The Sensitivity of Life Office Simulation Outcomes to Differences in Asset Model Structure

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THE SENSITIVITY OF LIFE OFFICE SIMULATION OUTCOMES TO DIFFERENCES IN ASSET MODEL STRUCTURE

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Abstract

The aim of this work is to see whether the choice of stochastic asset model structure is a material factor in determining the output and hence interpretation of stochastic actuarial asset-liability modelling experiments, or whether such differences can effectively be removed by appropriate parameterisation and hence remain under simple effective control by the user. In this study, an Accumulating With-Profits liability model is used.

Various simulation output measures related to both solvency and policy payout levels are tested for sensitivity to the choice of asset model structure. Three different stochastic asset models are tested: the Wilkie (1995) model, a Vector auto-regressive (VAR) model, and a so-called simple first-order auto-regression (AR1) model. Two levels of parameter standardisation of the models are attempted: (1) by fitting the models to the same data set, and (2) by attempting to make the mean and variance of the simulated 10-year asset returns of the VAR and AR1 models equal to those of the Wilkie model.

Simulated probabilities of ruin appear to be reasonably robust to model structure. Other output measures, particularly the simulated regular bonus rates and policy maturity values, are distinctly sensitive to model structure, reflecting residual differences which exist in the simulated asset returns from the different models over shorter durations, even after the models are 'fully' standardised. It is shown that very different conclusions can be drawn, for example regarding the effectiveness of different investment management strategies, when the different models are used.

It is concluded that the choice of asset model structure is of significant importance in determining the results (and hence interpretation) of stochastic asset-liability model output. It is therefore recommended that users of such models must identify and take into account, as a minimum, the distribution of the mean and variance of the simulated asset returns by duration when interpreting simulation results.

1. *Introduction*

The potential actuarial applications of stochastic simulation work have long been recognised by the actuarial profession. However, the incorporation of these techniques into actuarial practice continues to be dogged by the inability to find a stochastic asset model whose results can be handled confidently by practitioners as the basis of their actuarial advice and decision-making.

The difficulty lies with the difficulty of economic modelling in general. The economy is potentially influenced by an extremely large number of factors and the relationships between these factors are usually complex. As a result, few economic theories have been found to be well substantiated and those that have tend to be imprecise. This makes it particularly difficult to model the economic system. Furthermore, these difficulties are compounded by the limited amount of data available to fit and test possible long-term models.

A model can be used with more confidence, however, if the assumptions (including the economic theory) underlying the model are fully appreciated by the practitioner. An understanding of the degree of dependence between the model assumptions and the results delivered by the model is crucial to the satisfactory use of the model as a basis for formulating actuarial advice. It is this aspect of the problem which is addressed by this paper.

Smith (1996) described several of the stochastic asset models which are currently documented and are possibly of actuarial use, the most familiar of which is that of Wilkie (1986, 1995). Huber (1996) performed a particularly thorough review of the existing asset models, and identified the major elements of their structures and their consequent properties. Huber's work will clearly go some way to enabling the assumptions of these asset models to be more readily appreciated by practitioners, as this information becomes more widely disseminated (see, for example, Huber's (1997) review of the Wilkie model, and his (1998) review of Smith's (1996) model).

However, what is not clear from the existing work is the degree of sensitivity of any outcome variable, such as the probability of ruin from a life office model, to variations in particular aspects of the asset model structure. In this regard a distinction needs to be made between asset model differences which are due to differences of parameterisation (that is due to the assumed values of the models' parameters) and those which are due to differences in the way the model is constructed (that is due to the formulation of the model). Cairns (1995) and Hooker *et al* (1996), for example, refer to these as respectively the parameter and specification errors associated with the modelling process.

Model users have a fair amount of control over model parameterisation. Where models are of simple construction, for example where the behaviour of a variable is described solely by the mean and variance of its assumed distribution, then those parameters would usually have clear interpretation. Hence users can change the parameter values of simple models and have a clear appreciation of the assumptions that they are making when doing so, thereby establishing the parameter risk and enabling the monitoring and updating of assumptions through the control cycle process (see Goford, 1985).

The parameters of more complex models are much less easy to interpret intuitively, with the result that the effect of changing a parameter's value on the behaviour of the variable being modelled cannot be predicted easily by the user. Much of the Wilkie model, for example, would be subject to this problem. However, even in these cases users can exercise some control over model parameterisation. One way of doing this is to simulate distributions of the variable being modelled (for example, of the return on a particular asset class over a year) and calculate readily interpretable summary statistics of this distribution (for example, its mean and variance from a random sample of 1000 simulations). Parameter values of the asset model can then be varied, and the summary statistics re-simulated. The changes to the parameter values can then be translated into more readily interpreted effects (in this example the effect on the mean and variance of annual returns). Using this technique it is therefore possible to vary the parameters of these more complex models in a reasonably informed way.

Of course, part of the behaviour of these more complex models (at least those models which are defined by more than two parameters) will not be appreciated from observing simulated mean and variance. For example, a variable defined by 3 parameters can produce identical mean and variance for an infinite variety of parameter values. However the values of the third moment of the simulated distribution will differ. Hence in theory users could simulate moments to as high an order as necessary in order to encompass fully the effect of the parameter assumptions. However, this becomes increasingly cumbersome and, indeed, the higher order moments themselves become progressively harder to interpret.

For models which have originally been parameterised by reference to historical data, such as the Wilkie model, a more scientific approach to reparameterisation is to refit the model to a different data set (for example, perhaps using a more recent data series). However it should be noted, and fully appreciated by the users of all such models, that the future predictions from these models are essentially alternative realisations of the actual realisation represented by the data set used. Hence if the future is "expected" to differ from the past, then an historically fitted model may not be appropriate for the purpose in mind. Furthermore, even if a past period of behaviour of a variable is considered to be entirely typical of its expected future behaviour, refitting existing models to new data is not an entirely straightforward process. For example, if the behaviour of the variable is now different from its behaviour over the period which was used for fitting the original model, then by simply varying the parameters of the model it may not be possible to obtain an adequate fit. This in turn may indicate that the underlying economic basis for the model, which is essentially what dictates its specification (see Huber, 1997), may be wrong.

Those who create stochastic asset models have, to varying degrees, attempted to justify their models by reference both to economic theory and to empirical data. The Wilkie model, while incorporating some economic theory, is more heavily weighted towards empirical justification; whereas the models of Smith (1996), and Smith and Speed (1998), are much more theoretically based. It is these differences in rationale that lead to the different model specifications referred to earlier, and it is these differences which the user finds hardest to quantify in terms of the assumptions being made when a particular asset model is used. These differences are also part of the reason why two asset models, which a user might standardise to produce identical mean and variance of some simulated variable, might still differ to some significant degree with regard to the total distribution of this variable. (The remainder of this effect can be attributed to the model's parameterisation, as explained above).

The aim of this paper is to explore the effect of the variation in model specification upon a number of important output variables from model life-office simulation experiments. In doing so we take account of a user's ability to control model parameterisation, both by refitting different models to the same data set, and by standardising the first two moments of the simulated distributions of an aspect of the models' returns, as described above. In this way we hope to identify the importance of different model specifications, in relation to the totality of the assumptions made by any user of an asset model in a life-office context.

The rest of the paper is organised as follows.

- Section 2: Choice and description of the asset models used.
- Section 3: Description of the life office liability model used.
- Section 4: Description of the outcome measures considered.
- Section 5: Description of the parameterisation of the models.
- Section 6: Results.
- Section 7: Conclusions.

2. *Choosing the stochastic asset models*

2.1. *Wilkie model*

The most familiar stochastic asset model in the U.K. is the Wilkie model. This simple time-series model was first proposed in 1986 and was extended and refitted in 1995. The model uses a multivariate auto-regressive approach, based on the Box-Jenkins (1976) methodology, to project the future values of key economic variables such as the annual force of price inflation, the share dividend yield, the annual force of share dividend growth and the redemption yield on long-dated fixed-interest gilts (for this purpose, Wilkie used the yield on undated 2.5% Consols).

Whilst the model was formulated using certain basic economic theories, the structure was largely determined using empirical evidence and did not attempt to incorporate economic theories such as the efficient market hypothesis or the rational expectations hypothesis.

A consequence of using the standard auto-regressive techniques proposed by Box-Jenkins is that the series generated by the model are stationary (*ie* the mean and variance of the resulting series are constant over time). Geoghegan *et al* (1992) and Huber (1997), amongst others, argue that empirical evidence suggests that economic variables such as price inflation are not, in fact, stationary and, as a result, the standard Wilkie model "does not produce either future changes in the mean or future shocks, rather it produces a spurious tendency for the series to revert to the mean."

Also, Kitts (1990) criticises the use of independent Normal error terms in the standard Wilkie model. And, indeed, Wilkie (1995) admits that, for the key force of price inflation model, the residuals show "substantial positive skewness" and signs of "quite heavy 'tails' in the distribution".

In an attempt to allow for this observed skewness and kurtosis of the residuals, Daykin *et al* (1994) proposed the use of a three-parameter (or translated) Gamma distribution for the error terms of the force of price inflation residuals rather than a standard Normal distribution. This refinement substantially reduces the probability of negative price inflation under the model, which was another of the criticisms put forward by Geoghegan *et al.*

Whilst this adjustment to the model is theoretically attractive, initial investigations using the liability cashflow model for our life office suggested that the results obtained showed little or no sensitivity to this refinement in the structure of the underlying stochastic asset model and, as a result, this alternative to the standard Wilkie model was not considered any further in the following sensitivity analysis.

2.2. *Vector auto-regression model*

The second stochastic asset model considered also attempts to allow for this observed non-Normality in the error terms of certain of the key economic variables by incorporating the auto-regressive conditional heteroskedastic (or ARCH) techniques proposed by Engle (1982), whereby the variance of an innovation of the series at some time (t+1) is not constant but depends on the progress of the series up to time t.

In particular, Wilkie (1995) proposes an ARCH model for the force of price inflation which will allow for the "existence of large, irregular shocks", as suggested by Geoghegan *et al*. In this model, the variance of the error term for the force of price inflation at time $(t+1)$ is directly proportional to the difference between the force of price inflation at time t and the proposed long-term mean force of price inflation. As a result, the series generated by the model contains the, arguably, intuitive feature of long periods of stable price inflation punctuated by short bursts of very high volatility.

The vector auto-regressive (or VAR) model proposed by Wright (1998) makes direct use of this ARCH model for the force of price inflation. In addition to the force of price inflation, the model generates the same series as the original Wilkie (1986) model, namely the share dividend yield, the force of share dividend growth and the redemption yield on long-dated fixed-interest gilts.

As with the Wilkie model, the variance-covariance structure of the VAR model is determined by empirical evidence. The share dividend yield, force of share dividend growth and yield on long-dated fixed-interest gilts at any particular time is treated as a vector and modelled as a simple first-order multivariate auto-regressive process, with contemporaneous cross-correlation between the error terms where appropriate. The current force of price inflation is modelled separately using the ARCH model proposed by Wilkie and included within the VAR model as an exogenous variable.

Whilst this model is also a stationary time series model based largely on empirical evidence, the incorporation of the heteroskedastic price inflation model and the simplification of the inter-dependency between the series as a result of the use of a first-order multivariate auto-regressive process, mean that the underlying structure of the model is very different from that of the Wilkie model and that, as a result, the series generated display very distinct properties.

2.3. *Developing a simple stochastic asset model*

2.3.1. *A model based on total returns*

For the purposes of analysing sensitivity, it was also considered appropriate to explore the effects of using as simple a stochastic asset model as was possible given the demands of the liability cashflow projection model.

In the past, many practitioners have made use of very simple asset models based on total returns. These models are based, essentially, on the efficient market hypothesis (*ie* returns on securities from one year to the next cannot be forecast). This type of model proposes that the total nominal return (from both income and capital gain) on a particular asset class over time form a series of independent and identically distributed random variables. Empirical evidence suggests that total nominal returns are somewhat positively skewed, so that a log-Normal, rather than a Normal, distribution is often assumed. Contemporaneous cross-correlations between the total nominal returns on different asset classes and with the force of price inflation can be easily incorporated into the model.

Whilst this model has the advantage of simplicity, it has a number of serious flaws when used in conjunction with any reasonably sophisticated asset-liability modelling exercise.

The main disadvantage of this model is the requirement to separate the total nominal return on a particular asset in any given year into income and capital gain components to enable the derivation of the change in, for example, the share dividend yield or the redemption yield on long-dated fixed-interest gilts over the year. As will be shown later, the liability cashflow model requires the current value of the share dividend yield for the purposes of statutory solvency assessment and the current value of the yield on long-dated gilts is used in determining the reversionary bonus rate awarded.

For equity-type assets, it is common to assume that the cash return (or dividend yield) is constant and to determine the capital return accordingly. However, for gilt-type assets, this approach is inappropriate as there are analytical relationships between the total return, the cash return and the capital return. Unfortunately, this is where the simple total return model runs into problems.

Using the notation suggested by Wilkie (1995), suppose that $X(t) = \dfrac{CR(t)}{CR(t-1)} - 1$ is a random variable with a log-Normal distribution denoting the total return achieved on undated fixed-interest gilts between time (t-1) and time t, where $CR(t)$ is the value of a 'rolled-up' undated fixed-interest gilt at time t. Then, if $C(t)$ is the yield on undated fixed-interest gilts at time t, it is straightforward to show that the relationship between $X(t)$ and $C(t)$ is :

$$1 + X(t) = \left[\frac{1}{C(t)} + 1\right].C(t-1) \Rightarrow C(t) = \frac{C(t-1)}{\left[1 + X(t)\right] - C(t-1)}$$

Deriving the gilt yield in this way does not, however, lead to a stationary series. In fact, the series will behave in a totally unrealistic fashion, either decaying gradually over time towards zero or increasing exponentially to give impossibly large values.

This problem is also evident if a simpler, but less realistic, distribution (*eg* Normal) is used to model the total nominal annual return on gilts. As a result of this problem, it was concluded that a simple total-return model could not be used for the purposes of projecting the future cashflows of the model life office.

2.3.2. *A simple first-order auto-regression model*

The complexity of many of the stochastic asset models proposed has prevented many practitioners from developing a level of understanding necessary if such models are to be used confidently as decision-making tools. Thus, the development of a simple stochastic asset model, which can be readily understood and manipulated, could be very useful if it enables stochastic simulation techniques to be utilised to their full potential by the wider actuarial profession.

However, simplicity is not, of course, the only issue. The model proposed must generate the series required for projecting and capitalising the future cashflows and must, at least, be consistent with very basic economic theory and empirical evidence. The sensitivity testing performed in this paper will also help to assess the consequences of adopting a simple model specification on the output variables produced.

As a result, it was considered appropriate to develop and test a simple stochastic asset model using the Box-Jenkins methodology adopted by Wilkie (1995) but without any of the refinements subsequently incorporated.

The model covers only the four basic economic variables required for projecting the future cashflows of the model life office (namely, force of price inflation, share dividend yield, force of share dividend growth and redemption yield on long-dated fixed-interest gilts), although the techniques involved can be easily extended to cover other asset classes (*eg* overseas equities, property, cash, index-linked gilts) and economic variables (*eg* salary inflation, G.D.P. growth *etc.*).

The force of price inflation is modelled as a first-order auto-regressive process, in the same way as the standard Wilkie (1995) model, as no further simplification seemed either necessary or justified. Despite the criticisms of non-Normality and non-stationarity put forward by Kitts and Huber respectively, it is likely that such a model will be suitable for many practical asset-liability modelling exercises, provided that the effects of the limitations inherent in the model are appreciated by the user, and it has the key advantage of simplicity.

The share dividend yield is also modelled simply as a first-order auto-regressive process, although the error term involved is contemporaneously correlated with the error term in the price inflation model. The correlation coefficient in this case is high and positive, implying that, whilst nominal share dividend payout can be expected to rise broadly in line with inflation (although see comments below on a model for the force of share dividend growth), share prices will not rise to the same extent (perhaps as the market views on expected dividend growth in future are revised in the light of the rise in inflation). In contrast, Wilkie (1995) uses a log transformation for modelling the share dividend yield, which is likely to eliminate some of the skewness and kurtosis which is observed in the model residuals. However, in an attempt to keep the proposed model as simple as possible, this refinement has been omitted.

The real force of share dividend growth is modelled simply as a noise series and the current force of price inflation added to give the nominal force of share dividend growth. Wilkie (1995) and Daykin *et al* (1994) suggest that it may be more appropriate to model the real force of share dividend growth as a first-order moving average process. The rationale behind this is that company directors can be expected to smooth out dividend payments to some extent by paying out only part of any rise in company earnings in one year, with a further part in the following year. Again, in the interests of keeping the proposed model as simple as possible, this has not been investigated further. As for the share dividend yield series, the error term for the real force of dividend growth is contemporaneously correlated with the error term in the price inflation model. The correlation coefficient in this case is negative, implying that if company profitability rises as a result of an increase in inflation, company directors will be expected to hold back some of this increase as retained profits rather than distributing the full amount in the form of increased dividends.

Finally, the redemption yield on long-dated fixed-interest gilts is modelled in exactly the same way as the share dividend yield, using a first-order auto-regressive process where the error term is contemporaneously correlated with the error term in the price inflation model. Not surprisingly, the correlation coefficient in this case is very high and positive, implying that the yield on long-dated fixed-interest gilts can be expected to increase as the current rate of inflation increases. As the rate of inflation rises, expectations of future inflation rates are likely to rise and the size of the inflation risk premium (essentially reflecting the uncertainty in the expectations of future inflation rates) may also rise, so that, all other things being equal, the gilt yield would be expected to rise accordingly.

Whilst this model is somewhat more complicated than the total-returns model discussed above, it is readily usable for asset-liability modelling purposes (as the returns on the major asset classes are split into income and capital gain). Also, the model is easier to understand and manipulate than the Wilkie model (*eg* the likely effect on the model output of a change in one of the parameter values of the model should be relatively easy to predict as the model uses the minimum number of parameters possible and the purpose of each in determining the model output should be readily apparent).

The rationale behind the construction of this simple AR(1) model and a detailed analysis of the output obtained from the model can be found in Wright (1999). In addition, a summary of the model is given in Appendix A.

2.4. *Non-stationary models*

Whilst the models considered previously are very different, they were all constructed based on the stationary auto-regressive techniques proposed by Box-Jenkins and using empirical evidence to refine the underlying structure. As a result, they are inconsistent with certain prominent economic hypotheses, as was discussed previously in the case of the Wilkie model.

Thus, it was considered appropriate to include within the sensitivity analysis a model constructed using a completely different methodology with a structure dependent not on empirical evidence but rather on the construction of a hypothetical environment consistent with particular economic theories.

The jump-equilibrium model proposed by Smith (1996) incorporates a risk-neutral law, which can be used to solve certain optimisation problems, and assumes that asset markets are truly efficient. Smith argues that, for most actuarial applications, it is prudent to assume that markets are efficient as historical inefficiencies are likely to be exploited in future by rational investors.

The model attempts to describe the behaviour of the nominal and real annual returns on the major asset classes (*ie* equities, fixed-interest gilts, index-linked gilts and property) and the yield curves for both fixed-interest and index-linked gilts. However, the model is limited in that it allows only for 'parallel shifts' in these yield curves over time.

The yield curve processes, which are the basis for the other series generated by the model, contain no mean-reverting effect and, as a result, can be thought of, essentially, as non-stationary random walks. Thus, the model allows for the possibility of negative yields. Also, the price inflation process, which is defined by the difference between the nominal and real yield curves, is also a non-stationary process.

Clearly, this model is based on a fundamentally different methodology to the stationary Box-Jenkins type models discussed previously where the structure was based largely on empirical evidence, and as such it was considered important to examine the sensitivity of the output of the life office liability cashflow model to the use of such a distinct asset model.

However, practical problems were encountered in this regard. Firstly, the Smith jump-equilibrium model does not distinguish between return in the form of income and capital gain for the different asset classes. This was the fundamental reason why the very simple total-return model was rejected previously. In addition, to remove the model parameterisation effects (as will be discussed later), it was necessary firstly to fit the different models to the same data set. Whilst this was a relatively straightforward task for the Box-Jenkins type models above, the same cannot be said for the non-stationary jump-equilibrium model. The purpose of this paper is not to judge the strengths and weaknesses of the different asset models available, so that, rather than use the Smith model in a manner inconsistent with that in which the other models were used, it was decided not to use this model in the initial sensitivity analysis and to concentrate simply on the stationary models discussed previously.

3. *The Liability model*

It was decided to use a model office consisting of Accumulating With-Profits (AWP) contracts for the liability model. AWP was chosen for the analysis partly because of the rapidly increasing importance of the business in the UK market. With-profits liabilities were also chosen in preference to non-profit liabilities, because of the additional interest, in the present context, provided by the dynamic links which are necessary between the asset and liability components of a with-profits model office. Hence outcomes from both the liability and asset components of the model office will be sensitive to variations in the asset model.

The general nature of with-profits life insurance, and of AWP in particular, can be found described in Booth *et al* (1999). A brief summary will be given here. Note: readers who are familiar with UK Accumulating With-Profits contracts can omit the following sections marked *.

*3.1. *Accumulating With-Profits Contracts*

Under AWP contracts, policyholders' returns are passed back to them in the form of a notional accumulating fund, augmented by discretionary bonus additions. The precise form taken by AWP contracts varies from company to company (see Headdon *et al*, 1996a). The following describes a simplified benefit structure which retains the main features of the AWP contracts met in practice. Certain simplifying assumptions have been made. These include:

(1) premiums are assumed to be level and paid annually at the start of each year;

(2) charges are assumed to be deducted at the start of each policy year; and

(3) fund increases are assumed to take place annually at the end of each year.

*3.1.1. *Contractual benefit structure*

If a denotes the exact time of policy issue, the contractual benefit payable under an AWP contract for a given history of premium contributions exactly t integer years after the issue date can be taken to be:

$$B_{a,t} = F_{a,t}.(1 + TB_{a,t}) \tag{1}$$

where $F_{a,t}$ = the policy's accumulated fund to time $a+t$

and $TB_{a,t}$ = the terminal bonus rate payable on a contractual claim under the contract at time $a+t$.

The policyholder's fund accumulates according to the following relation:

$$F_{a,t} = (F_{a,t-1} + P_{a,t} - CH_{a,t})(1 + g)(1 + RB_{a+t}) \tag{2}$$

where $P_{a,t}$ = amount of premium received at the beginning of policy year t

$CH_{a,t}$ = amount of charges deducted at the beginning of policy year t

g = guaranteed annual rate of fund increase

and RB_{a+t} = discretionary annual rate of fund increase for the year ending at time $a+t$.

The charges cover the cost of meeting any additional insurance benefits and expenses.

The value of $TB_{a,t}$ is at the discretion of the company, and can be raised or lowered from time to time. Changes to $TB_{a,t}$ are usually made at discrete time intervals (two or three times a year or often less), and negative terminal bonus rates are not permitted. The nature of the terminal bonus is therefore a final single discretionary payment, made only at the date of claim, at the rate then currently payable by the company.

The value of RB_{a+t} is also discretionary but, once the bonus increase has been made to the fund, it cannot be removed. The same rates of regular bonus are generally applied to all policies in force at the time of the bonus declaration, irrespective of policy duration. The nature of the regular bonus is therefore a regular discretionary increase, at rates which are not guaranteed in advance, but which once added serve to increase the level of guaranteed benefit payable under the contract. The value of $F_{a,t}$ is the amount of guaranteed benefit payable on a contractual claim at time $a+t$.

*3.1.2. *Bonus distribution philosophy*

Participating policyholders in the UK share in the profits of the company through this combination of regular and terminal bonus additions to the policies' contractual benefits. The value of the total contractual claim benefit $B_{a,t}$ (equation 1) can be generally considered to be some smoothed asset share (see Needleman and Roff, 1995; Booth *et al*, 1999) though it may not be communicated as such to the policyholders. The bonus distribution philosophy is the approach that companies take to their distributions, which can be conveniently subdivided into the following three components:

(i) the extent of any guaranteed returns;

(ii) the relative proportion of total payout which is in the form of terminal bonus; and

(iii) the overall degree of smoothing employed, in relation to the underlying asset share.

Variations in these three aspects of bonus philosophy all contribute to the overall level of guarantee provided under contracts in force at any point in time, which in turn affects the solvency risk. Any model which is used to simulate solvency risk for with-profits funds therefore needs to include a dynamic mechanism by which a credible bonus distribution philosophy is incorporated within the model.

3.2. *The Model Office: assumptions*

3.2.1. *Experience assumptions*

Investment returns and inflation were generated by the particular stochastic investment model assumed. Deterministic mortality was assumed according to the A1967/70 ultimate table. Annual per-policy expenses were assumed, of £24 per annum as at the projection date, inflating according to the simulated rate of inflation produced by the investment model each year. (The projection date is the point in time which represents the current conditions of the model life office, from which future outcomes are projected). For simplicity charges were always assumed to equal expenses, and surrenders were ignored, so that no expense or surrender profit could arise.

3.2.2. *Historical experience assumptions and the existing liabilities*

10,000 15-year term level annual premium policies of identical size, in real terms, were assumed to have been issued each year over the past 15 years to policyholders aged 35 at entry. Assets were assumed to have earned a constant annual rate of return in the past of 10% per annum, broadly equal to the mean rate of return from the Wilkie (1995) stochastic investment model assuming an asset mix of 75% equities and 25% Consols. New premiums, expenses and charges were assumed to have inflated at a constant annual rate, broadly equal to the average annual rate of inflation predicted by the Wilkie (1995) model. Over the 15 years prior to the projection date, policyholders' funds are assumed to have been augmented at a rate of 7% per annum, which comprises both guaranteed and bonus annual increases. The overall result was to produce an existing profile of liabilities which is stationary in real terms as at the projection date, providing benefits which currently incorporate an overall terminal bonus of about 13%. It should be noted that most UK companies would probably have terminal bonus proportions currently somewhat less even than 13%, although terminal bonuses on conventional contracts might be significantly higher. This is mainly due to the relative lack of maturity of AWP business in the UK at the present time, but also partly due to a different bonus philosophy for AWP in which, it appears, a very high proportion of the earned rate of return is being distributed through annual fund increases rather than as terminal bonus (Headdon *et al*, 1996a).

3.2.3. *Future new business*

The model office is assumed to be closed to new business from one year after the projection date, with the last tranche of business being issued at the start of the projection period. Using a closed fund has the advantage of enabling the frequency of absolute ruin to be calculated, as well as the frequency of statutory ruin (see section 4 below), but may suffer from lack of relevance for the majority of offices which are open to new business.

Nevertheless a consideration of model outcomes which relate solely to the in-force business is of very relevant interest to all companies, and it can be noted that this is also the assumption made for a company's statutory valuation.

3.2.4. *Bonus philosophy*

Policyholders' funds are assumed to accumulate according to equation (2). The rate of declared bonus (RB_{a+t}) each year in each simulation is calculated according to the rules set out in Appendix B. This approach was developed by an Institute of Actuaries working party on Unitised With-Profits to represent a 'realistic' model of a 'typical' company's behaviour with respect to its bonus philosophy (see Headdon *et al*, 1996b). The qualities 'realistic' and 'typical' are subjective, but the model allows for variation in the choice of parameter values to reflect different philosophies if required.

The main features of the assumed bonus philosophy are to declare a regular bonus $RB_{a,t}$ which, together with the guaranteed rate of fund increase, broadly reflects the yield on consols, subject to the policyholders' funds remaining lower than the value of a 'reduced policy asset share' ($RAS_{a,t}$). $RAS_{a,t}$ accumulates at some proportion X $(0 \le X \le 1)$ of the total rate of return on assets, after charges. Should $F_{a,t}$ exceed $RAS_{a,t}$, pressure to cut the bonus rate will be generated in the model. Hence the difference between $RAS_{a,t}$ and the full policy asset share ($AS_{a,t}$) effectively represents a target minimum value for the terminal bonus payable under the contract. Constraints are also enforced over the extent to which the declared bonus can change from year to year, resulting in a considerable smoothing of bonus rates over time.

The calculation of the total benefit payout is also described in Appendix B. The benefit is essentially the value of a smoothed asset share, or the value of $F_{a,t}$, if higher. The smoothed asset share is calculated by accumulating policy cash-flows (premiums less charges) at the geometric average of the rates of return earned over the previous three years.

The guaranteed rate of fund increase was assumed to be 3% per annum (i.e. $g =$ .03 in equation 2).

3.2.5. *Investment policy*

The assets were assumed to be a mix of equities and Consols. The proportions in the two classes normally remain fixed throughout the projection period, at a level which is fixed from outset. Rebalancing of assets is assumed at the beginning of each year in order to re-establish the fixed proportion at the start of each year. The long-term proportion invested in fixed interest can be varied in order to test the sensitivity of model outcomes under different asset allocation strategies.

This temporally static asset allocation strategy would not appear to be consistent with a closed fund of liabilities. As a result, frequencies of insolvency, in particular of actual insolvency (that is, when negative assets remain at the time when all liabilities have terminated) are likely to be mis-represented. However, as the emphasis here will be to consider relative results, not their absolute values, then useful comparisons will be possible despite these simplifying assumptions made.

3.2.6. *Calculation of the statutory solvency ratio*

The total assets at time t years after the projection date (A_t) were generated by the following recurrence relation:

$$A_t = (A_{t-1} + P_t - E_t)(1 + roa_t) - C_t \quad (3)$$

where $P_t =$ total premiums received at the start of year t under all policies in force

$E_t =$ total expenses incurred at the start of year t

$roa_t =$ total return on assets over the year $[t\text{-}1, t]$

and $C_t =$ total benefits paid under maturity and death claims during the year $[t\text{-}1, t]$.

For simplicity, premiums and expenses were assumed to be incurred at the beginning of each year, with claims incurred at the end of each year. The statutory value of the assets in the UK is essentially a market valuation: hence A_t is also taken to be the statutory value of the assets at time t.

The valuation of AWP liabilities in the UK is currently under review. The method adopted for this investigation was consistent with that of Wright *et al* (1998). It was felt that the relative sensitivity of outcomes to variations in the asset model would not be materially affected by whether or not resilience reserves or the EU statutory solvency margin were included in the value of the liabilities: they were therefore excluded for simplicity.

Therefore for the present investigation the statutory valuation of the liabilities reduces to considering the higher of the following two values as at the valuation date:

(i) the lower of:

  (a) the policy fund value; and

  (b) the amount that policyholders' would reasonably expect to be currently payable on surrender;

and

(ii) the prospective value of the guaranteed benefit on a future contractual claim, at the valuation rate of interest.

The valuation rate of interest (broadly consistent with regulations current at the time this work was carried out) was calculated as a weighted average of the redemption yield of the company's fixed interest assets, and the dividend yield for all other assets, with an overall maximum rate of 6% per annum. For modelling purposes the surrender value in (i)(b) is assumed to be equal to the policy's asset share.

The resulting total value of the liabilities is denoted by V_t. The statutory solvency ratio at time t is therefore:

$$S_t = A_t / V_t \tag{4}$$

3.2.7. *Initial Estate*

The difference between the value of the total assets at time t, and the total amount of the asset shares under all policies in force at time t, is defined as the amount of capital, or estate, held within the fund at that time. The initial amount of the estate at $t = 0$ was assumed to be equal to 6% of the asset shares, an amount which was deliberately chosen to be quite low so that a non-negligible proportion of simulations would result in statutory and/or actual insolvency.

4. *Output measures tested for sensitivity*

The stochastic simulation of a model with-profits fund produces a plethora of output measurements. In order to make the investigation manageable it was therefore necessary to select examples of key output measures which would serve to illustrate the nature of the sensitivity experienced overall. The following measures were chosen for the initial investigations, based upon 500 or 1000 simulations, although these measures were subject to some further refinement ultimately. Throughout t is taken to be the number of years from the projection date.

(i) *Frequency of actual insolvency*

The frequency of actual insolvency was calculated. A simulation of the model life office which produces negative assets at the time when liabilities have run-off (at $t = 15$) is defined as resulting in actual insolvency.

(ii) *Frequency of statutory insolvency*

The frequency of statutory insolvency was calculated. Statutory insolvency is deemed to occur in any simulation when S_t first becomes less than zero. This naturally includes all simulations which remain solvent over the whole 15 years but are actually insolvent at the end.

(iii) *The statutory solvency ratio*

The mean and standard deviation of the statutory solvency ratio were calculated at the end of each year t ($t = 1, 2, .. 14$).

(iv) *Maturity payout*

The mean and standard deviation were calculated of the maturity payout per policy maturing during the year $[t\text{-}1, t]$, ($t = 1, 2, .. 15$).

(v) *Regular bonus rate*

The mean and standard deviation were calculated of the *total* annual rate of increase (that is of $(g+RB_t)$) to policyholders' funds over the year $[t\text{-}1, t]$, ($t = 1, 2, .. 15$).

5. *Parameterisation of the chosen stochastic asset models*

This process was performed at two levels.

5.1. *The 'before standardisation' basis*

The aim here was broadly to fit the three models to the same data set. As the Wilkie (1995) model was fitted to the data for 1923-1994, it seemed natural to use this base period and hence it was necessary to fit the VAR and the AR(1) models to the same data set.

It can be argued that, for some of the data series, the assumption of stationarity is invalid over this period and it may be more appropriate to use a shorter base period for parameter estimation, thereby giving more weight to recent data. While this is an important consideration were the models to be used primarily for future prediction, it is less important here, bearing in mind the aims of this investigation (see Section 1).

While the models were broadly fitted to the same data set before standardisation, there were aspects of the parameterisation which were subject to manual adjustment.

The Wilkie (1995) model was adjusted by making a 25% reduction in the value proposed for the *YSD* parameter, as suggested by Headdon *et al* (1996b). This change reduces the volatility of the share dividend yield.

Also, the *QSA* parameter in the ARCH model for the force of price inflation, utilised in the VAR model, was reduced from the fitted value of 0.0256 to 0.0190. This change reduces the volatility of the force of price inflation to be more consistent with that for the standard Wilkie (1995) model. Despite the fact that both the standard and the ARCH models for price inflation are fitted to the same 1923-1994 data set, the long-term standard deviation for the ARCH model is almost double that for the standard model (10.0% *pa* compared to 5.2% *pa*). For both the Wilkie model and the VAR model, the force of price inflation filters through to affect the other key variables (*eg* the annual equity return and the yield on long-dated fixed-interest gilts), so that using the ARCH model unadjusted would be a very significant source of distortion in the results obtained for the VAR model relative to the Wilkie model which was due entirely to differences in the model parameterisation rather than in the model structure.

The mean and standard deviation of the annualised rate of return on equities (based on 1,000 simulations) for the three models before standardisation are shown in Figures 1 and 2.

The mean and standard deviation of the annualised rate of return on long-dated fixed-interest gilts (based on 1,000 simulations) for the three models before standardisation are shown in Figures 3 and 4.

The means and standard deviations of the gross redemption yield on long-dated fixed-interest gilts (based on 1,000 simulations) for the three models before standardisation are shown in Figures 5 and 6.

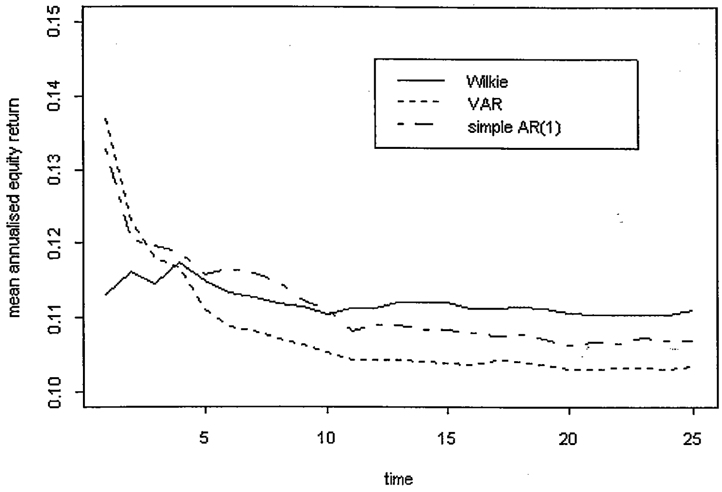

Figure 1. Mean annualised equity return – before standardisation.

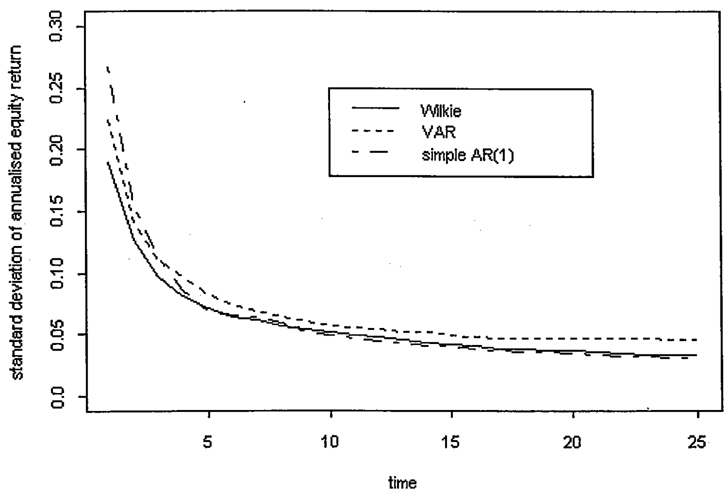

Figure 2. Standard deviation of annualised equity return – before standardisation.

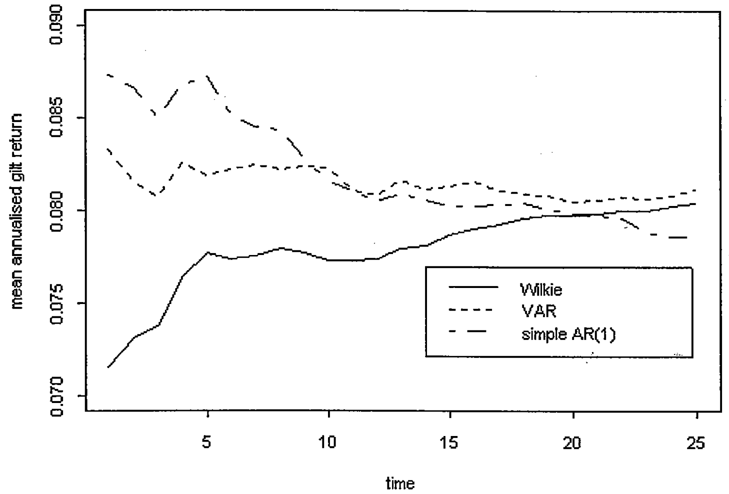

Figure 3. Mean annualised return on long-dated fixed-interest gilts – before standardisation.

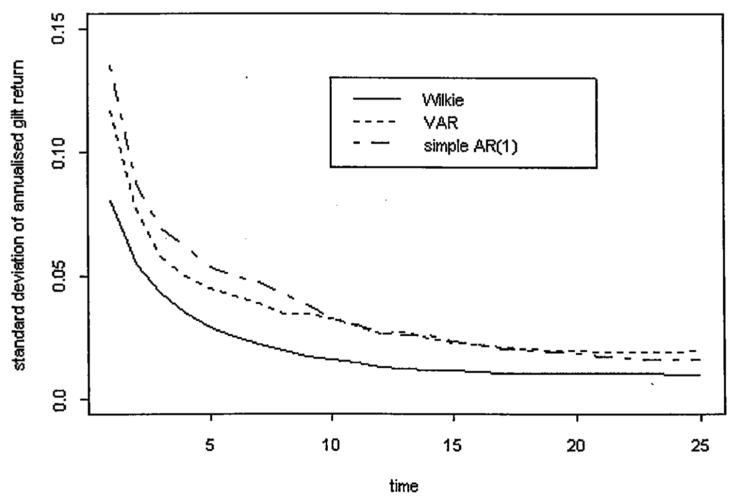

Figure 4. Standard deviation of annualised return on long-dated fixed-interest gilts – before standardisation.

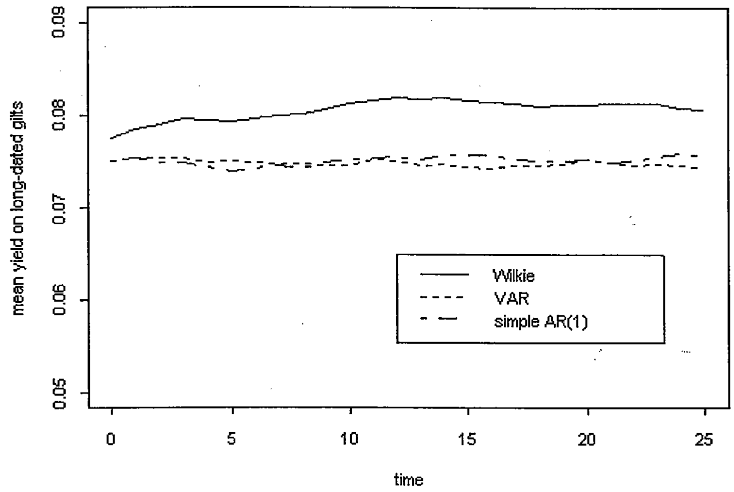

Figure 5. Mean yield on long-dated fixed-interest gilts – before standardisation.

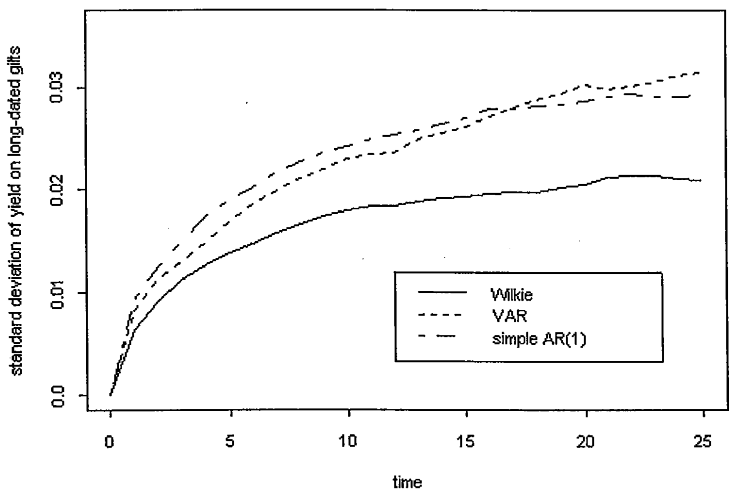

Figure 6. Standard deviation of yield on long-dated fixed-interest gilts – before standardisation.

It is clear from Figures 1-6 that there are some considerable differences in the simulated distributions of the asset returns and of the gilt yields between the three models. Given the rather ad-hoc nature of the adjustments described previously, further standardisation was considered appropriate.

5.2. *The 'after standardisation' basis*

Further standardisation was made by attempting to equate manually the means and variances of some of the key output variables from the three different models. The aim of this was to eliminate any distortions in the output from the AWP model office as a result of the model parameterisation, so that any remaining distortions will be a result of differences in the underlying model structures.

The key output variables used as a basis for standardisation were :

- the mean annualised rate of price inflation over a 10 year period
- the mean annualised rate of return on equities over a 10 year period
- the long-term gross redemption yield on long-dated fixed-interest gilts
- the mean annualised rate of return on long-dated fixed-interest gilts over a 10 year period

A period of 10 years was used for this purpose as it is approximately equal to the mean term of the AWP liabilities, which are run-off over a 15 year period.

Again, for simplicity, the Wilkie (1995) model was used as the basis for standardisation. The relevant output measures for the three models before standardisation (based on 1,000 simulations) are shown in Table 1.

	Wilkie	VAR	AR(1)
annualised rate of price inflation over 10 years			
mean	5.0%	4.3%	4.5%
standard deviation	2.9%	2.5%	3.2%
annualised rate of return on equities over 10 years			
mean	11.1%	10.4%	10.8%
standard deviation	4.7%	5.5%	4.6%
long-term gry on long-dated fixed-interest gilts			
mean	8.0%	7.5%	6.7%
standard deviation	2.1%	3.4%	3.2%
annualised rate of return on gilts over 10 years			
mean	7.7%	8.1%	8.1%
standard deviation	1.5%	2.9%	3.0%

Table 1. Output measures before standardisation.

Thus, for standardisation of the VAR model with the Wilkie model, the main changes to be made to the model were :

a) increase the mean annualised rate of price inflation
   - *QMU* parameter, the mean force of price inflation, is increased from 0.0400 to the corresponding value used in the Wilkie model of 0.0470

b) increase the mean annualised rate of return on equities
   - *YMU* parameter, the mean share dividend yield, is reduced from 0.0423 to the corresponding value used in the Wilkie model of 0.0375
   - *KMU* parameter, the mean force of share dividend growth, is increased from 0.0560 to the corresponding value used in the Wilkie model of $QMU + DMU = 0.0470 + 0.0160 = 0.0630$

c) reduce the standard deviation of the annualised rate of return on equities
   - the parameter defining the standard deviation of the force of share dividend growth, σ_2, is reduced from 0.0728 to 0.0500

d) increase the mean gross redemption yield on long-dated fixed-interest gilts
   - *CMU* parameter, the mean gross redemption yield on long-dated fixed-interest gilts, is increased from 0.0750 to the corresponding value used in the Wilkie model of $QMU + CMU = 0.0470 + 0.0305 = 0.0775$

e) reduce the standard deviation of the gross redemption yield on long-dated fixed-interest gilts

- the parameter defining the standard deviation of the gross redemption yield on long-dated fixed-interest gilts, σ_3, is reduced from 0.0079 to 0.0040
- this will then filter through to give a corresponding reduction in the mean annualised rate of return on gilts

Similarly, for standardisation of the simple AR(1) model with the Wilkie model, the main changes to be made to the model are :

a) use the Wilkie AR(1) model for the force of price inflation

- *QMU* parameter, the mean force of price inflation, is increased from 0.0433 to the corresponding value used in the Wilkie model of 0.0470
- *QA* parameter, the auto-regression parameter in the force of price inflation, is reduced from 0.6057 to the corresponding value used in the Wilkie model of 0.58
- *QSD* parameter, the parameter defining the standard deviation of the force of price inflation, is reduced from 0.0453 to the corresponding value used in the Wilkie model of 0.0425

b) increase the mean gross redemption yield on long-dated fixed-interest gilts

- *CMU* parameter, the mean gross redemption yield on long-dated fixed-interest gilts, is increased from 0.0664 to the corresponding value used in the Wilkie model of $QMU + CMU = 0.0470 + 0.0305 = 0.0775$

c) reduce the standard deviation of the gross redemption yield on long-dated fixed-interest gilts

- *CSD* parameter, the parameter defining the standard deviation of the gross redemption yield on long-dated fixed-interest gilts is reduced from 0.0085 to 0.0045
- this will then filter through to give a corresponding reduction in the mean annualised rate of return on gilts

The simulated model output after standardisation is shown in Table 2.

	Wilkie	VAR	AR(1)
annualised rate of price inflation over 10 years			
mean	5.0%	5.0%	5.0%
standard deviation	2.9%	2.7%	2.9%
annualised rate of return on equities over 10 years			
mean	11.1%	10.9%	10.9%
standard deviation	4.7%	4.9%	4.5%
long-term gry on long-dated fixed-interest gilts			
mean	8.0%	7.7%	7.8%
standard deviation	2.1%	2.4%	2.0%
annualised rate of return on gilts over 10 years			
mean	7.7%	7.9%	7.9%
standard deviation	1.5%	1.4%	1.4%

Table 2. Output measures after standardisation.

Thus, much of the difference in the mean and variance of the key output variables has been eliminated.

Note that, as the Wilkie model and the simple AR(1) model now use exactly the same model for the force of price inflation, any difference in the mean and/or standard deviation of the annualised rate of price inflation will simply be due to sampling errors (rather than any structural difference in the underlying models).

The mean and standard deviation of the annualised rate of return on equities (based on 1,000 simulations) for the three models after standardisation are shown in Figures 7 and 8.

The mean and standard deviation of the annualised rate of return on long-dated fixed-interest gilts (based on 1,000 simulations) for the three models after standardisation are shown in Figures 9 and 10.

The means and standard deviations of the gross redemption yield on long-dated fixed-interest gilts (based on 1,000 simulations) for the three models after standardisation are shown in Figures 11 and 12.

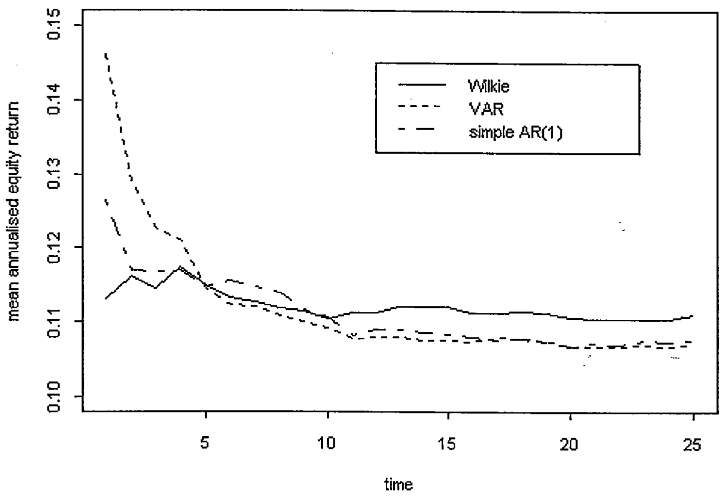

Figure 7. Mean annualised equity return – after standardisation.

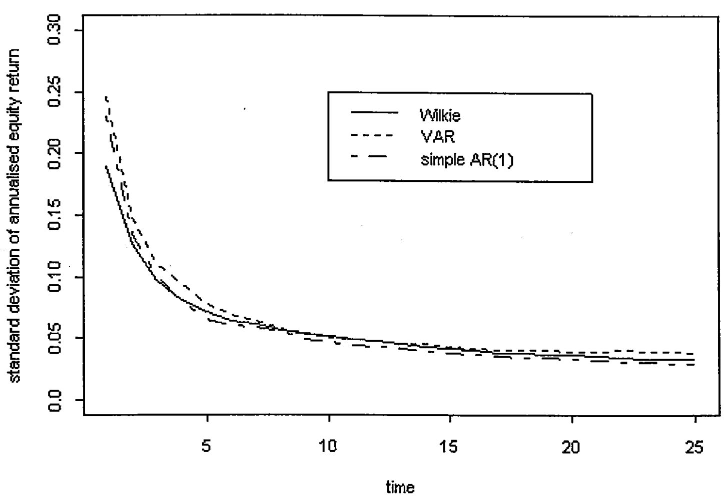

Figure 8. Standard deviation of annualised equity return – after standardisation.

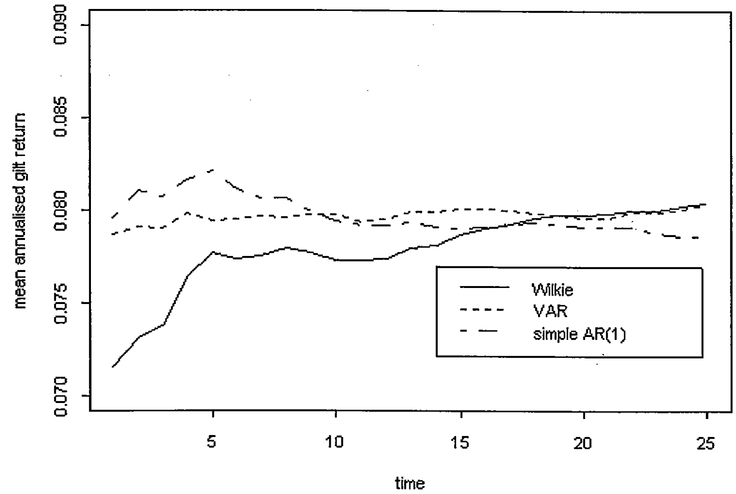

Figure 9. Mean annualised return on long-dated fixed-interest gilts – after standardisation.

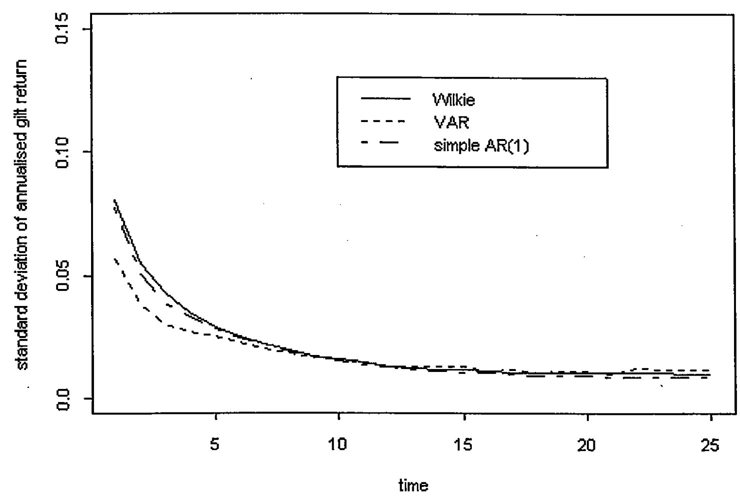

Figure 10. Standard deviation of annualised return on long-dated fixed-interest gilts – after standardisation.

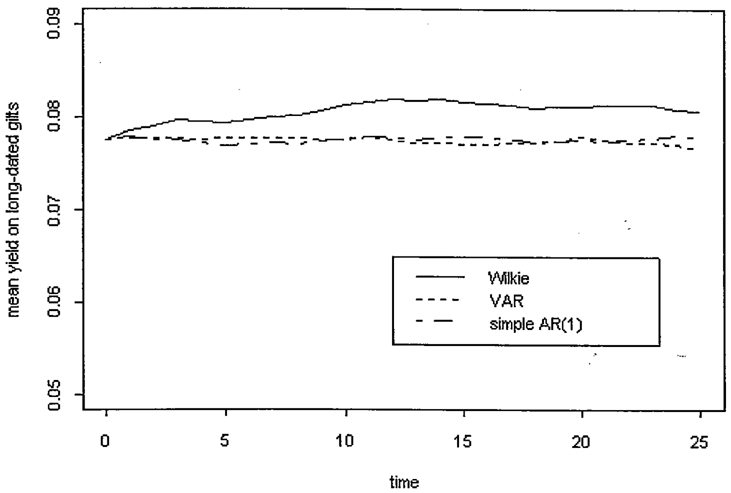

Figure 11. Mean yield on long-dated fixed-interest gilts – after standardisation.

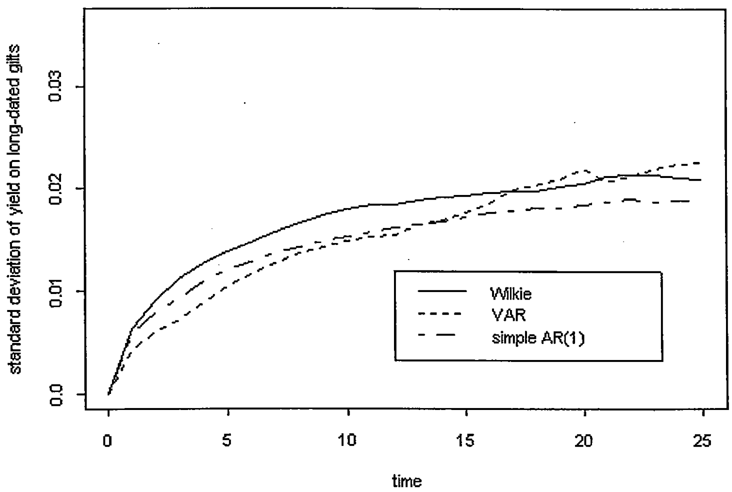

Figure 12. Standard deviation of yield on long-dated fixed-interest gilts – after standardisation.

It can be seen that, despite much effort, standardisation has not been able to achieve complete equality between the output measures of the three different models. The main remaining differences between the Wilkie model and the other two models are :

- in the mean and standard deviation of the equity return at short durations
- in the mean of the yield on fixed-interest gilts and on the return on fixed-interest gilts at short durations

However, it should be noted that the absolute size of the differences in (2) are small compared with (1).

It was felt that this was a reasonable attempt at standardisation given the different structures of the three models.

6. *Results*

6.1. *Assuming constant investment and bonus management strategies*

The results in this section assume an annually rebalanced asset mix of 75% in equities and 25% in long-term fixed interest securities.

6.1.1. *Frequency of actual insolvency (absolute ruin)*

The estimates of the frequency of actual insolvency and their 95% confidence intervals (assuming the normal approximation to the Binomial distribution when calculating the standard errors of the estimates) are shown in Table 3 (before standardisation).

Table 3: Estimates of the frequency of actual insolvency (before standardisation)

Model	Frequency	95% confidence interval
Wilkie	.032	(.017, .047)
VAR	.094	(.068, .120)
AR(1)	.048	(.029, .067)

The frequency of absolute ruin is significantly higher for the VAR model than for the other two models. This appears to reflect the higher standard deviation of annualised equity returns for the VAR model at durations over 4 years (see Figure 2). After standardisation, however, the ruin frequencies become almost identical (Table 4). This appears to reflect a convergence of the standard deviation of the equity returns after about duration 7 for the standardised models (Figure 8). Hence the sensitivity of this output measure seems to be reasonably explained by differences in the standard deviation of asset returns over long durations, which would appear intuitive.

Table 4: Estimates of the frequency of actual insolvency (after standardisation)

Model	Frequency	95% confidence interval
Wilkie	.032	(.017, .047)
VAR	.034	(.018, .050)
AR(1)	.036	(.020, .052)

The frequency of actual insolvency therefore appears to be robust to the underlying model structure.

6.1.2. *Frequency of statutory insolvency (statutory ruin)*

The estimates and their 95% confidence intervals for the three models before standardisation are shown in Table 5.

Table 5: Estimates of the frequency of statutory insolvency (before standardisation)

Model	Frequency	95% confidence interval
Wilkie	.142	(.111, .173)
VAR	.240	(.203, .277)
AR(1)	.172	(.149, .205)

These results also show significant differences between the Wilkie and VAR models, which broadly parallel the differences in absolute ruin (before standardisation) shown in 6.1.1. Again, standardisation removes the model dependency, so that the frequency of statutory insolvency also appears to be robust to the underlying model structure (Table 6).

Table 6: Estimates of the frequency of statutory insolvency (after standardisation)

Model	Frequency	95% confidence interval
Wilkie	.142	(.111, .173)
VAR	.142	(.111, .173)
AR(1)	.146	(.115, .177)

6.1.3. *Statutory solvency ratio*

The means and standard deviations of the statutory solvency ratio are shown in Figures 13 and 14 respectively.

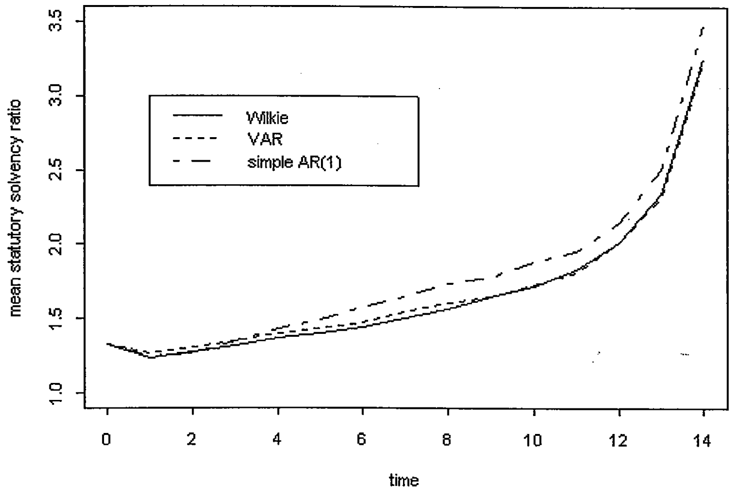

Figure 13. Mean statutory solvency ratio – before standardisation.

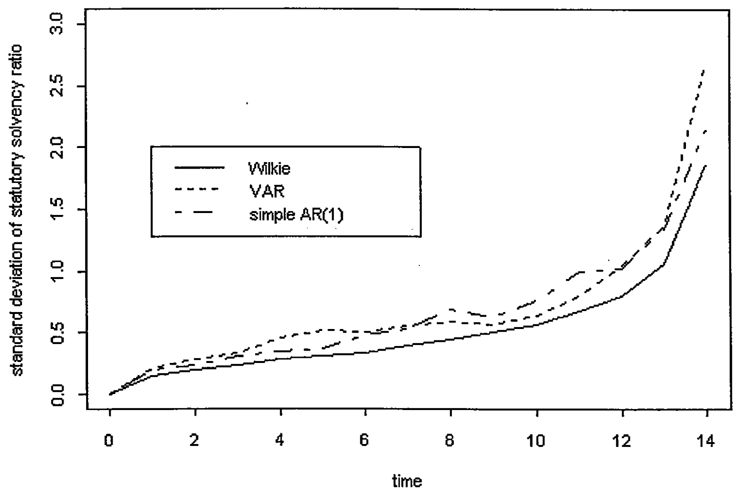

Figure 14. Standard deviation of statutory solvency ratio – before standardisation.

The tendency of both the mean and standard deviation of the solvency ratio to rise steeply from about year 11 is a result of using a closed fund model. If the initial estate can be considered as notionally allocated between each year's group of maturing policies, after each year the part of the estate relating to policies which have just matured is notionally reallocated to the remaining policyholders. Hence the estate would be expected to increase as a proportion of the total assets as the liabilities run-off, thereby increasing the statutory solvency ratio. The ratio reaches a value of (plus or minus) infinity at $t = 15$ when the value of the liabilities becomes zero.

The AR(1) model generates a somewhat higher mean ratio than the other two models, while both the VAR and AR(1) models produce higher variability of the solvency ratio than the Wilkie model. As the solvency ratio is a function both of asset values and valuation reserves then clearly this is a complex output measure, and differences will have a variety of interacting causes. There seems to be some relation to mean equity returns, which tend to be highest for AR(1) over the first ten years' duration (Figure 1) and which could partly explain the higher solvency ratio for the model. The higher mean gilt returns for AR(1) also at durations up to about 10 years (Figure 3) may also contribute. Both AR(1) and VAR models have higher equity return variability than the Wilkie model, at least for short durations (Figure 2), and also higher gilt yield variability at longer durations (Figure 6), both of which could be contributing to the higher variability of the solvency ratio observed (Figure 14). Other factors may operate through differences in guaranteed benefit levels and hence in the valuation reserves generated by the three models: some insight into this will be gained in section 6.1.4 below.

The equivalent results after standardisation are shown in Figures 15 and 16.

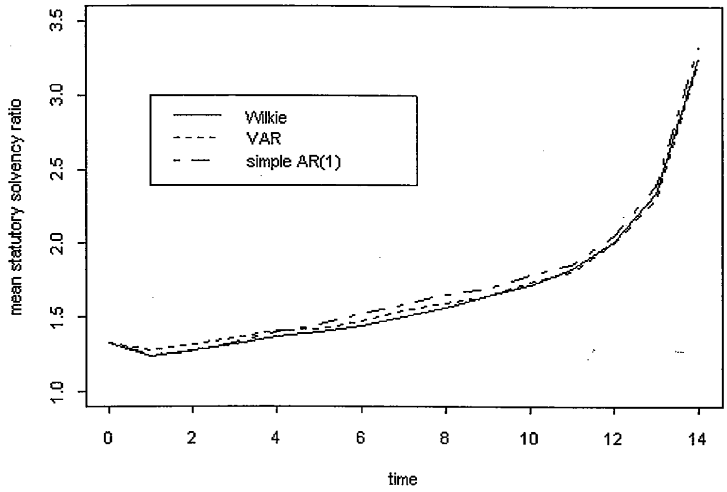

Figure 15. Mean statutory solvency ratio – after standardisation.

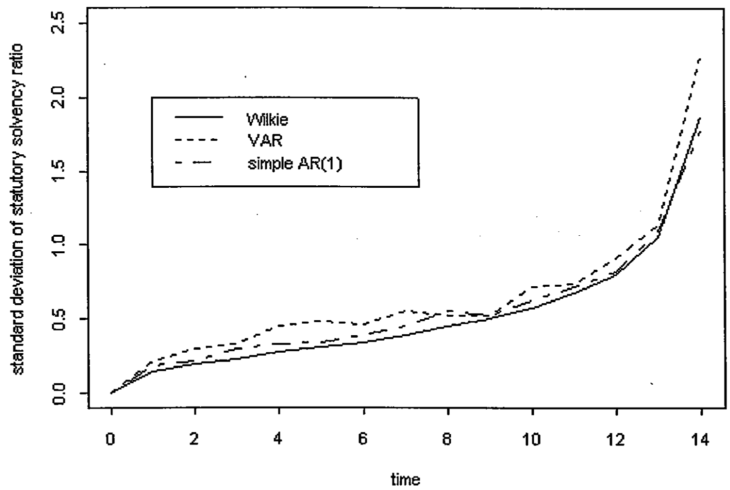

Figure 16. Standard deviation of statutory solvency ratio – after standardisation.

The differences in the mean solvency ratios have therefore been largely removed by standardisation, which appears consistent with a convergence of the mean equity returns, at least by the AR(1) model in relation to the Wilkie model (Figure 7); the differences in the standard deviations have been reduced, but not entirely eliminated, by standardisation. This could again be explained by the fact that, even after standardisation, the standard deviation of the short duration equity returns remained higher under the VAR and AR(1) models than for the Wilkie model. The fact that the gilt yield variability after standardisation (Figure 12) is lower for AR(1) and VAR than for the Wilkie model does not appear to have had a major impact on the variability of the solvency ratio.

The VAR and AR(1) models, both before and after standardisation, also show a less stable progression of standard deviation of solvency ratio over time than the Wilkie model.

These results show that standardisation of the first two moments of the ten-year asset returns has not been entirely sufficient in removing model sensitivity in the simulated distributions of the statutory solvency ratio, which is a complex output measure. This appears to result from residual differences in the variability of asset returns persisting after standardisation, which is a consequence of differences in structure between the three models. Hence there is therefore evidence of some sensitivity to asset model structure in the simulation of this variable.

6.1.4. *Regular bonus rate*

The means and standard deviations of the regular bonus rates for the three models, before standardisation, are shown in Figures 17 and 18.

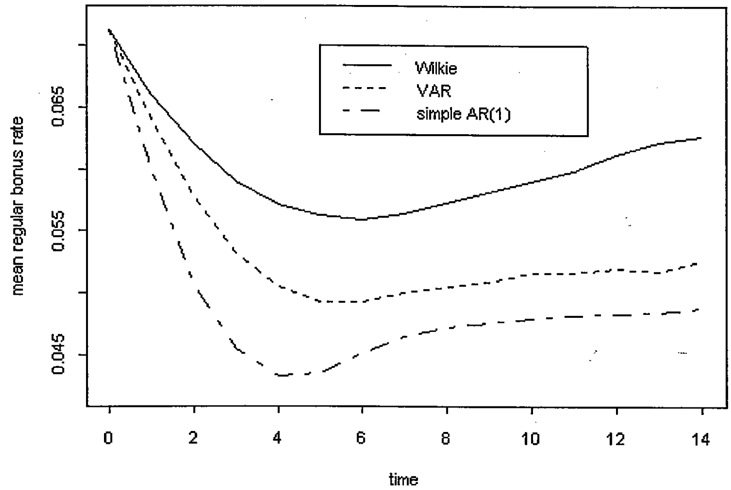

Figure 17. Mean regular bonus rate – before standardisation.

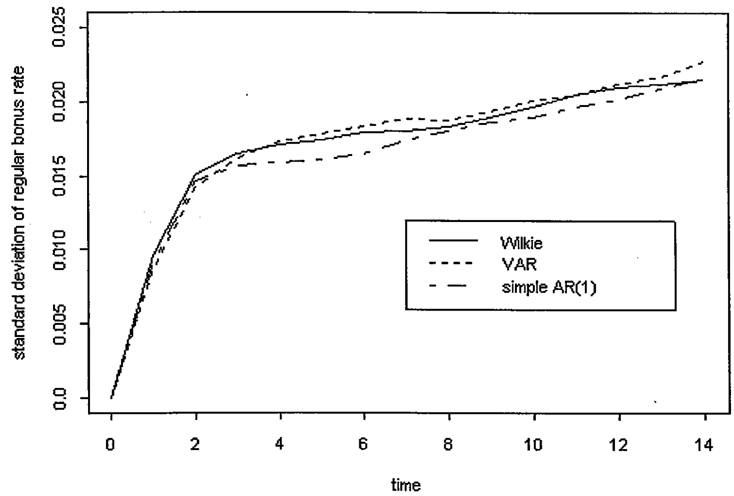

Figure 18. Standard deviation of regular bonus rate – before standardisation.

The standard deviations of bonus rates for the three models are not dissimilar.

All three models produce a similar progression of mean bonus rate over time, in that the mean rate declines steeply from the projection date, reaching a minimum within 4 to 6 years, then gradually rising (and levelling out, in some cases) thereafter. This indicates that, under all asset models, the levels of guarantee assumed prior to the projection date are actually inconsistent with the bonus distribution philosophy assumed in the stochastic asset-liability model. This does not, however, invalidate a comparison between models, and it is clear that there is a significant model-dependent effect here.

Because the modelled bonus rates are heavily smoothed over time, an increase in the variability in returns, and/or a reduction in mean returns, could produce the initial reduction in bonus rates observed. It will be recalled that the bonus rate is driven by a combination of the gilt yield and also by reference to a reduced asset share calculation, which imposes a target maximum policy fund value at any time. While there are differences in the mean gilt yields in most projection years before standardisation (see Table 1 and Figures 5 and 6), the gilt yields for the AR(1) and VAR models are very similar and cannot, therefore, explain the marked difference in mean bonus rates between the VAR and AR(1) models. It would therefore appear that the mean gilt yield is not a significant driver of bonus rate differences between the three models.

However, the mean bonus rates are, for all models, considerably lower than the mean gilt yields produced by each model. This must therefore be due to the combined effects of asset return variability and the bonus smoothing algorithm incorporated in the model. Asset return variability could affect simulated bonus rates by either or both of the two mechanisms referred to in the previous paragraph: (a) through variability in the gilt yield itself, and (b) through variability of the reduced policy asset share, which will be a function of the variability of the overall asset returns produced by the model. Experiments easily confirmed that (b) was by far the most important mechanism in bringing about the observed reduction in bonus rates over the first 4-6 projection years. Reference to Figure 2 shows that equity return variability over the shortest durations are clearly ranked, from highest to lowest, in the order AR(1), VAR and Wilkie. The standard deviations of the gilt yield (Figure 6) are also ranked in the same order. There is therefore clear evidence that differences in short term asset return variability have driven the differences in mean bonus rates through the dynamic smoothing mechanism.

Standardisation reduces, but does not eliminate, the model dependencies (see Figures 19 and 20), and a substantial difference in mean bonus rate between the Wilkie model, and the other two models, remains.

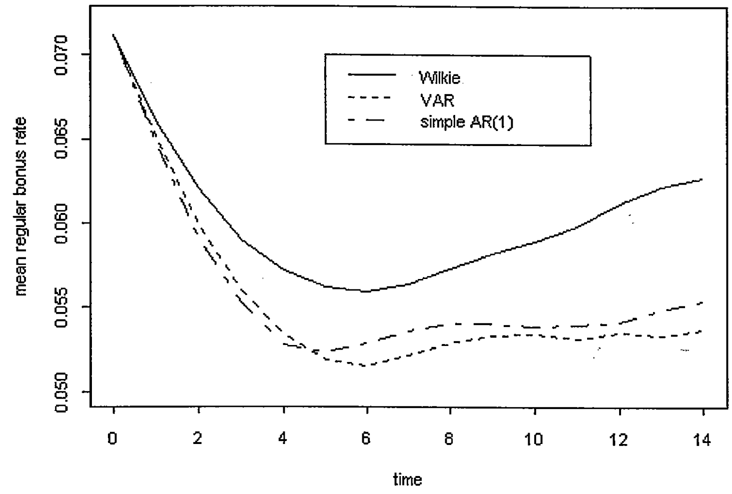

Figure 19. Mean regular bonus rate – after standardisation.

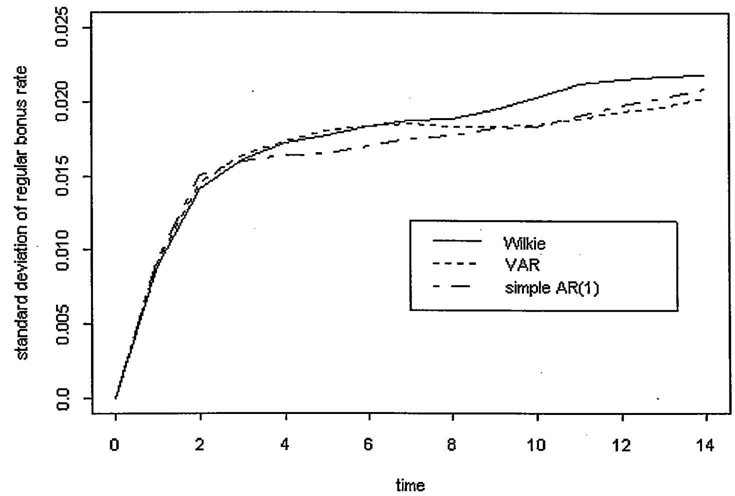

Figure 20. Standard deviation of regular bonus rate – after standardisation.

Figure 8 indicates that differences in the short duration equity return variability after standardisation closely mirrors the new behaviour of the mean bonus rates, whereas there seems little relation with the gilt yield variability (Figure 12). It is clear, therefore, that the *change* in mean bonus rates following standardisation closely mirrors the change in the variability of equity returns over short durations rather than relating to any changes to the distributions of the gilt yield. This being so, it must be remembered that the only driver for *increasing* bonus rates in the model is the gilt yield, which must therefore be material in the behaviour of mean bonus rates following about year 5 of the projection, on average. It is clear, therefore, that simulated bonus rates are quite sensitive to model differences in asset return variability.

6.1.5. *Maturity payout*

The means and standard deviations of the maturity payouts, before standardisation, are shown in Figures 21 and 22.

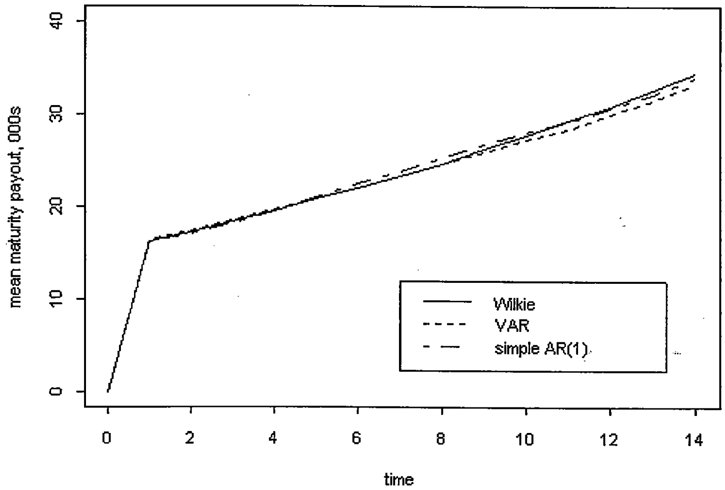

Figure 21. Mean maturity payout – before standardisation.

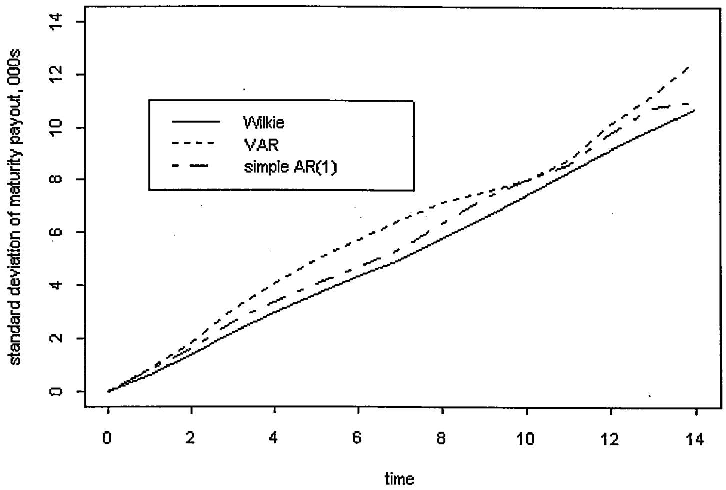

Figure 22. Standard deviation of maturity payout – before standardisation.

The mean maturity payout appears to be curiously robust to the assumed asset model, even before standardisation. The variability of the maturity payouts, on the other hand, appears to be more sensitive to the asset model assumed, with the VAR model producing noticeably higher variability than the other two models for almost all projection years. Differences in variability might certainly be anticipated from differences in the variability of the asset models' returns, given that the overall benefit level (ignoring the guarantee) is based on geometric averages of the simulated asset returns. The most significant contributor to this effect appears to be the differences in the standard deviations of the annualised equity return, which is somewhat higher for the VAR model than the others, as shown in Figure 2. This observation appears intuitive based on the fact that maturity values are a function of the model's asset returns between the projection date ($t = 0$) and the maturity date. However the equality of the mean payouts is not so immediately intuitive, bearing in mind that the mean asset returns are not identical under the three models (see Figures 1 and 3). Indeed, were there no guarantees on maturity, then the direct relation between the maturity value and the simulated asset return must lead to differences in mean maturity payout. As the latter is not actually the case, then the conclusion must be that the assumed guarantees under the contracts are significantly influencing the mean payouts in some way.

As we have seen, mean bonus rates differ considerably between the three models. Higher mean bonus rates imply a higher mean level of guarantee, and hence mean guaranteed payouts at maturity are highest under the Wilkie model and lowest under the AR(1) model. The effect of the guaranteed benefit is to increase the payout in years when asset share values are very low, which will serve also to increase the simulated mean maturity payout. A model producing high asset return variability, on the other hand, will tend to produce higher maturity payouts in 'good' years than the less volatile models, because the guaranteed benefit will not affect payouts under any of the models in such years. The robustness of the mean maturity payout between the models must therefore reflect the operation of this compensating effect. This mechanism should therefore also lead to sensitivity in the variability of maturity payouts (that is the effect will here be compounding rather than compensating) and this does indeed appear to be the case (Figure 22).

The equivalent results based on the standardised models are shown in Figures 23 and 24.

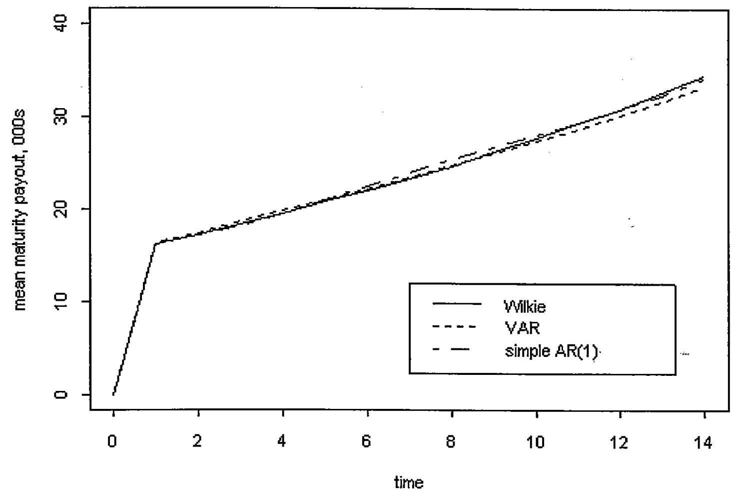

Figure 23. Mean maturity payout – after standardisation.

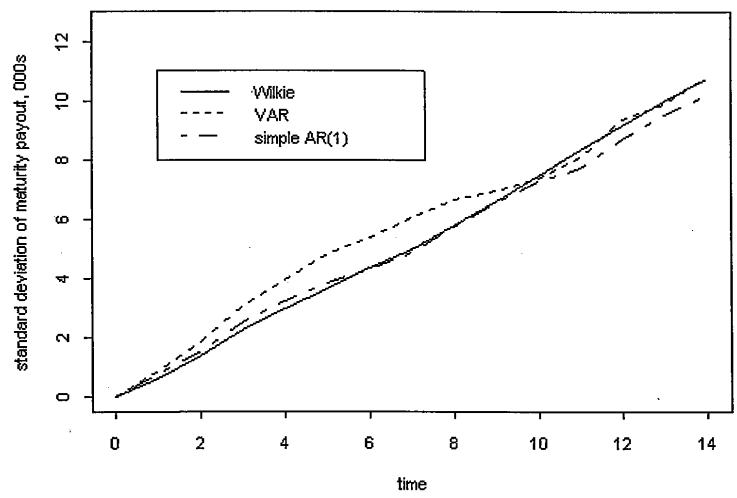

Figure 24. Standard deviation of maturity payout – after standardisation.

As might be expected, the mean payouts after standardisation remain model independent. Standardisation has, however, achieved relatively little towards reducing model dependence with regard to the variability in maturity values: both AR(1) and VAR models show slightly reduced variability (with the result that AR(1) variability is now very close to that of the Wilkie model up to about year 10), but the difference between the VAR and AR(1) models remains essentially as large as prior to standardisation. Again, the compounding effect of the mechanism for determining maturity values may well be part of the reason for this continued sensitivity.

Figure 24 gives a useful insight into the shortfalls of this (and arguably other) approaches to standardisation. Standardisation, as applied here, has achieved broad equality in the mean and variance of asset returns over ten years for the three models. In Figure 24 it can be seen that, for policies maturing in year 10, the standard deviations of payouts almost exactly coincide. These payouts will, necessarily, reflect the distribution of returns over 10 years, hence the convergence of the results of the three models. The divergence between the models at other time points, therefore, reflects residual differences between the models which have not been removed by standardisation. As we have seen, the most important reasons for this effect are the significant residual differences which remain between the standard deviations of the annualised equity returns after standardisation (Figure 8), in which the VAR model is shown to have highest variability until about year 6 or 7. These differences reflect structural differences between the models which may be difficult, if not impossible, to remove by standardising any two moments of the distribution.

6.2. *Effect of variations in management strategies on model sensitivity*

The differences between the VAR and Wilkie models after standardisation were re-tested under a number of different management assumptions, viz.:

(i) under different level asset allocation strategies;

(ii) in the presence of a dynamic crisis-response investment strategy; and

(iii) in the presence of a dynamic crisis-response bonus distribution philosophy.

The first of these is self-explanatory. The assumptions for (ii) and (iii) are described below.

Dynamic crisis investment strategy

Incorporating the dynamic crisis investment strategy will imply that, if at the end of a year the ratio of the value of the assets to the statutory value of the liabilities (the 'solvency ratio') is less than 1.15, equities are progressively moved into gilts until there is 100% investment in gilts at a solvency ratio or 1.05 or less. This rule, when it applies, overrides the normal rebalancing rule, described in section 3.2.5. This is similar to an assumption first suggested by Ross (1989), except that under his assumptions the strategy took effect over the range of solvency values down from 1.25. The lower value of 1.15 chosen here was felt to be more in keeping with the nature of Accumulating With-Profits, although any level chosen is essentially arbitrary.

Crisis bonus distribution strategy

If a crisis of solvency exists, then the office can reduce its risk by withholding bonus distributions which it would otherwise have made, and only reinstating those bonuses if and when solvency improves subsequently. A simple yet reasonably realistic mechanism was required for the crisis bonus strategy. (This mechanism was first used and described in Chadburn, 1997). It was felt that policyholders would reasonably expect normal bonuses to be withheld only when solvency was becoming critical. It was therefore decided that the crisis bonus response would be effective only where the solvency ratio was less than 1.04. The assumed response is:

(1) to remove all terminal bonuses from benefit payments; and

(2) to withhold the addition of the regular bonus that would otherwise have been declared if there were no solvency crisis.

As soon as the crisis is resolved, the total benefit levels return to the normal levels, and the next year's regular bonus is calculated as if there had been no interruption. However, the missed bonus (or bonuses) are not replaced, resulting in a lower level of guarantee existing for some years after the crisis has been resolved. The crisis bonus policy will therefore have both immediate and longer term benefits for the solvency of the office.

We focus here on the sensitivity of the estimates of the probabilities of absolute and statutory ruin, which for the base model assumptions were found to be very robust to the choice of model (after standardisation).

6.2.1. *The effect of different level asset allocation strategies*

The results are shown in Table 7. For ease of comparison, the results from the base model are repeated in Table 7(b).

Table 7. Estimates of the probabilities of actual and statutory ruin using the Wilkie and VAR models, under different level asset allocation strategies.

Table 7(a): 100% equities.

Model	Pr(abs. ruin)	95% Conf. Int.	Pr(stat ruin)	95% Conf. Int.
Wilkie	.110	(.083, .137)	.332	(.291, .373)
VAR	.128	(.099, .157)	.354	(.312, .396)

Table 7(b): 75% equities, 25% gilts (= base model).

Model	Pr(abs. ruin)	95% Conf. Int.	Pr(stat ruin)	95% Conf. Int.
Wilkie	.032	(.017, .047)	.142	(.111, .173)
VAR	.034	(.018, .050)	.142	(.111, .173)

Table 7(c): 50% equities, 50% gilts.

Model	Pr(abs. ruin)	95% Conf. Int.	Pr(stat ruin)	95% Conf. Int.
Wilkie	.006	(0, .013)	.044	(.026, .062)
VAR	.002	(0, .006)	.024	(.011, .037)

Table 7(d): 100% gilts.

Model	Pr(abs. ruin)	95% Conf. Int.	Pr(stat ruin)	95% Conf. Int.
Wilkie	.008	(0, .016)	.072	(.049, .095)
VAR	0	n/a	.010	(.001, .019)

With the exception of 100% gilts, we can observe only small absolute differences in the estimated ruin probabilities, none of which are statistically significant. However, there is a significant difference in the estimation of the probability of statutory ruin when 100% gilts are assumed. This seems to indicate that the standardisation of the gilt model has been less successful than for the equity model, but hitherto its effects have not been so obvious due to the greater influence of the equity return in the other scenarios used so far.

6.2.2. *The effect of incorporating a dynamic investment strategy*

The estimates of the probabilities of ruin under the two models (after standardisation), when a dynamic investment strategy is incorporated into the model, are shown in Table 8.

Table 8. Estimates of the probabilities of actual and statutory ruin using the Wilkie and VAR models, when a dynamic investment strategy is incorporated.

Model	Pr(abs. ruin)	95% Conf. Int.	Pr(stat ruin)	95% Conf. Int.
Wilkie	.036	(.020, .052)	.122	(.093, .151)
VAR	.016	(.005, .027)	.066	(.044, .088)

Here there is a significant difference in the estimate of the probability of statutory ruin, and the estimates for the actual ruin probability are almost significantly different, despite their small absolute values.

There are two possible reasons for this increased sensitivity:

(a) incorporation of the dynamic strategy implies an overall mix of equities and gilts which could be highly weighted towards 100% gilts, bearing in mind the level of initial estate assumed, so the model outcomes will reflect the higher sensitivity already observed to the gilt model; and

(b) the additional dynamic link in the asset-liability model could lead to a compounding of the effects caused by the existing differences in the asset models, thereby increasing the observed differences in the outcome variable.

Note that the reductions in the estimates of the statutory ruin probabilities are 2% for the Wilkie model (which is a fall of 14% of its original value), and 5.6% for the VAR model (a fall of 39% of its original value). Hence a user of the VAR model would conclude that the crisis investment strategy reduces the probability of statutory ruin by almost three times that which would be concluded by a user of the Wilkie model. Bearing in mind these models have been standardised, this is an important observation and shows how sensitive even relative results can be to the choice of asset model.

6.2.3. *The effect of incorporating a dynamic crisis bonus distribution strategy*

The results are shown in Table 9.

Table 9. Estimates of the probabilities of actual and statutory ruin using the Wilkie and VAR models, when a dynamic bonus distribution strategy is incorporated.

Model	Pr(abs. ruin)	95% Conf. Int.	Pr(stat ruin)	95% Conf. Int.
Wilkie	.026	(.012, .040)	.138	(.108, .168
VAR	.024	(.011, .037)	.130	(.101, .159)

The inclusion of this strategy has not actually reduced the probabilities of ruin by very much (in the case of absolute ruin a reduction of .006 for the Wilkie model and of .010 for VAR, and equivalently in the case of statutory ruin by .004 and .012 respectively). The net results are therefore still very similar between the two models, but there is a suggestion of greater sensitivity when this additional dynamic structure is included in the asset-liability model.

7. *Conclusions*

The models before standardisation were broadly parameterised to the same data set. Nevertheless significant differences in several of the output measures were observed (the frequencies of absolute and statutory ruin, the statutory solvency ratio, the variability of the maturity payouts, and (in particular) the simulated regular bonus rates). These results indicate that differences in model structure have significant implications for simulation results. However the robustness of the mean maturity payout was a significant feature of the models' behaviour before standardisation.

An attempt was made to try to eliminate these model-dependent effects, by standardising the mean and variance of the 10-year simulated asset returns of the VAR and AR(1) models to those produced by the Wilkie model. The aim was to see whether the control of the parameterisation by the user, based on just two moments of the distribution for each asset class, might be sufficient to remove the sensitivity to the asset model structure. This was indeed found to be the case for some of the simulated variables under certain conditions, in particular the frequencies of absolute and statutory ruin, and (to an extent) the value of the statutory solvency ratio.

However, clear evidence remained of significant sensitivity persisting after standardisation, exampled by the residual differences in the mean bonus rates and their corresponding effect on the variability of maturity payouts. This was an effect of substantial differences in short duration asset return variability still remaining between the standardised models. Clearly, the short duration returns were not explicitly included in the criteria for standardisation, and shows that standardising only the ten-year asset returns did not achieve particularly good standardisation of returns at shorter durations.

It is clear that a greater extent of standardisation would be necessary in order to achieve more similar output behaviour. For different asset models, this would probably be impossible to achieve, other than by changing the model structures themselves to be more similar to each other. For example, if attempts were to be made to standardise the models' asset returns at shorter durations, it is likely that the longer duration asset returns would diverge, reflecting differences in the assumed correlations in asset returns over time.

It was argued that the existence of dynamic asset-liability structures in the model can lead to a compounding of underlying asset model differences. This appears to have occurred in the case of the simulated variability of maturity values. It should be noted, however, that no indication of this potential sensitivity would have been obtained had only the mean maturity values been monitored in these experiments.

This therefore begs the question as to whether any of the other variables, which were seemingly robust to model structure, could actually prove not to be so robust in different circumstances. This refers, therefore, to the estimates of the probabilities of absolute and statutory ruin. We observed that ruin frequencies began to differ significantly when much of the asset portfolio was assumed to be in gilts. This was argued to be a result of less effective standardisation for this asset class, which might well be improved upon. The robustness of the frequencies of ruin must also be considered in the light of the dynamic structures which exist in the asset-liability model assumed, which in particular related to the bonus-distribution model. This model was shown to respond significantly to the assumed asset-return variability, by reducing bonus rates and therefore reducing policy benefit guarantees. Hence while a model with high asset return variability would (for a given level of policy guarantee) naturally lead to a higher ruin frequency, in the present models the ruin probability was reduced by the lower guarantees which built up under the policies over time. The relative robustness of the simulated ruin frequency must therefore, to some extent, reflect this built-in feedback mechanism. Hence while we may (arguably) accept that ruin frequency is broadly insensitive to model structure differences for a (properly structured) with-profits asset-liability model, this is much less likely to be the case for a non-profits model, for example.

Intrinsically, however, dynamic links serve to increase, or gear up, model output variability compared with that of the underlying asset models. Prime examples of this were found in the simulation of the mean bonus rate and of the variability of the maturity values by the present asset-liability model. Other possible examples were seen by the inclusion of dynamic crisis management strategies. Hence, notwithstanding the compensatory effects of the with-profits model described in the previous paragraph, greater sensitivity of output measures would be expected the greater the complexity of dynamic links incorporated in the asset-liability model.

A salutary example of the dangers of drawing absolute conclusions from stochastic modelling was identified in the case where the effect of adopting a particular dynamic investment strategy was being considered. Here the strategy would have been judged as almost three times more effective at reducing the risk of statutory ruin under one asset model compared with another, even where those two asset models had been standardised.

The results of this work have important implications for the use of stochastic asset-liability models as a basis for actuarial advice in a life-office context. Clearly model output must be considered sensitive to model structure as well as to model parameterisation. The user must be aware of the implications he or she is making through both the choice of model and the choice of parameter values made. It appears from this work that life office model behaviour can be understood by reference to simulated summary output measures from the asset models used. Provided the user is to assume a static asset allocation in all projection years, we suggest that it may be sufficient to perform an analysis of the mean and standard deviation of the annualised overall returns, calculated up to all future projection years ($t = 1, 2, ...$), for the particular asset mix being assumed. The effects of the autocorrelations between the returns of the different asset classes over time, and also of the correlations in the returns between the different asset classes, are thereby incorporated into summary measures which are relatively easily interpreted by the user. The effects of the autocorrelations and cross-correlations would usually be unfathomable by looking at the model's mathematical structure and parameter values.

In the case where a non-static asset mix is being assumed, a number of alternatives might be proposed. The duration analysis of annualised returns might be performed for each asset class separately, as done in this study, but then this would require further identification of and allowance for the effect of the cross-correlations between asset classes. Another, more pragmatic, method might be to perform the duration analysis over a range of typical asset mixes, thereby giving the user an indication of the relationship between the overall asset return behaviour and the underlying asset mix. Admittedly, this would become rather complicated where more than two asset classes are involved, but may be sufficient for many practical applications.

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Appendix A - *A simple first-order auto-regression model*

A.1. *Force of price inflation*

The force of price inflation at time t, denoted by $I(t)$, is modelled as a first-order auto-regressive process :

$$I(t) - QMU = QA.\left[I(t-1) - QMU\right] + QE(t)$$

where $QE(t) = QSD.QZ(t)$, and $QZ(t)$ is a series of *i.i.d.* $N(0,1)$ random variables.

Then, fitting this process to U.K. annual price inflation data over the period 1923-94 using the Yule-Walker equations gives the following parameter estimates :

$$\begin{aligned} QMU &= 0.0433 \\ QA &= 0.6057 \\ QSD &= 0.0453 \end{aligned}$$

A.2. *Share dividend yield*

The share dividend yield at time t, denoted by $Y(t)$, is modelled as a first-order auto-regressive process :

$$Y(t) - YMU = YA.\left[Y(t-1) - YMU\right] + YE(t)$$

where $\{YE(t)\}$ is a noise series with zero mean.

Also, as the error term for the share dividend yield is contemporaneously cross-correlated with the error term for the force of price inflation, we have :

$$YE(t) = YQ.QE(t) + YQE(t)$$

where $YQE(t) = YSD.YZ(t)$, and $YZ(t)$ is a series of *i.i.d.* $N(0,1)$ random variables.

Then, fitting this process to U.K. annual share dividend yield data over the period 1923-94 using the Yule-Walker equations gives the following parameter estimates :

$$\begin{aligned} YMU &= 0.0423 \\ YA &= 0.6443 \\ YQ &= 0.0825 \\ YSD &= 0.0069 \end{aligned}$$

A.3. *Force of share dividend growth*

The real force of share dividend growth at time t, denoted by $KQ(t)$, is modelled as a noise series :

$$KQ(t) = KMU + KE(t)$$

where $\{KE(t)\}$ is a noise series with zero mean.

Then, the nominal force of share dividend growth a time t, denoted by $K(t)$, is given by $K(t) = I(t) + KQ(t)$, where $I(t)$ is the force of price inflation at time t.

Also, as the error term for the force of share dividend growth is contemporaneously cross-correlated with the error term for the force of price inflation, we have :

$$KE(t) = KQ.QE(t) + KQE(t)$$

where $KQE(t) = KSD.KZ(t)$, and $KZ(t)$ is a series of *i.i.d.* $N(0,1)$ random variables.

Then, fitting this process to U.K. annual share dividend data over the period 1923-94 using the Yule-Walker equations gives the following parameter estimates :

$$\begin{aligned} KMU &= 0.0123 \\ KQ &= -0.3455 \\ KSD &= 0.0896 \end{aligned}$$

A.4. *Yield on long-dated fixed-interest gilts*

The yield on long-dated fixed-interest gilts at time t, denoted by $C(t)$, is modelled as a first-order auto-regressive process :

$$C(t) - CMU = CA.[C(t-1) - CMU] + CE(t)$$

where $\{CE(t)\}$ is a noise series with zero mean.

Also, as the error term for the yield on long-dated fixed-interest gilts is contemporaneously cross-correlated with the error term for the force of price inflation, we have :

$$CE(t) = CQ.QE(t) + CQE(t)$$

where $CQE(t) = CSD.CZ(t)$, and $CZ(t)$ is a series of *i.i.d.* $N(0,1)$ random variables.

Then, fitting this process to U.K. long-dated fixed-interest gilt yield data over the period 1923-94 using the Yule-Walker equations gives the following parameter estimates :

$$\begin{aligned} CMU &= 0.0764 \\ CA &= 0.9601 \\ CQ &= 0.0652 \\ CSD &= 0.0085 \end{aligned}$$

APPENDIX B - *Model office routines for calculating benefits*

(Subscript t replaces a,t as used in the main text, for convenience)

<u>*Calculating total benefit levels*</u>

Let roa_t = the return on assets between times $[t\text{-}1, t]$, where t=0 is the projection date;

i_t = the return attributed to policyholders over $[t\text{-}1, t]$

= $roa_t - CC - CG$

where CC is the annual charge to meet the cost of capital, and CG is the annual charge to meet the cost of the guarantees (see paragraphs 4.4 and 4.5).

Let ir_t = notional reduced return over $[t\text{-}1, t]$

= $.75 \times i_t$ for $i_t > 0$

= $i_t / .75$ for $i_t < 0$

is_t = the geometrically smoothed policyholders' return over times $[t\text{-}1,t]$, calculated as the geometric mean of i_t over the previous 3 years, ie over times $[t\text{-}3, t]$.

Then AS_t = policy asset share at time t (calculated by accumulating cash-flows over times $[0, t]$ at the rates i_k, $k = 1,2, \ldots t$);

RAS_t = reduced policy asset share at time t (calculated by accumulating cash-flows over times $[0, t]$ at the rates ir_k, $k = 1,2, \ldots t$);

SAS_t = smoothed policy asset share at time t (calculated by accumulating cash-flows over times $[0, t]$ at the rates is_k, $k = 1,2, \ldots t$);

F_t = amount of policyholder's fund at time t (calculated by accumulating cash-flows over times $[0, t]$ at the rates

$(1 + g)(1 + RB_k) - 1$, $k = 1,2, \ldots t$);

where g = guaranteed annual rate of fund increase (assumed to be .03); and

Rb_k = declared regular bonus fund increase over $[k\text{-}1, k]$, declared annually in advance.

The total benefit level at time t is calculated as:

$$B_t \quad = \quad \max\{SAS_t\,, F_t\}$$

so that the terminal bonus is:

$$TB_t \quad = \quad B_t - F_t$$

Determining the regular bonus

Define $DRT_{t\text{-}1} = \quad RB_t - RB_{t\text{-}1}$

The regular bonus rate for the year [t, t+1] is determined at time t as follows.

1. Calculate: $DRB_t = 0.5 \times \left\{ \max\left(\dfrac{ic_t - g}{1+g}, 0 \right) - RB_t \right\}$

   and $DRU_t = 0.25 \times \min\left(\dfrac{RAS_t - F_t}{RAS_t}, 0 \right)$

   where ic_t is the yield on consols at time t.

2. Calculate: $drt_t \quad = \quad DRB_t + DRU_t$
   rounded down to the next lower .0025.

3. drt_t is then constrained to lie within [-.02, +.01].

4. If drt_t is non-zero, compare it with $DRT_{t\text{-}1}$:
   - if drt_t and $DRT_{t\text{-}1}$ are of the same sign, then $DRT_t = drt_t$
   - if drt_t and $DRT_{t\text{-}1}$ are of different sign, then $DRT_t = 0$
   - if $DRT_{t\text{-}1} = 0$, then compare drt_t with $DRT_{t\text{-}2}$:

- if drt_t and DRT_{t-2} are of different sign, then .005 is deducted from the absolute value of drt_t to produce DRT_t, subject to retaining the same sign or zero.
- otherwise $DRT_t = drt_t$

5. If the return on assets is negative, then DRT_t is constrained not to exceed zero.

6. Calculate $RB_{t+1} = RB_t + DRT_t$

   with a minimum of zero.

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