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The Probationary Period as a Screening Device

Jaap Spreeuw

Abstract

In this paper, the effectiveness of the probationary period as a screening device in case of asymmetric information in an insurance market is investigated. This will be done within the classical framework of two risk classes. Both the monopolistic and the competitive insurance market will be analyzed. It will be shown that, if there is a so called partial stochastic order between the distributions of time-of-loss of the respective risk types, the equilibria resulting have at least some properties in common, compared with the corresponding equilibria related to the monetary deductible as a screening device.

Keywords: *Probationary periods, adverse selection, monopoly, competitive market, separating equilibrium, pooling equilibrium*

1 Introduction

The models discussed in the classical papers on the monetary deductible as a screening device in case of asymmetric information (Rothschild and Stiglitz [1976], Wilson [1977], Miyazaki [1977] and Spence [1978] for a fully competitive insurance market; Stiglitz [1977] in case of a monopolistic insurer) rely on the assumption that the loss in case of an accident is non-random. Offering a contract with a monetary deductible then actually means the same as offering a contract with partial coverage. Since this implies non-linear pricing, in a competitive insurance market this would require the insurers to share information with one another in a perfect way, a condition which may not always be satisfied. However, in non-life insurance, the insurer's loss in case of a claim is almost always random. Besides, in non-life business contracts usually remain in force after incurring a loss, and this gives insurers the opportunity to offer multiperiod contracts, where both premiums and coverage may be adjusted according to the claims experience. For an overview of such strategies, the reader is referred to Dionne and Doherty [1992].

On the other hand, life insurance contracts concern fixed benefits, to be paid out in case of death. Hence, there is actually no difference between imposing a monetary

deductible and lowering the benefit due in case of death. As explained above, in non-monopolistic markets such strategies require perfect sharing of knowledge among insurers concerning the purchase of insurance. Furthermore, offering multi-period contracts in the way described above is not possible, since each individual dies, i.e. produces a loss, only once.

Around 1990, some papers appeared considering different screening devices for life companies. One of these is Venezia [1993], dealing with the so-called cash-value life insurance contracts. He introduces a three-period model and argues that the savings element and its rate of return can be used as a screening device when dealing with a population consisting of two different risk classes.

An alternative method which can be used more generally, and which will be the theme of this paper, is the probationary period. Such a period excludes coverage for events that occur during some period after the inception of the policy. The method, aiming to rule out preexisting conditions, has found applications in some dental or medical policies, so the method is not exclusively available to life insurers. Moreover, recently it has gained popularity among Dutch group life companies, as a consequence of new legislation concerning medical examination of employees. By the new laws which came into force at the beginning of 1998, insurance companies are strongly restricted in their possibilities to test individual members of a group life scheme medically. Therefore they in general cannot determine whether any member has a serious disease. A probationary period may then be an appropriate instrument to keep out individuals who are likely to make a claim soon after issue. It can be compared with the money back guarantee on consumer goods, offered by a seller, see e.g. Moorthy and Srinivasan [1995]. However, most models on this signalling device lack a time dimension and are therefore not very useful in this context.

Several aspects of the probationary period, among others its impact on the expected utility of individuals, have been investigated in Eeckhoudt et al. [1988]. They prove that, in a fully competitive market, the solution under symmetric information involves full coverage for any risk class, similar to the monetary deductible case, provided that the probability of incurring an accident is smaller than one. The authors' main conclusion is that most of the basic properties of the above mentioned monetary deductibles do not carry over to probationary periods.

The aim of this paper is to investigate the effectiveness of the probationary period as a screening device in case information is asymmetric and hence adverse selection is a threat. In this sense, the approach is similar to the one in Fluet [1992] who considers the combination of a probationary period and a time-dependent monetary deductible. By allowing varying monetary deductibles, his model is more general than the model considered in Eeckhoudt et al. [1988] and in this paper. However, starting points of his analyses are a fixed contract of full coverage and an actuarially fair premium for the high risks and the assumption that the proportion of high risks among the entire population is not too low, just as in Rothschild and Stiglitz [1976]. Such restrictions are not imposed in this paper. Moreover, we will not restrict ourselves to a competitive market but will deal with the case of a monopolistic insurer as well.

The model will be described in Section 2, where also the basic assumptions will be listed. These will be the foundation for the two sections following thereafter. It will turn out that the assumption of a so-called partial stochastic order between the distribution functions of time-at-accident of the low and high risks – which means that the probability to make an accident before any relevant point of time is for the high risks at least as great as for the low risks – is crucial. In Section 3, we will deal with the case of a monopolistic insurer, while in Section 4, a fully competitive market will be considered. Both situations will deal both with the cases of symmetric and asymmetric information. Regarding the latter topic, it will be investigated under which circumstances the equilibria resulting are comparable to those related to the monetary deductible as a screening device. For the competitive market, it is assumed that companies behave *with foresight*, i.e. they foresee their competitors' response to their own policy offers. This implies that policies which are loss-making in the long run will not be offered. In both sections, it turns out that conclusions are relatively easy to draw if the utility function is taken to be exponential. Both sections will also be illustrated by a numerical example. The characteristics of the equilibria and its corresponding conditions will be compared with one another in Section 5. Conclusions and recommendations for further research will be given in Section 6.

This article is to a large extent based on Chapter 5 of the author's PhD thesis (Spreeuw [1999]).

2 The basic assumptions and the nature of a probationary period

The basic assumptions, mainly derived from Fluet [1992], are listed below:

- A population consists of two risk classes, namely the high risks and the low risks. In the remainder of this paper all variables pertaining to high risk and low risk individuals will be accompanied by the subscripts H and L, respectively. All individuals have an initial, nonrandom, wealth of W. All individuals within the population are identical, except with respect to the probability of having an accident in the period $[0, n]$, where 0 is the current time (by "accident" a certain unfavorable event is meant, which can e.g. be injury or death) and n is real valued and positive. In case an individual is faced with an accident, there is a welfare loss of D. This probability for an individual of risk class i is denoted by η_i, $i \in \{H, L\}$, with $\eta_L < \eta_H$. It is assumed that to each individual an accident can occur at most once.
- All risks are insurable.
- The proportion of high risks among the entire population is denoted by ρ.
- The time at which any accident occurs is perfectly observable to both the individual concerned and the company.

- The probability for an individual of risk class i, $i \in \{H, L\}$, of facing an accident before time t $(0 \leq t \leq n)$ is denoted by $F_i(t)$ (hence $F_i(n) = \eta_i$), being continuous in $[0, n]$. All individuals fully know these probabilities. These probabilities are exogenous, so that the risk of moral hazard is non-existent.
- To each individual, the same utility function $U(\cdot)$ applies, being increasing, strictly concave, twice continuously differentiable and independent of time.
- Only expected wealth at time n matters and all interest rates are equal to zero. Any individual's concern is to maximize expected utility.
- Insurance companies are risk neutral (implying that each company's utility function is linear). Their concern is to maximize expected profit.
- Insurance companies have enough resources to offer any set of contracts, which result in a nonnegative expected profit.
- There are no transaction costs in the supply of insurance and no costs of administering the insurance business. Nor are there costs of obtaining classification information on a potential insured when it is possible to do so.

In addition, for the sake of simplicity, we will assume that a *partial stochastic order* holds between the above introduced distributions $F_H(t)$ and $F_L(t)$:

$$F_H(t) \geq b(t) F_L(t), \text{ with } b(t) \geq 1; \quad 0 < t \leq n, \tag{1}$$

while for $t = n$:

$$\eta_H = F_H(n) = b(n) F_L(n) \text{ with } b(n) > 1. \tag{2}$$

The function $b(t)$ is assumed to be twice differentiable on the interval $[0, n]$. Later on in this paper it will turn out that, by imposing the above restrictions, for both insurance markets the resulting equilibria will have some properties in common with the corresponding equilibria related to the monetary deductible as a screening instrument (i.e. the equilibria analyzed in Rothschild and Stiglitz [1976], Wilson [1977], Miyazaki [1977] and Spence [1978] for a fully competitive insurance market and in Stiglitz [1977] for the case of a monopolistic insurer).

Remark 1 (Stochastic order) *The conclusions drawn in this chapter will not change if, instead of a partial stochastic order, one assumes the order between the distribution functions of the high risks and low risks also to hold outside the interval* $[0, n]$*, i.e.*

$$F_H(t) \geq F_L(t), \quad \forall t \geq 0. \tag{3}$$

For details about this so called stochastic order, see Kaas et al. [1994].

Remark 2 (Additional restriction) *Without additional restrictions regarding the distributions of time-at-accident, it is hard to draw conclusions in general. This was also notified by Fluet [1992]. At a certain stage in his analyses he assumes a strict inequality with respect to the corresponding hazard rates:*

$$\mu_H(t) = -\frac{d\ln(1-F_H(t))}{dt} > -\frac{d\ln(1-F_L(t))}{dt} = \mu_L(t), \quad \forall t \geq 0. \tag{4}$$

If the $>$-sign in (4) were replaced by a $\geq$-sign, then we would deal with a mortality order, as defined in Kaas et al. [1994]. These authors also prove that (4) is a more severe restriction than the partial stochastic order defined above.

In this paper, contracts will be specified by (t, P) with t and P denoting the probationary period and the premium, respectively. For the given contract, no indemnity is paid if an accident occurs in the period $[0, t]$, nor will the premium P, to be paid at time 0, be refunded to the insured. If on the other hand, an accident occurs somewhere during the period $(t, n]$, the insured will get a benefit equal to D.

In the next two sections, equilibria concerning the offering of insurance contracts in case of asymmetric information will be derived. These contracts are related to a monopolistic and a fully competitive insurance market, respectively. By "asymmetry of information", we mean that the insurer knows the proportion of both types of homogeneous subgroups with respect to the whole population but cannot monitor the risk type to which an individual belongs. In order to be able to interpret the equilibria in a useful way, each of the next two sections will start with the case of symmetry of information, where the insurer also knows the probabilities of facing an accident per individual.

As mentioned before, variables pertaining to high risk and low risk class individuals, will be accompanied by the subscript H or L, respectively. In the subsections on symmetry of information, such subscripts will not be used as the equalities and inequalities stated in these subsections apply to both risk types.

Remark 3 (Binding reservation and self-selection constraints) *In the next analyses it will turn out that any feasible contract is always subject to one or more constraints. One restriction involves that individuals from any of the two risk classes will value the contract designed for them at least as high as the contract designed for the individuals of the other risk class (the self-selection constraint). Besides, if any individual purchases the contract designed for him, his expected utility resulting must be at least as high as it would have been had he not bought any insurance at all (the reservation constraint). In some cases, one or both of these constraints are binding. Actually, for the self-selection constraint this would mean that each individual concerned is indifferent between choosing the contract designed for him or the contract designed for the other risk class. The consequence would be that, on average, half of those individuals would buy the contract designed for them and the other half would purchase the other contract. Something similar would apply for a binding reservation constraint: on average, half of the individuals considered would purchase the contract designed for them while the other*

half would not insure themselves. We, however, assume that each individual will buy the contract designed for him if the above restrictions are satisfied, whether they are binding or not.

3 Monopolistic insurer

In case the insurance market is monopolistic, the equilibrium resulting yields maximal expected profit for the insurer, since a monopolist is not hindered by (re)actions of competing firms. First it will be shown that, in case of perfect information, this aim is achieved by offering, to any individual of a certain risk class, full coverage against the maximal premium such an individual is willing to pay. In Subsection 3.2, the case of imperfect information will be discussed. Considering the exponential utility function

$$U(x) = -\alpha e^{-\alpha x}, \quad \alpha > 0, \tag{5}$$

it will turn out that only in particular cases the equilibrium resulting is similar to the ones derived in Stiglitz (1977) where the monetary deductible acts as the screening device.

3.1 Symmetry of information

Adopting the definitions stated in the previous section, the insurer's expected profit for an individual contract (t, P), denoted by $\Gamma(t, P)$, is equal to

$$\Gamma(t, P) = P - (\eta - F(t))\, D. \tag{6}$$

Regarding (t, P), we have the following *reservation* restriction for an individual with accident probability η (i.e. a condition that has to be satisfied in order to let such an individual purchase insurance):

$$F(t)U(W - P - D) + (1 - F(t))\, U(W - P) \geq \eta U(W - D) + (1 - \eta)\, U(W). \tag{7}$$

Since the left hand side of (7) decreases as either t or P increases, while

$$\frac{\partial \Gamma(t, P)}{\partial P} > 0 \text{ and } \frac{\partial \Gamma(t, P)}{\partial t} > 0, \tag{8}$$

equation (7) is binding:

$$F(t)U(W - P - D) + (1 - F(t))\, U(W - P) = \eta U(W - D) + (1 - \eta)\, U(W). \tag{9}$$

Remark 4 (Binding reservation constraint) *The second proposition is an example of a binding reservation constraint as explained in Remark 3. Recall that it has been assumed (see the same remark) that an individual will always buy the contract designed for him. If this were not the case, the contract designed for the individual should be designed in such a way that the expected utility resulting from buying that contract is an infinitesimally small bit higher than the expected utility resulting from not purchasing any insurance at all. The same will apply in the next subsection concerning a binding reservation constraint for the low risks.*

In the remainder of this paper, the right hand side of equation (7) will be denoted by E, so

$$E = \eta U(W-D) + (1-\eta)\, U(W), \tag{10}$$

with an appropriate subscript added if an individual belonging to a certain risk class is meant. It follows that t can be written as a function of P. This function will be denoted by $\varphi(P)$:

$$t = \varphi(P) = F^{-1}\left(\frac{U(W-P)-E}{U(W-P)-U(W-P-D)}\right). \tag{11}$$

The right hand side of (6) can also be written as a function of P alone, and hence we get:

$$\begin{aligned}\widehat{\Gamma}(P) &= P-(\eta - F(\varphi(P)))\, D \\ &\quad P-\left(\eta - \frac{U(W-P)-E}{U(W-P)-U(W-P-D)}\right) D.\end{aligned} \tag{12}$$

For full coverage the optimal premium is equal to

$$P = \varphi^{-1}(0) = W - U^{-1}(E). \tag{13}$$

In view of this, the premium for a contract with any nonnegative probationary period can be written as

$$P = \varphi^{-1}(t) = W - U^{-1}(E) - \epsilon D,\ \epsilon \geq 0. \tag{14}$$

Substituting into (12) gives, after some rewriting, the following result:

$$\begin{aligned}\widehat{\Gamma}(P) &= \widehat{\Gamma}(\varphi^{-1}(t)) = \widehat{\Gamma}\left(W - U^{-1}(E) - \epsilon D\right) \\ &= W - U^{-1}(E) - \eta D \\ &\quad + \left(\frac{(1-\epsilon)\, U(U^{-1}(E)+\epsilon D) + \epsilon U(U^{-1}(E)+(\epsilon-1)D) - E}{U(W-P)-U(W-P-D)}\right) D \\ &\leq W - U^{-1}(E) - \eta D \\ &\quad + \left(\frac{U(U^{-1}(E)+\epsilon D + ((1-\epsilon)\,0-\epsilon)D) - E}{U(W-P)-U(W-P-D)}\right) D \\ &= W - U^{-1}(E) - \eta D = \widehat{\Gamma}\left(W - U^{-1}(E)\right) = \widehat{\Gamma}(\varphi^{-1}(0)).\end{aligned} \tag{15}$$

The inequality sign follows from Jensen's inequality (which can be applied because $U(\cdot)$ is concave). It can only be replaced by an equality sign for $\epsilon = 0$. Hence offering full coverage is optimal.

We will, however, now note another peculiar thing. Notice that

$$\begin{aligned}&\frac{d\widehat{\Gamma}(P)}{dP} \\ &= 1 + \frac{\left\{\begin{array}{c} U'(W-P)\,U(W-P-D) - U'(W-P-D)\,U(W-P) \\ +E\,(U'(W-P-D) - U'(W-P)) \end{array}\right\}}{(U(W-P)-U(W-P-D))^2} D.\end{aligned} \tag{16}$$

The right hand side of (16) may be negative, contrary to the case of a monetary deductible. Hence there may be contracts with a certain strictly positive probationary period which give the individual an expected utility equal to the one corresponding to the situation of no insurance, but which will never be offered by an insurer since they yield a negative profit, as the next example shows:

Example 5 *Let $U(\cdot)$ be an exponential utility function with absolute risk aversion coefficient α:*

$$U(x) = -\alpha e^{-\alpha x}, \quad \alpha > 0. \tag{17}$$

Then regarding the contracts with an expected profit equal to no insurance, the following relationship between probationary period t and premium P holds:

$$\begin{aligned} F(t) &= \frac{U(W-P) - (\eta U(W-D) + (1-\eta)U(W))}{U(W-P) - U(W-P-D)} \\ &= \frac{\left(\eta e^{\alpha D} + (1-\eta)\right) - e^{\alpha P}}{e^{\alpha P}\left(e^{\alpha D} - 1\right)}, \end{aligned} \tag{18}$$

So the expected profit, written as a function of P only, proves to be equal to:

$$\widehat{\Gamma}(P) = P - \left(\eta - \frac{\left(\eta e^{\alpha D} + (1-\eta)\right) - e^{\alpha P}}{e^{\alpha P}\left(e^{\alpha D} - 1\right)}\right) D. \tag{19}$$

Differentiating with respect to P results in

$$\frac{d\widehat{\Gamma}(P)}{dP} = 1 - \frac{\alpha\left(\eta e^{\alpha D} + (1-\eta)\right) e^{-\alpha P}}{\left(e^{\alpha D} - 1\right)} D. \tag{20}$$

Hence, if

$$\eta > \frac{e^{\alpha D} - 1 - \alpha D}{\alpha D\left(e^{\alpha D} - 1\right)}, \tag{21}$$

there are contracts with relatively low P (or relatively high t), resulting in $\frac{d\widehat{\Gamma}(P)}{dP} < 0$, which will never be offered, because they will provide the insurer with losses.

It should be emphasized, however, that there are always contracts with a nonnegative probationary period yielding a positive profit if offered. This property always holds for contracts with full coverage, because of Jensen's inequality:

$$\begin{aligned} \widehat{\Gamma}(\varphi^{-1}(0)) &= \widehat{\Gamma}\left(W - U^{-1}(E)\right) \\ &= W - U^{-1}(\eta_H U(W-D) + (1-\eta_H)U(W)) - \eta_H D \\ &\geq W - (\eta_H(W-D) + (1-\eta_H)W) - \eta_H D \\ &= 0. \end{aligned} \tag{22}$$

3.2 Asymmetry of information

The profit optimization problem becomes more complex if the insurer does not have complete information at its disposal. Although, as assumed before, the firm still knows the relative weight of each of the two risk classes in the population, it is constrained in its facilities to offer contracts if it cannot be observed whether an individual is of a low or a high risk type. More formally stated, in case of imperfect information, the contracts to be offered to low risk and high risk individuals, in the remainder of this section denoted by (t_L, P_L) and (t_H, P_H), respectively, are not only subject to the *reservation constraint,* i.e. the restriction that they are purchased. The two contracts will have to be constructed such that, just as in the case of a monetary deductible, a self-selection mechanism is induced, which will only happen if each individual purchases the contract designed for the risk class he belongs to. This implies that *self-selection constraints* act upon these contracts: an individual belonging to the low risk class must value (t_L, P_L) as least as high as (t_H, P_H) and vice versa. Adopting the following notation for the expected value resulting from buying a certain contract (with $i \in \{L, H\}$):

$$E_i(t, P) = F_i(t)\, U(W - P - D) + (1 - F_i(t))\, U(W - P); \quad (23)$$

$$E_i = E_i(n, 0), \quad (24)$$

the Lagrangian has the following form:

$$\begin{aligned} \pounds = {} & \rho(P_H - (\eta_H - F_H(t_H))\, D) + (1 - \rho)(P_L - (\eta_L - F_L(t_L))\, D) \\ & + \lambda_L \{E_L(t_L, P_L) - E_L\} + \lambda_H \{E_H(t_H, P_H) - E_H\} \\ & + \gamma_L \{E_L(t_L, P_L) - E_L(t_H, P_H)\} + \gamma_H \{E_H(t_H, P_H) - E_H(t_L, P_L)\}, \end{aligned} \quad (25)$$

with λ_i and γ_i representing the multipliers associated with the reservation constraint and the self-selection constraint, respectively, of group i ($i \in \{H, L\}$).

In the remainder of this subsection we will compare the results of our investigations with the four main propositions derived by Stiglitz [1977] concerning the monopolist's optimal strategy in case the monetary deductible, instead of the probationary period, is the screening device. These propositions are displayed below:

1. The optimal contract for the high risks involves full coverage.
2. If a low risk individual purchases insurance, the utility of such a person is the same as the expected utility would have been, had he not purchased any insurance at all.
3. High and low risk individuals never purchase the same policy.
4. There exists a critical proportion of high risk individuals within the population, such that if the actual proportion exceeds the given critical one, the low risk individuals do not buy insurance.

Note that the first and third proposition imply that low risk individuals never buy full coverage. If the actual proportion of high risks within the population exceeds the critical one mentioned in the fourth proposition, this implies that it is optimal to offer only one contract, namely full coverage against the maximal premium the high risks are willing to pay. This premium is higher than the maximal premium the low risks are willing to pay and this is the reason why individuals of that risk type are then excluded from coverage. Offering only this contract - that is, full coverage against the maximal premium mentioned - gives rise to a nonnegative expected profit anyway. The same conclusion applies for a probationary period. We will denote the maximal premium by P_H^0, being equal to:

$$P_H^0 = W - U^{-1}(E_H). \tag{26}$$

Furthermore we will specify the maximal premium the low risks are willing to pay for full coverage as P_L^0. This quantity proves to be

$$P_L^0 = W - U^{-1}(E_L). \tag{27}$$

Recalling assumption (1) concerning a partial stochastic order, it will be shown that one or two of the given four Lagrange multipliers are equal to zero while the remaining ones are strictly positive.

First it will be shown that under a partial stochastic order, Proposition 1 of Stiglitz [1977] holds, that is that for any given feasible contract (t, P) for the low risks, the second contract, to be purchased by the high risks, always involves full coverage, both when the contract (t, P) will be purchased by the high risks if no other contract is offered, or not. Later on in this section, it will be clarified that contracts may exist which will be purchased by the low risks but not by the high risks.

If the given contract (t, P) is not purchased by the high risks, the self-selection constraint for the high risks is not binding, so by offering additionally the contract $(0, P_H^0)$ (a contract which will not be purchased by the low risks) maximal profit is achieved.

If, on the other hand, (t, P) will be purchased by the high risks if no other contract is offered, then again offering a second contract with full coverage for the high risks is optimal. This will be demonstrated by first ignoring the self-selection constraint for the low risks and then showing that for the given optimal solution this constraint is not binding. We have the following self-selection constraint for the high risks:

$$E_H(t_H, P_H) \geq E_H(t, P). \tag{28}$$

Just as in Stiglitz [1977], in order to achieve a maximal profit, the above equation must hold with equality, otherwise a contract could always be found with higher expected profit than the given (t_H, P_H) and still satisfying the inequality. Equality however implies that offering full coverage is optimal as argued in the previous subsection concerning symmetry of information.

This contract $(0, P_H)$, satisfying (28) with equality, so

$$E_H(0, P_H) = E_H(t, P), \tag{29}$$

does not violate the self-selection constraint for the low risks. The reason is that the resulting P_H is equal to:

$$P_H = W - U^{-1}(E_H(t, P)) \geq W - U^{-1}(E_L(t, P)), \tag{30}$$

where the right hand side of the above inequality represents the premium corresponding to the full coverage contract giving the low risks the same expected utility as (t, P). Note that the above inequality follows from the partial stochastic order assumed.

The conclusion is that the resulting self-selection constraint for the low risks, in this case being

$$E_L(t, P) \geq E_L(0, P_H), \tag{31}$$

is never binding, hence $\gamma_L = 0$. The Lagrangian reduces to the following expression:

$$\begin{aligned} \pounds \;=\; & \rho(P_H - \eta_H D) + (1-\rho)(P_L - (\eta_L - F_L(t_L))D) \\ & + \lambda_L \{E_L(t_L, P_L) - E_L\} + \lambda_H \{E_H(0, P_H) - E_H\} \\ & + \gamma_H \{E_H(0, P_H) - E_H(t_L, P_L)\}. \end{aligned} \tag{32}$$

As can be read in the appendix, it follows from the first order conditions that both the following restrictions on the Lagrange multipliers hold:

I) $\lambda_L > 0$;
II) $\lambda_H > 0$ and/or $\gamma_H > 0$.

Inequality I) indicates that, under the given conditions, Proposition 2 of Stiglitz [1977] also holds: whether a low risk individual purchases insurance or not, his expected utility remains the same. The pair of inequalities given in II) points out that, for the high risks, the option of purchasing the contract designed for them can never be strictly preferable to both the option of purchasing the contract for the low risks and the option of not insuring at all.

Hence there are only three different combinations to be taken into account. Next it will be shown that, contrary to the case of a monetary deductible where the self-selection constraint is always binding, there may be contracts acceptable to the low risks but not to the high risks. We will restrict ourselves to contracts which give the former class an expected utility equal to the situation of no insurance at all.

Such contracts (t_L, P_L) satisfy the following relationship:

$$t_L = F_L^{-1}\left(\frac{U(W - P_L) - E_L}{U(W - P_L) - U(W - P_L - D)}\right). \tag{33}$$

Hence

$$\begin{aligned}
& E_H(t_L, P_L) \\
= \; & F_H\left(F_L^{-1}\left(\frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)\right)U(W-P_L-D) \\
& +\left(1-F_H\left(F_L^{-1}\left(\frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)\right)\right)U(W-P_L) \\
\leq \; & b(t_L)\frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}U(W-P_L-D) \\
& +\left(1-b(t_L)\frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)U(W-P_L) \\
= \; & E_H+(b(n)-b(t_L))\,\eta_L\,(U(W)-U(W-D)) \\
& -(1-b(t_L))\,(U(W)-U(W-P_L)). \qquad (34)
\end{aligned}$$

For low values of P_L (and hence large values of t_L) and $b(t_L) > b(n)$, the following inequality may hold:

$$\begin{aligned}
& (b(n)-b(t_L))\,\eta_L\,(U(W)-U(W-D)) \\
& -(1-b(t_L))\,(U(W)-U(W-P_L)) \quad < 0. \qquad (35)
\end{aligned}$$

Contracts satisfying this inequality would be purchased by the low but not the high risks.

Example 6 *Consider the following extreme case, also dealt with in Fluet [1992]:*

$$\begin{array}{lll}
F_H(t) & = G(t), & 0 \leq t \leq t' \\
 & = G(t'), & t > t' \\
F_L(t) & = 0 & 0 \leq t < t' \\
 & = H(t) & t \geq t'
\end{array}$$

In the given case, both $G(t)$ and $H(t)$ are specified to be continuous and non-decreasing functions on the intervals $[0,t']$ and $[t',n]$, respectively, with $G(0) = H(t') = 0$, and $G(t') > H(n)$. Figure 1 gives an illustration for $t' = 10$. Consider the contract $(t', P_L(t'))$, where $P_L(t')$ is the maximal premium the low risks are willing to pay for a contract with probationary period t' (actually for full coverage). Note that this contract will never be bought by the high risks. An optimal solution involves offering $(t', P_L(t'))$ together with $(0, P_H^0)$.

The existence of such contracts may provide an opportunity for the insurer to increase profits since the optimal contract for the high risks is then always $(0, P_H^0)$.

Hence, in order to find an optimal solution, one should separate all contracts to be offered to the low risks (where the contract $(n, 0)$, involving no insurance at all, is also to be considered a contract) into two classes:

1. Contracts also acceptable to the high risks.

2. Contracts not acceptable to the high risks.

One then selects from all the contracts belonging to class 2 the one that is most profitable to the low risks. (As Example 5 indicates, this is the one with the lowest probationary period.) Then one compares the strategy of offering the combination of this contract and $(0, P_H^0)$ with the most profitable combination of the contracts belonging to class 1.

In the analyses below, for the sake of simplicity it will be assumed that (1) (the inequality expressing the partial stochastic order) holds with equality.

Remark 7 *Note that the consequence of this assumption is that cases such as Example 6 cannot occur, because then*

$$F_L(t) = 0 \Rightarrow F_H(t) = 0. \tag{36}$$

After some substitutions, the expected profit then turns out to be equal to equation (150) of the appendix:

$$\begin{aligned} & \widehat{\Gamma}(P_L, \rho) \\ = \; & \rho \left\{ W - U^{-1} \left\{ \begin{array}{c} U(W-P_L) - b\left(F_L^{-1}\left(\frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)\right) \\ \cdot \left(U(W-P_L) - E_L\right) \end{array} \right\} \right. \\ & \left. \begin{array}{c} \\ -D\eta_H \end{array} \right\} \\ & + (1-\rho)\left(P_L - D\left(\eta_L - \frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)\right). \end{aligned} \tag{37}$$

For any real valued interval of values P_L such that (t_L, P_L) belongs to class 1, differentiating the equation above with respect to P_L results in:

$$\begin{aligned} & \frac{\partial \widehat{\Gamma}(P_L, \rho)}{\partial P_L} \\ = \; & -\rho\left(U^{-1}\right)'\left(U(W-P_L) - b\left(F_L^{-1}(g(P_L))\right)\left(U(W-P_L)-E_L\right)\right) \\ & \cdot \left\{ \begin{array}{c} -U'(W-P_L) \\ -b'\left(F_L^{-1}(g(P_L))\right)\left(F_L^{-1}\right)'(g(P_L)) \cdot g'(P_L)\left(U(W-P_L)-E_L\right) \\ +b\left(F_L^{-1}(g(P_L))\right)U'(W-P_L) \end{array} \right\} \\ & + (1-\rho)\left(1 + Dg'(P_L)\right), \end{aligned} \tag{38}$$

with

$$g(P_L) = \frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}. \tag{39}$$

Local extreme values of the expected profit can be found by setting this derivative equal to zero and solving for P_L. If this gives a root falling between 0 and P_L^0, and besides the

second derivative is negative for any P_L falling within the critical region, then a global maximum is achieved. This second derivative turns out to be equal to:

$$\begin{aligned}
&\frac{\partial^2 \widehat{\Gamma}(P_L,\rho)}{\partial P_L^2} \\
= \; & -\rho\left(U^{-1}\right)''\left(U(W-P_L)-b\left(F_L^{-1}(g(P_L))\right)A(P_L)\right) \\
& \cdot\left\{\begin{array}{c} -U'(W-P_L) \\ -b'\left(F_L^{-1}(g(P_L))\right)\left(F_L^{-1}\right)'(g(P_L)) \\ \cdot g'(P_L)\,A(P_L) \\ +b\left(F_L^{-1}(g(P_L))\right)U'(W-P_L) \end{array}\right\}^2 \\
& -\rho\left(U^{-1}\right)'\left(U(W-P_L)-b\left(F_L^{-1}(g(P_L))\right)A(P_L)\right) \\
& \cdot\left\{\begin{array}{c} U''(W-P_L) \\ -A(P_L)\left\{\begin{array}{c} b''\left(F_L^{-1}(g(P_L))\right)\left(\left(F_L^{-1}\right)'(g(P_L))\right)^2(g'(P_L))^2 \\ +b'\left(F_L^{-1}(g(P_L))\right)\left(F_L^{-1}\right)''(g(P_L))(g'(P_L))^2 \\ +b'\left(F_L^{-1}(g(P_L))\right)\left(F_L^{-1}\right)'(g(P_L))\,g''(P_L) \end{array}\right\} \\ +2b'\left(F_L^{-1}(g(P_L))\right)\left(F_L^{-1}\right)'(g(P_L))\,g'(P_L)\,U'(W-P_L) \\ -b\left(F_L^{-1}(g(P_L))\right)U''(W-P_L) \end{array}\right\} \\
& +(1-\rho)\,Dg''(P_L), \qquad (40)
\end{aligned}$$

with

$$A(P_L) = U(W-P_L) - E_L. \qquad (41)$$

This expression is quite cumbersome, due to the fact that $b(\cdot)$ has not been specified, beyond the property of being twice differentiable in $[0,n]$. In the next subsection, an example will be worked out based on the assumption that $b(\cdot)$ is constant, implying that (1) reduces to

$$F_H(t) = bF_L(t), 0 < t \leq n. \qquad (42)$$

Then the formulas (37), (38) and (40) reduce to

$$\begin{aligned}
& \widehat{\Gamma}(P_L,\rho) \\
= \; & \rho\left\{\begin{array}{c} W - U^{-1}\left(bE_L-(b-1)U(W-P_L)\right) \\ -D\eta_H \end{array}\right\} \\
& +(1-\rho)\left(P_L - D\left(\eta_L - \frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)\right), \qquad (43)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} \\
= \; & -\rho(b-1)U'(W-P_L)\left(U^{-1}\right)'\left(U(W-P_L)-b\cdot\left(U(W-P_L)-E_L\right)\right) \\
& +(1-\rho)\left(1+Dg'(P_L)\right), \qquad (44)
\end{aligned}$$

and

$$\frac{\partial^2 \widehat{\Gamma}(P_L,\rho)}{\partial P_L^2} = \rho(b-1)\left\{\begin{array}{c} U''(W-P_L) \\ \cdot (U^{-1})'[bE_L-(b-1)U(W-P_L)] \\ -(b-1)(U'(W-P_L))^2 \\ \cdot (U^{-1})''[bE_L-(b-1)U(W-P_L)] \end{array}\right\}$$
$$+(1-\rho)Dg''(P_L), \tag{45}$$

respectively.

Remark 8 *Note that for constant b, (34) reduces to*

$$E_H(t_L,P_L) = E_H + (b-1)(U(W)-U(W-P_L)) > E_H, \tag{46}$$

so all contracts purchased by the low risks are then also acceptable to the high risks.

Still, in general it is hard to derive an optimal strategy. The only exception, at least after imposing a few additional restrictions, is the exponential utility function which will be discussed below.

3.3 Application: the exponential utility function

If $U(\cdot)$ has constant absolute risk aversion coefficient α, so

$$U(x) = -\alpha e^{-\alpha x}, \quad \alpha > 0, \tag{47}$$

the formulas (43), (44) and (45) become:

$$\begin{aligned}\widehat{\Gamma}(P_L,\rho) &= \rho\left(\frac{1}{\alpha}\ln\left[(b-1)e^{\alpha P_L} - b\left(\eta_L\left(e^{\alpha D}-1\right)+1\right)\right] - D\eta_H\right) \\ &\quad +(1-\rho)\left(P_L + D\left(\frac{\eta_L\left(e^{\alpha D}-1\right)+1-e^{\alpha P_L}}{e^{\alpha P_L}\left(e^{\alpha D}-1\right)} - \eta_L\right)\right),\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} &= -\rho\frac{(b-1)e^{\alpha P_L}}{b\left(\eta_L\left(e^{\alpha D}-1\right)+1\right)-(b-1)e^{\alpha P_L}} \\ &\quad +(1-\rho)\left(1-\alpha D\frac{\eta_L\left(e^{\alpha D}-1\right)+1}{e^{\alpha P_L}\left(e^{\alpha D}-1\right)}\right),\end{aligned} \tag{49}$$

$$\frac{\partial^2 \widehat{\Gamma}(P_L,\rho)}{\partial P_L^2} = \left(\eta_L\left(e^{\alpha D}-1\right)+1\right)\left\{\begin{array}{c} -\rho\alpha\frac{b(b-1)e^{\alpha P_L}}{\left(b(\eta_L(e^{\alpha D}-1)+1)-(b-1)e^{\alpha P_L}\right)^2} \\ +(1-\rho)\alpha\frac{\alpha D}{e^{\alpha P_L}(e^{\alpha D}-1)} \end{array}\right\}. \tag{50}$$

Note that the second derivative is decreasing in P_L. We now define

$$\rho^* = \frac{\alpha D}{(b-1)\, b\left(e^{\alpha D}-1\right)+\alpha D}, \tag{51}$$

and

$$\rho^{**} = \frac{\alpha D\left(b\eta_L\left(e^{\alpha D}-1\right)+1\right)^2}{(b-1)\, b\left(e^{\alpha D}-1\right)+\alpha D\left(b\eta_L\left(e^{\alpha D}-1\right)+1\right)^2}, \tag{52}$$

being the solutions of

$$\left.\frac{\partial^2 \widehat{\Gamma}\left(P_L,\rho\right)}{\partial P_L^2}\right|_{\left(P_L=P_L^0\right)} = 0, \tag{53}$$

and

$$\left.\frac{\partial^2 \widehat{\Gamma}\left(P_L,\rho\right)}{\partial P_L^2}\right|_{\left(P_L=0\right)} = 0, \tag{54}$$

respectively. We have $\rho^* < \rho^{**}$. Next let ρ_0 and $\rho_{P_L^0}$ be the solutions of

$$\left.\frac{\partial \widehat{\Gamma}\left(P_L,\rho\right)}{\partial P_L}\right|_{\left(P_L=0\right)} = 0, \tag{55}$$

and

$$\left.\frac{\partial \widehat{\Gamma}\left(P_L,\rho\right)}{\partial P_L}\right|_{\left(P_L=P_L^0\right)} = 0. \tag{56}$$

They are equal to

$$\rho_0 = \frac{\left(b\eta_L\left(e^{\alpha D}-1\right)+1\right)\left(e^{\alpha D}-1-\alpha D-\eta_L\left(e^{\alpha D}-1\right)\alpha D\right)}{\left(\eta_L\left(e^{\alpha D}-1\right)+1\right)\left(b\left(e^{\alpha D}-1\right)-\alpha D-\eta_L b\alpha D\left(e^{\alpha D}-1\right)\right)} \tag{57}$$

and

$$\rho_{P_L^0} = \frac{\left(e^{\alpha D}-1\right)-\alpha D}{b\left(e^{\alpha D}-1\right)-\alpha D}. \tag{58}$$

We will now introduce two theorems, proofs of which can be found in the appendix. In the next theorem a critical value of ρ is

$$\rho_1 = \frac{\ln\left[\eta_L\left(e^{\alpha D}-1\right)+1\right]-\alpha D\eta_L}{\ln\left[b\eta_L\left(e^{\alpha D}-1\right)+1\right]-\alpha D\eta_L}. \tag{59}$$

For $\rho < \rho_1$, the strategy of offering only $(0, P_L^0)$ is to be preferred to the strategy of offering only $(0, P_H^0)$, while for $\rho > \rho_1$, the opposite holds.

Theorem 9 *If* $b\eta_L < 1$ *as well as*

$$b < \frac{\alpha D}{e^{\alpha D} - 1 - \alpha D}, \tag{60}$$

then in case $\rho < \rho_1$*, it is optimal to offer only* $(0, P_L^0)$ *(a pooling contract) while in case* $\rho > \rho_1$*, the optimal strategy involves offering only* $(0, P_H^0)$ *(i.e. excluding the low risks from coverage).*

This theorem states that, if b, being the fixed proportion between $F_H(t)$ and $F_L(t)$, is lower than a certain critical level, then offering a contract with a probationary period is never optimal. Instead, it is always optimal to offer only one contract involving full coverage. Which contract this should be depends on the proportion of high risks within the population. If this proportion is lower than the given critical level ρ_1, defined in (59), a maximal profit is achieved by charging the maximal premium the *low risks* are willing to pay. On the other hand, if the critical ratio is exceeded, this premium should be the maximal one the *high risks* are willing to pay.

Note that, as $b > 1$, restriction (60) requires that $\frac{\alpha D}{e^{\alpha D} - 1 - \alpha D} > 1$. This is the case for $\alpha D \leq 1.2564$.

Theorem 10 *If*

$$b > \frac{\alpha D}{e^{\alpha D} - 1 - \alpha D}, \tag{61}$$

and

$$\eta_L < \frac{b\left(e^{\alpha D} - 1\right) - \alpha D}{2b\alpha D\left(e^{\alpha D} - 1\right)}, \tag{62}$$

the optimal strategy depends on ρ *in the following way:*

1. $\rho < \rho_{P_L^0}$*: it is optimal to offer only* $(0, P_L^0)$;
2. $\rho > \rho_0$*: it is optimal to offer only* $(0, P_H^0)$;
3. $\rho_{P_L^0} < \rho < \rho_0$*: it is optimal to offer* (t_L^*, P_L^*) *together with* $(0, P_H^*)$*, where*

$$P_L^* = P_L^0 + \frac{1}{\alpha} \ln \left[\frac{\begin{array}{c}(1-\rho)\left(b\left(e^{\alpha D} - 1\right) + (b-1)\alpha D\right) \\ + \sqrt{(1-\rho)\left\{\begin{array}{c}(1-\rho)\left(b\left(e^{\alpha D} - 1\right) + (b-1)\alpha D\right)^2 \\ -4(b-1)\alpha D b\left(e^{\alpha D} - 1\right)\end{array}\right\}}\end{array}}{2(b-1)\left(e^{\alpha D} - 1\right)} \right], \tag{63}$$

$$t_L^* = F_L^{-1}\left(\frac{U(W - P_L^*) - E_L}{U(W - P_L^*) - U(W - P_L^* - D)}\right), \tag{64}$$

and

$$P_H^* = W - U^{-1}\left(bE_L - (b-1)\,U\left(W - P_L^*\right)\right). \tag{65}$$

This solution satisfies

$$E_H\left(t_L^*, P_L^*\right) = E_H\left(0, P_H^*\right). \tag{66}$$

This theorem shows the conditions under which the optimal solution is most similar to the one derived in Stiglitz [1977]: if η_L falls below a certain upper bound and b exceeds a certain lower bound, and moreover, ρ belongs to a certain interval, then the insurer maximizes expected profits by offering insurance with a deductible to the low risks and full coverage to the high risks, thus satisfying the first two propositions of Stiglitz. Besides, there is a critical value of ρ, namely ρ_0, such that if this value is exceeded, the low risks will not purchase any coverage. Because of this, the *fourth* proposition of Stiglitz is satisfied as well. Note, however, an essential difference with Stiglitz' *third* proposition: it may be optimal to let the high and low risks buy the same contract. This is the case if ρ falls below another critical level.

In Figure 2, the upper bound of η_L as a function of αD, expressed in (62), is displayed graphically for $b = 2$ and $\alpha D \geq 0.25$. For $\alpha D \leq 0.2838$, this upper bound is greater than 1, so in that case there is in fact no restriction at all.

Example 11 *We take* $n = 30$, $\alpha = 1 \cdot 10^{-5}, D = 2 \cdot 10^5$, $b = 3$, *and*

$$F_L(t) = 1 - \exp\left(-\frac{0.003 \cdot 1.01^{35}}{\ln(1.01)}\left(1.01^t - 1\right)\right) \tag{67}$$

(so the mortality law is Gompertz (x, δ, κ) *with parameters* $x = 35, \delta = 0.003, \kappa = 1.01$*). Note that*

$$b = 3 > 0.45568 = \frac{\alpha D}{e^{\alpha D} - 1 - \alpha D} \tag{68}$$

and

$$\eta_L = F_L(n) = 0.13806 \leq 0.22391 = \frac{b\left(e^{\alpha D} - 1\right) - \alpha D}{2b\alpha D\left(e^{\alpha D} - 1\right)}, \tag{69}$$

so the conditions of the previous theorem are satisfied. It turns out that $\rho_0 = 0.42825$ *and* $\rho_{P_L^0} = 0.25567$. *Hence the optimal solution is similar to the one given in Stiglitz (1977) for* $\rho \in (0.25567, 0.42825)$. *In Figure 3, the resulting optimal values for* P_H *and* P_L *are displayed. The corresponding values for* t_L *are as given in Figure 4.*

4 Fully competitive market

A monopolistic insurer does not have to take into account possible actions of competitors and therefore the equilibrium in a monopolistic insurance market is obtained as a result of maximizing the aggregate profit. This is not the case for a fully competitive market.

Instead each company has to consider the possibility that a competitor in the same market may develop a strategy, which results in attracting the risks away of the former firm. In this section, we assume that there is freedom of entry and exit for insurance companies. Just as for the monopolistic case, we will first deal with perfect and then with imperfect information.

4.1 Symmetry of information

If an insurer has complete information on any individual's risk profile, the equilibrium resulting requires offering to any individual an actuarially fair priced contract. The reason lies in the fact that under competition firms are constrained to earn zero profits.

So for any probationary period t, the premium P is equal to

$$P = (\eta - F(t))\, D. \tag{70}$$

Another property of an equilibrium is that each individual's expected utility is maximal. If existing companies fail to meet this criterion regarding members of any risk type, any insurer would have the opportunity to offer a contract to that type with higher expected utility for the individual and positive expected profit for the firm. Just as in the case of a monopolistic insurer, this requirement is satisfied by offering full coverage, i.e. no probationary period, to any risk type, as shown in Eeckhoudt et al. [1988]:

$$\begin{aligned}
& E(t, (\eta - F(t))\, D) \\
= \;& F(t)\, U(W - D - P) + (1 - F(t))\, U(W - P) \\
= \;& F(t)\, U(W - D - (\eta - F(t))\, D) + (1 - F(t))\, U(W - (\eta - F(t))\, D) \\
\leq \;& U(W - (\eta - F(t))\, D - F(t)\, D) \\
= \;& U(W - \eta D) \\
= \;& F(0)\, U(W - D - (\eta - F(0))\, D) + (1 - F(0))\, U(W - (\eta - F(0))\, D) \\
= \;& E(0, \eta D)\,.
\end{aligned} \tag{71}$$

Note that

$$\begin{aligned}
& \frac{\partial E(t, (\eta - F(t))\, D)}{\partial t} \\
= \;& F'(t) \left\{ \begin{array}{c} U(W - D - (\eta - F(t))\, D) - U(W - (\eta - F(t))\, D) \\ +F(t)\, DU'(W - D - (\eta - F(t))\, D) \\ +(1 - F(t))\, DU'(W - (\eta - F(t))\, D) \end{array} \right\}.
\end{aligned} \tag{72}$$

The right hand side of (72) does not have to be negative. This result, also observed in Eeckhoudt et al. [1988], is in contrast with the case of the monetary deductible, where the individual's expected utility always decreases with increasing monetary deductible.

Example 12 *Let $U(\cdot)$ be an exponential utility function with absolute risk aversion coefficient α:*

$$U(x) = -\alpha e^{-\alpha x}, \quad \alpha > 0. \tag{73}$$

Then (72) reduces to

$$\begin{aligned} &\frac{\partial E(t, (\eta - F(t))D)}{\partial t} \\ &= F'(t)\, e^{\alpha(W-(\eta-F(t))D)} \left(F(t)\,\alpha D\left(e^{\alpha D} - 1\right) - \left(e^{\alpha D} - 1 - \alpha D\right)\right), \end{aligned} \tag{74}$$

which is positive for at least some t if

$$\eta > \frac{e^{\alpha D} - 1 - \alpha D}{\alpha D\left(e^{\alpha D} - 1\right)}. \tag{75}$$

The inequality derived is exactly equal to the one resulting in Example 5, being its monopolistic counterpart. This is just as one would expect. In Example 5 it has been shown that, if the above inequality holds, there are contracts which result in the same expected utility the individual would have had without purchasing any insurance. This example serves to show an equivalent conclusion: that there are contracts with a positive probationary period for which the insurer will break even on average, but which will not be bought by the insured, since the expected utility is lower than the one corresponding to not insuring oneself at all.

4.2 Asymmetry of information

If an insurer cannot distinguish between the different risk types, it makes no sense, of course, to offer the two actuarially fair contracts with full coverage to anybody, as all individuals will prefer the one with the lowest premium (the actuarial premium concerning the low risks), resulting in losses for each insurer.

Rothschild & Stiglitz [1976], considering the monetary deductible as a screening device, assume that each insurer follows a pure Cournot-Nash strategy. This means that all firms behave *without foresight*, i.e. they do not take into account their competitors' responses to their own strategy. The authors prove that, if the proportion of high risks in the population exceeds a certain critical level, there is a separating equilibrium with the following properties:

1. The high risks buy full coverage against an actuarially fair premium.

2. The low risks buy partial coverage, also against an actuarially fair premium.

For the probationary period, a similar strategy exists. It will be formulated in Definition 15 and we will call it the *Rothschild-Stiglitz strategy.*

If, on the other hand, the actual proportion of high risks in the portfolio falls below the critical one, there is no equilibrium. Rothschild & Stiglitz [1976] also show that an

equilibrium can never involve offering one pooling contract, where the low risks subsidize the high risks, because this would induce a new entrant to offer a contract attracting the low risks, but not the high risks, away from the insurer with the pooling contract. The latter policy would then result in losses.

In this section, however, we assume that any company in the market behaves *with foresight*: it explicitly considers the reactions of competitors to its own strategy. The consequence is that, as shown in Wilson [1977], an equilibrium always exists. Such an equilibrium is called a *Wilson equilibrium*, defined in Spence [1978] as follows:

Definition 13 (Wilson equilibrium) *A set of contracts offered is called a Wilson equilibrium if no firm can offer a different set that a) earns positive expected profits right away and b) continues to be profitable after competitors have dropped all unprofitable policies in response to the original firm's move.*

A difference with the situation of myopic firms just considered is that an equilibrium may involve a pooling contract, as shown in Wilson [1977]. We furthermore assume that the distribution of individuals is stable over time and that the probability distributions $F_H(\cdot)$ and $F_L(\cdot)$ remain unaltered as well.

In this section, Wilson equilibria will be derived and their characteristics compared with those involving the monetary deductible as a screening device. The main properties of the Wilson equilibrium are stated below:

1. The Wilson equilibrium results as a solution of the problem of maximizing the expected utility for the low risks under certain constraints.
2. Each firm's expected profit is equal to zero.
3. High risk and low risk individuals never purchase the same policy.
4. The high risks purchase full coverage against a price which is at most actuarially fair.
5. The low risks purchase partial coverage and pay a price which is at least actuarially fair.
6. There exists a critical ratio of high risk individuals to the entire population. If the actual ratio exceeds the critical one, the Rothschild-Stiglitz strategy applies. Otherwise, the low risks subsidize the high risks, but the low risks' deductible is lower than in the case of actuarially fair pricing.

Next, it will be shown that the *first* and *fourth* criterion also apply to the case of a probationary period, at least under the given assumptions. In a Wilson equilibrium, the final expected utility for the high risks should be at least equal to $U(W - \eta_H D)$, being the utility resulting from purchasing full coverage against the actuarially fair premium

$\eta_H D$. To verify this, suppose that the maximal utility for the high risks were equal to $U(W - \eta_H D - \beta D)$, with $\beta > 0$ such that

$$(\eta_H + \beta) D \leq W - U^{-1}(E_H), \tag{76}$$

where E_H is as defined in Subsection 3.2. (Recall that the right hand side of the above inequality represents the maximal premium the high risks are willing to pay in return for full coverage, so the above inequality means that the maximal possible expected utility for the high risks is higher than the expected utility related to no insurance.) Then a new entrant to the insurance market could make profits by offering the contract $(0, (\eta_H + \beta) D)$, to all individuals, regardless of whether the low risks would purchase it. Competitors would respond by offering some contract $(0, (\eta_H + \gamma) D)$ with $\gamma < \beta$, still resulting in profits. New series of premium cutting would follow, resulting finally in the situation where a high risk individual will obtain at least $U(W - \eta_H D)$ as expected utility.

The consequence of this result is that, since, for any $0 < t \leq n$,

$$\begin{aligned} U(W - \eta_H D) \geq\ & F_H(t)\, U(W - D - (\eta_H - F_H(t))\, D) \\ & + (1 - F_H(t))\, U(W - (\eta_H - F_H(t))\, D), \end{aligned} \tag{77}$$

the high risks pay a premium smaller than or equal to the actuarially fair one. Hence, since a company cannot afford making losses, the low risks should pay premiums at least as high as the actuarially fair ones. This proves the *fifth* criterion with respect to the premium the low risks pay (higher than actuarially fair). So companies rely on the low risks for their profits and therefore have to make the conditions for individuals of this type as favorable as possible in order to be able to attract them, by offering a contract maximizing their expected utility. This proves the *first* criterion.

The Lagrangian of the optimization problem is therefore equal to:

$$\begin{aligned} \pounds =\ & E_L(t_L, P_L) + \nu(\rho(P_H - (\eta_H - F_H(t_H))\, D) + (1 - \rho)(P_L - (\eta_L - F_L(t_L))\, D)) \\ & + \mu(E_H(t_H, P_H) - U(W - \eta_H D)) + \gamma_L\{E_L(t_L, P_L) - E_L(t_H, P_H)\} \\ & + \gamma_H\{E_H(t_H, P_H) - E_H(t_L, P_L)\}, \end{aligned} \tag{78}$$

with ν, γ_L and γ_H acting as the Lagrange multipliers corresponding to the nonnegative profit constraint, the self-selection constraint for the low risks and the self-selection constraint for the high risks, in this order. Finally, μ denotes the Lagrange multiplier specifying the constraint considered above (expected utility for the high risks at least equal to $U(W - \eta_H D)$).

Recall, however, that we assumed the partial stochastic order

$$F_H(t) \geq b(t)\, F_L(t), \text{ with } b(t) \geq 1; \quad 0 < t \leq n, \tag{79}$$

with equality in case $t = n$. As a consequence, the contract in the equilibrium for the high risks, should involve full coverage. If this were not the case, but instead a contract

(t_H, P_H), with $t_H > 0$, were offered, a firm could make profits by offering $(0, P_H^*)$, with P_H^* such that

$$U(W - P_H^*) = E_H(t_H, P_H), \tag{80}$$

as argued in Subsection 3.2, where it was also shown that this contract does not violate the self-selection constraint for the low risks. So the high risks always purchase full coverage and, since the premium they pay is actuarially fair at highest, this proves the *fourth* criterion of the Wilson equilibrium. It follows that $\gamma_L = 0$, and the Lagrangian reduces to

$$\begin{aligned} \pounds &= E_L(t_L, P_L) + \nu(\rho(P_H - \eta_H D) + (1-\rho)(P_L - (\eta_L - F_L(t_L))D)) \\ &\quad + \mu(\eta_H D - P_H) + \gamma_H\{E_H(0, P_H) - E_H(t_L, P_L)\}. \end{aligned} \tag{81}$$

As proved in the appendix, the following restrictions hold:

I) $\nu > 0 \Leftrightarrow \rho(P_H - \eta_H D) + (1-\rho)(P_L - (\eta_L - F_L(t_L))D) = 0$, and
II) $\mu > 0$ and/or $\gamma_H > 0$.

So in an equilibrium, as I) indicates, the insurer breaks even on average. The interpretation of II) is as follows: the optimal contract for the high risks is never a policy with premium strictly lower than $\eta_H D$ which is at the same time strictly preferable to the contract offered to the low risks.

Just as in the case of a monopolistic insurer, there are three different combinations to be taken into account. Again, this is contrary to the monetary deductible case, where the self-selection constraint for the high risks is always binding. In case of a probationary period, however, things may be different. The next example shows that actually equilibria satisfying the pair of constraints:

$$\mu > 0; \gamma_H = 0, \tag{82}$$

(maximal premium constraint binding, self-selection constraint non-binding) may exist.

Example 14 *Consider, just as in Example 6:*

$$\begin{array}{lll} F_H(t) & = G(t), & 0 \le t \le t' \\ & = G(t'), & t > t' \\ F_L(t) & = 0 & 0 \le t < t' \\ & = H(t) & t \ge t' \end{array}$$

Again, both $G(t)$ and $H(t)$ are specified to be continuous and non-decreasing functions on the intervals $[0, t']$ and $[t', n]$, respectively, with $G(0) = H(t') = 0$, and $G(t') > H(n)$. The contract $(t', H(n)D)$, on the one hand gives the low risks maximal expected utility (for them it is actually the same as full coverage) and on the other hand will be considered by the high risks to be strictly inferior to $(0, G(t')D)$. The former contract will never even be purchased by them since it actually involves no coverage at all. So an optimal solution involves offering $(t', H(n)D)$ together with $(0, G(t')D)$, having as property that the self-selection constraint for the high risks is not binding.

For the sake of simplicity, we will restrict ourselves in the analyses that follow to cases where the "$\geq$"-sign in (79) can be replaced by the "="-sign.

$$F_H(t) = b(t) F_L(t), \text{ with } b(t) \geq 1; \quad 0 < t \leq n, \tag{83}$$

The probationary period corresponding to the Rothschild-Stiglitz strategy, the latter having the property that both the self-selection constraint and maximal premium constraint are binding, is found as a solution of the equality:

$$\begin{aligned} U(W - \eta_H D) &= b(t_L) F_L(t_L) U(W - (\eta_L - F_L(t_L)) D - D) \\ &\quad + (1 - b(t_L) F_L(t_L)) U(W - (\eta_L - F_L(t)) D). \end{aligned} \tag{84}$$

This solution may not be unique but for the moment we assume it is the case. Later on in this section we will prove that it actually is unique if $b(\cdot)$ is constant and an exponential utility function applies.

Definition 15 (Rothschild–Stiglitz strategy) *We assume that the solution of (84) is unique, regarding the interval $[0, n]$, and specify it by $\widetilde{t_L}$. Now the Rothschild-Stiglitz strategy is defined as the strategy of offering $(0, \eta_H D)$ together with $\left(\widetilde{t_L}, (\eta_L - F_L(\widetilde{t_L}))\, D\right)$.*

Then for all potential solutions of the optimization problem (81) with $t_L < \widetilde{t_L}$, the self-selection constraint is binding, but the maximal premium constraint is not, implying that the low risks subsidize the high risks. On the other hand, for all potential solutions of the optimization problem (81) with $t_L > \widetilde{t_L}$, the maximal premium constraint is binding but the self-selection constraint is not: each contract pays the actuarially fair premium. The reason is, in graphical terms, that the high risks' *indifference curve* (an individual of a certain risk type is indifferent between two contracts lying on the same indifference curve applying to the same risk type) passing through $(0, \eta_H D)$ lies above the low risks' *zero-profit curve* (the curve consisting of contracts breaking even on average) for $t_L < \widetilde{t_L}$ and below it for $t_L > \widetilde{t_L}$. This will be illustrated in Example 17.

Hence, the optimization problem involves comparing:

1. the contract (t_L, P_L) of all the contracts satisfying the constraints:

$$\rho(P_H - \eta_H D) + (1 - \rho)(P_L - (\eta_L - F_L(t_L)) D) = 0; \tag{85}$$

$$E_H(t_L, P_L) = E_H(0, P_H); \tag{86}$$

$$t_L \in \left[0, \widetilde{t_L}\right], \tag{87}$$

   which maximizes the low risk class individuals' expected utility, with

2. the contract (t_L, P_L) of all the contracts satisfying the constraints:

$$P_L = (\eta_L - F_L(t_L)) D, \tag{88}$$

$$t_L \in \left[\widetilde{t_L}, n\right], \tag{89}$$

which maximizes the low risk class individuals' expected utility.

Remark 16 *In the monetary deductible case, the second partial optimization problem is trivial as the contract maximizing expected utility for the low risks is always the one with the lowest deductible. This may not be the case for the probationary period, as was argued by equation (72) and Example 12.*

Next, the optimal contract of the first partial optimization problem will be derived. Combining the corresponding constraints (85) and (86) results in the following relationship between t_L and P_L:

$$U\left(W-\eta_H D+\frac{(1-\rho)}{\rho}\left(P_L-\left(\eta_L-F_L\left(t_L\right)\right)D\right)\right)=E_H\left(t_L,P_L\right). \tag{90}$$

Concerning the above inequality, in general neither one of the two variables t_L and P_L can be written as a function of only the other one and hence in general the objective function can not be written as a function of only one variable. An exception is the exponential utility function, which will be considered in the next subsection.

4.3 Application: the exponential utility function

In this subsection, $U(\cdot)$ is specified to be exponential with α being the coefficient of absolute risk aversion:

$$U(x)=-\alpha e^{-\alpha x}, \quad \alpha>0. \tag{91}$$

Furthermore, $b(\cdot)$ is assumed to be constant. Then the contract satisfying both the maximal-premium and the self-selection constraint is unique. This will be verified by substituting (91) into (84), yielding:

$$e^{\alpha\eta_H D}=e^{\alpha(\eta_L-F_L(t_L))D}\left(bF_L\left(t_L\right)\left(e^{\alpha D}-1\right)+1\right). \tag{92}$$

The derivative of the right hand side of the above equality to t_L is equal to

$$F_L'\left(t_L\right)e^{\alpha(\eta_L-F_L(t_L))D}\left(b\left(e^{\alpha D}-1\right)-\alpha D\left(bF_L\left(t_L\right)\left(e^{\alpha D}-1\right)+1\right)\right). \tag{93}$$

It follows that, if

$$\eta_L\leq\frac{b\left(e^{\alpha D}-1\right)-\alpha D}{b\alpha D\left(e^{\alpha D}-1\right)}, \tag{94}$$

the right hand side of (92) is monotonously increasing as a function of t_L, varying from $e^{\alpha\eta_L D}\left(<e^{\alpha\eta_H D}\right)$ to $\left(b\eta_L\left(e^{\alpha D}-1\right)+1\right)\left(>e^{\alpha\eta_H D}\right)$. On the other hand, if

$$\eta_L>\frac{b\left(e^{\alpha D}-1\right)-\alpha D}{b\alpha D\left(e^{\alpha D}-1\right)}, \tag{95}$$

the right hand side of (92) is monotonously increasing in t_L for

$$t_L \in \left[0, F_L^{-1}\left(\frac{b\left(e^{\alpha D}-1\right)-\alpha D}{b\alpha D\left(e^{\alpha D}-1\right)}\right)\right], \tag{96}$$

varying from $e^{\alpha\eta_L D}\ \left(< e^{\alpha\eta_H D}\right)$ to

$$\left(\frac{b\left(e^{\alpha D}-1\right)}{\alpha D}\right) e^{\alpha\left(\eta_L - \frac{b\left(e^{\alpha D}-1\right)-\alpha D}{b\alpha D\left(e^{\alpha D}-1\right)}\right)D} \tag{97}$$

and monotonously decreasing in t_L varying from the last mentioned expression to $\left(b\eta_L\left(e^{\alpha D}-1\right)+1\right)\ \left(> e^{\alpha\eta_H D}\right)$. Hence the solution t_L, again denoted by $\widetilde{t_L}$, is in any case unique. As a consequence, in the equilibrium for all contracts with a probationary period $t_L < \widetilde{t_L}$ the self-selection constraint is binding, and the maximal premium constraint is not. Contracts (t_L, P_L) designed for the low risks then satisfy (90), which in the given case of an exponential utility function reduces to

$$P_L = \rho\left(\eta_H D + \frac{1-\rho}{\rho}\left(\eta_L - F_L(t_L)\right)D - \frac{1}{\alpha}\ln\left(bF_L(t_L)\left(e^{\alpha D}-1\right)+1\right)\right). \tag{98}$$

The opposite applies for $t_L > \widetilde{t_L}$: the maximal premium constraint is binding and the self-selection constraint is not. The low risks then buy a contract satisfying:

$$P_L = \left(\eta_L - F_L\left(t_L\right)\right)D, \tag{99}$$

while the high risks will buy $(0, \eta_H D)$. The above results will be illustrated graphically by means of the following example:

Example 17 *We take* $n = 30$, $\alpha = 1 \cdot 10^{-5}$, $D = 2 \cdot 10^5$, $\rho = 0.4$, $b = 3$, *and*

$$F_L(t) = 1 - \exp\left(-\frac{1.01 \cdot 0.003^{35}}{\ln\left(0.003\right)}\left(0.003^t - 1\right)\right). \tag{100}$$

Figure 5 illustrates the given case. It turns out that $\widetilde{t_L}$, *being the probationary period applying to the contract designed for the low risks in case of the Rothschild-Stiglitz strategy, is equal to* 10.8416. *Furthermore* $\widetilde{P_L} = 18,114$, *where* $\widetilde{P_L}$ *denotes the premium corresponding to* $\widetilde{t_L}$. *The figure displays the set of contracts* (t_L, P_L) *satisfying the constraints (solid curve). For* $t_L \leq \widetilde{t_L}$ *and* $t_L \geq \widetilde{t_L}$, *these constraints involve the equalities (98) and*

(99), respectively. The indifference curve for the high risks, passing through $(0, \eta_H D)$ *and* $\left(\widetilde{t_L}, \widetilde{P_L}\right)$ *(dashed), has been drawn as well. Besides, another high risks indifference curve (dotted) is given, corresponding to a possible strategy of offering* $(0, P_H^*)$ *together with* (t_L^*, P_L^*)*, with* $t_L^* < \widetilde{t_L}$*. The figure shows that in that case the self-selection constraint (for the high risks) is binding, but not the maximal premium constraint (also for the high risks). It should be noticed, however, that this strategy is just a feasible one, and not necessarily equal to the equilibrium.*

Let us consider the contracts (t_L, P_L) with $t_L \in \left[\widetilde{t_L}, n\right]$. Recall from Example 12 that

$$\begin{aligned} & \frac{\partial E\left(t_L, (\eta - F\left(t_L\right)) D\right)}{\partial t_L} \\ = \; & F'\left(t_L\right) e^{\alpha(W-(\eta-F(t_L))D)} \left(F\left(t_L\right) \alpha D \left(e^{\alpha D} - 1\right) - \left(e^{\alpha D} - 1 - \alpha D\right)\right), \end{aligned} \tag{101}$$

which is either positive for any value $t_L \in \left[\widetilde{t_L}, n\right]$, or negative for any value $t_L \in \left[\widetilde{t_L}, n\right]$, or negative for $t_L \in \left[\widetilde{t_L}, t_L'\right)$ and positive for $t_L \in (t_L', n]$, with $t_L' \in \left(\widetilde{t_L}, n\right)$. It follows that, when restricting oneself to contracts with probationary period $t_L \in \left[\widetilde{t_L}, n\right]$, a low risk class individual's expected utility is maximized either by offering $\left(\widetilde{t_L}, \left(\eta_L - F_L(\widetilde{t_L})\right) D\right)$ or by offering $(n, 0)$, which means no insurance at all. So contracts (t_L, P_L) in the equilibrium set with $t_L \in \left(\widetilde{t_L}, n\right)$ do not exist.

The conclusion is that the optimization problem involves deriving the optimal solution with constraint $0 \leq t_L \leq \widetilde{t_L}$, and then investigating whether the contract designed for the low risks (this might also be a pooling contract with full coverage) will actually be purchased by individuals of that type.

So we can use a *constrained* objective function (constrained, since it is subject to the constraint $0 \leq t_L \leq \widetilde{t_L}$) which can be written as a function of either t_L or P_L. This is derived by substituting (98) into the *unconstrained* objective function given in (81). Denote this function by $\widehat{V}\left(t_L, \rho\right)$. It turns out to be equal to

$$\begin{aligned} \widehat{V}\left(t_L, \rho\right) \; = \; & e^{-\alpha W} \left(bF_L(t_L) \left(e^{\alpha D} - 1\right) + 1\right)^{-\rho} \left(F_L(t_L) \left(e^{\alpha D} - 1\right) + 1\right) \\ & \cdot e^{\alpha D(\rho\eta_H + (1-\rho)(\eta_L - F_L(t_L)))}, \end{aligned} \tag{102}$$

with first derivative

$$\begin{aligned} & \frac{\partial \widehat{V}\left(t_L, \rho\right)}{\partial t_L} \\ = \; & e^{-\alpha W} \left(bF_L(t_L) \left(e^{\alpha D} - 1\right) + 1\right)^{-\rho} e^{\alpha D(\rho\eta_H + (1-\rho)(\eta_L - F_L(t_L)))} F_L'(t_L) \\ & \cdot \left\{ \begin{array}{c} \left(F_L(t_L) \left(e^{\alpha D} - 1\right) + 1\right) \left(\rho \frac{b\left(e^{\alpha D} - 1\right)}{bF_L(t_L)(e^{\alpha D} - 1) + 1} + (1 - \rho) \alpha D\right) \\ - \left(e^{\alpha D} - 1\right) \end{array} \right\}. \end{aligned} \tag{103}$$

The sign of this derivative is determined by the expression between large curly brackets. Therefore we will consider this expression, defined as $s(t_L, \rho)$. So

$$s(t_L,\rho) = \left(F_L(t_L)\left(e^{\alpha D}-1\right)+1\right)\left(\rho\frac{b\left(e^{\alpha D}-1\right)}{bF_L(t_L)\left(e^{\alpha D}-1\right)+1}+(1-\rho)\alpha D\right) - \left(e^{\alpha D}-1\right). \tag{104}$$

Taking the derivative with respect to t_L gives:

$$\frac{\partial s(t_L,\rho)}{\partial t_L} = F_L'(t_L)\left(e^{\alpha D}-1\right)\left((1-\rho)\alpha D-\rho\frac{b(b-1)\left(e^{\alpha D}-1\right)}{\left(bF_L(t_L)\left(e^{\alpha D}-1\right)+1\right)^2}\right). \tag{105}$$

The factor between large parentheses in the right hand side of the above equation increases monotonously in t_L. We now define

$$\rho^* = \frac{\alpha D}{(b-1)\,b\left(e^{\alpha D}-1\right)+\alpha D}, \tag{106}$$

and

$$\rho^{**} = \frac{\alpha D\left(bF_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1\right)^2}{(b-1)\,b\left(e^{\alpha D}-1\right)+\alpha D\left(bF_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1\right)^2}, \tag{107}$$

as the solutions of

$$\left.\frac{\partial s(t_L,\rho)}{\partial t_L}\right|_{(t_L=0)} = 0, \tag{108}$$

and

$$\left.\frac{\partial s(t_L,\rho)}{\partial t_L}\right|_{(t_L=\widetilde{t_L})} = 0, \tag{109}$$

respectively. We have $\rho^* < \rho^{**}$. Next let ρ_{POOL} and ρ_{RS} be the solutions of

$$\left.\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L}\right|_{(t_L=0)} = 0 \tag{110}$$

and

$$\left.\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L}\right|_{(t_L=\widetilde{t_L})} = 0, \tag{111}$$

respectively. They prove to be equal to

$$\rho_{POOL} = \frac{\left(e^{\alpha D}-1\right)-\alpha D}{b\left(e^{\alpha D}-1\right)-\alpha D}, \tag{112}$$

and

$$\rho_{RS} = \frac{\left(bF_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1\right)\left(\left(e^{\alpha D}-1\right)-\alpha D\left(F_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1\right)\right)}{\left(F_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1\right)\left(b\left(e^{\alpha D}-1\right)-\alpha D\left(bF_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1\right)\right)}. \tag{113}$$

We now derive the following two theorems, proofs of which can be found in the appendix. In the first of these, a critical value is

$$\rho_2 = \frac{\ln\left[F_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1\right]-F_L\left(\widetilde{t_L}\right)\alpha D}{\left(b-1\right)\eta_L \alpha D}. \tag{114}$$

For $\rho < \rho_2$, offering the pooling contract $(0, \rho\eta_H D + (1-\rho)\eta_L D)$ is to be preferred to the Rothschild-Stiglitz strategy, while for $\rho > \rho_2$, the opposite holds.

Theorem 18 *If* $b\eta_L < 1$ *and*

$$b < \frac{\alpha D}{e^{\alpha D}-1-\alpha D}, \tag{115}$$

the following conclusions hold concerning ρ*:*

1. *If* $\rho < \rho_2$*, it is optimal to offer only the contract* $(0, \rho\eta_H D + (1-\rho)\eta_L D)$*, provided that*

$$\rho < \frac{\ln\left[\eta_L\left(e^{\alpha D}-1\right)+1\right]-\eta_L \alpha D}{\left(b-1\right)\eta_L \alpha D} \tag{116}$$

   If the above condition is not satisfied, it is optimal to offer only the contract $(0, \eta_H D)$*.*

2. *In case* $\rho > \rho_2$*, the optimal strategy involves offering* $(0, \eta_H D)$ *anyway. Besides,* $\left(\widetilde{t_L}, \left(\eta_L - F_L\left(\widetilde{t_L}\right)\right) D\right)$ *will be offered, provided that*

$$e^{\alpha D\left(\eta_L - F_L\left(\widetilde{t_L}\right)\right)} < \frac{\eta_L\left(e^{\alpha D}-1\right)+1}{F_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1}. \tag{117}$$

This theorem indicates that, if the fixed proportion between $F_H(t)$ and $F_L(t)$ is lower than a certain critical level, then there are only three strategies to be considered: a) the Rothschild-Stiglitz strategy; b) offering only the pooling contract $(0, \rho\eta_H D + (1-\rho)\eta_L D)$; c) offering only $(0, \eta_H D)$, implying that the low risks are excluded from coverage.

Note, finally, that if

$$\eta_L < \frac{e^{\alpha D}-1-\alpha D}{\left(e^{\alpha D}-1\right)\alpha D}, \tag{118}$$

the equalities (116) and (117) hold anyway.

Theorem 19 *If*

$$b > \frac{\alpha D}{e^{\alpha D}-1-\alpha D}, \tag{119}$$

and

$$F_L\left(\widetilde{t_L}\right) < \frac{b\left(e^{\alpha D}-1\right)-\alpha D}{2b\alpha D\left(e^{\alpha D}-1\right)}, \tag{120}$$

the optimal strategy depends on ρ *in the following way:*

1. $\rho < \rho_{POOL}$: *it is optimal to offer only* $(0, \rho\eta_H D + (1-\rho)\eta_L D)$, *provided that*

$$\rho < \frac{\ln\left[\eta_L\left(e^{\alpha D}-1\right)+1\right] - \eta_L \alpha D}{(b-1)\eta_L \alpha D}; \tag{121}$$

2. $\rho > \rho_{RS}$: *it is optimal to offer* $(0, \eta_H D)$ *together with* $\left(\widetilde{t_L}, \left(\eta_L - F_L\left(\widetilde{t_L}\right)\right) D\right)$, *provided that*

$$e^{\alpha D\left(\eta_L - F_L\left(\widetilde{t_L}\right)\right)} < \frac{\eta_L\left(e^{\alpha D}-1\right)+1}{F_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1}. \tag{122}$$

3. $\rho_{POOL} < \rho < \rho_{RS}$: *it is optimal to offer* (t_L^*, P_L^*) *together with* $(0, P_H^*)$, *where*

$$t_L^* = F_L^{-1}\left(\frac{\left\{\begin{array}{c}(1-\rho)\left(b\left(e^{\alpha D}-1\right)-(b+1)\alpha D\right) \\ -\sqrt{(1-\rho)\left\{\begin{array}{c}\left((b-1)\alpha D - b\left(e^{\alpha D}-1\right)\right)^2 \\ -\rho\left((b-1)\alpha D + b\left(e^{\alpha D}-1\right)\right)^2\end{array}\right\}}\end{array}\right\}}{2\left(e^{\alpha D}-1\right)(1-\rho)b\alpha D}\right), \tag{123}$$

$$P_L^* = \rho\left(\eta_H D + \frac{1-\rho}{\rho}\left(\eta_L - F_L(t_L^*)\right) D - \frac{1}{\alpha}\ln\left(bF_L(t_L^*)\left(e^{\alpha D}-1\right)+1\right)\right), \tag{124}$$

and

$$P_H^* = \eta_H D - \frac{(1-\rho)}{\rho}\left(P_L^* - \left(\eta_L - F_L\left(t_L^*\right)\right) D\right), \tag{125}$$

provided that

$$e^{\alpha P_L^*}\left(F_L\left(t_L^*\right)\left(e^{\alpha D}-1\right)+1\right) < \eta_L\left(e^{\alpha D}-1\right)+1. \tag{126}$$

If the condition related to any of the three cases listed is not satisfied, then it is optimal to offer only $(0, \eta_H D)$.

The last theorem shows the conditions under which the optimal solution is most similar to the one derived in Miyazaki [1977] and Spence [1978]: if $F_L\left(\widetilde{t_L}\right)$ falls below a certain upper bound and b exceeds a certain lower bound, and moreover, ρ belongs to a certain interval, then a low risk class member's expected utility is optimized by offering insurance with a deductible to the low risks, which is smaller than the deductible corresponding to the Rothschild-Stiglitz strategy. In addition the high risks are subsidized by the low risks, thus satisfying the *fourth* and *fifth* property of the Wilson equilibrium. Besides, there is a critical value of ρ, namely ρ_{RS}, such that if this value is exceeded, then the Rothschild-Stiglitz strategy applies. Because of this, the *sixth* property of the Wilson equilibrium

is satisfied as well. Note however an essential difference with the *third* characteristic: it may be optimal to let the high and low risks buy the same contract. This is the case if ρ falls below another critical level, namely ρ_{POOL}.

The optima considered are only the real ones if they improve the low risks' situation compared with no insurance. This is the reason why extra conditions concerning b and $F_L\left(\widetilde{t_L}\right)$ are imposed. Note however, just as in the previous theorem, that if

$$\eta_L < \frac{e^{\alpha D} - 1 - \alpha D}{\left(e^{\alpha D} - 1\right)\alpha D}, \tag{127}$$

these conditions are always satisfied.

Example 20 *As in Example 11, we take* $n = 30$, $\alpha = 1 \cdot 10^{-5}, D = 2 \cdot 10^5$,

$$F_L(t) = 1 - \exp\left(-\frac{1.01 \cdot 0.003^{35}}{\ln(0.003)}\left(0.003^t - 1\right)\right) \tag{128}$$

(so the mortality law is Gompertz (x, δ, κ) *with parameters* $x = 35, \delta = 0.003, \kappa = 1.01$*), and* $b = 3$. *The actuarially fair premium corresponding to the full coverage contract for the high risks is equal to*

$$\eta_H D = 82,836. \tag{129}$$

The Rothschild-Stiglitz strategy comprises the contract

$$\widetilde{t_L} = 10.8416, \widetilde{P_L} = \left(\eta_L - F_L\left(\widetilde{t_L}\right)\right) D = 18,114. \tag{130}$$

We have that

$$F_L\left(\widetilde{t_L}\right) = 0.047487 < 0.22391 = \frac{b\left(e^{\alpha D} - 1\right) - \alpha D}{2b\alpha D\left(e^{\alpha D} - 1\right)}, \tag{131}$$

and

$$b = 3 > .45568 = \frac{\alpha D}{e^{\alpha D} - 1 - \alpha D}, \tag{132}$$

so the first two conditions of Theorem 19 are satisfied. Furthermore:

$$\eta_L = .13806 < \frac{e^{\alpha D} - 1 - \alpha D}{\left(e^{\alpha D} - 1\right)\alpha D}, \tag{133}$$

so the extra conditions regarding the three different cases considered in Theorem 19 are satisfied anyway. It turns out that $\rho_{RS} = 0.25567$ *and* $\rho_{POOL} = 0.36119$. *Hence the optimal solution is similar to the one given in Miyazaki (1977) and Spence (1978) for* $\rho \in (0.25567, 0.36119)$. *In Figure 6, the optimal values for* P_H *and* P_L *are displayed. The corresponding values for* t_L *are as in Figure 7.*

The reader may have noticed that the equilibria derived in this section have some properties in common with the optimal strategies for the monopolistic insurer dealt with in the previous section. This is the reason why we will next spend an extra section on the properties both equilibria share.

5 Comparison between equilibria for both insurance markets

We will first make some comments about the properties that the equilibria for the monopolistic and competitive insurance markets have in common in general. (In this respect, the solution yielding maximal profit for the monopolistic insurer is also considered to be an equilibrium.) Then we will discuss the special cases of the exponential utility function, which were taken as the leading example throughout the whole paper.

5.1 In general

It was shown that, if an insurer is able to monitor the individuals' accident probabilities, for both a monopolistic and a competitive insurance market, equilibria involve full coverage for all risk types.

In this paper, a partial stochastic order between the c.d.f.'s of the time-at-accident of low risks and high risks was assumed. This implies that, for both types of insurance markets, an equilibrium always involves full coverage for the high risks. Furthermore, the high risks are never worse off compared with symmetry of information (this also applies for a monetary deductible). As a consequence we get that the Lagrange function concerns three constraints in both cases, one of which (the low risks reservation constraint in case of a monopolistic market, the nonnegative profit constraint if the market is fully competitive) is always binding, just like at least one of the other two.

Irrespective of the market form, the Lagrange function applying to the monetary deductible as a screening device has the property that the self-selection constraint for the high risks is always binding. This does not have to be the case for the probationary period, as was illustrated by one and the same extreme example for both market types (namely Examples 6 and 14, respectively). In other words: contracts with partial coverage may exist which will be purchased by the low risks, but which will be considered to be inferior by the high risks, when compared to a contract with full coverage against the maximal premium a member of the latter type can be charged. (This maximal premium is the actuarial premium in case of full competition and the maximal premium the high risks are willing to pay for full coverage if the insurance market is monopolistic.)

5.2 The special case of constant absolute risk aversion

The case of an exponential utility function was considered extensively in this paper. It was shown that if the entire accident probability η exceeds a certain bound, depending on the welfare loss, denoted by D, and on the risk aversion coefficient, denoted by α, then there are contracts, with a relatively high probationary period, which will never be offered, either because the individual considered will ultimately be worse off (in case of a fully competitive market) or because they will contribute to losses from the company's point of view (in case of a monopolistic insurer).

Such contracts will never be part of an equilibrium. A monopolistic insurer has always a more profitable strategy at its disposal of excluding the low risks from coverage and charging the high risks the maximal premium they are willing to pay, while, on the other hand, the high risks, always buying full coverage, never subsidize the low risks.

In both sections two theorems were derived, being in pairs similar and based on a constant proportion between the c.d.f.'s of both risk types, denoted by b.

We will first focus on the Theorems 9 and 18. It was shown that, if

$$b < \frac{\alpha D}{e^{\alpha D} - 1 - \alpha D}, \tag{134}$$

then offering a contract with a probationary period is never optimal if the insurer is a monopolist, while in a competitive market, offering a contract with a probationary period is never optimal if different from the one resulting from the Rothschild-Stiglitz strategy. If the proportion of high risks within the entire period is low enough, it is optimal to offer a pooling contract, a solution which never applies in case of a monetary deductible.

Theorems 10 and 19 will now be compared. Suppose that the conditions of Theorem 10 apply, so:

$$b > \frac{\alpha D}{e^{\alpha D} - 1 - \alpha D}, \tag{135}$$

and

$$\eta_L < \frac{b\left(e^{\alpha D} - 1\right) - \alpha D}{2b\alpha D\left(e^{\alpha D} - 1\right)}. \tag{136}$$

This implies

$$F_L\left(\widetilde{t_L}\right) < \frac{b\left(e^{\alpha D} - 1\right) - \alpha D}{2b\alpha D\left(e^{\alpha D} - 1\right)}, \tag{137}$$

so the conditions of Theorem 19 are satisfied as well. Besides, it implies (127), hence in an equilibrium of a competitive market the low risks will always get coverage, with probationary period at most equal to $\widetilde{t_L}$. Solving $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} = 0$ (cf. (103)) for ρ results in:

$$\rho = \frac{\left(bF_L\left(t_L\right)\left(e^{\alpha D} - 1\right) + 1\right)\left(\left(e^{\alpha D} - 1\right) - \alpha D\left(F_L\left(t_L\right)\left(e^{\alpha D} - 1\right) + 1\right)\right)}{\left(F_L\left(t_L\right)\left(e^{\alpha D} - 1\right) + 1\right)\left(b\left(e^{\alpha D} - 1\right) - \alpha D\left(bF_L\left(t_L\right)\left(e^{\alpha D} - 1\right) + 1\right)\right)}. \tag{138}$$

And solving $\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} = 0$ (with $\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L}$ as in (44)) for ρ yields:

$$\rho = \frac{\beta_1\beta_2}{\beta_3\beta_4}, \tag{139}$$

with

$$\begin{aligned}
\beta_1 &= e^{\alpha P_L}\left(e^{\alpha D} - 1\right) - \alpha D\left(\eta_L\left(e^{\alpha D} - 1\right) + 1\right); && (140)\\
\beta_2 &= b\left(\eta_L\left(e^{\alpha D} - 1\right) + 1\right) - (b-1)\,e^{\alpha P_L}; && (141)\\
\beta_3 &= \eta_L\left(e^{\alpha D} - 1\right) + 1; && (142)\\
\beta_4 &= \left(e^{\alpha P_L}\left(b\left(e^{\alpha D} - 1\right) + (b-1)\,\alpha D\right) - b\alpha D\left(\eta_L\left(e^{\alpha D} - 1\right) + 1\right)\right). && (143)
\end{aligned}$$

which by applying (33) results in (138).

So if the given conditions hold, the following can be concluded concerning the equilibria for monopolistic and competitive insurance markets:

1. For $\rho \leq \rho_{POOL} = \rho_{P_L^0}$, it is optimal for both market types to offer only one contract of full coverage.

2. For $\rho_{POOL} < \rho \leq \rho_{RS}$, it is optimal in both cases to offer two contracts, one of which having a probationary period t_L^* equal to

$$t_L^* = F_L^{-1}\left(\frac{\left\{(1-\rho)\left(b\left(e^{\alpha D}-1\right)-(b+1)\alpha D\right) - \sqrt{(1-\rho)\left\{\begin{array}{c}\left((b-1)\alpha D - b\left(e^{\alpha D}-1\right)\right)^2 \\ -\rho\left((b-1)\alpha D + b\left(e^{\alpha D}-1\right)\right)^2\end{array}\right\}}\right\}}{2\left(e^{\alpha D}-1\right)(1-\rho)\, b\alpha D}\right). \tag{144}$$

3. For $\rho > \rho_{RS}$, the optimal strategy for a monopolist involves offering two contracts, one of which has a probationary period greater than $\widetilde{t_L}$ if $\rho < \rho_0$, and no coverage at all if $\rho \geq \rho_0$. On the other hand, if the market is fully competitive, the probationary period for the low risks will always be equal to $\widetilde{t_L}$.

So if $\rho \leq \rho_{RS}$, for both market types the equilibria coincide with respect to the probationary period, provided the given conditions hold.

6 Conclusions, final comments and recommendations for further research

Insurance companies have always been faced with the problem that they know less about individual persons than individuals know about themselves. As a consequence, they have to cope with the threat of adverse selection. In this paper, the probationary period has been introduced as a method to design contracts in such a way that each individual person in a population selects the contract designed for him. This has been done for two extreme insurance market types, namely the monopolistic insurer and a fully competitive market. It has been compared with the more common screening instrument of a monetary deductible. This has been done under simplified assumptions such as the existence of only two risk classes, no discounting of interest and a partial stochastic order between the distribution functions of time-at-accident of a low and a high risk class individual.

In order to be able to draw conclusions in an appropriate way, the case of symmetric information has been considered as well. For both market types it has been shown that in that case any individual will purchase full coverage, i.e. no probationary period. This will be against either the actuarial premium, in case the market is competitive, or against the maximal premium the individual considered is willing to pay, if there is only one insurer. It has also been shown however, that there may be contracts with a certain probationary period which are actuarially fair with respect to an individual but will not be purchased by him. It is difficult to explain this phenomenon. To our opinion, the reason seems to be the fact that an insured gets less certainty from a contract with a probationary period, compared to a contract with a monetary deductible (which will always be purchased by a risk averse individual if actuarially fair). In the latter case, the insured at least always gets some compensation for a welfare loss, whereas in the former case there is the chance of getting no compensation at all. For other possible reasons, we refer to Eeckhoudt et al. [1988].

The assumption of a partial stochastic order between the distribution functions of time-of-accident of the two respective risk classes has proved to be an essential one, since the equilibria resulting have at least some properties in common when compared with the corresponding ones for the monetary deductible. One characteristic is that the high risks will always purchase full coverage, i.e. insurance without a probationary period. Just as in the case of a monetary deductible, they are also at least as well off as they would have been in case of symmetric information. On the other hand, the low risks, and this is again the same as with the monetary deductible case, are either as well off (monopolistic insurer) or worse off (competitive market) compared with the situation of perfect availability of information.

For both market types considered, using the probationary period as a screening device can give the insurer extra opportunities, since contrary to the monetary deductible, contracts may exist which will be purchased by the low risks, but not by the high risks. In general, however, it is difficult to draw conclusions, partly due to the fact that so many different specifications of the distribution functions of time-of-accident are possible. Some analyses have been carried out with the exponential utility function and an additional assumption concerning a fixed proportion between a high risk's probability and a low risk's probability of facing an accident before any point of time. It has been shown that, for a monopolistic insurance market, the strategy yielding a maximal profit is comparable with the one derived by Stiglitz [1977], provided that some restrictions hold. Under similar circumstances, the equilibrium in case of a competitive market has much in common with the one derived in Miyazaki [1977] and Spence [1978]. For both market types, however, an equilibrium may involve a pooling contract with full coverage, where both risk types pay the same price. This can never happen in case of a monetary deductible.

The approach in this paper has been analytic. This has yielded some unambiguous results, though at the price of rather severe restrictions. Many extensions of the approach in this paper are possible. One can, for instance, add interest as an additional parameter

or relax the assumption of a partial stochastic order between the two above mentioned distribution functions of the low and high risks. It may be interesting to investigate how the conclusions derived in this paper may then change. Other possibilities are the addition of more risk classes, the study of other utility functions than the exponential one, consideration of more moderate and more realistic insurance market forms (e.g. an oligopolistic market), and the addition of costs due to underwriting. Last but not least, one can study the consequences in a competitive market if the assumption used in this paper that companies behave with foresight, is replaced by the one that they are myopic, i.e. do not take not account the competitors' reactions to their own strategy.

The analyses in this paper rely on the expected utility hypothesis, an assumption which is very often used in the literature, but which may contradict reality. Another critical assumption is that individuals have perfect information on themselves. One can make this restriction more lenient by supposing that an individual's perception of his own profile is to be compared with the realization of a random variable, or more loosely stated, a "move of Nature".

It should be noted that most (if not: all) of these extensions require thorough simulation studies in order to obtain well-founded conclusions

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Appendix A: First order conditions of optimization problem (32)

The first order conditions of the reduced Lagrangian, displayed in (32) are:

Table A.1

First order conditions of maximization problem (32)

A.1	$\frac{\partial \mathcal{L}}{\partial P_H} = 0:$	$\rho - (\lambda_H + \gamma_H) U'(W - P_H) = 0.$
A.2	$\frac{\partial \mathcal{L}}{\partial P_L} = 0:$	$(1-\rho) - (\lambda_L - \gamma_H) \left[\begin{array}{c} F_L(t_L) U'(W - P_L - D) \\ + (1 - F_L(t_L)) U'(W - P_L) \end{array} \right] = 0.$
A.3	$\frac{\partial \mathcal{L}}{\partial t_L} = 0:$	$(1-\rho) F_L'(t_L) D + [U(W - P_L) - U(W - P_L - D)] \cdot (\gamma_H F_H'(t_L) - \lambda_L F_L'(t_L)) = 0.$
A.4	$\lambda_L, \lambda_H, \gamma_H \geq 0.$	
A.5	$\frac{\partial \mathcal{L}}{\partial \lambda_L} \geq 0:$	$E_L(t_L, P_L) \geq E_L.$
A.6	$\frac{\partial \mathcal{L}}{\partial \lambda_H} \geq 0:$	$E_H(0, P_H) \geq E_H.$
A.7	$\frac{\partial \mathcal{L}}{\partial \gamma_H} \geq 0:$	$E_H(0, P_H) \geq E_H(t_L, P_L).$

From (A.2) or (A.3) we have that λ_L cannot be equal to zero. Furthermore it follows from (A.1) that the following combination of values of multipliers cannot be part of an optimal solution:

$$\lambda_H = 0, \gamma_H = 0. \tag{145}$$

As the constraint corresponding to λ_L is binding, the following relation between t_L and P_L holds in any case (identical to (7) with "$\geq$"-sign replaced with the "$=$"-sign):

$$E_L(t_L, P_L) = E_L, \tag{146}$$

yielding the following relationship (writing t_L as a function of P_L):

$$t_L = F_L^{-1}\left(\frac{U(W - P_L) - E_L}{U(W - P_L) - U(W - P_L - D)}\right). \tag{147}$$

We now assume that the constraint corresponding to γ_H is binding. This is e.g. the case for b constant. Then the following relation between P_H, t_L and P_L applies:

$$U(W - P_H) = b(t_L) F_L(t_L) U(W - P_L - D) + (1 - b(t_L) F_L(t_L)) U(W - P_L). \tag{148}$$

Substituting (147) results in:

$$P_H = W - U^{-1}\left\{ \begin{array}{c} U(W - P_L) \\ -b\left(F_L^{-1}\left(\frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)\right)(U(W - P_L) - E_L) \end{array} \right\}. \tag{149}$$

The expected profit resulting from offering the contracts, satisfying the constraints (147) and (149) is equal to:

$$\begin{aligned}
&\widehat{\Gamma}(P_L,\rho)\\
&= \rho\left\{W-U^{-1}\left\{\begin{array}{c} U(W-P_L)\\ -b\left(F_L^{-1}\left(\frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)\right)\\ \cdot\left(U(W-P_L)-E_L\right)\end{array}\right\}\right\}\\
&\quad -D\eta_H\\
&\quad +(1-\rho)\left(P_L-D\left(\eta_L-\frac{U(W-P_L)-E_L}{U(W-P_L)-U(W-P_L-D)}\right)\right). \qquad (150)
\end{aligned}$$

Appendix B: Proof of the theorems of Subsection 3.2

B.1. Proof of Theorem 9

We have

$$b<\frac{\alpha D}{e^{\alpha D}-1-\alpha D}\Leftrightarrow \rho_{P_L^0}<\rho^*. \qquad (151)$$

For $\rho\leq\rho^*$, $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}$ is monotonously increasing in P_L, so it is optimal to offer either only $(0,P_L^0)$ or only $(0,P_H^0)$, depending on whether $\rho<\rho_1$ or $\rho>\rho_1$, respectively. Regarding the case $\rho>\rho^*$, note that the equation $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}=0$ has no roots for

$$\rho>\left(\frac{b\left(e^{\alpha D}-1\right)-(b-1)\alpha D}{b\left(e^{\alpha D}-1\right)+(b-1)\alpha D}\right)^2. \qquad (152)$$

In the given case, under the given assumption (i.e. $b<\frac{\alpha D}{e^{\alpha D}-1-\alpha D}$):

$$\begin{aligned}
&\rho^*-\left(\frac{b\left(e^{\alpha D}-1\right)-(b-1)\alpha D}{b\left(e^{\alpha D}-1\right)+(b-1)\alpha D}\right)^2\\
&= -\frac{(b-1)\,b\left(e^{\alpha D}-1\right)\left(\left(b\left(e^{\alpha D}-1\right)-(b-1)\alpha D\right)^2-(2\alpha D)^2\right)}{\left(b\left(e^{\alpha D}-1\right)+(b-1)\alpha D\right)^2\left((b-1)\,b\left(e^{\alpha D}-1\right)+\alpha D\right)}\\
&\geq 0. \qquad (153)
\end{aligned}$$

Hence, for $\rho>\rho^*$, $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}$ is strictly negative, so it is optimal to offer only $(0,P_H^0)$. This proves the theorem.

B.2. Proof of Theorem 10

We have

$$\left(b > \frac{\alpha D}{e^{\alpha D} - 1 - \alpha D} \wedge \eta_L < \frac{b\left(e^{\alpha D} - 1\right) - \alpha D}{2b\alpha D\left(e^{\alpha D} - 1\right)}\right) \Longrightarrow \rho_0 > \rho^{**} \wedge \rho^* < \rho_{P_L^0} < \rho_0. \quad (154)$$

We now consider the several scenarios and distinguish between the cases 1: $\rho_{P_L^0} < \rho^{**}$ and 2: $\rho_{P_L^0} > \rho^{**}$.

1. $\rho_{P_L^0} < \rho^{**}$:

   (a) $\rho \leq \rho^*$: $\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} > 0$ at $P_L = 0$, monotonously increasing in P_L, so $\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} > 0$ at $P_L = P_L^0$. The optimal solution involves offering only the contract $(0, P_L^0)$.

   (b) $\rho^* < \rho \leq \rho_{P_L^0}$: $\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} > 0$ at $P_L = 0$, monotonously increasing in P_L varying from 0 to the solution of

   $$\frac{\partial^2 \widehat{\Gamma}\left(P_L, \rho\right)}{\partial P_L^2} = 0, \quad (155)$$

   and then decreasing, $\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} > 0$ at $P_L = P_L^0$. The optimal solution involves offering only the contract $(0, P_L^0)$.

   (c) $\rho_{P_L^0} < \rho \leq \rho^{**}$: $\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} > 0$ at $P_L = 0$, monotonously increasing in P_L varying from 0 to the solution of

   $$\frac{\partial^2 \widehat{\Gamma}\left(P_L, \rho\right)}{\partial P_L^2} = 0, \quad (156)$$

   and then decreasing, $\frac{\partial \widehat{\Gamma}(P_L,\rho)}{\partial P_L} < 0$ at $P_L = P_L^0$. The optimal solution is found as follows. In this case, the equation

   $$\frac{\partial \widehat{\Gamma}\left(P_L, \rho\right)}{\partial P_L} = 0 \quad (157)$$

   has a zero root falling in the interval $[0, P_L^0]$ which at the same time is the optimal value of P_L. This equality can be rewritten as follows:

   $$\frac{\left\{\begin{array}{c}(b-1)\left(e^{\alpha D}-1\right)e^{2\alpha P_L} \\ +(1-\rho)\left\{\begin{array}{c}\left(b\left(e^{\alpha D}-1\right)+(b-1)\alpha D\right)e^{\alpha P_L}e^{\alpha P_L^0} \\ -b\alpha D e^{2\alpha P_L^0}\end{array}\right\}\end{array}\right\}}{e^{\alpha P_L}\left(be^{\alpha P_L^0}-(b-1)e^{\alpha P_L}\right)\left(e^{\alpha D}-1\right)} = 0, \quad (158)$$

   resulting in two roots, namely

   $$P_L = P_L^0 + \frac{1}{\alpha}\ln\left[A\left(\rho; b; \alpha; D\right)\right], \quad (159)$$

with

$$A(\rho;b;\alpha;D)=\frac{\left\{\begin{array}{c}(1-\rho)\left(b\left(e^{\alpha D}-1\right)+(b-1)\alpha D\right)\\ \pm\sqrt{(1-\rho)\left\{\begin{array}{c}(1-\rho)\left\{\begin{array}{c}b\left(e^{\alpha D}-1\right)\\ +(b-1)\alpha D\end{array}\right\}^{2}\\ -4(b-1)\alpha D b\left(e^{\alpha D}-1\right)\end{array}\right\}}\end{array}\right\}}{2(b-1)\left(e^{\alpha D}-1\right)}. \tag{160}$$

The root with the '+' sign turns out to be the relevant one. Denote this root by P_L^*. Then the corresponding values for t_L and P_H are found by substituting P_L^* in (147) and (149), respectively.

(d) $\rho^{**}<\rho<\rho_0$: just as in case 1.c.

(e) $\rho>\rho_0$: $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}<0$ at $P_L=0$, monotonously decreasing in P_L, $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}<0$ at $P_L=P_L^0$. So $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}<0$ for each $P_L\in[0,P_L^0]$, and therefore the optimal solution involves offering only $(0,P_H^0)$.

2. $\rho_{P_L^0}>\rho^{**}$:

(a) $\rho<\rho^*$: $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}>0$ at $P_L=0$, monotonously increasing in P_L, so $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}>0$ at $P_L=0$. The optimal solution involves offering only the contract $(0,P_L^0)$.

(b) $\rho^*<\rho<\rho^{**}$: $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}>0$ at $P_L=0$, monotonously increasing in P_L varying from 0 to the solution of

$$\frac{\partial^2\widehat{\Gamma}(P_L,\rho)}{\partial P_L^2}=0, \tag{161}$$

and then decreasing, $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}>0$ at $P_L=P_L^0$. The optimal solution involves offering only the contract $(0,P_L^0)$.

(c) $\rho^{**}<\rho<\rho_{P_L^0}$: $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}>0$ at $P_L=0$, monotonously decreasing in P_L , $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}>0$ at $P_L=P_L^0$. The optimal solution involves offering only the contract $(0,P_L^0)$.

(d) $\rho_{P_L^0}<\rho<\rho_0$: $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}>0$ at $P_L=0$, monotonously decreasing in P_L , $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}<0$ at $P_L=P_L^0$. The optimal solution is as in case 1.d.

(e) $\rho>\rho_0$: $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}<0$ at $P_L=0$, monotonously decreasing in P_L, $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}<0$ at $P_L=P_L^0$. So $\frac{\partial\widehat{\Gamma}(P_L,\rho)}{\partial P_L}<0$ for each $P_L\in[0,P_L^0]$, and therefore the optimal solution involves offering only $(0,P_H^0)$.

Hence the theorem is proved.

Appendix C: First order conditions of optimization problem (81)

The first order conditions of the reduced Lagrangian displayed in (81) are:

Table C.1

First order conditions of maximization problem (32)

C.1	$\frac{\partial \mathcal{L}}{\partial P_H} = 0:$	$\nu\rho - \mu - \gamma_H U'(W - P_H) = 0.$
C.2	$\frac{\partial \mathcal{L}}{\partial P_L} = 0:$	$(1-\rho)\nu + \gamma_H \left\{ \begin{array}{c} F_H(t_L) U'(W - P_L - D) \\ +(1 - F_H(t_L)) U'(W - P_L) \end{array} \right\}$ $- \left\{ \begin{array}{c} F_L(t_L) U'(W - P_L - D) \\ +(1 - F_L(t_L)) U'(W - P_L) \end{array} \right\} = 0.$
C.3	$\frac{\partial \mathcal{L}}{\partial t_L} = 0:$	$(U(W - P_L) - U(W - P_L - D))(\gamma_H F'_H(t_L) - F'_L(t_L))$ $+\nu(1-\rho) F'_L(t_L) D = 0.$
C.4	$\lambda_L, \lambda_H, \gamma_H \geq 0.$	
C.5	$\frac{\partial \mathcal{L}}{\partial \nu} \geq 0:$	$\rho(P_H - \eta_H D) + (1-\rho)(P_L - (\eta_L - F_L(t_L)) D) \geq 0.$
C.6	$\frac{\partial \mathcal{L}}{\partial \mu} \geq 0:$	$P_H \leq \eta_H D.$
C.7	$\frac{\partial \mathcal{L}}{\partial \gamma_H} \geq 0:$	$E_H(0, P_H) \geq E_H(t_L, P_L).$

From (C.2) or (C.3) we have that at least one of the variables ν and γ_H must be non-zero. But then $\nu = 0$ would imply that (C.1) has no solution. It follows that $\nu > 0$ (implying that in an equilibrium the insurer breaks even on average). As the constraint corresponding to ν is binding, the following relation between t_L and P_L and P_H holds anyway:

$$\rho(P_H - \eta_H D) + (1-\rho)(P_L - (\eta_L - F_L(t_L)) D) = 0. \tag{162}$$

Furthermore we have from (C.1) that the following combination of values of multipliers cannot be part of an optimal solution:

$$\mu = 0, \gamma_H = 0. \tag{163}$$

Appendix D: Proof of the theorems of Subsection 4.2

D.1 Proof of Theorem 18

Consider the case

$$b < \frac{\alpha D}{e^{\alpha D} - 1 - \alpha D}. \tag{164}$$

This implies that:

$$\rho_{POOL} < \rho^*. \tag{165}$$

For all $\rho < \rho^*$, we have that $\frac{\partial s(t_L,\rho)}{\partial t_L} > 0$ for $t_L \in \left[0, \widetilde{t_L}\right]$, so $s\left(t_L, \rho\right)$, determining the sign of $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L}$, is increasing everywhere. The consequence is that either:

- $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L}$ is positive for all $t_L \in \left[0, \widetilde{t_L}\right]$, or
- $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L}$ is negative for all $t_L \in \left[0, \widetilde{t_L}\right]$, or
- there is a value $t_L^\circ \in \left[0, \widetilde{t_L}\right]$, such that $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L}$ is negative for $t_L \in [0, t_L^\circ)$ and positive for $t_L \in \left(t_L^\circ, \widetilde{t_L}\right]$.

It follows that for $\rho < \rho^*$, the equilibrium involves one of the following two strategies:

1. For $\rho < \rho_2$: offering only the pooling contract $(0, \rho\eta_H D + (1-\rho)\eta_L D)$, provided that this will actually be purchased by the low risks, which is the case if

$$E_L\left(0, \rho\eta_H D + (1-\rho)\eta_L D\right) > E_L, \tag{166}$$

implying

$$\rho < \frac{\ln\left[\eta_L\left(e^{\alpha D} - 1\right) + 1\right] - \eta_L \alpha D}{(b-1)\eta_L \alpha D}. \tag{167}$$

2. For $\rho > \rho_2$: the Rothschild-Stiglitz strategy, provided that the contract $\left(\widetilde{t_L}, \left(\eta_L - F_L\left(\widetilde{t_L}\right)\right) D\right)$ will actually be bought by the low risks, which is the case if

$$E_L\left(\widetilde{t_L}, \left(\eta_L - F_L\left(\widetilde{t_L}\right)\right) D\right) > E_L, \tag{168}$$

implying

$$e^{\alpha D\left(\eta_L - F_L\left(\widetilde{t_L}\right)\right)} < \frac{\eta_L\left(e^{\alpha D} - 1\right) + 1}{F_L\left(\widetilde{t_L}\right)\left(e^{\alpha D} - 1\right) + 1}. \tag{169}$$

If, for any of the above cases, the condition related is not satisfied (implying that the low risks will not purchase the contract), then the equilibrium involves offering only $(0, \eta_H D)$, hence excluding the low risks from coverage.

Regarding the case $\rho > \rho^*$, note that the equality

$$\frac{\partial \widehat{V}\left(t_L, \rho\right)}{\partial t_L} = 0 \tag{170}$$

has no solutions for

$$\rho > \left(\frac{b\left(e^{\alpha D} - 1\right) - (b-1)\alpha D}{b\left(e^{\alpha D} - 1\right) + (b-1)\alpha D}\right)^2. \tag{171}$$

If ρ exceeds this upper bound, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L}$ is strictly positive for $t_L \in \left[0, \widetilde{t_L}\right]$. In the proof in Appendix B.1, it was shown that, under the given conditions,

$$\rho^* \geq \left(\frac{b\left(e^{\alpha D}-1\right)-(b-1)\alpha D}{b\left(e^{\alpha D}-1\right)+(b-1)\alpha D} \right)^2. \tag{172}$$

So for $\rho > \rho^*$, the equilibrium involves the Rothschild-Stiglitz strategy, provided that condition (169) holds. Otherwise, the equilibrium involves offering only $(0, \eta_H D)$.

D.2 Proof of Theorem 19

We have that

$$b > \frac{\alpha D}{e^{\alpha D}-1-\alpha D}, \tag{173}$$

and

$$F_L\left(\widetilde{t_L}\right) < \frac{b\left(e^{\alpha D}-1\right)-\alpha D}{2b\alpha D\left(e^{\alpha D}-1\right)}, \tag{174}$$

imply

$$\rho^{**} < \rho_{RS}, \tag{175}$$

$$\rho_{RS} > \rho_{POOL}, \tag{176}$$

and

$$\rho_{POOL} > \rho^*. \tag{177}$$

As the proof of Theorem 18 showed, each equilibrium derived is always accompanied by a certain condition indicating that the low risk class individuals actually buy the contract designed for them. This will also be the case in this proof. It will turn out from the analyses that follow, that there are three kinds of equilibria, all subject to the condition just described. Just as in the proof of Theorem 18, for the pooling equilibrium and the Rothschild-Stiglitz strategy, these conditions are

$$\rho < \frac{\ln\left[\eta_L\left(e^{\alpha D}-1\right)+1\right]-\eta_L\alpha D}{(b-1)\eta_L\alpha D} \tag{178}$$

and

$$e^{\alpha D(\eta_L - F_L(\widetilde{t_L}))} < \frac{\eta_L\left(e^{\alpha D}-1\right)+1}{F_L\left(\widetilde{t_L}\right)\left(e^{\alpha D}-1\right)+1}. \tag{179}$$

The equilibrium resulting from the next cases 1.c, 1.d and 2.d are subject to a condition which will be clarified in case 1.c.

Next we consider the several scenarios and distinguish between the cases 1: $\rho_{POOL} < \rho^{**}$ and 2: $\rho_{POOL} > \rho^{**}$.

1. $\rho_{POOL} < \rho^{**}$:

(a) $\rho \leq \rho^*$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = 0$, $s(t_L,\rho)$ monotonously increasing for $t_L \in [0, \widetilde{t_L}]$, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = \widetilde{t_L}$. The equilibrium involves offering the pooling contract $(0, \rho\eta_H D + (1-\rho)\eta_L D)$.

(b) $\rho^* < \rho \leq \rho_{POOL}$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = 0$, $s(t_L,\rho)$ monotonously decreasing at t_L varying from 0 to the solution of

$$\frac{\partial s(t_L,\rho)}{\partial t_L} = 0, \tag{180}$$

and then increasing, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = \widetilde{t_L}$. The equilibrium involves offering the pooling contract $(0, \rho\eta_H D + (1-\rho)\eta_L D)$.

(c) $\rho_{POOL} < \rho \leq \rho^{**}$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} > 0$ at $t_L = 0$, $s(t_L,\rho)$ monotonously decreasing in t_L varying from 0 to the solution of

$$\frac{\partial s(t_L,\rho)}{\partial t_L} = 0, \tag{181}$$

and then increasing, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = \widetilde{t_L}$. In this case, the equation

$$\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} = 0 \tag{182}$$

has a root falling in the interval $[0, \widetilde{t_L}]$ which at the same time is the optimal value of t_L. The equilibrium is found as follows: the two roots of the equation $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} = 0$ are

$$t_L = F_L^{-1}\left(\frac{(1-\rho)\left(b\left(e^{\alpha D}-1\right)-(b+1)\alpha D\right) \pm \sqrt{(1-\rho)\left\{\begin{array}{c}\left((b-1)\alpha D - b\left(e^{\alpha D}-1\right)\right)^2 \\ -\rho\left((b-1)\alpha D + b\left(e^{\alpha D}-1\right)\right)^2\end{array}\right\}}}{2\left(e^{\alpha D}-1\right)(1-\rho)\, b\alpha D} \right). \tag{183}$$

The root with the '$-$'-sign turns out to be the relevant one. Denote this root by t_L^*. Then the corresponding values for P_L and P_H are found by first substituting t_L^* in (98), yielding P_L^* as defined in the theorem, and then substituting t_L^* and P_L^* into (162). This yields P_H^* as defined in the theorem. This results in the strategy of offering (t_L^*, P_L^*) together with $(0, P_H^*)$ which is the equilibrium if purchasing (t_L^*, P_L^*) gives the low risks a higher expected utility than not purchasing any insurance at all. This is the case if

$$e^{\alpha P_L^*}\left(F_L(t_L^*)\left(e^{\alpha D}-1\right)+1\right) < \eta_L\left(e^{\alpha D}-1\right)+1. \tag{184}$$

(d) $\rho^{**} < \rho < \rho_{RS}$: as in case 1.c.

(e) $\rho > \rho_{RS}$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} > 0$ at $t_L = 0$, $s(t_L,\rho)$ monotonously decreasing in t_L, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} > 0$ at $t_L = \widetilde{t_L}$. The equilibrium involves the Rothschild-Stiglitz strategy.

2. $\rho_{P_L^0} > \rho^{**}$:

(a) $\rho \leq \rho^*$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = 0$, $s(t_L,\rho)$ monotonously increasing in P_L, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = \widetilde{t_L}$. The equilibrium involves offering the pooling contract $(0, \rho\eta_H D + (1-\rho)\eta_L D)$.

(b) $\rho^* < \rho \leq \rho^{**}$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = 0$, $s(t_L,\rho)$ monotonously increasing at t_L varying from 0 to the solution of

$$\frac{\partial^2 s(t_L,\rho)}{\partial t_L} = 0, \tag{185}$$

and then decreasing, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = \widetilde{t_L}$. The equilibrium involves offering the pooling contract $(0, \rho\eta_H D + (1-\rho)\eta_L D)$.

(c) $\rho^{**} < \rho < \rho_{POOL}$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = 0, s(t_L,\rho)$ monotonously decreasing at t_L, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = \widetilde{t_L}$. The equilibrium involves offering the pooling contract $(0, \rho\eta_H D + (1-\rho)\eta_L D)$.

(d) $\rho_{POOL} < \rho < \rho_{RS}$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} > 0$ at $t_L = 0$, $s(t_L,\rho)$ monotonously decreasing in t_L , $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} < 0$ at $t_L = \widetilde{t_L}$. The equilibrium is as in case 1.c.

(e) $\rho > \rho_{RS}$: $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} > 0$ at $t_L = 0$, $s(t_L,\rho)$ monotonously decreasing in t_L, $\frac{\partial \widehat{V}(t_L,\rho)}{\partial t_L} > 0$ at $t_L = \widetilde{t_L}$. The equilibrium involves the Rothschild-Stiglitz strategy.

Hence the theorem is proved.

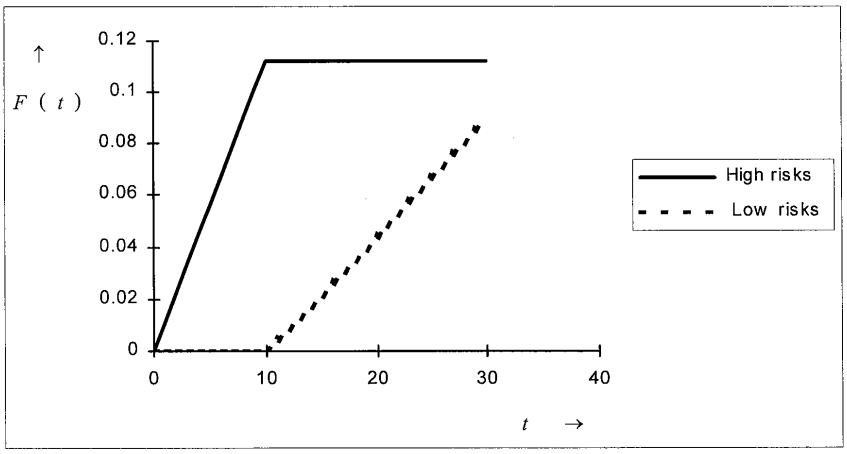

Figure 1: Illustration of Example 3 with $F_H(\cdot)$ solid, $F_L(\cdot)$ dotted, $n = 30$, and $t' = 10$.

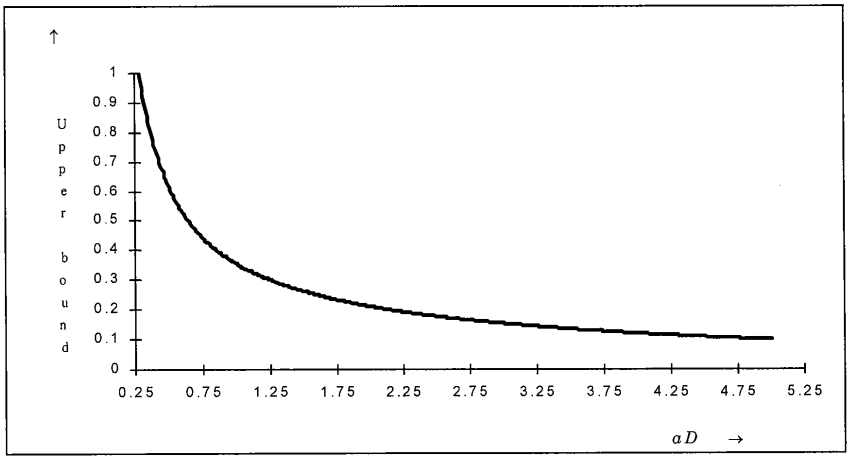

Figure 2: Upper bound of η_L for $b = 2$ as a function of αD.

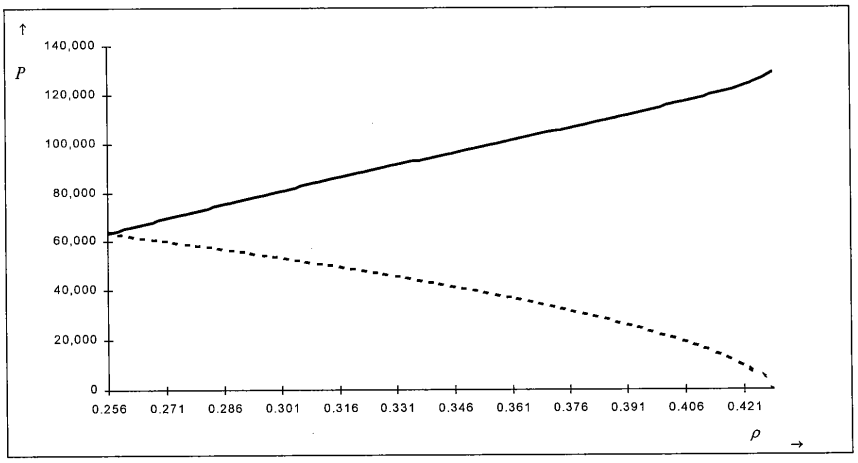

Figure 3: Optimal values for P_L (dotted) and P_H (solid) as a function of ρ.

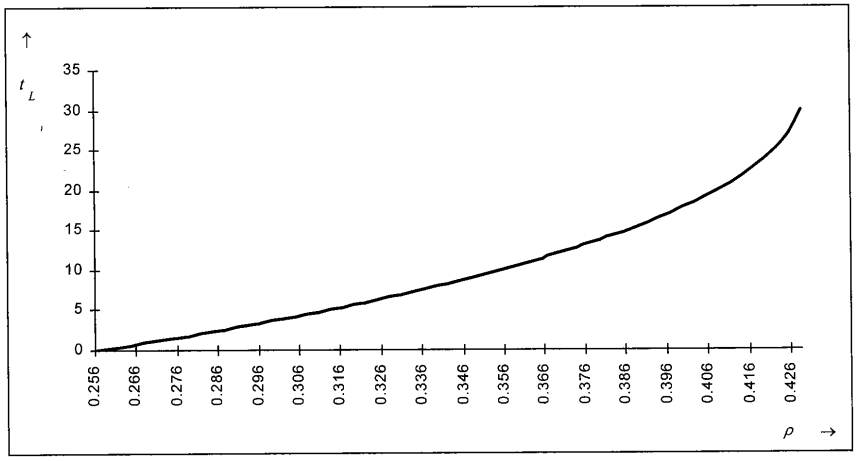

Figure 4: Optimal values of t_L as a function of ρ.

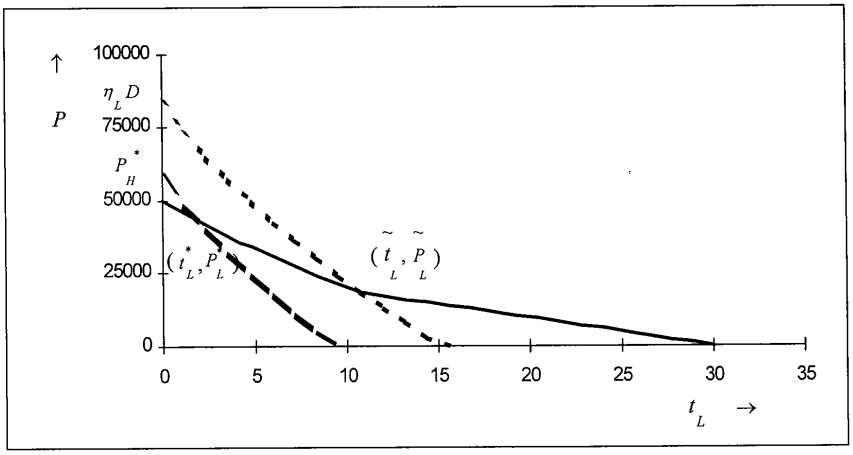

Figure 5: Illustration of case of an exponential utility function.

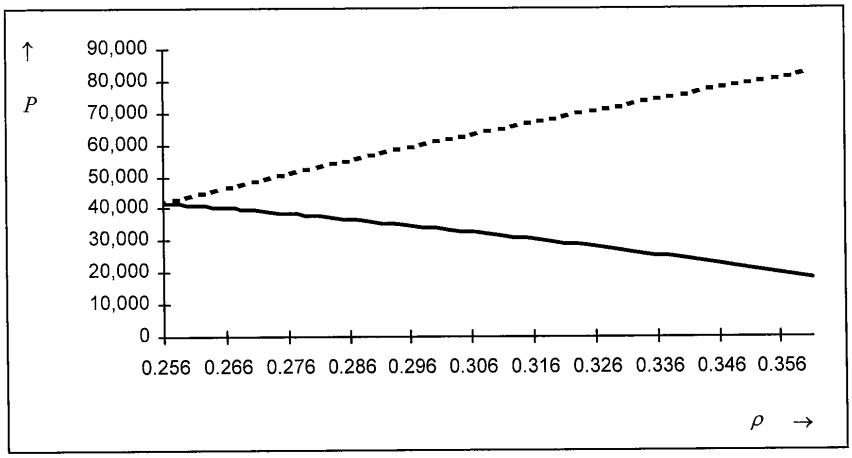

Figure 6: Optimal values for P_L (solid) and P_H (dotted) as a function of ρ.

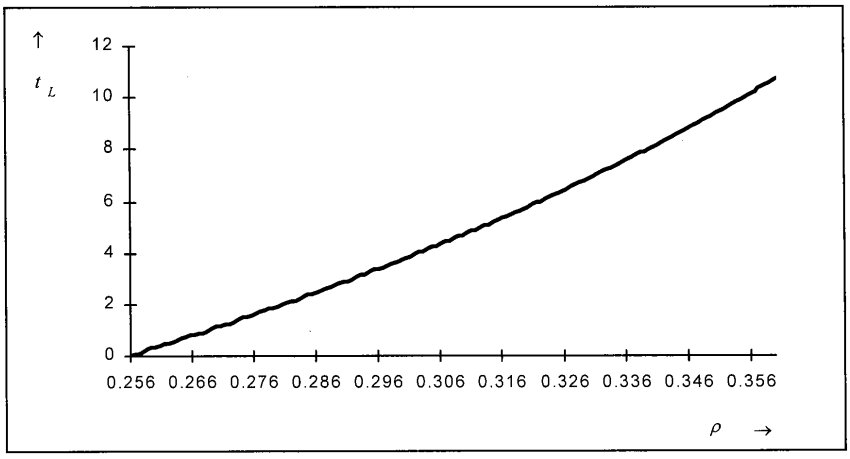

Figure 7: Optimal values of t_L as a function of ρ.

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