ and } \to)\\
&= \inf\{M_{i,i} \mid g \to g' \to i \to h' \to h\} && \text{(definition of } \bar{M})\\
&= \inf\{M_{i,i} \mid g \to i \to h\} && \text{(since } g \to g' \text{ and } h' \to h)\\
&= \bar{M}_{g,h}\ . && \text{(definition of } \bar{M})
\end{aligned}$$

Let us show that $\bar{M}$ and M coincide over the diagonal. Let j be an index, we have $\bar{M}_{j,j} = \inf\{M_{i,i} \mid i \leftrightarrow j\} = M_{j,j}$ (using Corollary 2).

Let us show now that $\bar{M}$ is a normalized idempotent matrix. We use the second characterization of Lemma 20. Let g, h be such that $g \to h$. By definition of $\bar{M}$, there is some i such that $g \to i \to h$ and $\bar{M}_{g,h} = M_{i,i}$, and this is equal to $\bar{M}_{i,i}$ by the fact that M and $\bar{M}$ coincide over the diagonal. Hence $\bar{M}$ is a normalized idempotent matrix.

Let us show now $M \leqslant \bar{M}$. This simply comes from the fact that for $g \to h$ there is i such that $g \to i \to h$ and $\bar{M}_{g,h} = M_{i,i}$. Since furthermore $M_{g,h} \leqslant M_{i,i}$ by Lemma 18. Hence $M \leqslant \bar{M}$.

Let us finally establish uniqueness. For this, we show that whenever two normalized idempotent matrices M, N coincide over the diagonal, then these are equal. Indeed, given g, h such that $g \to h$, there is some i such that $g \to i \to h$ and $M_{g,h} = M_{i,i}$. Thus $M_{g,h} = M_{i,i} = N_{i,i} \geqslant N_{g,h}$ (using the fact that M and N coincide over the diagonal and Lemma 18). Since M and N play a symmetric role, we finally get $M = N$.

The second statement of the lemma, stating monotonicity with respect to the diagonal, is immediate from the definition of $\bar{M}$.

Let us establish now the third statement of the lemma. Given $\varepsilon > 0$ and $a > 0$, we fix some r such that

$$r\varepsilon \geqslant na \ . \tag{2}$$

Let now M be some uniform matrix with finite entries not exceeding a. Let g, h be such that $g \to h$. By definition of the product, there exists a sequence $g = i_0, i_1, \ldots, i_r = h$ such that $(M^r)_{g,h} = M_{i_0,i_1} + \cdots + M_{i_{r-1},i_r}$. Note first that since $g \to h$, $(M^r)_{g,h} < \infty$. Hence $M_{i_{j-1},i_j} < \infty$ for all $j = 1 \ldots r$, i.e., $i_{j-1} \to i_j$. Let us now consider the indices j in $1 \ldots r$. For each of them, two cases may occur:

- $i_{j-1} \leftrightarrow i_j$. In this case, we have $g \to i_j \to h$ and $M_{i_{j-1},i_j} = M_{i_j,i_j}$ by Lemma 18. This means that M_{i_j,i_j} appears in the infimum defining $\bar{M}_{g,h}$. Hence $M_{i_{j-1},i_j} = M_{i_j,i_j} \geqslant \bar{M}_{g,h}$.
- or (i_{j_1}, i_j) is transient, i.e., $i_{j-1} \to i_j$ but $i_j \not\to i_{j-1}$. By transitivity of $\to$, there are at most n such indices. For each of these indices, we use the inequality $M_{i_{j-1},i_j} \geqslant 0$.

Combining these inequalities, we get

$$\begin{aligned}
(M^r)_{g,h} + r\varepsilon &= M_{i_0,i_1} + \cdots + M_{i_{r-1},i_r} + r\varepsilon && \text{(by choice of } i_0, \ldots, i_r) \\
&\geqslant (r-n)M_{i,i} + na && \text{(above remarks and inequality (2))} \\
&\geqslant r\bar{M}_{g,h} && \text{(since } a \geqslant M_{i,i} \text{ and by Lemma 18)}
\end{aligned}$$

Thus, we have the expected $r\bar{M} \leqslant M^r + r\varepsilon$. □

We are now ready to establish the key result of this section, that we restate for the sake of completeness.

Lemma 11 *For all $\varepsilon > 0$ and all finitely presented sets of uniform matrices $X \subseteq \phi^{-1}(E)$, there exists effectively a finitely presented set Z such that*

$$Z \approx_\varepsilon \langle X \rangle \ .$$

Proof Let $\varepsilon > 0$ and X be a finitely presented set that can be written

$$X = \bigcup_{1 \leqslant i \leqslant p} \{(x_i M_i, x_i)\} \cup \bigcup_{p+1 \leqslant i \leqslant m} \{(xM_i, x) \mid x \geqslant x_i\} \ .$$

As before, we will approximate the set $\langle X \rangle$ by the union of two sets: the set of exact products up to some length, and the set of asymptotic matrices that will be the products of 'barycenters' of matrices in X. Our finitely presented set is the set Z defined in equation (5) below. Some parameters have to be introduced beforehand.

Construction of the finitely presented set Z**.** Let a be the largest finite coefficient in the matrices M_i. From Lemma 21 applied with parameters $\frac{\varepsilon}{2}$ for the precision and a, we obtain an integer r. We choose now some integer z sufficiently large for having

$$rx_i \leqslant \frac{\varepsilon z}{2am} , \tag{3}$$

and also some $\gamma > 0$, inverse of an integer, such that

$$ma\gamma \leqslant \varepsilon . \tag{4}$$

We are now ready to define our finitely presented set Z as follows:

$$Z = Z_1 \cup Z_2 \tag{5}$$

$$\text{where} \quad Z_1 = \{(M,\ell) \in \langle X\rangle \mid \ell < z\} \tag{6}$$

$$\text{and} \quad Z_2 = \bigcup_{\substack{\lambda_1,\dots,\lambda_m \in [0,1]\cap\gamma\mathbb{N} \\ \lambda_1+\dots+\lambda_m=1}} \left\{\left(\ell\ \overline{\lambda_1 M_1 \otimes \dots \otimes \lambda_m M_m}, \ell\right) \mid \ell \geqslant z\right\} . \tag{7}$$

The first thing to note is that the set Z is effectively finitely presented. We prove below that $Z \approx_\varepsilon \langle X\rangle$. We study successively the two directions of this equivalence.

First direction: $Z \preccurlyeq_\varepsilon \langle X\rangle$**.** Note to begin that since $Z_1 \subseteq \langle X\rangle$, we have $Z_1 \preccurlyeq_\varepsilon \langle X\rangle$. Hence, we just have to establish $Z_2 \preccurlyeq_\varepsilon \langle X\rangle$. Consider a matrix $(M,\ell) \in Z_2$. By definition of Z_2, M is of the form:

$$M = \ell\ \overline{\lambda_1 M_1 \otimes \dots \otimes \lambda_m M_m} ,$$

for some integer $\ell \geqslant z$ and non-negative coefficients $\lambda_1,\dots,\lambda_m$ of unit sum (the fact that these are multiples of γ is not used in this direction of the proof). Our goal is to construct a weighted matrix $(N,\ell') \in \langle X\rangle$ such that $(M,\ell) \preccurlyeq_\varepsilon (N,\ell')$. For this, we construct a matrix in $\langle X\rangle$ that is close to $\frac{\ell}{r}(\lambda_1 M_1 \otimes \dots \otimes \lambda_m M_m)$ on its diagonal. Then, we iterate it r times using Lemma 21 in order to make it similar to $\ell\overline{\lambda_1 M_1 \otimes \dots \otimes \lambda_m M_m}$, i.e., M. Let us implement these ideas.

Since $\ell \geqslant z$ and by (3), we have $rx_i \leqslant \frac{\varepsilon z}{2am} \leqslant \frac{\varepsilon\ell}{2am}$. Hence there exist non-negative integers $y_1,\dots,y_m$ such that $ry_ix_i \in \left[\lambda_i\ell - \frac{\varepsilon\ell}{2am}, \lambda_i\ell\right]$ for all $i = 1\dots m$. These integers have the properties that:

$$ry_ix_i + \dots + ry_mx_m \leqslant \ell \quad \text{and} \quad \lambda_i\ell \leqslant ry_ix_i + \frac{\varepsilon\ell}{2am} \quad \text{for all } i = 1\dots m. \tag{8}$$

Let us consider now the weighted matrix

$$(K,k) = (x_1M_1, x_1)^{y_1} \otimes \dots \otimes (x_mM_m, x_m)^{y_m} .$$

Clearly, $(K,k) \in \langle X\rangle$ since $(x_1M_1,x_1),\dots,(x_mM_m,x_m)$ belong to X. Another consequence is that (K,k) is a uniform matrix.

Let us show that $(M,\ell) \preccurlyeq_\varepsilon (K,k)^r$. As far as the weight is concerned, this is straightwforward since $rk \leqslant \ell$ from (8). What remains to be shown is

$M \leqslant K^r + \varepsilon\ell$. We prove it first for the diagonal coefficients, and for increased precision $\frac{\varepsilon}{2}$. For all $i = 1 \ldots m$ we have

$$\begin{aligned} M_{i,i} &= \lambda_1 \ell (M_1)_{i,i} + \cdots + \lambda_m \ell (M_m)_{i,i} && \text{(by Lemma 19)} \\ &\leqslant \left(r x_1 y_1 + \frac{\varepsilon\ell}{2am} \right) (M_1)_{i,i} + \cdots + \left(r x_m y_m + \frac{\varepsilon\ell}{2am} \right) (M_m)_{i,i} && \text{(by (8))} \\ &\leqslant r \left(x_1 y_1 (M_1)_{i,i} + \cdots + x_m y_m (M_m)_{i,i} \right) + \frac{\varepsilon\ell}{2} \\ &\leqslant r K_{i,i} + \frac{\varepsilon\ell}{2} \; . && \text{(by Lemma 19)} \end{aligned}$$

We shall now use this in combination with Lemma 21, and get that

$$M = \bar{M} \leqslant r\bar{K} + \frac{\varepsilon\ell}{2} \leqslant \left(K^r + \frac{\varepsilon\ell}{2} \right) + \frac{\varepsilon\ell}{2} = K^r + \varepsilon\ell \; .$$

To conclude, we have proved $(M, \ell) \preccurlyeq_\varepsilon (K, k)^r \in \langle X \rangle$. Hence $Z \preccurlyeq_\varepsilon \langle X \rangle$.

Second direction: $\langle X \rangle \preccurlyeq_\varepsilon Z$**.** This part of the proof deals with the uniform structure of matrices. Let us consider a matrix (M, ℓ) in $\langle X \rangle$:

$$(M, \ell) = (\ell_1 M_{i_1}, \ell_1) \otimes \cdots \otimes (\ell_k M_{i_k}, \ell_k) \; .$$

By definition of Z_1, if $\ell < z$ then $(M, \ell) \in Z_1$. Let us concentrate our attention to the case $\ell \geqslant z$, and show that $(M, \ell) \preccurlyeq_\varepsilon (P, \ell)$ for some $(P, \ell) \in Z_2$.

Each matrix M_i for $i = 1 \ldots m$ may appear or not, once ore more, in the product defining (M, ℓ). For each $i = 1 \ldots m$, we define β_i to be the ratio of the weight corresponding to the weighted matrices in which M_i is involved, with respect to the total weight. This is formalized as follows:

$$\beta_i = \frac{1}{\ell} \sum_{\substack{j \in \{1, \ldots, k\} \\ i_j = i}} \ell_j \; .$$

Note that the sum of all the β_i's is naturally 1. Our goal is to construct a matrix (P, ℓ) from Z_2 such that $(M, \ell) \preccurlyeq_\varepsilon (P, \ell)$. Two independent arguments are involved in this proof: 1) show that the above product can be turned into a more regular one (i.e., a repetition of always the same pattern), and 2) show that the β_i's can be approximated by multiples of γ, yielding the λ_i parameters in the definition of Z_2. The proof now proceeds in two steps that correspond respectively to the two above points.

Our first step is to note that

$$M \leqslant \ell \; \overline{\beta_1 M_1 \otimes \cdots \otimes \beta_m M_m} \; .$$

This directly comes from Lemma 21, since $M = \bar{M}$ and $\ell(\beta_1 M_1 \otimes \cdots \otimes \beta_m M_m)$ coincide over the diagonal by Lemma 19. This presentation is already very close to the definition of Z_2. The only detail is that the coefficients β_i need not be multiples of γ. For correcting this, we choose for all $i = 1 \ldots m$ some λ_i's, multiple of γ, of unit sum, such that $\beta_i \leqslant \lambda_i + \gamma$ for all $i = 1 \dots m$. (This is possible simply by choosing $\lambda_i = \left\lfloor \frac{\beta_i}{\gamma} \right\rfloor \gamma$ for $i = 1 \dots m-1$, and λ_m such that $\lambda_1 + \cdots + \lambda_m = 1$.) We now have

$$\begin{aligned} \beta_1 M_1 \otimes \cdots \otimes \beta_m M_m &\leqslant (\lambda_1 + \gamma) M_1 \otimes \cdots \otimes (\lambda_m + \gamma) M_m && \text{(choice of } \lambda\text{'s)} \\ &\leqslant (\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m) + ma\gamma \\ &\leqslant (\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m) + \varepsilon\ . && \text{(by (4))} \end{aligned}$$

Hence $M \leqslant \ell\ \overline{\beta_1 M_1 \otimes \cdots \otimes \beta_m M_m} \leqslant \ell\ \overline{\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m} + \varepsilon\ell$ (using Lemma 21). Overall we have $(M, \ell) \preccurlyeq_\varepsilon Z_2$, and hence $\langle X \rangle \preccurlyeq_\varepsilon Z$ □

4.3.4 Finding uniform matrices in a long product: proof of Lemma 14

We shall now establish Lemma 14, which states that any product of weighted matrices in X can be decomposed according to a simple pattern. It states formally that given η, we can find p such that

$$\langle X \rangle = \langle X \rangle_{p,\eta} \cup \langle X \rangle_{p,\eta} \otimes \langle \langle X \rangle^u_{p,\eta} \rangle \otimes \langle X \rangle_{p,\eta}\ .$$

Technically, this result should be understood as a decomposition lemma (to some extent a Ramsey-like statement). It expresses that given a product of weighted matrices from X, either it belongs to $\langle X \rangle_{p,\eta}$, or it can be factorized as $(N_1, n_1) \otimes \cdots \otimes (N_m, n_m)$ such that $(N_1, n_1), (N_m, n_m) \in \langle X \rangle_{p,\eta}$ and $(N_2, n_2), \dots, (N_{m-1}, n_{m-1}) \in \langle X \rangle^u_{p,\eta}$. We can dive even further into this statement, and note that the property for a product of weighted matrices to belong to $\langle X \rangle_{p,\eta}$ or $\langle X \rangle^u_{p,\eta}$ is a property that does only involve the weight of the weighted matrices, and not at all the content of the matrices themselves. This means that the problem can be restated simply as a simplified one that involve only a sequence of positive integers. For this, let us redefine the notions of smallness to our case: a sequence of numbers $\ell_1, \dots, \ell_k$ of sum ℓ is *p, η-small* if there are $1 \leqslant i_1 < \cdots < i_r \leqslant k$ with $r \leqslant p$ such that $\sum_{j=1}^r \ell_{j_i} \geqslant (1 - \eta)\ell$. It is *uniform p, η-small* if $1 < i_1 < \cdots < i_r < k$ in the above definition.

We can now restate our problem as follows: for all $\eta > 0$ we have to find an integer p such that given a sequence of positive integers,

$$\bar{\ell} = \ell_1, \dots, \ell_k\ ,$$

either it is p, η-small, or it can be be factorized into subsequences as $\overline{\ell}^1, \dots, \overline{\ell}^m$ such that $\overline{\ell}^1, \overline{\ell}^m$ are p, η-small, and $\overline{\ell}^2, \dots, \overline{\ell}^{m-1}$ are uniform p, η-small.

Our first result in this direction is a criterion for proving that a product is p, η-small.

Lemma 22 *Let $\eta > 0$, there exists p such that for all sequences of positive integers $\bar{\ell} = \ell_1, \dots, \ell_k$ such that for all $i = 1 \dots k$,*

$$\frac{\ell_i}{\ell_1 + \cdots + \ell_i} \geqslant \eta\ , \qquad \left(\textit{or equivalently} \quad \ell_i \geqslant \frac{\ell_1 + \cdots + \ell_{i-1}}{1 - \eta} \right)$$

then $\bar{\ell}$ is p, η-small.

Proof Given η, let us fix some $p \geqslant \frac{1}{\eta}$. Consider now a sequence of positive integers $\bar{\ell} = \ell_1, \ldots, \ell_k$. Note first that if $k \leqslant p$, then the conclusion obviously holds. Otherwise, we factorize this sequence into $\bar{\ell} = \bar{\ell}^1, \bar{\ell}^2$ where $\bar{\ell}^2$ has length p. Let s_1 be the sum of the sequence $\bar{\ell}^1$ and s_2 be the sum of $\bar{\ell}_2$. From the hypothesis, we know that each integer in $\bar{\ell}^2$ is at least equal to $\frac{s_1}{1-\eta}$. Thus $s_2 \geqslant \frac{ps_1}{1-\eta}$, which means

$$s_1 \leqslant \frac{1-\eta}{p} s_2 \leqslant \eta s_2 \leqslant \eta(s_1 + s_2) \ .$$

Hence $\bar{\ell}$ is p, η-small. □

Let us advance toward the factorisation we aim at. We want to extract uniform p, η-small sequences of positive integers. Uniformity is a two-sided notion, since it requires that both the first and last integers in the sequence are 'small'. That is why, as an intermediate step, we consider the one-sided versions of uniform p, η-smallness. Formally, a sequence of positive integers $\ell_1, \ldots, \ell_k$ of sum ℓ is *right-uniform* p, η*-small* (*resp. left-uniform* p, η*-small*) if there exist $1 \leqslant i_1 < \cdots < i_m < k$ (*resp.* $1 < i_1 < \cdots < i_m \leqslant k$) for some $m \leqslant p$ such that $\sum_{j=1}^{m} \ell_{i_j} \leqslant \eta\ell$.

We are now ready to prove a one-sided variant of Lemma 14 we aim at.

Lemma 23 *Let* $\eta > 0$*, there exists an integer* p *such that all sequences of positive integers* $\bar{\ell}$ *can be factorized as* $\bar{\ell} = \bar{\ell}^1, \ldots, \bar{\ell}^k$ *such that*

- $(\bar{\ell}^1), \ldots, (\bar{\ell}^{k-1})$ *are right-uniform* p, η*-small, and*
- $\bar{\ell}^k$ *is* p, η*-small.*

Proof Let $\eta > 0$ and p be obtained from Lemma 22 for the value $\frac{\eta}{2}$.
We prove first claim $(\star)$. Given any sequence $\bar{\ell}$ of positive integers, then

- either $\bar{\ell}$ is p, η-small, or
- it has a non-empty prefix which is right-uniform p, η-small.

Let $\bar{\ell}$ be a sequence of positive integers. Two cases may arise, either for all $k \geqslant 1$

$$\frac{\ell_k}{\ell_1 + \cdots + \ell_k} \geqslant \frac{\eta}{2} \ .$$

and then by Lemma 22, $\bar{\ell}$ is $p, \frac{\eta}{2}$-small and hence p, η-small,

Otherwise, let $\ell_1, \ldots, \ell_k$ be the shortest non-empty prefix such that

$$\frac{\ell_k}{\ell_1 + \cdots + \ell_k} < \frac{\eta}{2} \ .$$

Note first that, by minimality in its construction, the sequence $\ell_1, \ldots, \ell_{k-1}$ satisfies the hypothesis of Lemma 22 for the value $\frac{\eta}{2}$. Hence, it is $p, \frac{\eta}{2}$-small. Since furthermore $\ell_k \leqslant \frac{\eta}{2}(\ell_1 + \cdots + \ell_k)$, it follows that $\ell_1, \ldots, \ell_k$ is right-uniform p, η-small. Claim $(\star)$ is proved.

The lemma itself is obtained by induction on the length of the sequence, using Claim $(\star)$. □

Let us now extend the above result into a two-sided version.

Lemma 24 *Let $\eta > 0$, there exists an integer p such that all sequences of positive integers $\bar{\ell}$ can be factorized as $\bar{\ell}^1, \dots, \bar{\ell}^k$ such that*

- *$\bar{\ell}^1$ and $\bar{\ell}^k$ are p, η-small, and*
- *$(\bar{\ell}^2), \dots, (\bar{\ell}^{k-1})$ are uniform p, η-small.*

Proof The principle is to compose Lemma 23 with itself, or more precisely, with its symmetric variant. For this, we need to be able to compose several use of such lemmas. This is the subject of the following claim.

We first Claim ($\star$): Given a sequence $\bar{\ell}$ factorized into $\bar{\ell} = \bar{\ell}^1, \dots, \bar{\ell}^k$, of respective sums $s_1, \dots, s_k = \bar{s}$,

- if the sequences $(\bar{\ell}^1), \dots, (\bar{\ell}^k)$ are all p, η-small, and $\bar{s}$ is p, η-small, then $\bar{\ell}$ is $p^2, 2\eta$-small,
- if the sequences $(\bar{\ell}^1), \dots, (\bar{\ell}^k)$ are right-uniform p, η-small, and $\bar{s}$ is left-uniform p, η-small, then $\bar{\ell}$ is uniform $p^2, 2\eta$-small.

For the first item. Let $i_1 < \dots < i_r$ with $r \leqslant p$ be the indices witnessing that $\bar{s}$ is p, η-small. Since each $\bar{\ell}^i$ is p, η-small, there exists a sub-sequence $\bar{b}^i$ of $\bar{\ell}^i$ of length at most p and sum at least $(1-\eta)s_i$. Now, consider the sequence

$$\bar{b} = \bar{b}^{i_1}, \dots, \bar{b}^{i_r} \ .$$

This is a sub-sequence of $\bar{\ell}$, of length at most p^2, and its sum is at most

$$\begin{aligned}(1-\eta)s_{i_1} + \dots + (1-\eta)s_{i_r} &\leqslant (1-\eta)(s_{i_1} + \dots + s_{i_r}) \\ &\leqslant (1-\eta)^2 t \leqslant (1-2\eta)t \ ,\end{aligned}$$

where t is the total sum of the sequence $\bar{\ell}$. Hence $\bar{b}$ is a witness that $\bar{\ell}$ is $p^2, 2\eta$-small. For the second item, this is the same proof, with just the extra remark that under the stronger assumptions that the sequences $(\bar{\ell}^1), \dots, (\bar{\ell}^k)$ are right-uniform p, η-small and $\bar{s}$ is left-uniform p, η-small, then neither the first element of $\bar{\ell}^1$, nor the last element of $\bar{\ell}^k$ are used in the construction of $\bar{b}$. Hence this time, the same sequence $\bar{b}$ is a witness that $\bar{\ell}$ is uniform $p^2, 2\eta$-small. Claim ($\star$) is established.

Consider now a sequence $\bar{\ell}$. According to Lemma 23 used with parameter $\frac{\varepsilon}{2}$, it can be decomposed as $\bar{\ell} = \bar{\ell}^1, \dots, \bar{\ell}^m$ where $(\bar{\ell}^1), \dots, (\bar{\ell}^{m-1})$ are right-uniform $p, \frac{\eta}{2}$-small, and $\bar{\ell}^m$ is $p, \frac{\eta}{2}$-small. Let $s_1, \dots, s_m$ be the respective sums of $(\bar{\ell}^1), \dots, (\bar{\ell}^m)$. We apply now Lemma 23, but this time in a mirrored version, to the sequence $\bar{s} = s_1, \dots, s_m$, and get $\bar{s}^1, \dots, \bar{s}^n$, where $\bar{s}^1$ is p, η-uniform, and $\bar{s}^2, \dots, \bar{s}^n$ are left-uniform p, η-small. Now, let us recall that each $\bar{s}^i$ is of the form $s_x, s_{x+1}, \dots, s_y$. Let $\bar{t}^i$ be the sequence $\bar{\ell}^x, \bar{\ell}^{x+1}, \dots, \bar{\ell}^y$. Clearly, $t^1, \dots, t^n = \bar{\ell}$. Using now the first item of Claim ($\star$), we get that $\bar{t}^1$ and $\bar{t}^n$ are p^2, η-small. Using finally the second item of Claim ($\star$), we get that $(\bar{t}^2), \dots, (\bar{t}^{n-1})$ are uniform p^2, η-small. $\square$

Let us recall now Lemma 14.

Lemma 14 *For all $X \subseteq \phi^{-1}(E)$ and all $\eta > 0$ there is an integer p such that:*

$$\langle X \rangle = \langle X \rangle_{p,\eta} \cup \langle X \rangle_{p,\eta} \otimes \langle \langle X \rangle^{u}_{p,\eta} \rangle \otimes \langle X \rangle_{p,\eta} \ .$$

Proof Given a product of weighted matrices resulting in a weighted matrix in $\langle X \rangle$, it is sufficient to apply Lemma 24 to the sequence of the weights of the weighted matrices. The resulting decomposition exactly matches the conclusion of the lemma. □

5 Conclusion and further remarks

In this paper, we have provided an algorithm for deciding the approximate comparison of distance automata. This algorithm involves the computation of the closure under product of sets of weighted matrices. This result can be of independent interest.

The main open question is the complexity of the problem. It is clear that the problem is at least PSPACE hard. A correct implementation of the arguments in this paper shows that EXSPACE is an upper bound. We conjecture that the exact complexity is PSPACE.

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