ANNUITY CHOICES FOR PENSIONERS

by

M Z KHORASANEE

Actuarial Research Paper No. 90

Department of Actuarial Science and Statistics
City University
London

August 1996

ISBN 1 874 770 94 8

Annuity choices for pensioners

Abstract

Deterministic and stochastic models are used to assess the risks and benefits of obtaining a pension from a retirement fund by means of:

1) the purchase of a life annuity providing a level monetary income;

2) the withdrawal of income from a fund invested in equities.

In each case the projected cash-flows are compared with those from a life annuity providing an income linked to price inflation.

It is concluded that although the each of the two options considered involve significant risks, they may nevertheless be attractive to certain groups of pensioners, in particular those with additional savings held outside the retirement fund.

1. Introduction

Certain types of pension plan provide a lump sum benefit, rather than a pension, from which the retiring member must secure an income.[1] The manner in which the lump sum is invested may be restricted by legislation; in the United Kingdom there are several permitted methods by which a retirement fund may be used to provide an income.[2]

The safest option might be to buy a life annuity providing an income guaranteed to increase in step with an index of consumer price inflation. In the UK, such a product is referred to as an *index-linked annuity*.

In this paper we shall consider the merits of two alternatives to an index-linked annuity currently available in the UK:

1) a *level annuity*, providing a stable monetary income, which remains the most popular option;

2) the *income withdrawal* option, in which no annuity is purchased, but instead the pensioner simply draws an income from the retirement fund.

We shall examine the implications of each of the above options by comparing the projected cash flows obtained with those from an index-linked annuity.

1.1 Why not choose an index-linked annuity?

Index-linked annuities are relatively unpopular with UK consumers, principally because the income obtained is initially much lower than the income provided by a level annuity. The following table shows the initial income available from the most competitive UK insurer in July 1996 for each type of annuity.

Table 1: Annual income to a male pensioner from a fund of £10,000

Pensioner's age	Level annuity	Index-linked annuity
60	£1,016	£683
65	£1,119	£797
70	£1,280	£951

Source: Annuity Direct, July 19 1996

1 These include individual insured plans, defined contribution plans, and defined benefit plans in which the benefit formula is for a lump sum at retirement.

2 A portion of the fund, usually not exceeding 25%, may be taken as cash.

At the time the figures in Table 1 were obtained, consumer price inflation was running at approximately 3% per annum in UK. If this rate of inflation were to continue, an index-linked annuity would provide a lower income for a period of 13 years from age 60, 11 years from age 65, and 10 years from age 70. Pensioners might feel that, even if they lived a reasonable lifespan beyond these durations, they would probably receive a higher aggregate income from a level annuity, and we shall show that this opinion is indeed justified.

However, the purchase of a level annuity has two potential hazards:

1) that the pensioner lives well beyond his/her life expectancy;

2) that inflation turns out to be higher than anticipated.

For this reason, purchasers of level annuities are sometimes advised to save a substantial part of the income they obtain in the early years, to build up a fund which can later provide some protection against inflation and longevity. In Section 2, we shall examine the efficacy of such a policy by assuming that pensioners save (or dis-save) the difference between the income from their level annuity and the income which would have been obtained from an index-linked annuity. It will be shown that even such a conservative policy does not immunize the pensioner against risk.

1.2 The income withdrawal option

The income withdrawal option was introduced in the UK in 1995, and is also a feature of the funded Social Security plan in Chile.[3] This option allows the pensioner to defer the purchase of an annuity, instead drawing an income by realising assets in the fund.

In the UK certain restrictions apply: the income drawn must lie between 35% and 100% of the income which otherwise could have been obtained from a level annuity, and the pensioner cannot delay the purchase of a life annuity beyond age 75.

The income withdrawal option has two principal attractions:

1) on death, the capital remaining in the fund remains part of the pensioner's estate;

2) the fund can be invested in assets which may provide a higher return to the pensioner than available from a life annuity.

The first characteristic is sometimes described as *capital protection.* In a life annuity, the absence of a death payout allows a higher level of income to be provided for the living. Thus, the income from a life annuity fund will exceed that from an income withdrawal fund if both are invested in the same assets and subject to the same expenses. This difference will increase as the pensioner ages, because of the increasing mortality strain.

---

[3] A description of this plan is given in the August 1996 issue of *The Actuary* magazine in an article entitled: "Chile - the pensions panacea?".

It follows that an income withdrawal fund must earn a higher return than a life annuity fund if the pensioner is to maintain an equivalent income. In practice, this means that an income withdrawal fund is likely to be invested wholly or partly in equities, as equities are expected to outperform the government bonds held by insurers' annuity funds. We shall show, however, that the market price volatility associated with equities creates additional risks for the pensioner.

2. Level annuity versus Index-linked annuity

2.1 Assets of life annuity fund

The income received from a life annuity policy depends on the interest, mortality and expenses assumed by the insurer for its annuity portfolio.

The interest assumption depends on the assets held by the insurer. In the case of level annuities, the insurer normally holds a fixed interest bond portfolio roughly matching the mean term of its liabilities. In the case of index-linked annuities, the insurer normally holds *index-linked bonds*, providing interest and principal payments which increase in step with consumer price inflation. Both types of security have been issued by the British government to finance the UK national debt, although the total market value of fixed interest bonds currently in issue is roughly eight times as great as that of index-linked bonds.[4]

The prospective return on a fixed interest bond is measured by its *gross redemption yield* - the interest rate at which the present value of future income and capital payments equals the market value of the bond. The prospective return offered by an index-linked bond is measured relative to future price inflation, and is referred to as its *real gross redemption yield.* This real yield can be thought of as the interest rate at which the present value of future income and capital payments from the bond would equal its current market value if future inflation were zero. The interest rates used to price level and index-linked annuities closely follow the average yields available on fixed interest and index-linked bonds, respectively.

We now define:

j = ***interest rate used to price level annuities***

r = ***real interest rate used to price index-linked annuities***

c = ***rate of price inflation***

---

[4] Total market capitalizations were £217bn ($337bn) for fixed interest bonds and £26bn ($41bn) for index-linked bonds. Source: *London Times* July 29, 1996.

And for each of the above, the instantaneous force of interest is given by:

$$\delta_j = \ln(1+j)$$
$$\delta_r = \ln(1+r)$$
$$\delta_c = \ln(1+c)$$

In a scenario of uniform and predictable price inflation, we would expect the total return on fixed interest and index-linked bonds to be identical, thus:

$$\delta_j = \delta_r + \delta_c \qquad (1)$$

Price inflation is, of course, neither uniform nor predictable. However, throughout this Section we shall assume that the difference between the nominal yield on fixed interest bonds and the real yield on index-linked bonds gives an unbiased estimate of future price inflation.

2.2 Deterministic comparison

We now compare the projected income from both types of annuity assuming a constant force of price inflation.

Notation

x = *age of pensioner at retirement*

$\bar{a}^j_x$ = *price of level annuity of* 1 *unit per annum*

$\bar{a}^r_x$ = *price of index-linked annuity of* 1 *unit per annum*

S_t = *sinking fund accumulated t years after purchase of level annuity*

We assume the purchaser of a level annuity spends income equal to that from an index-linked annuity, the difference being saved in (or drawn from) a sinking fund. It follows that the present value of the projected sinking fund per unit of retirement fund is given by:

$$PV[S_t] = \int_0^t (1/\bar{a}^j_x - e^{\delta_c z}/\bar{a}^r_x).e^{-\delta_j z}.dz \qquad (2)$$

Initially, the sinking fund grows because the money spent by the pensioner is less than the income from the level annuity. Eventually, the index-linked income required will exceed the level annuity payments, and the pensioner will have to draw money from the sinking fund.

At some post-retirement duration, N, the sinking fund falls to zero. This is the *break-even* duration: the level annuity provides a more valuable overall income for a pensioner who dies before this duration, and the index-linked annuity provides a more valuable overall income for a pensioner who survives beyond this duration.

It follows that:

$$PV[S_N] = 0 \tag{3}$$

And we can deduce from equations (2) and (3) that N must satisfy:

$$\frac{\bar{a}^{r}_{\overline{N|}}}{\bar{a}^{j}_{\overline{N|}}} = \frac{\bar{a}^{r}_{x}}{\bar{a}^{j}_{x}} \tag{4}$$

We now choose the following basis for pricing the annuities:

- $j = 7\%$ net of all expenses
- $r = 3\%$ net of all expenses
- mortality follows male PA(90) life table.[5]

Equation (4) can be solved for N, at any retirement age x, by the use of a suitable iterative method. Table 2 below shows the life expectancy at retirement, the value of N, and the probability of surviving to age $x+N$, for different values of x.

Table 2: Duration at which sinking fund is exhausted

x	$\mathring{e}_x$	N	${}_Np_x$
50	26.1	27.8	0.465
55	21.8	24.1	0.433
60	18.0	20.7	0.401
65	14.6	17.5	0.368
70	11.6	14.5	0.342
75	9.0	11.8	0.303

5 The PA(90) life table is based on the mortality experience of pensioners in employer-sponsored plans administered by UK insurance companies.

Table 2 shows that, in conditions of uniform and predictable inflation, the sinking fund is exhausted a few years after the pensioner's life expectancy at retirement. The final column of Table 2 shows that the probability of surviving to the age at which the sinking fund is exhausted is always less than 0.5, and falls with the age at retirement.

It therefore appears that a pensioner will *probably* obtain more value from a level annuity than from an index-linked annuity. However, the risk of a dramatic fall in income on surviving beyond the break-even duration is significant at all ages.

In the above comparison, the rate of inflation is given by:

$1.07/1.03 - 1 = 3.88\%$ per annum

Thus if a male pensioner retiring at age 65 survives to the break-even duration, his income would be reduced by the following factor:

$1/1.0388^{17.5} = 0.514$

2.3 Effect of uncertain rate of inflation

We now use a very simple model to investigate the effect of uncertainty in the rate of inflation on our previous results.

The expected force of inflation implied by our pricing basis has been defined as:

$$\delta_c = \delta_j - \delta_r$$

We now assume that the actual and expected inflation rates can differ, by defining:

$$\delta_p = \textit{actual force of price inflation}$$

Hence equation (2) is modified as follows:

$$PV[S_t] = \int_0^t (1/\bar{a}_x^j - e^{\delta_p z}/\bar{a}_x^r).e^{-\delta_j z}.dz \qquad (5)$$

And equation (4) becomes:

$$\frac{\bar{a}_{\overline{N}|}^{r'}}{\bar{a}_{\overline{N}|}^{j}} = \frac{\bar{a}_x^r}{\bar{a}_x^j} \qquad (6)$$

where: $\delta_{r'} = \delta_j - \delta_p$

and: $\delta_r = \delta_j - \delta_c$

Let us now examine the following case:

- $x = 65$
- $j = 7\%$
- $c = 3.88\%$
- $p = c+1\%$ with probability 0.5
- $p = c - 1\%$ with probability 0.5

On solving equation (6) the two possible values of N are as follows:

$N_1 = 26.0$, for $p = 2.88\%$

$N_2 = 13.3$, for $p = 4.88\%$

The probability of surviving beyond exhaustion of the sinking fund:

$= 0.5*_{N1}p_{65} + 0.5*_{N2}p_{65}$

$= 0.5*0.093 + 0.5*0.543$, using the PA(90) life table

$= 0.314$

This probability of 0.314 is lower than the corresponding figure in Table 2, because the reduction in the survival probability for the low inflation scenario is greater than the increase in the survival probability for the high inflation scenario. This is a consequence of the fact that mortality increases with age, resulting in a skewed probability distribution for the age at death. It follows that uncertainty in the rate of inflation might *reduce* the risk associated with a level annuity, which is perhaps a surprising result.

2.4 Stochastic comparison

In the real world, the rate of inflation is both variable and unpredictable, which makes it more difficult to quantify the risk associated with a level annuity. In this Section, we shall modify the results obtained in Sections 2.2 and 2.3 by means of simulation, using a stochastic model for the rate of price inflation in retirement.

In Sections 2.3 and 2.4 it was convenient to use a continuous-time model: as most annuities bought in the UK provide a monthly income, a continuous-time model is a reasonably good approximation to reality. In this Section, however, we switch to a discrete-time model, as it is easier to apply stochastic methods in a discrete-time framework. We shall therefore assume that the annuities provide an income payable annually in advance.

2.4.1 Inflation history of the United Kingdom

Parsons (1990) has examined the history of UK consumer price inflation since 1810, using the most representative index available in each era. From these data, he concluded that persistent positive inflation rates, as now exist in the UK and other industrialized economies, have only been observable from after World War II. He therefore did not reject the possibility that inflation might eventually revert to its earlier pattern of behaviour, in which the price level appeared as likely to fall as to rise in any year and the average rate of inflation measured over long periods was very small.

Another feature of the data is the existence of inflation shocks, which appeared to arise from the economic consequences of major historical events. Brief periods of high inflation occurred during all of the following crises:

- World War I
- World War II
- the rise in oil prices following the 1973 Arab/Israeli war
- the rise in oil prices at the start of Iran/Iraq war in 1980.

Currently, the rate of inflation is running at 3% per annum in the UK, an historically low figure for the post-war era, and most other industrialised economies are in a similar position.

2.4.2 Scenarios for future inflation

For a pensioner retiring in current conditions there are three possible scenarios:

1) the rate of inflation will continue to fall, and the economy will revert to long-term price stability;

2) the economy will experience an inflation shock;

3) the rate of inflation will vary more or less randomly around an average figure close to the current rate.

The implications of scenarios (1) and (2) for the choice of annuity are fairly clear. Under scenario (1) fixed interest bonds would offer excellent real returns, and a level annuity would offer considerable extra value when compared with an index-linked annuity. Under scenario (2) an index-linked annuity would be the only safe way to protect the real value of the pensioner's income.

It is only in scenario (3) that the choice of annuity is still finely balanced, and we shall argue that this is also the easiest scenario to model stochastically. We also make the subjective assessment that this is the most likely of the three scenarios.

2.4.3 Stochastic model for price inflation

Stochastic models for the rate of inflation have been suggested by Wilkie (1986) and Clarkson (1991).

Wilkie used a first order autoregressive model for price inflation, as follows:

$$\ln(1+e_{t+1}) = \mu_e + k_e.[\ln(1+e_t) - \mu_e] + \sqrt{1-k_e^2}.\sigma_e.Z_t \qquad (7)$$

where: e_t *= annual rate of inflation over year t*

μ_e, σ_e, *and* k_e *are parametric constants*

Z_t *is an independent normal random variable with zero mean and unit variance*

This model assumes that inflation will vary around a long term average, in accordance with the third scenario described in Section 2.4.2. It has been criticized by Clarkson for failing to allow for inflation shocks and other non-linear effects, who has proposed an alternative model which purports to do so. However, the difficulty with incorporating shock terms is that the assumed average size and frequency of these shocks will be arbitrary. The inflation shocks of the past, which arose from disparate and unique historical events, offer us little help in modelling the future.

Huber (1995) has also pointed out that the data used by Wilkie to parameterize his model included inflation shocks, which is inappropriate given that the model assumes a stationary mean and variance for the rate of inflation. As the assumption of stationarity is only really valid for periods of moderate inflation, the parameters should be estimated from periods which exclude inflation shocks.

We shall therefore use Wilkie's approach to model the third inflation scenario described earlier, but we shall estimate values for the parameters from UK inflation rates since 1982, excluding the inflation shock of 1980/81. This gives the following estimates:

$\mu_e = 0.047$
$\sigma_e = 0.019$
$k_e = 0.58$

The first parameter is the mean of the annual force of inflation, the second is its standard deviation, and the third is the correlation between the force of inflation in adjacent years.

However, we also require that the expected force of inflation is consistent with the bases used for pricing level and index-linked annuities, as implied by equation (1). Thus we shall ignore the estimate from past data and assume, instead, that the mean force of inflation is given by:

$$\mu_e = \delta_j - \delta_r \qquad (8)$$

2.4.4 Method of comparing level and index-linked annuities

As for the deterministic comparison, we shall assume that the pensioner buys a level annuity but only spends the income which would have been provided by an index-linked annuity, the difference being saved (or drawn from) a sinking fund.

We now consider what rate of interest is earned on the sinking fund. It would be prudent to place these savings in a secure, interest-bearing deposit account. It would also be preferable for the interest to be linked to the rate of inflation, as the purpose of the sinking fund is to provide long-term protection against inflation. In the UK, an appropriate instrument would be *Index-linked National Savings Certificates*, which provide a fixed compound return above the rate of inflation over 5-year periods. The real interest rate offered for the 7th issue of these certificates in 1993 was 3% per annum, close to the average real yield on index-linked bonds. We shall therefore assume that the sinking fund assets achieve a fixed real yield equivalent to that assumed in the pricing basis for index-linked annuities.

To simplify our simulations, we assume the pensioner receives an income payable annually in advance. It follows that the sinking fund per unit of retirement fund must satisfy the following recurrence formula:

$$S_{t+1} = (S_t + 1/\ddot{a}_x^j - E_t/\ddot{a}_x^r)\times(1+e_{t+1})\times(1+r) \tag{9}$$

where:

e_t = *random annual rate of inflation over year t*

$$E_t = \prod_{z=1}^{t}(1+e_z)$$

r = *annual real interest earned on sinking fund*

The present value of the sinking fund will be evaluated using the actual interest rate earned on the sinking fund assets, as follows:

$$PV[S_t] = \frac{S_t}{E_t\times(1+r)^t} \tag{10}$$

As e_t is random, it follows that all of E_t, S_t and $PV[S_t]$ will also be random.

2.4.5 Results of simulations

1000 simulations for projected values of S_t against t, were carried out for selected retirement ages, again setting $j=7\%$ and $r=3\%$ and using the stochastic inflation model described in Section 2.4.3. For the purpose of these projections, we assume that the sinking fund can go into debt when its assets have been exhausted.

The simulations were used to estimate for each integer value of t:

$$Pr(t,y) = Probability\{PV[S_t] < -y\}$$

for fixed values of y equal to 0, 0.1, 0.2, 0.3.

When $y=0$ $Pr(t,y)$ is the probability that the sinking fund is exhausted for a pensioner who has survived t years into retirement. When y takes a non-zero value, $Pr(t,y)$ is the probability that the present value of the sinking fund debt would have exceeded y after t years. But this is also the probability that a sinking fund with *initial assets* at retirement equal to y would have been exhausted after t years. Thus, $Pr(t,y)$ is the probability that, even with additional savings at retirement equal to y, the pensioner would not be able to maintain an inflation-linked income for t years.

We now define:

$$Pr(y) = Probability\{PV[Sinking\ Fund] < -y \text{ at start of year of death}\}$$

Assuming that the year of death does not depend on the rate of inflation, we can write:

$$Pr(y) = \sum_{t=0}^{\infty} Pr(t,y) \times \frac{d_{x+t}}{l_x} \tag{11}$$

Table 3, below, shows the values of $Pr(y)$ estimated from our simulations:

Table 3: Probability of not being able to maintain index-linked pension

Retirement age	$y=0$	$y=0.1$	$y=0.2$	$y=0.3$
$x=50$	0.38	0.17	0.05	0.01
$x=55$	0.33	0.15	0.05	0.01
$x=60$	0.31	0.14	0.05	0.02
$x=65$	0.29	0.14	0.05	0.02
$x=70$	0.24	0.11	0.05	0.02
$x=75$	0.22	0.10	0.05	0.02

The column of figures in Table 3 under $y=0$ gives the probability of exhausting the sinking fund for each retirement age. The probabilities are lower than those obtained for the deterministic comparison, as we might have expected from our earlier result in Section 2.3. The other columns show the probability of exhausting the sinking fund when it contains assets at retirement equal to y multiplied by the pensioner's retirement fund. An interesting feature of these simulations is that initial sinking fund assets equal to 20% of the pensioner's retirement fund would reduce the probability of exhaustion to 5% for all the retirement ages examined. Thus, for pensioners contemplating the purchase of a level annuity, additional liquid assets of roughly 20% of the retirement fund might ensure reasonable protection against inflation in retirement, if one ignores the possibility of a severe inflation shock.

2.5 Summary

The models we have used to compare the merits of level and index-linked annuities are based on the following premises:

1) the difference between the nominal yield on fixed interest bonds and the real yield on index-linked bonds gives an unbiased estimate of future price inflation;

2) the rate of inflation is either constant, or varies randomly about a stationary average.

The main conclusions to be drawn from our comparisons are:

- a pensioner will probably obtain more value from a level annuity than from an index-linked annuity;

- the longevity and inflation risks associated with a level annuity are significant, but reduce with the age of retirement;

- a pensioner who buys a level annuity might reduce the risk to an acceptable level by:

    a) initially spending no more income than could have been obtained from an index-linked annuity;

    b) investing additional savings of roughly 20% of the retirement fund in deposits providing a guaranteed real return.

Another factor to consider is the possibility that the nominal yield on fixed interest bonds exceeds the real yield on index-linked bonds by the expected rate of inflation *plus a risk premium*. Since their introduction in 1981, index-linked bonds have underperformed relative to fixed interest bonds.[6] Although it would be premature to infer the existence of a risk premium, it is a possibility of which those advising pensioners should be aware.

---

[6] The difference in cumulative returns over the decade ending in 1995 was approximately 2.5% per annum. Source: Bacon & Woodrow.

3. Income withdrawal versus Index-linked annuity

The income withdrawal option allows a pensioner draw income directly from the retirement fund instead of purchasing a life annuity. In the UK, a pensioner who opts for income withdrawal can defer the purchase of a life annuity to age 75 at the latest.

The income withdrawal option is a means by which pensioners can avoid being "locked into" an asset offering an income linked to government bond yields at retirement, which many individuals may find unduly restrictive.[7] It follows that most pensioners opting for income withdrawal would choose to invest their fund in assets believed to offer higher returns, such as equities.

In this Section we investigate the risks and benefits of drawing income from a fund invested entirely in equities, by comparing the projected cash flows with those from an index-linked annuity. As in Section 2, we shall first adopt a deterministic approach using a continuous-time model, and then move on to stochastic projections using a discrete-time model.

3.1 Risk premium on equity portfolio

By how much is the expected real return on a diversified equity portfolio expected to exceed the real yield on index-linked bonds? Wilkie (1994) considered this question, and concluded that a figure of 3% per annum would be reasonable for the long-term risk premium on equities. Thus, if we assume a real yield of 3% in pricing index-linked annuities, it would be reasonable to assume an expected real return of 6% on our equity portfolio.

3.2 Dividend yield and dividend growth

An equity portfolio is expected to produce an increasing stream of dividend income. As equities are not redeemable, the expected return can be determined by evaluating the present value of the projected income stream over an infinite time horizon. We now derive a formula for the real return from an equity portfolio in terms of the dividend yield and the rate of dividend growth, both of which are assumed to be stationary. We first define:

d = *stationary dividend yield on equity portfolio*

g = *stationary real annual dividend growth*

q = *real annual return earned on portfolio*

$\delta_g = \ln(1+g)$

$\delta_q = \ln(1+q)$

---

7 A.D. Wilkie said that he had generally avoided investing in pension contracts for this very reason (*The Actuary*, December 1995 issue).

As the dividend yield, d, is the ratio of dividend income to market value, we can write:

$$\int_0^{\infty} d.e^{(\delta_g - \delta_q).z}.dz = 1$$

From which it follows that:

$$\delta_q = d + \delta_g \tag{12}$$

Equation (12) implies that the real force of interest earned on an equity portfolio can be split into two components: dividend yield and dividend growth. For the UK equity market as a whole, Thornton and Wilson (1992a) have shown that real dividend growth has historically averaged approximately 1-2% per annum. However, the balance between yield and growth depends on the stocks selected: many equity funds have invested specifically to provide above average growth (hence lower yield) or above average yield (hence lower growth).

3.3 Deterministic comparisons

3.3.1 Pensioner who lives off dividends

A pensioner drawing income from an equity fund may wish to live off the dividends alone, to avoid having to sell stocks in order to meet the need for income. For such a pensioner, a high yielding equity portfolio with zero real dividend growth[8] might be preferable as the closest alternative to an index-linked annuity.

If the real dividend growth is zero, equation (12) becomes:

$$\delta_q = d \tag{13}$$

Taking the risk premium on the equity portfolio to be 3%, we now assume:

$$\delta_r = \ln(1.03)$$

$$\delta_q = \ln(1.06)$$

Thus, it follows from equation (13) that the income from the equity portfolio per unit of retirement fund is given by:

$d = \ln(1.06) = 5.8\%$

---

8 An example of such a portfolio is the M&G Equity Income Fund, which in providing an above average income yield for its investors has achieved income growth roughly line with price inflation since its formation in 1972. M&G is one of the leading Unit Trusts (mutual funds) in the UK.

The comparable income yield from an index-linked annuity is:

$$1/\bar{a}_x^r$$

which is shown below for different retirement ages, using the male PA(90) life table.

Table 4: Income yield from an index-linked annuity

Retirement age	45	50	55	60	65	70	75
Income yield	5.2%	5.8%	6.6%	7.6%	8.9%	10.8%	13.4%

Table 4 shows that the dividend income from a high-yielding equity portfolio is unlikely to match the income from an index-linked annuity at retirement ages above 50, the shortfall becoming greater as the age of retirement increases.

After retirement the real value of the fund will not change, given our assumption of a stationary dividend yield and zero real dividend growth. It follows that when the pensioner later buys an annuity, the same fund (in real terms) will be available to purchase a cheaper annuity (because the pensioner is older). If the annuity is bought m years after retirement, the income (compared with buying an annuity at retirement) increases by the proportion:

$$\Delta_m = \frac{\bar{a}_x^r}{\bar{a}_{x+m}^r} - 1 \qquad (14)$$

Table 5 , below, shows the percentage increase in income obtained by deferring the purchase of an annuity, for selected values of x and m.

Table 5: Increase in income for pensioner living off dividends

Retirement age	m=5	m=10	m=15	m=20	m=25	m=30
x=45	11.6%	26.8%	46.2%	72.4%	108.4%	158.6%
x=50	13.7%	31.1%	54.5%	86.8%	131.8%	
x=55	15.4%	36.0%	64.4%	103.9%		
x=60	17.9%	42.5%	76.8%			
x=65	20.9%	50.0%				
x=70	24.1%					

As would be expected from equation (14), a greater increase in income is achieved for a pensioner who retires later and defers the annuity purchase for longer. However, the sacrifice of income before the annuity purchase also increases with retirement age (see Table 4). In the next Section, we shall use a method of comparison to determine when a higher overall income can be obtained by drawing income from an equity fund.

3.3.2 Pensioner who matches annuity income

We now assume the pensioner draws an income from the fund which exactly matches the income from an index-linked annuity, selling off assets (or re-investing surplus dividends) as required. If the pensioner can now obtain a higher income after the purchase of an annuity, the income withdrawal option would have proved to be advantageous.

Using currency units which are linked to price inflation, we now define:

f_t = *fund at time t per unit of fund at retirement*

As the pensioner retires at $t=0$, $f_0=1$.

If assets are continually sold to maintain the same income as from an index-linked annuity, f_t must satisfy the following differential equation:

$$-\frac{df_t}{dt} + f_t.d = 1/\bar{a}_x^r \qquad (15)$$

Equation (15) yields the following solution for f_t:

$$f_t = e^{d.t} - \frac{\bar{s}_{\overline{t|}}^{\,d}}{\bar{a}_x^r} \quad , \quad where\ \bar{s}_{\overline{t|}}^{\,d} = \frac{e^{d.t}-1}{d} \qquad (16)$$

And the proportionate change in income on purchasing an annuity at age $x+m$ is given by:

$$\Delta_m = f_m.\bar{a}_x^r/\bar{a}_{x+m}^r - 1 \qquad (17)$$

Table 6, below, shows the increase in income on buying an annuity for different x and m.

Table 6: Increase in income for pensioner who matches annuity income

Retirement age	m=5	m=10	m=15	m=20	m=25	m=30
x=45	15.8%	38.1%	69.3%	115.4%	185.9%	297.1%
x=50	14.0%	32.2%	56.8%	91.0%	139.6%	
x=55	10.5%	22.6%	35.7%	47.7%		
x=60	6.0%	8.9%	3.3%			
x=65	-0.8%	-12.8%				
x=70	-11.5%					

Table 6 suggests there is a critical retirement age above which the pensioner is worse off as a result of taking the income withdrawal option. For pensioners who defer the purchase of the annuity until age 75, the projected increase in income is only 3% for a retirement age of 60, and falls to a reduction of 13% for a retirement age of 65. Thus, it appears that drawing income from an equity fund cannot be expected to provide a greater overall income for retirement ages much above 60, using the male PA(90) life table to price annuities.

It does not follow, however, that pensioners retiring at younger ages should always opt for income withdrawal, because the investment risk involved may be unacceptable. In the next Section, we attempt to quantify this risk through simulation, using a stochastic model for the return obtained from the equity portfolio.

3.4 Stochastic comparison

Deterministic projections, as used in Section 3.3, can be used to determine when a fund invested in equities is likely to provide more income than a life annuity, and the expected amount of this extra income. In this Section we use stochastic projections to estimate:

1) the probability that the pensioner is unable to match the income of an index-linked life annuity, as a result of poor investment experience over the period in which income is drawn from the equity fund;

2) the amount of additional savings at retirement that would give reasonable assurance of maintaining an inflation-linked income in spite of poor investment experience.

3.4.1 Formulae for projections

We require a stochastic model that will enable simulation of the market value of our equity portfolio and its dividend income. As in Section 2.4, we switch to a discrete-time framework in which a stochastic approach is more easily accommodated.

We define the following variables:

d_t = dividend yield on equity portfolio t years after retirement

g_t = real growth in dividend income between t-1 and t

i_t = real actuarial return on fund between t-1 and t

c_t = real growth in market value of assets between t-1 and t

We can immediately write down the following formula for c_t:

$$c_t = (1+g_t)\times\frac{d_{t-1}}{d_t} - 1 \tag{18}$$

We now assume that:

1) dividend income is distributed annually in advance, and the pensioner draws (or re-invests) assets to match the income from an index-linked annuity;

2) the sale or purchase of assets does not alter the composition of the portfolio (ie the relative weighting given to each individual stock does not change).

This leads to the following recurrence formula for f_t:

$$f_{t+1} = [f_t \times (1+d_t) - 1/\ddot{a}_x^r] \times (1+c_{t+1}) \tag{19}$$

Following Thornton and Wilson (1992b) we define the real actuarial return on the fund as:

$$i_t = (1+d_{t-1}) \times (1+g_t) - 1 \tag{20}$$

Using equations (18) and (20), equation (19) can be re-written as:

$$f_{t+1} = [f_t \times (1+d_t) - 1/\ddot{a}_x^r] \times \frac{(1+i_{t+1})}{(1+d_t)} \times \frac{d_t}{d_{t+1}} \tag{21}$$

3.4.2 Stochastic model

Our stochastic model for the equity portfolio consists of the following two components:

1) i_t is an independent, identically distributed, log-normal random variable, ie:

$$\ln(1+i_t) \sim N(\mu_i, \sigma_i) \tag{22}$$

2) $\ln(d_t)$ follows a first order autoregressive process with a normal residual, ie:

$$\ln(d_{t+1}) = \mu_d + k_d.[\ln(d_t) - \mu_d] + \sqrt{1-k_d^2}.\sigma_d.Z_t \tag{23}$$

The first component of the model focuses on the actuarial return rather then the return on market value, as historic data for the UK equity market shows that actuarial returns have been much less variable. Thus, a model based on actuarial returns is likely to be a better description of the behaviour of the UK equity market. The second component of the model is similar to the approach used by Wilkie in assuming that the dividend yield on UK equities tends to revert to a long-term average. This implies that the equity market tends to correct itself when stock prices are overvalued or undervalued relative to some par dividend yield, an assumption which is well supported by historic data for the UK equity market.

As in Section 3.3, we assume the pensioner invests in a portfolio of high-yielding stocks from which the expected real dividend growth is zero. If real dividend growth is zero and dividends are payable annual in advance, the real return on the equity portfolio, q, must be related to the initial dividend yield in the following manner:

$$d_0 \times \sum_{t=0}^{\infty} (1+q)^{-t} = 1$$

$$\Rightarrow \quad d_0 = q/(1+q)$$

And we must also assume that the par dividend yield is consistent with q, i.e.:

$$\mu_d = \ln(d_0) = \ln[q/(1+q)] \tag{24}$$

Equation (20) implies that when real dividend growth is zero, the actuarial return over any year is equal to the dividend yield at the start of the year. We therefore use the following estimate for the mean of the probability distribution given in equation (22):

$$\mu_i = \ln(1+d_0) = \ln[1+q/(1+q)] \tag{25}$$

Values for the other parameters used in our stochastic model, were estimated from representative UK equity indices covering the period 1919-1995, as follows:

$\sigma_i = 0.0675$
$\sigma_d = 0.24$
$k_d = 0.50$

The first parameter is the standard deviation of the logarithm of the actuarial return, the second is the standard deviation of the logarithm of the dividend yield at each year-end, and the third is the correlation between the logarithm of dividend yields in adjacent years.

3.4.3 Present value of surplus

We shall use the model outlined above to simulate values for the fund, f_t, at different durations from retirement. For each projection, we are interested in comparing the market value of the fund at any chosen duration with the money required to maintain an unchanged income after the purchase of an index-linked annuity.

In carrying out these simulations there are two complications:

1) the pricing basis for index-linked annuities may change over time;

2) the pensioner can time the purchase of the annuity to exploit favourable changes in investment yields.

Strictly, we also require a stochastic model for the real yield on index-linked bonds to allow for random fluctuations in the pricing basis. However, given that index-linked yields have been more stable than equity dividend yields, and that most of the uncertainty is believed to arise from variability in equity returns, it may be adequate for our purpose to assume that the pricing basis does not alter between retirement and the purchase of the annuity.

We shall also ignore the second complication, for it assumes that pensioners can judge when equities are overpriced relative to index-linked bonds, something that even experienced fund managers may find difficult. As in Section 3.3, we shall carry out projections assuming that the pensioner defers the purchase of an annuity for a fixed period.

On purchasing an annuity m years after retirement the surplus assets, u_m, are given by:

$$u_m = f_m - \frac{\ddot{a}^r_{x+m}}{\ddot{a}^r_x} \qquad (26)$$

If $u_m=0$ the fund would be just sufficient to purchase an index-linked annuity providing the same real income. We assume that u_m can be negative as well as positive, which is clearly implied from the use of our stochastic model.

We shall calculate the present value at retirement of the projected surplus or deficit, using the same real interest used to price the index-linked annuities, ie:

$$PV[u_m] = \frac{u_m}{(1+r)^m} \qquad (27)$$

3.4.4 Results of simulations

1000 simulations were carried out, for selected values of x and m, using the stochastic model described in Section 3.4.2, and assuming:

1) $r = 3\%$

2) $q = 6\%$

The initial dividend yield for each simulation and its long-term average value were as given in equation (24). Following a similar method to that employed in Section 2.4.5, the simulations were used to estimate:

$$Pr(m,y) = Probability\{PV[u_m] < -y\} \qquad (28)$$

for fixed values of y equal to 0, 0.1, 0.2, 0.3.

When $y=0$ $Pr(m,y)$ gives the probability (at any given retirement age) of not being able to maintain the same real income after using the remaining fund to purchase an index-linked annuity m years after retirement. For the purpose of this comparison we assume that the pensioner survives to this duration.

Table 7, below, shows the estimated probabilities for $y=0$.

Table 7: Estimated probabilities of $PV[u_m] < 0$

	$m=5$	$m=10$	$m=15$	$m=20$	$m=25$	$m=30$
$x=45$	0.31	0.22	0.18	0.15	0.14	0.16
$x=50$	0.33	0.26	0.23	0.20	0.24	
$x=55$	0.40	0.33	0.34	0.36		
$x=60$	0.44	0.43	0.50			
$x=65$	0.49	0.59				

Table 7 is consistent with Table 6 in showing that the income withdrawal option becomes more risky as the age of retirement increases. It is noticeable, however, that the risk involved is significant even at a retirement age as low as 45. Tables 8, 9, and 10 show the same estimated probabilities for $y=0.1$, 0.2, and 0.3, respectively.

Table 8: Estimated probabilities of $PV[u_m] < -0.1$

	$m=5$	$m=10$	$m=15$	$m=20$	$m=25$	$m=30$
$x=45$	0.16	0.11	0.08	0.06	0.06	0.08
$x=50$	0.18	0.13	0.13	0.09	0.14	
$x=55$	0.23	0.17	0.19	0.23		
$x=60$	0.28	0.27	0.34			
$x=65$	0.32	0.41				

Table 9: Estimated probabilities of $PV[u_m] < -0.2$

	$m=5$	$m=10$	$m=15$	$m=20$	$m=25$	$m=30$
$x=45$	0.07	0.03	0.02	0.02	0.02	0.04
$x=50$	0.07	0.04	0.04	0.03	0.08	
$x=55$	0.08	0.07	0.08	0.13		
$x=60$	0.12	0.13	0.18			
$x=65$	0.17	0.24				

Table 10: Estimated probabilities of $PV[u_m] < -0.3$

	m=5	m=10	m=15	m=20	m=25	m=30
x=45	0.02	0.01	0.00	0.00	0.01	0.01
x=50	0.02	0.01	0.01	0.01	0.03	
x=55	0.02	0.01	0.02	0.04		
x=60	0.03	0.04	0.08			
x=65	0.06	0.10				

Following the same reasoning used Section 2.4.5, Tables 8, 9 and 10 give probabilities of not being able to maintain an inflation-linked income for a pensioner who had additional savings at retirement equal to y times the retirement fund, and invested these savings in assets giving a guaranteed real return of 3% per annum.

Tables 7-10 show that each increment of 10% in the additional savings held at retirement significantly reduces the risk for any combination of x and m. For initial savings of 30% of the retirement fund, the risk is very small for the younger retirement ages.

3.5 Summary

In comparing income withdrawal from an equity fund with an index-linked annuity we have assumed that:

- the average return on equities will exceed the average real yield on index-linked bonds by 3% per annum;
- the pensioner is primarily interested in which option gives the higher overall income, rather than the capital protection provided by the income withdrawal option.

We can summarize the results of our comparison as follows:

- the expected overall income from the equity fund is greater for retirement ages below a critical threshold - using the male PA(90) life table, this critical age is somewhere between 60 and 65;
- the younger the retirement age, the greater the expected extra income from the equity fund, and the lower the risk of not being able to match the income from an index-linked annuity;
- additional savings invested in assets providing a guaranteed real return can significantly reduce the risk of not being able to maintain a level real income - savings of 30% of the retirement fund would reduce the risk to below 5% for most retirement ages at which a higher overall income was expected.

4. Conclusions

We have identified three important factors which affect how a pensioner's retirement fund should be invested to provide an income:

1) expected future price inflation;

2) the pensioner's expected remaining lifespan;

3) additional savings held by the pensioner.

4.1 Level annuities

The existence of price inflation has meant that the traditional life annuity, providing a level monetary income, is no longer a risk-free option. Thus, we have adopted the newer index-linked annuity as the benchmark against which other options should be measured.

However, most UK pensioners still opt for a level annuity, and this is not an irrational preference: we have shown that the odds are in favour of obtaining a higher aggregate income from a level annuity, especially at older retirement ages. However, we have also shown that the existence of price inflation poses a significant longevity risk to the recipient of a level annuity, irrespective of whether the rate of inflation is stable and predictable or variable and uncertain. Thus, the purchase of a level annuity is perhaps only advisable for individuals with additional savings: we have estimated that further assets of at least 20% of the retirement fund are necessary.

4.2 Income withdrawal

The expected remaining lifespan determines the expected term of the cash-flows required by the pensioner, and this has an important bearing on the risk/benefit trade off between the different types of asset that may be used to provide these cash-flows. We have shown that the greater the life expectancy at retirement, the greater the advantages of drawing income from an equity fund compared with a life annuity providing an income linked to bond yields. This accords with the actuarial viewpoint that equities are suitable assets to match against longer term liabilities.

There is still a significant risk, however, that an equity fund may not be able to match the income from an index-linked annuity, even at young retirement ages. Again, it appeared necessary that the pensioner should have additional savings at retirement, the minimum savings required varying between 20% and 30% of the retirement fund.

As a pensioner's actual lifespan is uncertain, the longevity risk associated with drawing income from a fund will eventually dominate other considerations. Thus, at some age the purchase of a life annuity becomes necessary. UK legislation does not allow the purchase of an annuity to be deferred beyond age 75; we have shown that a male pensioner might be unwise to defer the purchase beyond age 65.

5. Acknowledgement

The author would like to acknowledge the helpful comments made on earlier drafts of this paper by Steven Haberman, which have helped him to improve the final version.

References

CLARKSON, R S (1991) - *A non-linear stochastic model for inflation*
Proceedings of the 2nd AFIR International Colloquium, Vol 3.

HUBER, P (1995) - *A review of Wilkie's stochastic investment model*
Transactions of International Congress of Actuaries 1995, Vol 3, pages 333-364.

PARSONS, D J (1990) - *Pension schemes and best estimates*
Monograph presented to Staple Inn Actuarial Society, May 8 1990.

THORNTON P N & WILSON A F (1992a), *A realistic approach to pension funding*
Journal of the Institute of Actuaries, Vol 119, Part II, pages 229-286.

THORNTON P N & WILSON A F (1992b), *Note on the emergence of surplus in occupational pension schemes in the 1980s*
Institute of Actuaries monograph.

WILKIE, A D (1986) - *A stochastic investment model for actuarial use*
Transactions of the Faculty of Actuaries, Vol 39, pages 341-403.

WILKIE, A D (1994) - *The risk premium on ordinary shares*
British Actuarial Journal, Vol 1, Part II, pages 251-293.

CITY University

ACTUARIAL RESEARCH CENTRE

DEPARTMENT OF ACTUARIAL SCIENCE AND STATISTICS

Actuarial Research Papers

1. Haberman S. Long Term Prognosis after Cerebral Infarction. 78 pages, 1984 (out of print).

2. Haberman S. Long Term Prognosis after Subarachnoid Haemorrhage. 43 pages, 1984 (out of print).

3. Haberman S. Age Variation in Incidence of Cerebrovascular Disease. 36 pages, 1985 (out of print).

4. Kaye G.D. Taxation of Pension Funds. 11 pages, 1985.

   [Also: Haberman S. and Bloomfield D.S.F. Social Class Differences in Mortality Around 1981. SSRU Working Paper No. 55. 41 pages, 1987.]

5. Kaye G.D. Life Offices' Expenses Regression Analyses I, 98 pages, 1988. (Limited circulation).

6. Kaye G.D. Life Assurance Taxation - Comments in Response to the Board of the Inland Revenue's Consultative Document Concerning the Taxation of Life Insurance. 11 pages. 1988.

7. Verrall R.J. Bayesian Linear Models and the Claims Run-Off Triangle. 32 pages. 1988.

8. Kaye G.D. Life Offices' Expenses Regression Analyses II. 12 pages. 1989. (Limited circulation).

9. Bloomfield D.S.F. and Haberman S. The Assessment of Damages and the Measurement of Work time Lost to Sickness, Unemployment and Stoppages. 100 pages. 1989.

10. Adams A.T. Pricing Property Investment Company Shares. 14 pages. 1989.

11. Booth P.M. History of Exchange Controls in the EC with Particular Reference to the United Kingdom. 39 pages. 1989.

12. Verrall R.J. Chain Ladder and Maximum Likelihood. 12 pages. 1989.

13. Verrall R.J. On the Unbiased Estimation of Reserves from Loglinear Models. 13 pages, 1989.

14. Adams A.T. An Analysis of Investment Trust Shares, 1990. 31 pages.

15. Haberman S. Demographic Trends Towards 2000. 16 pages. 1989.

16. Haberman S. British Life Insurance: The Role of the Actuary and 1992. 30 pages. 1990.

17. Verrall R.J. Bootstrapping, Saddlepoint Approximations and the Probability of Ruin. 10 pages. July 1990.

18. Booth P.M. Theoretical Aspects of the Taxation of Real Returns from Short Term Deposits. 26 pages. June 1990.

19. Booth P.M. Practical Aspects of the Taxation of Real Interest. 38 pages, 1990.

20. Puzey A.S. Actuarial Funding in Unit-linked Life Assurance - Derivation of the Actuarial Funding Factors. 19 pages. July 1990.

21. Kaye G.D. Economies of Scale in the United Kingdom Insurance Industry. A report for the Association of British Insurers. Limited Circulation. October 1990.

22. Adams A.T. & Venmore-Rowland P. Direct or Indirect Property Investment. 71 pages. September 1990.

23. England P.D. Statistical Modelling of Excess Mortality. 124 pages. September 1990.

24. Verrall R.J. & Li Z. Negative Incremental Claims: Chain Ladder and Linear Models. November 1990.

25. Haberman S. State Pension Provision and Social Security in the United Kingdom. Presented at a Conference in Lisbon on "Social Security and Private Provision; What are the Prospects?". 40 pages. October 1990.

26. Renshaw A.E. Graduation by Generalized Linear Modelling Techniques. 42 pages. February 1991.

27. England P.D. Statistical Modelling of Excess Mortality No. 2. 170 pages. June 1991.

28. Adams, A.T. The Market Pricing of Composite Insurance Shares. 37 pages. July 1991.

29. Haberman, S. The Effect of Autoregressive Rates of Return on the Variability of Pension Contributions and Fund Levels. 58 pages. September 1991.

30. Puzey, A.S. An Investigation into the Effect of Averaging Age Nearest Birthday at the Beginning of a Rate Year to Exact Age. October 1991. 33 pages.

31. Chadburn, R.G. Bias in Select Mortality Investigations where Data are Subdivided by Age at Death. November 1991. 69 pages.

32. England P.D. and Verrall R.J. Dynamic Estimation for Models of Excess Mortality. January 1992. 19 pages.

33. Verrall R.J. A State Space Formulation of Whittaker-Henderson Graduation, with Extensions. January 1992. 13 pages.

34. Verrall R.J. Graduation by Dynamic Regression Methods. January 1992. 37 pages.

35. Gerrard R.G. and Haberman S. Stability of Pension Systems when Gains/Losses are Amortized and Rates of Return are Autoregressive. March 1992. 12 pages.

36. Haberman S. HIV, AIDS and the Approximate Calculation of Life Insurance Functions, Annuities and Net Premiums. April 1992. 28 pages.

37. Cooper D.R. Savings and Loans: An Historical Perspective. May 1992. 29 pages.

38. Verrall R.J. Dynamic Bayes Estimation for AIDS Delay Tables. May 1992. 16 pages.

39. Kaye G.D. Data Sources Relating to Life Assurance Expenses. May 1992. 39 pages. Presented to the Staple Inn Actuarial Society and Royal Statistical Society - February 1992.

40. Renshaw A.E., and Haberman S. On the Graduation Associated with a Multiple State Model for Permanent Health Insurance. May 1992.

41. England P.D. Statistical Modelling of Excess Mortality Number 3. June 1992. 163 pages.

42. Bloomfield D.S.F. and Haberman S. Male Social Class Mortality Differences Around 1981: An Extension to Include Childhood Ages. 21 pages. June 1992.

43. Berg M.P and Haberman S. Trend Analysis and Prediction Procedures for Time Nonhomogeneous Claim Processes. 33 pages. June 1992.

44. Booth P.M. The Single Market for Insurance, Free Capital Movements and attempts to Fix Exchange Rates. October 1992. 28 pages.

45. Verrall R.J. Chain Ladder with Varying Run-off Evolutions. February 1993. 15 pages.

46. Gavin J., Haberman S. and Verrall R.J. Moving Weighted Average Graduation using Kernel Estimation. November 1992. 14. pages.

47. Gavin J., Haberman S. and Verrall R.J. On the Choice of Bandwidth for Kernel Graduation. November 1992. 21 pages.

48. S. Haberman. Pension Funding with Time Delays and the Optimal Spread Period. May 1993. 13 pages.

49. S. Haberman. Stochastic Investment Returns and the Present Value of Future Contributions in a Defined Benefit Pension Scheme. May 1993. 22 pages.

50. A. Zimbidis and S. Haberman. Delay, Feedback and Variability of Pension Contributions and Fund Levels. May 1993. 19 pages.

51. S. Haberman. Pension Funding: The Effect of Changing The Frequency Valuations. June 1993. 19 pages.

52. S Haberman. HIV, AIDS Markov Chains and PHI. June 1993. 15 pages.

53. S Haberman. A Consideration of Pension Credit and Termination Insurance. June 1993. 22 pages.

54. M Z Khorasanee. Survey of Actuarial Practice in the Funding of UK Defined Benefit Pension Schemes. July 1993. 19 pages.

55. P M Booth, R G Chadburn and A S K Ong. A Utility Maximisation Approach to Asset Allocation. September 1993. 40 pages.

56. R G Chadburn. Bias in Select Mortality Investigations. August 1993. 62 pages.

57. M Z Khorasanee. A Comparison of Pension Funding Strategies by means of Simulations for a Model Scheme. August 1993. 43 pages.

58. A E Renshaw, P Hatzopolous and S Haberman. Recent Mortality Trends in Male Assured Lives. June 1993. 23 pages.

59. E Pitacco. Disability Risk Models: Towards A Unifying Approach. September 1993. 33 pages.

60. M Boskov and R J Verrall. Premium Rating by Geographic Area Using Spatial Models. September 1993. 14 pages.

61. R G Chadburn. Managing Mutual Life Offices: The Use of an Expense Ratio in New Business Decision Making and Expense Control. October 1993. 21 pages.

62. Haberman S. Pension Funding Modelling and Stochastic Investment Returns. 56 pages. March 1994.

63. Renshaw A E and Verrall R J. The Stochastic Model Underlying the Chain-Ladder Technique. 25 pages. April 1994.

64. Haberman S and Sung J-H. Dynamic Approaches to Pension Funding. 22 pages. April 1994.

65. Renshaw A.E. On the Second Moment Properties and the Implementation of Certain GLIM Based Stochastic Claims Reserving Models. 36 pages. September 1994.

66. Booth P.M., J.N. Allan, and J.W. Jang. An Evaluation of the UK Life Insurance Mismatching Test. September 1994.

67. Booth P.M. and Stroinski K. Insurance and Investment Markets in Poland. September 1994. 35 pages.

68. Ong A. A Stochastic Model for Treasury-Bills: An Extension to Wilkie's Model. September 1994. 12 pages.

69. Bloomfield D.S.F. Moving on from Undergraduate Exams to Professional Exams: Actuaries. November 1994. 22 pages.

70. Huber P. A Review of Wilkie's Stochastic Investment Model. January 1995. 22 pages.

71. Renshaw A.E. On the Graduation of 'Amounts'. January 1995. 24 pages.

72. Renshaw A.E. Claims Reserving by Joint Modelling. December 1994. 26 pages.

73. Renshaw A.E. Graduation and Generalised Linear Models: An Overview. February 1995. 40 pages.

74. Khorasanee M.Z. Simulation of Investment Returns for a Money Purchase Fund. June 1995. 20 pages.

75. Owadally M.I. and Haberman S. Finite-time Pension Fund Dynamics with Random Rates of Return. June 1995. 28 pages.

76. Owadally M.I. and Haberman S. Stochastic Investment Modelling and Optimal Funding Strategies. June 1995. 25 pages.

77. Khorasanee M.Z Applying the Defined Benefit Principle to a Defined Contribution Scheme. August 1995. 30 pages.

78. Sebastiani P. and Settimi R. Experimental Design for Non-Linear Problems. September 1995. 13 pages.

79. Verrall R.J. Whittaker Graduation and Parametric State Space Models. November 1995. 23 pages.

80. Verrall R.J. Claims Reserving and Generalised Additive Models. November 1995. 17 pages.

81. Nelder J.A. and Verrall R.J. Credibility Theory and Generalized Linear Models. November 1995. 15 pages.

82. Renshaw A.E., Haberman S. and Hatzopoulos P. On The Duality of Assumptions Underpinning The Construction of Life Tables. December 1995. 17 Pages.

83. Chadburn R.G. Use of a Parametric Risk Measure in Assessing Risk Based Capital and Insolvency Constraints for With Profits Life Insurance. March 1996. 17 Pages.

84. Haberman S. Landmarks in the History of Actuarial Science (up to 1919). March 1996. 62 Pages.

85. Renshaw A.E. and Haberman S. Dual Modelling and Select Mortality. March 1996. 30 Pages.

86. Booth P.M. Long-Term Care for the Elderly: A Review of Policy Options. April 1996. 45 Pages.

87. Huber P.P. A Note on the Jump-Equilibrium Model. April 1996. 17 Pages.

88. Haberman S and Wong L.Y.P. Moving Average Rates of Return and the Variability of Pension Contributions and Fund Levels for a Defined Benefit Pension Scheme. May 1996. 51 Pages.

89. Cooper D.R. Providing Pensions for Employees with Varied Working Lives. June 1996. 25 Pages.

90. Khorasanee M.Z. Annuity Choices for Pensioners. August 1996. 25 Pages.

Statistical Research Papers

1. Sebastiani P. Some Results on the Derivatives of Matrix Functions. December 1995. 17 Pages.

2. Dawid A.P. and Sebastiani P. Coherent Criteria for Optimal Experimental Design. March 1996. 35 Pages.

3. Sebastiani P. and Wynn H.P. Maximum Entropy Sampling and Optimal Bayesian Experimental Design. March 1996. 22 Pages.

4. Sebastiani P. and Settimi R. A Note on D-optimal Designs for a Logistic Regression Model. May 1996. 12 Pages.

ARU-ARP/arcdsk

ISBN 1 874 770 94 8