A STOCHASTIC ASSET MODEL USING VECTOR AUTO-REGRESSION

by

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ABSTRACT

In this paper, a stochastic asset model is constructed using vector auto-regression. The model covers the share dividend yield, the force of share dividend growth and the long-term interest rate. In addition, the force of price inflation is modelled using the auto-regressive conditional heteroskedastic (or ARCH) model proposed by Wilkie [1995], and is included as an exogenous variable in the vector auto-regressive (or VAR) model. The model parameters are estimated using UK annual data over the period 1946-1994.

The output from this model is then compared with that obtained using the Wilkie [1995] model, the best-known stochastic investment model in use in the UK at present.

Finally, using each of the two models in turn, the sensitivity of the results obtained from a simple stochastic asset-liability modelling exercise for a large final-salary pension scheme to changing the underlying stochastic investment model used is examined.

1. INTRODUCTION

Many actuaries remain sceptical as to the use of stochastic asset-liability modelling, believing that the results obtained owe more to the specific model used than to any underlying reality. Thus, it is very important to explore the robustness of the results of such an exercise to a change in the stochastic investment model used (preferably by means of a fundamental change in the structure of the model used, rather than just a change in the model parameters).

The principal stochastic investment model in use in the UK at present is the Wilkie model (originally proposed in 1986, and revised in 1995). This model has attracted a great deal of criticism (see, for example, Kitts [1990], Clarkson [1991], Geoghegan *et al* [1992] and Huber [1995]). However, little of this criticism has been backed up with the suggestion of a credible alternative model.

As a result, in order to explore the sensitivity of the results of a stochastic asset-liability modelling exercise to a change in the stochastic investment model used, it was necessary to develop a new model.

Subsequently, alternative stochastic asset models have been proposed by, amongst others, Dyson & Exley [1995] and Smith [1996].

2. A STOCHASTIC ASSET MODEL USING VECTOR AUTO-REGRESSION

2.1 *Introduction*

The original stochastic investment model proposed by Wilkie [1986] covered only the following four series :

1. the force of price inflation,
2. the share dividend yield,
3. the force of share dividend growth, and
4. the long-term interest rate.

Similarly, the vector auto-regressive (or VAR) model proposed below covers only the two main UK asset classes (namely, equities and long-dated fixed-interest gilts). However, it could be easily extended to include other asset classes (for example, overseas equities, UK property, UK index-linked gilts, UK cash deposits and short-dated fixed-interest gilts etc.) and other economic variables (for example, salary growth, GDP growth etc.), provided that suitable data exists from which to estimate the required parameters values.

2.2 *Model specification*

Price inflation is modelled using the auto-regressive conditional heteroskedastic (or ARCH) model proposed by Wilkie [1995] and is treated as an exogenous variable in the VAR model.

Thus, we consider :

1. the share dividend yield at time t, denoted by $Y(t)$;
2. the force of share dividend growth in year t, denoted by $K(t)$; and
3. the long-term interest rate at time t, denoted by $C(t)$.

as a vector, denoted by $\underline{\boldsymbol{X}}(t)$.

The current value of $\underline{\boldsymbol{X}}(t)$ is dependent on :

1. previous values of $\underline{X}$, given by $\{\underline{X}(s) : s = t\text{-}1, t\text{-}2, ...\}$; and
2. current and, possibly, previous values of the exogenous variable representing the force of price inflation, given by $\{I(s) : s = t, t\text{-}1, t\text{-}2, ...\}$.

More formally, a VAR(m) model of the proposed form is given by :

$$\underline{X}(t) = \underline{M} + \sum_{i=1}^{m} \Theta^i . \left[\underline{X}(t-i) - \underline{M} \right] + \sum_{i=0}^{m} \Phi^i . \left[I(t-i) - QMU \right] + \underline{\varepsilon}(t)$$

Equation 1- VAR(m) model

where

1. $\underline{X}(t) = \begin{pmatrix} Y(t) \\ K(t) \\ C(t) \end{pmatrix}$ is the (3×1) vector of series values at time t;
2. $\underline{M} = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}$ is the (3×1) vector of unconditional series means;
3. $\Theta^i = \begin{pmatrix} \theta_{11}^i & \theta_{12}^i & \theta_{13}^i \\ \theta_{21}^i & \theta_{22}^i & \theta_{23}^i \\ \theta_{31}^i & \theta_{32}^i & \theta_{33}^i \end{pmatrix}$ is the (3×3) matrix of auto-regression parameters;
4. $\Phi^i = \begin{pmatrix} \phi_1^i \\ \phi_2^i \\ \phi_3^i \end{pmatrix}$ is the (3×1) vector of price inflation influence parameters;
5. $I(t)$ is the force of price inflation in year t;
6. QMU is the mean force of price inflation (from the Wilkie [1995] ARCH model);
7. $\underline{\varepsilon}(t) = \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \varepsilon_3(t) \end{pmatrix} = L.\underline{Z}(t)$ is the (3×1) vector of multi-variate Normal random errors at time t, with mean $\underline{0}$ and variance-covariance matrix Σ;
8. $\underline{Z}(t) = \begin{pmatrix} Z_1(t) \\ Z_2(t) \\ Z_3(t) \end{pmatrix}$ is an *i.i.d.* standard multi-variate Normal random variable with mean $\underline{0}$ and variance-covariance matrix $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

9. L is the *Choleski decomposition* of $\Sigma \Rightarrow L.L^T = \Sigma$.

2.3 *A model for the force of price inflation*

2.3.1 The Wilkie [1995] AR(1) model

The widely-used AR(1) model proposed by Wilkie [1995] for the force of price inflation in year t, $I(t)$, is as follows :

$$I(t) = QMU + QA.\left[I(t-1) - QMU\right] + QE(t)$$

where

1. $I(t)$ is the force of price inflation in year t;
2. QMU is the mean force of price inflation;
3. QA is the parameter controlling the strength of the auto-regression (i.e. a high value of QA implies that the series can be expected to move slowly back towards the mean value over time, and vice-versa);
4. $QE(t) = QSD.QZ(t)$ is the random component of the force of price inflation in year t; and
5. $QZ(t)$ is a $N(0,1)$ white noise series.

The parameter values suggested by Wilkie [1995], which are obtained by fitting the proposed model to UK annual data over the proposed period 1923-1994, are :

$$\begin{aligned} QMU &= 0.047 \\ QA &= 0.58 \\ QSD &= 0.0425 \end{aligned}$$

Note that, since the force of price inflation in year t, $I(t)$, can be expressed as a linear function of the Normal random variables $\{QE(s)\}_{s=1}^{t}$, $I(t)$ has a Normal distribution.

Also, by expressing $I(t)$ as a linear function, it can be shown that the *mean function* for the process $I(t)$ is given by :

$$\mathrm{E}\left[I(t)\right] = QMU + QA^t.\left[I(0) - QMU\right]$$

Hence, letting $t \rightarrow \infty$, we have :

$$E[I(t \to \infty)] = QMU = 4.70\%$$

Also, it can be shown that the *variance function* for the process *I(t)* is given by :

$$V[I(t)] = QSD^2 . \ddot{a}(QA^2, t)$$

Again, letting $t \to \infty$, we have :

$$V[I(t \to \infty)] = \frac{QSD^2}{1 - QA^2} = 5.22\%^2$$

Figure 1 gives an empirical distribution (obtained by means of 5,000 simulations) of the shape of the probability distribution function of the force of price inflation using the standard Wilkie [1995] model after the equilibrium position has been reached. The expected normality is clearly evident.

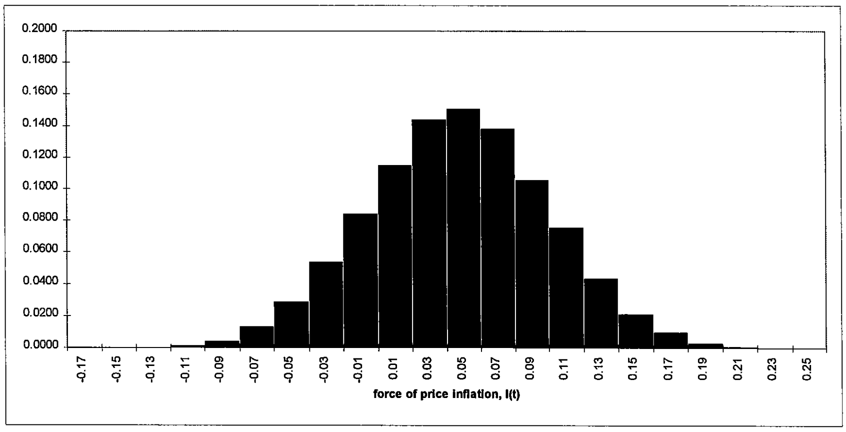

Figure 1- Empirical distribution function of long-term force of price inflation under the standard Wilkie [1995] model

Figure 2 shows 10 independent simulations of the future progress of the force of price inflation using the standard Wilkie [1995] model. The 'neutral' starting conditions suggested by Wilkie [1995] are used.

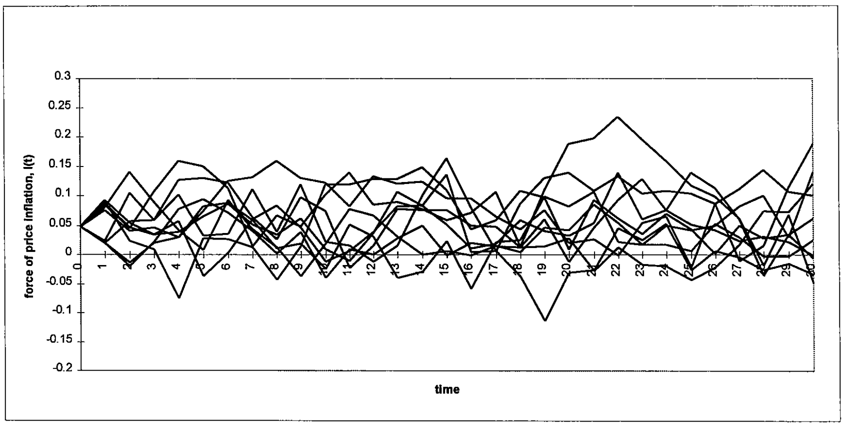

Figure 2- Ten simulations of the force of price inflation under the standard Wilkie [1995] model

2.3.2 The Wilkie [1995] ARCH model

Auto-regressive conditional heteroskedastic (or ARCH) models were introduced by Engle [1982]. The essential feature of such models is that the variance of the error component at time t is not constant (as for the standard auto-regression model above), but makes use of the information available on the progress of the series up to time t-1.

Thus, the ARCH model for the force of price inflation proposed by Wilkie [1995] reflects the (arguably sensible) notion that the variance of the innovation in year t, $I(t)$, is directly proportional to the absolute difference between the actual force of price inflation in the previous year, $I(t$-1), and the mean force of price inflation, QMU.

Thus, the ARCH model proposed by Wilkie [1995] for the force of price inflation in year t, $I(t)$, is as follows :

$$I(t) = QMU + QA.\left[I(t-1) - QMU\right] + QE(t)$$

where

1. $QE(t) = QSD(t).QZ(t)$ is the random component of the force of price inflation in year t;
2. $QSD(t) = QSA + QSB.\left[I(t-1) - QSC\right]^2$; and
3. $QZ(t)$ is a $N(0,1)$ white noise series.

The parameter values suggested by Wilkie [1995], which are again obtained by fitting the proposed model to UK annual data over the proposed period 1923-1994, are :

$$\begin{aligned} QMU &= 0.04 \\ QA &= 0.62 \\ QSA &= 0.0256^2 \\ QSB &= 0.55 \\ QSC &= 0.04 \end{aligned}$$

Note that use of an ARCH model means that the force of price inflation in year *t*, *I*(*t*), no longer has a Normal distribution.

As for the standard AR(1) model, by expressing *I*(*t*) as a linear function, it can be shown that the *mean function* for the process *I*(*t*) is given by :

$$\mathrm{E}\big[I(t)\big] = QMU + QA^t.\big[I(0) - QMU\big]$$

Hence, letting $t \to \infty$, we have :

$$\mathrm{E}\big[I(t \to \infty)\big] = QMU = 4.00\%$$

Also, as $QSC = QMU$, it can be shown that the *variance function* for the process *I*(*t*) is given by :

$$\mathrm{V}\big[I(t)\big] = QSD^2.\ddot{a}(QSB + QA^2, t)$$

Again, letting $t \to \infty$, we have :

$$\mathrm{V}\big[I(t \to \infty)\big] = \frac{QSD^2}{1 - \left(QSB + QA^2\right)} = 10.00\%^2$$

Hence, the variance of the long-term (and, indeed, the short-term) force of price inflation is significantly increased using the ARCH model.

Figure 3 gives an empirical distribution of the shape of the probability distribution function of the force of price inflation using the ARCH model after the equilibrium position has been reached. This empirical distribution is obtained using 5,000 simulations of the future progress of the force of price inflation. In comparison with the observed normality for the AR(1) model above, the probability distribution function for the ARCH model can be seen to have :

1. fatter tails; and
2. a greater concentration around the mean.

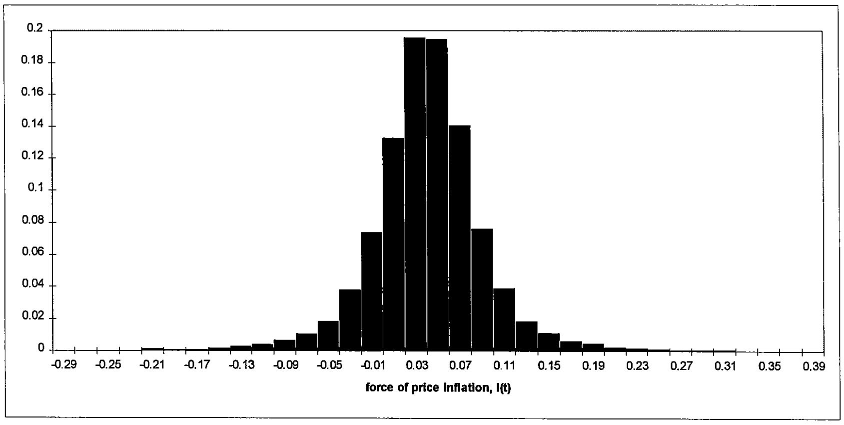

Figure 3- Empirical distribution function of long-term force of price inflation under the Wilkie [1995] ARCH model

This is a consequence of the non-stationary variance of the error term, so that the process can be expected to contain long periods when values are close to the long-term (and, consequently, the variance of the error term is low), but also occasional bursts of very high positive or negative inflation.

Figure 4 shows 10 independent simulations of the future progress of the force of price inflation using the ARCH model. As above, the 'neutral' starting conditions proposed by Wilkie [1995] are used, so that $I(0) = QMU = 4.00\%$. Comparing with Figure 2, the concentration around the long-term mean and the existence of occasional very large jumps (both upwards and downwards) can be seen.

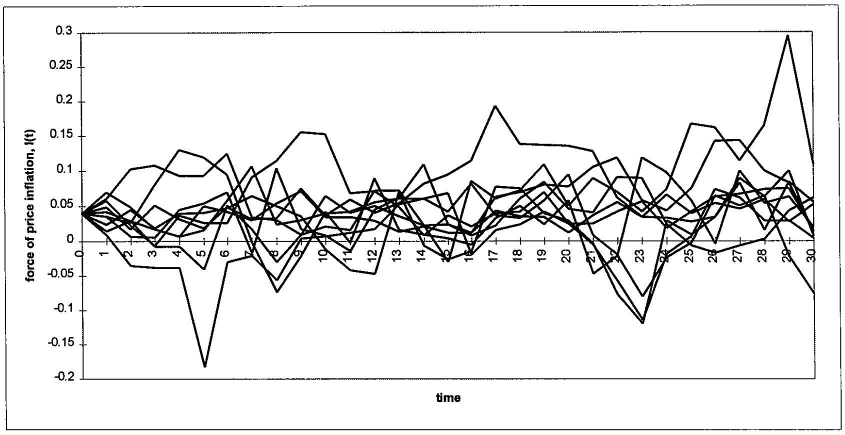

Figure 4- Ten simulations of the force of price inflation under the Wilkie ARCH model

2.4 *Parameter estimation*

Choosing the base period over which to fit any stochastic model is a far from trivial exercise. The resulting parameters may well be highly dependent on the period chosen. The base period should be long enough to give credible parameter estimates but not too long (since the further back we go, the less relevant the data is likely to be).

Generally, Wilkie [1995] uses a base period of 1923-1994 for estimating the required parameters. However, there is some evidence that the annual return achieved on equities and, to a lesser extent, the yield on long-dated fixed-interest gilts have not been stationary over this period.

Equity returns have, historically, been one of the most volatile investment variables. However, there is a strong suggestion that the annual return on equities in the last 30 years or so have been significantly higher and more volatile than previously. Whether this is a short-term phenomenon (and equity returns in future can be expected to return to past levels) or an indication that the underlying mean and variance of equity returns have changed fundamentally is still unclear.

However, as a result of this observed (potential) non-stationarity, it was decided to ignore the pre-1946 data in the parameter estimation for the VAR model, and use a base period of 1946-1994. The use of a shorter base period effectively means that greater weight is being placed on the data gathered in recent years.

A model of the form proposed in Equation 1 was fitted to UK annual data over the period 1946-1994 using least squares regression.

The model obtained is a first-order, or VAR(1) model given by :

$$
\begin{aligned}
\underline{X}(t) \quad = & \begin{pmatrix} 0.0450 \\ 0.0790 \\ 0.0800 \end{pmatrix} \\
& + \begin{pmatrix} 0.32 & 0.00 & 0.11 \\ 0.00 & 0.35 & 0.00 \\ -0.63 & 0.00 & 1.05 \end{pmatrix} . \left[\underline{X}(t-1) - \begin{pmatrix} 0.0450 \\ 0.0790 \\ 0.0800 \end{pmatrix} \right] \\
& + \begin{pmatrix} 0.06 \\ 0.30 \\ 0.07 \end{pmatrix} . \left[I(t) - 0.040 \right] + \underline{\varepsilon}(t)
\end{aligned}
$$

Equation 2- Proposed VAR(1) model

Appendix A shows the process by which the above VAR(1) model was arrived at.

It is worth reiterating at this stage that the force of price inflation in year *t*, *I(t)*, is based on the ARCH model proposed by Wilkie [1995], which was fitted to UK annual data over the period 1923-94 (rather than 1946-94 which was used to estimate the parameters of the VAR(1) model). It may be more appropriate to re-fit the ARCH model to 1946-94 data only for use with the above model, however this has not been done at this stage.

An earlier version of this model (with the external price inflation modelled using the standard Wilkie AR(1) model) was used by the Joint Actuarial Working Party in the stochastic modelling of a life office (see JAWP [1995]).

2.5 *Analysis of residuals*

Figure 5, Figure 6 and Figure 7 show the three sets of residuals obtained by fitting the VAR(1) model shown in Equation 2 to UK annual data over the period 1946-1994.

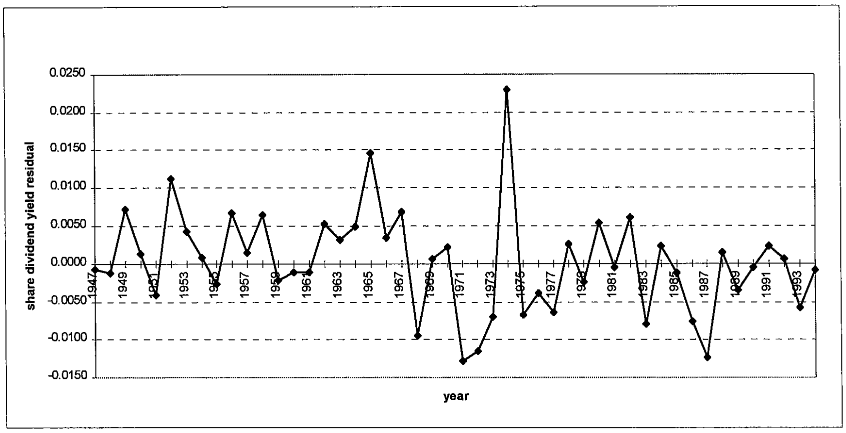

Figure 5- Residuals for the share dividend yield model

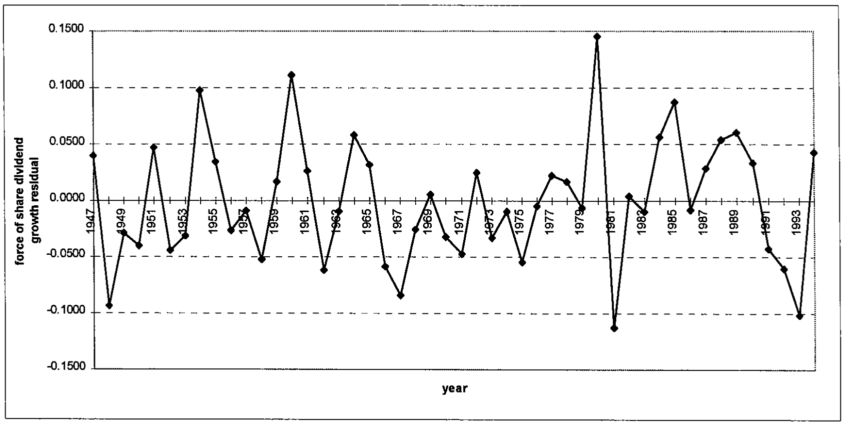

Figure 6- Residuals for the force of share dividend growth model

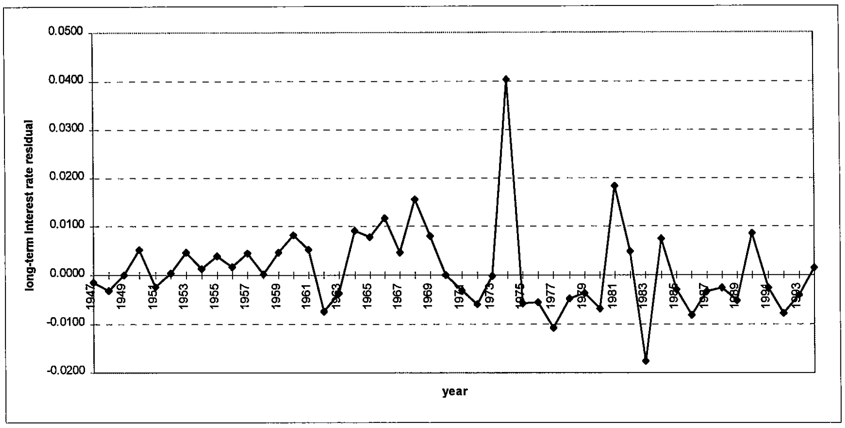

Figure 7- Residuals for the long-term interest rate model

Examination of the each of the three sets of standardised residuals reveals no significant serial correlations. None of the auto-correlation coefficients differ significantly from zero, given a standard error of $\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{49}} = 0.143$ under the null hypothesis of independence over time. Also, there is no significant cross-correlation with the previous years' residuals of any of the other series.

However, as would be expected, the contemporaneous cross-correlation between the series of residuals is significant (particularly between the share dividend yield residuals and the long-term interest rate residuals, with a correlation coefficient of 0.5).

With only 49 observations in each series, it is difficult to test satisfactorily for non-normality. One possible method is to use the Jacques-Bera test statistic (see, for example, Kendall & Stuart [1977]).

Under the null hypothesis of normality, the skewness, denoted by b_1, has the following distribution :

$$b_1 \sim N\left(0, \tfrac{6}{n}\right)$$

and the kurtosis (or fat-tailedness), denoted by b_2, has the following distribution :

$$b_2 \sim N\left(3, \tfrac{24}{n}\right)$$

Then, the Jacques-Bera test statistic, J, is given by :

$$J = n.\left[\frac{b_1^2}{6} + \frac{(b_2 - 3)^2}{24}\right] \sim \chi_2^2$$

From tables, the 95% point of the χ_2^2 distribution is 5.992.

Considering each of the three sets of residuals in turn, we have :

share dividend yield		
skewness, b_1	=	0.54
kurtosis, b_2	=	4.27
test statistic, J	=	5.67
force of share dividend growth		
skewness, b_1	=	0.31
kurtosis, b_2	=	3.10
test statistic, J	=	0.78
long-term interest rate		
skewness, b_1	=	1.80
kurtosis, b_2	=	9.96
test statistic, J	=	122.73

Thus, the residuals for both the share dividend yield and the force of share dividend growth show no significant evidence of non-normality.

However, on first inspection, the residuals for the long-term interest rate appear to be significantly positively skewed and leptokurtic (i.e. fat-tailed) in comparison with a standard Normal distribution.

But, from Figure 7, it can be seen that there is one very significant outlier. This outlier is for the year 1974 when, fuelled by very high price inflation (of well over 20%), the long-term interest rate rose sharply (from 10.4% in 1973 to 15.6% in 1974).

Removing this outlier from the data gives a test statistic of 0.92, and so we can conclude that the residuals for the long-term interest rate show little significant evidence of non-normality.

The variance-covariance matrix Σ is constructed as follows :

$$\Sigma = \begin{pmatrix} \sigma_1^2 & c_{12}.\sigma_1.\sigma_2 & c_{13}.\sigma_1.\sigma_3 \\ c_{21}.\sigma_2.\sigma_1 & \sigma_2^2 & c_{23}.\sigma_2.\sigma_3 \\ c_{31}.\sigma_3.\sigma_1 & c_{32}.\sigma_3.\sigma_2 & \sigma_3^2 \end{pmatrix}$$

where

1. σ_i is the standard deviation of residual series i; and
2. c_{ij} is the correlation coefficient between residual series i and residual series j.

Clearly, $c_{ij} = c_{ji}$.

The observed parameter values are as follows :

$$\sigma_1 = 0.007$$
$$\sigma_2 = 0.055$$
$$\sigma_3 = 0.009$$

$$c_{12} = 0.0$$
$$c_{13} = 0.5$$
$$c_{23} = 0.0$$

Hence, the variance-covariance matrix of the model residuals, Σ, is given by :

$$\Sigma = \begin{pmatrix} 0.000049 & 0.000000 & 0.000032 \\ 0.000000 & 0.003025 & 0.000000 \\ 0.000032 & 0.000000 & 0.000081 \end{pmatrix}$$

Then, the model residuals at time t, $\underline{\varepsilon}(t)$, are given by :

$$\underline{\varepsilon}(t) = L.\underline{Z}(t)$$

where

1. L is the *Choleski decomposition* of $\Sigma \Rightarrow L.L^T = \Sigma$

$$\Rightarrow L = \begin{pmatrix} 0.007000 & 0.000000 & 0.000000 \\ 0.000000 & 0.055000 & 0.000000 \\ 0.004500 & 0.000000 & 0.000061 \end{pmatrix}$$

2. $\underline{\mathbf{Z}}$(t) is an *i.i.d.* standard multi-variate Normal random variable with mean $\underline{0}$ and variance-covariance matrix I_3.

3. Comparing the Model Output from the Proposed Model with that from the Wilkie [1995] Model

In this section, we consider, in isolation, the series generated by the VAR(1) model and the Wilkie [1995] model for :

1. the share dividend yield,
2. the nominal annual return on equities,
3. the long-term interest rate, and
4. the nominal annual return on long-dated fixed-interest gilts.

However, for asset-liability modelling purposes (see Section 4), it is important to appreciate that both models are multi-variate, and so the interaction between each of the various series (in particular, the contemporaneous cross-correlations) will affect the results obtained as well as the absolute values of the series themselves.

The latter effect can be referred to as 'model parameter sensitivity', and the former effect as 'model structure sensitivity'.

3.1 *Share dividend yield*

3.1.1 Wilkie [1995] model

The model proposed by Wilkie [1995] for the share dividend yield at time t, $Y(t)$, is as follows :

$$\ln Y(t) = YW.I(t) + YN(t)$$

where

1. $Y(t)$ is the share dividend yield at time t;
2. $I(t)$ is the force of price inflation in year t, according the standard Wilkie [1995] model;
3. $YN(t) = \ln YMU + YA.\left[YN(t-1) - \ln YMU\right] + YE(t)$ is an AR(1) process independent of the price inflation process;
4. YMU is the mean share dividend yield;
5. $YE(t) = YSD.YZ(t)$ is the random component of the share dividend yield at time t; and
6. $YZ(t)$ is a $N(0,1)$ noise series.

The parameter values suggested by Wilkie [1995] are :

$$\begin{aligned} YW &= 1.8 \\ YMU &= 0.0375 \\ YA &= 0.55 \\ YSD &= 0.155 \end{aligned}$$

Note that, since $\ln Y(t)$ can be expressed as a linear function of the Normal random variables $\{QE(s)\}_{s=1}^{t}$ and $\{YE(s)\}_{s=1}^{t}$, $\ln Y(t)$ has a Normal distribution. Thus, the share dividend yield at time t, $Y(t)$, has a log-Normal distribution.

By expressing $\ln Y(t)$ as a linear function, it can be shown that the *mean function* for the process $\ln Y(t)$ is given by :

$$E[\ln Y(t)] = YW.\{QMU + QA^t.[I(0) - QMU]\} + \ln YMU + YA^t.[YN(0) - \ln YMU]$$

Hence, letting $t \to \infty$, we have :

$$E[\ln Y(t \to \infty)] = YW.QMU + \ln YMU = -3.1988$$

Also, it can be shown that the *variance function* for the process $\ln Y(t)$ is given by :

$$V[\ln Y(t)] = YW^2.QSD^2.\ddot{a}(QA^2, t) + YSD^2.\ddot{a}(YA^2, t)$$

Again, letting $t \to \infty$, we have :

$$V[\ln Y(t \to \infty)] = \frac{YW^2.QSD^2}{1 - QA^2} + \frac{YSD^2}{1 - YA^2} = 0.2080^2$$

Now, if a random variable X is such that :

$$\ln X \sim N(\mu, \sigma^2)$$

then, X has a log-Normal distribution with :

$$\begin{aligned} E[X] &= \exp(\mu + \tfrac{1}{2}\sigma^2) \\ V[X] &= \{E[X]\}^2.[\exp(\sigma^2) - 1] \end{aligned}$$

Hence, we have :

$$\begin{aligned} E[Y(t \to \infty)] &= \exp\left[YW.QMU + \ln YMU + \frac{1}{2}.\left(\frac{YW^2.QSD^2}{1 - QA^2} + \frac{YSD^2}{1 - YA^2}\right)\right] \\ &= 4.17\% \\ V[Y(t \to \infty)] &= 0.0417^2.\left[\exp\left(\frac{YW^2.QSD^2}{1 - QA^2} + \frac{YSD^2}{1 - YA^2}\right) - 1\right] \\ &= 0.88\%^2 \end{aligned}$$

Figure 8 shows the 1st, 5th, 10th, 25th, 50th, 75th, 90th, 95th and 99th percentiles of the distribution of the share dividend yield using the standard Wilkie [1995] model at annual intervals.

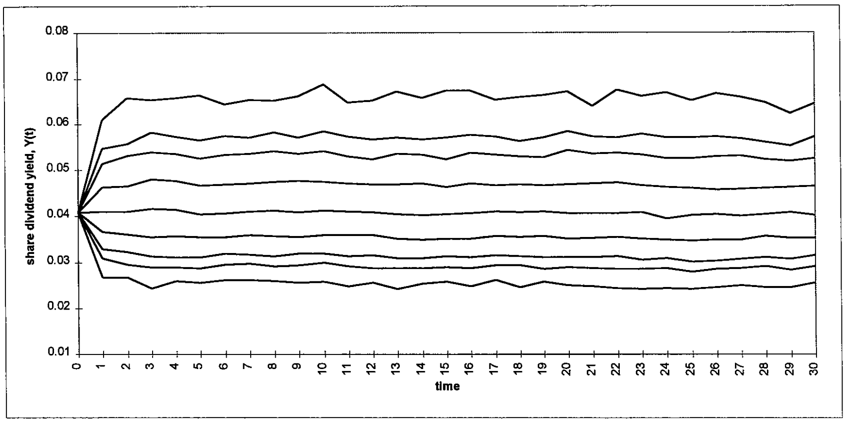

Figure 8- Percentiles of the share dividend yield distribution for the standard Wilkie [1995] model

Figure 9 gives an empirical distribution of the shape of the probability distribution function of the share dividend yield using the standard Wilkie [1995] model after the equilibrium position has been reached. The distribution is positively skewed, as would be expected for a log-Normal random variable.

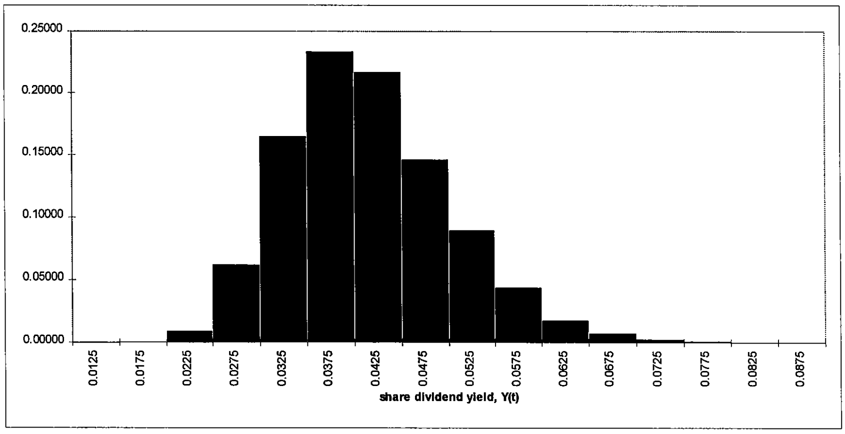

Figure 9- Empirical equilibrium distribution of the share dividend yield for the standard Wilkie [1995] model

Figure 10 shows 10 independent simulations of the future progress of the share dividend yield using the standard Wilkie [1995] model. The 'neutral' starting conditions are used.

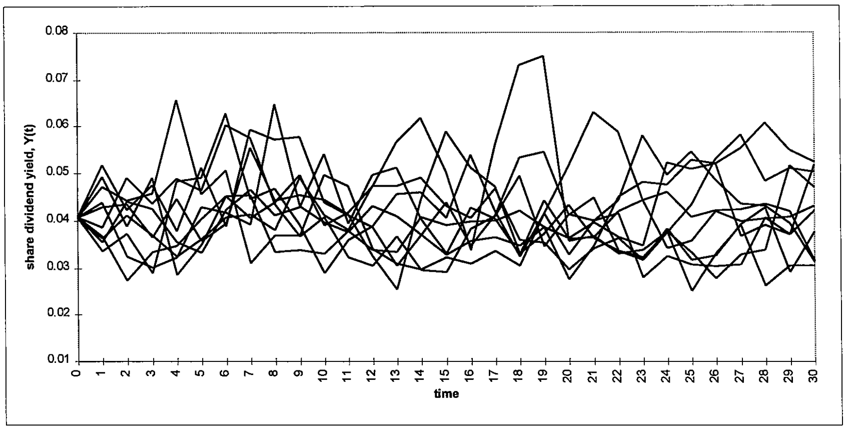

Figure 10- Ten simulations of the share dividend yield using the standard Wilkie [1995] model

3.1.2 VAR(1) model

Because of the ARCH nature of the price inflation influence, the distribution of the share dividend yield at time t, $Y(t)$, is not Normal (or log-Normal, as for the Wilkie [1995] model above) but has a non-standard distribution.

By expressing the vector $\underline{\boldsymbol{X}}(t)$ as a linear function (of previous values of $\underline{\boldsymbol{X}}$, and current and previous values of the force of price inflation), it can be shown that the *mean function* for the process $\underline{\boldsymbol{X}}(t)$ is given by :

$$\begin{aligned}\mathrm{E}[\underline{X}(t)] &= \underline{M} + \Theta^{t}.[\underline{X}(0) - \underline{M}] \\ &\quad + QA.\Theta^{t-1}.\left[I - \left(QA.\Theta^{-1}\right)\right]^{-1}.\left[I - \left(QA.\Theta^{-1}\right)^{t}\right].\Phi.[I(0) - QMU]\end{aligned}$$

where $\Theta \equiv \Theta^{1}$ and $\Phi \equiv \Phi^{0}$.

Hence, letting $t \to \infty$, we have :

$$\mathrm{E}[\underline{X}(t \to \infty)] = \underline{M} \Rightarrow \exp[Y(t \to \infty)] = M_1 = 4.50\%$$

The *variance function* for the process $\underline{X}(t)$ can be obtained similarly, and letting $t \to \infty$, it can be shown that :

$$V[Y(t \to \infty)] = 1.04\%^2$$

Thus, the difference in the mean of the equilibrium distribution of the share dividend yield for the two models is not very significant, and can be largely attributed to the different length of the base period used for parameter estimation in each case.

The increase in the variance can be attributed to :

1. the use of a shorter base period, during which the share dividend yield has been more volatile than previously; and
2. the higher variance in the ARCH model for the force of price inflation, which has filtered through (as a result of the ϕ_1^0 term) to give a corresponding increase (albeit, small) in the variance of the equilibrium distribution of the share dividend yield.

Figure 11 shows the 1^{st}, 5^{th}, 10^{th}, 25^{th}, 50^{th}, 75^{th}, 90^{th}, 95^{th} and 99^{th} percentiles of the distribution of the share dividend yield using the proposed VAR(1) model at annual intervals. These percentiles are obtained using 5,000 simulations of the future progress of the share dividend yield.

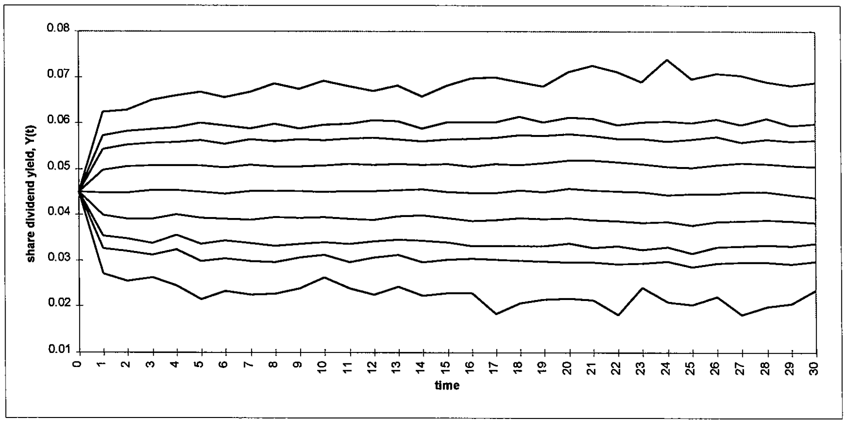

Figure 11- Percentiles of the share dividend yield distribution for the proposed VAR(1) model

Figure 12 gives an empirical distribution of the shape of the probability distribution function of the share dividend yield using the proposed VAR(1) model after the equilibrium position has been reached.

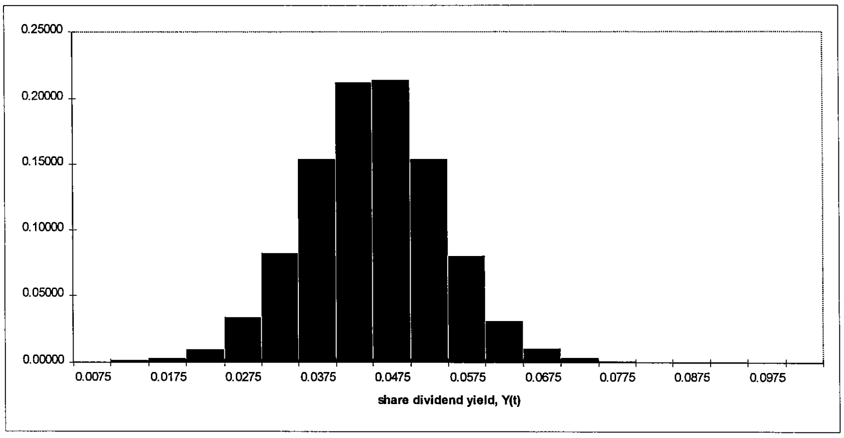

Figure 12- Empirical equilibrium distribution of the share dividend yield for the proposed VAR(1) model

From Figure 12, the distribution of the share dividend yield using the VAR(1) model appears to be approximately Normal in shape (so that the distribution is symmetrical, rather than positively skewed as was the case for the standard Wilkie [1995] model above). The reason for this is that the strength of the exogenous price inflation influence on the share dividend yield is low ($\phi_1^0 = 0.06$), and so the standard first-order vector auto-regression effect dominates.

Figure 13 shows 10 independent simulations of the future progress of the share dividend yield using the proposed VAR(1) model. The 'neutral' starting conditions are used, so that :

$$I(0) = QMU = 4.00\% \text{ and } \underline{X}(0) = \underline{M} = \begin{pmatrix} 4.50\% \\ 7.90\% \\ 8.00\% \end{pmatrix}$$

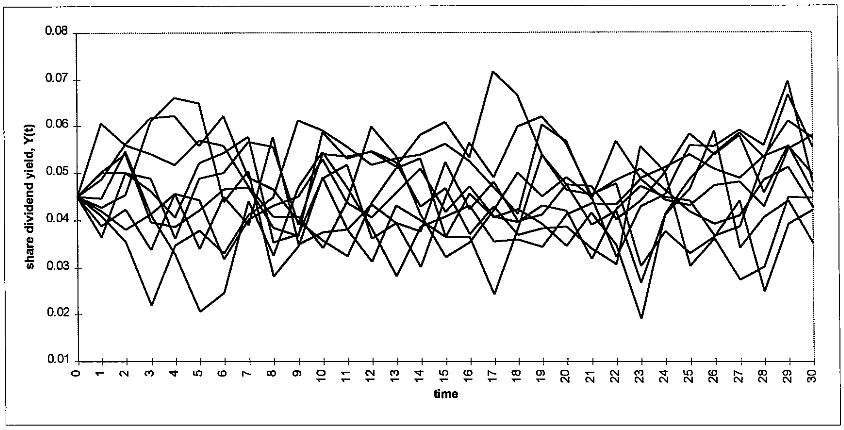

Figure 13- Ten simulations of the share dividend yield using the proposed VAR(1) model

Comparing Figure 11, Figure 12 and Figure 13 with the corresponding Figure 8, Figure 9 and Figure 10 respectively for the standard Wilkie [1995] model, it can be seen that, although the two share dividend yield models are very different in structure (in particular, the share dividend yield has a standard log-Normal distribution under the Wilkie [1995] model and an non-standard heteroskedastic - although approximate Normal - distribution under the VAR(1) model) and the parameters are estimated using different length base periods, the processes generated in each case have similar moments (both in the short-term and in the long-term).

Share dividend yields in the UK are not particularly volatile from year to year and have remained relatively stable for many years. Thus, it is not surprising that, despite the differences both in the structure of the two models and in the length of the base period used for parameter estimation, the share dividend yield series generated in each case is similar.

3.2 *Annual share index return*

The annual return on equities is likely to be one the most influential variables in any stochastic asset-liability modelling exercise for a large on-going final-salary pension scheme. This is because a substantial proportion of the assets of the scheme (possibly more than 80%) are likely to be invested in UK equities or other similar assets (e.g. overseas equities).

Furthermore, equity returns have, historically, been highly volatile in comparison with the other random variables underlying the future progress of a pension fund (for example, price inflation, earnings inflation and gilt yields - which are likely to be used in the basis for valuing discontinuance liabilities and calculating transfer values).

The nominal annual return on equities in year t, denoted by $LPR(t)$, is given by :

$$LPR(t) = \frac{PR(t)}{PR(t-1)} - 1$$

where

1. $PR(t) = PR(t-1).\frac{P(t)+D(t)}{P(t-1)}$ is the value of the index representing the nominal return achieved on a portfolio of equities up to time t;
2. $P(t) = \frac{D(t)}{Y(t)}$ is the value of the share price index at time t;
3. $D(t) = D(t-1).\exp[K(t)]$ is the value of the share dividend index at time t; and
4. $K(t)$ is the force of share dividend growth in year t.

3.2.1 Wilkie [1995] model

The distribution of the nominal annual return on equities using the Wilkie [1995] model is non-standard, and so it is much easier to estimate the parameters defining the distribution using a simulation approach.

Figure 14 shows the 1st, 5th, 10th, 25th, 50th, 75th, 90th, 95th and 99th percentiles of the distribution of the nominal annual return on equities using the Wilkie [1995] model at annual intervals. These percentiles are obtained using 5,000 simulations of the future progress of the nominal annual return on equities.

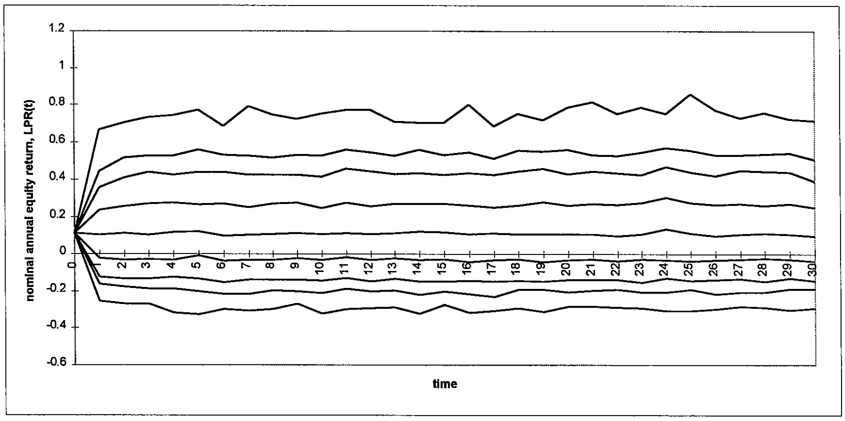

Figure 14- Percentiles of the nominal annual equity return distribution for the standard Wilkie [1995] model

From Figure 14, it can be seen that the equilibrium position is reached almost immediately. The reason for this is that, according to the Wilkie [1995] model, annual equity returns in consecutive years are largely independent (and so the equity market is considered to be highly efficient with the effect of the, essentially arbitrary, starting conditions quickly eliminated).

Figure 15 gives an empirical distribution of the shape of the probability distribution function of the nominal annual return on equities using the Wilkie [1995] model after the equilibrium position has been reached.

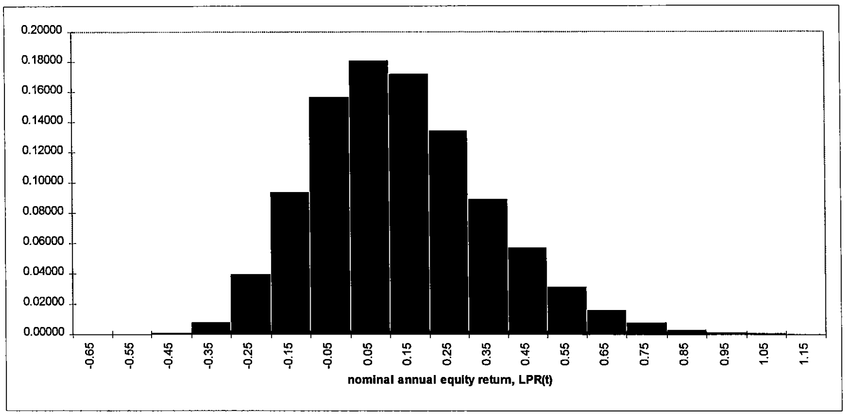

Figure 15- Empirical equilibrium distribution of the nominal annual equity return for the standard Wilkie [1995] model

The distribution is positively skewed, and it appears that the nominal annual return on equities using the Wilkie [1995] model has an approximate log-Normal distribution.

Figure 16 shows a quantile-quantile plot for the empirical equilibrium distribution of $\ln[1 + LPR(t)]$ against a standard Normal distribution. Figure 16 is approximately linear, which implies that the distribution of $\ln[1 + LPR(t)]$ is similar in shape to that of a Normal distribution. Thus, the (unknown and non-standard) distribution of the nominal annual return on equities, $LPR(t)$, is similar in shape to a log-Normal distribution.

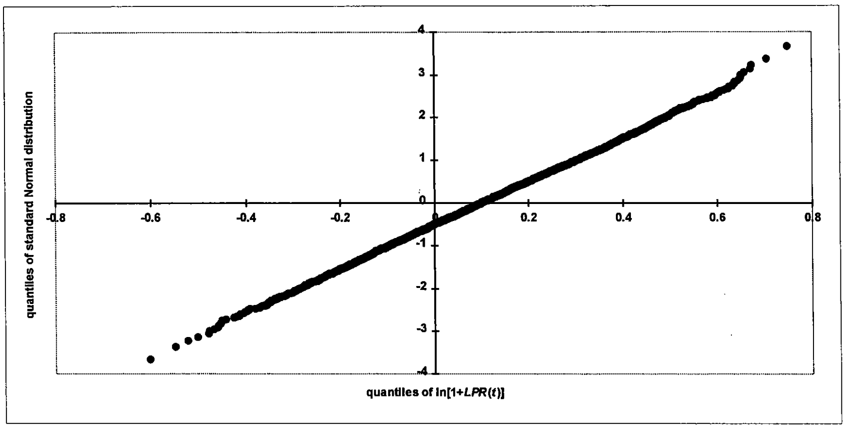

Figure 16- Quantile-quantile plot of the logarithm of the empirical equilibrium distribution of the nominal annual equity return for the standard Wilkie [1995] model

The empirical equilibrium distribution function obtained for the nominal annual return on equities using the Wilkie [1995] model is summarised in Table 1.

mean	13.1%
standard deviation	22.8%
skewness	0.60
kurtosis	3.60
1st percentile	-29.0%
5th percentile	-20.0%
10th percentile	-14.0%
25th percentile	-3.0%
50th percentile	11.0%
75th percentile	27.0%
90th percentile	43.0%
95th percentile	53.0%
99th percentile	75.0%

Table 1- Empirical equilibrium distribution of the nominal annual equity return for the standard Wilkie [1995] model

Figure 17 shows 10 independent simulations of the future progress of the nominal annual return on equities using the Wilkie [1995] model. The 'neutral' starting conditions are used.

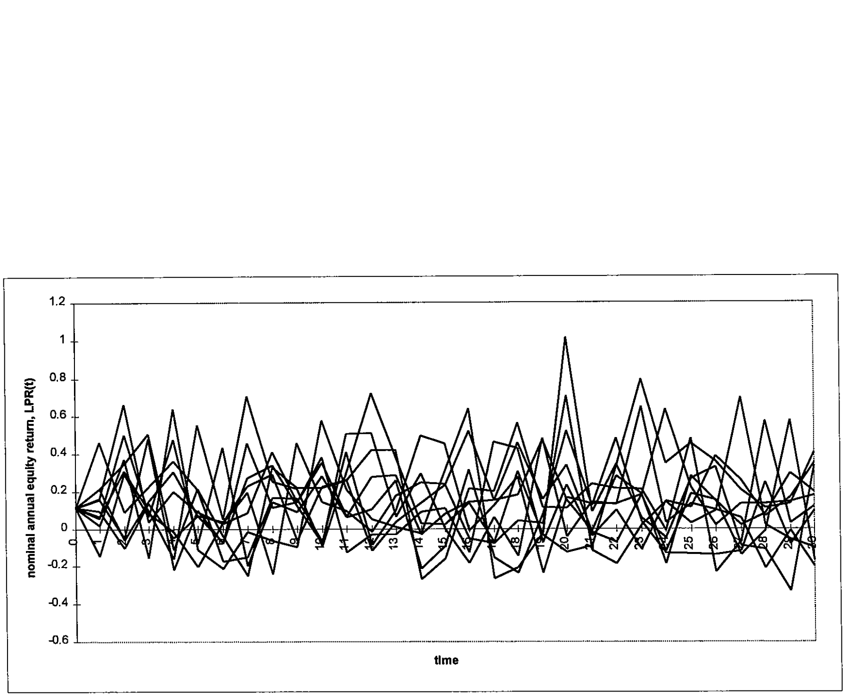

Figure 17- Ten simulations of the nominal annual equity return using the standard Wilkie [1995] model

3.2.2 VAR(1) model

The distribution of the nominal annual return on equities using the VAR(1) model is also non-standard, and so it is much easier to estimate the parameters defining the distribution using a simulation approach.

Figure 18 shows the 1^{st}, 5^{th}, 10^{th}, 25^{th}, 50^{th}, 75^{th}, 90^{th}, 95^{th} and 99^{th} percentiles of the distribution of the nominal annual return on equities using the VAR(1) model at annual intervals. These percentiles are obtained using 5,000 simulations of the future progress of the nominal annual return on equities.

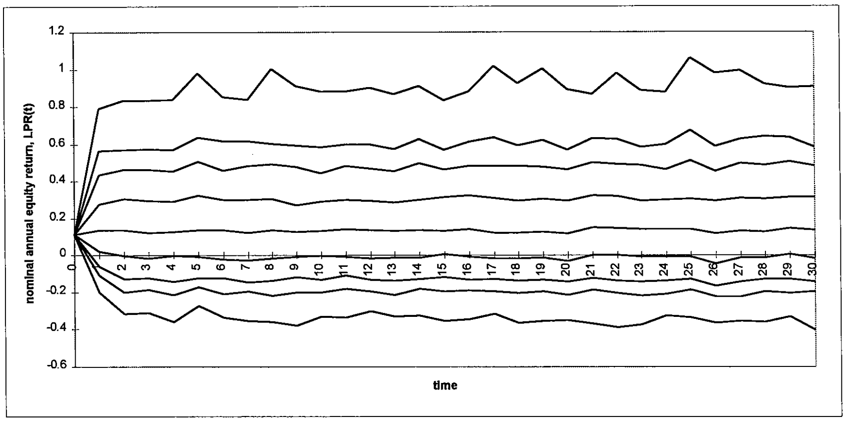

Figure 18- Percentiles of the nominal annual equity return distribution for the proposed VAR(1) model

From Figure 18, it can be seen that, as for the Wilkie [1995] model, the equilibrium position is reached almost immediately, implying that, according to the VAR(1) model, annual equity returns in consecutive years are largely independent.

Figure 19 gives an empirical distribution of the shape of the probability distribution function of the nominal annual return on equities using the VAR(1) model after the equilibrium position has been reached. As for the Wilkie [1995] model, the distribution is positively skewed.

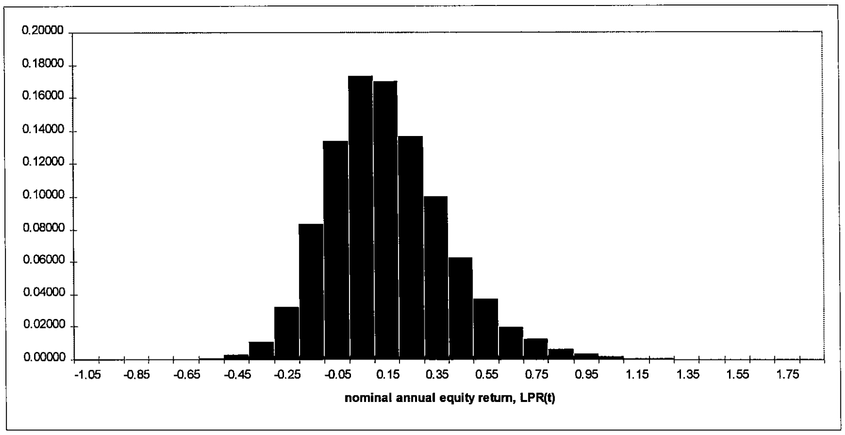

Figure 19- Empirical equilibrium distribution of the nominal annual equity return for the proposed VAR(1) model

Unlike the Wilkie [1995] model, however, it can be shown that the nominal annual return on equities using the VAR(1) model does not have an approximate log-Normal distribution. As a result of the ARCH nature of the price inflation influence on the annual equity return, the distribution is fat-tailed in comparison to a standard log-Normal distribution.

Figure 20 shows a quantile-quantile plot for the empirical equilibrium distribution of $\ln[1 + LPR(t)]$ against a standard Normal distribution. Figure 20 is has an slight 'S' shape, which implies that the distribution of $\ln[1 + LPR(t)]$ is fat-tailed in comparison to that of a Normal distribution.

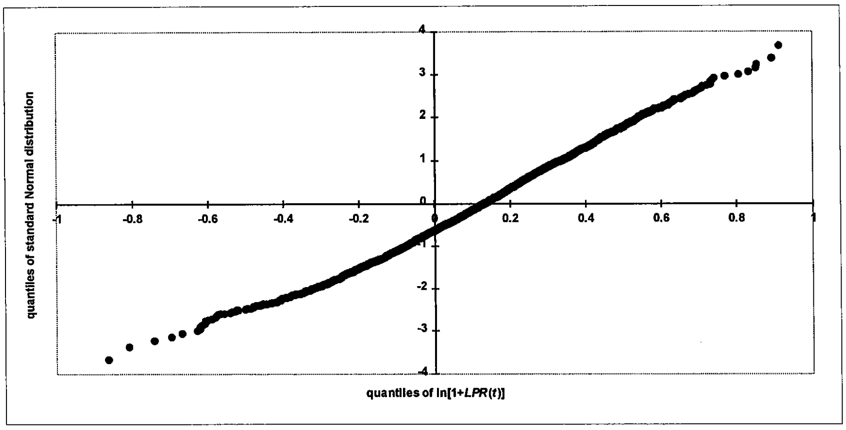

Figure 20- Quantile-quantile plot of the logarithm of the empirical equilibrium distribution of the nominal annual equity return for the proposed VAR(1) model

The empirical equilibrium distribution function obtained for the nominal annual return on equities using the VAR(1) model is summarised in Table 2. The corresponding values obtained using the Wilkie [1995] model are shown for comparison.

	VAR(1) model	Wilkie [1995] model
mean	16.1%	13.1%
standard deviation	27.4%	22.8%
skewness	2.40	0.60
kurtosis	33.00	3.60
1^{st} percentile	-35.0%	-29.0%
5^{th} percentile	-20.0%	-20.0%
10^{th} percentile	-13.0%	-14.0%
25^{th} percentile	-1.0%	-3.0%
50^{th} percentile	13.0%	11.0%
75^{th} percentile	30.0%	27.0%
90^{th} percentile	48.0%	43.0%
95^{th} percentile	61.0%	53.0%
99^{th} percentile	91.0%	75.0%

Table 2- Empirical equilibrium distribution of the nominal annual equity return for the proposed VAR(1) model

From Table 2, it can be seen that both the mean and the variance of nominal annual return on equities are significantly higher using the VAR(1) model rather than the

Wilkie [1995] model (although the mean-variance ratio is reasonably robust). The main reason for this difference is the shorter base period used for parameter estimation in the VAR(1) model which, as mentioned previously, gives more weight to recent experience (when annual rates of equity return have been higher and more volatile, principally as a result of higher and more volatile annual rates of dividend growth).

A difference of 3% in the expected annual return on equities means that, all other things being equal, for a pension fund investing predominantly in equities, the size of the fund expected to have been built up at any future date will be (significantly) higher if investment experience is assumed to follow the series generated using the VAR(1) model (rather than the Wilkie [1995] model).

Whether this difference will have a material impact on the results of an asset-liability modelling exercise will depend both on the purpose of the exercise and on the interaction of the annual equity return with the other investment variables modelled.

Figure 21 shows 10 independent simulations of the future progress of the nominal annual return on equities using the VAR(1) model. The 'neutral' starting conditions are used.

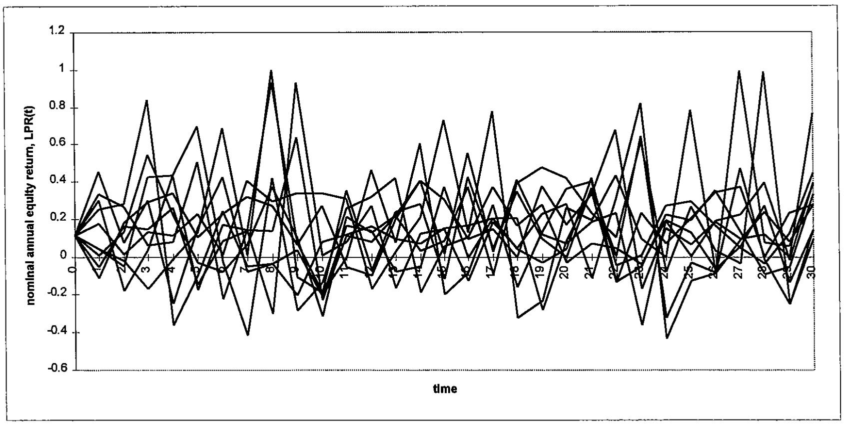

Figure 21- Ten simulations of the nominal annual equity return using the proposed VAR(1) model

3.3 *Long-term interest rate*

The expressions 'long-term interest rate' and 'yield on long-dated fixed-interest gilts' are used interchangeably.

3.3.1 Wilkie [1995] model

The model proposed by Wilkie [1995] for the long-term interest rate at time t, $C(t)$, is as follows :

$$C(t) = CW.CM(t) + CMU.\exp[CN(t)]$$

where

1. $C(t)$ is the long-term interest rate at time t;
2. $CM(t) = CD.I(t) + (1 - CD).CM(t-1)$;
3. $CN(t) = CA.CN(t-1) + CY.YE(t) + CE(t)$ is a process independent of the price inflation process;
4. CMU is the mean *real* long-term interest rate;
5. $CE(t) = CSD.CZ(t)$ is the random component of the long-term interest rate at time t; and
6. $CZ(t)$ is a $N(0,1)$ noise series.

The parameter values suggested by Wilkie [1995] are :

$$\begin{aligned} CW &= 1.0 \\ CD &= 0.045 \\ CMU &= 0.0305 \\ CA &= 0.9 \\ CY &= 0.34 \\ CSD &= 0.185 \end{aligned}$$

Wilkie [1995] shows that the process $C(t)$ can be expressed as the sum of two independent processes, $CM(t)$ and $CS(t)$, where :

$$\begin{aligned} CM(t) &= CD.I(t) + (1 - CD).CM(t-1) \\ \ln CS(t) &= \ln CMU + CN(t) \end{aligned}$$

Note that :

1. since $CM(t)$ can be expressed as a linear function of the Normal random variables $\{QE(s)\}_{s=1}^{t}$, $CM(t)$ has a Normal distribution; and
2. since $\ln CS(t)$ can be expressed as a linear function of the Normal random variables $\{CE(s)\}_{s=1}^{t}$ and $\{YE(s)\}_{s=1}^{t}$, $CS(t)$ has a log-Normal distribution.

Now, according to Wilkie [1995], "... the of sum of a Normal and a log-Normal (random variable) is distributed neither normally or log-normally, but presumably somewhere in between ...", so that the distribution of the long-term interest rate is non-standard.

By expressing *CM(t)* and $\ln CS(t)$ as linear functions, it can be shown that the *mean function* for each process is given by :

$$\begin{aligned} \mathrm{E}\left[CM(t)\right] &= CD.\ddot{a}(CDC,t).QMU + CDC^{t}.CM(0) \\ &\quad + CD.CDC^{t-1}.QA.\ddot{a}(QC,t).\left[I(0) - QMU\right] \\ \mathrm{E}\left[\ln CS(t)\right] &= CA^{t}.CN(0) + \ln CMU \end{aligned}$$

where $CDC = 1 - CD$ and $QC = \dfrac{QA}{CDC}$.

Also, it can be shown that the *variance function* for each process is given by :

$$\begin{aligned} \mathrm{V}\left[CM(t)\right] &= \frac{CD^2.QSD^2}{1-QC^2}.\left[\ddot{a}(CDC^2,t) - 2.QC.\ddot{a}(CDC.QA,t) + QC^2.\ddot{a}(QA^2,t)\right] \\ \mathrm{V}\left[\ln CS(t)\right] &= \left(CY^2.YSD^2 + CSD^2\right).\ddot{a}(CA^2,t) \end{aligned}$$

Then, as $\ln CS(t)$ has a Normal distribution, we have :

$$\begin{aligned} \mathrm{E}\left[CS(t)\right] &= \exp\left\{\mathrm{E}\left[\ln CS(t)\right] + \frac{1}{2}.\mathrm{V}\left[\ln CS(t)\right]\right\} \\ \mathrm{V}\left[CS(t)\right] &= \left\{\mathrm{E}\left[CS(t)\right]\right\}^2.\left(\exp\left\{\mathrm{V}\left[\ln CS(t)\right]\right\} - 1\right) \end{aligned}$$

As *CM(t)* and $\ln CS(t)$ are independent, we have :

$$\mathrm{E}\left[C(t)\right] = \mathrm{E}\left[CM(t)\right] + \mathrm{E}\left[CS(t)\right] \text{ and } \mathrm{V}\left[C(t)\right] = \mathrm{V}\left[CM(t)\right] + \mathrm{V}\left[CS(t)\right]$$

Hence, letting $t \to \infty$, it can be easily shown that :

$$\mathrm{V}\left[C(t \to \infty)\right] = 8.06\% \text{ and } \mathrm{V}\left[C(t \to \infty)\right] = 2.15\%^2$$

Figure 22 shows the 1st, 5th, 10th, 25th, 50th, 75th, 90th, 95th and 99th percentiles of the distribution of the long-term interest rate using the standard Wilkie [1995] model at annual intervals.

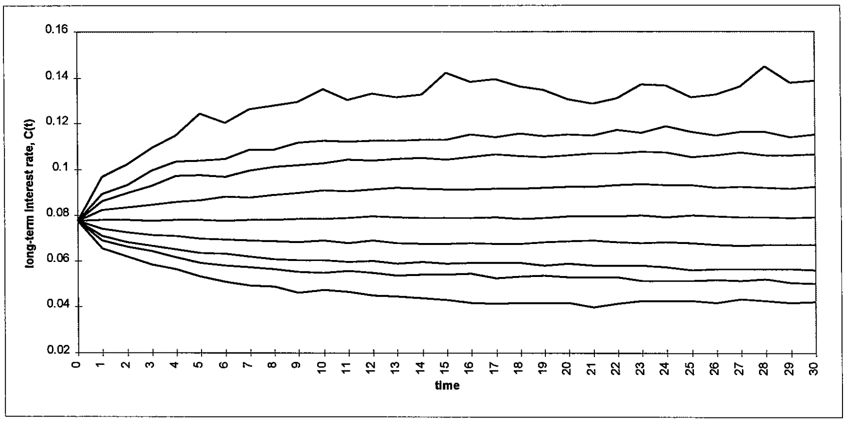

Figure 22- Percentiles of the long-term interest rate distribution for the standard Wilkie [1995] model

Figure 23 gives an empirical distribution of the shape of the probability distribution function of the long-term interest rate using the standard Wilkie [1995] model after the equilibrium position has been reached.

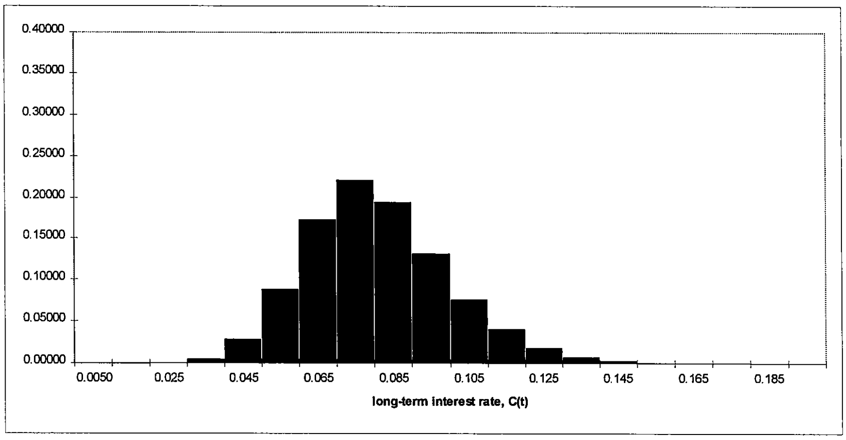

Figure 23- Empirical equilibrium distribution of the long-term interest rate for the standard Wilkie [1995] model

From Figure 23, it can be seen that the distribution of the long-term interest rate for the Wilkie [1995] model is positively skewed. However, the distribution can be shown to be less fat-tailed than a standard log-Normal distribution. This suggests that Wilkie's assumption as to the shape of a distribution made up from the sum of a Normal and a log-Normal random variable is reasonable.

Figure 24 shows 10 independent simulations of the future progress of the long-term interest rate using the standard Wilkie [1995] model. The 'neutral' starting conditions are used.

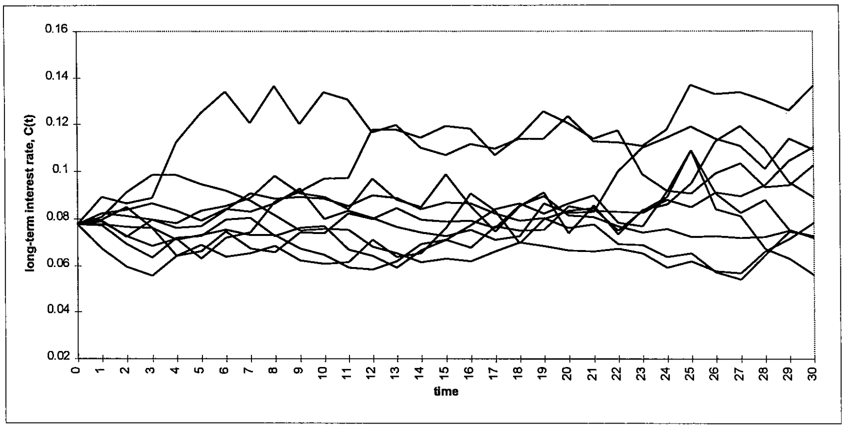

Figure 24- Ten simulations of the long-term interest rate using the standard Wilkie [1995] model

3.3.2 VAR(1) model

Because of the ARCH nature of the price inflation influence, the distribution of the long-term interest rate at time t, $C(t)$, has a non-standard distribution.

From Section 3.1.2, the *mean function* for the process $\underline{X}(t)$ is given by :

$$\begin{aligned} E[\underline{X}(t)] &= \underline{M} + \Theta^t.[\underline{X}(0) - \underline{M}] \\ &\quad + QA.\Theta^{t-1}.[I - (QA.\Theta^{-1})]^{-1}.[I - (QA.\Theta^{-1})^t].\Phi.[I(0) - QMU] \end{aligned}$$

where $\Theta \equiv \Theta^1$ and $\Phi \equiv \Phi^0$.

Hence, letting $t \to \infty$, we have :

$$\mathrm{E}\big[\underline{X}(t \to \infty)\big] = \underline{M} \Rightarrow \mathrm{E}\big[C(t \to \infty)\big] = M_3 = 8.00\%$$

Also, from Section 3.1.2, the *variance function* for the process $\underline{X}(t)$ can be obtained similarly, and letting $t \to \infty$, it can be shown that :

$$\mathrm{V}\big[C(t \to \infty)\big] = 1.20\%^2$$

Thus, as for the share dividend yield series, the difference in the mean of the equilibrium distribution of the long-term interest rate for the two models is not very significant, and can be largely attributed to the different length of the base period used for parameter estimation in each case.

However, in this case, the variance of the equilibrium distribution of the long-term interest rate is significantly lower using the VAR(1) model.

The main reason for this is the shorter base period used for parameter estimation. Up until the mid-1950s, government intervention ensured that the long-term interest rate in the UK was constrained largely between 3% and 4.5% (i.e. well below the observed long-term mean level). Thus, all other things being equal, using this data in the parameter estimation process (as Wilkie has done) would be expected to lead to a higher estimate of variance of the long-term interest rate.

Figure 25 shows the 1st, 5th, 10th, 25th, 50th, 75th, 90th, 95th and 99th percentiles of the distribution of the long-term interest rate using the proposed VAR(1) model at annual intervals. These percentiles are obtained using 5,000 simulations of the future progress of the long-term interest rate.

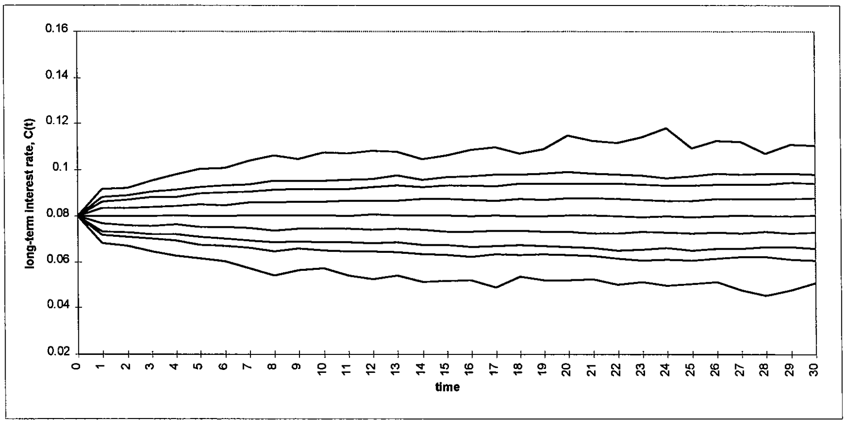

Figure 25- Percentiles of the long-term interest rate distribution for the proposed VAR(1) model

Figure 26 gives an empirical distribution of the shape of the probability distribution function of the long-term interest rate using the proposed VAR(1) model after the equilibrium position has been reached.

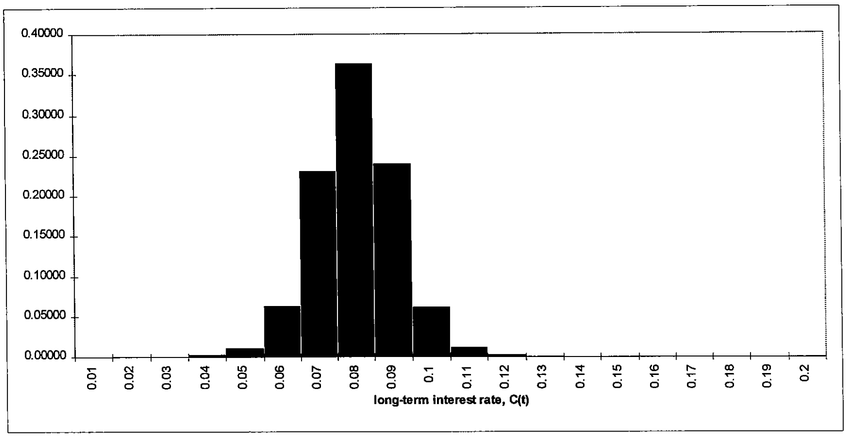

Figure 26- Empirical equilibrium distribution of the long-term interest rate for the proposed VAR(1) model

As for the share dividend yield in Section 3.1.2, the distribution of the long-term interest rate using the VAR(1) model appears to be approximately Normal in shape (so that, again, the distribution is symmetrical, rather than positively skewed as was the case for the standard Wilkie [1995] model above). The reason for this is that the strength of the exogenous price inflation influence on the long-term interest rate is low ($\phi_3^0 = 0.07$), and so the standard first-order vector auto-regression effect dominates.

Figure 27 shows 10 independent simulations of the future progress of the long-term interest rate using the proposed VAR(1) model. The 'neutral' starting conditions are used.

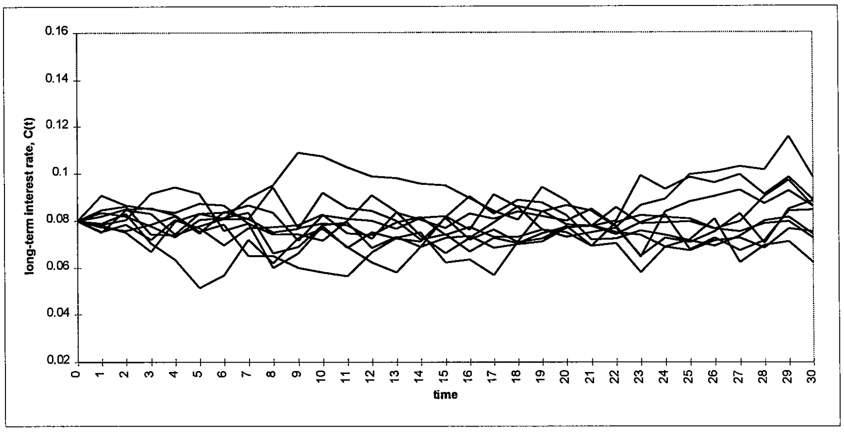

Figure 27- Ten simulations of the long-term interest rate using the proposed VAR(1) model

Although not immediately apparent from Figure 27, because of the ARCH nature of the price inflation influence on the long-term interest rate using the VAR(1) model, the resulting series will contain occasional large jumps (both up and down) in the long-term interest rate series (as a result of corresponding large jumps in the force of price inflation). This effect is damped somewhat due to the size of the ϕ_3^0 parameter, but will still be significant.

As a result, although the long-term variance of the long-term interest rate is lower using the VAR(1) model than the Wilkie [1995] model, the one-step ahead variance of the long-term interest rate (i.e. $\mathrm{V}[C(t+1)/C(t)]$) is significantly higher. The effect of this on the variance of the annual return on long-dated fixed-interest gilts is considered below.

3.4 *Annual long-dated fixed-interest gilt index return*

The nominal annual return on long-dated fixed-interest gilts in year t, denoted by $LCR(t)$, is given by :

$$LCR(t) = \frac{CR(t)}{CR(t-1)} - 1$$

where $CR(t) = CR(t-1).\left[\frac{1}{C(t)} + 1\right].C(t-1)$ is the value of the index representing the nominal return achieved on a portfolio of long-dated fixed-interest gilts up to time t.

3.4.1 Wilkie [1995] model

The distribution of the nominal annual return on long-dated fixed-interest gilts using the Wilkie [1995] model is non-standard, and so it is much easier to estimate the parameters defining the distribution using a simulation approach.

Figure 28 shows the 1^{st}, 5^{th}, 10^{th}, 25^{th}, 50^{th}, 75^{th}, 90^{th}, 95^{th} and 99^{th} percentiles of the distribution of the nominal annual return on long-dated fixed-interest gilts using the Wilkie [1995] model at annual intervals. These percentiles are obtained using 5,000 simulations of the future progress of the nominal annual return on long-dated fixed-interest gilts.

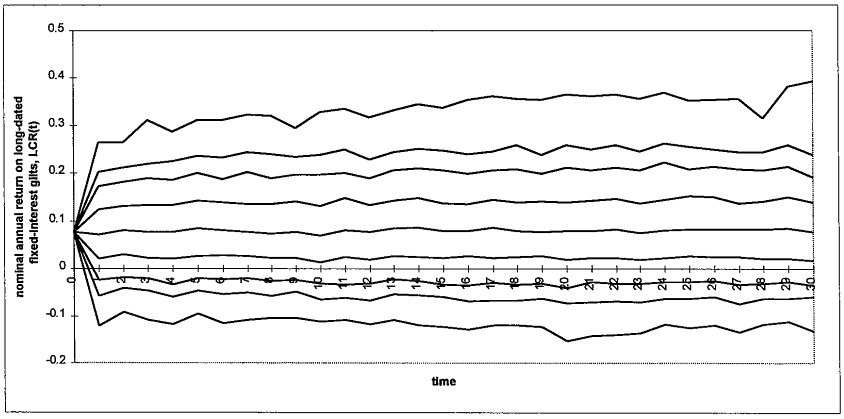

Figure 28- Percentiles of the nominal annual long-dated fixed-interest gilt return distribution for the standard Wilkie [1995] model

From Figure 28, it can be seen that, as in Section 3.2.1 for the nominal annual return on equities, the equilibrium position is reached almost immediately. The reason for this is that, according to the Wilkie [1995] model, annual long-dated fixed-interest gilt returns in consecutive years are largely independent (and so the gilt market is also considered to be highly efficient with the effect of the, essentially arbitrary, starting conditions quickly eliminated).

Figure 29 gives an empirical distribution of the shape of the probability distribution function of the nominal annual return on long-dated fixed-interest gilts using the Wilkie [1995] model after the equilibrium position has been reached. As for equities, the distribution is positively skewed.

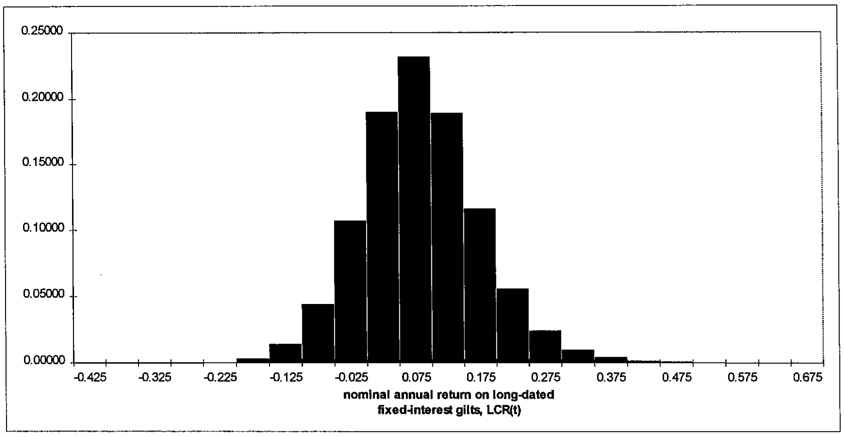

Figure 29- Empirical equilibrium distribution of the nominal annual long-dated fixed-interest gilt return for the standard Wilkie [1995] model

Figure 30 shows a quantile-quantile plot for the empirical equilibrium distribution of $\ln[1+LCR(t)]$ against a standard Normal distribution. Figure 30 has an 'S' shape (particularly in the upper tail), which implies that the distribution of $\ln[1+LCR(t)]$ is slightly fat-tailed (particularly in the upper tail) in comparison to that of a Normal distribution. Thus, the distribution of the nominal annual return on long-dated fixed-interest gilts, *LCR*(*t*), is slightly fat-tailed in comparison that of a log-Normal distribution.

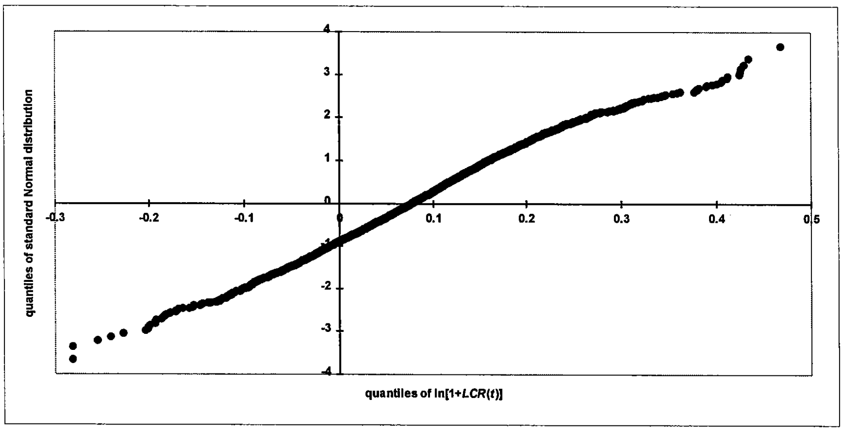

Figure 30- Quantile-quantile plot of the logarithm of the empirical equilibrium distribution of the nominal annual long-dated fixed-interest gilt return for the standard Wilkie [1995] model

The empirical equilibrium distribution function obtained for the nominal annual return on long-dated fixed-interest gilts using the Wilkie [1995] model is summarised in Table 3.

mean	8.4%
standard deviation	9.4%
skewness	0.40
kurtosis	4.20
1st percentile	-12.0%
5th percentile	-6.0%
10th percentile	-3.0%
25th percentile	2.0%
50th percentile	8.0%
75th percentile	14.0%
90th percentile	20.0%
95th percentile	24.0%
99th percentile	34.0%

Table 3- Empirical equilibrium distribution of the nominal annual long-dated fixed-interest gilt return for the standard Wilkie [1995] model

Comparing Table 3 with the corresponding Table 1 obtained for the nominal annual return on equities, it can be seen that, according to the Wilkie [1995] model :

1. equities can be expected to give significantly higher annual returns than long-dated fixed-interest gilts (13.1% p.a. compared with 8.4% p.a.); and
2. the annual return on equities is significantly more volatile than the annual return on long-dated fixed-interest gilts (standard deviation of 22.8% p.a. compared with 9.4% p.a.).

These results are intuitive (i.e. more volatile, high-risk asset classes would be expected to give a higher return) and are supported by past experience.

Figure 31 shows 10 independent simulations of the future progress of the nominal annual return on long-dated fixed-interest gilts using the Wilkie [1995] model. The 'neutral' starting conditions are used.

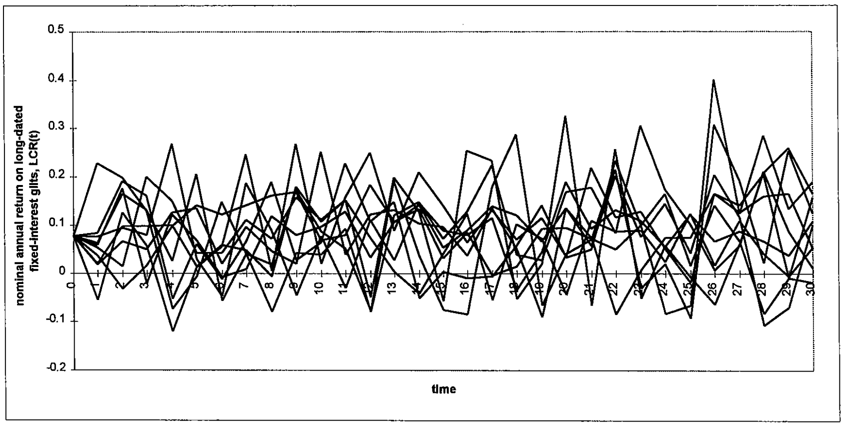

Figure 31- Ten simulations of the nominal annual long-dated fixed-interest gilt return using the standard Wilkie [1995] model

3.4.2 VAR(1) model

The distribution of the nominal annual return on long-dated fixed-interest gilts using the VAR(1) model is also non-standard, and so it is much easier to estimate the parameters defining the distribution using a simulation approach.

Figure 32 shows the 1st, 5th, 10th, 25th, 50th, 75th, 90th, 95th and 99th percentiles of the distribution of the nominal annual return on long-dated fixed-interest gilts using the VAR(1) model at annual intervals. These percentiles are obtained using 5,000 simulations of the future progress of the nominal annual return on long-dated fixed-interest gilts.

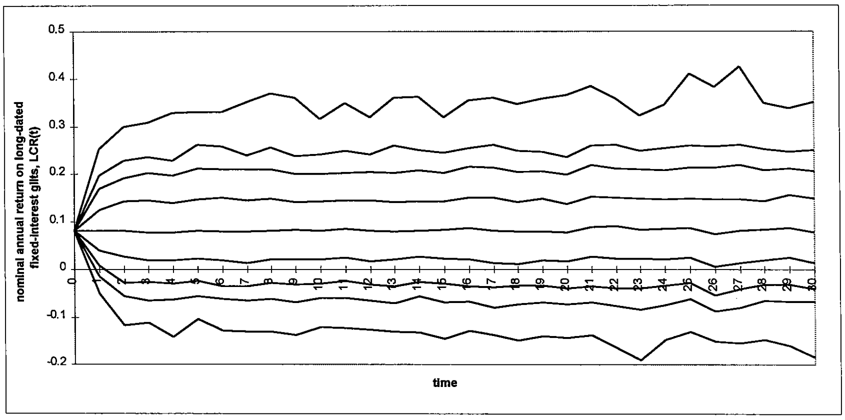

Figure 32- Percentiles of the nominal annual long-dated fixed-interest gilt return distribution for the proposed VAR(1) model

From Figure 32, it can be seen that the equilibrium position is reached fairly quickly (although not as quickly as was the case for equity return according to the VAR(1) model), implying that, annual long-term fixed-interest gilt returns in consecutive years are largely independent.

Figure 33 gives an empirical distribution of the shape of the probability distribution function of the nominal annual return on long-dated fixed-interest gilts using the VAR(1) model after the equilibrium position has been reached. As for the Wilkie [1995] model, the distribution is positively skewed.

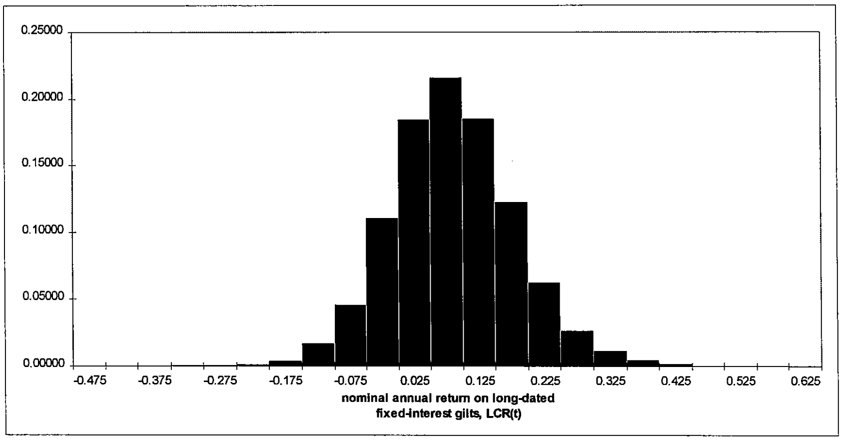

Figure 33- Empirical equilibrium distribution of the nominal annual long-dated fixed-interest gilt return for the proposed VAR(1) model

Figure 34 shows a quantile-quantile plot for the empirical equilibrium distribution of $\ln[1+LCR(t)]$ against a standard Normal distribution. Figure 34 has a very obvious 'S' shape, which implies that the distribution of $\ln[1+LCR(t)]$ is substantially fat-tailed in comparison to that of a Normal distribution. Thus, the distribution of the nominal annual return on long-dated fixed-interest gilts, *LCR*(*t*), is fat-tailed in comparison to that of a log-Normal distribution.

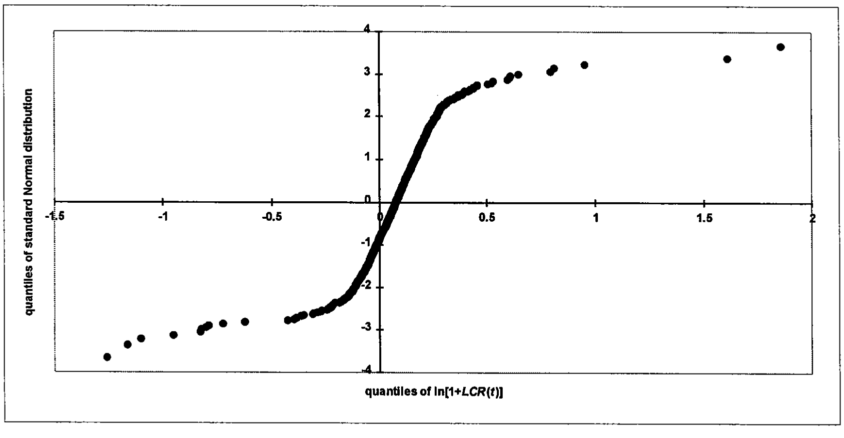

Figure 34- Quantile-quantile plot of the logarithm of the empirical equilibrium distribution of the nominal annual long-dated fixed-interest gilt return for the proposed VAR(1) model

As a result of the ARCH nature of the price inflation influence on the annual long-dated fixed-interest gilt return and the increased year-on year volatility of the long-term interest rate, the distribution is particularly fat-tailed.

The empirical equilibrium distribution function obtained for the nominal annual return on long-dated fixed-interest gilts using the VAR(1) model is summarised in Table 4. The corresponding values obtained using the Wilkie [1995] model are shown for comparison.

	VAR(1) model	Wilkie [1995] model
mean	8.7%	8.4%
standard deviation	12.6%	9.4%
skewness	10.0	0.40
kurtosis	345.0	4.20
1^{st} percentile	-14.0%	-12.0%
5^{th} percentile	-7.0%	-6.0%
10^{th} percentile	-3.0%	-3.0%
25^{th} percentile	2.0%	2.0%
50^{th} percentile	8.0%	8.0%
75^{th} percentile	15.0%	14.0%
90^{th} percentile	21.0%	20.0%
95^{th} percentile	25.0%	24.0%
99^{th} percentile	35.0%	34.0%

Table 4- Empirical equilibrium distribution of the nominal annual long-dated fixed-interest gilt return for the proposed VAR(1) model

From Table 4, it can be seen that both the mean and the variance of nominal annual return on long-dated fixed-interest gilts are higher using the VAR(1) model rather than the Wilkie [1995] model.

As for equities, the main reason for the difference in the means is the shorter base period used for parameter estimation in the VAR(1) model which, as mentioned previously, gives more weight to recent experience (when annual rates of gilt return have been slightly higher than previously).

However, the higher variance in the nominal annual return on long-dated fixed-interest gilts using the VAR(1) model is largely due to the higher one-step variance in the long-term interest rate (as noted above). From the equation for $LCR(t)$, it can be seen that the nominal annual return on long-dated fixed-interest gilts in any given year will depend only on the change in the long-term interest rate over that year.

Also, comparing Table 4 with the corresponding Table 2 obtained for the nominal annual return on equities, it can be seen that, as for the Wilkie [1995] model, according to the proposed VAR(1) model :

1. equities can be expected to give significantly higher annual returns than long-dated fixed-interest gilts (16.1% p.a. compared with 8.7% p.a.); and
2. the annual return on equities is significantly more volatile than the annual return on long-dated fixed-interest gilts (standard deviation of 27.4% p.a. compared with 12.6% p.a.).

The fact that the relative difference between the two asset classes is maintained in the VAR(1) model is likely to be crucial when examining the robustness of a stochastic asset-liability modelling exercise to a change in the stochastic asset model used.

Indeed, Wilkie [1995] states that the purpose of his stochastic asset model is "to provide a realistic variance and covariance structure for many years ahead, to quantify the expanding funnel of doubt".

Figure 35 shows 10 independent simulations of the future progress of the nominal annual return on long-dated fixed-interest gilts using the VAR(1) model. The 'neutral' starting conditions are used.

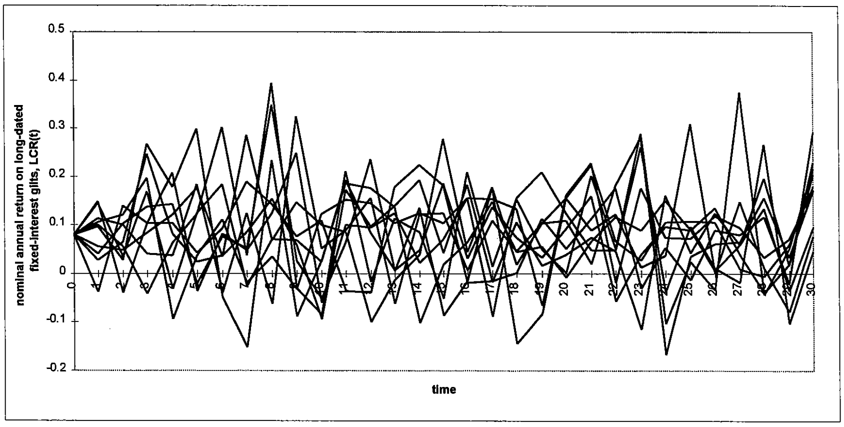

Figure 35- Ten simulations of the nominal annual long-dated fixed-interest gilt return using the proposed VAR(1) model

4. A SIMPLE EXAMPLE OF STOCHASTIC ASSET-LIABILITY MODELLING

Any stochastic asset-liability modelling exercise involves projecting the *actual* cashflows to and from the fund during each future time period. These cashflows will depend, amongst other things, on the *actual* investment experience (as generated by the underlying stochastic asset model).

Also, at any future date, by discounting the *expected* future cashflows in and out of the fund, a value can be placed on both the assets and the liabilities. This enables the current levels of funding and the required future contribution rate to be determined.

For a final-salary pension scheme, appropriate measures of the level of solvency would be :

1. the discontinuance funding level; or
2. the statutory Minimum Funding Requirement (or MFR) funding level.

The discontinuance funding level is explored below as this gives an explicit measure of the actual solvency (or otherwise) of the scheme in the event of immediate wind-up.

The discontinuance funding level is given by :

$$\frac{\text{market value of the scheme assets at the valuation date}}{\text{value of accrued benefits on immediate discontinuance}}$$

For the model pension scheme used, it is assumed that, on discontinuance, each active member is granted a transfer value (calculated in accordance with Guidance Note 11) in lieu of a deferred pension.

In line with the requirements of GN11, the rate of interest used to value the accrued benefits is market-related and is based on the current yield available on long-dated fixed-interest gilts, given by $C(t)$. Allowance must also be made for statutory revaluation in the period up to retirement and Limited Price Indexation (or LPI) increases in payment.

Thus, the size of the discontinuance funding level at each future date is a random variable. Other factors which will have a significant effect on the size of this random variable are :

1. The growth in retail prices prior to the valuation date.

   Because of the cascade structure of the two models, this series filters through to affect all the other series generated;

   The growth in retail prices up to time t is given by :

$$\frac{Q(t)}{Q(0)} - 1$$

where $Q(t) = Q(t-1).\exp[I(t)]$ and $I(t)$ is the force of price inflation in year t.

2. The return achieved on investments prior to the valuation date.

   In particular, the return achieved on equities as the bulk of the assets of the fund are likely to invested in equities;

   The return achieved on equities up to time t is given by :

$$\frac{PR(t)}{PR(0)} - 1 = \left\{ \prod_{s=1}^{t} [1 + LPR(s)] \right\} - 1 ;$$

   Similarly, the return achieved on long-dated fixed-interest gilts up to time t is given by :

$$\frac{CR(t)}{CR(0)} - 1 = \left\{ \prod_{s=1}^{t} [1 + LCR(s)] \right\} - 1 ;$$

   Of particular importance is the *real* return on investments (i.e. the investment return in excess of salary growth).

3. The growth in average earnings prior to the valuation date.

   This will affect the level of the accrued benefits at the valuation date;

   The growth in average earnings up to time t is given by :

$$\frac{W(t)}{W(0)} - 1$$

   where $W(t) = W(t-1).\exp[J(t)]$ and $J(t)$ is the force of average earnings growth in year t;

   Appendix B outlines the model for average earnings growth proposed by Wilkie [1995] and also suggests a very simple alternative model for use with the VAR(1) model.

The structure of the model pension scheme used is assumed to remain stable in future with respect to age, pensionable salary (in real terms) and past pensionable service.

The fund is assumed to be invested 80% in equities and 20% in long-dated fixed-interest gilts. These proportions are fixed during the projection period and the portfolio is re-balanced annually.

Annual valuations of the scheme are conducted using the Projected Unit Method and a typical 'prudent' on-going valuation basis for the purpose of recommending a future contribution rate to be paid. The contribution rate is calculated in accordance with both the Inland Revenue surplus regulations introduced in the Finance Act 1986 and the Minimum Funding Requirement regulations introduced in the Pensions Act 1995.

The initial discontinuance funding level is set equal to 150%.

By carrying out a large number of such simulations of the future progress of the model scheme, each of which is considered equally likely, an empirical distribution function for the discontinuance funding level at each future date can be obtained.

This empirical distribution function can then be used, amongst other things, to estimate the probability of insolvency on wind-up at the particular future date.

This probability of insolvency can then be used as an objective measure to compare the level of security provided by different investment or funding strategies.

It is important to explore the sensitivity of the results obtained to the underlying stochastic asset model. As mentioned previously, many actuaries remain sceptical as to the power of stochastic asset-liability modelling and believe that the results obtained are highly dependent on the particular asset model used.

Figure 36 and Figure 37 show the 1st, 5th, 10th, 25th, 50th, 75th, 90th, 95th and 99th percentiles of the distribution of the discontinuance funding level at annual intervals using the Wilkie [1995] model and the VAR(1) model respectively. These percentiles are obtained using 1,000 simulations of the future progress of the model pension scheme.

As both the Wilkie [1995] model and the VAR(1) model are stationary time series models, the process defining the discontinuance funding level is also stationary (i.e. the process tends towards a stable equilibrium distribution).

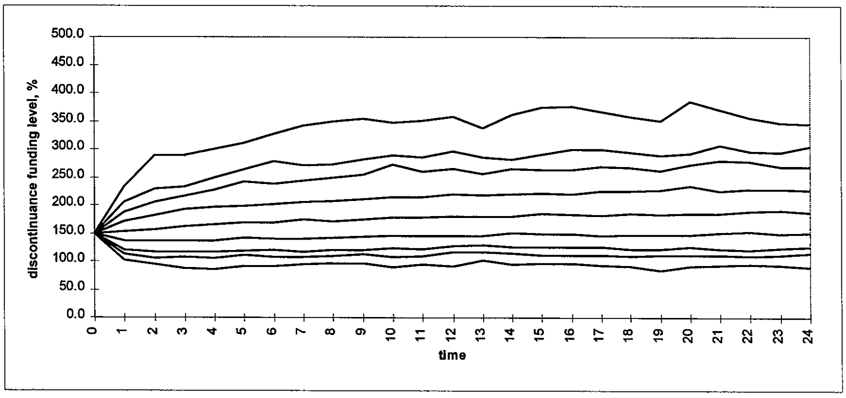

Figure 36- Percentiles of the discontinuance funding level distribution using the standard Wilkie [1995] model

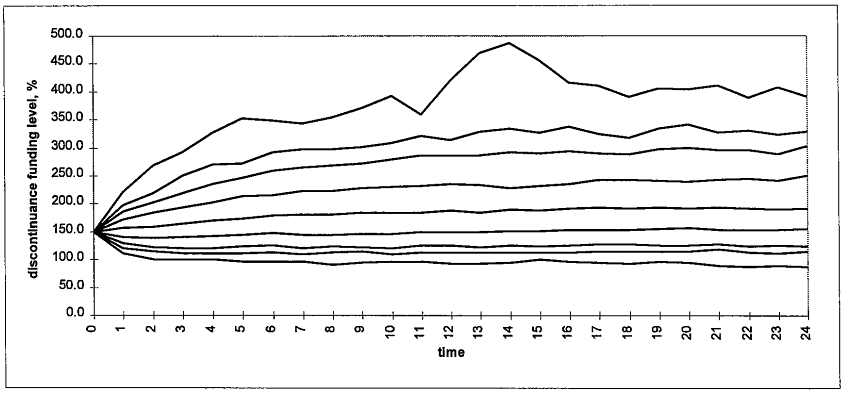

Figure 37- Percentiles of the discontinuance funding level distribution using the proposed VAR(1) model

Figure 38 and Figure 39 give an empirical distribution of the shape of the probability distribution function of the discontinuance funding level after the equilibrium position has been reached using the Wilkie [1995] and the VAR(1) model respectively.

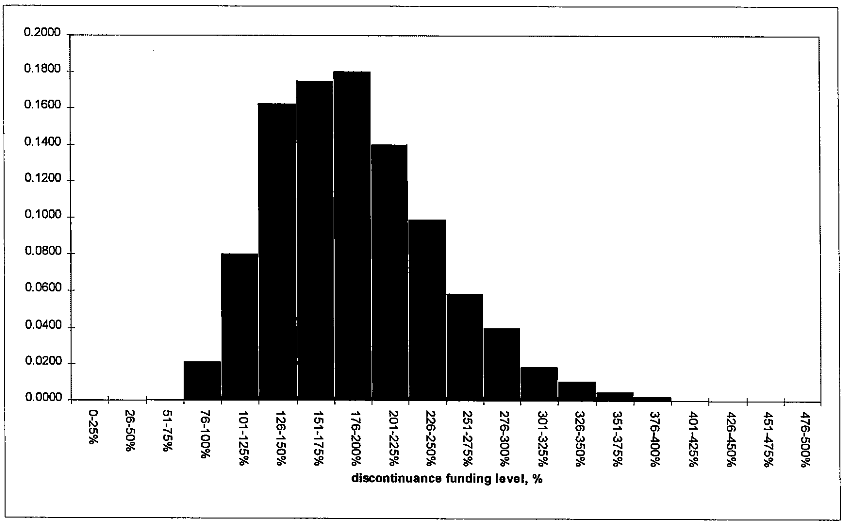

Figure 38- Empirical equilibrium distribution of the discontinuance funding level using the standard Wilkie [1995] model

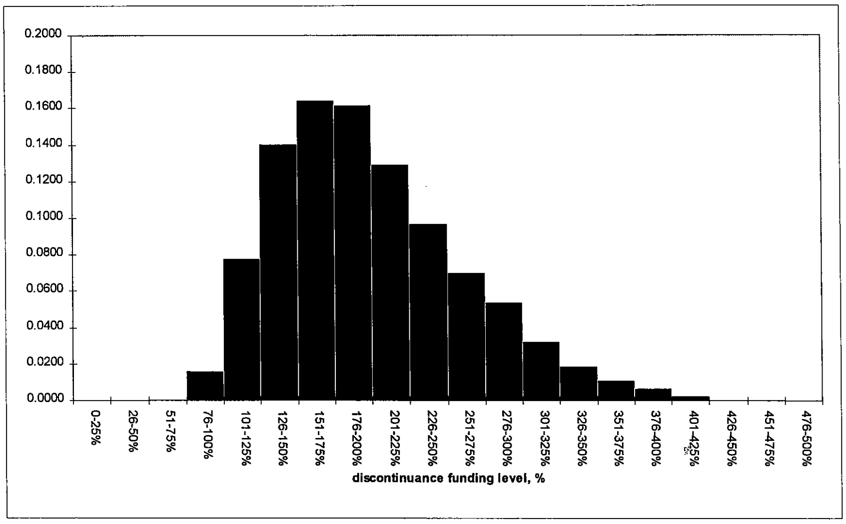

Figure 39- Empirical equilibrium distribution of the discontinuance funding level using the proposed VAR(1) model

For both models, the distribution of the discontinuance funding level is positively skewed (not unsurprising given that the discontinuance funding level is the ratio of two non-negative random variables). Further, though, it can be shown that, for both models, the distribution of the discontinuance funding level can be shown to strongly resemble a log-Normal distribution.

Figure 40 and Figure 41 show a quantile-quantile plot for the empirical equilibrium distribution of the logarithm of the discontinuance funding level against a standard Normal distribution for the Wilkie [1995] model and the VAR(1) model respectively. Both Figure 40 and Figure 41 are approximately linear, so that the (unknown and non-standard) distribution of the discontinuance funding level in each case is similar in shape to a log-Normal distribution.

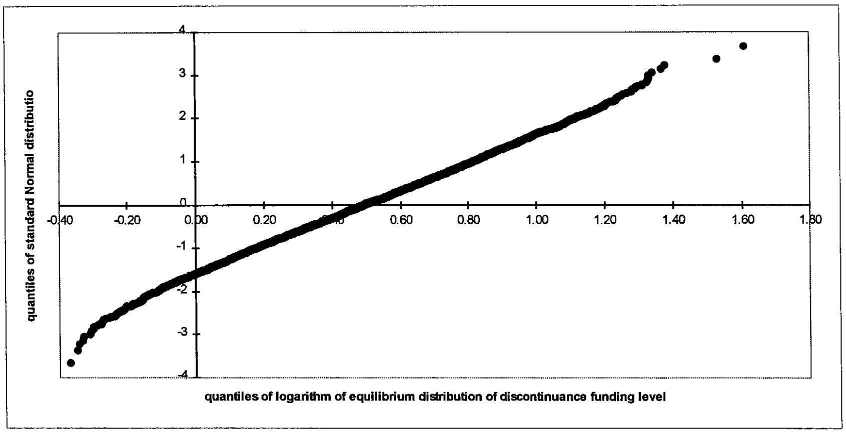

Figure 40- Quantile-quantile plot of the logarithm of the empirical equilibrium distribution of the discontinuance funding level using the standard Wilkie [1995] model

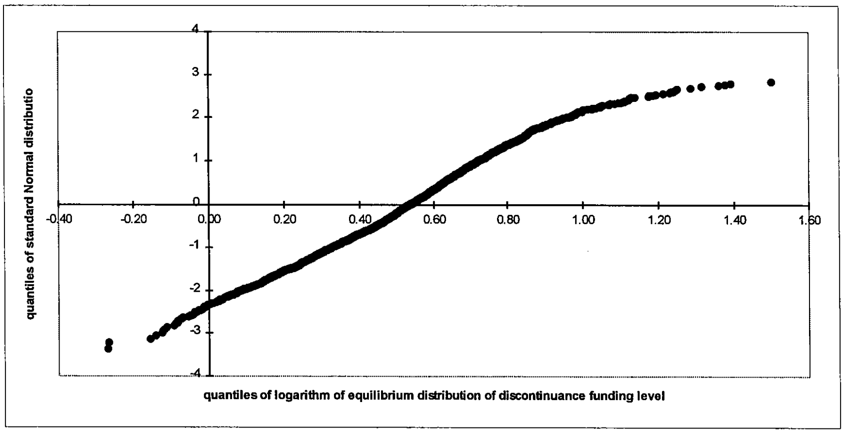

Figure 41- Quantile-quantile plot of the logarithm of the empirical equilibrium distribution of the discontinuance funding level using the proposed VAR(1) model

Thus, the ARCH influence on the price inflation used in the VAR(1) model (which filtered through to give an ARCH effect in the key variables affecting the discontinuance funding level, namely the annual return on equities and the long-term interest rate) does not appear to have a significant effect on the shape of the discontinuance funding level.

The empirical equilibrium distribution functions obtained for the discontinuance funding level using the Wilkie [1995] model and the VAR(1) model are summarised in Table 5.

	Wilkie [1995] model	VAR(1) model
mean	193.0%	203.0%
standard deviation	59.0%	71.0%
1^{st} percentile	93.0%	89.0%
5^{th} percentile	111.0%	113.0%
10^{th} percentile	123.0%	125.0%
25^{th} percentile	149.0%	154.0%
50^{th} percentile	186.0%	192.0%
75^{th} percentile	227.0%	240.0%
90^{th} percentile	270.0%	295.0%
95^{th} percentile	297.0%	330.0%
99^{th} percentile	360.0%	405.0%

Table 5- Mean, standard deviation and percentiles of equilibrium distribution of discontinuance funding level

From Table 5, it can be seen that the mean and variance of the discontinuance funding level is slightly higher using the VAR(1) model. The main reason for this is the higher mean and variance of the annual return on equities under the VAR(1) model (as a result of the use of a shorter base period for parameter estimation).

However, from Table 5, it is also apparent that the lower tail of the distribution (which, for the discontinuance funding level, is of most interest) is very robust to the change in the stochastic asset model.

The long-term probability of insolvency on wind-up can be estimated as the proportion of the total number of simulations which gives a discontinuance funding level of below 100%.

Then, for the Wilkie [1995] model, the long-term probability of insolvency is estimated as 2.6% whereas, for the VAR(1) model, the long-term probability of insolvency is estimated as 2.9%. Such a small difference is likely to be insignificant in practice.

Figure 42 and Figure 43 show 10 independent simulations of the future progress of the discontinuance funding level using the Wilkie [1995] model and the VAR(1) model respectively.

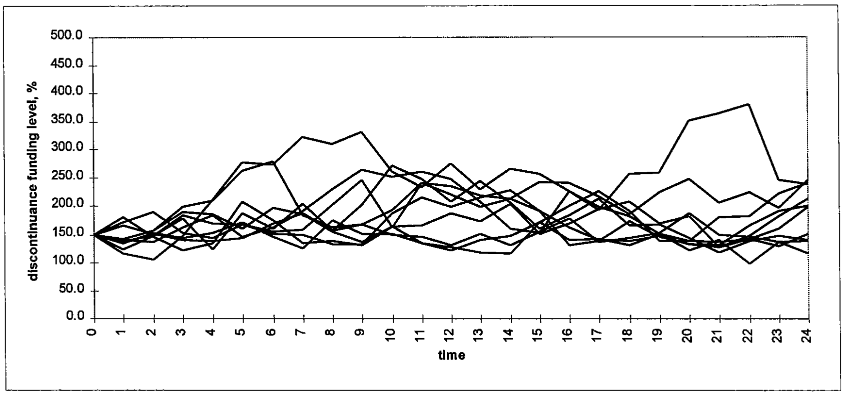

Figure 42- Ten simulations of the discontinuance funding level using the standard Wilkie [1995] model

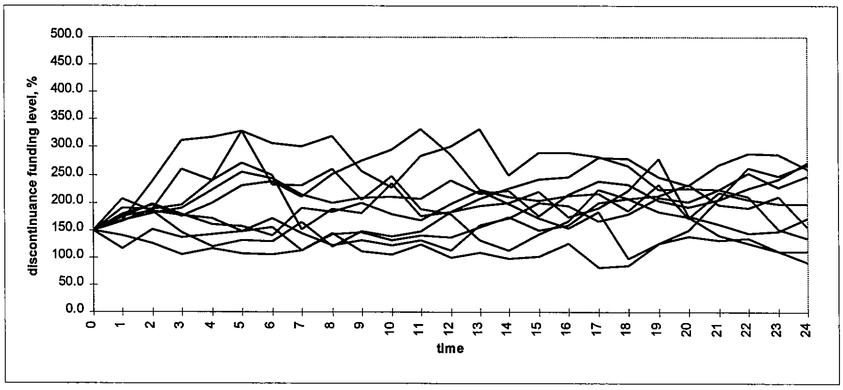

Figure 43- Ten simulations of the discontinuance funding level using the proposed VAR(1) model

It is possibly to largely eliminate model parameter sensitivity by (approximate) standardisation of the first and second moments of the key investment variables (e.g. force of price inflation, annual return on equities, long-term interest rate etc.). This can be done very simply by changing the appropriate parameters in the Wilkie [1995] model so as to more accurately reflect the experience of recent years (e.g. higher and more volatile equity returns than previously etc.).

However, given the robustness of the distribution of the discontinuance funding level (particularly in the vital lower tail) when the two original models are used, no attempt at standardisation to eliminate model parameter sensitivity has been carried out at this stage.

5. CONCLUSIONS

The series generated by the two stochastic asset models considered differ significantly, in particular :

(i) force of price inflation

> under the Wilkie [1995] model, the force of price inflation has a Normal distribution with a long-term mean of 4.7% p.a. and a long-term standard deviation of 5.2% p.a.;
>
> whereas, under the proposed VAR(1) model, the force of price inflation has an unknown heteroskedastic distribution with a lower long-term mean of 4.0% p.a. and a significantly higher long-term standard deviation of 10.0% p.a.;

(ii) nominal annual return on equities

> under the Wilkie [1995] model, the nominal annual return on equities was shown to be largely independent in successive years with an unknown (but approximately log-Normal) distribution with a mean of 13.1% p.a. and a standard deviation of 22.8% p.a.;
>
> whereas, under the proposed VAR(1) model, the nominal annual return on equities was again shown to be largely independent in successive years but with an unknown distribution with a significantly higher mean of 16.1% p.a. and a significantly higher standard deviation of 27.4% p.a. which was shown to be fat-tailed in comparison with a log-Normal distribution;

(iii) long-term interest rate

> under the Wilkie [1995] model, the long-term interest rate has an unknown distribution (somewhere between a Normal and a log-Normal distribution) with a long-term mean of 8.1% p.a. and a long-term standard deviation of 2.2% p.a.;
>
> whereas, under the proposed VAR(1) model, the long-term interest rate has an unknown heteroskedastic distribution with with a significantly higher short-term variance, but a lower long-term mean of 8.0% p.a. and a significantly lower long-term standard deviation of 1.2% p.a.;

(iv) nominal annual return on long-dated fixed-interest gilts

> under the Wilkie [1995] model, the nominal annual return on long-dated fixed-interest gilts was shown to be largely independent in successive years with an unknown distribution with a mean of 8.4% p.a. and a standard deviation of 9.4% p.a. which was shown to be slightly fat-tailed (particularly in the upper tail) in comparison with a log-Normal distribution;
>
> whereas, under the proposed VAR(1) model, the nominal annual return on equities was again shown to be largely independent in successive years but with an unknown distribution with a slightly higher mean of 8.7% p.a. and a significantly higher standard deviation of 12.4% p.a.

which was shown to be substantially fat-tailed in comparison with a log-Normal distribution;

However, from Section 4, the distribution of the discontinuance funding level, both in the short term and in the long term, appears to be highly robust, with regard to both the location and, in particular, the shape, to the change in the underlying stochastic asset model.

This suggests that the results obtained for more sophisticated stochastic asset-liability modelling exercises (e.g. to determine an optimal asset allocation) would be similarly robust.

It should be pointed out at this stage that, because both models are constructed from past experience (albeit, using different base periods), the variance and covariance structures underlying the two models will depend on past experience and so would not be expected to be fundamentally different.

The models proposed by Dyson & Exley [1995] and Smith [1996] are based on the theoretical construction of a hypothetical environment. Also, these two models are non-stationary (i.e. there is no mean-reversion element within the series generated), which can lead to implausible long-term results. Thus, the variance and covariance structures underlying these models differ significantly from historical reality, and the robustness of a stochastic asset-liability modelling exercise to a change in the underlying stochastic asset model is reduced somewhat..

However, it is the view of this author that, rather than try to construct a model based on how it is felt that the market 'should' behave, it is better to make use of the past investment experience available (which can easily be adjusted to reflect any changes expected in future), in conjunction, of course, with accepted basic investment theory and principles.

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A. Fitting a vector auto-regression model

A.1 *VAR(2) model*

Fitting the VAR model shown in Equation 1 with m=2 to UK annual data over the period 1946-1994 using least-squares regression, it can be shown that none of the second-order parameters estimates is significantly different from zero (even at the 10% level), which suggests that a first-order model should be fitted.

A.2 *VAR(1) model*

Re-fitting a first-order model to UK annual data over the period 1946-94 using least-squares regression gives the parameter estimates shown in Table 6.

		standard error	t-statistic	p-value
θ^1_{11}	0.3205	0.1518	2.11	0.04
θ^1_{12}	0.0180	0.0180	1.00	0.32
θ^1_{13}	0.1018	0.0447	2.28	0.03
θ^1_{21}	-1.8650	1.1635	-1.60	0.12
θ^1_{22}	0.3113	0.1381	2.25	0.03
θ^1_{23}	0.6497	0.3426	1.90	0.06
θ^1_{31}	-0.6335	0.2032	-3.12	0.00
θ^1_{32}	0.0078	0.0241	0.32	0.75
θ^1_{33}	1.0467	0.0598	17.50	0.00
ϕ^0_1	0.0566	0.0246	2.30	0.03
ϕ^0_2	0.5133	0.1956	2.62	0.01
ϕ^0_3	0.0848	0.0328	2.59	0.01
ϕ^1_1	-0.0006	0.0243	-0.02	0.98
ϕ^1_2	-0.3534	0.1931	-1.83	0.07
ϕ^1_3	-0.0290	0.0323	-0.90	0.37

Table 6- Parameter estimates for first-order vector auto-regression model

A.3 *Reduced VAR(1) model*

Re-fitting the VAR(1) model above, but setting equal to zero all of the parameters in Table 6 which do not differ significantly from zero (at the 5% level) gives the parameter estimates shown in Table 7.

		standard error	t-statistic	p-value
θ_{11}^{1}	0.3203	0.1518	2.11	0.04
θ_{12}^{1}	0.0000			
θ_{13}^{1}	0.1095	0.0440	2.49	0.02
θ_{21}^{1}	0.0000			
θ_{22}^{1}	0.3476	0.1366	2.54	0.01
θ_{23}^{1}	0.0000			
θ_{31}^{1}	-0.6336	0.2012	-3.15	0.00
θ_{32}^{1}	0.0000			
θ_{33}^{1}	1.0501	0.0583	18.00	0.00
ϕ_{1}^{0}	0.0562	0.0197	2.85	0.01
ϕ_{2}^{0}	0.3039	0.1626	1.87	0.07
ϕ_{3}^{0}	0.0848	0.0328	2.59	0.01
ϕ_{1}^{1}	0.0000			
ϕ_{2}^{1}	0.0000			
ϕ_{3}^{1}	0.0000			

Table 7- Parameter estimates for reduced first-order vector auto-regression model

B. A STOCHASTIC MODEL FOR EARNINGS INFLATION

B.1 *Wilkie [1995] model*

The model proposed by Wilkie [1995] for the force of earnings inflation in year *t*, *J*(*t*), is as follows :

$$J(t) = WW1.\,I(t) + (1 - WW1).\,I(t-1) + WN(t)$$

where

1. *J*(*t*) is the force of earnings inflation in year *t*;
2. *I*(*t*) is the force of price inflation in year *t*, according the standard Wilkie [1995] model;
3. $WN(t) = WMU + WA.\left[WN(t-1) - WMU\right] + WE(t)$ is an AR(1) process independent of the price inflation process;
4. *WMU* is the mean share dividend yield;
5. $WE(t) = WSD.WZ(t)$ is the random component of the share dividend yield at time *t*; and
6. *WZ*(*t*) is a *N*(0,1) noise series.

The parameter values suggested by Wilkie [1995] are :

$$\begin{aligned} WW1 &= 0.69 \\ WMU &= 0.016 \\ WA &= 0.0 \\ WSD &= 0.0244 \end{aligned}$$

Then, the index of earnings inflation at time *t*, denoted by *W*(*t*), is given by :

$$W(t) = W(t-1).\exp\left[J(t)\right]$$

B.2 *Alternative model for use with VAR(1) model*

Historical evidence suggests that earnings inflation can be expected to exceed price inflation by about 1.0% to 2.5% p.a. in the long term (see Thornton & Wilson [1992]).

Hence, the (very simple) model proposed for the force of earnings inflation in year *t*, *J*(*t*), is given by :

$$J(t) = I(t) + WE(t)$$

where

1. $I(t)$ is the force of price inflation in year t, according to the Wilkie [1995] ARCH model; and
2. $WE(t)$ is a $U(0.01,0.025)$ noise series.

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