Antilinear deformations of Coxeter groups with application to Hamiltonian systems

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Declaration

The work presented in this thesis was carried out in collaboration with Prof. Andreas Fring at City University London. The work is believed to be original. No other universities or educational institutions have been approached to gain a degree or diploma for this work. I have clearly referenced all contributions made by other people, through their own publications or joint works with me.

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Abstract

In this thesis we provide several different systematic methods for constructing complex root spaces that remain invariant under an antilinear transformation. The first method is based on any element of the Weyl group, which is extended to factorizations of the Coxeter element and a reduced Coxeter element thereafter. An antilinear deformation method for the longest element of the Weyl group is given as well. Our last construction method leads to an alternative construction for q-deformed roots. For each of these construction methods we provide examples. In addition, we show a method of construction that for some special cases leads to rotations in the dual space and vice versa, starting from a rotation we find the root space involved. We then continue to apply these deformations to a generalized Calogero model and Affine Toda field theory. We provide a general solution for the ground state wave function of the Calogero model that is independent of a root representation and we extend this to the deformed case. An important property of this deformed Calogero model is that the amount of singularities in its potential is significantly reduced. We find that the exchange of particles in this model then leads to anyonic exchange factors. Following this we solve the model and find the ground state eigenvalues and eigenfunctions for the deformed Calogero model. We apply the q-deformed roots to an Affine Toda field theory and find that one may formulate a classical theory respecting the mass renormalisation of the quantum case.

Introduction

For physically meaningful systems the standard assumption in Quantum Mechanics textbooks [4, 5] is that the operators in those systems have to be Hermitian. An operator is referred to as being Hermitian, often referred to as Dirac Hermitian, if it is equal to its own adjoint, or conjugate transpose when it is a matrix. In general, when an operator H is Hermitian it satisfies the condition

$$\int \psi_1^*(H\psi_2)dt = \int (H\psi_1)^*\psi_2 dt, \tag{1}$$

for time t. The integration takes place over the entire domain on which the spectrum is defined and * indicates complex conjugation. ψ_n are wavefunctions [5], which are used to describe the states of a particle in a system and also to describe its particular behaviour. Hermiticity is a very strong constraint and its usefulness stems from the fact that it guarantees real eigenvalues, which are the energy levels of the system.

$$H = H^\dagger \implies E_n = E_n^* \tag{2}$$

The preservation of probability densities ρ are also guaranteed by the Hermiticity in a system, i.e.,

$$\frac{\partial\rho}{\partial t} = 0, \text{ with } \rho = |\psi_n|^2 . \tag{3}$$

$\frac{\partial\rho}{\partial t}$ is the change in probability of the observed quantities with time and ρ is the probability of ψ_n having an eigenvalue E_n. To prove the reality of the eigenvalues of a Hermitian Hamiltonian (2), we start with equation (1), and insert into it the Schrödinger equation $H\psi_n = E_n\psi_n$, which gives

$$\begin{aligned}
\int \psi_n^*(H\psi_n)dt &= \int (H\psi_n)^*\psi_n dt, \qquad (4)\\
\int \psi_n^*(E_n\psi_n)dt &= \int (E_n\psi_n)^*\psi_n dt,\\
E_n \int \psi_n^*\psi_n dt &= E_n^* \int \psi_n^*\psi_n dt,\\
(E_n - E_n^*) \int |\psi_n|^2 dt &= 0.
\end{aligned}$$

Since $\int |\psi_n|^2 dt$ can never be negative and since $\int |\psi_n|^2 dt = 0$ can only occur for $|\psi_n| = 0$, which is not permissible for a favourable wavefunction, we must therefore have that $E_n = E_n^*$. To prove equation (3) we start with the left hand side

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t}|\psi_n|^2, \qquad (5)\\
&= \frac{\partial}{\partial t}\psi_n^*\psi_n,\\
&= \frac{\partial\psi_n^*}{\partial t}\psi_n + \frac{\partial\psi_n}{\partial t}\psi_n^*,\\
&= (\frac{-i}{\hbar}H\psi_n)^*\psi_n + \psi_n^*(\frac{-i}{\hbar}H\psi_n),\\
&= (\frac{i}{\hbar})\psi_n^* H\psi_n + (\frac{-i}{\hbar})\psi_n^* H\psi_n,\\
&= 0.
\end{aligned}$$

Since Hermiticity preserves the probability of the system, conventionally it was assumed that non-Hermitian systems contained dissipation, which is the permanent loss of some of the energy as the system evolves with time. However, in 1960 Eugene Wigner [6] made the observation that operators that are invariant under an antilinear transformation possess real eigenvalues if their eigenfunctions are also invariant under the same transformation. These operators are not necessarily Hermitian. An antilinear transformation ω is a transformation that satisfies

$$\omega(ax_1 + bx_2) = a^*\omega x_1 + b^*\omega x_2, \tag{6}$$

where a, b are complex numbers and $x_{1,}$ x_2 are vectors. If the operator H, not necessarily Hermitian, is invariant under the antilinear operator ω, then H will commute with ω. As proof of the reality of the spectrum of H, we assume we have a Hamiltonian H which is invariant under the antilinear transformation ω, and we have an eigenfunction ψ of H with the eigenvalue E

$$H\psi = E\psi \quad \text{and} \quad [H, \omega] = 0 \tag{7}$$

Assuming that H and ω commute, ψ has to be also an eigenfunction of ω and we assume it has eigenvalue γ, then

$$\omega\psi = \gamma\psi. \tag{8}$$

As ω is an antilinear operator and making the assumption that $\omega^2 = 1$, then ω is an isometry. However, when (8) is not true then one encounters what is known as broken $\mathcal{PT}$-symmetry, which leads to eigenvalues that appear in complex conjugate pairs. Now acting on (8) with ω, gives

$$\omega\omega\psi = \omega\gamma\psi \Rightarrow \psi = \gamma^*\omega\psi = \gamma^*\gamma\psi, \tag{9}$$

and $\psi = \gamma^*\gamma\psi$ is solved by $\gamma = e^{i\phi}$ for $\phi \in \mathbb{R}$, therefore

$$\omega\psi = e^{i\phi}\psi. \tag{10}$$

This means that since ω is an isometry, its eigenvalue is merely a phase. If we now act on $H\psi = E\psi$ with ω and use the fact that H and ω commute, we have

$$\omega H\psi = \omega E\psi \Rightarrow H\omega\psi = E^*\omega\psi. \tag{11}$$

Using (10) and the second equation in (7), we arrive at

$$He^{i\phi}\psi = E^*e^{i\phi}\psi \Rightarrow e^{i\phi}H\psi = e^{i\phi}E^*\psi \Rightarrow H\psi = E^*\psi \Rightarrow E\psi = E^*\psi. \tag{12}$$

From this we deduce that $E = E^*$, i.e., the eigenvalue E of the Hamiltonian H is real, $E \in \mathbb{R}$.

There are many operators with exactly these properties and a good example of this is the $\mathcal{PT}$-operator [1]. Here $\mathcal{P}$ is the parity operator and $\mathcal{T}$ is the time reversal operator. These operators have the actions on the momentum p and coordinate x as follows:

$$\begin{aligned} \mathcal{P} &: \; x \to -x, \; p \to -p; \\ \mathcal{T} &: \; x \to x, \; p \to -p, \; \imath \to -\imath. \end{aligned} \tag{13}$$

A very important constraint on these operators is that one wants to preserve the commutation relation between the position and momentum operators $\hat{x}$ and $\hat{p}$, $[\hat{x}, \hat{p}] = i$. It is easy to see that if you act on the left hand side of the relation with $\mathcal{T}$ this is equal to $-i$, so to preserve the relation when acting on the right hand side of this relation with $\mathcal{T}$, it should conjugate it. In other words we have that $\mathcal{PT}$: $[\hat{x}, \hat{p}] \to -[\hat{x}, \hat{p}] \Rightarrow \mathcal{PT}$: $i \to -i$. Therefore most often one finds the $\mathcal{T}$-operator employed as complex conjugation. There are however cases where it is not the case that $\mathcal{T}$ is used as complex conjugation. In [7] Bender and Mannheim showed that one can formulate a relativistic $\mathcal{PT}$-symmetric quantum theory, where instead of $\mathcal{T}$ being complex conjugation, it directly sends the time coordinate t to $-t$. In addition to this, in most cases one find that it is assumed that $\mathcal{T}^2 = 1$, for time reversal being even. However, it has been shown that it is possible to formulate a consistent theory for the case of odd time reversal, i.e., $\mathcal{T}^2 = -1$, see [8].

Since the $\mathcal{PT}$-operator is an antilinear operator, having a Hamiltonian that possesses $\mathcal{PT}$-symmetry will ensure real eigenenergies when the eigenfunctions of the Hamiltonian possess the same symmetry. This means that not only standardly acceptable Hermitian models are candidates for having physical interpretation, but also a broad set of non-Hermitian Hamiltonians. Thus non-Hermitian models that have previously been disregarded can be made sense of by using their inherent $\mathcal{PT}$-symmetry and/or broader antilinear symmetry, or by employing deformations that will give these properties.

Over the years many methods have been built or borrowed to gather some physical meaning from non-Hermitian Hamiltonians, such as quasi-Hermiticity and pseudo-Hermiticity.

Consider a non-Hermitian Hamiltonian H and a Hermitian Hamiltonian

h and relate them to each other via a similarity transformation such that

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \Longleftrightarrow H^\dagger \rho = \rho H \text{ with } \rho = \eta^\dagger \eta. \qquad (14)$$

If there exists such a metric ρ which is not necessarily positive definite, although invertible, linear and Hermitian then this is known as pseudo-Hermiticity, see for instance [9]-[14]. When the metric ρ is positive definite but not invertible, this is known as quasi-Hermiticity. Quasi-Hermiticity was first introduced in 1960 by Dieudonné [15] and has been studied in various contexts, see [13][15]-[17].

Since h is a Hamiltonian that is Hermitian, it has real eigenvalues and the Hamiltonians H and h now have the same eigenvalues as they are in the same similarity class which then in turn implies the reality of the eigenvalues of H. If we now define a new inner product as [17]

$$(\psi_1, \psi_2)_\rho \equiv (\psi_1, \rho\psi_2), \qquad (15)$$

then H is Hermitian with respect to this new inner product since

$$(\psi_1, H\psi_2)_\rho = (\psi_1, \rho H \psi_2) = (\psi_1, H^\dagger \rho \psi_2) = (H\psi_1, \psi_2)_\rho. \qquad (16)$$

Originally when Bender et al, [1] first started investigating $\mathcal{PT}$-symmetry, they were examining the Hamiltonian $H = p^2 + x^2 + \imath x^3$, which has eigenenergies that are both real and positive. It is claimed in [1] that the reality of the spectrum of this Hamiltonian is due to it having $\mathcal{PT}$-symmetry. It is not difficult to establish that this Hamiltonian is indeed invariant under the $\mathcal{PT}$-transformation (7). However, Bender et al, extended this to the investigation of the whole class of Hamiltonians $H = p^2 + m^2x^2 - (\imath x)^N$ for $N \in \mathbb{R}$, m the mass. They find that for $N \geq 2$ the energies are completely real and also positive, where for $N = 2$, the Hamiltonian is the harmonic oscillator. For $1 < N < 2$ there are a finite number of eigenenergies that are real and the rest appear in complex conjugate pairs. For $N \leq 1$ there are only complex eigenvalues. These results are depicted in a well-known figure, see Figure 1. One can now adopt this idea and use it to construct new models that will

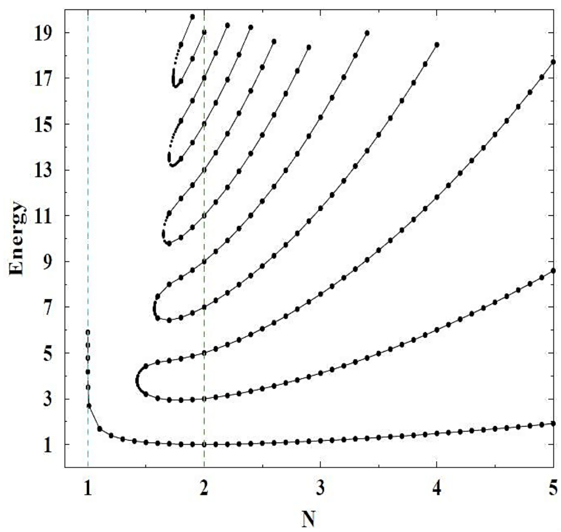


Figure 1: [1] Graphical representation of the eigenvalues of the set of Hamiltonians $H = p^2 + m^2x^2 - (\imath x)^N$

have real energies once it has been established that these models respect the conditions mentioned above. For instance, one can investigate spin chains such as was done in [18], where Korff and Weston analyzed the $\mathcal{PT}$-symmetry and quasi-Hermiticity of the XXZ spin chain. Another possibility is to investigate Kortweg de Vries type equations, which has been done by several different people. For example in [19] the author proposed a $\mathcal{PT}$-symmetric extension of the KdV equation which he then related to some non-Hermitian Hamiltonians and constructed the first few charges of the deformed model or in [20] Bender et.al. also constructs extensions of the KdV equation which then results in a new family of nonlinear wave equations. The quantum brachistochrone problem has also been investigated [14][21]. Bender et.al. [21] showed that when the operator that governs time-evolution in the quantum brachistochrone problem is a $\mathcal{PT}$-symmetric non-Hermitian operator, one can make the evolution of time from a specified initial to a specified final state arbitrarily small. Assis and Fring [14] showed that this can also be achieved for a non-Hermitian Hamiltonian with no $\mathcal{PT}$-symmetry.

$\mathcal{PT}$-symmetry has even been observed and proved to be a useful concept in the field of optics [22][23][24][25]. In order to draw an analogy between optics and Quantum mechanics one starts with a set of equations which are crucial to the field of optics, namely the Maxwell's equations [26][27]

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{17}$$

$$\nabla \times B = \mu_0 E + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}, \tag{18}$$

$$\nabla . E = \frac{\sigma_0}{\varepsilon_0}, \tag{19}$$

$$\nabla . B = 0, \tag{20}$$

where E is an electric field, B is a magnetic field, J is the current density, μ_0 is the electric conductivity and ε_0 is the magnetic permeability and σ_0 is the charge density. Equation (17) is derived from Ampere's law, equation (18) is derived from Faraday's law, the third equations (19) is derived from the Gaussian theorem for a magnetic field and the last equation is derived from the Gaussian theorem for an electric field. From these equations one notes that the magnetic and electric fields are coupled to each other [28]. [27].If one now takes the curl of equation (17) or (18) and use the identity

$\nabla \times \nabla \times \hat{V} = \nabla(\nabla.\hat{V}) - \nabla^2\hat{V}$ for any vector field $\hat{V}$, we will obtain a wave equation of the form

$$\nabla^2 E - \sigma_0\mu_0\frac{\partial E}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} = 0, \tag{21}$$

where ∇^2 is the Laplacian operator [28]. If we then make an assumption on the form of $E = E_0e^{-i\omega t}$ [27] and differentiate this twice we get

$$\begin{aligned} E &= E_0e^{-\iota\omega t}, \\ \frac{\partial E}{\partial t} &= -i\omega E_0e^{-\iota\omega t} = -\iota\omega E, \\ \frac{\partial^2 E}{\partial t^2} &= -\omega^2E_0e^{-\iota\omega t} = -\omega^2E, \end{aligned} \tag{22}$$

and substitute this back into (21) we obtain

$$\begin{aligned} \nabla^2 E + \sigma_0\mu_0\iota\omega E + \mu_0\varepsilon_0\omega^2 E &= 0, \\ \nabla^2 E + \mu_0\omega(\sigma_0\iota + \varepsilon_0\omega)E &= 0, \end{aligned} \tag{23}$$

which is known as the Helmholtz equation [28][29]. Often the wave number is abbreviated as $k^2 = \mu_0\omega(\sigma_0\iota + \varepsilon_0\omega)$. One can make a paraxial approximation to equation (23), if we consider an element of the field having the form $E = Ae^{-\iota kz}$, assuming we have a wave traveling in the, z-direction for instance. To proceed any further it is intuitive to decompose the differential operators in (23) by using the identity $\nabla\phi = \nabla_T\phi + \hat{z}\frac{\partial\phi}{\partial z}$, where $\nabla_T\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y}$ is the transverse gradient of unspecified ϕ. Equation (21) then becomes

$$\nabla_T^2 A - \iota k\frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial z^2} = 0.$$

Now making the assumption that $\frac{\partial^2 A}{\partial z^2} \ll k\frac{\partial A}{\partial t}$, which is the condition necessary for making the paraxial approximation as this means that $\frac{\partial A}{\partial z} \ll A$, so we ignore the third term leading to [27][29]

$$\nabla_T^2 A - \iota k \frac{\partial A}{\partial z} = 0.$$

The paraxial approximation of the Helmholtz equation (23) leads to a Schrödinger type equation that has the form [30, 31]

$$i\frac{\partial}{\partial z}\psi = -(\frac{\partial^2}{\partial x^2} + V(x))\psi \tag{24}$$

where ψ is the amplitude of the electric field, z is the propagation distance and $V(x)$ is the optical potential. An interesting difference between $\mathcal{PT}$-symmetric quantum physics and optics is that instead of seeking a system with unbroken $\mathcal{PT}$-symmetry, i.e., one where the eigenvalues are completely real, in optics one is at present usually interested in systems or regions of systems where the eigenvalues appear in complex conjugate pairs. This originates from the idea that when the optical potential $V(x)$ is complex, it represents a complex refractive index and the imaginary part of this is either the loss or gain of the system. Recently it has been found and shown in experiments that for certain $\mathcal{PT}$-symmetric systems that as one increases the loss in the system past a specific point known as an exceptional point, the transmission in the system starts to increase, even though it was decreasing before the loss passed through the exceptional point. The results of this experiment is depicted in Figure 2 [2].

Another example of how $\mathcal{PT}$-symmetry was used in the field of optics is the idea of optical solitons [24] and some more recent investigations have been into the concept of stabilizing the soliton solutions of particular systems [32], as well as solitons in $\mathcal{PT}$-invariant dimers [33].

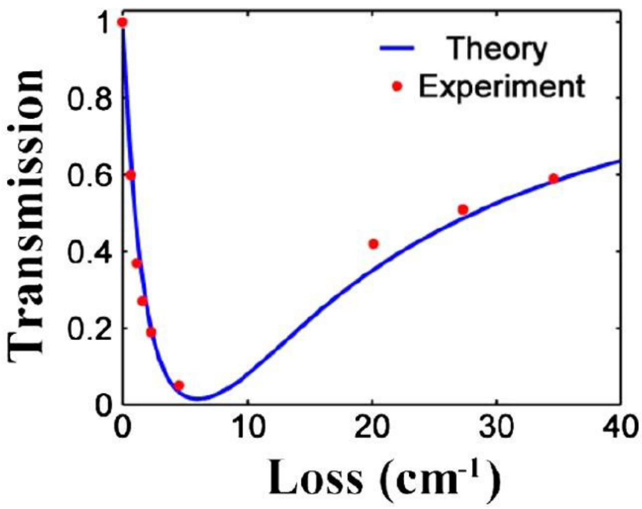


Figure 2: Passive $\mathcal{PT}$-symmetry breaking observed in an experiment [2]. The graph depicts how the transmission decreases as the loss of the system is increased up until an exceptional point is reached, where the transmission starts to increase.

Assis and Fring [34] showed how the Benjamin-Ono[1] equation

$$u_t + uu_x + \lambda \tilde{H} u_{xx} = 0, \tag{25}$$

where $\tilde{H} \equiv$ Hilbert transform, i.e., $\tilde{H}u(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{u(z)}{z-x} dz$, can be related to Calogero systems

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{i<j}(x_i - x_j)^2 + \sum_{i<j}\frac{g}{(x_i - x_j)^2}, \tag{26}$$

if the poles of the solutions of the Benjamin-Ono equation satisfy the Calogero equations of motion . i.e., z_k satisfies the A_n-Calogero equation of motion

$$\ddot{z}_k = \frac{\lambda^2}{2}\sum_{k\neq j}(z_j - z_k)^{-3} \tag{27}$$

[1]The Boussinesq equation $u_{tt} + (\alpha u_{xx} + \beta u^2 - \gamma u)_{xx} = 0$ and the KdV equation $u_t + (\alpha u_{xx} + \beta u^2)_x = 0$ and the Burgers' equation $u_t + \alpha u_{xx} + \beta(u^2)_x = 0$ were also investigated in the same publication and related to the Calogero model in a similar fashion with additional constraints.

for $u(x,t)$ being a solution to (25)

$$u(x,t) = \frac{\lambda}{2} \sum_{k=1}^{\ell} \left(\frac{i}{x - z_k} - \frac{i}{x - z_k^*} \right) \in \mathbb{R} \tag{28}$$

with g a coupling constant. They explore how $\mathcal{PT}$-symmetric Calogero systems emerge naturally from solutions of these equations without having to deform the Calogero systems themselves. Some of these systems have even been found to have solitons and compactons [35][36]. In [37] the integrability of $\mathcal{PT}$-symmetric deformed models were investigated by use of the Painlevé test and it was found that the Burgers' equation allows for a large amount deformations that indeed pass this test, but the Korteweg de Vries equation does not pass the test in total generality.

Systems such as Calogero models and Toda field theories can be related to root systems coupled to Coxeter groups or Weyl groups in complete generality [38][39][40]. In single particle systems it is easy enough to obtain the symmetry of the system, which can and has been used to construct new models that are physically meaningful. However, in field theories and multi-particle systems it is not always a straight forward procedure to observe the symmetries involved in these systems. Often it will involve elaborate transformations on the level of the dynamical variables. The generalized Calogero model takes the form

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\alpha \in \Delta} (\alpha.x)^2 + \sum_{\alpha \in \Delta} \frac{g}{(\alpha.x)^2}, \tag{29}$$

where α are the roots of the Coxeter group, $x = \{x_1, x_2, ..., x_n\}$ the position coordinates, $p = \{p_1, p_2, ..., p_n\}$ is the momentum and n is the number of particles in the system. For example one could choose to deform the variables of the A_2-Calogero model as [41]

$$
\begin{aligned}
x_1 &\rightarrow \tilde{x}_1 = x_1 \cosh\varepsilon + i\sqrt{3}(x_2 - x_3)\sinh\varepsilon, \\
x_2 &\rightarrow \tilde{x}_2 = x_2 \cosh\varepsilon + i\sqrt{3}(x_3 - x_1)\sinh\varepsilon, \\
x_3 &\rightarrow \tilde{x}_3 = x_3 \cosh\varepsilon + i\sqrt{3}(x_1 - x_2)\sinh\varepsilon,
\end{aligned}
\tag{30}
$$

and then using the standard three dimensional representation of the simple A_2-roots, we would compute

$$
\begin{aligned}
\alpha_1.\tilde{x} &= x_{12}\cosh\varepsilon - \frac{i}{\sqrt{3}}(x_{13} + x_{23})\sinh\varepsilon, \\
\alpha_2.\tilde{x} &= x_{23}\cosh\varepsilon - \frac{i}{\sqrt{3}}(x_{21} + x_{31})\sinh\varepsilon, \\
\alpha_3.\tilde{x} &= x_{13}\cosh\varepsilon - \frac{i}{\sqrt{3}}(x_{12} + x_{32})\sinh\varepsilon,
\end{aligned}
\tag{31}
$$

where we abbreviate $x_{ij} := x_i - x_j$ and then the symmetries would be

$$
\begin{aligned}
S_1 &: x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_3, i \rightarrow -i, \\
S_2 &: x_2 \leftrightarrow x_3, x_1 \leftrightarrow x_1, i \rightarrow -i.
\end{aligned}
\tag{32}
$$

However, even though this deformation does work, there is no obvious reasoning as to why one would choose to deform the variables x_i in this particular fashion. Since the root systems remain invariant under the action of the whole Weyl group, they possess a natural symmetry. So in the A_2-Calogero model one can deform the simple roots instead as

$$
\begin{aligned}
\alpha_1 &\rightarrow \tilde{\alpha}_1 = \alpha_1 \cosh\varepsilon + i\sqrt{3}(\lambda_2)\sinh\varepsilon, \\
\alpha_2 &\rightarrow \tilde{\alpha}_2 = \alpha_2 \cosh\varepsilon - i\sqrt{3}(\lambda_1)\sinh\varepsilon,
\end{aligned}
\tag{33}
$$

where the λ_i are the fundamental weights. It is a far less involved task to identify these symmetries in these root spaces than it is in the dual space on the level of the dynamical variables, the symmetries in our example are

$$\begin{aligned} \sigma_1^{\varepsilon} &: \tilde{\alpha}_1 \leftrightarrow -\tilde{\alpha}_1, \tilde{\alpha}_2 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2; \\ \sigma_2^{\varepsilon} &: \tilde{\alpha}_1 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2, \tilde{\alpha}_2 \leftrightarrow -\tilde{\alpha}_2. \end{aligned} \tag{34}$$

Once the symmetry has been identified one can easily change it over to the dual space by using the identity $\alpha.\tilde{x} = \tilde{\alpha}.x$, which would lead to the symmetry in the dual space

$$\begin{aligned} \sigma_1^{\varepsilon} &: x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_3, i \to -i, \\ \sigma_2^{\varepsilon} &: x_2 \leftrightarrow x_3, x_1 \leftrightarrow x_1, i \to -i. \end{aligned} \tag{35}$$

We will show that one can eliminate the singularities that exist in the undeformed Calogero Hamiltonian (26) by deforming the root space, that the Calogero model is related to, in an antilinear fashion. An interesting result of this procedure is that, even though there are no more singularities in the deformed model, one picks up a phase when some particles are exchanged and we will identify this as anyonic behaviour. With this in mind we construct a completely general, systematic method for deforming Coxeter groups in an antilinear fashion. The aim is to make use of the inherent symmetry that already exists in the Coxeter groups. Deforming the Coxeter groups in such a way will ensure that the deformed groups will remain invariant under an antilinear symmetry, which we will employ as analogues of the $\mathcal{P}$-operator. We begin by selecting any involutory element $\hat{\omega} \in \mathcal{W}$, $\hat{\omega}^2 = \mathbb{I}$, and deform it antilinearly. There are several different possible choices for $\hat{\omega}$. In [41] the authors make the choice of directly deforming the Weyl reflections $\sigma_i \in \mathcal{W}$ themselves, however for this particular choice, it is only possible to deform the rank 2 algebras consistently. This was done explicitly in [34][41].

Another natural choice would be to deform the longest element $\omega_0 \in \mathcal{W}$ of the Weyl group. This deformation is more general than the deformation of the Weyl reflections themselves, but is still restricted to particular Coxeter groups, A_l, D_{2l} and E_6.

One can also deform the factors of the Coxeter element $\sigma_\pm \in \mathcal{W}$, however for some groups this deformation results in trivial deformations. To address these groups that had trivial deformations one can deform a new modified Coxeter element which has a lower order than the original Coxeter element.

Except for the deformation of the Weyl reflections, we show the generalized constructions for these deformations. For the Weyl reflection deformations we give an explicit argument as to why this only works for the rank 2 algebras. We give case-by-case solutions in support of the other deformations.

We generalize the solution of the Calogero model that was originally constructed by Calogero [42][43][44] in 1969, such that it is independent of its root space representation. Thereafter we extend this generalized solution to that of a deformed model based on deformations of the root spaces of the model. Additionally we calculate the groundstate eigenvalues and eigenfunctions for the deformed model and give explicit examples of the symmetries in the dual space after deforming the root systems of the model.

As mentioned above affine Toda field theories can also be related to Coxeter groups in complete generality. In 1+1 dimensions an affine Toda field theory is a theory whose Lagrangian is of the following form [45][39][46]

$$\mathcal{L} := \frac{1}{2}\sum_{i=1}^{\ell} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{m^2}{\beta^2}\sum_{i=0}^{\ell} n_i e^{\beta \alpha_i \cdot \phi}, \qquad \alpha_i \in \Delta. \tag{36}$$

A key feature of affine Toda field theories is that after renormalizing the model, the classical mass spectrum is preserved in quantum field theory, when the affine Toda field theory is related to a simply laced Lie algebra [47][48]-[55]. For the non simply laced cases this property no longer holds [47][56]-[62] and one has to consider pairs of dual algebras [63], in order to formulate a consistent theory. Another property of the affine Toda field theory (36) is that this theory has an infinite amount of conserved charges that commute such that the theory is classically integrable. On the quantum level this implies a factorisable S-matrix. When the coupling constant β is real this leads to diagonal scattering matrices or S-matrices. A scattering matrix is a matrix of a system, in the process of being scattered, that relates final and initial states of the particles being scattered [64]. For integral systems this implies that the n-particle S-matrix factorises into 2-particle S-matrices. Scattering refers to the result of when two particles in a system collide [4]. The S-matrices of the Toda field theories based on simply laced algebras were constructed in [48]-[55] where the authors use the building block

$$\begin{aligned} \{x\} &= \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}, \\ (x) &= \frac{\sinh\frac{1}{2}(\theta+\frac{i\pi x}{h})}{\sinh\frac{1}{2}(\theta-\frac{i\pi x}{h})}\ , \end{aligned} \tag{37}$$

where the h is the Coxeter number, θ is the difference in rapidity of the two particle scattering and $B(\beta) = \frac{2\beta^2}{\beta^2+4\pi}$ for $0 \leq B \leq 2$ as conjectured in [48]-[55]. The elements of the scattering matrix then take the form

$$S_{ab}(\theta) = \prod_{x=1}^{h} \{x\}^{m_{ab}(x)}, \tag{38}$$

where $m_{ab}(x)$ are the multiplicities of $\{x\}$ whose explicit form is known and yet another application of Coxeter elements. For the theories based on non simply laced algebras there are many constructions for specific examples [56]-[62] however a general construction was only found in 1997 by Oota [47][62]. Oota constructed a general S matrix for an unspecified dual pair of non simply laced Coxeter groups $(X_N^{(1)}, Y_M^{(l)})$. He uses the generalisation of the building block [56]-[62][65] similar in form to (37), i.e.,

$$\begin{aligned}
\{x,y\} &= \frac{< x-1, y-1 >< x+1, y+1 >}{< x-1, y+1 >< x+1, y-1 >}, \\
< x, y > &= < \frac{(2-B)x}{2h} + \frac{By}{2h^{(l)\vee}} >, \\
< x > &= \frac{\sinh \frac{1}{2}(\theta + i\pi x)}{\sinh \frac{1}{2}(\theta - i\pi x)},
\end{aligned} \tag{39}$$

here $h^{(l)\vee}$ is the l-th dual Coxeter number and $h^\vee$ is the dual Coxeter number of the group X_N and B now takes the form

$$B(\beta) = \frac{2\beta^2}{\beta^2 + \frac{4\pi h}{h^\vee}} \quad \text{for} \quad 0 \leq B \leq 2. \tag{40}$$

Using (39) and (40) Oota then proposed the general elements of the S matrix for the pair $(X_N^{(1)}, Y_M^{(l)})$ as [62]

$$S_{ab}(\theta) = \prod_{x=1}^{h} \prod_{y=1}^{h^{(l)\vee}} \{x,y\}^{m_{ab}(x,y)}, \tag{41}$$

with the multiplicities of $\{x,y\}$ being $m_{ab}(x,y)$ [47].

The construction we propose here is a classical version to that proposed in [47, 62]. We give a concrete example for the q-deformed Coxeter dual pair $\left(C_2^{(1)}, D_3^{(2)}\right)$ after which we apply the deformation to our affine Toda field theory.

Outline

In chapter 1 we construct a general mathematical framework for the construction of antilinear deformations of root spaces and we extend the general framework to specific choices of elements in the Coxeter group. Section one is dedicated to deforming factors of the Coxeter element, in section 2 we extend this to a modified Coxeter element, we demonstrate in section 3 how the Weyl reflections can only be consistently deformed for rank 2 algebras, section 4 contains the deformation of two arbitrary elements of the Coxeter group and in section 5 we obtain deformations from rotations in the dual space. We present examples in each section.

In chapter 2 we investigate the deformation of the longest element of a Coxeter group and present some examples of this.

In chapter 3 we build a q-deformation of Coxeter groups and calculate a concrete example.

In chapter 4 we generalize the solution of the standard Calogero model to be independent of the root space involved. Thereafter we apply the deformation of the Coxeter groups to the generalized Calogero model and present concrete examples.

In chapter 5 we apply the q-deformation to affine Toda field theories and compute the mass spectra of the deformed models, which we demonstrate with an example.

Chapter 6 is dedicated to some concluding remarks.

Chapter 1

Root spaces invariant under antilinear involutions

We start by defining some key concepts that are used throughout this thesis. If we have Euclidean vector space E with a subset Δ satisfying some specific properties [66][3], then Δ is known as a root space. The first property is that the root system Δ is finite and spans the Euclidean space. It also does not include 0. Secondly if a nonzero vector $\alpha \in \Delta$, then the only multiples of α in Δ are $\pm\alpha$. The third property is that the reflection σ_α leaves the root system invariant if $\alpha \in \Delta$, where $\sigma_\alpha(\beta) = \beta - 2(\beta \cdot \alpha)/\alpha^2\alpha$ is known more commonly as a Weyl reflection. The last property states that if $\alpha, \beta \in \Delta$ then the quantity $2(\beta \cdot \alpha)/\alpha^2 \in \mathbb{Z}$ for the crystallographic groups, for the non-crystallographic groups this is not an integer. This quantity is often abbreviated to $2(\beta \cdot \alpha)/\alpha^2 = <\beta, \alpha>$ in the literature and is known as the

Cartan integers which become the elements of the Cartan matrix

$$K_{ij} = \frac{2(\alpha_i.\alpha_j)}{\alpha_i^2}. \tag{1.1}$$

Here the α are the roots of the system and one refers to them as simple roots if they cannot be written as the sum of other roots in the system. Generally we labeled the roots of a system by subscripts, i.e., $\alpha_i \in \Delta$. The simple roots of a root space form a basis from which every other non-simple root can be calculated.

A Weyl group $\mathcal{W}$ is a group that is generated by the Weyl reflections σ_α with $\alpha \in \Delta$. By the third property mentioned above, the set Δ is permuted by the Weyl group $\mathcal{W}$. If one has a pair consisting of a group $\mathcal{U} \subset V$ where V is a vector space and a set of generators $S = \{s_k\} \subset \mathcal{U}$ such that

$$(s_i s_j)^{m(s_i,s_j)} = 1, \tag{1.2}$$

where $m(s_i, s_i) = 1$, $m(s_i, s_j) = m(s_j, s_i) \geq 2 \in \mathbb{Z}$ for $s_i \neq s_j$ and $s_k \in S$, then such a system is called a Coxeter system and $\mathcal{U}$ is called a Coxeter group [3]. One can redefine the generators s_i to be reflections given by the simple roots of the vector space V as

$$\sigma_{s_i}(\alpha_j) = \alpha_j - < \alpha_i, \alpha_j > \alpha_i \tag{1.3}$$

which can be identified with the Weyl reflections of a Weyl group. The Coxeter groups consists of the groups $A_n, B_n, C_n, D_n, G_2, E_6, E_7, E_8, F_4, H_3$ and H_4. All of these groups are Weyl groups except for the groups H_3 and H_4. The groups A_n, E_6, E_7, E_8 and D_n are known as the simply-laced Coxeter groups as their root systems consist of roots of only one length.

These groups can be classified by their Dynkin diagrams. A Dynkin diagram has the same number of vertices as there are simple roots in the system and they are labeled by these roots. Each vertex is connected by a line which corresponds to a generator of the system. If two roots have different lengths then an arrow is drawn on the line pointing in the direction of the shorter root. The number of lines drawn between 2 vertices corresponds to the angle between the roots that are represented by those two vertices. There are only three possible angles $\frac{2\pi}{3}$, $\frac{3\pi}{4}$ and $\frac{4\pi}{5}$. One line is drawn for the angle $\frac{2\pi}{3}$, two lines for $\frac{3\pi}{4}$ and three for $\frac{4\pi}{5}$. For a complete set of Dynkin diagrams please see the appendix. A Coxeter element σ of a specific Coxeter group is defined as the product of all the simple Weyl reflections in that Coxeter group i.e.,

$$\sigma = \prod_{k=1}^{l} \sigma_k \quad \text{for } l \equiv \text{ the rank of } \mathcal{U}. \tag{1.4}$$

The Coxeter number of a Coxeter group is the order h for which the Coxeter element is equal to 1 i.e., $\sigma^h = 1$[1]. The number of roots in a Coxeter group is the rank of the group times its Coxeter number, $N = l \times h$.

First of all we would like to present a mathematical framework that is completely general at the onset, but which may be applied in a different setting than outlined in this thesis. We, as of yet, are not considering any concrete physical models but they serve as a guide in our construction. In this section we mainly aim to construct a complex extended root system $\Delta(\varepsilon)$ which remains invariant under a newly defined antilinear involutory map. To start off we deform the real roots $\alpha_i \in \Delta \subset \mathbb{R}^n$ in such a way that we can represent them in a complex space depending on some deformation

[1] We provide a table of the different Coxeter numbers of the Coxeter groups in the appendix, as well as the number of roots in each group.

parameter $\varepsilon \in \mathbb{R}$ as $\alpha_i(\varepsilon) \in \Delta(\varepsilon) \subset \mathbb{R}^n \oplus \imath\mathbb{R}^n$. We define a linear deformation map as

$$\delta : \ \Delta \to \Delta(\varepsilon), \tag{1.5}$$

and this will relate the simple roots and deformed simple roots as

$$\alpha \mapsto \alpha(\varepsilon) = \theta_\varepsilon \alpha. \tag{1.6}$$

where α is the column vector made up of all simple roots $\alpha = \{\alpha_1, ..., \alpha_\ell\}$, θ_ε is an $\ell \times \ell$ matrix and ℓ is the rank of the group $\mathcal{W}$. In addition we want to find an antilinear involutory map ζ which leaves this complex root space invariant under its action

$$\zeta : \Delta(\varepsilon) \to \Delta(\varepsilon), \qquad \alpha(\varepsilon) \mapsto \lambda\alpha(\varepsilon). \tag{1.7}$$

This means the map satisfies (1.5) in an antilinear fashion (6), i.e., $\zeta : \alpha(\varepsilon) = \mu_1\alpha_1(\varepsilon) + \mu_2\alpha_2(\varepsilon) \mapsto \mu_1^*\lambda\alpha_1(\varepsilon) + \mu_2^*\lambda\alpha_2(\varepsilon)$ for $\mu_1, \mu_2 \in \mathbb{C}$ and $\zeta^2 = \mathbb{I}$.

Assuming that λ can be decomposed into an element of the Weyl group $\hat{\omega} \in \mathcal{W}$ with $\hat{\omega}^2 = \mathbb{I}$ and a complex conjugation τ, $\lambda = \tau\hat{\omega} = \hat{\omega}\tau$. The presence of τ ensures the antilinearity of ζ. In some concrete applications it is understood that the maps $\hat{\omega}$ and τ correspond to analogues of the parity operator $\mathcal{P}$ and time reversal operator $\mathcal{T}$, respectively. Candidates for $\hat{\omega}$ are simple Weyl reflections σ_i [41], the two factors $\sigma_\pm$ of the Coxeter element [67], the longest element w_0 of the Weyl group [67] and some more general elements in $\mathcal{W}$ for the example of E_8 in [68] and the other groups in [69].

Concretely we assume here that we have at least two different involutions ζ of the type (1.7) at our disposal, say ζ_i with $i = 1, 2, \ldots$ With our application in mind, namely to construct physically viable self-consistent non-Hermitian multi-particle systems, one such map would in principle be sufficient. However, the presence of two maps leads immediately to some extremely useful constraints. We take the associated rules of correspondence to be of the form

$$\lambda_i := \theta_\varepsilon \hat{\omega}_i \theta_\varepsilon^{-1} = \tau \hat{\omega}_i, \qquad \text{for } i = 1, \ldots, \kappa \geq 2. \tag{1.8}$$

here both sides of the equality act on $\alpha(\varepsilon)$ so the equality actually reads as $\theta_\varepsilon \hat{\omega}_i \theta_\varepsilon^{-1} \alpha(\varepsilon) = \tau \hat{\omega}_i \alpha(\varepsilon)$. For a detailed example as to how this equation works please refer to 1.1.1.

Then by

$$\lambda_i \lambda_j = \tau \hat{\omega}_i \tau \hat{\omega}_j = \tau^2 \hat{\omega}_i \hat{\omega}_j = \hat{\omega}_i \hat{\omega}_j = \theta_\varepsilon \hat{\omega}_i \hat{\omega}_j \theta_\varepsilon^{-1}, \tag{1.9}$$

it follows directly that the composition $\Omega_{ij} := \hat{\omega}_i \hat{\omega}_j$ of any two elements $\hat{\omega}_i$ and $\hat{\omega}_j$ of the Weyl group commutes with the deformation matrix θ_ε

$$[\Omega_{ij}, \theta_\varepsilon] = 0. \tag{1.10}$$

Note that in general $\Omega_{ij} \neq \Omega_{ji}$. Since by construction $\Omega_{ij} \in \mathcal{W}$ we can expand θ_ε in all elements $\breve{\omega}_i \in \mathcal{W}$ which commute with Ω_{ij}, i.e., $[\Omega_{ij}, \breve{\omega}_i] = 0$,

$$\theta_\varepsilon = \sum_k r_k(\varepsilon) \breve{\omega}_k \qquad \text{for } r_k(\varepsilon) \in \mathbb{C}, \tag{1.11}$$

and subsequently determine the coefficient functions $r_k(\varepsilon)$ from additional constraints. One further natural constraint, from a physical and mathematical point of view, is to assume the preservation of the dot products on $\Delta(\varepsilon)$, and we do this by assuming that θ_ε is an orthogonal matrix. So we have that

$$\alpha_i \cdot \alpha_j = (\theta_\varepsilon \alpha_i) \cdot (\theta_\varepsilon \alpha_j), \tag{1.12}$$

which means that θ_ε is an isometry, since by definition an isometry is an operator that preserves the inner product [70]. Since we are assuming that θ_ε is an orthogonal matrix we must have the property [70]

$$\theta_\varepsilon^* \theta_\varepsilon = \theta_\varepsilon \theta_\varepsilon^* = \mathbb{I} \Longrightarrow \theta_\varepsilon^* = \theta_\varepsilon^{-1}. \tag{1.13}$$

Now using the second equation in (1.13) we prove that

$$\begin{aligned} \det(\theta_\varepsilon \theta_\varepsilon^*) &= \det(\mathbb{I}), \\ \det(\theta_\varepsilon) \det(\theta_\varepsilon^*) &= 1, \\ \det(\theta_\varepsilon)^2 &= 1, \\ \det(\theta_\varepsilon) &= \pm 1. \end{aligned} \tag{1.14}$$

Acting on 1.8 with τ from the left and then using 1.14 gives

$$\theta_\varepsilon \hat{\omega}_i \theta_\varepsilon^{-1} \alpha(\varepsilon) = \tau \hat{\omega}_i \alpha(\varepsilon) \Rightarrow \theta_\varepsilon \hat{\omega}_i \theta_\varepsilon^{-1} \theta_\varepsilon \alpha = \tau \hat{\omega}_i \theta_\varepsilon \alpha \Rightarrow \theta_\varepsilon \hat{\omega}_i = \hat{\omega}_i \theta_\varepsilon^*.$$

In summary, the task is to pick κ elements of the Weyl group $\hat{\omega}_i$, expand the deformation matrix θ_ε in terms of the elements commuting with the products of these elements, and finally determine the coefficient functions $r_k(\varepsilon)$ in these expansions from the constraints

$$\theta_\varepsilon^* \hat{\omega}_i = \hat{\omega}_i \theta_\varepsilon, \quad [\hat{\omega}_i \hat{\omega}_j, \theta_\varepsilon] = 0, \quad \theta_\varepsilon^* = \theta_\varepsilon^{-1}, \quad \det \theta_\varepsilon = \pm 1, \quad \lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}, \tag{1.15}$$

or possibly in reverse, that is for given θ_ε to identify meaningful involutions $\hat{\omega}_i$. It turns out that these constraints are quite restrictive and often allows one to determine θ_ε with only very few free parameters left. In some situations it might not be desirable to preserve the inner products (1.12) after the deformation, in which case one may give up (1.14).

With our applications to physical models of Calogero or Toda type in mind, we may then easily construct a dual map $\delta^\star$ for δ, meaning the deformation map associated to δ that acts on the coordinates of the Calogero model, for example, once we have related the model to a Coxeter group

$$\delta^\star : \ \mathbb{R}^n \to \Delta^\star(\varepsilon) = \mathbb{R}^n \oplus i\mathbb{R}^n, \qquad x \mapsto \tilde{x} = \theta_\varepsilon^\star x; \tag{1.16}$$

i.e., this map acts on the coordinate space with $x = \{x_1, \ldots, x_n\}$ or possibly fields as we will see below. Throughout the manuscript we will denote quantities in and acting on the dual space by $\star$, which is of course not to be confused with the complex conjugation denoted by $*$. Given θ_ε we construct $\theta_\varepsilon^\star$ by solving the ℓ equations

$$(\alpha_i(\varepsilon) \cdot x) = ((\theta_\varepsilon \alpha)_i \cdot x) = (\alpha_i \cdot \theta_\varepsilon^\star x) = (\alpha_i \cdot \tilde{x}), \quad \text{for } i = 1, \ldots, \ell, \tag{1.17}$$

involving the standard inner product. This means $(\theta_\varepsilon^\star)^{-1}\alpha_i = (\theta_\varepsilon \alpha)_i$. Note that in general $\theta_\varepsilon^\star \neq \theta_\varepsilon^*$. Naturally we can also identify an antilinear involutory map

$$\zeta^\star : \Delta^\star(\varepsilon) \to \Delta^\star(\varepsilon), \qquad \tilde{x} \mapsto \lambda^\star \tilde{x}. \tag{1.18}$$

corresponding to ζ but acting in the dual space. Concretely we solve for this the $\kappa \times \ell$ relations

$$(\lambda_i \alpha(\varepsilon))_j \cdot x = \alpha_j \cdot \lambda_i^\star \tilde{x}, \quad \text{for } i = 1, \ldots \kappa; \, j = 1, \ldots, \ell, \tag{1.19}$$

for $\lambda_i^\star$ with given λ_i.

Let us now look at different ways in which we can choose $\hat{\omega}$ and what the various solutions look like.

1.1 Deformations of Coxeter group factors

We will start with a Coxeter element of the Weyl group $\sigma \in \mathcal{W}$. The Coxeter element can, by definition, always be expressed as a product over ℓ simple Weyl reflections $\sigma = \prod_{i=1}^{\ell} \sigma_i$, where the Weyl reflections are defined as

$$\sigma_i(x) = x - 2(x \cdot \alpha_i)/\alpha_i^2 \alpha_i, \quad 1 \leq i \leq \ell, \tag{1.20}$$

with ℓ being the rank of the group.

Due to the fact that the Weyl reflections, in general, do not commute, the Coxeter element (1.2) is not unique and only defined up to conjugacy. A useful connection one can make is to assign the values $c_i = \pm 1$ to the vertices of the Coxeter graphs in such a fashion that every pair of linked vertices does not have the same value. Consequently we are left with two disjoint sets of the simple roots that are associated to each vertex, say $V_\pm$, which means that the Coxeter element can now be defined uniquely as

$$\sigma = \sigma_- \sigma_+ \qquad \text{with} \quad \sigma_\pm := \prod_{i \in V_\pm} \sigma_i, \tag{1.21}$$

[66, 3, 71, 52, 54, 72]. Since all elements in the same set now commute, i.e., $[\sigma_i, \sigma_j] = 0$ for $i, j \in V_+$ or $i, j \in V_-$, and $\sigma_i^2 = \mathbb{I}$, the only remaining task is to choose the ordering of the σ_+ and σ_-. Because of this we ensure that the property that $\sigma_\pm^2 = \mathbb{I}$ is maintained and therefore we can use σ_- or σ_+ as candidates for the analogue to the parity operator $\mathcal{P}$ which is what we want to deform antilinearly to construct the map λ in (1.7).

We achieve this by defining the antilinear deformations of the factors of the Coxeter element in the following way:

$$\sigma_{\pm}^{\varepsilon} := \theta_{\varepsilon}\sigma_{\pm}\theta_{\varepsilon}^{-1} = \tau\sigma_{\pm}, \tag{1.22}$$

with complex conjugation τ, which we employ as the time reversal operator and θ_{ε} being the deformation matrix introduced in (1.6).

Defining the deformed Coxeter element in this way we ensure the invariance of the root space under the action of this operator i.e.,

$$\sigma_{\pm}^{\varepsilon} : \Delta(\varepsilon) \rightarrow \Delta(\varepsilon). \tag{1.23}$$

Our construction ensures that the deformed Coxeter element σ_{ε} acts on the deformed root space $\Delta(\varepsilon)$ in the same way as the undeformed Coxeter element σ acting on the undeformed root space Δ, eg. :

$$\begin{aligned} \sigma &: \beta_1 \in \Delta \rightarrow \beta_1 + \beta_2 \in \Delta, \\ \sigma_{\varepsilon} &: \beta_1^{\varepsilon} \in \Delta(\varepsilon) \rightarrow \beta_1^{\varepsilon} + \beta_2^{\varepsilon} \in \Delta(\varepsilon), \end{aligned} \tag{1.24}$$

for β_i being some element in Δ and β_i^{ε} being some element in $\Delta(\varepsilon)$.

Therefore our deformation map commutes with the Coxeter element

$$[\sigma, \theta_{\varepsilon}] = 0. \tag{1.25}$$

From (1.25) it follows that one equation in (1.22) implies the other, the σ_- deformation will give the σ_+ deformation and the other way around.

The undeformed root space Δ can be constructed by using the quantity $\gamma_i = c_i\alpha_i$ with $c_i = \pm 1$ as introduced on page 25 and acting on it consecutively with the powers of σ. We want the deformed root space to be constructed in an analogous way by using $\gamma_i(\varepsilon) = c_i\alpha_i(\varepsilon) = c_i\theta_{\varepsilon}\alpha_i$, therefore we can construct the deformed Coxeter orbits as

$$\Omega_i^\varepsilon := \left\{\gamma_i, \sigma_\varepsilon\gamma_i, \sigma_\varepsilon^2\gamma_i, \ldots, \sigma_\varepsilon^{h-1}\gamma_i\right\} = \theta_\varepsilon\Omega_i, \tag{1.26}$$

such that the deformed root space is given by

$$\Delta(\varepsilon) = \bigcup_{i=1}^{\ell}\Omega_i^\varepsilon = \theta_\varepsilon\Delta. \tag{1.27}$$

By this we have that the entire deformed root space remains invariant under the action of the deformed Coxeter element $\sigma_\varepsilon : \Delta(\varepsilon) \to \Delta(\varepsilon)$. Crucial to our intentions, that our root spaces are $\mathcal{PT}$-symmetric, i.e., the root space is invariant under the action of our map defined in (1.22)

$$\sigma_\pm^\varepsilon : \Delta(\varepsilon) \to \theta_\varepsilon\sigma_\pm\theta_\varepsilon^{-1}\Delta(\varepsilon) = \theta_\varepsilon\sigma_\pm\Delta = \theta_\varepsilon\Delta = \Delta(\varepsilon) \tag{1.28}$$

Because of the way we construct the deformed root space, we demand a one-to-one relation between the individual roots of the undeformed and deformed root spaces such that $\Delta(\varepsilon)$ is isomorphic to Δ. To guarantee this we impose the limit

$$\lim_{\varepsilon\to 0}\theta_\varepsilon = \mathbb{I}, \tag{1.29}$$

which then gives $\lim_{\varepsilon\to 0}\Delta(\varepsilon) = \Delta$, i.e., $\lim_{\varepsilon\to 0}\alpha_i(\varepsilon) = \alpha_i$.

Provided that we can construct θ_ε, we can now formulate $\mathcal{PT}$-symmetric physical models based on root systems by using the deformation map (1.6), $\delta : \alpha \mapsto \alpha(\varepsilon)$. However, there is still a large amount of free parameters. To remedy this we can impose some more constraints.

As mentioned before we keep physical applications in mind when performing this construction, so we would like it that the kinetic energy and possibly other terms to remain invariant under this deformation. This will be guaranteed by demanding that the inner products of the corresponding root spaces remain invariant, i.e.,

$$\alpha_i \cdot \alpha_j = \alpha_i(\varepsilon) \cdot \alpha_j(\varepsilon), \tag{1.30}$$

which is the same as saying,

$$\theta_\varepsilon^* = \theta_\varepsilon^{-1} \qquad \text{and} \qquad \det \theta_\varepsilon = \pm 1. \tag{1.31}$$

Now we have several constraints that will aid in the actual construction of the deformation map θ_ε, i.e., (1.24), (1.25), (1.31) and (1.29). We will summarize them as follows

$$\theta_\varepsilon^* \sigma_\pm = \sigma_\pm \theta_\varepsilon, \quad [\sigma, \theta_\varepsilon] = 0, \quad \theta_\varepsilon^* = \theta_\varepsilon^{-1}, \quad \det \theta_\varepsilon = \pm 1, \quad \lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}. \tag{1.32}$$

Considering now the fact the θ_ε and the Coxeter element σ commute, together with the last equation in (1.32), we make the following ansatz

$$\theta_\varepsilon = \sum_{k=0}^{h-1} c_k(\varepsilon) \sigma^k, \qquad \text{with } \lim_{\varepsilon \to 0} c_k(\varepsilon) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}, \; c_k(\varepsilon) \in \mathbb{C} \tag{1.33}$$

Using equation (1.32) and the relations $\sigma_- \sigma^{-1} = \sigma \sigma_-$ and $\sigma^h = 1$, we try to satisfy the first relation in (1.33). The left hand side gives

$$\theta_\varepsilon^* \sigma_- = \sum_{k=0}^{h-1} c_k^*(\varepsilon) \sigma^k \sigma_- = \sum_{k=0}^{h-1} c_k^*(\varepsilon) \sigma_- \sigma^{h-k} \tag{1.34}$$

and the right hand side equals

$$\sigma_- \theta_\varepsilon = \sum_{k=0}^{h-1} c_k(\varepsilon) \sigma_- \sigma^k \tag{1.35}$$

These two equations are equal when

$$c_{h-k}(\varepsilon) = c_k^*(\varepsilon) \text{ and } c_0(\varepsilon) = c_0^*(\varepsilon), \tag{1.36}$$

One can obtain the relation (1.25) from the constraint $\theta_\varepsilon^* \sigma_+ = \sigma_+ \theta_\varepsilon$ and using, $\sigma_+ \sigma = \sigma^{-1} \sigma_+$ instead of $\sigma_- \sigma^{-1} = \sigma \sigma_-$, since one can obtain one equation from the other in (1.22). Since $c_0(\varepsilon) \in \mathbb{R}$ from the second equation in (1.36), we insert this into the first equation in (1.36) and deduce that $c_h(\varepsilon) = c_0(\varepsilon)$. We set $c_0(\varepsilon) =: r_0(\varepsilon) \in \mathbb{R}$ and also deduce that $c_{h/2}(\varepsilon) =: r_{h/2}(\varepsilon) \in \mathbb{R}$ when h is even. Finally by taking $c_k(\varepsilon)$ to be purely imaginary $c_k(\varepsilon) = \imath r_k(\varepsilon)$ with $r_k(\varepsilon) \in \mathbb{R}$, we reduced the number of free parameters even more. This then leads to the equation for the deformation map θ_ε

$$\theta_\varepsilon = \begin{cases} r_0(\varepsilon)\mathbb{I} + \imath \sum_{k=1}^{(h-1)/2} r_k(\varepsilon)(\sigma^k - \sigma^{-k}) & \text{for } h \text{ odd}, \\ r_0(\varepsilon)\mathbb{I} + r_{h/2}(\varepsilon)\sigma^{h/2} + \imath \sum_{k=1}^{h/2-1} r_k(\varepsilon)(\sigma^k - \sigma^{-k}) & \text{for } h \text{ even}. \end{cases} \tag{1.37}$$

Diagonalizing θ_ε by recalling [3] the eigenvalue equation for the Coxeter element

$$\sigma v_n = e^{2\pi i s_n/h} v_n, \tag{1.38}$$

with s_n being the exponents of a particular Coxeter group $\mathcal{W}$. Defining the matrix $\vartheta = \{v_1, v_2, \dots, v_\ell\}$, we diagonalize the Coxeter element simply as $\sigma = \vartheta \hat{\sigma} \vartheta^{-1}$ with $\hat{\sigma}_{nn} = e^{2\pi \imath s_n/h}$, such that the deformation matrix diagonalizes as

$$\theta_\varepsilon = \vartheta \hat{\theta}_\varepsilon \vartheta^{-1}, \tag{1.39}$$

with eigenvalues

$$(\hat{\theta}_\varepsilon)_{nn} = \begin{cases} r_0(\varepsilon) - 2 \sum_{k=1}^{(h-1)/2} r_k(\varepsilon) \sin\left(\frac{2\pi k}{h} s_n\right) & \text{for } h \text{ odd}, \\ r_0(\varepsilon) + (-1)^{s_n} r_{h/2}(\varepsilon) - 2 \sum_{k=1}^{h/2-1} r_k(\varepsilon) \sin\left(\frac{2\pi k}{h} s_n\right) & \text{for } h \text{ even}. \end{cases} \tag{1.40}$$

This means that the constraint $\det\theta_\varepsilon = \pm 1$ in (1.32) is equivalent to $\det\hat{\theta}_\varepsilon = \pm 1$ and therefore

$$\begin{aligned} \pm 1 &= \prod_{n=1}^{\ell}\left[r_0(\varepsilon) - 2\sum_{k=1}^{(h-1)/2} r_k(\varepsilon)\sin\left(\tfrac{2\pi k}{h}s_n\right)\right] && \text{for } h \text{ odd,} \\ \pm 1 &= \prod_{n=1}^{\ell}\left[r_0(\varepsilon) + (-1)^n r_{h/2}(\varepsilon) - 2\sum_{k=1}^{h/2-1} r_k(\varepsilon)\sin\left(\tfrac{2\pi k}{h}s_n\right)\right] && \text{for } h \text{ even.} \end{aligned} \tag{1.41}$$

Next we implement the third relation in (1.32), which, using (1.39), corresponds to the ℓ equations

$$\vartheta^{-1}\vartheta^*\hat{\theta}_\varepsilon(\vartheta^*)^{-1}\vartheta = \hat{\theta}_\varepsilon^{-1}. \tag{1.42}$$

What is left is to find the $(h-1)/2$ or $h/2+1$ unknown functions $r_i(\varepsilon)$ when h is odd or even, respectively, from the $\ell+1$ equations (1.41) and (1.42). This task we carry out case-by-case.

1.1.1 Case-by-case solutions

Deformed root spaces, $\Delta(\varepsilon)$, for A_ℓ

$\Delta(\varepsilon)$ for A_2 The simple roots for A_2 are $\alpha_1 = \{1,-1,0\}$ and $\alpha_2 = \{0,1,-1\}$, with Cartan matrix

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{1.43}$$

Then we find the Weyl reflections for each root, $\sigma_1(\alpha_1) = -\alpha_1$, $\sigma_1(\alpha_2) = \alpha_1+\alpha_2$, $\sigma_2(\alpha_1) = \alpha_1+\alpha_2$ and $\sigma_2(\alpha_2) = -\alpha_2$. Then we write the coefficients in matrix form leading to the matrix form of the Weyl reflection σ_i and

Coxeter element σ as follows

$$\sigma_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = \sigma_-, \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \sigma_+, \tag{1.44}$$

$$\sigma = \sigma_1\sigma_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_-\sigma_+. \tag{1.45}$$

Inserting these into equation (1.33), then the deformation matrix becomes

$$\theta_\varepsilon = r_0(\varepsilon)\mathbb{I} + \imath r_1(\varepsilon)(\sigma - \sigma^{-1}), \tag{1.46}$$

$$= \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 \\ 2\imath r_1 & r_0 + \imath r_1 \end{pmatrix}, \tag{1.47}$$

where we abbreviate the coefficients as $r_i = r_i(\varepsilon)$. Then solving the constraint $\det \theta_\varepsilon = 1$, (1.41) with $s_n = n$ and $h = 3$ yields $r_0^2 - 3r_1^2 = 1$ with solutions $r_0 = \cosh \varepsilon, r_1 = -1/\sqrt{3} \sinh \varepsilon$ leading to a deformation matrix

$$\theta_\varepsilon = \begin{pmatrix} \cosh \varepsilon - \imath/\sqrt{3} \sinh \varepsilon & -2\imath/\sqrt{3} \sinh \varepsilon \\ 2\imath/\sqrt{3} \sinh \varepsilon & \cosh \varepsilon + \imath/\sqrt{3} \sinh \varepsilon \end{pmatrix}. \tag{1.48}$$

Next we inspect that all our constraints hold and we start by examining equation (1.22), we have

$$\begin{aligned} \theta_\varepsilon \sigma_- \theta_\varepsilon^{-1} \alpha(\varepsilon) &= \tau \sigma_- \alpha(\varepsilon), \\ \theta_\varepsilon \sigma_- \theta_\varepsilon^{-1} \theta_\varepsilon \alpha &= \tau \sigma_- \theta_\varepsilon \alpha, \\ \theta_\varepsilon \sigma_- \alpha &= \sigma_- \theta_\varepsilon^* \alpha, \end{aligned} \tag{1.49}$$

The left hand side of this is

$$\theta_\varepsilon\sigma_-\alpha = \begin{pmatrix} -\cosh(\epsilon) - \frac{i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} & \cosh(\epsilon) - \frac{i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} & \frac{2i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} \\ \cosh(\epsilon) - \frac{i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} & \frac{2i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} & -\cosh(\epsilon) - \frac{i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} \end{pmatrix}, \tag{1.50}$$

and this is equal to the right hand side. There are no further constraints resulting from the equations (1.42) as with $\vartheta = \{(e^{\imath\pi/3}, e^{-\imath\pi/3}), e^{\imath\pi 2/3}, e^{-\imath\pi 2/3})\}$ it is trivially satisfied when $r_0^2 - 3r_1^2 = 1$. With (1.6) we obtain from this exactly the roots presented later on in (2.18) and (2.19).

Next we apply the deformation to our undeformed simple roots

$$\theta_\varepsilon\alpha = \alpha(\varepsilon) = \begin{pmatrix} \cosh(\epsilon) - \frac{i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} & -\cosh(\epsilon) - \frac{i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} & \frac{2i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} \\ \frac{2i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} & \cosh(\epsilon) - \frac{i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} & -\cosh(\epsilon) - \frac{i\sqrt{\sinh^2(\epsilon)}}{\sqrt{3}} \end{pmatrix}, \tag{1.51}$$

where we have that $\alpha(\varepsilon) = \{\alpha_1(\varepsilon), \alpha_2(\varepsilon)\}$.

Note that in this case the constraint even holds for the individual Weyl reflections, i.e., $\sigma_1\theta_\varepsilon = (\theta_\varepsilon\sigma_1)^*$ and $\sigma_2\theta_\varepsilon = (\theta_\varepsilon\sigma_2)^*$ as $\sigma_1 = \sigma_-$ and $\sigma_2 = \sigma_+$. This means we can view this deformation in an alternative way as deformations across every hyperplane in the A_2-root system. The latter was the constraint imposed in [41], which explains that (2.18) and (2.19) are precisely the deformations constructed therein.

The remaining positive nonsimple root is simply $\alpha_1(\varepsilon) + \alpha_2(\varepsilon)$ as we demand a one-to-one relationship between the deformed and undeformed root spaces.

$\Delta(\varepsilon)$ **for** A_3 For A_3 the Weyl reflections σ_i, the factors of the Coxeter element $\sigma_\pm$, the Coxeter element σ and the diagonalising matrix ϑ takes the form

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \tag{1.52}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \sigma_- = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \tag{1.53}$$

$$\sigma = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \vartheta = \begin{pmatrix} 1 & -1 & 1 \\ -(1+\imath) & 0 & \imath - 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{1.54}$$

Here $\sigma_2 = \sigma_+$ and $\sigma_- = \sigma_1\sigma_3$. The ansatz (1.33) reads now

$$\theta_\varepsilon = r_0\mathbb{I} + r_2\sigma^2 + \imath r_1\left(\sigma - \sigma^3\right) \tag{1.55}$$

$$= \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 & -\imath r_1 - r_2 \\ 2\imath r_1 & r_0 - r_2 + 2\imath r_1 & 2\imath r_1 \\ -\imath r_1 - r_2 & -2\imath r_1 & r_0 - \imath r_1 \end{pmatrix} \tag{1.56}$$

The constraints (1.41) and (1.42) yield

$$(r_0 + r_2)\left[(r_0 + r_2)^2 - 4r_1^2\right] = 1, \tag{1.57}$$

$$r_0 - r_2 + 2r_1 = (r_0 - r_2 + 2r_1)(r_0 + r_2), \tag{1.58}$$

$$(r_0 + r_2) = (r_0 - r_2)^2 - 4r_1^2, \tag{1.59}$$

with $s_n = n$ and $h = 4$. This is solved for instance by

$$r_0(\varepsilon) = \cosh\varepsilon, \quad r_1(\varepsilon) = \pm\sqrt{\cosh^2\varepsilon - \cosh\varepsilon} \quad \text{and} \quad r_2(\varepsilon) = 1 - \cosh\varepsilon. \tag{1.60}$$

Then the three remaining positive nonsimple roots are $\alpha_4(\varepsilon) := \alpha_1(\varepsilon) + \alpha_2(\varepsilon)$, $\alpha_5(\varepsilon) := \alpha_2(\varepsilon) + \alpha_3(\varepsilon)$, $\alpha_6(\varepsilon) := \alpha_1(\varepsilon) + \alpha_2(\varepsilon) + \alpha_3(\varepsilon)$.

$\Delta(\varepsilon)$ **for** A_4 For A_4 the Weyl reflections σ_i and Coxeter element σ are

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.61}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.62}$$

$$\sigma = \sigma_1\sigma_3\sigma_2\sigma_4 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The ansatz (1.33) then reads

$$\theta_\varepsilon = r_0(\varepsilon)\mathbb{I} + \imath r_1(\varepsilon)(\sigma - \sigma^4) + \imath r_2(\varepsilon)(\sigma^2 - \sigma^3), \tag{1.63}$$

The constraints (1.41) and (1.42) yield now

$$
\begin{aligned}
r_0^4 - 5r_0^2(r_1^2 + r_2^2) + 5(r_2^2 + r_2 r_1 - r_1^2)^2 &= 1, && (1.64)\\
2r_0^2 + \left(-5+\sqrt{5}\right) r_1^2 - \left(5+\sqrt{5}\right) r_2^2 + 4\sqrt{5} r_1 r_2 - 2 &= 0, && (1.65)\\
2r_0 + \sqrt{2\left(5+\sqrt{5}\right)} r_1 + \sqrt{10 - 2\sqrt{5}} r_2 &\neq 0, && (1.66)
\end{aligned}
$$

with $s_n = n$ and $h = 5$, which is solved for instance by

$$
r_0(\varepsilon) = \cosh\varepsilon, \quad r_1(\varepsilon) = \frac{1}{5}\sqrt{5 - 2\sqrt{5}}\sinh\varepsilon, \quad r_2(\varepsilon) = \frac{1}{5}\sqrt{5 + 2\sqrt{5}}\sinh\varepsilon. \tag{1.67}
$$

Explicitly this yields the deformation matrix

$$
\theta_\varepsilon = \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 & -\imath r_1 - \imath r_2 & -2\imath r_2 \\ 2\imath r_1 & r_0 + 2\imath r_1 + \imath r_2 & 2\imath r_1 + 2\imath r_2 & \imath r_1 + \imath r_2 \\ -\imath r_1 - \imath r_2 & -2\imath r_1 - 2\imath r_2 & r_0 - 2\imath r_1 - \imath r_2 & -2\imath r_1 \\ 2\imath r_2 & \imath r_1 + \imath r_2 & 2\imath r_1 & r_0 + \imath r_1 \end{pmatrix}. \tag{1.68}
$$

Notice that in this case we also have $w_0\theta_\varepsilon = (\theta_\varepsilon w_0)^*$.

$\Delta(\varepsilon)$ for A_{4n-1} For A_{4n-1} we find a closed formula. Setting in (1.33) all $r_k = 0$, except for $k = 0, n, 2n$, the determinant in (1.41) takes on the simple form

$$
\det\theta_\varepsilon = (r_0 + r_{2n})^{2n-1}\left(r_0 - 4r_n^2 - 2r_0 r_{2n} + r_{2n}^2\right)^n, \tag{1.69}
$$

which equals one for $r_{2n} = 1 - r_0$ and $r_n = \pm\sqrt{r_0^2 - r_0}$. We have verified up to rank 11 that for these values

$$
\theta_\varepsilon = r_0\mathbb{I} + r_{2n}\sigma^{2n} + \imath r_n\left(\sigma^n - \sigma^{-n}\right), \tag{1.70}
$$

also satisfies the first and fourth constraint (1.32). Once again $r_0 = \cosh\varepsilon$ is a useful choice to guarantee also the last constraint in (1.32).

Deformed root spaces, $\Delta(\varepsilon)$, for B_ℓ

$\Delta(\varepsilon)$ for B_2 For B_2 the ansatz (1.33) becomes

$$\theta_\varepsilon = r_0\mathbb{I} + r_2\sigma^2 + \imath r_1\left(\sigma - \sigma^{-1}\right). \tag{1.71}$$

The first four constraints in (1.32) are satisfied for $r_0 = r_2 \pm \sqrt{1+4r_1^2}$, which in turn is conveniently solved for $r_0 = \cosh\varepsilon$, $r_2 = 0$ and $r_1 = 1/2\sinh\varepsilon$, such that

$$\theta_\varepsilon = \begin{pmatrix} \cosh\varepsilon - \imath\sinh\varepsilon & -2\imath\sinh\varepsilon \\ \imath\sinh\varepsilon & \cosh\varepsilon + \imath\sinh\varepsilon \end{pmatrix}. \tag{1.72}$$

$\Delta(\varepsilon)$ for B_3 For B_3 the ansatz (1.33) gives

$$\theta_\varepsilon = r_0\mathbb{I} + r_3\sigma^3 + \imath r_1\left(\sigma - \sigma^{-1}\right) + \imath r_2\left(\sigma^2 - \sigma^{-2}\right), \tag{1.73}$$

which is solving the first four constraints in (1.32) when $r_0 = r_3 - 1$ and $r_1 = -r_2$. However this corresponds to a trivial real solution with $(\theta_\varepsilon)_{ii} = -1$ for $i = 1, 2, 3$.

$\Delta(\varepsilon)$ for B_4 For B_4 the ansatz (1.33) yields

$$\theta_\varepsilon = r_0\mathbb{I} + r_4\sigma^4 + \imath r_1\left(\sigma - \sigma^{-1}\right) + \imath r_2\left(\sigma^2 - \sigma^{-2}\right) + \imath r_3\left(\sigma^3 - \sigma^{-3}\right), \tag{1.74}$$

solving the first four constraints in (1.32) when $r_0 = r_4 \pm \sqrt{1+4r_2^2}$ and $r_1 = -r_3$. We may incorporate the last constraint in (1.32) by solving this with $r_0 = \cosh\varepsilon$, $r_4 = 0$ and $r_2 = 1/2\sinh\varepsilon$, such that

$$\theta_\varepsilon = \begin{pmatrix} \cosh\varepsilon & 0 & -\imath\sinh\varepsilon & -2\imath\sinh\varepsilon \\ 0 & \cosh\varepsilon + \imath\sinh\varepsilon & 2\imath\sinh\varepsilon & 2\imath\sinh\varepsilon \\ -\imath\sinh\varepsilon & -2\imath\sinh\varepsilon & \cosh\varepsilon - 2\imath\sinh\varepsilon & -2\imath\sinh\varepsilon \\ \imath\sinh\varepsilon & \imath\sinh\varepsilon & \imath\sinh\varepsilon & \cosh\varepsilon + \imath\sinh\varepsilon \end{pmatrix}. \tag{1.75}$$

$\Delta(\varepsilon)$ for B_{2n} For B_{2n} we conjecture a closed formula

$$\theta_\varepsilon = r_0\mathbb{I} + \frac{\imath}{2} r_n \left(\sigma^n - \sigma^{-n}\right), \tag{1.76}$$

for the solution of the first four constraints in (1.32). It is easily seen from (1.41) that the determinant of θ_ε in (1.76) results to

$$\det\theta_\varepsilon = \prod_{k=1}^{n} \left[r_0 - 2r_n \sin\left(\frac{2\pi n}{4n} s_k\right)\right] = \left(r_0^2 - 4r_n^2\right)^n, \tag{1.77}$$

when using the fact that $h = 4n$ and $s_k = 2k - 1$. Choosing $r_0 = \cosh\varepsilon$ and $r_n = 1/2\sinh\varepsilon$ will then ensure that the last two constraints in (1.32) are satisfied. It turns out that the remaining equations are also solved, which we verified on a case-by-case basis up to rank 8.

$\Delta(\varepsilon)$ for B_{2n+1} Based on the example (1.73) and supplemented with several for higher rank, not reported here, we conjecture that there are no complex solutions for our constraints in the case of odd rank B_{2n+1}

Deformed root spaces, $\Delta(\varepsilon)$, for C_ℓ

This case can be solved in a completely analogous way to the B_n-case. Equation (1.77) is completely identical to B_{2n} and we find that the ansatz (1.76) together with the relevant r_n also solves the remaining constraints, which we have verified up to rank 8. Once again we did not find any complex solutions up to that order of the rank for C_{2n+1} and conjecture that also in this case they do not exist.

Deformed root spaces, $\Delta(\varepsilon)$, for D_ℓ

For the odd-rank subseries, that is D_{2n+1}, we find a closed formula very similar to the one for A_{4n-1}. This is not surprising given the fact that these two groups are embedded into each other as $D_{2n+1} \hookrightarrow A_{4n-1}$. We find that the deformation matrix of the form

$$\theta_\varepsilon = r_0 \mathbb{I} + r_{2n}\sigma^{2n} + \imath r_n \left(\sigma^n - \sigma^{-n}\right), \qquad (1.78)$$

solves the first four constraints in (1.32).

There are no complex solutions for D_{2n} based on ansatz (1.33). For instance, considering the ansatz for D_4 the constraining equations force us to take $r_1 = -r_2$ and $r_3 = r_0 - 1$, which reduces θ_ε to the identity matrix $\mathbb{I}$. Similarly the constraints (1.32) for D_6 demand that $r_1 = -r_4$, $r_2 = -r_3$ and $r_5 = r_0 - 1$, which reduces θ_ε to the identity matrix $\mathbb{I}$.

Deformed root spaces, $\Delta(\varepsilon)$, for E_n

$\Delta(\varepsilon)$ for E_6 As we have seen in the previous examples we have usually more parameters at our disposal than we need to solve the constraining equations. Thus instead of finding the most general solution we will be content here to solve (1.33) for some restricted set of values and attempt to solve the constraints in (1.32) for

$$\theta_\varepsilon = r_0 \mathbb{I} + \imath r_k \left(\sigma^k - \sigma^{-k}\right). \tag{1.79}$$

Considering (1.41) for this ansatz yields

$$1 = \prod_{n=1}^{6} \left[r_0 - 2r_k \sin\left(\frac{\pi k}{6} s_n\right)\right] \qquad \text{with } s_n = 1,4,5,7,8,11, \tag{1.80}$$

which reduces to

$$1 = \left(r_0^2 - 3r_k^2\right)^3 \qquad \text{for } \; k = 2,4. \tag{1.81}$$

It turns out that in both cases the solution $r_0 = \pm\sqrt{1+3r_k^2}$ for (1.81) also solves the first three constraints in (1.32). For the deformation matrix we then obtain for $k=2$

$$\theta_\varepsilon = \begin{pmatrix} r_0 & -2\imath r_2 & 0 & -2\imath r_2 & -2\imath r_2 & -\imath r_2 \\ 2\imath r_2 & r_0 + \imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 \\ 0 & 2\imath r_2 & r_0 + 2\imath r_2 & 4\imath r_2 & 3\imath r_2 & 2\imath r_2 \\ -2\imath r_2 & -2\imath r_2 & -4\imath r_2 & r_0 - 5\imath r_2 & -4\imath r_2 & -2\imath r_2 \\ 2\imath r_2 & 2\imath r_2 & 3\imath r_2 & 4\imath r_2 & r_0 + 2\imath r_2 & 0 \\ -\imath r_2 & -2\imath r_2 & -2\imath r_2 & -2\imath r_2 & 0 & r_0 \end{pmatrix}, \tag{1.82}$$

and for $k=4$

$$\theta_\varepsilon = \begin{pmatrix} r_0 - \imath r_4 & -2\imath r_4 & -2\imath r_4 & -2\imath r_4 & 0 & 0 \\ 2\imath r_4 & r_0 + \imath r_4 & 2\imath r_4 & 2\imath r_4 & 2\imath r_4 & 2\imath r_4 \\ 2\imath r_4 & 2\imath r_4 & r_0 + 3\imath r_4 & 4\imath r_4 & 2\imath r_4 & 0 \\ -2\imath r_4 & -2\imath r_4 & -4\imath r_4 & r_0 - 5\imath r_4 & -4\imath r_4 & -2\imath r_4 \\ 0 & 2\imath r_4 & 2\imath r_4 & 4\imath r_4 & r_0 + 3\imath r_4 & 2\imath r_4 \\ 0 & -2\imath r_4 & 0 & -2\imath r_4 & -2\imath r_4 & r_0 - \imath r_4 \end{pmatrix}. \tag{1.83}$$

In each case we may specify further $r_0 = \cosh\varepsilon$ and $r_k = 1/\sqrt{3}\sinh\varepsilon$ in order to ensure also the right limiting behaviour, i.e., the last constraint in (1.32).

$\Delta(\varepsilon)$ for E_7 Our convention for labeling of the roots is the same as for E_6 by linking the additional root α_7 to α_6. There exists no complex solution to (1.32) based on the ansatz (1.33) with $h = 18$. Together with the explicit representation for σ we substitute this into constraints (1.32) and find the unique real solution for the unknown functions $r_0 = 1 + r_5, r_1 = -r_4 - r_5 - r_8, r_2 = -r_4 - r_5 - r_7$ and $r_3 = -r_6$, which reduced the deformation matrix θ_ε to the identity matrix $\mathbb{I}$.

$\Delta(\varepsilon)$ for E_8 Our convention for labeling of the roots is the same as for E_7 by linking the additional root α_8 to α_7. There exists no complex solution to (1.32) based on the ansatz (1.33) with $h = 30$. Together with the explicit representation for σ we substitute this into constraints (1.32) and find the unique real solution for the unknown functions $r_0 = 1 + r_5, r_1 = -r_5 - r_6 - r_9 - r_{10} - r_{14}, r_2 = -2r_5 - r_7 - r_8 - 2r_{10} - r_{13}, r_3 = -r_5 - r_7 - r_8 - r_{10} - r_{12}$ and $r_4 = r_5 - r_6 - r_9 + r_{10} - r_{11}$. However, this simply corresponds to $\theta_\varepsilon = \mathbb{I}$.

$\Delta(\varepsilon)$ for F_4

In the F_4-ansatz (1.33)

$$\theta_\varepsilon = r_0\mathbb{I} + r_6\sigma^6 + \imath\sum_{k=1}^{5} r_k\left(\sigma^k - \sigma^{-k}\right), \tag{1.84}$$

we have seven unknown quantities left. We find two inequivalent solutions for the first four constraints in (1.32)

$$r_1 = -2r_3 - r_5 \pm \sqrt{(r_0 - r_6)^2 - 1} \qquad \text{and} \qquad r_2 = -r_4, \tag{1.85}$$

and

$$r_1 = -2r_3 - r_5 \qquad \text{and} \qquad r_2 = -r_4 \pm \frac{1}{\sqrt{3}}\sqrt{(r_0 - r_6)^2 - 1}. \tag{1.86}$$

This leaves five functions at our disposal, which we may choose in accordance with the last constraint in (1.32). Taking for instance $r_3 = r_4 = r_5 = r_6 = 0$ in (1.85) yields

$$\theta_\varepsilon = \begin{pmatrix} r_0 - \imath a_0 & -2\imath a_0 & -2\imath a_0 & 0 \\ 2\imath a_0 & r_0 + 3\imath a_0 & 4\imath a_0 & 2\imath a_0 \\ -\imath a_0 & -2\imath a_0 & r_0 - 3\imath a_0 & -2\imath a_0 \\ 0 & \imath a_0 & 2\imath a_0 & r_0 + \imath a_0 \end{pmatrix}, \tag{1.87}$$

for the deformation matrix where we used $a_0 = \sqrt{r_0^2 - 1}$ for a more compact matrix. One may now choose $r_0 = \cosh \varepsilon$, which will then satisfy all the constraints in (1.32).

$\Delta(\varepsilon)$ for G_2

As mentioned, this case has been solved before [41], but nonetheless we report it here for completeness and to demonstrate that it fits well into the framework provided. The ansatz (1.33) with $h = 6$ solves the first four constraints (1.32) uniquely with $r_3 = 0$ and $r_0 = \pm\sqrt{1 + 3(r_1 + r_2)^2}$. The choice $r_1 = 1/\sqrt{3}\sinh \varepsilon - r_2$ reproduces the result of [41].

This concludes the study of all crystallographic Coxeter groups. We will also consider one noncrystallographic example.

$\Delta(\varepsilon)$ for H_3

In this case there are no complex solutions of the type we are seeking here. Substituting the ansatz (1.33) with $h = 6$ into the constraints (1.32) leads to the unique solution $r_0 = 1$, $r_5 = 0$ and $r_1 + r_4 = -\phi(r_2 + r_3)$ with ϕ being the golden ratio $\phi = (1 + \sqrt{5})/2$ appearing in the off-diagonal of the H_3-Cartan matrix. However, this solution simply corresponds to $\theta_\varepsilon = \mathbb{I}$.

1.2 Deformations of modified Coxeter elements

As explained in the beginning of this chapter, in principle the involution $\hat{\omega}_i$ could be *any* element in the Weyl group. We will now present a construction based on the selection of two specific, albeit still fairly generic, elements $\hat{\omega}_1 = \tilde{\sigma}_-$ and $\hat{\omega}_2 = \tilde{\sigma}_+$ defined as

$$\tilde{\sigma}_\pm := \prod_{i \in \tilde{V}_\pm} \sigma_i. \tag{1.88}$$

The σ_i in (1.88) are simple Weyl reflections (1.20). The sets $V_\pm$ are still defined via the bi-colouration of the Dynkin diagram as explained above. The difference towards the treatment above is that the products in (1.88) do not have to extend over all possible elements in $V_\pm$, such that $\tilde{V}_\pm \subseteq V_\pm$.

Denoting by $\sigma_\pm$ the factors of $\tilde{\sigma}$ when $\tilde{\sigma} = \sigma$, we may therefore express the reduced elements as $\tilde{\sigma}_\pm := \sigma_\pm \prod_{j \in \check{V}_\pm} \sigma_i$ for some values j, which follows by recalling $[\sigma_i, \sigma_j] = 0$ for $i, j \in V_+$ or $i, j \in V_-$ and $\sigma_i^2 = 1$. Thus $\check{V}_\pm$ is the complement of $\tilde{V}_\pm$ in $V_\pm$, that is $V_\pm = \check{V}_\pm \cup \tilde{V}_\pm$. This ensures that we have maintained the crucial involutory property $\tilde{\sigma}_\pm^2 = 1$

From the above follows that the element Ω_{ij} in (1.10) can be viewed as a modified Coxeter element $\tilde{\sigma} := \tilde{\sigma}_- \tilde{\sigma}_+$ with property

$$\tilde{\sigma}^{\tilde{h}} = \mathbb{I}, \qquad\qquad \text{with } \tilde{h} \leq h. \tag{1.89}$$

Therefore $\tilde{\sigma}$ equals a Coxeter element σ when the order $\tilde{h}$ becomes the Coxeter number h.

The reduced root space $\tilde{\Delta}$ is then constructed by acting with $\tilde{\sigma}$ on representatives $\tilde{\gamma}_i = c_i \tilde{\alpha}_i$ of a particular orbit $\tilde{\Omega}_i$ containing now $\tilde{h}$ instead of h roots

$$\tilde{\Omega}_i := \left\{ \gamma_i, \tilde{\sigma}\gamma_i, \tilde{\sigma}^2\gamma_i, \ldots, \tilde{\sigma}^{\tilde{h}-1}\gamma_i \right\}. \tag{1.90}$$

The corresponding entire root space containing $\ell \times \tilde{h}$ roots is the union of all orbits

$$\tilde{\Delta} = \bigcup_{i=1}^{\ell} \tilde{\Omega}_i. \tag{1.91}$$

In analogy to the deformations defined to before we construct therefore the map ϖ as

$$\tilde{\sigma}_\pm^\varepsilon := \theta_\varepsilon \tilde{\sigma}_\pm \theta_\varepsilon^{-1} = \tilde{\sigma}_\pm \tau, \tag{1.92}$$

where we assumed an additional property with θ_ε being the deformation matrix as introduced in (1.6). Defining the deformed reduced Coxeter element as $\tilde{\sigma}^\varepsilon := \tilde{\sigma}_-^\varepsilon \tilde{\sigma}_+^\varepsilon$ we use a similar line of reasoning as in the deduction of (1.10) to show that $[\tilde{\sigma}, \theta_\varepsilon] = 0$. Therefore we make the following ansatz for the deformation matrix

$$\theta_\varepsilon = \sum_{k=0}^{\tilde{h}-1} \mu_k(\varepsilon)\tilde{\sigma}^k, \qquad \text{with } \lim_{\varepsilon\to 0} \mu_k(\varepsilon) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}, \ \mu_k(\varepsilon) \in \mathbb{C}. \quad (1.93)$$

The assumption for the coefficients $\mu_k(\varepsilon)$ ensures the appropriate limit $\lim_{\varepsilon\to 0}\theta_\varepsilon = \mathbb{I}$. Equation (1.92) yields the constraint $\theta_\varepsilon^*\tilde{\sigma}_\pm = \tilde{\sigma}_\pm\theta_\varepsilon$, from which we deduce with (1.93)

$$\theta_\varepsilon = \begin{cases} r_0(\varepsilon)\mathbb{I} + \iota \sum_{k=1}^{(\tilde{h}-1)/2} r_k(\varepsilon)(\tilde{\sigma}^k - \tilde{\sigma}^{-k}) & \text{for } \tilde{h} \text{ odd,} \\ r_0(\varepsilon)\mathbb{I} + r_{\tilde{h}/2}(\varepsilon)\tilde{\sigma}^{\tilde{h}/2} + \iota \sum_{k=1}^{\tilde{h}/2-1} r_k(\varepsilon)(\tilde{\sigma}^k - \tilde{\sigma}^{-k}) & \text{for } \tilde{h} \text{ even,} \end{cases} \quad (1.94)$$

where $\mu_0(\varepsilon) =: r_0(\varepsilon) \in \mathbb{R}$, $\mu_{\tilde{h}/2}(\varepsilon) =: r_{\tilde{h}/2}(\varepsilon) \in \mathbb{R}$ when $\tilde{h}$ is even. In addition we defined $\mu_k(\varepsilon) = \iota r_k(\varepsilon)$. Demanding next that θ_ε is an isometry, we invoke the constraint $\det\theta_\varepsilon = 1$. By means of the eigenvalue equations for $\tilde{\sigma}$

$$\tilde{\sigma}\tilde{v}_n = e^{2\pi i\tilde{s}_n/\tilde{h}}\tilde{v}_n \qquad \text{with } n = 1, \ldots \ell, \quad (1.95)$$

we define a set of "modified exponents" $\tilde{s} = \{\tilde{s}_1, \ldots, \tilde{s}_\ell\}$. Unlike as for the standard case, the eigenvalues may be degenerate in the modified scenario. In general, they take the values

$$\tilde{s} = \left\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, (\tilde{h}-1)^{\lambda_{\tilde{h}-1}}, \tilde{h}^{\lambda_{\tilde{h}}}\right\} \qquad \text{with } \sum_{k=1}^{\tilde{h}} \lambda_k = \ell, \quad (1.96)$$

with λ_i indicating the degeneracy of certain eigenvalues in (1.95). Due to the degeneracy there could be several solutions to (1.95) with different elements $\tilde{\sigma}^{(i)}$ for $i = 1, \ldots m$ forming a similarity class

$$\Sigma_{\tilde{s}} = \left\{\tilde{\sigma}^{(1)}, \tilde{\sigma}^{(2)}, \ldots, \tilde{\sigma}^{(m)}\right\}. \quad (1.97)$$

Similar to before we demand the preservation of the inner product between the original and deformed roots, which implies that $\det \theta_\varepsilon = 1$ and $\theta_\varepsilon^* = \theta_\varepsilon^{-1}$. Diagonalizing (1.94) the constraint $\det \theta_\varepsilon = 1$ simply becomes

$$\begin{aligned} 1 &= \prod_{n=1}^{\ell} \left[r_0(\varepsilon) - 2 \sum_{k=1}^{(\tilde{h}-1)/2} r_k(\varepsilon) \sin\left(\tfrac{2\pi k}{\tilde{h}} \tilde{s}_n\right) \right] && \text{for } \tilde{h} \text{ odd,} \\ 1 &= \prod_{n=1}^{\ell} \left[r_0(\varepsilon) + (-1)^{\tilde{s}_n} r_{\tilde{h}/2}(\varepsilon) - 2 \sum_{k=1}^{\tilde{h}/2-1} r_k(\varepsilon) \sin\left(\tfrac{2\pi k}{\tilde{h}} \tilde{s}_n\right) \right] && \text{for } \tilde{h} \text{ even.} \end{aligned} \tag{1.98}$$

Solving these constraints for θ_ε allows us to construct the simple roots $\tilde{\alpha}_i$ and therefore the entire deformed reduced root space $\tilde{\Delta}(\varepsilon)$. Note that just as in (1.26) for simplicity we use the same notation for the undeformed and deformed root space, distinguishing the latter always by the explicit mentioning of the deformation parameter ε. Hence we have

$$\tilde{\Omega}_i^\varepsilon = \theta_\varepsilon \tilde{\Omega}_i, \tag{1.99}$$

and therefore

$$\tilde{\Delta}(\varepsilon) = \bigcup_{i=1}^{\ell} \tilde{\Omega}_i^\varepsilon = \theta_\varepsilon \tilde{\Delta}. \tag{1.100}$$

This construction guarantees that the $\tilde{\sigma}_\pm^\varepsilon$ are indeed representations of the map ϖ in (1.7). Evidently it leaves the root space invariant

$$\tilde{\sigma}_\pm^\varepsilon : \tilde{\Delta}(\varepsilon) \to \theta_\varepsilon \tilde{\sigma}_\pm \theta_\varepsilon^{-1} \tilde{\Delta}(\varepsilon) = \theta_\varepsilon \tilde{\sigma}_\pm \tilde{\Delta} = \theta_\varepsilon \tilde{\Delta} = \tilde{\Delta}(\varepsilon). \tag{1.101}$$

For the latter property to hold we may also exclude some of the orbits $\tilde{\Omega}_i^\varepsilon$ in the union $\bigcup_{i=1}^{\ell}$, whenever they are mapped into themselves $\tilde{\sigma}_\pm^\varepsilon : \tilde{\Omega}_i^\varepsilon \to \tilde{\Omega}_i^\varepsilon$.

1.2.1 Antilinearly deformed A_ℓ root systems

When engaging into a case-by-case description previously mentioned, we characterized different solutions group by group. Here we will take equation

(1.98) as more fundamental and classify the solutions according to different values of the modified Coxeter number. In this manner different types of solutions to (1.98) are then characterized by different sets of modified exponents (1.96). This means we need to verify subsequently whether a corresponding $\tilde{\sigma}$ really exists.

We find various similarity classes $\Sigma_{\tilde{s}}$ characterized by different sets of modified exponents $\tilde{s}$.

The class with modified exponents $\{1,2,3,4^{\ell-3}\}$ and $\tilde{\mathbf{h}}$=4

We find that the simplest similarity class Σ for which $x^4 = 1$ when $x \in \Sigma$ is

$$\Sigma_{\{1,2,3,4^{\ell-3}\}} = \left\{ \tilde{\sigma}^{(1)}, \ldots, \tilde{\sigma}^{(\ell-2)} \right\}, \tag{1.102}$$

where the elements of that class are defined as

$$\tilde{\sigma}^{(i)} := (\sigma_{i+1}\sigma_i\sigma_{i+2})^{c_i} \qquad \text{for } i = 1, \ldots, \ell-2. \tag{1.103}$$

It is clear that each element $\tilde{\sigma}^{(i)}$ in (1.103) has order 4, since it is formed from three consecutive elements on the Dynkin diagram and thus being isomorphic to the Coxeter element of A_3 when acting on the three corresponding roots.

Furthermore, by definition all elements of Σ have to be related by a similarity transformation. Indeed we find:

Proposition 1 Two consecutive elements in $\Sigma_{\{1,2,3,4^{\ell-3}\}}$ are related as

$$\varkappa_i \tilde{\sigma}^{(i)} = \tilde{\sigma}^{(i+1)} \varkappa_i \qquad \textit{with } \varkappa_i := \sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}\sigma_{i+1}. \tag{1.104}$$

Therefore all elements in Σ can be related to each other by an adjoint action simply by successive applications of (1.104).

Proof. *Let us now prove the relation (1.104). The starting point is the identity*

$$\sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_{i-1}\sigma_i\sigma_{i+1}, \tag{1.105}$$

which follows by applying the left and right hand side to some arbitrary x *using the definition of the simple Weyl reflection (1.20) consecutively. Normalizing the length of the roots to be* 2*, we find in both cases*

$$x - [(x\cdot\alpha_{i-1}) + (x\cdot\alpha_i) + (x\cdot\alpha_{i+1})]\,\alpha_{i-1} - [(x\cdot\alpha_i) + (x\cdot\alpha_{i+1})]\,(\alpha_i + \alpha_{i+1}). \tag{1.106}$$

Multiplying (1.105) from the left by $\prod_{k=1}^{i-2}\sigma_k$ *and* $\prod_{k=i+2}^{\ell}\sigma_k$ *from the right and noting that for* A_ℓ *we have* $[\sigma_i,\sigma_j]=0$ *for* $|i-j|\geq 2$*, it follows*

$$\hat{\sigma}\sigma_i = \sigma_{i+1}\hat{\sigma}, \qquad \textit{with } \hat{\sigma} := \prod_{k=1}^{\ell}\sigma_k. \tag{1.107}$$

The element $\hat{\sigma}$ *is the standard Coxeter element. Multiplying next the identity (1.104) from the left by* $\prod_{k=1}^{i-1}\sigma_k$ *and* $\sigma_{i+1}\prod_{k=i+4}^{\ell}\sigma_k$ *from the right and recalling that* $\sigma_i^2=1$ *yields*

$$\hat{\sigma}\left(\sigma_i\sigma_{i+2}\sigma_{i+1}\right)^{c_i} = \left(\sigma_{i+1}\sigma_{i+3}\sigma_{i+2}\right)^{c_i}\hat{\sigma}. \tag{1.108}$$

This relation is now easily established by commuting all three simple Weyl reflections through the Coxeter element using the identity (1.107), which in turn also proves (1.104). ■

Proposition 2 *Some special elements in* Σ *are related by the adjoint action of the Coxeter element* σ*. We find:* The first and the last element in

$\Sigma_{\{1,2,3,4^{\ell-3}\}}$ are related as

$$\tilde{\sigma}^{(\ell-2)}\sigma^{\frac{h-c_\ell}{2}} = \sigma^{\frac{h-c_\ell}{2}}\tilde{\sigma}^{(1)}, \qquad (1.109)$$

Proof. *We prove (1.109) by using the more elementary relations*

$$\sigma_{\ell+1-i}\sigma^{\frac{h}{2}+\frac{c_i+c_i c_\ell}{4}} = \sigma^{\frac{h}{2}+\frac{c_i+c_i c_\ell}{4}}\sigma_i. \qquad (1.110)$$

For even h we compute by a successive use of (1.110)

$$\tilde{\sigma}^{(\ell-2)}\sigma^{\frac{h}{2}} = \sigma_{\ell-2}\sigma_\ell\sigma_{\ell-1}\sigma^{\frac{h}{2}} = \sigma_{\ell-2}\sigma_\ell\sigma^{\frac{h}{2}}\sigma_2 = \sigma_{\ell-2}\sigma^{\frac{h}{2}}\sigma_1\sigma_2 = \sigma^{\frac{h}{2}}\sigma_3\sigma_1\sigma_2 = \sigma^{\frac{h}{2}}\tilde{\sigma}^{(1)}. \qquad (1.111)$$

Similarly we compute for odd h

$$\begin{aligned}\tilde{\sigma}^{(\ell-2)}\sigma^{\frac{h-1}{2}} &= \sigma_{\ell-1}\sigma_{\ell-2}\sigma_\ell\sigma^{\frac{h-1}{2}} = \sigma_{\ell-1}\sigma_{\ell-2}\sigma^{\frac{h-1}{2}}\sigma_1 = \sigma_{\ell-1}\sigma^{\frac{h-1}{2}}\sigma_3\sigma_1, \qquad (1.112)\\ &= \sigma_{\ell-1}\sigma^{\frac{h+1}{2}}\sigma^{-1}\sigma_3\sigma_1 = \sigma^{\frac{h+1}{2}}\sigma_2\sigma^{-1}\sigma_3\sigma_1 = \sigma^{\frac{h-1}{2}}\sigma_-\sigma_+\sigma_2\sigma_+\sigma_-\sigma_3\sigma_1,\\ &= \sigma^{\frac{h-1}{2}}\sigma_3\sigma_1\sigma_2 = \sigma^{\frac{h-1}{2}}\tilde{\sigma}^{(1)}.\end{aligned}$$

Thus we have established that the first element $\tilde{\sigma}^{(1)}$ in the similarity class Σ is related via the similarity transformation (1.109) to the last element $\tilde{\sigma}^{(\ell-2)}$ in this class. In comparison to one rank less the last element is the only additional one. For the other elements we can use the same argumentation but employing the Coxeter element for one rank less. ■

Expl.: A_8 We illustrate now the working of these formulae for a concrete example. We consider A_8 and generate the entire root space $\tilde{\Delta}$ as described in (1.99) from $\tilde{\sigma}^{(1)}$. The results are depicted in Table 2.1.

For convenience we used the following conventions: For any non-simple root $\beta = \sum_i \mu_i\alpha_i$ we present only the non-vanishing coefficients μ_i in the table

$(\tilde{\sigma}^{(1)})^j \backslash \alpha_i$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde{\sigma}^{(1)}$	$\mathbf{-1,2}$	$\mathbf{1,2,3}$	$\mathbf{-2,3}$	$2,3,4$	5	6	7	8
$\tilde{\sigma}^{(1)}\tilde{\sigma}^{(1)}$	$\mathbf{-3}$	$\mathbf{-2}$	$\mathbf{-1}$	$1,2,3,4$	5	6	7	8
$\tilde{\sigma}^{(1)}\tilde{\sigma}^{(1)}\tilde{\sigma}^{(1)}$	$\mathbf{2,3}$	$\mathbf{-1,2,3}$	$\mathbf{1,2}$	$3,4$	5	6	7	8

Table 1.1: The reduced A_8-root space $\tilde{\Delta}$ generated from the orbits of $\tilde{\sigma}^{(1)}$.

with the overall sign written in front, e.g. $\alpha_1+\alpha_2+\alpha_3$ is represented as $1,2,3$ and $-\alpha_1-\alpha_2$ as $-1,2$. We indicate the A_3 substructure in bold. Further examples for root spaces obtained from different elements in $\Sigma_{\{1,2,3,4^{\ell-3}\}}$ are presented in appendix A.

Crucial to our construction is the invariance under the action of $\tilde{\sigma}^{(1)}_{\pm}$. Acting on the roots as depicted in table 1 with $\tilde{\sigma}^{(1)}_{\pm}$ we recover all the elements in table 1, albeit in a permuted way as indicated in Table 2.2.

$\tilde{\sigma}^{(1)}_{-}(\tilde{\Delta})$	-1	$1,2,3$	-3	$3,4$	5	6	7	8
	$-2,3$	2	$-1,2$	$1,2,3,4$	5	6	7	8
	3	$-1,2,3$	-1	$2,3,4$	5	6	7	8
	$1,2$	-2	$2,3$	4	5	6	7	8

$\tilde{\sigma}^{(1)}_{+}(\tilde{\Delta})$	$1,2$	-2	$2,3$	4	5	6	7	8
	-1	$1,2,3$	-3	$3,4$	5	6	7	8
	$-2,3$	2	$-1,2$	$1,2,3,4$	5	6	7	8
	3	$-1,2,3$	1	$3,4$	5	6	7	8

Table 1.2: The invariance of the A_8-root space $\tilde{\Delta}$ generated from $\tilde{\sigma}^{(1)}$ under the action of $\tilde{\sigma}^{(1)}_{\pm}$.

The class with modified exponents $\{1, 2^2, 3, 4^{\ell-3}\}$ and $\tilde{\mathrm{h}}$=4

Other classes become considerably more complicated. We present here only some examples to indicate this. For instance in the class

$$\Sigma_{\{1,2^2,3,4^{\ell-4}\}} = \left\{\tilde{\sigma}^{(1,1,1)}, \tilde{\sigma}^{(1,1,2)}, \ldots, \tilde{\sigma}^{(2,1,\ell-4)}\right\}, \tag{1.113}$$

we have to label the elements by three indices

$$\tilde{\sigma}^{(1,i,j)} := \sigma_i \sigma_{i+2} \sigma_{i+3+j} \sigma_{i+1} \qquad \text{and} \qquad \tilde{\sigma}^{(2,i,j)} := \sigma_i \sigma_{i+1+j} \sigma_{i+3+j} \sigma_{i+j+2}, \tag{1.114}$$

with $i = 1, \ldots, \ell - j - 3$ and $j = 1, \ldots, \ell - 4$. It is easy to convince oneself that these elements have order 4. In both types of labeling we have three consecutive elements and one additional factor which commutes with all the other elements, that is σ_{i+3+j} in $\tilde{\sigma}^{(1,i,j)}$ and σ_i in $\tilde{\sigma}^{(2,i,j)}$, respectively. Thus by the same argument as in the previous class and the fact that $\sigma_i^2 = 1$ it follows that the order of all elements in (1.114) is 4.

Arguing along similar lines as for the class presented in the previous subsection, we can also show that all elements in $\Sigma_{\{1,2^2,3,4^{\ell-4}\}}$ are indeed related by a similarity transformation. We will not present this proof here.

The similarity class structure with $\tilde{\mathrm{h}}$=4

It is clear that for higher ranks more and more possible sets of exponents characterizing different classes may exist. Here we only indicate in table 3 the general structure but do not report a detailed construction of the elements of these classes and their interrelations as the argumentation goes along the same lines as in the two previous subsections. By inspection of the table we notice the onset of two new classes when we increase the rank by two, that is the number of classes increases by 2 for $\ell = 2n+5$ for $n = 1, 2, \ldots$ We also observe that the number of classes for $\ell = 2n+1$ and $\ell = 2n+2$ is the same.

ℓ					
3	$\{1,2,3\}$				
4	$\{1,2,3,4\}$				
5	$\{1,2,3,4^2\}$	$\{1,2^2,3,4\}$			
6	$\{1,2,3,4^3\}$	$\{1,2^2,3,4^2\}$			
7	$\{1,2,3,4^4\}$	$\{1,2^2,3,4^3\}$	$\{1,2^3,3,4^2\}$	$\{1^2,2^2,3^2,4\}$	
8	$\{1,2,3,4^5\}$	$\{1,2^2,3,4^4\}$	$\{1,2^3,3,4^3\}$	$\{1^2,2^2,3^2,4^2\}$	
9	$\{1,2,3,4^6\}$	$\{1,2^2,3,4^5\}$	$\{1,2^3,3,4^4\}$	$\{1^2,2^2,3^2,4^3\}$	$\{1^2,2^3,3^2,4^2\}$
10	$\{1,2,3,4^7\}$	$\{1,2^2,3,4^6\}$	$\{1,2^3,3,4^5\}$	$\{1^2,2^2,3^2,4^4\}$	$\{1^2,2^3,3^2,4^3\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
ℓ	$\{1,2,3,4^{\ell-3}\}$	$\{1,2^2,3,4^{\ell-4}\}$	$\{1,2^3,3,4^{\ell-5}\}$	$\{1^2,2^2,3^2,4^{\ell-6}\}$	$\ldots$

Table 1.3: Similarity classes in A_ℓ with $\tilde{h} = 4$.

In addition we note that the number of factors in the elements of a similarity class increases by one in the table in each column from the left to the right, starting with three factors on the very left.

The class with modified exponents $\left\{1, 2, \ldots, 4n-1, 4n^{\ell-4n+1}\right\}$ and $\tilde{\mathbf{h}}$=4n

Let us now generalize the previous considerations towards classes with larger amounts of eigenvalues, such that they are related to modified Coxeter numbers of higher powers. The class (1.102) acquires the more general form

$$\Sigma_{\{1,2,\ldots,4n-1,4n^{\ell-4n+1}\}} = \left\{\tilde{\sigma}^{(n,1)}, \ldots, \tilde{\sigma}^{(n,\ell+2-4n)}\right\}, \qquad (1.115)$$

when $x^{4n} = 1$ for $x \in \Sigma$. In this case the elements of the class

$$\begin{aligned}\tilde{\sigma}^{(n,i)} \quad : \quad = & \left[\left(\prod_{k=2}^{n} \sigma_{i-1+4(k-1)} \sigma_{i+1+4(k-1)} \right) \right. \\ & \left. \sigma_{i+1} \left(\prod_{k=1}^{n} \sigma_{i+4(k-1)} \sigma_{i+2+4(k-1)} \right) \right]^{c_i},\end{aligned} \tag{1.116}$$

are characterized by two indices, n distinguishing the particular type of class and $i = 1, \ldots, \ell + 2 - 4n$ labeling the individual elements in that class. The case $n = 1$ reduces to our previous simpler example with $\tilde{\sigma}^{(n,i)} = \tilde{\sigma}^{(i)}$ as defined in (1.103). Evidently the element $\tilde{\sigma}^{(n,i)}$ contains the $4n-1$ consecutive factors σ_i to σ_{i+4n-3} separated into odd and even indices. This means each element can be viewed as a Coxeter element for the A_{4n-1}-Weyl group and therefore the order of $\tilde{\sigma}^{(n,i)}$ is $\tilde{h} = 4n$.

In this case we will also establish that all elements in Σ are indeed related by a similarity transformation. Two consecutive elements in this class are related as

$$\varkappa_i^{(n)} \tilde{\sigma}^{(n,i)} = \tilde{\sigma}^{(n,i+1)} \varkappa_i^{(n)} \quad \text{with} \quad \varkappa_i^{(n)} := \prod_{k=1}^{4n} \sigma_{i+k-1} \prod_{k=1}^{2n-1} \sigma_{i+2k-1}, \tag{1.117}$$

which in turn implies that all elements in Σ are related by a similarity transformation. The proof for this identity goes along the same line as the one for the particular case $n = 1$ of the identity (1.104).

The class with modified exponents $\left\{1, 2^2, \ldots, 4n-1, 4n^{\ell-4n}\right\}$ and $\tilde{\mathbf{h}}$=4n

For higher order the similarity class (1.113) generalizes to

$$\Sigma_{\{1,2^2,\ldots,4n-1,4n^{\ell-4n+1}\}} = \left\{ \tilde{\sigma}^{(1,1,1,1)}, \tilde{\sigma}^{(1,2,1,1)}, \ldots \right\}, \tag{1.118}$$

where we label its elements

$$\tilde{\sigma}^{(1,n,i,j)} \quad : \quad = \prod_{k=1}^{n} \sigma_{i+4(k-1)}\sigma_{i+2+4(k-1)}\sigma_{i+j+(\tilde{h}-1)}\sigma_{i+1} \times \prod_{k=2}^{n} \sigma_{i-1+4(k-1)}\sigma_{i+1+4(k-1)}, \tag{1.119}$$

$$\tilde{\sigma}^{(2,n,i,j)} \quad : \quad = \sigma_{i+j+2} \prod_{k=1}^{n} \sigma_{i+j+4(k-1)}\sigma_{i+j+2+4(k-1)}\sigma_{i} \times \prod_{k=2}^{n} \sigma_{i+j+1+4(k-1)}\sigma_{i+j+3+4(k-1)}, \tag{1.120}$$

now by four indices with $j = 1, \cdots, \ell - 4n$ and $i = 1, \cdots, \ell - j - (4n - 1)$. We recover the case discussed in the previous section for $n = 1$. Using similar arguments as before we can show that all elements in this class have order $\tilde{\text{h}}$=4n. For instance for the element $\sigma_{i+j+(\tilde{h}-1)}$ in (1.119) the subscript obeys $i + j + (\tilde{h} - 1) > i + \tilde{h}$, which means that the element may be commuted to the left. Taking then the $\tilde{h}$-th power of the entire expression we find

$$\Big[(\sigma_{i+j+(\tilde{h}-1)})^{\tilde{h}} \big[\Big(\prod_{k=1}^{n} \sigma_{i+4(k-1)}\sigma_{i+2+4(k-1)} \Big) \sigma_{i+1} \Big(\prod_{k=2}^{n} \sigma_{i-1+4(k-1)}\sigma_{i+1+4(k-1)} \Big)\big]^{\tilde{h}}\Big]. \tag{1.121}$$

Since $\tilde{h}$ is even we have $(\sigma_{i+j+(\tilde{h}-1)})^{\tilde{h}} = 1$ and since the expression in the bracket is a reduced Coxeter element for $A_{\tilde{h}=4n}$ the expression in (1.121) equals 1, thus establishing the order of $\tilde{\sigma}^{(1,n,i,j)}$ to be $\tilde{h} = 4n$. Similar arguments can be used for $\tilde{\sigma}^{(2,n,i,j)}$ to prove that this element has the same order.

Antilinearly invariant complex root spaces

Based on the various classes constructed in the previous sections we may now compute the deformation matrix with the help of (1.37) subject to the mentioned constraints. As reported above we found some relatively simple solutions for $h = 4n$. We present now similar solutions for $\tilde{h} = 4n$. Taking in (1.98) all but three coefficients to be zero

$$r_i(\varepsilon) = 0 \qquad \text{for } i \neq 0, n, 2n, \tag{1.122}$$

the equation reduces with the help of (1.96) to

$$1 = (r_0 + r_{2n})^{2\sum_{k=1}^{n}\lambda_{2k}} \left[(r_0 - r_{2n})^2 - 4r_n^2\right]^{\sum_{k=1}^{n}\lambda_{2k-1}}. \tag{1.123}$$

As can be seen directly, this equation is solved by

$$r_{2n} = 1 - r_0 \qquad \text{and} \qquad r_n = \sqrt{r_0(r_0 - 1)} =: \vartheta. \tag{1.124}$$

Thus the corresponding deformation matrix resulting from (1.94) reads

$$\theta_\varepsilon = r_0(\varepsilon)\mathbb{I} + [1 - r_0(\varepsilon)]\,\tilde{\sigma}^{2n} + i\vartheta(\tilde{\sigma}^n - \tilde{\sigma}^{-n}). \tag{1.125}$$

All what remains left to establish whether the set of modified exponents in (1.96) really exists for some concrete elements of $\tilde{\sigma} \in \mathcal{W}$ of order $\tilde{h} = 4n$ and possibly to specify the function $r_0(\varepsilon)$.

It is useful to consider a concrete example. For instance, the deformed roots resulting from $\tilde{\sigma}^{(3)}$ of the class $\Sigma_{\{1,2,3,4^{\ell-3}\}}$ for A_8 according to (1.125) are The θ_ε resulting from different elements in the same class have a similar form with the A_3-substructure displaced similarly as for the undeformed roots. We do not report these solutions here. Unlike as in (1.126) all eight roots are deformed when constructing θ_ε (1.127) for instance from $\tilde{\sigma}^{(2,1)}$ as specified

$$
\begin{aligned}
\tilde{\alpha}_1 &= \alpha_1,\ \tilde{\alpha}_7 = \alpha_7,\ \tilde{\alpha}_8 = \alpha_8, \\
\tilde{\alpha}_2 &= \alpha_2 + (1-r_0)\alpha_3 + (1-r_0+i\vartheta)\alpha_4 + (1-r_0)\alpha_5, \\
\tilde{\alpha}_3 &= (r_0-i\vartheta)\alpha_3 - 2i\vartheta\alpha_4 + (r_0-i\vartheta-1)\alpha_5, \\
\tilde{\alpha}_4 &= 2i\vartheta\alpha_3 + (2r_0+2i\vartheta-1)\alpha_4 + 2i\vartheta\alpha_5, \\
\tilde{\alpha}_5 &= (r_0-i\vartheta-1)\alpha_3 - 2i\vartheta\alpha_4 + (r_0-i\vartheta)\alpha_5, \\
\tilde{\alpha}_6 &= (1-r_0)\alpha_3 + (1-r_0+i\vartheta)\alpha_4 + (1-r_0)\alpha_5 + \alpha_6.
\end{aligned}
\tag{1.126}
$$

in (1.116). Here we abbreviate $\kappa_0 = r_0 - 1$ and $\lambda_0 = r_0 - i\vartheta$ to achieve a

$$
\theta_\varepsilon = \begin{pmatrix}
r_0 & 0 & i\vartheta & 2i\vartheta & i\vartheta & 0 & \kappa_0 & 0 \\
0 & \lambda_0 & -2i\vartheta & -2i\vartheta & -2i\vartheta & \kappa_0 - i\vartheta & 0 & 0 \\
i\vartheta & 2i\vartheta & r_0 + 2i\vartheta & 2i\vartheta & \kappa_0 + 2i\vartheta & 2i\vartheta & i\vartheta & 0 \\
-2i\vartheta & -2i\vartheta & -2i\vartheta & 2\lambda_0 - 1 & -2i\vartheta & -2i\vartheta & -2i\vartheta & 0 \\
i\vartheta & 2i\vartheta & \kappa_0 + 2i\vartheta & 2i\vartheta & r_0 + 2i\vartheta & 2i\vartheta & i\vartheta & 0 \\
0 & \kappa_0 - i\vartheta & -2i\vartheta & -2i\vartheta & -2i\vartheta & \lambda_0 & 0 & 0 \\
\kappa_0 & 0 & i\vartheta & 2i\vartheta & i\vartheta & 0 & r_0 & 0 \\
-\kappa_0 & -\kappa_0 & -\kappa_0 & -\kappa_0 - i\vartheta & -\kappa_0 & -\kappa_0 & -\kappa_0 & 1
\end{pmatrix}.
\tag{1.127}
$$

compact notation. The dual map $\delta^\star$ is obtained by solving (1.17) for the dual deformation matrix $\theta_\varepsilon^\star$ (1.2.1) with the explicit form for θ_ε. Taking the latter to be given by (1.127) we compute for the standard $(\ell+1)$-dimensional representation of A_ℓ $(\alpha_i)_j = \delta_{ij} - \delta_{(i+1)j}$, $i = 1, 2, \ldots, \ell$, $j = 1, 2, \ldots, \ell+1$. By construction the corresponding dual root space $\tilde{\Delta}^\star(\varepsilon)$ is invariant under the action of some antilinear maps $\varpi^\star$, obtained by solving (1.19). For antilinear symmetry $\omega_1 = \tau\sigma_2\sigma_4\sigma_6$ we compute the dual antilinear transformation to (1.129)

where we abbreviated $\pi = \vartheta\nu$, $\nu := 2r_0 - 1$ and $\mu := 4(r_0 - r_0^2)$. The action

$$\theta_\varepsilon^\star = \begin{pmatrix} r_0 & 0 & 0 & i\vartheta & -i\vartheta & 0 & 0 & 1-r_0 & 0 \\ 0 & r_0 & -i\vartheta & 0 & 0 & i\vartheta & 1-r_0 & 0 & 0 \\ 0 & i\vartheta & r_0 & 0 & 0 & 1-r_0 & -i\vartheta & 0 & 0 \\ -i\vartheta & 0 & 0 & r_0 & 1-r_0 & 0 & 0 & i\vartheta & 0 \\ i\vartheta & 0 & 0 & 1-r_0 & r_0 & 0 & 0 & -i\vartheta & 0 \\ 0 & -i\vartheta & 1-r_0 & 0 & 0 & r_0 & i\vartheta & 0 & 0 \\ 0 & 1-r_0 & i\vartheta & 0 & 0 & -i\vartheta & r_0 & 0 & 0 \\ 1-r_0 & 0 & 0 & -i\vartheta & i\vartheta & 0 & 0 & r_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.128}$$

$$\omega_1^\star = \tau \begin{pmatrix} \nu^2 & 0 & 0 & -2i\pi & 2i\pi & 0 & 0 & \mu & 0 \\ 0 & -2i\pi & \nu^2 & 0 & 0 & \mu & 2i\pi & 0 & 0 \\ 0 & \nu^2 & 2i\pi & 0 & 0 & -2i\pi & \mu & 0 & 0 \\ -2i\pi & 0 & 0 & \mu & \nu^2 & 0 & 0 & 2i\pi & 0 \\ 2i\pi & 0 & 0 & \nu^2 & \mu & 0 & 0 & -2i\pi & 0 \\ 0 & \mu & -2i\pi & 0 & 0 & 2i\pi & \nu^2 & 0 & 0 \\ 0 & 2i\pi & \mu & 0 & 0 & \nu^2 & -2i\pi & 0 & 0 \\ \mu & 0 & 0 & 2i\pi & -2i\pi & 0 & 0 & \nu^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.129}$$

on the deformed and original variables amounts with (1.129) simply to

$$\omega_1^\star \; : \; \tilde{\Delta}^*(\varepsilon) \to \tilde{\Delta}^*(\varepsilon), \tag{1.130}$$

$$\tilde{x}_1 \mapsto \tilde{x}_1, \tilde{x}_2 \leftrightarrow \tilde{x}_3, \tilde{x}_4 \leftrightarrow \tilde{x}_5, , \tilde{x}_6 \leftrightarrow \tilde{x}_7, \tilde{x}_8 \mapsto \tilde{x}_8, \tilde{x}_9 \mapsto \tilde{x}_9, \tag{1.131}$$

$$x_1 \mapsto x_1, x_2 \leftrightarrow x_3, x_4 \leftrightarrow x_5, , x_6 \leftrightarrow x_7, x_8 \mapsto x_8, x_9 \mapsto x_9, i \mapsto -i.$$

A similar computation leads to the dual antilinear symmetry corresponding for $\omega_2 = \tau\sigma_1\sigma_3\sigma_5\sigma_7$.

Obviously these solutions only capture part of all possibilities as we may of course also consider the cases $\tilde{h} = 4n$ and since (1.125) is a restriction of the most general ansatz (1.93). Some solutions filling these gaps were presented in [67]. Having been fairly detailed for the A_ℓ-Weyl group, we will only indicate some selected examples for reference for the other cases.

1.2.2 Antilinearly deformed B_ℓ root systems

For the deformation in section 1.1 we conjectured that odd ranking B_ℓ Coxeter groups does not admit a non trivial solution for the deformation matrix (1.37). We will show that this is remedied for the current deformation method. We will report an explicit example of this.

The simplest class for $\tilde{h} = 4$ contains only one element comprised of two Weyl reflections

$$\Sigma_{\{1,3,4^{\ell-2}\}} = \{\tilde{\sigma} = \sigma_{\ell-1}\sigma_\ell\}. \tag{1.132}$$

The next class with $\tilde{h} = 4$ contains $2\ell - 6$ elements

$$\Sigma_{\{1,2,3,4^{\ell-3}\}} = \left\{\tilde{\sigma}^{(1,1)}, \ldots, \tilde{\sigma}^{(1,\ell-3)}, \tilde{\sigma}^{(2,1)}, \ldots, \tilde{\sigma}^{(2,\ell-3)}\right\}, \tag{1.133}$$

build from a composition of three Weyl reflections

$$\tilde{\sigma}^{(1,i)} := \sigma_i\sigma_{i+2}\sigma_{i+1} \quad \text{and} \quad \tilde{\sigma}^{(2,i)} := \sigma_\ell\sigma_{\ell-i-2}\sigma_{\ell-1} \quad \text{for} \quad i = 1, \cdots, \ell-3. \tag{1.134}$$

In table 4 we indicate the different types of classes with increasing rank ℓ. We note that whenever the rank increases by one, a new type of class emerges with one additional Weyl reflection in the element $\tilde{\sigma}$.

Using for B_5 the same general ansatz as for the A_ℓ-case in (1.125), we

ℓ					
3	$\{1,3,4\}$				
4	$\{1,3,4^2\}$	$\{1,2,3,4\}$			
5	$\{1,3,4^3\}$	$\{1,2,3,4^2\}$	$\{1,2^2,3,4\}$		
6	$\{1,3,4^4\}$	$\{1,2,3,4^3\}$	$\{1,2^2,3,4^2\}$	$\{1^2,2,3^2,4\}$	
7	$\{1,3,4^5\}$	$\{1,2,3,4^4\}$	$\{1,2^2,3,4^3\}$	$\{1^2,2,3^2,4^2\}$	$\{1,2^3,3,4^2\}$
8	$\{1,3,4^6\}$	$\{1,2,3,4^5\}$	$\{1,2^2,3,4^4\}$	$\{1^2,2,3^2,4^3\}$	$\{1,2^3,3,4^3\}$
9	$\{1,3,4^7\}$	$\{1,2,3,4^6\}$	$\{1,2^2,3,4^5\}$	$\{1^2,2,3^2,4^4\}$	$\{1,2^3,3,4^4\}$
10	$\{1,3,4^8\}$	$\{1,2,3,4^7\}$	$\{1,2^2,3,4^6\}$	$\{1^2,2,3^2,4^5\}$	$\{1,2^3,3,4^5\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\dots$
ℓ	$\{1,3,4^{\ell-2}\}$	$\{1,2,3,4^{\ell-3}\}$	$\{1,2^2,3,4^{\ell-4}\}$	$\{1^2,2,3^2,4^{\ell-5}\}$	$\dots$

Table 1.4: Similarity classes in B_ℓ with $\tilde{h}=4$.

obtain for the antilinearly deformed symmetry of $\tilde{\sigma}^{(1,1)}$ the solution

$$\theta_\varepsilon = \begin{pmatrix} r_0 - i\vartheta & -2i\vartheta & r_0 - i\vartheta - 1 & 0 & 0 \\ 2i\vartheta & 2r_0 + 2i\vartheta - 1 & 2i\vartheta & 0 & 0 \\ r_0 - i\vartheta - 1 & -2i\vartheta & r_0 - i\vartheta & 0 & 0 \\ 1 - r_0 & 1 - r_0 + i\vartheta & 1 - r_0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.135}$$

We compute the dual map $\delta^\star$ by solving (1.17) for the dual deformation matrix $\theta_\varepsilon^\star$ with the explicit form for θ_ε as in (1.135). For the standard root representation of the B_ℓ-roots $(\alpha_i)_j = \delta_{ij} - \delta_{(i+1)j}$, $(\alpha_\ell)_j = \delta_{\ell j}$, $i = 1, 2, \ldots, \ell - 1$, $j = 1, 2, \ldots, \ell$ we obtain

$$\theta_\varepsilon^\star = \begin{pmatrix} r_0 & -i\vartheta & i\vartheta & 1 - r_0 & 0 \\ i\vartheta & r_0 & 1 - r_0 & -i\vartheta & 0 \\ -i\vartheta & 1 - r_0 & r_0 & i\vartheta & 0 \\ 1 - r_0 & i\vartheta & -i\vartheta & r_0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.136}$$

By construction the corresponding root space $\tilde{\Delta}^{\star}(\varepsilon)$ is invariant under the action of some antilinear maps $\varpi^{\star}$, obtained by solving (1.19). For $\omega_1 = \tau\sigma_1\sigma_3$ we compute the dual antilinear transformation to

$$\omega_1^{\star} = \tau \begin{pmatrix} -2i\vartheta\mu & (-\mu)^2 & -4\nu r_0 & 2i\vartheta\mu & 0 \\ (-\mu)^2 & 2i\vartheta\mu & -2i\vartheta\mu & -4\nu r_0 & 0 \\ -4\nu r_0 & -2i\vartheta\mu & 2i\vartheta\mu & (-\mu)^2 & 0 \\ 2i\vartheta\mu & -4\nu r_0 & (-\mu)^2 & -2i\vartheta\mu & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.137}$$

As abbreviation we use $\nu = r_0 - 1$ and $\mu = 2r_0 - 1$. The action on the variables amounts with (1.137) simply to

$$\omega_1^{\star} : \tilde{\Delta}^{*}(\varepsilon) \to \tilde{\Delta}^{*}(\varepsilon), \qquad \tilde{x}_1 \leftrightarrow \tilde{x}_2, \tilde{x}_3 \leftrightarrow \tilde{x}_4, \tilde{x}_5 \mapsto \tilde{x}_5, \tag{1.138}$$

$$x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4, x_5 \mapsto x_5, i \mapsto -i. \tag{1.139}$$

A similar computation leads to the dual antilinear symmetry corresponding to $\omega_2 = \tau\sigma_2$.

1.2.3 Antilinearly deformed C_ℓ root systems

A simple class for $\tilde{h} = 4$ with only one element $\tilde{\sigma} = \sigma_1\sigma_3\sigma_2$ is the case $\Sigma_{\{1,2,3,4^{\ell-3}\}}$. We present the deformation matrix for the C_4-case resulting from this element

$$\theta_\varepsilon = \begin{pmatrix} r_0 - i\vartheta & -2i\vartheta & r_0 - i\vartheta - 1 & 0 \\ 2i\vartheta & 2r_0 + 2i\vartheta - 1 & 2i\vartheta & 0 \\ r_0 - i\vartheta - 1 & -2i\vartheta & r_0 - i\vartheta & 0 \\ 2(1-r_0) & 2(1-r_0) + 2i\vartheta & 2(1-r_0) & 1 \end{pmatrix}. \tag{1.140}$$

Note that the classes for C_ℓ are the same as those for B_ℓ, albeit the deformation matrices are different due to the difference of the Weyl reflections.

1.2.4 Antilinearly deformed D_ℓ root systems

In this case a simple class for $\tilde{h} = 4$ contains $\ell - 1$ elements

$$\Sigma_{\{1,3,4^{\ell-3}\}} = \left\{ \tilde{\sigma}^{(1)}, \tilde{\sigma}^{(2)}, \ldots, \tilde{\sigma}^{(\ell=2)}, \tilde{\sigma}^{(\ell)} \right\}, \tag{1.141}$$

with

$$\tilde{\sigma}^{(i)} = \sigma_i \sigma_{i+2} \sigma_{i+1} \quad \text{and} \quad \tilde{\sigma}^{(\ell)} = \sigma_{\ell-3} \sigma_\ell \sigma_{\ell-2} \quad \text{for} \quad i = 1, \cdots, \ell - 2. \tag{1.142}$$

As an example for a deformation matrix for D_4 we present the one resulting from $\tilde{\sigma}^{(1)} = \sigma_1 \sigma_3 \sigma_2$

$$\theta_\varepsilon = \begin{pmatrix} r_0 - i\vartheta & -2i\vartheta & r_0 - i\vartheta - 1 & 0 \\ 2i\vartheta & 2r_0 + 2i\vartheta - 1 & 2i\vartheta & 0 \\ r_0 - i\vartheta - 1 & -2i\vartheta & r_0 - i\vartheta & 0 \\ 1 - r_0 - i\vartheta & 2 - 2r_0 & 1 - r_0 - i\vartheta & 1 \end{pmatrix}. \tag{1.143}$$

1.2.5 Antilinearly deformed E_{6+n} root systems

We may treat the exceptional algebras together using for the labeling the E_8-convention as explained above and in the appendix and removing vertices from the long end Dynkin diagram to obtain the E_7 and E_6-cases. A simple class for $\tilde{h}=4$ contains $n+5$ elements

$$\Sigma_{\{1,2,3,4^3\}} = \left\{\sigma_1\sigma_3\sigma_4, \sigma_1\sigma_5\sigma_4, \sigma^{(2)}, \sigma^{(3)} \ldots, \sigma^{(n+4)}\right\}, \tag{1.144}$$

with $\sigma^{(i)} = \sigma_i\sigma_{i+2}\sigma_{i+1}$ for $i = 2, .., n+4$. The deformation matrix for $\sigma^{(2)} = \sigma_3\sigma_2\sigma_4$ is computed to be

$$\theta_\varepsilon = \begin{pmatrix} 1 & 1-r_0 & -r_0-i\vartheta+1 & 1-r_0 & 0 & 0 & \cdots \\ 0 & r_0+i\vartheta & 2i\vartheta & r_0+i\vartheta-1 & 0 & 0 & \cdots \\ 0 & -2i\vartheta & 2r_0-2i\vartheta-1 & -2i\vartheta & 0 & 0 & \cdots \\ 0 & r_0+i\vartheta-1 & 2i\vartheta & r_0+i\vartheta & 0 & 0 & \cdots \\ 0 & 1-r_0 & -r_0-i\vartheta+1 & 1-r_0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 1 & \end{pmatrix}. \tag{1.145}$$

A further class is $\Sigma_{\{1,2^2,3,4^{2+n}\}}$ with elements $\tilde{\sigma} = \sigma_1\sigma_4\sigma_2\sigma_5, \ldots$

1.2.6 Antilinearly deformed F_4 root systems

The simplest class for $\tilde{h}=4$ contains only one element

$$\Sigma_{\{1,3,4^2\}} = \{\sigma_3\sigma_2\}. \tag{1.146}$$

The deformation matrix is computed to

$$\theta_\varepsilon = \begin{pmatrix} 1 & 2(1-r_0) & 2(1-r_0)-2i\vartheta & 0 \\ 0 & 2r_0+2i\vartheta-1 & 4i\vartheta & 0 \\ 0 & -2i\vartheta & 2r_0-2i\vartheta-1 & 0 \\ 0 & 1-r_0+i\vartheta & 2(1-r_0) & 1 \end{pmatrix}. \qquad (1.147)$$

1.3 Deformations of two arbitrary elements in $\mathcal{W}$

The procedure outlined in section 2 is entirely generic and may of course also be carried out by starting from any arbitrary elements in $\mathcal{W}$ different from σ_+ and σ_-. Due to the random choice we allow for the symmetries, we have to consider now concrete cases. It is instructive to discuss some examples for which no nontrivial solutions were found in the previous sections.

Let us therefore consider B_3. As an abstract Coxeter group, B_3 is fully characterized by three involutory generators $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{I}$ together with the three relations $\sigma_1\sigma_3 = \sigma_3\sigma_1$, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and $\sigma_2\sigma_3\sigma_2\sigma_3 = \sigma_3\sigma_2\sigma_3\sigma_2$. Choosing now in (1.8) the involutions different from the previous section as $\hat{\omega}_1 = \sigma_1$ and $\hat{\omega}_2 = \sigma_1\sigma_3$ yields $\Omega_{12} = \sigma_3$. Thus we have taken $\hat{\omega}_1$ and $\hat{\omega}_2$ both to be factors in σ_-. According to (1.11) we have to identify next all elements in B_3 commuting with σ_3. Using the three relations and the three generators we find $\{\mathbb{I}, \sigma_1, \sigma_3, \sigma_1\sigma_3, \sigma_2\sigma_3\sigma_2, \sigma_1\sigma_2\sigma_3\sigma_2, \sigma_2\sigma_3\sigma_2\sigma_1, \sigma_2\sigma_3\sigma_2\sigma_3\}$ leading to the ansatz

$$\theta_\varepsilon = r_0(\varepsilon)\mathbb{I}+r_1(\varepsilon)\sigma_1 + r_2(\varepsilon)\sigma_3 + r_3(\varepsilon)\sigma_1\sigma_3 + r_4(\varepsilon)\sigma_2\sigma_3\sigma_2 \quad (1.148)$$
$$+r_5(\varepsilon)\sigma_1\sigma_2\sigma_3\sigma_2 + r_6(\varepsilon)\sigma_2\sigma_3\sigma_2\sigma_1 + r_7(\varepsilon)\sigma_2\sigma_3\sigma_2\sigma_3.$$

which unfortunately, leads to a trivial solution when solved for the constraints (1.15).

The same ansatz (1.148) can be used for the choice $\hat{\omega}_1 = \sigma_2$ and $\hat{\omega}_2 = \sigma_2\sigma_3$, but in that case we are led to the trivial solution $\theta_\varepsilon = \mathbb{I}$.

1.4 Deformations from rotations in the dual space

So far we have started with a given antilinear involution ϖ_i and constructed the deformation map δ by solving the constraints (1.15) for a given Weyl group, i.e., given some ω_i we determined the deformation matrix θ_ε. Subsequently we constructed the corresponding maps $\delta^\star$ and $\varpi^\star$acting in the dual spaces. We may also try to reverse the procedure and start with the dual space with given maps $\delta^\star$ and $\varpi^\star$ and determine the maps ϖ_i and δ thereafter. In view of the last section it is natural to assume the $\theta_\varepsilon^\star$ to be an element of the special orthogonal group. We define therefore the $(2n+1)\times(2n+1)$-matrix

$$\theta_\varepsilon^\star = \begin{pmatrix} R & & & & \\ & R & & 0 & \\ & & \ddots & & \\ & 0 & & R & \\ & & & & 1 \end{pmatrix} \qquad \text{with } R = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}, \tag{1.149}$$

and construct the deformation matrix θ_ε by solving (1.17). We note that this constraint might not admit any solutions for certain Weyl groups. In fact for the standard representation for A_ℓ it is easy to verify that indeed there exists no solution. However, for the special orthogonal Weyl groups $B_\ell \equiv SO(2\ell+1)$ and $D_\ell \equiv SO(2\ell)$ one can solve (1.17). Since previously in section 1.1 we did not find solutions for odd ranks in the B-series based on the assumptions made in 1.1, we present here some solutions for B_{2n+1}. Solving (1.19) for θ_ε using the standard representation for the B_ℓ-roots we compute the deformed roots to

$$\begin{aligned}
\tilde{\alpha}_{2j-1} &= \cosh\varepsilon\alpha_{2j-1} + i\sinh\varepsilon\left(\alpha_{2j-1} + 2\sum_{k=2j}^{\ell}\alpha_k\right) \text{ for } j=1,\ldots,n, \\
\tilde{\alpha}_{2j} &= \cosh\varepsilon\alpha_{2j} - i\sinh\varepsilon\left(\sum_{k=2j}^{2j+2}\alpha_k + 2\sum_{k=2j+3}^{\ell}2\alpha_k\right) \text{ for } j=1,\ldots,n-1, \\
\tilde{\alpha}_{\ell-1} &= \cosh\varepsilon(\alpha_{\ell-1}+\alpha_\ell) - \alpha_\ell - i\sinh\varepsilon\left(\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell\right), \\
\tilde{\alpha}_\ell &= \alpha_\ell.
\end{aligned} \tag{1.150}$$

By construction we have satisfied the last three constraints in (1.15). Furthermore, we find that $\theta_\varepsilon^*\sigma_- = \sigma_-\theta_\varepsilon$ but $\theta_\varepsilon^*\sigma_+ \neq \sigma_+\theta_\varepsilon$ with $\sigma_- = \prod_{k=1}^{n+1}\sigma_{2k-1}$ and $\sigma_+ = \prod_{k=1}^{n}\sigma_{2k}$. Thus in this case $\tau\sigma_+$ does not constitute an antilinear symmetry which implies that $[\sigma, \theta_\varepsilon] \neq 0$. This is the reason why this solution has escaped the above mentioned analysis in 1.1. However, besides under the action of $\omega_- := \tau\sigma_- = \sigma_-^\varepsilon$ the root space $\tilde{\Delta}(\varepsilon)$ remains invariant under various other antilinear maps which consist of subfactors of σ_-. For B_3 we observed this in section 4 with $\sigma_- = \sigma_1\sigma_3$ and σ_3 being the additional symmetry. A generalization to B_{2n} is straightforward simply by starting in (1.149) with an $(2n) \times (2n)$-matrix of the form (1.149) without the entry 1.

Similarly as for B_{2n+1} we may also solve (1.17) for the D_{2n} Weyl group for which we demonstrated in [67] that no solution to the constraining equations (1.15) based on (1.94) could exist, that is for a given invariance σ_-^ε and σ_+^ε. Starting with $\theta_\varepsilon^\star$ in the form (1.149) we construct the deformed roots with standard representation for the D_ℓ-roots $(\alpha_i)_j = \delta_{ij} - \delta_{(i+1)j}$, $(\alpha_\ell)_j = \delta_{j(\ell-1)} + \delta_{j\ell}$, $i = 1, 2, \ldots, \ell - 1$, $j = 1, 2, \ldots, \ell$ as

$$\begin{aligned}
\tilde{\alpha}_{\ell-(2j+1)} &= \cosh\varepsilon\alpha_{\ell-(2j+1)} + i\sinh\varepsilon\left[\sum_{k=\ell-(2j+1)}^{\ell}\alpha_k + \sum_{\ell-2j}^{\ell-2}\alpha_k\right], \\
\tilde{\alpha}_{\ell-(2j+2)} &= \cosh\varepsilon\alpha_{\ell-(2j+2)} - i\sinh\varepsilon\left[\sum_{k=\ell-2j-3}^{\ell}\alpha_k + \sum_{\ell-2j}^{\ell-2}\alpha_k\right], \\
\tilde{\alpha}_{\ell-2} &= \cosh\varepsilon\alpha_{\ell-2} - i\sinh\varepsilon(\alpha_{\ell-3} + \alpha_{\ell-2} + \alpha_\ell), \\
\tilde{\alpha}_{\ell-1} &= \cosh\varepsilon\alpha_{\ell-1} + i\sinh\varepsilon\alpha_\ell, \\
\tilde{\alpha}_\ell &= \cosh\varepsilon\alpha_\ell - i\sinh\varepsilon\alpha_{\ell-1}.
\end{aligned} \tag{1.151}$$

Similarly as for the B_{2n+1} case we find that $\theta_\varepsilon^*\sigma_- = \sigma_-\theta$ whereas $\theta_\varepsilon^*\sigma_+ \neq \sigma_+\theta$ with $\sigma_- = \prod_{k=1}^{n}\sigma_{2k-1}$ and $\sigma_+ = \prod_{k=1}^{n-2}\sigma_{2k}$. Again it is easy enough to generalize this to the D_{2n+1} case.

For the standard $(n+1)$-dimensional representation of A_ℓ a rotation on a subspace of $\tilde{\Delta}^*(\varepsilon)$ for the first two coordinates and its conjugate momenta was suggested in [73, 74]. In that case, and for its generalization (1.149), the corresponding deformation $\tilde{\Delta}(\varepsilon)$ cannot be constructed since (1.17) admits no solution.

Chapter 2

Deformations of the longest element

Intuitively it would be more natural to have just one deformed involutory map from the start instead of two. In fact there exist one very distinct involution in $\mathcal{U}$, called the longest element. The length of an element in the Coxeter group $\mathcal{U}$ is defined as the smallest amount of simple Weyl reflections σ_i needed to express that element, see e.g. [3]. Since Coxeter groups are finite, there exists an element in $\mathcal{U}$ of maximal length, i.e., the longest element, which we denote as w_0. The length of this element equals the number of positive roots $h\ell$, with h being the Coxeter number of $\mathcal{U}$ and ℓ is the rank of the group. The map w_0 is involutive, mapping the set of positive roots $\Delta_+ \subset \mathbb{R}^n$ to negative ones $\Delta_- \subset \mathbb{R}^n$ and vice versa

$$w_0 : \quad \Delta_\pm \to \Delta_\mp, \tag{2.1}$$

where $w_0^2 = 1$. Two specific simple roots $\alpha_i, \alpha_{\bar{\imath}}$ are linearly related by w_0 as

$$\alpha_i \mapsto -\alpha_{\bar{\imath}} = (w_0 \alpha)_i. \tag{2.2}$$

Here we have borrowed the notation from the context of affine Toda field theories, where it was found [72] that the longest element serves as charge conjugation operator $\mathcal{C}$, mapping a particle of type i to its anti-particle $\bar{\imath}$. From a mathematical point of view this map is a particular symmetry of the Dynkin diagrams, see e.g. [40].

The longest element admits a concrete realization in terms of products of Coxeter transformations σ. The unique longest element can be expressed as [72]

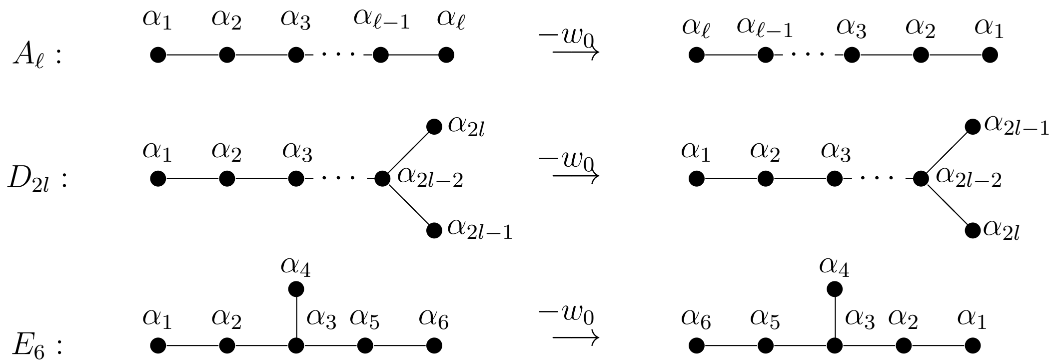
$$w_0 = \begin{cases} \sigma^{h/2} & \text{for } h \text{ even,} \\ \sigma_+ \sigma^{(h-1)/2} & \text{for } h \text{ odd.} \end{cases} \tag{2.3}$$

For the individual algebras the roots $\alpha_{\bar{\imath}}$ in (2.2) are calculated directly or identified from the symmetries of the Dynkin diagrams [40] as

$$\begin{aligned} A_\ell: &\quad \alpha_{\bar{\imath}} = \alpha_{\ell+1-i}, \\ D_\ell: &\quad \begin{cases} \alpha_{\bar{\imath}} = \alpha_i \text{ for } 1 \leq i \leq \ell, & \text{when } \ell \text{ odd} \\ \alpha_{\bar{\imath}} = \alpha_i \quad \text{for } 1 \leq i \leq \ell - 1,\ \alpha_{\bar{\ell}} = \alpha_{\ell-1}, & \text{when } \ell \text{ even,} \end{cases} \\ E_6: &\quad \alpha_{\bar{1}} = \alpha_6, \alpha_{\bar{2}} = \alpha_5, \alpha_{\bar{3}} = \alpha_3, \alpha_{\bar{4}} = \alpha_4, \\ B_\ell, C_\ell, E_7, E_8: &\quad \\ F_4, G_2: &\quad \alpha_{\bar{\imath}} = \alpha_i. \end{aligned} \tag{2.4}$$

Defining then a $\mathcal{CT}$-operator in analogy to (1.22) in two alternative ways,

Figure 2.1: The action of $-w_0$ on the Dynkin diagrams.



we have

$$w_0^\varepsilon = \theta_\varepsilon w_0 \theta_\varepsilon^{-1} = \tau w_0. \tag{2.5}$$

When $[\sigma, \theta_\varepsilon] = 0$ this equation has no solution for even h, since $w_0^\varepsilon = \theta_\varepsilon \sigma^{h/2} \theta_\varepsilon^{-1} = \sigma^{h/2} = \tau \sigma^{h/2}$, which is evidently a contradiction. Whereas for odd h the realization (2.3) in (2.5) yields $\theta_\varepsilon \sigma_+ \sigma^{(h-1)/2} \theta_\varepsilon^{-1} = \theta_\varepsilon \sigma_+ \theta_\varepsilon^{-1} \sigma^{(h-1)/2} = \tau \sigma_+ \sigma^{(h-1)/2}$, which equals (1.22) when canceling $\sigma^{(h-1)/2}$, such that this case is equivalent to the one described in the first section of the previous chapter. This means to obtain a new solution from (2.5) we need to assume $[\sigma, \theta_\varepsilon] \neq 0$.

This fact implies immediately that we have now two options to construct the remaining nonsimple roots. We may either define in complete analogy to (1.26) and (1.27) a root space which remains invariant under the action of the deformed Coxeter transformation. This root space is then also $\mathcal{CT}$-symmetric

$$w_0^\varepsilon : \tilde{\Delta}(\varepsilon) \to \theta_\varepsilon w_0 \theta_\varepsilon^{-1} \tilde{\Delta}(\varepsilon) = \theta_\varepsilon w_0 \Delta(\varepsilon) = \theta_\varepsilon \Delta(\varepsilon) = \tilde{\Delta}(\varepsilon). \tag{2.6}$$

Alternatively we could also define

$$\hat{\Omega}_i^\varepsilon := \left\{ \tilde{\gamma}_i, \sigma \tilde{\gamma}_i, \sigma^2 \tilde{\gamma}_i, \ldots, \sigma^{h-1} \tilde{\gamma}_i \right\}, \tag{2.7}$$

and the entire root space as $\tilde{\Delta}(\varepsilon) := \bigcup_{i=1}^{\ell} \hat{\Omega}_i^{\varepsilon}$. However, this root space will only remain invariant under the action of σ instead of σ^{ε} and in addition it will not be $\mathcal{CT}$-symmetric. This definition is therefore unsuitable for our purposes here.

Using the two definitions in (2.5) leads on one hand to

$$w_0^{\varepsilon}\tilde{\alpha} = \theta_{\varepsilon} w_0 \theta_{\varepsilon}^{-1} \theta_{\varepsilon} \alpha = \theta_{\varepsilon} w_0 \alpha = -\theta_{\varepsilon}\bar{\alpha}, \tag{2.8}$$

and on the other to

$$w_0^{\varepsilon}\tilde{\alpha} = \tau w_0 \tilde{\alpha} = -\tau \bar{\tilde{\alpha}} = -\bar{\tilde{\alpha}}^*, \tag{2.9}$$

such that

$$(\theta_{\varepsilon})_{ij} = (\theta_{\varepsilon}^*)_{\bar{\imath}\bar{\jmath}}. \tag{2.10}$$

As in the first chapter we require the inner products to be preserved (1.12), such that in summary the set of determining equations results to

$$\theta_{\varepsilon}^* w_0 = w_0 \theta_{\varepsilon}, \quad [\sigma, \theta_{\varepsilon}] \neq 0, \quad \theta_{\varepsilon}^* = \theta_{\varepsilon}^{-1}, \quad \det \theta_{\varepsilon} = \pm 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \theta_{\varepsilon} = \mathbb{I}. \tag{2.11}$$

In this case it is instructive to separate θ_{ε} into its real and imaginary part $(\theta_{\varepsilon})_{ij} = R_i^j(\varepsilon) + \imath I_i^j(\varepsilon)$ and therefore expand an arbitrary simple deformed root in terms of the ℓ simple roots as

$$\tilde{\alpha}_i(\varepsilon) := \sum_{j=1}^{\ell} \left(R_i^j(\varepsilon)\alpha_j + \imath I_i^j(\varepsilon)\alpha_j \right), \tag{2.12}$$

with $R_i^j(\varepsilon)$ and $I_i^j(\varepsilon)$ being some real valued functions satisfying

$$\lim_{\varepsilon \to 0} R_i^j(\varepsilon) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad \text{and} \quad \lim_{\varepsilon \to 0} I_i^j(\varepsilon) = 0. \tag{2.13}$$

The relation (2.10) then implies that

$$R_i^j(\varepsilon) = R_{\bar{\imath}}^{\bar{\jmath}}(\varepsilon) \qquad \text{and} \qquad I_i^j(\varepsilon) = -I_{\bar{\imath}}^{\bar{\jmath}}(\varepsilon). \tag{2.14}$$

This means a nontrivial complex $\mathcal{CT}$-symmetric deformation of the longest element does not exist for Coxeter groups in which for all simple roots are self-conjugate $\alpha_i = \alpha_{\bar{\imath}}$.

2.1 Case by case solutions for the Longest element deformation

2.1.1 ω_0 for A_2

Let us start with the construction of a $\mathcal{CT}$-symmetric deformation as outlined in section 2. According to (2.4) the two simple roots are conjugate to each other in the A_2-case, i.e., $\bar{1} = 2$. Using the expansion (2.12) and the constraints (2.14) the deformed roots acquire the form

$$\tilde{\alpha}_1 = R_1^1(\varepsilon)\alpha_1 + R_1^2(\varepsilon)\alpha_2 + \imath(I_1^1(\varepsilon)\alpha_1 + I_1^2(\varepsilon)\alpha_2), \tag{2.15}$$

$$\tilde{\alpha}_2 = R_1^2(\varepsilon)\alpha_1 + R_1^1(\varepsilon)\alpha_2 - \imath(I_1^2(\varepsilon)\alpha_1 + I_1^1(\varepsilon)\alpha_2). \tag{2.16}$$

Demanding next that the inner products are preserved (1.12) amounts to three further constraint, such that the four free functions in (2.15), (2.16) are reduced to only one. We obtain the two solutions

$$R_1^2 = 0, \quad I_1^2 = 2I_1^1, \quad \left(R_1^1\right)^2 - \frac{3}{4}\left(I_1^2\right)^2 = 1 \quad \text{and} \quad 1 \leftrightarrow 2. \tag{2.17}$$

The third relation in (2.17) is solved for instance by $R_1^1 = \cosh\varepsilon$, $I_1^2 = 2/\sqrt{3}\sinh\varepsilon$ satisfying also the limiting constraint (2.13) for $\varepsilon \to 0$. The deformed simple roots are therefore

$$\tilde{\alpha}_1 = \cosh\varepsilon\alpha_1 + \imath\frac{1}{\sqrt{3}}\sinh\varepsilon(\alpha_1 + 2\alpha_2), \qquad (2.18)$$

$$\tilde{\alpha}_2 = \cosh\varepsilon\alpha_2 - \imath\frac{1}{\sqrt{3}}\sinh\varepsilon(2\alpha_1 + \alpha_2). \qquad (2.19)$$

If we compare these deformed simple roots (2.18) and (2.19) with the deformed simple roots in section 1.1.1, we find that they are exactly the same deformed simple roots.

2.1.2 ω_0 for A_3

We obtain an additional non-equivalent solution when $[\sigma, \theta_\varepsilon] \neq 0$ by solving (2.5). For A_3 we read off from (2.4) that $\bar{1} = 3$, $\bar{2} = 2$, such that (2.10) leads to the deformation matrix

$$\theta_\varepsilon = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} = \theta_{22}^* & \theta_{21}^* \\ \theta_{13}^* & \theta_{12}^* & \theta_{11}^* \end{pmatrix}. \qquad (2.20)$$

Substituting this into (2.11) yields a set of constraining equations. Assuming θ_{12} to vanish they simplify to

$$\theta_{22} = |\theta_{11}|^2 - |\theta_{13}|^2, \quad \theta_{22}^2 = 1, \quad |\theta_{11}|^2 - \theta_{13}^2 = 1, \qquad (2.21)$$

$$\theta_{11}\theta_{21}^* = \theta_{21}(\theta_{22} + \theta_{13}^*), \quad \theta_{11}\,\mathrm{Re}\,\theta_{13} = 0. \qquad (2.22)$$

Making now only the one further assumption that $\theta_{11} = \cosh\varepsilon$ all remaining entries are fixed by (2.21) and (2.22). We obtain

$$\theta_\varepsilon = \begin{pmatrix} \cosh\varepsilon & 0 & \imath\sinh\varepsilon \\ (-\sinh^2\frac{\varepsilon}{2} + \frac{\imath}{2}\sinh\varepsilon) & 1 & (-\sinh^2\frac{\varepsilon}{2} - \frac{\imath}{2}\sinh\varepsilon) \\ -\imath\sinh\varepsilon & 0 & \cosh\varepsilon \end{pmatrix}. \tag{2.23}$$

It is easily verified that the corresponding roots have the desired behaviour under the $\mathcal{CT}$-transformation, namely $\tilde{w}_0(\tilde{\alpha}_1) = -\tilde{\alpha}_3$, $\tilde{w}_0(\tilde{\alpha}_2) = -\tilde{\alpha}_2$. This solution does not correspond to a deformation of $\sigma_\pm$ as now $\theta_\varepsilon^*\sigma_\pm \neq \sigma_\pm\theta_\varepsilon$.

In this case the nonsimple roots cannot be constructed from a simple analogy to the undeformed case as $\sigma_\varepsilon \neq \sigma$. Instead we have to act successively with σ_ε on the simple deformed roots. In this way the set of all positive deformed roots results to

$$\tilde{\alpha}_1 = \cosh\varepsilon\alpha_1 + \imath\sinh\varepsilon\alpha_3, \tag{2.24}$$

$$\tilde{\alpha}_2 = \alpha_2 - \sinh^2\frac{\varepsilon}{2}(\alpha_1 + \alpha_3) + \frac{\imath}{2}\sinh\varepsilon(\alpha_1 - \alpha_3), \tag{2.25}$$

$$\tilde{\alpha}_3 = \cosh\varepsilon\alpha_3 - \imath\sinh\varepsilon\alpha_1, \tag{2.26}$$

$$\tilde{\alpha}_4 = \cosh\varepsilon(\alpha_1 + \alpha_2) - \imath\sinh\varepsilon(\alpha_2 + \alpha_3), \tag{2.27}$$

$$\tilde{\alpha}_5 = \cosh\varepsilon(\alpha_2 + \alpha_3) + \imath\sinh\varepsilon(\alpha_1 + \alpha_2), \tag{2.28}$$

$$\tilde{\alpha}_6 = \cosh\varepsilon\alpha_2 + \cosh^2\frac{\varepsilon}{2}(\alpha_1 + \alpha_3) + \frac{\imath}{2}\sinh\varepsilon(\alpha_3 - \alpha_1). \tag{2.29}$$

Notice that the nonsimple roots no are no longer just simple roots added together.

2.1.3 ω_0 for A_4

For this case we have that the longest element is

$$\omega = \sigma_2\sigma_4\sigma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.30}$$

and it turns out that the deformation of the longest element is of the form $\omega_0\theta_\varepsilon = (\theta_\varepsilon\omega_0)^*$ where θ_ε is the deformation matrix in (1.68).

2.1.4 ω_0 for $D_{2\ell+1}$

For the odd rank subseries we should also be able to construct an alternative solution by solving (2.11). As a special solution valid for the entire subseries we find

$$\theta_\varepsilon = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \hat{\theta}_\varepsilon \end{pmatrix}, \tag{2.31}$$

with

$$\hat{\theta}_\varepsilon = \begin{pmatrix} 1 & (-\sinh^2\frac{\varepsilon}{2} - \frac{\imath}{2}\sinh\varepsilon) & (-\sinh^2\frac{\varepsilon}{2} + \frac{\imath}{2}\sinh\varepsilon) \\ 0 & \cosh\varepsilon & -\imath\sinh\varepsilon \\ 0 & -\imath\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}, \tag{2.32}$$

and $\mathbb{I} \equiv (2\ell{-}2) \times (2\ell{-}2)$ unit matrix. The solutions (1.78) and (2.31) do not coincide.

2.1.5 ω_0 for E_6

We obtain an additional solution by means of the construction laid out in section 2. As a particular solution we find

$$\theta_\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \theta_\varepsilon^{A_3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.33}$$

with $\theta_\varepsilon^{A_3}$ given in (2.23). This means the fact that the subsystem made from the vertices $3, 4$ and 5 is identical to A_3 also reflects in the solution for the deformation matrix. Clearly this solution is different from (1.82) as well as (1.83).

We have constructed a general deformation based on the longest element of the Coxeter group. However, this deformation is only possible for the simply-laced algebras. For the $A_{2,4}$-longest element we have that $\omega_0\theta_\varepsilon = (\theta_\varepsilon\omega_0)^*$ where θ_ε is the deformation matrix constructed in chapter 1.1 for each case. This is however, not true for the other cases.

2.2 Solutions from folding

One deficiency of the above constructions is that in some cases they do not lead to any complex solution for $\tilde{\Delta}$. However, we demonstrate now that in these cases one may still construct higher dimensional solutions by means of the so-called folding procedure, see e.g. [40, 75, 76, 77, 78]. This construction makes use of the fact that some root systems are embedded into larger ones.

Identifying roots which are related by the involution (2.4), one obtains a root system associated to a different type of Coxeter group. At the same time we may use the folding procedure for consistency checks.

$B_n \hookrightarrow A_{2n}$ We showed that there exist no complex deformations for the B_{2n-1}-series based on the ansatz (1.33). However, making use of the embedding $B_n \hookrightarrow A_{2n}$ we demonstrate now that one can construct higher dimensional solutions from the reduction of A_{4n-2} to B_{2n-1}. We illustrate this in detail for the particular case of $B_3 \hookrightarrow A_6$. Starting with the solution to the constraints (1.32) for A_6-deformation matrix

$$\theta_\varepsilon = r_0 \mathbb{I} + \imath r_1 \left(\sigma - \sigma^{-1}\right) + \imath r_2 \left(\sigma^2 - \sigma^{-2}\right) + \imath r_3 \left(\sigma^3 - \sigma^{-3}\right), \qquad (2.34)$$

with $r_0 = \cosh\varepsilon$ and $r_1 = r_2 = -r_3 = 1/\sqrt{7}\cosh\varepsilon$, we employ the explicit form for σ to obtain the simple deformed A_6-roots from (1.6)

$$\begin{aligned}
\tilde{\alpha}_1 &= \cosh\varepsilon\alpha_1 - \imath/\sqrt{7}\sinh\varepsilon(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 2\alpha_6), \qquad (2.35)\\
\tilde{\alpha}_2 &= \cosh\varepsilon\alpha_2 + \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4),\\
\tilde{\alpha}_3 &= \cosh\varepsilon\alpha_3 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6),\\
\tilde{\alpha}_4 &= \cosh\varepsilon\alpha_4 + \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 2\alpha_6),\\
\tilde{\alpha}_5 &= \cosh\varepsilon\alpha_5 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6),\\
\tilde{\alpha}_6 &= \cosh\varepsilon\alpha_6 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6).
\end{aligned}$$

Following the folding procedure we can now define deformed simple B_3-roots as

$$
\begin{aligned}
\tilde{\beta}_1 &= \tilde{\alpha}_1 + \tilde{\alpha}_6 \qquad (2.36) \\
&= \cosh\varepsilon(\alpha_1+\alpha_6) - \imath/\sqrt{7}\sinh\varepsilon[3(\alpha_1-\alpha_6)+2(\alpha_2-\alpha_5)], \\
\tilde{\beta}_2 &= \tilde{\alpha}_2 + \tilde{\alpha}_5 \\
&= \cosh\varepsilon(\alpha_2+\alpha_5) + \imath/\sqrt{7}\sinh\varepsilon[2(\alpha_1-\alpha_6+\alpha_3-\alpha_4)+\alpha_2-\alpha_5], \\
\tilde{\beta}_3 &= \tilde{\alpha}_3 + \tilde{\alpha}_4 \\
&= \cosh\varepsilon(\alpha_1+\alpha_6) - \imath/\sqrt{7}\sinh\varepsilon[2(\alpha_2-\alpha_5)+\alpha_3-\alpha_4].
\end{aligned}
$$

When substituted into (1.1), these roots reproduce the B_3-Cartan matrix, but it is not possible to express the imaginary part in terms of the undeformed B_3-roots. As expected from section 1.1.1, it is therefore impossible to find a three dimensional deformation matrix of the type (1.6). When identifying the undeformed A_6-roots related by the involution (2.4) according to $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_2 \leftrightarrow \alpha_5$ and $\alpha_3 \leftrightarrow \alpha_4$, the deformed B_3-roots will all become real.

$F_4 \hookrightarrow E_6$ Having found some new solutions for a case which could not be solved previously, let us see next how some solutions we have found are related to each other through the folding procedure. In analogy to the undeformed case we may define the deformed F_4-roots in terms of the deformed E_6-roots as

$$
\tilde{\beta}_1^{F_4} = \tilde{\alpha}_1^{E_6} + \tilde{\alpha}_6^{E_6}, \qquad \tilde{\beta}_2^{F_4} = \tilde{\alpha}_3^{E_6} + \tilde{\alpha}_5^{E_6}, \qquad \tilde{\beta}_3^{F_4} = \tilde{\alpha}_4^{E_6} \quad \text{and} \quad \tilde{\beta}_4^{F_4} = \tilde{\alpha}_3^{E_6}. \qquad (2.37)
$$

This means the F_4-deformation matrix is constructed as

$$\theta_\varepsilon^{F_4} = \begin{pmatrix} \frac{\theta_{11}^{E_6}+\theta_{61}^{E_6}+\theta_{16}^{E_6}+\theta_{66}^{E_6}}{2} & \frac{\theta_{13}^{E_6}+\theta_{63}^{E_6}+\theta_{15}^{E_6}+\theta_{65}^{E_6}}{2} & \theta_{14}^{E_6}+\theta_{64}^{E_6} & \theta_{12}^{E_6}+\theta_{62}^{E_6} \\ \frac{\theta_{31}^{E_6}+\theta_{51}^{E_6}+\theta_{36}^{E_6}+\theta_{56}^{E_6}}{2} & \frac{\theta_{33}^{E_6}+\theta_{53}^{E_6}+\theta_{35}^{E_6}+\theta_{55}^{E_6}}{2} & \theta_{34}^{E_6}+\theta_{54}^{E_6} & \theta_{32}^{E_6}+\theta_{52}^{E_6} \\ \frac{\theta_{41}^{E_6}+\theta_{46}^{E_6}}{2} & \frac{\theta_{43}^{E_6}+\theta_{45}^{E_6}}{2} & \theta_{44}^{E_6} & \theta_{42}^{E_6} \\ \frac{\theta_{21}^{E_6}+\theta_{26}^{E_6}}{2} & \frac{\theta_{23}^{E_6}+\theta_{25}^{E_6}}{2} & \theta_{24}^{E_6} & \theta_{22}^{E_6} \end{pmatrix}. \tag{2.38}$$

In this reduction the two inequivalent deformed E_6-root systems (1.82) and (1.83) produce the same solution for F_4

$$\theta_\varepsilon^{F_4} = \begin{pmatrix} r_0 - \imath r_k & -2\imath r_k & -4\imath r_k & -4\imath r_k \\ 2\imath r_k & r_0 + 5\imath r_k & 8\imath r_k & 4\imath r_k \\ -2\imath r_k & -4\imath r_k & r_0 - 5\imath r_k & -2\imath r_k \\ 2\imath r_k & 2\imath r_k & 2\imath r_k & r_0 + \imath r_k \end{pmatrix}. \tag{2.39}$$

This solution corresponds to a special solution we found in the context of F_4, namely (1.86) with $r_4 = r_6 = 0$.

Using the same identification between the F_4 and E_6 roots as in (2.37), we obtain from the solution based on the deformation of the longest element (2.33)

$$\begin{aligned} \tilde{\beta}_1^{F_4} &= \alpha_1^{E_6} + \alpha_6^{E_6}, \qquad (2.40) \\ \tilde{\beta}_2^{F_4} &= (\cosh\varepsilon - \imath\sinh\varepsilon)\alpha_3^{E_6} + (\cosh\varepsilon + \imath\sinh\varepsilon)\alpha_5^{E_6}, \\ \tilde{\beta}_3^{F_4} &= \frac{1}{2}(1 - \cosh\varepsilon + \imath\sinh\varepsilon)\alpha_3^{E_6} + \alpha_4^{E_6} + \frac{1}{2}(1 - \cosh\varepsilon - \imath\sinh\varepsilon)\alpha_5^{E_6} \\ \tilde{\beta}_4^{F_4} &= \cosh\varepsilon\,\alpha_3^{E_6} + \imath\sinh\varepsilon\,\alpha_5^{E_6}. \end{aligned}$$

When substituted into (1.1), these roots reproduce the F_4-Cartan matrix, but it is not possible to express them in terms of the undeformed F_4-roots.

This reflects the fact that the longest elements acts trivially in this case and therefore also no nontrivial deformation of this involution exists.

Chapter 3

Non-Hermitian Calogero models

We have constructed various different types of deformation maps δ which replace each root α by a deformed counterpart $\tilde{\alpha}$ as specified above. We will now employ these constructions and replace the set of n-dynamical variables $x = \{x_1, \ldots, x_n\}$ and their conjugate momenta $p = \{p_1, \ldots, p_n\}$ by means of one deformation maps $\delta : (x, p) \to (\tilde{x}, \tilde{p})$.

3.1 The groundstate wavefunctions and eigenenergies in the undeformed case

Let us first generalize Calogero's construction [42] for the solution of the $l = 0$ wavefunction to generic Coxeter groups $\mathcal{W}$. We consider the generalized

Calogero Hamiltonian

$$\mathcal{H}_0(p,x) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha\in\Delta^+}(\alpha\cdot x)^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot x)^2}, \tag{3.1}$$

with g_α being real coupling constants, which for the moment may be different for each positive root $\alpha \in \Delta^+$ associated to any Coxeter group $\mathcal{W}$. Generalizing [42] we define now the variables

$$z := \prod_{\alpha\in\Delta^+}(\alpha\cdot x) \qquad \text{and} \qquad r^2 := \frac{1}{\hat{h}t_\ell}\sum_{\alpha\in\Delta^+}(\alpha\cdot x)^2, \tag{3.2}$$

where $\hat{h}$ denotes the dual Coxeter number and t_ℓ is the ℓ-th symmetrizer of the incidence matrix I defined through the relation $I_{ij}t_j = t_i I_{ij}$ with ℓ being the rank of the Coxeter group. We now assume that the wavefunction can be separated in terms of these variables

$$\psi(x) \to \psi(z,r) = z^{\kappa+1/2}\varphi(r) \tag{3.3}$$

with κ being an undetermined constant for the moment. Using this ansatz we try to solve the n-body Schrödinger equation in position space $\mathcal{H}_0\psi(x) = E\psi(x)$ with $p^2 = -\sum_{i=1}^n \partial^2_{x_i}$. Changing variables for the Laplace operator then yields

$$\begin{aligned}\Bigg\{-\frac{1}{2}\sum_{i=1}^n\Bigg[\left(\kappa^2-\frac{1}{4}\right)\frac{1}{z^2}\left(\frac{\partial z}{\partial x_i}\right)^2 + \left(\kappa+\frac{1}{2}\right)\frac{1}{z}\left(\frac{\partial^2 z}{\partial x_i^2} + 2\frac{\partial z}{\partial x_i}\frac{\partial r}{\partial x_i}\frac{\partial}{\partial r}\right) \\ +\frac{\partial^2 r}{\partial x_i^2}\frac{\partial^2}{\partial r^2} + \left(\frac{\partial r}{\partial x_i}\right)^2\frac{\partial}{\partial r}\Bigg] + \frac{\omega^2}{4}\hat{h}t_\ell r^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot x)^2} - E\Bigg\}\varphi(r) \;=\; 0.\end{aligned} \tag{3.4}$$

Taking now $g_\alpha = g\alpha^2/2$ and using the identities (B.7)-(B.11) from appendix A this reduces to

$$\left\{-\frac{1}{2}\left[\frac{\partial^2}{\partial r^2}+\left[\left(\kappa+\frac{1}{2}\right)h\ell+(\ell+1)\right]\frac{1}{r}\frac{\partial}{\partial r}\right]+\frac{\omega^2}{4}\hat{h}t_\ell r^2\right\}\varphi(r)=E\varphi(r). \tag{3.5}$$

The key feature is that due to the identity (B.7) the first term in (3.4) combines with part of the potential term to

$$\left[\frac{g}{2}-\frac{1}{2}\left(\kappa^2-\frac{1}{4}\right)\right]\sum_{\alpha\in\Delta^+}\frac{\alpha^2}{(\alpha\cdot x)^2}. \tag{3.6}$$

This term vanishes when choosing the free parameter κ to $\kappa = \pm 1/2\sqrt{1+4g}$. The positive solution is the only physical acceptable one, as we would obtain singularities in (3.3) and therefore a nonnormalizable wavefunction otherwise.

The equation (3.5) is a second order differential equation which may be solved by standard methods. Imposing as usual the physical constraint that the wavefunction vanishes at infinity, the energy quantizes to

$$E_n = \frac{1}{4}\left[\left(2+h+h\sqrt{1+4g}\right)\ell+8n\right]\sqrt{\frac{\hat{h}t_\ell}{2}}\omega, \tag{3.7}$$

with corresponding wavefunctions

$$\varphi_n(r) = c_n \exp\left(-\sqrt{\frac{\hat{h}t_\ell}{2}}\frac{\omega}{2}r^2\right)L_n^a\left(\sqrt{\frac{\hat{h}t_\ell}{2}}\omega r^2\right). \tag{3.8}$$

Here $L_n^a(x)$ denotes the generalized Laguerre polynomial, c_n is a normalization constant and $a = \left(2+h+h\sqrt{1+4g}\right)\ell/4-1$.

A key feature of the model is that the last term in the potential in (3.1) becomes singular whenever $x_i = x_j$ for any $i, j \in \{1, 2, \ldots, n\}$. This means that the wavefunction is vanishing at these points and we may encounter nontrivial phases for any two particle interchange. In fact, as the variable z

defined in (3.2) is antisymmetric and r symmetric in all variables it is easy to see that the associated particles obey anyonic statistics

$$\psi(x_1, \ldots, x_i, x_j, \ldots x_n) = e^{\imath \pi s} \psi(x_1, \ldots, x_j, x_i, \ldots x_n), \quad \text{for } 1 \leq i, j \leq n, \tag{3.9}$$

with

$$s = 1/2 + 1/2\sqrt{1+4g}. \tag{3.10}$$

This feature will change in the deformed case.

When one has bosons the statistics of the particles is

$$\psi(x_1, \ldots, x_i, x_j, \ldots x_n) = \psi(x_1, \ldots, x_j, x_i, \ldots x_n), \quad \text{for } 1 \leq i, j \leq n, \tag{3.11}$$

and for fermions the statistics are

$$\psi(x_1, \ldots, x_i, x_j, \ldots x_n) = -\psi(x_1, \ldots, x_j, x_i, \ldots x_n), \quad \text{for } 1 \leq i, j \leq n. \tag{3.12}$$

Anyonic statistics (3.9) can be viewed as a continuous interpolation between bose and fermionic statistics [79]. When $g = 0$ then (3.9) gives fermions and when $g = 2$ then (3.9) gives bosons.

3.2 The groundstate wavefunctions and eigenenergies in the deformed case

Now we consider the antilinear deformation of the Hamiltonian $\mathcal{H}_0(p, x)$

$$\mathcal{H}_{\varepsilon,q}(p, x) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\tilde{\alpha} \in \tilde{\Delta}^+} (\tilde{\alpha} \cdot x)^2 + \sum_{\tilde{\alpha} \in \Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha} \cdot x)^2}, \tag{3.13}$$

where $\mathcal{H}_0(p,x)$ is the undeformed Calogero model. In analogy to the deformed case we attempt to solve this model by a similar ansatz, i.e., defining the variables

$$\tilde{z} := \prod_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot x) \qquad \text{and} \qquad \tilde{r}^2 := \frac{1}{\hat{h}t_\ell} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot x)^2, \tag{3.14}$$

and separating the wavefunction as

$$\psi(x) \to \psi(\tilde{z},\tilde{r}) = \tilde{z}^{1/2+1/2\sqrt{1+4g}}\varphi(\tilde{r}). \tag{3.15}$$

As a consequence of our construction for the deformed roots in which we demanded that inner products are preserved, we find that $\tilde{r} = r$. Furthermore, we observe that due to this fact the relations (B.2) and (B.4) also hold when replacing α by $\tilde{\alpha}$ and consequently the solution procedure for the eigenvalue equation does not change. We obtain

$$\psi(x) = \psi(\tilde{z},r) = \tilde{z}^{1/2+1/2\sqrt{1+4g}}\varphi_n(r), \tag{3.16}$$

as solution with $\varphi_n(r)$ given in (3.8) and unchanged energy eigenvalues (3.7). When generalizing the ansatz (3.15) to take also values for $l \neq 0$ into account the energy eigenvalues will, however, change, as was demonstrated in [41] for A_2 and G_2. The main difference between the deformed and undeformed case for the solution provided here is the occurrence of the variable $\tilde{z}$ instead of z. As a consequence the wavefunction (3.16) no longer vanishes when two x_i values coincide, which in turn is a reflection of the fact that all singularities resulting from a two-particle exchange have been regularized through the deformation. However, we still encounter singularities in the potential when all n values for the x_i coincide. The wavefunction vanishes in this case and we obtain nontrivial statistics exchange factors. Let us see in detail for some concrete models how to obtain nontrivial statistics for an n-particle exchange.

The deformed A_2-model

The potential in (3.13) and the variable $\tilde{z}$ in (3.14) are computed from the inner products of all 3 roots in $\tilde{\Delta}^+_{A_2}$ with the vector x. Using the standard three dimensional representation for the simple A_2-roots $\alpha_1 = \{1, -1, 0\}$ and $\alpha_2 = \{0, 1, -1\}$, we find with (2.18) and (2.19)

$$\tilde{\alpha}_1 \cdot x = x_{12} \cosh \varepsilon - \frac{\imath}{\sqrt{3}}(x_{13} + x_{23}) \sinh \varepsilon, \tag{3.17}$$

$$\tilde{\alpha}_2 \cdot x = x_{23} \cosh \varepsilon - \frac{\imath}{\sqrt{3}}(x_{21} + x_{31}) \sinh \varepsilon, \tag{3.18}$$

$$(\tilde{\alpha}_1 + \tilde{\alpha}_2) \cdot x = x_{13} \cosh \varepsilon + \frac{\imath}{\sqrt{3}}(x_{12} + x_{32}) \sinh \varepsilon. \tag{3.19}$$

For convenience we introduced the notation $x_{ij} := x_i - x_j$. The new feature of these models is that the last term in the potential (3.13) resulting from these products is no longer singular when the position of two particles coincides. It is easy to see that the $\mathcal{PT}$-symmetry constructed for the $\tilde{\alpha}$ may be realized alternatively in the dual space, that is on the level of the dynamical variables

$$\sigma_-^\varepsilon \; : \; \tilde{\alpha}_1 \leftrightarrow -\tilde{\alpha}_1, \, \tilde{\alpha}_2 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \;\Leftrightarrow\; x_1 \leftrightarrow x_2, \, x_3 \leftrightarrow x_3, \, \imath \to -\imath, \tag{3.20}$$

$$\sigma_+^\varepsilon \; : \; \tilde{\alpha}_2 \leftrightarrow -\tilde{\alpha}_2, \, \tilde{\alpha}_1 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \;\Leftrightarrow\; x_2 \leftrightarrow x_3, \, x_1 \leftrightarrow x_1, \, \imath \to -\imath. \tag{3.21}$$

A crucial difference to the undeformed case is that $\tilde{z}$ will, unlike z, not vanish in the two particle scattering process when two positions x_i and x_j coincide. In fact in that case $\tilde{z}$ will be purely imaginary as follows directly from the

$\mathcal{PT}$-symmetry

$$\begin{aligned}\sigma_-^\varepsilon \tilde{z}(x_1,x_2,x_3) &= \tilde{z}^*(x_2,x_1,x_3) = -\tilde{z}(x_1,x_2,x_3) \qquad (3.22)\\ &\Rightarrow \quad \tilde{z}(x_1,x_1,x_3) \in \imath\mathbb{R},\\ \sigma_+^\varepsilon \tilde{z}(x_1,x_2,x_3) &= \tilde{z}^*(x_1,x_3,x_2) = -\tilde{z}(x_1,x_2,x_3) \qquad (3.23)\\ &\Rightarrow \quad \tilde{z}(x_1,x_3,x_3) \in \imath\mathbb{R}.\end{aligned}$$

The remaining possibility $\tilde{z}(x_1,x_2,x_1) \in \imath\mathbb{R}$ follows from the previous cases together with the cyclic property $\tilde{z}(x_1,x_2,x_3) = \tilde{z}(x_2,x_3,x_1)$, which in turn results when combining (3.22) and (3.23). Under these circumstances a new symmetry arises

$$\alpha_1 = 0,\, \alpha_2 \to -\alpha_2 \quad \Leftrightarrow \quad \tilde{\alpha}_1 \to -\tilde{\alpha}_1\,,\, \tilde{\alpha}_2 \to -\tilde{\alpha}_2 \quad \Leftrightarrow \quad x_1 = x_2,\, x_2 \leftrightarrow x_3, \qquad (3.24)$$

leading to $\tilde{z}(x_2,x_2,x_3) = -\tilde{z}(x_3,x_3,x_2)$. By (3.3) this means

$$\psi(x_2,x_2,x_3) = e^{\imath\pi s}\psi(x_3,x_3,x_2), \qquad (3.25)$$

with s given in (3.10). Hence we obtain a nontrivial exchange factor in the three-particle scattering process when particle 1 and 2 have the same position and are simultaneously scattered with particle 3.

Similarly we observe

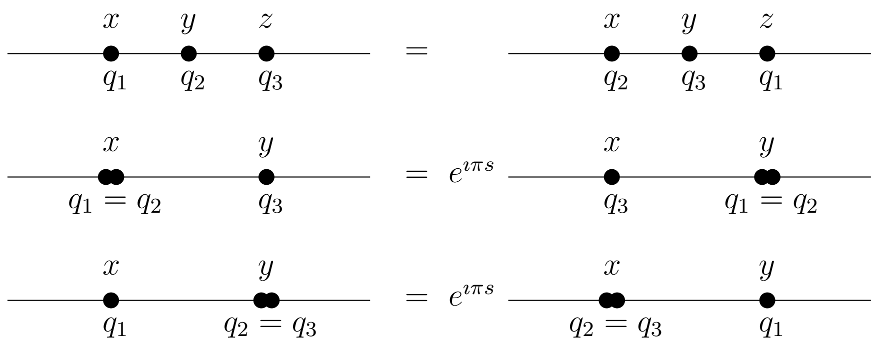
$$\alpha_2 = 0,\, \alpha_1 \to -\alpha_1 \quad \Leftrightarrow \quad \tilde{\alpha}_1 \to -\tilde{\alpha}_1,\, \tilde{\alpha}_2 \to -\tilde{\alpha}_2 \quad \Leftrightarrow \quad x_2 = x_3,\, x_1 \leftrightarrow x_2, \qquad (3.26)$$

leading to $\tilde{z}(x_1,x_2,x_2) = -\tilde{z}(x_2,x_2,x_1)$ and therefore

$$\psi(x_1,x_2,x_2) = e^{\imath\pi s}\psi(x_2,x_2,x_1). \qquad (3.27)$$

Now a nontrivial exchange factor emerges in the three-particle scattering process when particle 2 and 3 have the same position and are simultaneously scattered with particle 1. We depict various possibilities in Figure 3.1.

Figure 3.1: Anyonic exchange factors for the 3-particle scattering in the A_2-model.



Notice that the first case in Figure 3.1, leading to a bosons exchange possesses an analogue in the undeformed case. This process can be viewed in two alternative ways, either corresponding to two consecutive two particle exchanges, i.e., $1 \leftrightarrow 2$ and subsequently $1 \leftrightarrow 3$, or equivalently to a simultaneous three particle scattering process that is the ordering 123 goes to 231 in one scattering event. This is the typical factorization of an n-particle scattering process into a sequence of two-particle scattering encountered in integrable models, see e.g. [80]. In fact, as this feature is so central it is often used synonymously with integrability. In our deformed model we encounter new possibilities, namely that a compound particle can exist in the first place and then also scatter with a single particle; giving rise to anyonic exchange factors in this case.

Deformed A_3-models

Based on $\mathcal{PT}$-symmetrically deformed Coxeter group factors In this case the potential and $\tilde{z}$ are computed from the inner products of all 6 roots in $\tilde{\Delta}^+_{A_3}$ with x. Taking the simple roots in the standard four dimensional representation $\alpha_1 = \{1, -1, 0, 0\}$, $\alpha_2 = \{0, 1, -1, 0\}$, $\alpha_3 = \{0, 0, 1, -1\}$, we evaluate with (1.55) and (1.60)

$$\begin{aligned}
\tilde{\alpha}_1 \cdot x &= x_{43} + \cosh\varepsilon(x_{12} + x_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(x_{13} + x_{24}), && (3.28)\\
\tilde{\alpha}_2 \cdot x &= x_{23}(2\cosh\varepsilon - 1) + \imath 2\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}x_{14}, && (3.29)\\
\tilde{\alpha}_3 \cdot x &= x_{21} + \cosh\varepsilon(x_{12} + x_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(x_{13} + x_{24}), && (3.30)\\
\tilde{\alpha}_4 \cdot x &= x_{42} + \cosh\varepsilon(x_{13} + x_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(x_{12} + x_{34}), && (3.31)\\
\tilde{\alpha}_5 \cdot x &= x_{31} + \cosh\varepsilon(x_{13} + x_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(x_{12} + x_{34}), && (3.32)\\
\tilde{\alpha}_6 \cdot x &= x_{14}(2\cosh\varepsilon - 1) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}x_{23}. && (3.33)
\end{aligned}$$

Once again the last term in the potential (3.13) resulting from these products is no longer singular in any two particle exchange. However, in this case it could become singular in two simultaneous two-particle scattering processes, e.g. $x_{14} = x_{23} = 0$. We may realize the $\mathcal{PT}$-symmetry constructed for the $\tilde{\alpha}$

$$\sigma_-^\varepsilon \;:\; \tilde{\alpha}_1 \to -\tilde{\alpha}_1,\, \tilde{\alpha}_2 \to \tilde{\alpha}_6,\, \tilde{\alpha}_3 \to -\tilde{\alpha}_3, \tilde{\alpha}_4 \to \tilde{\alpha}_5,\, \tilde{\alpha}_5 \to \tilde{\alpha}_4,\, \tilde{\alpha}_6 \to \tilde{\alpha}_2, \quad (3.34)$$

$$\sigma_+^\varepsilon \;:\; \tilde{\alpha}_1 \to \tilde{\alpha}_4,\, \tilde{\alpha}_2 \to -\tilde{\alpha}_2,\, \tilde{\alpha}_3 \to \tilde{\alpha}_5,\, \tilde{\alpha}_4 \to \tilde{\alpha}_1,\, \tilde{\alpha}_5 \to \tilde{\alpha}_3,\, \tilde{\alpha}_6 \to \tilde{\alpha}_6, \quad (3.35)$$

also in the dual space

$$\sigma_-^\varepsilon \;:\; x_1 \to x_2,\, x_2 \to x_1,\, x_3 \to x_4,\, x_4 \to x_3,\, \imath \to -\imath, \quad (3.36)$$

$$\sigma_+^\varepsilon \;:\; x_1 \to x_1,\, x_2 \to x_3,\, x_3 \to x_2,\, x_4 \to x_4,\, \imath \to -\imath. \quad (3.37)$$

As in the A_2-case $\tilde{z}$ will not vanish when two positions x_i and x_j coincide, but once again we may pick up nontrivial exchange factors when involving all particles in the model in the scattering process. We observe

$$\sigma_{-}^{\varepsilon}\tilde{z}(x_1,x_2,x_3,x_4) \;=\; \tilde{z}^{*}(x_2,x_1,x_4,x_3)=\tilde{z}(x_1,x_2,x_3,x_4), \qquad (3.38)$$

$$\sigma_{+}^{\varepsilon}\tilde{z}(x_1,x_2,x_3,x_4) \;=\; \tilde{z}^{*}(x_1,x_3,x_2,x_4)=-\tilde{z}(x_1,x_2,x_3,x_4). \qquad (3.39)$$

Combining (3.38) and (3.39) then yields

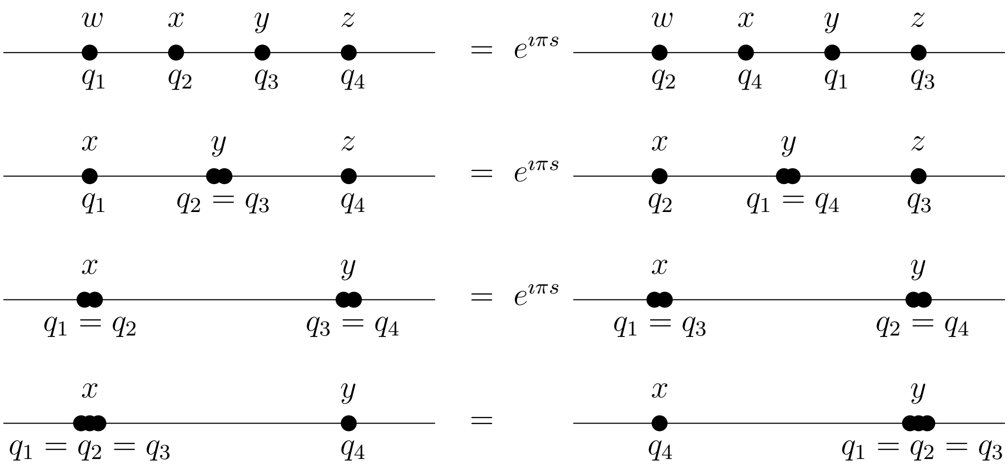
$$\tilde{z}(x_1,x_2,x_3,x_4)=-\tilde{z}(x_2,x_4,x_1,x_3), \qquad (3.40)$$

and therefore we will encounter nontrivial exchange factors in a 4-particle scattering process

$$\psi(x_1,x_2,x_3,x_4)=e^{\imath\pi s}\psi(x_2,x_4,x_1,x_3). \qquad (3.41)$$

We depict various possibilities in Figure 3.2.

Figure 3.2: Anyonic exchange factors for the 4-particle scattering in the A_3-model.



As in the previous case we encounter several possibilities which have no counterpart in the undeformed case.

Based on $\mathcal{CT}$-symmetrically deformed longest element We keep now the representation for the simple roots, but use the construction for the deformed roots as provided in the second part of section 1.1.1. The potential is obtained again by computing the inner product of all the roots with the position vector

$$\begin{aligned}
\tilde{\alpha}_1 \cdot x &= \cosh \varepsilon x_{12} + \imath \sinh \varepsilon x_{34}, && (3.42)\\
\tilde{\alpha}_2 \cdot x &= \cosh^2 \frac{\varepsilon}{2} x_{23} - \sinh^2 \frac{\varepsilon}{2} x_{14} + \frac{\imath}{2} \sinh \varepsilon (x_{12} + x_{43}), && (3.43)\\
\tilde{\alpha}_3 \cdot x &= \cosh \varepsilon x_{34} + \imath \sinh \varepsilon x_{21}, && (3.44)\\
\tilde{\alpha}_4 \cdot x &= \cosh \varepsilon x_{13} - \imath \sinh \varepsilon x_{24}, && (3.45)\\
\tilde{\alpha}_5 \cdot x &= \cosh \varepsilon x_{24} + \imath \sinh \varepsilon x_{13}, && (3.46)\\
\tilde{\alpha}_6 \cdot x &= \cosh^2 \frac{\varepsilon}{2} x_{14} + \sinh^2 \frac{\varepsilon}{2} x_{23} + \frac{\imath}{2} \sinh \varepsilon (x_{21} + x_{34}). && (3.47)
\end{aligned}$$

Clearly the potential is different from the one resulting from (3.28)-(3.33). Despite the fact that it is a simpler potential, it cannot be solved analogously to the previous case since the crucial relations (B.1)-(B.4) no longer hold.

The deformed F_4-model

In order to unravel any features which might differ in the non-simply laced case, which is usually the case, we also present here one example for a non-simply laced model. To allow a direct comparison with the previous 4-particle case, we have selected F_4. The positive root space $\tilde{\Delta}^+_{F_4}$ contains now 24 roots. Employing the simple roots in the standard four dimensional representation $\alpha_1 = \{0, 1, -1, 0\}$, $\alpha_2 = \{0, 0, 1, -1\}$, $\alpha_3 = \{0, 0, 0, 1\}$ and $\alpha_4 = \{1/2, -1/2, -1/2, -1/2\}$ we compute the following factorization for $\tilde{z}$, with each factor corresponding to one of the 24 products $\tilde{\alpha}_i \cdot x$:

$$
\begin{aligned}
&(x_1 \cosh\varepsilon + \imath \sinh\varepsilon x_4)\,(x_2 \cosh\varepsilon - \imath \sinh\varepsilon x_3) \\
&\times (x_3 \cosh\varepsilon + \imath \sinh\varepsilon x_2)\,(x_4 \cosh\varepsilon - \imath \sinh\varepsilon x_1) \\
&\times (x_{12} \cosh\varepsilon + \imath \sinh\varepsilon \hat{x}_{34})\,(x_{14} \cosh\varepsilon + \imath \sinh\varepsilon \hat{x}_{14}) \\
&\times (x_{34} \cosh\varepsilon + \imath \sinh\varepsilon \hat{x}_{12})\,(x_{23} \cosh\varepsilon - \imath \sinh\varepsilon \hat{x}_{23}) \\
&\times (\hat{x}_{13} \cosh\varepsilon + \imath \sinh\varepsilon \hat{x}_{24})\,(\hat{x}_{24} \cosh\varepsilon - \imath \sinh\varepsilon \hat{x}_{13}) \\
&\times (\hat{x}_{34} \cosh\varepsilon - \imath \sinh\varepsilon x_{12})\,(\hat{x}_{23} \cosh\varepsilon + \imath \sinh\varepsilon \hat{x}_{23}) \\
&\times (\hat{x}_{12} \cosh\varepsilon - \imath \sinh\varepsilon \hat{x}_{34})\,(\hat{x}_{14} \cosh\varepsilon - \imath \sinh\varepsilon x_{14}) \\
&\times (x_{24} \cosh\varepsilon + \imath \sinh\varepsilon x_{13})\,(x_{13} \cosh\varepsilon - \imath \sinh\varepsilon x_{24}) \\
&\times \left[\frac{\hat{x}_{12} + \hat{x}_{34}}{2} \cosh\varepsilon - \frac{\imath}{2} \sinh\varepsilon (x_{12} + x_{34})\right] \\
&\times \left[\frac{\hat{x}_{12} - x_{34}}{2} \cosh\varepsilon - \frac{\imath}{2} \sinh\varepsilon (\hat{x}_{12} + x_{34})\right] \\
&\times \left[\frac{x_{12} - \hat{x}_{34}}{2} \cosh\varepsilon + \frac{\imath}{2} \sinh\varepsilon (x_{12} + \hat{x}_{34})\right] \\
&\times \left[\frac{\hat{x}_{12} - \hat{x}_{34}}{2} \cosh\varepsilon + \frac{\imath}{2} \sinh\varepsilon (x_{12} - x_{34})\right] \\
&\times \left[\frac{x_{12} + \hat{x}_{34}}{2} \cosh\varepsilon - \frac{\imath}{2} \sinh\varepsilon (x_{12} - \hat{x}_{34})\right] \\
&\times \left[\frac{x_{12} - x_{34}}{2} \cosh\varepsilon - \frac{\imath}{2} \sinh\varepsilon (\hat{x}_{12} - \hat{x}_{34})\right] \\
&\times \left[\frac{x_{12} + x_{34}}{2} \cosh\varepsilon + \frac{\imath}{2} \sinh\varepsilon (\hat{x}_{12} + \hat{x}_{34})\right] \\
&\times \left[\frac{x_{12} + x_{34}}{2} \cosh\varepsilon + \frac{\imath}{2} \sinh\varepsilon (\hat{x}_{12} - x_{34})\right],
\end{aligned}
$$

where we used the abbreviation $\hat{x}_{ij} := q_i + q_j$. Once again, several singularities have disappeared through the deformation. The $\mathcal{PT}$-symmetry constructed for the simple deformed roots $\tilde{\alpha}$

$$\sigma_-^{\varepsilon}: \qquad \tilde{\alpha}_1 \to -\tilde{\alpha}_1,\ \tilde{\alpha}_2 \to \tilde{\alpha}_1 + \tilde{\alpha}_2 + 2\tilde{\alpha}_3, \tilde{\alpha}_3 \to -\tilde{\alpha}_3,\ \tilde{\alpha}_4 \to \tilde{\alpha}_3 + \tilde{\alpha}_4, \tag{3.48}$$

$$\sigma_+^{\varepsilon}: \qquad \tilde{\alpha}_1 \to \tilde{\alpha}_1 + \tilde{\alpha}_2,\ \tilde{\alpha}_2 \to -\tilde{\alpha}_2,\ \tilde{\alpha}_3 \to \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4,\ \tilde{\alpha}_4 \to -\tilde{\alpha}_4, \tag{3.49}$$

is now realized in the dual space as

$$\sigma_-^{\varepsilon}: \qquad x_1 \to x_1,\ x_2 \to x_3,\ x_3 \to x_2,\ x_4 \to -x_4,\ \imath \to -\imath, \tag{3.50}$$

$$\sigma_+^{\varepsilon}: \qquad x_1 \to \frac{1}{2}(\hat{x}_{12} + \hat{x}_{34}),\ x_2 \to \frac{1}{2}(\hat{x}_{12} - \hat{x}_{34}),\ x_3 \to \frac{1}{2}(x_{12} - x_{34}), \tag{3.51}$$

$$x_4 \to \frac{1}{2}(x_{12} + x_{34}),\ \imath \to -\imath. \tag{3.52}$$

Now we observe

$$\sigma_-^{\varepsilon}\tilde{z} = \tilde{z}^*(x_1, x_3, x_2, -x_4) = \tilde{z}(x_1, x_2, x_3, x_4), \tag{3.53}$$

$$\sigma_+^{\varepsilon}\tilde{z} = \tilde{z}^*\left[\frac{\hat{x}_{12} + \hat{x}_{34}}{2}, \frac{\hat{x}_{12} - \hat{x}_{34}}{2}, \frac{x_{12} - x_{34}}{2}, \frac{x_{12} + x_{34}}{2}\right] \tag{3.54}$$

$$= \tilde{z}(x_1, x_2, x_3, x_4). \tag{3.55}$$

A consequence of this we find the symmetry

$$\psi(x_1, x_2, x_3, x_4) = \psi(\frac{\hat{x}_{13} + x_{24}}{2}, \frac{\hat{x}_{13} - x_{34}}{2}, \frac{x_{13} - \hat{x}_{24}}{2}, \frac{x_{13} + \hat{x}_{24}}{2}). \tag{3.56}$$

3.3 A new metric and Hermitian counterpart

As was seen in the previous chapter we have various options for deforming the Calogero Hamiltonian (3.1). We may consider new types of non-Hermitian generalisations of Calogero models

$$\mathcal{H}_{0,\varepsilon,q}(p,x) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha}(\alpha \cdot x)^2 + \sum_{\alpha}\frac{g_\alpha}{(\alpha \cdot x)^2}, \qquad \alpha_i \in \Delta, \tilde{\Delta}(\varepsilon), \Delta_q, \tag{3.57}$$

or the analogues of Calogero-Moser-Sutherland models when replacing the rational potential by a trigonometric or elliptic one. The model $\mathcal{H}_\varepsilon$ for the rational potential has been investigated in the previous chapter and was found to have remarkable properties when compared with the standard undeformed models $\mathcal{H}_0$. As a result of the deformation into the complex domain the singularities in the potential are regularized. Therefore the models no longer have to be defined in separate disjointed regimes and continued by phase factors corresponding to some selected statistics. As was shown in [67], in the $\mathcal{H}_\varepsilon$-models the anyonic phase factors are automatically present and the models can be defined on the entire domain of the configuration space. As a consequence the energy spectra of these models will also be different. Various ground state wavefunctions and those corresponding to exited states were computed in [67] and [41], respectively. Since the Hamiltonians $\mathcal{H}_{\varepsilon,q}$ in (3.13) are not Hermitian the canonical variables p and x are non-observable in the standard Hilbert space. However, it is by now well understood how to reconcile this by constructing a well defined metric operator ρ [81, 82, 83, 84, 85, 86, 87, 88, 89, 13, 90]. One seeks a linear, invertible, Hermitian and positive operator acting in the Hilbert space, such that $\mathcal{H}_{\varepsilon,q}$ becomes a self-adjoint operator with regard to this metric such that p and x become observable in this space. For this purpose one constructs a so-called Dyson map η [17][91], which maps the non-Hermitian Hamiltonian H adjointly to a Hermitian Hamiltonian h

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \;\; \text{with } \rho = \eta^\dagger \eta. \tag{3.58}$$

Depending on the assumptions made on the metric such type of Hamiltonians are referred to with different terminology. When no assumption is made on the positivity of the ρ in (3.58), the relation on the right hand side constitutes the *pseudo-Hermiticity* condition, see e.g. [92, 93, 10], whenever the operator ρ is linear, invertible and Hermitian. In case the operator ρ is positive but not invertible this condition is usually referred to as *quasi-Hermiticity* [15, 17]. Different terminology is used at times with a less clear meaning.

In general we cannot map the Hamiltonians $\mathcal{H}_{\varepsilon,q}$ to some Hermitian counterparts in a very obvious way, but in some case we can provide the explicit transformation η. We recall that the rotations in (1.149) on two variables can be realized by means of the angular momentum operators $L_{ij} = x_i p_j - x_j p_i$

$$\begin{pmatrix} \tilde{z}_i \\ \tilde{z}_j \end{pmatrix} = R_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} = \eta_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} \eta_{ij}^{-1}, \qquad \text{for } z \in \{x, p\},\ \eta_{ij} = e^{\varepsilon(x_i p_j - x_j p_i)}. \tag{3.59}$$

Noting furthermore that

$$\mathcal{H}_0(\tilde{p}, \tilde{x}) = \mathcal{H}_\varepsilon(p, x), \tag{3.60}$$

we can find many explicit transformations of the type (3.58), which map these Hamiltonians to some isospectral Hermitian counterpart

$$\mathcal{H}_0(p, x) = \eta \mathcal{H}_\varepsilon(p, x) \eta^{-1}. \tag{3.61}$$

For instance for the B_ℓ-models based on the deformations (1.149) the Dyson map is simply

$$\eta = \eta_{12}^{-1} \eta_{34}^{-1} \eta_{56}^{-1} \ldots \eta_{(\ell-2)(\ell-1)}^{-1}. \tag{3.62}$$

In other cases based on special orthogonal groups the rotations involved might not commute. For instance, for the B_5-model based on the deformation (1.136) with $r_0 = \cosh^2 \varepsilon$ we find that

$$\tilde{x} = \theta_\varepsilon^\star x = R_{24}^{-1} R_{13} R_{34} R_{12}^{-1} x = \eta x \eta^{-1}, \qquad \text{with } \eta = \eta_{24}^{-1} \eta_{13} \eta_{34} \eta_{12}^{-1}. \quad (3.63)$$

When the deformation in the configuration space is not based on rotations such that inner products are not preserved it remains a challenge to find the corresponding Dyson maps and isospectral Hermitian counterparts. We also leave the investigation for the $\mathcal{H}_q(p, x)$-models for further investigations.

Chapter 4

Non-Hermitian affine Toda theories

One of the main obstacles to overcome when passing from a classical description of a field theory to a full-fledged quantum field theory is renormalisation. In 1+1 space-time dimensions many miracles occur which allow one to express a number of physical quantities in an exact, that is non-perturbative, manner. In particular it is possible to formulate classical Lagrangians which are in some sense exact from the quantum field theoretical point of view. The classical affine Toda field theory is a prototype for this kind of behaviour and has the remarkable property that its classical mass ratios that remain preserved in the quantum field theory after renormalisation, whenever the associated Lie algebra is simply laced [48, 49, 50, 51, 52, 53, 54, 55]. This property ceases to be valid when the algebra becomes non-simply laced [56, 57, 58, 59, 60, 61, 47, 62], in which case one has to consider a dual pair of affine Lie algebras [63] and the quantum mass ratios interpolate via an effective coupling constant between the values obtained from these two algebras. In the strong and weak limit of the coupling constant either of these two cases is obtained.

One may now pose the question whether it is also possible to formulate some naturally modified Lagrangians for non-simply laced algebras which already capture some exact features from the quantum level, such as preserving the classical mass ratios when renormalised. In addition, we may study models in which the roots are elements of the antilinearly invariant space. In terms of simple roots we consider now the three different versions of affine Toda field theories defined by the Lagrangians

$$\mathcal{L}_{0,\varepsilon,q} := \frac{1}{2}\sum_{i=1}^{\ell} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{m^2}{\beta^2}\sum_{i=0}^{\ell} n_i e^{\beta \alpha_i \cdot \phi}, \qquad \alpha_i \in \Delta, \tilde{\Delta}(\varepsilon), \Delta_q. \tag{4.1}$$

The Lagrangian $\mathcal{L}_0$ corresponds to the standard version whereas $\mathcal{L}_{\varepsilon,q}$ are newly proposed models. The ℓ components of ϕ are real scalar fields, m an overall mass scale and the β is the coupling constant. The α's are simple roots with α_0 being the negative of the longest root, whose expansion in terms of simple roots in the relevant spaces $\alpha_0 = -\sum_{i=1}^{\ell} n_i \alpha_i$ is the defining relation for the integers n_i, often referred to as Kac labels. The $\mathcal{L}_0$ theories are known to fall roughly into two different classes characterized by β taken to be either real or purely complex in which case the Yang-Baxter equation obeyed by the scattering matrix is either trivial or non-trivial, respectively. When $\beta \in i\mathbb{R}$ the theory is in general non-Hermitian, except for the A_2-case corresponding to the sine-Gordon model, but the classical mass spectra were still found to be real and stable with respect to small perturbations [94]. Here we conjecture that the $\mathcal{L}_{\varepsilon,q}$-models are also meaningful.

The classical mass matrix for the scalar fields is simply given by the quadratic term in the fields of the Lagrangian and is easily extracted from the formulation (4.1)

$$M_{ij}^2 = m^2 \sum_{a=0}^{\ell} n_a \alpha_a^i \alpha_a^j\,, \qquad \alpha_i \in \Delta, \tilde{\Delta}(\varepsilon), \Delta_q. \tag{4.2}$$

The mathematical fact that the overall length of the roots is a matter of convention is reflected in the physical property that the overall mass scale is not fixed. This is captured in the constant m.

4.1 Construction of Q-deformed Coxeter groups

Mainly motivated by an applications to affine Toda field theories in mind, we provide in this section a construction for q-deformed roots, meaning that we are seeking a map

$$\delta_q :\ \Delta \subset \mathbb{R}^n \to \Delta_q \subset \mathbb{R}^n[q], \qquad \alpha \mapsto \alpha_q = \Theta_q \alpha, \tag{4.3}$$

with $\mathbb{R}^n[q]$ denoting a polynomial ring in $q \in \mathbb{C}$. In this case the complex deformation matrix Θ_q depends on the deformation q in such a way that $\lim_{q\to 1} \Theta_q = \mathbb{I}$. Our construction is centered on a q-deformation of the Coxeter element in the factorized form already used in this manuscript $\sigma := \sigma_- \sigma_+$ as introduced in [47, 62]

$$\sigma_q := \sigma_-^q\, \tau_q\, \sigma_+^q\, \tau_q\ . \tag{4.4}$$

The deformations of the Coxeter factors $\sigma_\pm$ are defined by

$$\sigma_\pm^q := \prod_{i \in V_\pm} \sigma_i^q\ , \tag{4.5}$$

where the product is taken over q-deformed Weyl reflections, whose action on simple roots $\alpha_i \in \Delta$ is given as

$$\sigma_i^q(\alpha_j) := \alpha_j - (2\delta_{ij} - [I_{ji}]_q)\alpha_i \ . \tag{4.6}$$

We employed here one of the standard definition for a q-deformed integer[1]

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \ . \tag{4.7}$$

A further deformation in q results from the map τ_q also employed in (4.4)

$$\tau_q(\alpha_i) := q^{t_i}\alpha_i \ , \tag{4.8}$$

where $I = 2\mathbb{I} - K$ is the incidence matrix, $\mathbb{I}$ is the unit matrix and K is the Cartan matrix (1.1). The integers t_i are the symmetrizers of the incidence matrix I, i.e., $I_{ij}t_j = I_{ji}t_i$. From these definitions it is evident the q-deformed Coxeter element is only different from the ordinary one when the associated Weyl group is related to non-simply laced algebras.

Since σ_q is defined by its action on the simple roots α, it is natural to seek an operator $\mathcal{O}_q$ acting on elements $\alpha_q \in \Delta_q$ with the appropriate limit $\lim_{q\to 1}\mathcal{O}_q = \sigma$. Recalling that the order of σ is the Coxeter number h, i.e., $\sigma^h = 1$, whereas the order of σ_q is deformed $\sigma_q^h = q^{2H}$, it is obvious that the relation cannot be a simple similarity transformation. Here H is the ℓ-th Coxeter number of the dual algebra, see e.g. [63] for more details. Therefore we make the ansatz

$$\sigma_q\alpha = q^{2H/h}\Theta_q^{-1}\sigma\Theta_q\,\alpha \ = q^{2H/h}\Theta_q^{-1}\sigma\,\alpha_q. \tag{4.9}$$

and readily identify the operator $\mathcal{O}_q = q^{2H/h}\Theta_q^{-1}\sigma$. The relation (4.9) serves

[1] We will frequently use the identities $[1]_q = 1$, $[2]_q = q + q^{-1}$ and $[3]_q = 1 + q^2 + q^{-2}$.

as the defining relation for the q-deformed simple roots $\alpha_q = \Theta_q \, \alpha$.

In analogy to the undeformed situation we introduce a q-deformed simple root dressed by a colour value as a separate quantity $(\gamma_q)_i := c_i \, (\alpha_q)_i$. This serves as a representant to introduce the q-deformed Coxeter orbits

$$(\Omega_q)_i := \left\{ (\gamma_q)_i \, , \sigma \, (\gamma_q)_i \, , \ldots , \sigma^{h-1} \, (\gamma_q)_i \right\} \, . \tag{4.10}$$

The entire q-deformed root system Δ_q is then spanned by the union of all ℓ q-deformed Coxeter orbits

$$\Delta_q := \bigcup_{i=1}^{\ell} (\Omega_q)_i \, . \tag{4.11}$$

At this stage it is not obvious under which type of symmetry Δ_q remains invariant.

4.2 The q-deformed root space for $\left(C_2^{(1)}, D_3^{(2)}\right)$

Let is now illustrate the working of the above formulae with a simple explicit example. The incidence matrix for C_2 is in this case defined as $I_{12} = 1$, $I_{21} = 2$, such that the symmetrizers are $t_1 = 1$ and $t_2 = 2$. The Coxeter numbers are $h = 4$ and $H = 6$. Therefore we obtain

$$\begin{aligned} \sigma_-^q &= \begin{pmatrix} -1 & 0 \\ [2]_q & 1 \end{pmatrix} , \quad \sigma_+^q = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} , \quad \tau_q = \begin{pmatrix} q & 1 \\ 0 & q^2 \end{pmatrix} , \\ \sigma_q &= q^2 \begin{pmatrix} -1 & -q \\ [2]_q & 1 \end{pmatrix} . \end{aligned} \tag{4.12}$$

Solving equation (4.9) then yields the deformed roots

$$(\alpha_q)_1 = r_1\alpha_1 + \frac{q}{1+q}(r_1 - r_2)\alpha_2, \tag{4.13}$$

$$(\alpha_q)_2 = \frac{r_2 + (r_2 - 2r_1)q^2}{q+q^2}\alpha_1 + r_2\alpha_2, \tag{4.14}$$

where r_1, r_2 depend on q with the limiting behaviour $\lim_{q\to 1} r_1 = 1$ and $\lim_{q\to 1} r_2 = 1$.

We have now constructed a systematic method for q-deformations of Coxeter elements and substantiated the construction with a concrete example. We will apply this deformation to a physical model.

4.3 The mass spectrum of $\left(C_2^{(1)}, D_3^{(2)}\right)$-$\mathcal{L}_q$

Taking the two q-deformed simple roots to be of the form (4.13), (4.14), noting that the Kac labels for C_2 are $n_1 = 2$, $n_2 = 1$ and using the non-standard representation for the undeformed C_2-roots $\alpha_1 = \{0, 1\}$, $\alpha_2 = \{1, -1\}$ we compute the mass matrix in (4.2). The virtue of this basis is that in the limit $q \to 1$ the mass matrix is diagonal. For $q \neq 1$ the direct evaluation leads to a nondiagonal matrix. However, imposing the additional constraint

$$r_2 = r_1 q \frac{3q^2 - 5q + 2 + (q+1)\sqrt{(16-7q)q - 8}}{2\left(2q^3 - q^2 + q - 1\right)}, \tag{4.15}$$

eliminates the off-diagonal elements. We obtain

$$M_{11}^2 = r_1^2 q^3 \frac{2q^3 + 8q^2 - 7q + (1-2q^2)\,\xi}{\left(1 - 2q^3 + q^2 - q\right)^2}, \tag{4.16}$$

$$M_{22}^2 = r_1^2 q \frac{11q^5 - 18q^4 + 19q^3 - 10q^2 + q + (q^4 + 2q^3 - 3q^2 + 2q - 1)\,\xi}{\left(2q^3 - q^2 + q - 1\right)^2}, \tag{4.17}$$

with $m_1 = M_{11}$, $m_2 = M_{22}$ being the classical masses of the two scalar fields and we abbreviate $\xi = \sqrt{16q - 7q^2 - 8}$. As can be found in the above mentioned literature, the quantum mass ratios of the $\mathcal{L}_0$-theory are given by

$$\frac{m_1}{m_2} = \frac{\sin\left[\frac{1}{24}(6-B)\pi\right]}{\cos\left(\frac{B\pi}{12}\right)}, \qquad \text{with } B = \frac{2H\beta^2}{H\beta^2 + 4\pi\ell h}, \tag{4.18}$$

where $B \in [0,2]$ denotes the effective coupling constant. From (4.16), (4.17) and (4.18) we can therefore fix the deformation parameter such that the quantum mass ratios of $\mathcal{L}_0$ correspond to the classical mass ratios of $\mathcal{L}_q$. We find

$$q = \frac{1}{1+\sqrt{3\left(\cos\frac{B\pi}{24}+\sin\frac{B\pi}{24}\right)+2\sin\frac{B\pi}{12}-3}}, \tag{4.19}$$

$$= 1 - \frac{1}{2}\sqrt{\frac{7\pi}{6}}\sqrt{B} + \frac{7\pi B}{24} - \frac{193\pi^{3/2}B^{3/2}}{192\sqrt{42}} + \frac{95\pi^2 B^2}{1152} + O\left(B^{5/2}\right) \tag{4.20}$$

Notice that deformation parameter $q(B)$ is a decreasing real valued function of B taking values between 1 and ≈ 0.435936. Consequently the coefficients in (4.13) and (4.14) in front of the simple roots acquire a complex part when the effective coupling constant varies between 0 and 2. We find that the classical mass spectrum of $\mathcal{L}_q$ equals the quantum mass spectrum of $\mathcal{L}_0$. One may now seek to generalise this behaviour for other algebras.

Chapter 5

Conclusion

In this thesis we have systematically formulated several different construction methods for antilinearly deformed complex root spaces. Firstly we proposed a construction that is based on two arbitrary elements of the Coxeter group. These elements are then employed as analogues to the $\mathcal{P}$-operator, which together with complex conjugation constitutes our analogue $\mathcal{PT}$-operator. The construction is of such a nature that the entire root space remains invariant under the antilinear deformation. We then extend this construction to a specific choice for the elements of the Coxeter group, namely the factors of the Coxeter element.

After solving this particular choice on a case-by-case basis we found that there are some cases where the deformation leads to a trivial solution. To address the fact that there were some groups that only resulted in a trivial solution, we modified the formulation of the specific Coxeter element to that of a newly reduced Coxeter element, which is of a lower rank than the original element. This leads to a large amount of possible choices for the elements we want to employ as a $\mathcal{P}$-transformation, however, we did find that many of these elements lie in the same similarity class. By making use of some identities we even proved how the elements of some of these similarity classes are related to each other.

We found that after deforming certain orthogonal groups that we could identify in the dual space their corresponding rotations. In addition we found that it is possible to operate in the opposite direction, by starting with a rotation in the dual space and then identifying their corresponding roots.

Another method of construction of an antilinear operator is that of the longest element. A specific feature of this deformation type is that it leads to a unique $\mathcal{PT}$-symmetry. One drawback about this construction is that it is limited to only some of the Coxeter groups, namely A_n, D_{2n+1} and E_6.

We show that a construction based on the deformation of the Weyl reflections themselves, can only be consistently formulated for the rank 2 algebras and cannot be generalized to higher ranking algebras, this was explicitly done in [41][34].

The key point behind these constructions is that non-Hermitian Hamiltonians that admit antilinear symmetry will have real eigenvalues when their eigenfunctions also possess the same symmetry. Since models such as Calogero models and Toda field theories can be related to root spaces [38][39][40], the task of identifying the symmetries of the deformations of these models become significantly easier if one deforms the root spaces these models are related to.

After constructing the deformed root spaces, we applied them to some physical models, namely Calogero models and Toda field theories. For the

Calogero system this deformation eliminates the singularities that exist in the potential of the undeformed model when two particles' position coincide or are exchanged. However, one might still pick up a phase when the position of two particles are exchanged or coincide,which is due to anyonic exchange factors. After building a general solution for the undeformed Calogero model, that is independent of the root space it is based on, we extended this same general solution to the deformed case and were able to find the ground state eigenvalues of the deformed model. We were even able to construct a Dyson map η, that relates non-Hermitian Hamiltonian $\mathcal{H}_{\varepsilon,q}$ to some Hermitian counterpart. This is a very difficult task and we were able to do it for specific cases, however, formulating a construction for this Dyson map independent of the algebra representation remains an open problem.

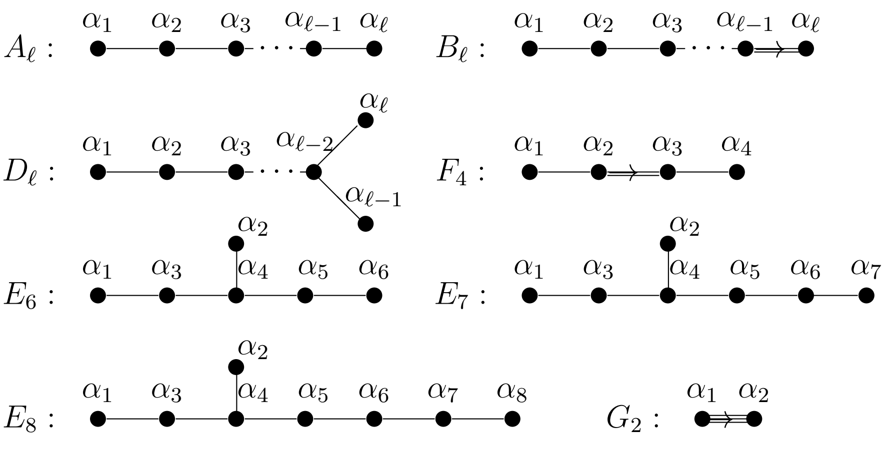
After analyzing the above mentioned constructions we turned our attention to q-deformed Coxeter elements, which we still aimed to be antilinearly deformed, however, with a different physical model in mind. We applied these q-deformed Coxeter elements to affine Toda field theories and we found that mass ratios between the classical case and the quantum case, for all orders of the coupling constant, are identical.

There are several questions that one can still ask about these models and the deformations. One of which is what will happen if we were to relax the constraint where we demanded the preservation of the inner products between the deformed and undeformed roots. Naturally this will lead to more free parameters in the deformation matrix, which might not be straightforward to solve, as one might have to make some choices for some. Another natural question to ask is can we find the eigenvalues of the excited states of the deformed Calogero model. We may also ask ourselves the Lax pairs for these models, so as to completely prove their integrability. Additionally, a rigorous, algebra independent proof for the identities used to construct the general solution of the Calogero Hamiltonian is yet to be formulated. Furthermore, one may ask how these deformations would act upon new other models that are based on roots.

Appendix A

Coxeter Groups

Figure A.1: The Dynkin diagrams for the Coxeter groups.[3]



A.1 Case-by-case data

For convenience we present in this appendix some numerical data for individual Coxeter groups. We present the values for the Coxeter number h defined as the total number of roots divided by the rank, the order of the Coxeter element σ or $1+\sum_{i=1}^{\ell} n_i$ when the highest root is expressed in terms of simple roots as $\sum_{i=1}^{\ell} n_i \alpha_i$. The dual Coxeter number is defined in the same way as the Coxeter number for the situation in which the arrows on the affine Diagram have been reversed. The exponents s_n are related to the eigenvalues of the Coxeter element as defined in (1.38) and t_ℓ is the ℓ-th symmetrizer of the incidence matrix I defined by means of the relation $I_{ij} t_j = t_i I_{ij}$. Additionally we give the number of roots N for each Coxeter group

$\mathcal{W}$	N	h	$\hat{h}$	s_n	t_ℓ
A_ℓ	$N(N+1)$	$\ell+1$	$\ell+1$	$1, 2, 3, ..., \ell$	1
B_ℓ	$2N^2$	2ℓ	$2\ell-1$	$1, 3, 5, ..., 2\ell-1$	1
C_ℓ	$2N^2$	2ℓ	$\ell+1$	$1, 3, 5, ..., 2\ell-1$	2
D_ℓ	$N(N-1)$	$2\ell-2$	$2\ell-2$	$1, 3, ..., \ell-1, \ldots, 2\ell-3$	1
E_6	72	12	12	$1, 4, 5, 7, 8, 11$	1
E_7	126	18	18	$1, 5, 7, 9, 11, 13, 17$	1
E_8	240	30	30	$1, 7, 11, 13, 17, 19, 23, 29$	1
F_4	48	12	9	$1, 5, 7, 11$	1
G_2	12	6	4	$1, 5$	3
H_3	30	10	10	$1, 5, 9$	1

Appendix B

Identities

We assemble here the crucial identities for the derivation of (3.5). Underlying are the generic relations which only involve roots and the dynamical variables $q = \{q_1, \ldots, q_n\}$

$$\sum_{\alpha,\beta\in\Delta^+} \frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)} = \sum_{\alpha\in\Delta^+} \frac{\alpha^2}{(\alpha\cdot q)^2}, \tag{B.1}$$

$$\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\frac{(\alpha\cdot q)}{(\beta\cdot q)} = \frac{\hat{h}h\ell}{2}t_\ell, \tag{B.2}$$

$$\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q) = \hat{h}t_\ell \sum_{\alpha\in\Delta^+} (\alpha\cdot q)^2, \tag{B.3}$$

$$\sum_{\alpha\in\Delta^+} \alpha^2 = \ell\hat{h}t_\ell. \tag{B.4}$$

At present we do not have a generic proof for these relations. Evidence on a case-by-case basis for the first identity was already provided in [95]. Here we have verified (B.2) and (B.3) for a large number of Coxeter groups. Denoting by n_s, α_s^2 and n_l, α_l^2 the number and length of the short and long roots, respectively, (B.4) follows from

$$\sum_{\alpha\in\Delta^+}\alpha^2=\frac{n_s}{2}\alpha_s^2+\frac{n_l}{2}\alpha_l^2=\frac{\alpha_l^2}{2}\left(n_s\frac{\alpha_s^2}{\alpha_l^2}+n_l\right)=\ell\hat{h}t_\ell, \tag{B.5}$$

where we used $n_s\alpha_s^2/\alpha_l^2+n_l=\ell\hat{h}$, which can be found for instance in [96] and $\alpha_l^2=2t_\ell$.

Accepting these relations the identities involving derivatives of r and z are easily derived. From (3.2) follows

$$\frac{\partial z}{\partial q_i}=z\sum_{\alpha\in\Delta^+}\frac{\alpha^i}{(\alpha\cdot q)}\qquad\text{and}\qquad\frac{\partial r}{\partial q_i}=\frac{1}{r\hat{h}t_\ell}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)\alpha^i. \tag{B.6}$$

Multiplying them and summing over the dynamical variables gives

$$\sum_{i=1}^n\left(\frac{\partial z}{\partial q_i}\right)^2 = z^2\sum_{\alpha,\beta\in\Delta^+}\frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)}=z^2\sum_{\alpha\in\Delta^+}\frac{\alpha^2}{(\alpha\cdot q)^2}, \tag{B.7}$$

$$\sum_{i=1}^n\frac{\partial z}{\partial q_i}\frac{\partial r}{\partial q_i} = \frac{z}{\hat{h}t_\ell r}\sum_{\alpha,\beta\in\Delta^+}(\alpha\cdot\beta)\frac{(\alpha\cdot q)}{(\beta\cdot q)}=\frac{h\ell}{2}\frac{z}{r}, \tag{B.8}$$

$$\sum_{i=1}^n\left(\frac{\partial r}{\partial q_i}\right)^2 = \frac{1}{r^2\hat{h}^2t_\ell^2}\sum_{\alpha,\beta\in\Delta^+}(\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q)=1, \tag{B.9}$$

where we have used (B.1) in (B.7), (B.2) in (B.8) and (B.3) in (B.9). Furthermore we need the sums over the second order derivatives. From (B.6) we obtain with the help of (B.1) and (B.2)

$$\sum_{i=1}^n\frac{\partial^2 z}{\partial q_i^2} = z\left(\sum_{\alpha,\beta\in\Delta^+}\frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)}-\sum_{\alpha\in\Delta^+}\frac{\alpha^2}{(\alpha\cdot q)^2}\right)=0, \tag{B.10}$$

$$\begin{aligned}\sum_{i=1}^n\frac{\partial^2 r}{\partial q_i^2} &= \frac{1}{r\hat{h}t_\ell}\sum_{\alpha\in\Delta^+}\alpha^2-\frac{1}{r^3\hat{h}t_\ell}\sum_{\alpha,\beta\in\Delta^+}(\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q)\\ &= \frac{\ell-1}{r}.\end{aligned} \tag{B.11}$$

Appendix C

Similarity Classes

In this appendix we provide more examples of reduced root spaces generated from different types of classes. We exhibit also the action of $\tilde{\sigma}_{\pm}$ on the simple roots from which one can easily infer the invariance of the entire root space. We use the same conventions as for the tables 2 and 3.

C.1 A_8-Root spaces based on the class $\Sigma_{\{1,2,3,4,^{\ell-3}\}}$ and their invariance

$\tilde{\sigma}^{(i)}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde{\sigma}^{(1)}$	$\mathbf{-1,2}$	$\mathbf{1,2,3}$	$\mathbf{-2,3}$	$2,3,4$	5	6	7	8
$\tilde{\sigma}^{(1)^2}$	$\mathbf{-3}$	$\mathbf{-2}$	$\mathbf{-1}$	$1,2,3,4$	5	6	7	8
$\tilde{\sigma}^{(1)^3}$	$\mathbf{2,3}$	$\mathbf{-1,2,3}$	$\mathbf{1,2}$	$3,4$	5	6	7	8
$\tilde{\sigma}_{-}^{(1)}$	-1	$1,2,3$	-3	$3,4$	5	6	7	8
$\tilde{\sigma}_{+}^{(1)}$	$1,2$	-2	$2,3$	4	5	6	7	8

$\tilde{\sigma}^{(2)}$	1, 2	**3, 4**	**−2, 3, 4**	**2, 3**	4, 5	6	7	8
$\tilde{\sigma}^{(2)^2}$	1, 2, 3, 4	**−4**	**−3**	**−2**	5	6	7	8
$\tilde{\sigma}^{(2)^3}$	1, 2, 3	−, **2, 3**	**2, 3, 4**	−, **3, 4**	3, 4, 5	6	7	8

$\tilde{\sigma}_{-}^{(2)}$	1	2, 3	−3	3, 4	5	6	7	8
$\tilde{\sigma}_{+}^{(2)}$	1, 2	−2	2, 3, 4	−4	4, 5	6	7	8

$\tilde{\sigma}^{(3)}$	1	2, 3, 4	**−3, 4**	**3, 4, 5**	**−4, 5**	4, 5, 6	7	8
$\tilde{\sigma}^{(3)^2}$	1	2, 3, 4, 5	**−5**	**−4**	**−3**	3, 4, 5, 6	7	8
$\tilde{\sigma}^{(3)^3}$	1	2, 3	**4, 5**	**−3, 4, 5**	**3, 4**	5, 6	7	8
$\tilde{\sigma}_{-}^{(3)}$	1	2, 3	−3	3, 4, 5	−5	5, 6	7	8
$\tilde{\sigma}_{+}^{(3)}$	1	2	3, 4	−4	4, 5	6	7	8

$\tilde{\sigma}^{(4)}$	1	2	3, 4	**5, 6**	**−4, 5, 6**	**4, 5**	6, 7	8
$\tilde{\sigma}^{(4)^2}$	1	2	3, 4, 5, 6	**−6**	**−5**	**−4**	4, 5, 6, 7	8
$\tilde{\sigma}^{(4)^3}$	1	2	3, 4, 5	**−4, 5**	**4, 5, 6**	**−5, 6**	5, 6, 7	8
$\tilde{\sigma}_{-}^{(4)}$	1	2	3	4, 5	−5	5, 6	7	8
$\tilde{\sigma}_{+}^{(4)}$	1	2	3, 4	−4	4, 5, 6	−6	6, 7	8

$\tilde{\sigma}^{(5)}$	1	2	3	4, 5, 6	**−5, 6**	**5, 6, 7**	**−6, 7**	6, 7, 8
$\tilde{\sigma}^{(5)^2}$	1	2	3	4, 5, 6, 7	**−7**	**−6**	**−5**	5, 6, 7, 8
$\tilde{\sigma}^{(5)^3}$	1	2	3	4, 5	**6, 7**	**−5, 6, 7**	**5, 6**	7, 8
$\tilde{\sigma}_{-}^{(5)}$	1	2	3	4, 5	−5	5, 6, 7	−7	7, 8
$\tilde{\sigma}_{+}^{(5)}$	1	2	3	4	5, 6	−6	6, 7	8

$\tilde{\sigma}^{(6)}$	1	2	3	4	$5,6$	$\mathbf{7,8}$	$\mathbf{-6,7,8}$	$\mathbf{6,7}$
$\tilde{\sigma}^{(6)^2}$	1	2	3	4	$5,6,7,8$	$\mathbf{-8}$	$\mathbf{-7}$	$\mathbf{-6}$
$\tilde{\sigma}^{(6)^3}$	1	2	3	4	$5,6,7$	$\mathbf{-6,7}$	$\mathbf{6,7,8}$	$\mathbf{-7,8}$
$\tilde{\sigma}^{(6)}_-$	1	2	3	4	5	$6,7$	-7	$7,8$
$\tilde{\sigma}^{(6)}_+$	1	2	3	4	$5,6$	-6	$6,7,8$	-8

C.2 A_8-Root spaces based on the class $\Sigma_{\{1,2^2,3,4,\ell-4\}}$ and their invariance

$\tilde{\sigma}^{(i,j)}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde{\sigma}^{(1,1)}$	$\mathbf{-1,2}$	$\mathbf{1,2,3}$	$\mathbf{-2,3}$	$2,3,4,5$	-5	$5,6$	7	8
$\tilde{\sigma}^{(1,1)^2}$	$\mathbf{-3}$	$\mathbf{-2}$	$\mathbf{-1}$	$1,2,3,4$	5	6	7	8
$\tilde{\sigma}^{(1,1)^3}$	$\mathbf{2,3}$	$\mathbf{-1,2,3}$	$\mathbf{1,2}$	$3,4,5$	-5	$5,6$	7	8
$\tilde{\sigma}^{(1,1)}_-$	-1	$1,2,3$	-3	$3,4,5$	-5	$5,6$	7	8
$\tilde{\sigma}^{(1,1)}_+$	$1,2$	-2	$2,3$	4	5	6	7	8
$\tilde{\sigma}^{(2,1)}$	$1,2$	$\mathbf{3,4}$	$\mathbf{-2,3,4}$	$\mathbf{2,3}$	$4,5,6$	-6	$6,7$	8
$\tilde{\sigma}^{(2,1)^2}$	$1,2,3,4$	$\mathbf{-4}$	$\mathbf{-3}$	$\mathbf{-2}$	$2,3,4,5$	6	7	8
$\tilde{\sigma}^{(2,1)^3}$	$1,2,3$	$\mathbf{-2,3}$	$\mathbf{2,3,4}$	$\mathbf{-3,4}$	$3,4,5,6$	-6	$6,7$	8
$\tilde{\sigma}^{(2,1)}_-$	1	$2,3$	-3	$3,4$	5	6	7	8
$\tilde{\sigma}^{(2,1)}_+$	$1,2$	-2	$2,3,4$	-4	$4,5,6$	-6	$6,7$	8

$\tilde{\sigma}^{(3,1)}$	1	$2,3,4$	$\mathbf{-3,4}$	$\mathbf{3,4,5}$	$\mathbf{-4,5}$	$4,5,6,7$	-7	$7,8$
$\tilde{\sigma}^{(3,1)^2}$	1	$2,3,4,5$	$\mathbf{-5}$	$-\mathbf{4}$	$-\mathbf{3}$	$3,4,5,6$	7	8
$\tilde{\sigma}^{(3,1)^3}$	1	$2,3$	$\mathbf{4,5}$	$\mathbf{-3,4,5}$	$\mathbf{3,4}$	$5,6,7$	-7	$7,8$
$\tilde{\sigma}_{-}^{(3,1)}$	1	$2,3$	-3	$3,4,5$	-5	$5,6,7$	-7	$7,8$
$\tilde{\sigma}_{+}^{(3,1)}$	1	2	$3,4$	-4	$4,5$	6	7	8

$\tilde{\sigma}^{(4,1)}$	1	2	$3,4$	$\mathbf{5,6}$	$\mathbf{-4,5,6}$	$\mathbf{4,5}$	$6,7,8$	-8
$\tilde{\sigma}^{(4,1)^2}$	1	2	$3,4,5,6$	$-\mathbf{6}$	$-\mathbf{5}$	$-\mathbf{4}$	$4,5,6,7$	8
$\tilde{\sigma}^{(4,1)^3}$	1	2	$3,4,5$	$\mathbf{-4,5}$	$\mathbf{4,5,6}$	$\mathbf{-5,6}$	$5,6,7,8$	-8
$\tilde{\sigma}_{-}^{(4,1)}$	1	2	3	$4,5$	-5	$5,6$	7	8
$\tilde{\sigma}_{+}^{(4,1)}$	1	2	$3,4$	-4	$4,5,6$	-6	$6,7,8$	-8

C.3 A_8-Root spaces based on the class $\Sigma_{\{1,2^2,3,4,\ell-4\}}$ and their invariance

$\tilde{\sigma}^{(i,j)}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde{\sigma}^{(1,2)}$	-1	$1,2,3,4$	$\mathbf{-3,4}$	$\mathbf{3,4,5}$	$\mathbf{-4,5}$	$4,5,6$	7	8
$\tilde{\sigma}^{(1,2)^2}$	1	$2,3,4,5$	$-\mathbf{5}$	$-\mathbf{4}$	$-\mathbf{3}$	$3,4,5,6$	7	8
$\tilde{\sigma}^{(1,2)^3}$	-1	$1,2,3$	$\mathbf{4,5}$	$\mathbf{-3,4,5}$	$\mathbf{3,4}$	$5,6$	7	8
$\tilde{\sigma}_{-}^{(1,2)}$	-1	$1,2,3$	-3	$3,4,5$	-5	$5,6$	7	8
$\tilde{\sigma}_{+}^{(1,2)}$	1	2	$3,4$	-4	$4,5$	6	7	8

$\tilde{\sigma}^{(2,2)}$	$1,2$	-2	$2,3,4$	$\mathbf{5,6}$	$\mathbf{-4,5,6}$	$\mathbf{4,5}$	$6,7$	8
$\tilde{\sigma}^{(2,2)^2}$	1	2	$3,4,5,6$	$\mathbf{-6}$	$\mathbf{-5}$	$\mathbf{-4}$	$4,5,6,7$	8
$\tilde{\sigma}^{(2,2)^3}$	$1,2$	-2	$2,3,4,5$	$\mathbf{-4,5}$	$\mathbf{4,5,6}$	$\mathbf{-5,6}$	$5,6,7$	8

$\tilde{\sigma}_{-}^{(2,2)}$	1	2	3	$4,5$	-5	$5,6$	7	8
$\tilde{\sigma}_{+}^{(2,2)}$	$1,2$	-2	$2,3,4$	-4	$4,5,6$	-6	$6,7$	8

$\tilde{\sigma}^{(3,2)}$	1	$2,3$	-3	$3,4,5,6$	$\mathbf{-5,6}$	$\mathbf{5,6,7}$	$\mathbf{-6,7}$	$6,7,8$
$\tilde{\sigma}^{(3,2)^2}$	1	2	3	$4,5,6,7$	$\mathbf{-7}$	$\mathbf{-6}$	$\mathbf{-5}$	$5,6,7,8$
$\tilde{\sigma}^{(3,2)^3}$	1	$2,3$	-3	$3,4,5$	$\mathbf{6,7}$	$\mathbf{-5,6,7}$	$\mathbf{5,6}$	$7,8$

$\tilde{\sigma}_{-}^{(3,2)}$	1	$2,3$	-3	$3,4,5$	-5	$5,6,7$	-7	$7,8$
$\tilde{\sigma}_{+}^{(3,2)}$	1	2	3	4	$5,6$	-6	$6,7$	8

$\tilde{\sigma}^{(4,2)}$	1	2	$3,4$	-4	$4,5,6$	$\mathbf{7,8}$	$\mathbf{-6,7,8}$	$\mathbf{6,7}$
$\tilde{\sigma}^{(4,2)^2}$	1	2	3	4	$5,6,7,8$	$\mathbf{-8}$	$\mathbf{-7}$	$\mathbf{-6}$
$\tilde{\sigma}^{(4,2)^3}$	1	2	$3,4$	-4	$4,5,6,7$	$\mathbf{-6,7}$	$\mathbf{6,7,8}$	$\mathbf{-7,8}$

$\tilde{\sigma}_{-}^{(4,2)}$	1	2	3	4	5	$6,7$	-7	$7,8$
$\tilde{\sigma}_{+}^{(4,2)}$	1	2	$3,4$	-4	$4,5,6$	-6	$6,7,8$	-8

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