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Set-theoretical reflection equation: Classification of reflection maps

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Abstract

The set-theoretical reflection equation and its solutions, the reflection maps, recently introduced by two of the authors, is presented in general and then applied in the context of quadrirational Yang-Baxter maps. We provide a method for constructing reflection maps and we obtain a classification of solutions associated to all the families of quadrirational Yang-Baxter maps that have been classified recently.

Keywords: Set-theoretical Yang-Baxter equation, set-theoretical reflection equation, quadrirational maps, reflection maps.

Introduction

The Yang-Baxter equation, introduced to study a gas of spin with a δ interaction [1] or a 2-dimensional solvable statistical model in [2], has been developed in many directions and found applications in different areas of mathematics and physics. Much of the effort has been concentrated on the so-called *quantum* Yang-Baxter equation which is at the heart of the field of quantum integrable systems. Although less well-known, the set-theoretical solutions of the Yang-Baxter equation have also attracted some attention. This question goes back to Drinfeld [3] (and even to [4] where a solution was given). Then, different aspects have been developed and numerous connections have been established with, for example, the geometric crystals [5, 6], the soliton cellular automatons [7, 8], the scattering of multicomponent solitons [9, 10, 11], the discrete integrable systems [12, 13] and the Lax or transfer matrices [14, 15]. In view of these applications, there is also a lot of effort put into the possibility of classifying the solutions of the set-theoretical Yang-Baxter equations [16, 17].

In the context of quantum integrable systems with boundaries, there exists, in addition to the Yang-Baxter equation, a second equation: the reflection equation. This equation was introduced to encode the reflection on the boundary of particles in quantum field theory [18] and to prove the integrability of quantum models with boundaries [19]. This equation appears also as the defining relations for subalgebras of quantum groups [20, 21] which are the symmetry of quantum models such as the BC_n Sutherland models [22, 23], as a relation to construct commuting family of time evolutions for box-ball system with reflectiong end [24] and as a consistency relation for the coordinate Bethe ansatz applied to the δ gas with boundaries [25] and impurity [26].

Motivated by all these fields of application of the set-theoretical Yang-Baxter equation and of the reflection equation, we propose in this article to explore the interplay between these two notions and to study the set-theoretical reflection equation. Surprisingly, this equation was introduced only very recently by two of the authors [27] in the context of factorization of vector soliton interactions on the half-line. There, it appeared naturally as the equation ensuring that the interactions of the polarizations of an N-soliton solution among themselves and with the boundary factorises consistently as a sequence of soliton-soliton and soliton-boundary interactions. At the origin of the study in [27] lie the results of [28]. The latter falls into the vast area of integrable PDEs on the half-line for which various related approaches can be found in [29] (classical r matrix approach), [30] (Bäcklund transformations approach), [31] (symmetry algebra approach) or [32] (inverse scattering method approach).

In the first section of this paper, after recalling the important facts about the set-theoretical Yang-Baxter (YB) equation, we give the definition of the set-theoretical reflection equation and its solutions, the reflection maps. We also give its parametric version and comment on its graphical representation. We discuss, in this context, the construction of the transfer map to compare with [14] in the case of the YB equation. The second section is dedicated to the construction of reflection maps. The argument is based on a folding technique. Using this for the set-theoretical reflection equation associated to the YB maps classified in [16, 17], we obtain a classification of solutions. Finally, we conclude with the numerous open problems and perspectives related to our study.

1 Set-theoretical reflection equation

We present some elements and facts of the theory of reflection maps and set-theoretical reflection equation first introduced at the end of the paper [27]. To do so, we first need to recall some definitions related to YB maps (see e.g. [14]).

1.1 Yang-Baxter maps

Let S be a set and $\mathcal{R} : S \times S \to S \times S$ a map from the Cartesian product of S onto itself,

$$\mathcal{R}(X, Y) = (U, V) \equiv (f(X, Y), g(X, Y)) . \tag{1.1}$$

Define $\mathcal{R}_{ij} : S^N \to S^N$ as the map acting as $\mathcal{R}$ on the ith and jth copies of S in the N-fold Cartesian product S^N and identically on the others. More precisely, for $i < j$,

$$\mathcal{R}_{ij}(X_1, \ldots, X_n) = (X_1, \ldots, X_{i-1}, f(X_i, X_j), \ldots, g(X_i, X_j), X_{j+1}, \ldots, X_n) , \tag{1.2}$$

$$\mathcal{R}_{ji}(X_1, \ldots, X_n) = (X_1, \ldots, X_{i-1}, g(X_j, X_i), \ldots, f(X_j, X_i), X_{j+1}, \ldots, X_n) . \tag{1.3}$$

In particular, for $N = 2$, $\mathcal{R}_{12} \equiv \mathcal{R}$. If $\mathcal{R}$ satisfies the following Yang-Baxter relation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} , \tag{1.4}$$

as an identity on $S \times S \times S$, then $\mathcal{R}$ is called a Yang-Baxter map. In addition, if $\mathcal{R}$ satisfies

$$\mathcal{R}_{21}\mathcal{R}_{12} = Id , \tag{1.5}$$

then $\mathcal{R}$ is called a reversible Yang-Baxter map.

It is useful to introduce the so-called parametric Yang-Baxter map $\mathcal{R}(a, b)$ which is an important special case obtained by considering $S \times \Sigma$, where Σ is some parameter set (usually $\mathbb{C}$), instead of only S as before. The corresponding map

$$\mathcal{R} : (X, a; Y, b) \mapsto (f(X, a; Y, b), a; g(X, a; Y, b), b) \equiv (f_{ab}(X, Y), a; g_{ab}(X, Y), b) , \tag{1.6}$$

is conveniently written as $\mathcal{R}(a, b)$ and seen as acting only on $S \times S$:

$$\mathcal{R}(a, b) : (X, Y) \mapsto (f_{ab}(X, Y), g_{ab}(X, Y)) . \tag{1.7}$$

It satisfies the parametric YB equation

$$\mathcal{R}_{12}(a, b)\mathcal{R}_{13}(a, c)\mathcal{R}_{23}(b, c) = \mathcal{R}_{23}(b, c)\mathcal{R}_{13}(a, c)\mathcal{R}_{12}(a, b) , \tag{1.8}$$

and the corresponding reversibility condition reads

$$\mathcal{R}_{21}(b, a)\mathcal{R}_{12}(a, b) = Id . \tag{1.9}$$

The notion of parametric maps is important, for example, for YB maps arising from the study of vector or matrix soliton solutions of certain classical integrable nonlinear equations [10, 11, 9].

There is a well-known pictorial description of YB maps (see e.g. [16]). A map $\mathcal{R}$ defined by (1.6) is associated to a quadrilateral as in Fig. 1. Then, the parametric YB equation (1.8) can be understood as the consistency condition ensuring that the two possible orders of action of the map $\mathcal{R}$ on $(X, a; Y, b; Z, c)$, described by both sides of the equation shown in Fig.2, provide the same result. Let us also remark the similarities between this graphical representation of the set-theoretical YB equation and the so-called face representation of the YB equation [33, 34, 35].
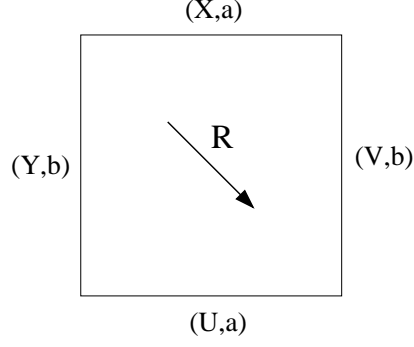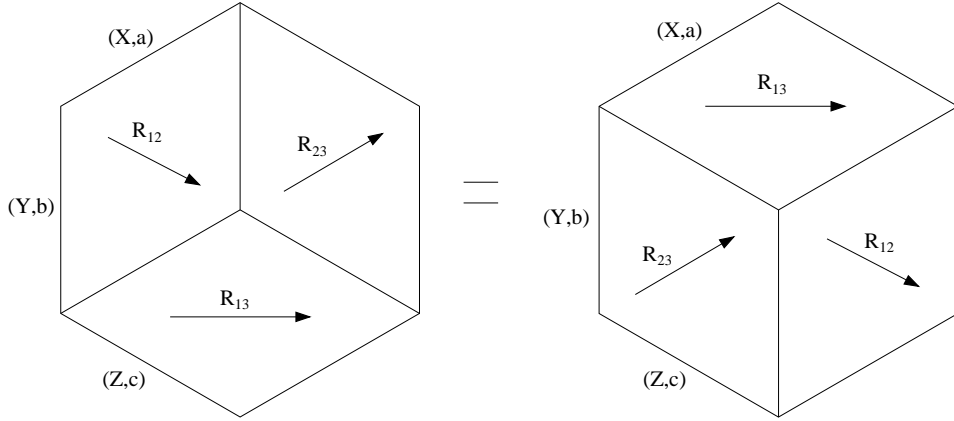
Figure 1: Map $\mathcal{R}$

Figure 2: Graphical representation of the set-theoretical Yang-Baxter equation

1.2 Reflection maps

Based on the form of the quantum reflection equation [18, 19] and on the important role played by its set-theoretical version in [27], we now introduce the following general definition of a reflection map.

Definition 1.1 *Given a Yang-Baxter map $\mathcal{R}$, a reflection map $\mathcal{B}$ is a solution of the set-theoretical reflection equation*

$$\mathcal{B}_1 \mathcal{R}_{21} \mathcal{B}_2 \mathcal{R}_{12} = \mathcal{R}_{21} \mathcal{B}_2 \mathcal{R}_{12} \mathcal{B}_1 \,, \tag{1.10}$$

as an identity on $S \times S$. The reflection map is called involutive if

$$\mathcal{B}\mathcal{B} = Id \,. \tag{1.11}$$

To introduce the notion of parametric reflection map, we need some extension of the above discussion for parametric YB maps in order to encompass known situations. Indeed, a parametric YB map has a trivial action on the parameter set Σ but it is known from the solutions found in [27] for reflection maps that a parametric reflection map can in general have a nontrivial action on the parameter set.

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In general this reads

$$\mathcal{B} : (X, a) \mapsto (h_a(X), \sigma(a)) \ , \quad X \in S \ , \quad a \in \Sigma, \tag{1.12}$$

for some maps h_a and σ. One can then use the convenient notation $\mathcal{B}(a)$ for a parametric reflection map but one has to keep in mind the nontrivial action of σ when composing a reflection map with other maps. To illustrate this, we give the definition of a parametric reflection map as follows:

Definition 1.2 *Given a parametric Yang-Baxter map* $\mathcal{R}_{12}(a, b)$, $a, b \in \Sigma$ *and a map* $\sigma : \Sigma \to \Sigma$, *a parametric reflection map* $\mathcal{B}(a)$ *is a solution of the parametric set-theoretical reflection equation*

$$\mathcal{B}_1(a)\mathcal{R}_{21}(\sigma(b), a)\mathcal{B}_2(b)\mathcal{R}_{12}(a, b) = \mathcal{R}_{21}(\sigma(b), \sigma(a))\mathcal{B}_2(b)\mathcal{R}_{12}(\sigma(a), b)\mathcal{B}_1(a) \tag{1.13}$$

as an identity on $S \times S$. *The reflection map is called involutive if* σ *is an involution in* Σ *and*

$$\mathcal{B}(\sigma(a))\mathcal{B}(a) = Id \,. \tag{1.14}$$

Note that classes of parametric reflection maps were constructed in [27] for $S = \mathbb{CP}^{n-1}$, $\Sigma = \mathbb{C}$ and $\sigma(k) = -k^*$ with $\mathcal{R}_{12}(k_1, k_2)$ being the parametric YB map corresponding to the vector nonlinear Schrödinger equation.

Analogous to the quadrilateral description of a YB map, we introduce a pictorial description of a reflection map as a half-quadrilateral shown in Fig. 3. The set-theoretical reflection equation
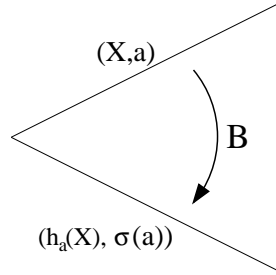

Figure 3: Map $\mathcal{B}$

can then be depicted as in Fig. 4. Let us remark that this visualization is reminiscent of the face representation of the reflection equation in [33, 34, 35]. There are also similitudes between Fig. 4 and half of the figure representing the tetrahedron equation for interaction round-a-cube [36, 37]. Explicitly, the left-hand side represents the following chain of maps

$$(X, a; Y, b) \xrightarrow{\mathcal{R}_{12}} (X_1, a; Y_1, b) \xrightarrow{\mathcal{B}_2} (X_1, a; Y_2, \sigma(b)) \xrightarrow{\mathcal{R}_{21}} (X_2, a; Y_3, \sigma(b)) \xrightarrow{\mathcal{B}_1} (X_3, \sigma(a); Y_3, \sigma(b))$$

with

$$X_1 = f_{ab}(X, Y) \ , \quad Y_1 = g_{ab}(X, Y) \ , \quad Y_2 = h_b(Y_1) \,,$$
$$X_2 = g_{\sigma(b)a}(Y_2, X_1) \ , \quad Y_3 = f_{\sigma(b)a}(Y_2, X_1) \ , \quad X_3 = h_a(X_2) \,.$$

Similarly, the right-hand side means

$$(X, a; Y, b) \xrightarrow{\mathcal{B}_1} (X_1', \sigma(a); Y, b) \xrightarrow{\mathcal{R}_{12}} (X_2', \sigma(a); Y_1', b) \xrightarrow{\mathcal{B}_2} (X_2', \sigma(a); Y_2', \sigma(b)) \xrightarrow{\mathcal{R}_{21}} (X_3, \sigma(a); Y_3, \sigma(b))$$
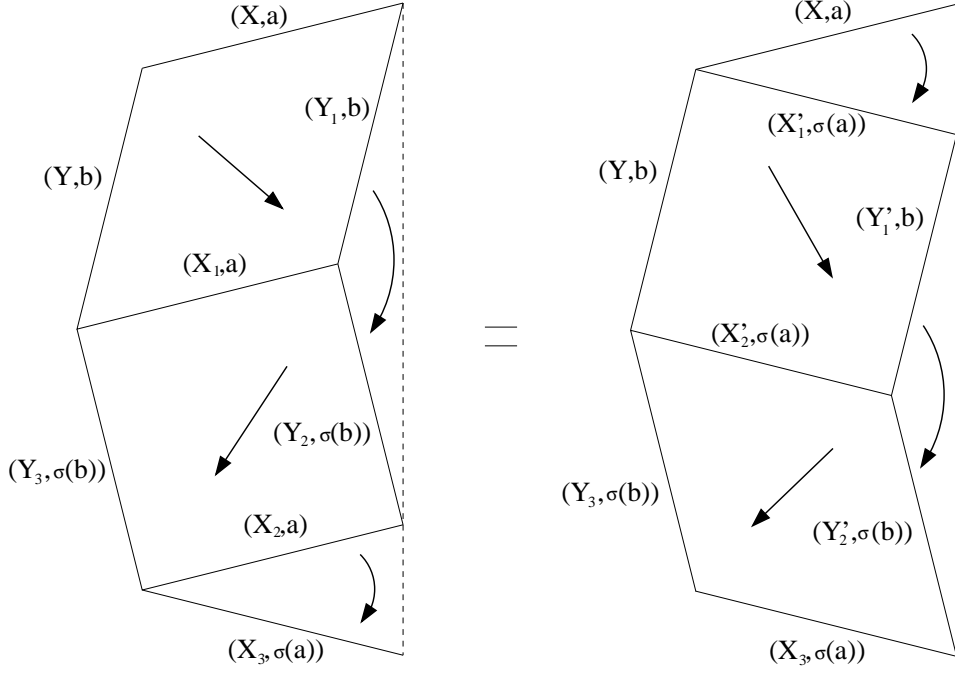
4

Figure 4: Graphical representation of the reflection equation

with

$$X'_1 = h_a(X) \,, \quad X'_2 = f_{\sigma(a)b}(X'_1, Y) \,, \quad Y'_1 = g_{\sigma(a)b}(X'_1, Y) \,,$$
$$Y'_2 = h_b(Y'_1) \,, \quad X_3 = g_{\sigma(b)\sigma(a)}(Y'_2, X'_2) \,, \quad Y_3 = f_{\sigma(b)\sigma(a)}(Y'_2, X'_2) \,.$$

So, Eq. (1.13) ensures that the two ways of obtaining (X_3, Y_3) from (X, Y) lead to the same result.

1.3 Transfer maps

We conclude this presentation on the theory of reflection maps by defining the notion of transfer maps in analogy with that introduced in [14]. Fix $N \geq 2$ and define for $j = 1, \ldots, N$ the following maps of S^N into itself,

$$\mathcal{T}_j = \mathcal{R}_{j+1j} \ldots \mathcal{R}_{Nj} \mathcal{B}_j^- \mathcal{R}_{jN} \ldots \mathcal{R}_{jj+1} \mathcal{R}_{jj-1} \ldots \mathcal{R}_{j1} \mathcal{B}_j^+ \mathcal{R}_{1j} \ldots \mathcal{R}_{j-1j} \,, \tag{1.15}$$

where $\mathcal{B}^+$ is a solution of

$$\mathcal{B}_1 \mathcal{R}_{21} \mathcal{B}_2 \mathcal{R}_{12} = \mathcal{R}_{21} \mathcal{B}_2 \mathcal{R}_{12} \mathcal{B}_1 \,, \tag{1.16}$$

and $\mathcal{B}^-$ a solution of

$$\mathcal{B}_1 \mathcal{R}_{12} \mathcal{B}_2 \mathcal{R}_{21} = \mathcal{R}_{12} \mathcal{B}_2 \mathcal{R}_{21} \mathcal{B}_1 \,. \tag{1.17}$$

Then one proves by direct (but long) calculation the following result

Proposition 1.3 *For any reversible Yang-Baxter map* $\mathcal{R}$, *the transfer maps* (1.15) *commute with each other*

$$\mathcal{T}_j \mathcal{T}_\ell = \mathcal{T}_\ell \mathcal{T}_j \ , \quad j, \ell = 1, \ldots, N \, . \tag{1.18}$$

Of course, this result transfers immediately to the case of parametric maps.

2   Reflection maps for quadrirational Yang-Baxter maps

In this section, we provide a classification of solutions for reflection maps associated to the 10 families of quadrirational YB maps found in [16, 17]. To achieve this, we consider a certain class of functions of rational type and use an adaptation of the folding argument used in [27]. This is further combined with a duality property which we derive from the particular properties of quadrirational maps exhibited in [16]. The overall amount of work required is reduced by applying the notion of symmetry as explained below.

2.1   Symmetry and quadrirational Yang-Baxter maps

We first need the following proposition shown in [17].

Proposition 2.1 *Let* $s(a)$ *be an involutive symmetry of the parametric Yang-Baxter map* $\mathcal{R}(a,b)$ *i.e.*

$$s(a)^2 = Id \quad and \quad s(a) \times s(b) \ \mathcal{R}(a,b) = \mathcal{R}(a,b) \ s(a) \times s(b) \, . \tag{2.1}$$

The map defined by

$$\mathcal{R}^s(a,b) = \big(s(a) \times Id\big) \ \mathcal{R}(a,b) \ \big(Id \times s(b)\big) \tag{2.2}$$

is a Yang-Baxter map.

Such symmetry maps $s(a)$ allow one to relate YB maps that look different a priori. This is in particular the case for quadrirational YB maps *i.e.* maps of a certain rational form acting on $S = \mathbb{C}\mathbb{P}^1$ with parameter set $\Sigma = \mathbb{C}$: the F families and the H families (in the notations of [16, 17]) are related by such symmetry maps. In general, in the context of YB maps, different maps may indeed be related to actual different underlying systems. But in the context of reflection maps, the following proposition shows that symmetries leave the set of solutions invariant.

Proposition 2.2 *The map* $\mathcal{B}(a)$ *is a parametric reflection map for the parametric Yang-Baxter map* $\mathcal{R}(a,b)$ *if and only if the map* $\mathcal{B}(a)$ *is a parametric reflection map for the parametric Yang-Baxter map* $\mathcal{R}^s(a,b)$.

<u>Proof:</u> The proof of this proposition consists in writing the set-theoretical reflection equation for $\mathcal{R}^s(a,b)$ expressed thanks to (2.2). Then, remarking for example that $\big(Id \times s(b)\big)K_1(a) = K_1(a)\big(Id \times s(b)\big)$ and using that $s(a)^2 = Id$, one proves that this equation is equivalent to the set-theoretical reflection equation for $\mathcal{R}(a,b)$. ∎

Our proposition therefore implies that the reflection maps for the F_I to F_V families are the same as the reflection maps for the corresponding families $H_I - H_V{}^1$. We can therefore restrict our attention

Type	$f_{ab}(X,Y)$	$g_{ab}(X,Y)$	P
F_I	aYP	bXP	$\frac{(1-b)X+b-a+(a-1)Y}{b(1-a)X+(a-b)XY+a(b-1)Y}$
F_{II}	$\frac{Y}{a}P$	$\frac{X}{b}P$	$\frac{aX-bY+b-a}{X-Y}$
F_{III}	$\frac{Y}{a}P$	$\frac{X}{b}P$	$\frac{aX-bY}{X-Y}$
F_{IV}	YP	XP	$1+\frac{b-a}{X-Y}$
F_V	$Y+P$	$X+P$	$\frac{a-b}{X-Y}$

Table 1: Families F for the quadrirational map

to the F families, which we reproduce in Table 1 for completeness. Finally, since these families involve parameters, we will work directly with parametric maps.

2.2 Method

We propose an adaptation of the folding method that was used in [27] to obtain families of reflection maps from the YB map associated to the vector nonlinear Schrödinger equation. Roughly speaking, the idea is to see a reflection map as a YB map modulo some folding applied on the variables on which the YB map acts. More precisely, here we start from Fig. 1 and impose that $b = \sigma(a)$ for some map σ and that $Y = \varphi_a(X)$ and $V = \varphi_a(U)$ for some map φ_a depending on a. For convenience, we assume that σ is an involution as well as φ_a, the latter meaning that

$$\varphi_{\sigma(a)}(\varphi_a(X)) = X. \tag{2.3}$$

Now, since (U,V) is related to (X,Y) by the YB map $\mathcal{R}(a,b)$, we obtain two possibilities for expressing V in terms of X

$$V = \varphi_a\left(f_{a\sigma(a)}(X,\varphi_a(X))\right) = g_{a\sigma(a)}(X,\varphi_a(X)). \tag{2.4}$$

This provides a functional equation constraining admissible functions φ_a and σ. So, for each such admissible pair (φ_a, σ), there is a well-defined map between V and X which we denote h_a

$$V = h_a(X), \tag{2.5}$$

and which can be computed using (2.4). The map h_a together with σ are our candidates to get a reflection map $\mathcal{B}$ as in (1.12).

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[1] With the understanding that F_{III}, H_{III}^A and H_{III}^B share the same solutions and recalling that H_{IV} does not exist.

To complete our set of functional equations, we can also look at the folding method from the point of view of the inverse map $\mathcal{R}^{-1}(a,b)$ which maps (U,V) to $(X,Y) = (g_{ba}(V,U), f_{ba}(V,U))$ (due to the reversibility of $\mathcal{R}$) and the companion map $\overline{\mathcal{R}}(b,a)$ which maps (V,X) to $(Y,U) \equiv (\bar{f}_{ba}(V,X), \bar{g}_{ba}(V,X))^2$. From the folding with the inverse map, we get the constraints

$$Y = \varphi_a\big(g_{\sigma(a)a}(\varphi_a(U),U)\big) = f_{\sigma(a)a}(\varphi_a(U),U). \tag{2.6}$$

Notice that, at first glance, this provides a new constraint for φ and σ. However, for the quadrirational maps we consider in this paper, listed in Table 1, we have $\mathcal{R}_{12}(a,b) = \mathcal{R}_{21}(b,a)$ i.e. $f_{ab}(X,Y) = g_{ba}(Y,X)$. Therefore, the constraint given by (2.6) is equivalent to (2.4). Now, provided that admissible functions (φ_a, σ) and the related h_a are found, we deduce that $Y = h_a(U)$. Using this last result and the folding for the companion map $\overline{\mathcal{R}}(b,a)$, we get

$$Y = h_a\big(\bar{g}_{\sigma(a)a}(h_a(X),X)\big) = \bar{f}_{\sigma(a)a}(h_a(X),X). \tag{2.7}$$

The quadrirational maps considered in this paper satisfy the following additional property

$$\bar{f}_{ab}(X,Y) = g_{ba}(Y,X), \quad \bar{g}_{ab}(X,Y) = f_{ba}(Y,X). \tag{2.8}$$

Taking account of these properties, one gets

$$Y = h_a\big(f_{a\sigma(a)}(X,h_a(X))\big) = g_{a\sigma(a)}(X,h_a(X)). \tag{2.9}$$

Comparing (2.4) and (2.9), we observe that φ_a and h_a play completely dual roles. Therefore, if φ_a is chosen to be of a certain form, then h_a should be of the same form and vice versa. This is reminiscent of the properties derived in [16] concerning the quadrirationality of a YB map, its companion map and their inverses. Also, the involution property (2.3) should hold for h_a.

Combining the ideas of the folding of YB map and the duality between φ_a and h_a, we are able to classify solutions of the set-theoretical reflection equation for the 5 families of quadrirational YB maps presented in [16]. In our context, it is natural to choose φ_a and h_a to be both Möbius maps with coefficients being first order polynomials in a:

$$\varphi_a(X) = \frac{p_1(a)X + p_2(a)}{p_3(a)X + p_4(a)} \quad , \quad h_a(X) = \frac{q_1(a)X + q_2(a)}{q_3(a)X + q_4(a)}, \tag{2.10}$$

where $p_j(a) = p_j^0 + p_j^1 a$ and $p_j(a) = q_j^0 + q_j^1 a$. This corresponds to 16 free parameters a priori. We also choose σ to be an involutive Möbius map, which gives 3 additional parameters:

$$\sigma(a) = \frac{c_1\, a + c_2}{c_3\, a - c_1}. \tag{2.11}$$

We then solve for these 19 parameters by inserting into the following constraints as explained above

$$h_a(X) = g_{a\sigma(a)}(X,\varphi_a(X)), \tag{2.12}$$

$$\varphi_{\sigma(a)}(\varphi_a(X)) = X, \quad h_{\sigma(a)}(h_a(X)) = X. \tag{2.13}$$

The resulting solutions are inserted in the set-theoretical reflection equation to check their validity.

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[2]Note that conventions for these maps may vary in the literature but results are unaffected by such variations since all four maps $\mathcal{R}, \mathcal{R}^{-1}, \overline{\mathcal{R}}, \overline{\mathcal{R}}^{-1}$ coincide as shown in [16].

2.3 Classification

It is straightforward to check that the identity map (*i.e.* both h_a and σ are the identity map) is a solution for all families so in the following, we only present nontrivial solutions, by which we mean that either σ or h_a or both are not the identity map. With the exception of one solution for the F_I solution, all the maps satisfying (2.12) and (2.13) are reflection maps as we found by inspection. Note that these operations have been performed with the help of symbolic computation softwares. The final results are reported in Table 2. Note that we did not find any solution for the F_V family. The duality property is explicitly shown in the table: the roles of h_a and φ_a can be swapped. All solutions depend on an arbitrary parameter μ. In view of the results in [27], this could be interpreted as the parameter controlling the boundary and, accordingly, the integrable boundary conditions of the underlying integrable systems associated to the quadrirational YB maps of the F families.

Type	$\sigma(a)$	$\varphi_a(X)$ (resp. $h_a(X)$)	$h_a(X)$ (resp. $\varphi_a(X)$)
F_I	$\frac{\mu^2}{a}$	$\frac{\mu X}{a}$	$\frac{(X(1+\mu)-\mu-a)\mu}{X(a+\mu)-(1+\mu)a}$
	$\frac{a+\mu^2-1}{a-1}$	$\frac{(a+\mu^2-1)X}{X(a-\mu-1)+a\mu}$	$\frac{a+\mu-X\mu-1}{a-1}$
	$\frac{1-a}{a(\mu^2-1)+1}$	$\frac{(a-1)X}{X(a+\mu a-1)-a\mu}$	$\frac{\mu a+1-X\mu-a}{1+a(\mu^2-1)}$
F_{II}	$\frac{\mu^2}{a}$	$\frac{a}{\mu}(X-1)+1$	$-\frac{aX}{\mu}$
	$-a+2\mu$	$\frac{X}{2X-1}$	$\frac{a-\mu-aX}{a-2\mu}$
F_{III}	$\frac{\mu^2}{a}$	$\frac{aX}{\mu}$	$-\frac{aX}{\mu}$
F_{IV}	$-a+2\mu$	$-X$	$X-a+\mu$

Table 2: Reflection maps for quadrirational map $F_I - F_{IV}$

In the geometric construction of the quadrirational maps $F_I - F_V$ in [16], the so-called singular points play an important role. It turns out that we can make use of these points in our construction to obtain other reflection maps which we call degenerate. This is achieved by allowing $\varphi_a(X)$ in (2.4) to take on the singular values of the corresponding family. In general, this can be $0, 1, \infty, \sigma(a)$. Each time a degenerate case is solution, the reflection equation is satisfied for *any* map σ. The duality

property still holds for these degenerate cases. We present the results in Table 3.

Type	$h_a(X)$ (resp. $\varphi_a(X)$)	$\varphi_a(X)$ (resp. $h_a(X)$)
F_I	$\frac{\sigma(a)(a-1)X}{a(X-1)+\sigma(a)(a-X)}$	∞
	$\frac{a-X+\sigma(a)(X-1)}{a-1}$	0
	$\frac{\sigma(a)}{a}X$	1
	X	$\sigma(a)$
F_{II}	X	∞
	$\frac{a}{\sigma(a)}(X-1)+1$	0
	$\frac{a}{\sigma(a)}X$	1
F_{III}	X	∞
	$\frac{a}{\sigma(a)}X$	0
F_{IV}	X	∞
	$X-a+\sigma(a)$	0
F_V	X	∞

Table 3: Degenerate reflection maps

Conclusions and perspectives

We introduced the general theory of the set-theoretical reflection equation. The associated notion of reflection maps originally introduced in [27] has been connected to the concept of quadrirational maps which is itself strongly related to the so-called Yang-Baxter maps and the idea of 3D consistency of equations on quad-graphs. We have used the quadrirational YB maps obtained in [16, 17] to

derive a classification of reflection maps associated to them, thus providing a large family of such maps in addition to the two classes originally found by two of the authors in [27] from the study of factorization of vector soliton interactions in the presence of an integrable boundary. Some ideas of the latter paper were adapted here to perform a rather "brute force" construction. It would be nice to reinterpret our results in more geometrical terms along the lines of the use of results on pencils of conics which led to the classification in [16]. More generally, one can also wonder how the general and abstract construction of YB maps as performed in [38, 39] can be adapted to yield a general construction of the associated reflection maps.

The results of this paper open the way to a wider programme of understanding reflection maps in the context of quad-graph systems and fully discrete integrable systems. Indeed, just like YB maps are one aspect of the idea of discrete integrability related to the 3D consistency approach, one can ask the important question of the meaning of a reflection map in the context of 3D consistency on a quad-graph. We are currently investigating this challenging issue and will report our results in a forthcoming paper. When this is understood, it will be interesting to see if the interpretation of a reflection map as an object encoding the integrable interaction between a soliton and a boundary which was found in [27] in the context of the vector nonlinear Schrödinger equation also holds in the realm of fully discrete integrable systems. We also believe that reflection maps can have a wide range of applications, for example in quantum algebra, in geometric crystals or in the soliton cellular automaton.

It seems also quite plain to us that different generalizations of the set-theoretical reflection equation introduced in this paper are possible, mimicking existing generalizations of the quantum reflection equation. Indeed, after the introduction of the quantum reflection equation in [18, 19], a more general structure, known as quadratic algebras, was introduced in [40]: the structure that is considered in that paper involves four different matrices subject to general Yang-Baxter like equations. It could be interesting to study the set-theoretical analog of these quadratic algebras. Following the same idea, it may be possible to define a reflection type equation associated to the dynamical set-theoretical Yang-Baxter equation [41]. In the quantum context, this type of equation has been studied e.g. in [42].

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