Appendix A: Long Replenishment Lead Time

We analyze the case when the replenishment lead time is long so that both (instead of one) retailers place their orders "before" the market size M is realized. We show numerically that the structural results continue to hold.

A.1. Setting 2: Selling two substitutable products through one retailer

Observe that Setting 2 (Figure 4-2) corresponds to the base case when $s_{m1} = s_{m2} = 0$. Hence, $D_1 = m \cdot \frac{(p_2 - s_{b_2}) - \delta(p_1 - s_{b_1})}{\delta - 1}$, and $D_2 = m \cdot [1 - \frac{(p_2 - s_{b_2}) - (p_1 - s_{b_1})}{\delta - 1}]$.

Retailer's pricing problem. Because the retailer's pricing problem occurs after the orders z_1 and z_2 are placed and the market size m is realized, the ordering costs (i.e., $w_1 \cdot z_1$, $w_2 \cdot z_2$) are "sunk" and the sales $S_1 = \min\{D_1, z_1\}$ and $S_2 = \min\{D_2, z_2\}$; respectively. Therefore, the retailer's problem is: $\max_{p_1,p_2} \{(p_1 + s_{r_1}) \cdot D_1 + (p_2 + s_{r_2}) \cdot D_2\}$, s.t. $D_1 \leq z_1, D_2 \leq z_2$. Let $\mathcal{M}_2 = \frac{2z_2 \cdot (\delta - 1)}{\delta - 1 - s_1 + s_2}$ and $\mathcal{M}_3 = \frac{2(z1+z2)}{1+s_1}$, we can show that:

$$p_1^* = \begin{cases} \frac{1}{2}(1 + s_{b_1} - s_{r_1}) & \text{if } m \leq \mathcal{M}_3 \\ 1 + s_{b_1} - \frac{z_1 + z_2}{m} & \text{if } m \geq \mathcal{M}_3 \end{cases}, \quad p_2^* = \begin{cases} \frac{1}{2}(\delta + s_{b_2} - s_{r_2}) & \text{if } m \leq \mathcal{M}_2 \\ \frac{m(2\delta - 1 - s_1 + 2s_{b_2}) - 2z_2(\delta - 1)}{2m} & \text{if } \mathcal{M}_2 < m < \mathcal{M}_3 \\ \delta + s_{b_2} - \frac{z_1 + \delta \cdot z_2}{m} & \text{if } m \geq \mathcal{M}_3 \end{cases}$$

$$S_1^* = \begin{cases} \frac{m \cdot (\delta \cdot s_1 - s_2)}{2(\delta - 1)} & \text{if } m \leq \mathcal{M}_2 \\ \frac{1}{2}m(1 + s_1) - z_2 & \text{if } \mathcal{M}_2 < m < \mathcal{M}_3 \\ z_1 & \text{if } m > \mathcal{M}_3 \end{cases}, \quad S_2^* = \begin{cases} \frac{m \cdot (\delta - 1 + s_2 - s_1)}{2(\delta - 1)} & \text{if } m \leq \mathcal{M}_2 \\ z_2 & \text{if } m > \mathcal{M}_2 \end{cases}$$

Retailer's ordering problem. By using (p_1^*, p_2^*) and (S_1^*, S_2^*), the retailer's problem is:

$$\max_{z_1, z_2} E_M[\Pi_r(m)] = \int_0^{\mathcal{M}_2} \Pi_{r,1}(m) \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}_2}^{\mathcal{M}_1} \Pi_{r,2}(m) \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}_1}^{\infty} \Pi_{r,3}(m) \cdot f(m)\mathrm{d}m, \text{ where}$$

$$\Pi_r(m) = (p_1^* + s_{r_1}) \cdot S_1^* + (p_2^* + s_{r_2}) \cdot S_2^* - w_1 \cdot z_1 - w_2 \cdot z_2 = \begin{cases} \Pi_{r,1}(m) & \text{if } m \leq \mathcal{M}_2 \\ \Pi_{r,2}(m) & \text{if } \mathcal{M}_2 < m < \mathcal{M}_1 \\ \Pi_{r,3}(m) & \text{if } m \geq \mathcal{M}_1 \end{cases}.$$

Donor's problem. When offering uniform subsidy $s_1 = s_2 = s$, the donor's problem is: $\max_s \quad E_M[S_1^* + S_2^*]$ s.t. $E_M[s \cdot (S_1^* + S_2^*)] \leq K$, where

$$E_M[S_1^* + S_2^*] = \int_0^{\mathcal{M}_2} \frac{m(\delta \cdot s + \delta - 1)}{2(\delta - 1)} \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}_2}^{\mathcal{M}_1} (\frac{m \cdot s}{2} + z_2^*) \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}_1}^{\infty} (z_1^* + z_2^*) \cdot f(m)\mathrm{d}m.$$

A.2. Setting 3: Two manufacturers sell two products separately through two retailers

We now consider Setting 3 (Figure 4-3) that corresponds to the base case when $s_{m1} = s_{m2} = 0$ and the wholesale price is exogenous.

Retailers' pricing problem. By using the same approach as before, each retailer solves:

$$\max_{p_1} \quad \{(p_1 + s_{r_1}) \cdot D_1\} \quad \text{s.t. } D_1 = m \cdot \frac{(p_2 - s_{b_2}) - \delta(p_1 - s_{b_1})}{\delta - 1} \leq z_1, \quad \text{and}$$
$$\max_{p_2} \quad \{(p_2 + s_{r_2}) \cdot D_2\} \quad \text{s.t. } D_2 = m \cdot [1 - \frac{(p_2 - s_{b_2}) - (p_1 - s_{b_1})}{\delta - 1}] \leq z_2.$$

Let $\mathcal{M}'_2 = \frac{z_2 \cdot (4\delta-1)\cdot(\delta-1)}{2\delta^2 - \delta(2+s_1) + (2\delta-1)s_2}$ and $\mathcal{M}'_3 = \frac{z_1(2\delta-1)+z_2\delta}{(1+s_1)\cdot\delta}$, we get:

$$p_1^* = \begin{cases} \frac{\delta-1-s_{b_1}-s_2+2\delta(s_{b_1}-s_{r_1})}{4\delta-1} & \text{if } m \leq \mathcal{M}'_2 \\ \frac{m((\delta-1)(1+s_{b_1})-\delta s_{r_1})-z_2(\delta-1)}{m(2\delta-1)} & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ 1 + s_{b_1} - \frac{z_1+z_2}{m} & \text{if } m \geq \mathcal{M}'_3 \end{cases}, \quad p_2^* = \begin{cases} \frac{2\delta^2-2\delta-s_{b_2}-\delta s_1+2\delta(s_{b_2}-s_{r_2})}{4\delta-1} & \text{if } m \leq \mathcal{M}'_2 \\ \frac{m(\delta(2\delta-2-s_1)+(2\delta-1)s_{b_2})-2z_2(\delta-1)\delta}{m(2\delta-1)} & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ \delta + s_{b_2} - \frac{z_1+z_2\delta}{m} & \text{if } m \geq \mathcal{M}'_3 \end{cases}$$

$$S_1^* = \begin{cases} m \cdot \frac{\delta^2(1+2s_1)-\delta(1+s_1+s_2)}{(4\delta-1)(\delta-1)} & \text{if } m \leq \mathcal{M}'_2 \\ \frac{\delta\cdot(m(1+s_1)-z_2)}{2\delta-1} & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ z_1 & \text{if } m \geq \mathcal{M}'_3 \end{cases}, \quad S_2^* = \begin{cases} m \cdot \frac{(2\delta-1)\cdot s_2+\delta(2\delta-2-s_1)}{(4\delta-1)(\delta-1)} & \text{if } m \leq \mathcal{M}'_2 \\ z_2 & \text{if } m > \mathcal{M}'_2 \end{cases}$$

Retailers' ordering problem. By using (p_1^*, p_2^*) and (S_1^*, S_2^*), each retailer's profit $\Pi_r^i(m), i = 1, 2$ is:

$$\Pi_r^1(m) = (p_1^* + s_{r_1}) \cdot S_1^* - w_1 \cdot z_1 = \begin{cases} \Pi_{r,1}^1(m) & \text{if } m \leq \mathcal{M}'_2 \\ \Pi_{r,2}^1(m) & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ \Pi_{r,3}^1(m) & \text{if } m \geq \mathcal{M}'_3 \end{cases}$$

$$\Pi_r^2(m) = (p_2^* + s_{r_2}) \cdot S_2^* - w_2 \cdot z_2 = \begin{cases} \Pi_{r,1}^2(m) & \text{if } m \leq \mathcal{M}'_2 \\ \Pi_{r,2}^2(m) & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ \Pi_{r,3}^2(m) & \text{if } m \geq \mathcal{M}'_3 \end{cases}$$

Hence, each retailer maximizes its own profit by solves:

$$\max_{z_1} E_M[\Pi_r^1(m)] = \int_0^{\mathcal{M}'_2} \Pi_{r,1}^1(m) \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}'_2}^{\mathcal{M}'_3} \Pi_{r,2}^1(m) \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}'_3}^{\infty} \Pi_{r,3}^1(m) \cdot f(m)\mathrm{d}m,$$
$$\max_{z_2} E_M[\Pi_r^2(m)] = \int_0^{\mathcal{M}'_2} \Pi_{r,1}^2(m) \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}'_2}^{\mathcal{M}'_3} \Pi_{r,2}^2(m) \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}'_3}^{\infty} \Pi_{r,3}^2(m) \cdot f(m)\mathrm{d}m$$

Donor's problem. When offering uniform subsidy $s_1 = s_2 = s$, the donor's problem is: $\max_s \quad E_M[S_1^* + S_2^*]$ s.t. $E_M[s \cdot (S_1^* + S_2^*)] \le K$, where

$$E_M[S_1^* + S_2^*] = \int_0^{\mathcal{M}_2'} \frac{m(s+\delta(3+2s))}{4\delta-1} \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}_2'}^{\mathcal{M}_3'} (\frac{\delta(m(1+s)-z_2^*)}{2\delta-1} + z_2^*) \cdot f(m)\mathrm{d}m + \int_{\mathcal{M}_3'}^{\infty} (z_1^* + z_2^*) \cdot f(m)\mathrm{d}m.$$

A.3. Numerical Analysis

We consider the market size $M \sim N(1, 0.04)$, set $w_1 = 0.5, w_2 = 0.8$, set $\delta = 1.2$, and we get Figure 1.

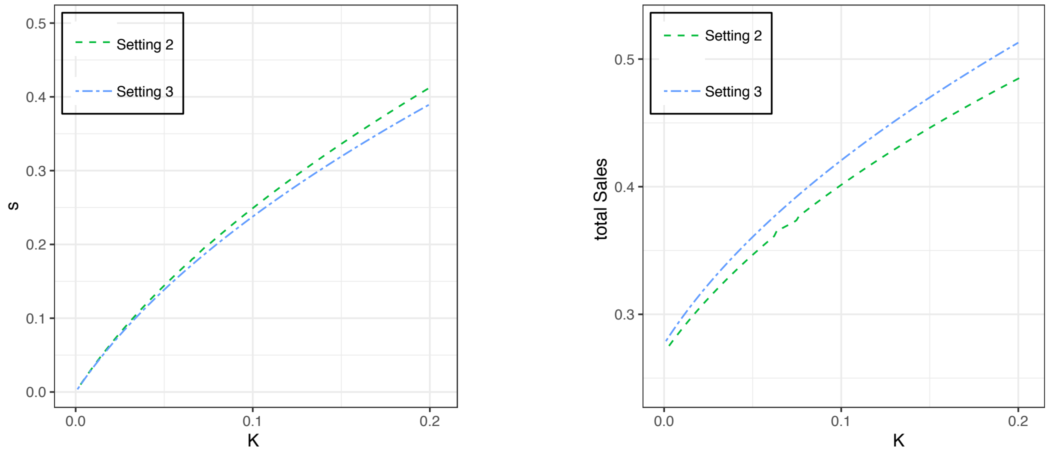

Figure 1 Optimal uniform subsidy (left) and the corresponding total sales (right)

From Figure 1, we find that the optimal per unit subsidy s^* is lower in setting 3, and the total sales $(S_1^* + S_2^*)$ is higher in setting 3. Hence, we can conclude that, by using the same budget K, having more retail-channel choice can increase product adoption. Therefore, our structural results obtained in Section 5 continue to hold even when the replenishment lead time is long so that both retailers have to place their orders before the market size is realized.

Appendix B: Proofs

Proof of Proposition 1 By considering the budget constraint, we can obtain that $D \le \frac{1-w+\sqrt{(1-w)^2+8K}}{4}$. As the objective function is increasing in D, we know that the optimal $D^* = \frac{1-w+\sqrt{(1-w)^2+8K}}{4}$. And we can then calculate the optimal s^* via substitution.

Proof of Proposition 2 By taking the first order derivative of $f_1(D_1, D_2)$ with respect to D_1, D_2, we get:

$$\frac{\partial f_1}{\partial D_1} = 4(D_1 + D_2) + (w_1 - 1) = 2s_1 + (1 - w_1) = 2(D_1 + D_2) + s_1 > 0,$$
$$\frac{\partial f_1}{\partial D_2} = 4(D_1 + \delta D_2) + (w_2 - \delta) = 2s_2 + (\delta - w_2) = 2(D_1 + \delta D_2) + s_2 > 0,$$

from which we know that $f_1(D_1, D_2)$ is increasing in both D_1 and D_2. As the objective function $D_1 + k \cdot D_2$ is also increasing in both D_1 and D_2, we know that the optimal D_1^* and D_2^* should satisfy the binding budget constraint (i.e., $f_1(D_1^*, D_2^*) = K$). Next, by considering the first order condition of the objective function of the donor's problem given by (10), we obtain $D_2^* = \frac{(\delta - w_2) - (1 - w_1)}{4(\delta - 1)}$. When $\delta - w_2 \geq 1 - w_1$, then D_2^* is feasible, else when $\delta - w_2 < 1 - w_1$, we can find that the objective function is always decreasing in D_2 when $D_2 > 0$, thus we can obtain $D_2^* = 0$. As such, we can get the corresponding D_1^* and optimal subsidy $(s_{b_i}^*, s_{r_i}^*)$ via substitution. Moreover, as $(D_1^*, D_2^*) = (\frac{1}{4}(1 - w_1 + \sqrt{8K + (1 - w_1)^2}), 0)$ is always a feasible solution of donor's problem in setting 2, we know that total demand in setting 2 $D_1^* + D_2^* \geq \frac{1}{4}(1 - w_1 + \sqrt{8K + (1 - w_1)^2})$.

Proof of Proposition 3 By denoting the subsidy cost (i.e., the left hand side of (13)) as $f_2(D_1, D_2)$ and by taking the first order derivative of $f_2(\cdot)$ with respect to D_1, D_2, we get:

$$\frac{\partial f_2}{\partial D_1} = 2D_1 \cdot \frac{2\delta - 1}{\delta} + 2D_2 + (w_1 - 1) = 2s_1 + (1 - w_1) = \frac{2\delta - 1}{2\delta} \cdot D_1 + D_2 + s_1 > 0,$$
$$\frac{\partial f_2}{\partial D_2} = 2(2\delta - 1)D_2 + 2D_1 + (w_2 - \delta) = 2s_2 + (\delta - w_2) = (2\delta - 1)D_2 + D_1 + s_2 > 0,$$

from which we know that $f_2(D_1, D_2)$ is increasing in both D_1 and D_2. As the objective function $D_1 + D_2$ is also increasing in both D_1 and D_2, we know that the optimal D_1^* and D_2^* should satisfy the binding budget constraint (i.e., $f_2(D_1^*, D_2^*) = K$). Also, from (12), we know that D_i only depends on the total subsidy s_i for each product so that we can solve out the unique s_i based on the binding budget constraint, while the optimal $s_{b_i}^*$ and $s_{r_i}^*$ are not uniquely determined.

Proof of Corollary 1 To achieve the same demand (D_1, D_2), the donor should spend $f_1(D_1, D_2) = 2D_1^2 + 2\delta D_2^2 + 4D_1 D_2 + (w_1 - 1)D_1 + (w_2 - \delta)D_2$ in setting 2 and spend $f_2(D_1, D_2) = \frac{2\delta - 1}{\delta} D_1^2 + 2D_1 D_2 + (2\delta - 1)D_2^2 + (w_1 - 1)D_1 + (w_2 - \delta)D_2$ in setting 3. By comparing $f_1(D_1, D_2)$ and $f_2(D_1, D_2)$, we obtain:

$$f_1(D_1, D_2) - f_2(D_1, D_2) = (2 - \frac{2\delta - 1}{\delta}) \cdot (D_1^2 + \delta D_2^2) + 2D_1 D_2 > 0.$$

Hence we know that to get the same (D_1, D_2), the donor needs to spend more money in a single retailer case (i.e., setting 2) than two competing retailers case (i.e., setting 3). Recall Proposition2 and 3, the optimal solutions of the donor's problem all satisfy the binding constraint. Therefore, we know that the optimal solution $(D_{1,1}^*, D_{1,2}^*)$ of setting 2 with a single retailer satisfies $f_1(D_{1,1}^*, D_{1,2}^*) = K$. Meanwhile, we also know that $f_2(D_{1,1}^*, D_{1,2}^*) < K$, which means $(D_{1,1}^*, D_{1,2}^*)$ is not the optimal solution of setting 3 with two competing retailers. As such, we know that the optimal solution of setting 3 yields a greater total demand (i.e., the objective function $D_1 + D_2$) than the optimal solution of setting 2.

Proof of Proposition 4 It is easy to check that the objective function $D = \frac{1-c}{4} + \frac{s'}{4}$ and the donor's subsidy cost $s' \cdot (\frac{1-c}{4} + \frac{s'}{4})$ are both increasing in s'. Hence we know that the budget constraint is binding at the optimal solution. By solving the binding budget constraint, we obtain $s' = \frac{-(1-c)+\sqrt{(1-c)^2+16K}}{2}$ and we then get $D^* = \frac{(1-c)+\sqrt{(1-c)^2+16K}}{8}$, $W^* = \frac{[(1-c)+\sqrt{(1-c)^2+16K}]^2}{128}$, $\pi_r^* = \frac{[(1-c)+\sqrt{(1-c)^2+16K}]^2}{64}$, $\pi_m^* = \frac{[(1-c)+\sqrt{(1-c)^2+16K}]^2}{32}$ via substitution.

Proof of Proposition 5 By denoting $f_1(D_1, D_2)$ as the subsidy cost (i.e., the left hand side of (19)) and taking the first order derivative, we obtain:

$$\frac{\partial f_1}{\partial D_1} = [-1 + c_1 + 4(D_1 + D_2)] + 4D_1 + 4D_2 = s_1' + 4(D_1 + D_2) > 0,$$
$$\frac{\partial f_2}{\partial D_2} = [-\delta + c_2 + 4(\delta D_2 + D_1)] + 4(D_1 + \delta D_2) = s_2' + 4(D_1 + \delta D_2) > 0.$$

Hence we know that for feasible s_1', s_2', D_1, D_2, the donor's expense $f_1(D_1, D_2)$ is increasing in D_1 and D_2. As the objective function $D_1 + D_2$ is also increasing in D_1 and D_2, we know the optimal (D_1^*, D_2^*) satisfies the binding budget constraint (i.e., $[-1 + c_1 + 4(D_1^* + D_2^*)] \cdot D_1^* + [-\delta + c_2 + 4(D_1^* + \delta D_2^*)] \cdot D_2^* = K$). Next, by considering the first order condition of donor's objective function given by (20), we obtain $D_2^* = \frac{\delta - c_2 - (1-c_1)}{\delta(\delta-1)}$. When $\delta - c_2 \geq 1 - c_2$, $D_2^* > 0$ so that we can further compute $D_1^* = \frac{c_2 - c_1\delta}{8(\delta-1)} + \frac{1}{8}\sqrt{c_1^2 - 2c_2 + 16K + \frac{(c_1-c_2)^2}{\delta-1} + \delta}$ via substitution. When $\delta - c_2 < 1 - c_2$, $\frac{\delta - c_2 - (1-c_1)}{\delta(\delta-1)} < 0$ so that the objective function is always increasing in D_2 when $D_2 > 0$. Hence we get the optimal $D_2^* = 0$ and $D_1^* = \frac{1}{8}(1 - c_1) + \sqrt{(1-c_1)^2 + 16K}$. And we can then further compute the optimal subsidy $(s_1'^*, s_2'^*)$, and the corresponding π_m^*, π_r^* and W^* via substitution.

Proof of Proposition 6 By denoting $f_2(D_1, D_2)$ as the subsidy cost (i.e., the left hand side of (22)) and taking the first order derivative of $f_2(D_1, D_2)$ with respect to D_1 and D_2, we get:

$$\frac{\partial f_2}{\partial D_1} = c_1 - 1 + 2D_2 + 2D1 \cdot (4 + \frac{1}{1-2\delta} - \frac{2}{\delta}) = 2s_1' + 1 - c_1 > 0$$
$$\frac{\partial f_2}{\partial D_2} = c_2 - \delta + 2D_1 + D_2 \cdot (-5 + \frac{1}{1-2\delta} + 8\delta) = 2s_2' + \delta - c_2 > 0$$

Therefore, for feasible s_1', s_2', D_1, D_2, the donor's expense $f_2(D_1, D_2)$ is increasing in D_1 and D_2. As the objective function $D_1 + D_2$ is also increasing in D_1 and D_2, we obtain that the optimal (D_1^*, D_2^*) should satisfy the binding budget constraint (i.e., $[c_1 - 1 + D_2^* + D_1^* \cdot (4 + \frac{1}{1-2\delta} - \frac{2}{\delta})] \cdot D_1^* + [c_2 - \delta + D_1^* + D_2^* \cdot (-\frac{5}{2} + \frac{1}{2-4\delta} + 4\delta)] \cdot D_2^* = K$), which is stated as the first statement of Proposition 6. Next, we know from (21) that D_i only depends on s_i', which also implies that the total subsidy per unit s_i' for product i is uniquely determined but the optimal subsidy $(s_{b_i}^*, s_{r_i}^*, s_{m_i}^*)$ are not unique. Then we can easily check that π_r^*, π_m^*, and W^* also only depend on s_i'. Finally, we show the third statement by the following. To achieve the same demand (D_1, D_2), the donor should spend $f_1(D_1, D_2)$ in the setting 2 and spend $f_2(D_1, D_2)$ in the setting 3. By comparing $f_1(D_1, D_2)$ and $f_2(D_1, D_2)$, we obtain:

$$f_1(D_1, D_2) - f_2(D_1, D_2) = \frac{1}{2}[D_2(12D_1 + 5D_2) + \frac{4D_1^2}{\delta} + \frac{2D_1^2 + D_2^2}{2\delta - 1}] > 0.$$

Hence we know that to get the same (D_1, D_2), the donor needs to spend more money in setting 2 than setting 3. As the optimal solutions of the donor's problem all satisfy the binding constraint, we know that the optimal solution $(D^*_{1,1}, D^*_{1,2})$ of setting 2 satisfies $f_1(D^*_{1,1}, D^*_{1,2}) = K$. Meanwhile, we also know that $f_2(D^*_{1,1}, D^*_{1,2}) < K$, which means $(D^*_{1,1}, D^*_{1,2})$ is not the optimal solution of setting 3. As such, we know that the optimal solution of setting 3 yields a greater total demand (i.e., $D_1 + D_2$) than setting 2.

Proof of Proposition 7 Then by taking the second order derivative of $E_m[\Pi_r(m)]$ with respect to z and using the Leibniz integral rule, we obtain

$$\frac{\partial^2 E_m[\Pi_r(m)]}{\partial z^2} = -\int_{\frac{2z}{1+s}}^{\infty} \frac{2}{m} \cdot f(m)\mathrm{d}m < 0$$

Hence we know the expected profit function of the retailer is concave. Hence the optimal z^* satisfies the first order condition (i.e., $\int_{\frac{2z^*}{1+s}}^{\infty}(1 + s - \frac{2z^*}{m}) \cdot f(m)\mathrm{d}m - w = 0$). We use $g(z, s, w)$ to represent the function $\int_{\frac{2z^*}{1+s}}^{\infty}(1+s-\frac{2z^*}{m}) \cdot f(m)\mathrm{d}m - w$, and we have $g(z^*, s, w) = 0$. By taking the first order derivative of $g(z, s, w)$ with respect to z, s and w, we get:

$$\frac{\partial g}{\partial z} = -\int_{\frac{2z}{1+s}}^{\infty} \frac{2}{m} \cdot f(m)\mathrm{d}m < 0, \quad \frac{\partial g}{\partial s} = \int_{\frac{2z}{1+s}}^{\infty} f(m)\mathrm{d}m > 0, \quad \frac{\partial g}{\partial w} = -1 < 0$$

From the above, we know that $g(z, s, w)$ is increasing in s and decreasing in z and w. Hence to ensure $g(z^*, s, w) = 0$, we can easily know that z^* is increasing in s and decreasing in w.

Proof of Proposition 8 By taking the first order derivative of $E_M[S]$ with respect to s, we get:

$$\begin{aligned}\frac{\partial E_M[S]}{\partial s} =& \frac{1+s}{2} \cdot \frac{2z^*}{1+s} \cdot f(\frac{2z^*}{1+s}) \cdot \partial(\frac{2z^*}{1+s})/\partial s + \int_0^{\frac{2z^*}{1+s}} \frac{m}{2} \cdot f(m)\mathrm{d}m \\ & - z^* f(\frac{2z^*}{1+s}) \cdot \partial(\frac{2z^*}{1+s})/\partial s + \int_{\frac{2z^*}{1+s}}^{\infty} \frac{\partial z^*}{\partial s} \cdot f(m)\mathrm{d}m \\ =& \int_0^{\frac{2z^*}{1+s}} \frac{m}{2} \cdot f(m)\mathrm{d}m + \int_{\frac{2z^*}{1+s}}^{\infty} \frac{\partial z^*}{\partial s} \cdot f(m)\mathrm{d}m\end{aligned}$$

From Proposition 7 we know that z^* is increasing in s. Hence we obtain that $\frac{\partial E_M[S]}{\partial s} > 0$, which indicates that the total sale is increasing in the donor's subsidy s. With the objective function $E_M[S]$ and the total subsidy cost $s \cdot E_M[S]$ both increasing in s, we know that the optimal solution will be achieved at the binding budget constraint. With the binding budget constraint, we know that when the budget K increase, the optimal s^* will increase.

By taking the first order derivative of the subsidy cost $s \cdot E_M[S]$ with respect to z^*, we get $\frac{\partial(s \cdot E_M[S])}{\partial z^*} = s \cdot (\int_{\frac{2z}{1+s}}^{\infty} f(m)\mathrm{d}m) > 0$, from which we know the cost is increasing in z^*. As we have shown in Proposition 7 that z^* is decreasing in the wholesale price w, we obtain that the cost is decreasing in w. To ensure budget constraint is binding, we get that when w increases, the optimal s^* will increase.

Proof of Proposition 9 By taking the second order derivative of $E_M[\Pi_r(m)]$, we get:

$$\frac{\partial E_M^2[\Pi_r(m)]}{\partial z_1^2} = \frac{\partial M_1}{\partial z_1} \cdot 0 + \int_{M_1}^{\infty} (\frac{-2(\delta-1)}{m\delta}) \cdot f(m)\mathrm{d}m < 0,$$

from which we know the retailer's expected profit by selling product 1 is a concave function of z_1. By considering the first order condition, we obtain that the optimal ordering decision for product 1 (i.e., z_1^*) satisfies $\int_{\frac{2z_1^*(\delta-1)}{\delta s_1+w_2-s_2}}^{\infty} [\frac{-2(\delta-1)z_1^*}{m\delta} + \frac{\delta s_1 - s_2 + w_2}{\delta}] \cdot f(m)\mathrm{d}m - w_1 = 0.$

We use $g(z_1, s_1, s_2, w_1, w_2)$ to represent $\int_{\frac{2z_1(\delta-1)}{\delta s_1+w_2-s_2}}^{\infty} [\frac{-2(\delta-1)z_1}{m\delta} + \frac{\delta s_1 - s_2 + w_2}{\delta}] \cdot f(m)\mathrm{d}m - w_1$, and we have shown that $g(z_1^*, s_1, s_2, w_1, w_2) = 0$. By taking the first order derivative of $g(z_1, s_1, s_2, w_1, w_2)$ with respect to z_1, s_1, s_2, w_1, w_2, we get:

$$\frac{\partial g}{\partial z_1} = \int_{M_1}^{\infty} (\frac{-2(\delta-1)}{m\delta}) \cdot f(m)\mathrm{d}m < 0, \quad \frac{\partial g}{\partial s_1} = \int_{M_1}^{\infty} f(m)\mathrm{d}m > 0,$$
$$\frac{\partial g}{\partial s_2} = \int_{M_1}^{\infty} -\frac{1}{\delta} f(m)\mathrm{d}m < 0, \quad \frac{\partial g}{\partial w_1} = -1 < 0, \quad \frac{\partial g}{\partial w_2} = \int_{M_1}^{\infty} \frac{1}{\delta} f(m)\mathrm{d}m > 0$$

To ensure $g(z_1^*, s_1, s_2, w_1, w_2) = 0$, we can easily obtain that z_1^* is increasing s_1 and w_2, while is decreasing in s_2 and w_1.

Proof of Proposition 10 We use $SS_1(m)$ and $SS_2(m)$ to represent the total sales (i.e., $S_1 + S_2$) under cases when $m \le M_1$ and $m \ge M_1$, respectively; and we have $SS_1(M_1) = SS_2(M_1)$. By taking the first order derivative of $E_M[S_1 + S_2]$ with respect to s, we obtain:

$$\begin{aligned} \frac{\partial E_M[S_1+S_2]}{\partial s} =& \frac{\partial M_1}{\partial s} \cdot SS_1(M_1) \cdot f(M_1) + \int_0^{M_1} \frac{m}{2} \cdot f(m)\mathrm{d}m \\ & - \frac{\partial M_1}{\partial s} \cdot SS_2(M_1) \cdot f(M_1) + \int_{M_1}^{\infty} (\frac{\partial z_1^*}{\partial s} + \frac{m}{2\delta}) \cdot f(m)\mathrm{d}m \\ =& \int_0^{M_1} \frac{m}{2} \cdot f(m)\mathrm{d}m + \int_{M_1}^{\infty} (\frac{\partial z_1^*}{\partial s} + \frac{m}{2\delta}) \cdot f(m)\mathrm{d}m \end{aligned}$$

When $s_1 = s_2 = s$, we know that the optimal order quantity z_1^* satisfies $g(z_1^*, s, w_1, w_2) = \int_{\frac{2z_1^*(\delta-1)}{\delta s+w_2-s}}^{\infty} [\frac{-2(\delta-1)z_1}{m\delta} + \frac{\delta s-s+w_2}{\delta}] \cdot f(m)\mathrm{d}m - w_1 = 0$. By taking the first order derivative of $g(\cdot)$, we find that $\frac{\partial g}{\partial z_1} < 0$ and $\frac{\partial g}{\partial s} > 0$, from which we can further know z_1^* is increasing in s so as to ensure $g(z_1^*, s, w_1, w_2) = 0$. As z_1^* is increasing in s, we can obtain that the total expected sales is increasing in s (i.e.,$\frac{\partial E_M[S_1+S_2]}{\partial s} > 0$). Moreover, it is obvious that the total expense $E_M[s \cdot (S_1 + S_2)] = s \cdot E_M[S_1 + S_2]$ is also increasing in s. Hence we know that the optimal per unit subsidy s^* should satisfy the binding budget constraint.

Proof of Proposition 11 By taking the first order derivative of $E_M[\Pi_{r_1}(M)]$ with respect to z_1, we get:

$$\begin{aligned}\frac{\partial E_M[\Pi_{r_1}(m)]}{\partial z_1} =& \frac{\partial M_2}{\partial z_1} \cdot \Pi_{r_1,1}(M_2) \cdot f(M_2) + \int_0^{M_2} (-w_1) \cdot f(m)\mathrm{d}m \\ &- \frac{\partial M_2}{\partial z_1} \cdot \Pi_{r_1,2}(M_2) \cdot f(M_2) + \int_{M_2}^{\infty} [-\frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1 + \frac{\delta - 1 - (s_2 - w_2)}{2\delta - 1} + s_1 - w_1] \cdot f(m)\mathrm{d}m \\ =& -w_1 + \int_{M_2}^{\infty} [-\frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1 + \frac{\delta - 1 - (s_2 - w_2)}{2\delta - 1} + s_1] \cdot f(m)\mathrm{d}m\end{aligned}$$

By checking the second order derivative of $E_M[\Pi_{r_1}(m)]$, we obtain: $\frac{\partial^2 E_M[\Pi_{r_1}(m)]}{\partial z_1^2} = \frac{\delta-1}{2\delta-1} \cdot [\frac{1}{\delta} \cdot f(M_2) - 4\int_{M_2}^{\infty} \frac{1}{m} f(m)\mathrm{d}m] < 0$ when $\frac{1}{\delta} \cdot f(M_2) < 4\int_{M_2}^{\infty} \frac{1}{m} f(m)\mathrm{d}m$. Hence we know that $E_M[\Pi_{r_1}(M)]$ is a concave function of z_1; and we can obtain Proposition 11 by considering the first order condition.

Proof of Proposition 12 By taking the first order derivative of $E_M[S_1 + S_2]$ with respect to s, we get:

$$\frac{\partial E_M[S_1 + S_2]}{\partial s} = \int_0^{M_2} \frac{1+2\delta}{4\delta - 1} \cdot m \cdot f(m)\mathrm{d}m + \int_{M_2}^{\infty} [\frac{2(\delta-1)}{2\delta-1} \cdot \frac{\partial z_1^*}{\partial s} + \frac{m}{2\delta-1}] \cdot f(m)\mathrm{d}m$$

From Proposition 11, we know that $-w_1 + \int_{M_2}^{\infty} [-\frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1^* + \frac{\delta-1-(s_2-w_2)}{2\delta-1} + s_1] \cdot f(m)\mathrm{d}m = 0$. Hence when $s_1 = s_2 = s$, we denote $g(s, z_1) = -w_1 + \int_{M_2}^{\infty} [-\frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1^* + \frac{\delta-1-(s-w_2)}{2\delta-1} + s] \cdot f(m)\mathrm{d}$ and we know $g(s, z_1^*) = 0$. It is easy to check that $\frac{\partial g}{\partial z} < 0$ and $\frac{\partial g}{\partial s} > 0$, from which we can obtain that z_1^* is increasing in s so as to ensure $g(s, z_1^*) = 0$. With $\frac{\partial z_1^*}{\partial s} > 0$, we can show $\frac{\partial E_M[S_1+S_2]}{\partial s} > 0$. Therefore, we obtain that both the objective function and the subsidy cost shown in the donor's problem (41) is increasing in s, from which we know that the budget constraint should be binding at the optimal solution.