Asset Valuation and the Dynamics of Pension Funding with Random Investment Returns

by

M. Iqbal Owadally and
Steven Haberman

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M. Iqbal Owadally* and Steven Haberman

Abstract

Various methods are used to value the assets of defined benefit pension plans when such plans undergo actuarial valuations. The motivation for special asset valuation methods in the context of funding valuations and the properties of these methods are described. Variants of the "Average of Market", "Weighted Average", "Deferred Recognition", "Adjusted Market" and "Write-up" methods that use exponential smoothing are defined and shown to be equivalent. The dynamics of the pension funding process is investigated in the context of a simple model where asset gains and losses emerge as a result of random rates of investment return and where the gains and losses are spread indirectly and in a proportional manner over a moving term. The valuation methods generate asset values that are stable and consistent. Smoothing market values up to a point does improve the stability of contributions but excessive smoothing is shown to be inefficient. It is also shown that consideration should be given to the combined effect of the asset valuation and gain and loss adjustment methods. Efficient combinations of spreading period (when the spread form of gain and loss adjustment is used) and weighting on the current market value of assets (when asset values are smoothed) are suggested.

*Correspondence: Department of Actuarial Science and Statistics, School of Mathematics, The City University, Northampton Square, London EC1V 0HB, England. Phone: +44 (0)20 7477 8953. Fax: +44 (0)20 7477 8838. E-mail: iqbal@city.ac.uk. This research was funded by the Actuarial Research Club of the Department of Actuarial Science and Statistics at The City University, to whom thanks are due.

1 Pension Plan Asset Valuation

Various methods are used by actuaries to value the assets of pension plans. The choice of method should be consistent with the aim of the valuation. Specific methods are often prescribed for valuations that are carried out to verify compliance with solvency or maximum funding regulations or for accounting valuations. Only funding valuations of defined benefit pension plans are considered here. The main aim of such a valuation is to compare assets and liabilities and to recommend contribution rates from a going-concern perspective. Practical methods of valuing pension plan assets for such purposes have been described and classified, notably by Jackson & Hamilton (1968), Trowbridge & Farr (1976, p. 88), Winklevoss (1993, p. 171) and by the Committee on Retirement Systems Research (1998).

Market-related methods are used most frequently. The value of pension plan assets may be taken to be the current market value of these assets, or it may be some average of current and past values in an attempt to remove short-term volatility. Market-related methods are based approximately on the economic valuation of both asset and liability cash flows by reference to the market. Pension liabilities are not generally traded and so pension liability cash flows must be priced by comparison with similar asset cash flows. This argument has been diversely expressed in the actuarial literature in terms of immunization (Vanderhoof, 1972; Milgrom, 1985), matching (Wise, 1987) and hedging (Tilley, 1988). Pension liabilities are therefore to be discounted at market discount rates, suitably risk-adjusted, or at the rates implied in asset portfolios that are dedicated or matched by cash flow to these liabilities. In practice, the pricing of pension liability cash flows is approximate. For example, pension liabilities which have a long duration are often discounted at a single term-independent (possibly duration-weighted) average discount rate. Adjustments for the riskiness of pension liabilities, stemming for instance from potential default by the sponsor or from the uncertainty in the amount and timing of benefit payments, are also made approximately (typically yields from corporate bonds rather than government securities are used). On the asset side of the pension plan balance sheet, market values may be volatile and are often smoothed over short intervals. Comparison of the pension plan liability and asset values provides a measure of the unfunded liability that is consistent (at least approximately) so that contribution rates may be set to secure the long-term funding of pension benefits.

An alternative approach is the discounted-income method which was popular in the United Kingdom until recently and is mentioned for completeness. The difference between income from assets and outgo (benefit payments and expenses) in each year is viewed as a surplus of cash that is reinvested in the asset portfolio held by the pension fund. Accumulating projected net proceeds at the rate of reinvestment return would give an accumulated value of the surplus in the plan. Asset and liability cash flows may instead be discounted at that rate yielding 'present values' of assets and liabilities, the difference between these values being the present value of the surplus of asset over liability cash flows (or an unfunded liability). Contributions may then be determined such that, when reinvested at the assumed rate of return, they liquidate the unfunded liability (Gilley & Funnell, 1958; Funnell & Morse, 1973). The assumed rate of reinvestment return is based on a notional portfolio representing the long term investment strategy of the fund so that the value of the unfunded liability does not change as pension fund asset allocation changes. The notional portfolio should also loosely match the liability cash flows so that, by way of an approximate immunization, the value of the unfunded liability is insensitive to small changes in the assumed rate of return and remains consistent. To reflect any mismatch between the actual and notional portfolios, approximate risk adjustments are apparently made to the assumed liability valuation discount rate and assets are hypothetically switched into the notional portfolio before the asset cash flows are discounted (Springbett, 1964; Wise, 1982;

Thornton & Wilson, 1992). Fixed income securities and any pension liability matched by these securities are thus valued by discounting cash flows at the same rate. Equities are typically valued using a Dividend Discount Model possibly with term-dependent dividend growth assumptions (Day & McKelvey, 1964).

2 Some Properties of Asset Valuation Methods

Asset valuation methods should satisfy certain desirable properties, irrespective of the methodology employed. *Consistency* between the valuation of plan assets and plan liabilities is one such property. Since pension fund valuation involves the comparison of asset and liability cash flows and the subsequent determination of an unfunded liability and contribution rate, it is important that the values placed on assets and liabilities be comparable and consistent. Asset values may be smoothed to remove the impermanent fluctuations in security prices driven by speculators and short-horizon investors because it is believed that such volatility is not reflected in pension liability values and that it is irrelevant to long-term planning for retirement benefits. The plan sponsor's own financial planning tends to be over a shorter term, however, and will be influenced by volatile market conditions. Fair pricing of assets and liabilities should be consistent by virtue of the no-arbitrage principle and smoothing asset prices and using practical approximations in the valuation of pension liabilities may arguably distort the comparison of asset and liability cash flows and the measurement of the unfunded liability. If an asset valuation method is not consistent with liability valuation, then systematic gains or losses will emerge, and such a method would not be acceptable, for example under the Standard of Practice for Valuation of Pension Plans of the Canadian Institute of Actuaries (1994, para. 5.01).

Pension plan assets should also be valued in an *objective* way. Market values of assets are objective in the sense that, absent accounting errors, two actuaries will employ the same asset value. The variety and opacity of some averaging techniques, especially if they are frequently changed, may appear to be somewhat arbitrary. (Smoothed asset values would certainly not be objective for the determination of solvency.) Off-market valuation methods are often more arbitrary. When both asset and liability cash flows are discounted and a Dividend Discount Model is used, asset values are highly sensitive to the equity dividend growth assumption and the smoothing effect may not be very transparent (Dyson & Exley, 1995). Details of any smoothing method should be disclosed according to Actuarial Standard of Practice No. 4 of the Actuarial Standards Board (1993) and should presumably be applied systematically and rationally.

Asset values must also be *realistic.* For funding purposes (as opposed to compliance purposes such as solvency and accounting), the primary objective is to determine a reasonable rate of contribution rather than to place an absolute value on assets. The asset value is not meant to be a fundamental measure of the worth of pension plan assets. Asset values may be smoothed, but should remain in some proximity to market values (Berin, 1989, p. 30; Winklevoss, 1993, p. 172). The U.S. Internal Revenue Service for example requires that the actuarial asset value be within a 20% corridor of market value. Market values are relevant because market conditions do affect the plan sponsor, who ultimately contributes to the pension plan. It is nevertheless sometimes argued that, when asset and liability cash flows are discounted on the same basis, the intermediate present values of assets and liabilities are superior to the corresponding market values (Day & McKelvey, 1964). Although they may satisfy some internal consistency, the resultant values of the unfunded liability and recommended contribution rates may be artificial, off-market and irrelevant to the financial management of corporate plan sponsors.

Asset valuation methods also *stabilize* and *smooth* the pension funding process. Actuarial valuations for funding purposes aim at measuring the shortfall (or surplus) of assets over liabilities so that contribution rates may be calculated in order to make good these shortfalls and secure assets to meet pension liabilities as and when they are due. Firms sponsor pension plans on a voluntary basis and also through competitive pressures in the employment market. But plan sponsors are motivated to fund defined retirement benefits in advance when the contributions required are stabilized and spread over time, so that the costs arising from the uncertainty of long-term pension provision are not immediately borne. (It is not generally desirable however that the accounting pension expense be smoothed.) Thus, an unfunded liability is generally defrayed over a number of years. Short-term variations in asset prices conflict with this stability objective. A fundamental reason for using special methods to value assets in the funding valuations of defined benefit pension plans is therefore to moderate volatility in *asset values* and generate a stable and smooth pattern of *contribution rates* (Anderson, 1992, p. 108; Ezra, 1979, p. 40; Winklevoss, 1993, p.171).

The effect of the asset valuation method on the *dynamics* of pension funding is significant. Asset-liability models turn out to be sensitive to the specification of the asset valuation method (Kingsland, 1982; Kemp, 1996). Asset allocation decisions should not be based on the outcome of a funding valuation but may be influenced by the assessment of pension plan liabilities and assets and the unfunded liability reported after such an actuarial valuation. The asset valuation method should not therefore lead to wrong investment decisions (Ezra, 1979, p. 110; Dyson & Exley, 1995). The way in which assets are valued certainly affects the timing of contributions, as the emergence of *asset* gains and losses depends on the value placed on plan assets. This has an indirect effect on the *ex post* cost of pension provision. Asset gains and losses are also amortized but it is often considered that this is not powerful enough to dampen their volatility and stabilize contributions, and consequently asset valuation methods themselves need to incorporate a smoothing quality (Anderson, 1992, p. 108). It is not meaningful, therefore, to consider the issue of asset valuation without addressing the method by which gains and losses are amortized. Trowbridge & Farr (1976, p. 73) thus refer to the "consistency between the asset valuation and the techniques of actuarial gain or loss adjustment".

3 A Simple Model of the Pension Funding Process

Market-related or smoothed market asset valuation methods employing exponential smoothing are considered in the following. Certain simplifying assumptions are required to study the effect of valuing the assets of pension plans according to this method when contribution rates are determined. A simple but mathematically tractable model of a defined benefit plan, providing a pension based on final salary upon retirement at a normal retirement age, is set up as follows.

The plan is valued at the beginning of every year and a contribution rate is determined. Plan assets are directly marketable, their market value $f(t)$ is instantly obtainable and an actuarial asset value $F(t)$ is calculated. The valuation of pension liabilities is not analyzed and it is assumed from the outset that the actuarial valuation basis is constant and that the actuarial cost method is not changed and generates an actuarial liability AL and a normal cost NC. The valuation discount rate i is chosen such that AL is a practical approximation to the fair value of pension liabilities. The actuarial assumption as to the projected long-term rate of return on plan assets is also i.

As regards the demographic experience of the plan, its membership is constant and evolves exactly according to the life table used for valuation purposes. No mortality gain or loss arises. The economic experience of the plan is such that only asset gains and losses emerge owing to a variable rate of return $r(t)$ in year $(t-1,\ t)$. For simplicity, neither salaries nor benefits are subject to economic inflation. The actuarial liability AL, normal cost NC and yearly benefit outgo B are constant as a result of the stationary nature of pension liabilities and of the fixed liability valuation method. Alternatively, all salaries and benefits (including pensions in payment) increase at the *same* rate of inflation, and all monetary quantities (including i, $r(t)$, $f(t)$ etc.) are then deflated. AL, NC, B are then constant in *real* terms.

This simple model resembles the model of Trowbridge (1952), who shows that when the liability structure of the pension plan is in equilibrium,

$$AL = (1+i)(AL + NC - B). \tag{1}$$

The model is also similar to those of Bowers *et al.* (1979) and Dufresne (1988), except that a smoothed actuarial asset value $F(t)$ is explicitly considered here.

All cash flows occur only at the start of the year, and if $c(t)$ is the contribution income at the beginning of year $(t,\, t+1)$,

$$f(t+1) = (1 + r(t+1))[f(t) + c(t) - B], \tag{2}$$

for $t \geq 0$. The unfunded liability based on the market value of plan assets and the smoothed unfunded liability based on the smoothed actuarial value of plan assets are respectively defined as (for $t \geq 0$)

$$ul(t) = AL - f(t), \tag{3}$$

$$UL(t) = AL - F(t). \tag{4}$$

The anticipated market value of plan assets at time $t \geq 1$, if the actuarial assumption as to returns on plan assets is borne out during year $(t-1,\, t)$, is

$$f^A(t) = (1+i)\left[f(t-1) + c(t-1) - B\right]. \tag{5}$$

The anticipated or written-up actuarial value of plan assets is, correspondingly,

$$F^A(t) = (1+i)\left[F(t-1) + c(t-1) - B\right]. \tag{6}$$

The unsmoothed anticipated unfunded liability is $ul^A(t) = AL - f^A(t)$ and the smoothed anticipated unfunded liability is $UL^A(t) = AL - F^A(t)$.

An asset gain or loss emerges in year $(t-1,\, t)$ as the actual return on plan assets $r(t)$ differs from the actuarial assumption i. A loss is defined as the difference between the unfunded liability and anticipated unfunded liability, and a gain is a negative loss. The unsmoothed and smoothed loss (based on whether market or smoothed asset values are used) are respectively, for $t \geq 1$,

$$l(t) = ul(t) - ul^A(t) = f^A(t) - f(t), \tag{7}$$

$$L(t) = UL(t) - UL^A(t) = F^A(t) - F(t). \tag{8}$$

From equations (2), (5) and (7), for $t \geq 1$,

$$l(t) = (i - r(t))[f(t-1) + c(t-1) - B]. \tag{9}$$

Contribution rates are adjusted to pay off asset gains and losses. The population-salary profile is fixed (in nominal or real terms) in this simplified model and we consider the contribution (in nominal or real) dollar terms rather than as a percentage of payroll:

$$c(t) = NC + adj(t), \tag{10}$$

where $adj(t)$ is a supplementary contribution paid to liquidate gains and losses as well as any initial unfunded liability in the plan.

4 Asset Valuation Methods

The method by which the actuarial asset value $F(t)$ of plan assets is determined is yet to be defined. For notational convenience, $u = 1 + i$ and $v = 1/(1+i)$ are used henceforth.

Define the present value of plan assets at time t written up over $j \geq 1$ years and allowing for cash flows to be

$$F_j(t) = u^j f(t-j) + \sum_{k=1}^{j} u^k [c(t-k) - B]. \tag{11}$$

From the definition of the loss $l(t)$ in equations (5) and (7),

$$f(t) + l(t) = u[f(t-1) + c(t-1) - B] \tag{12}$$

and, by recursion, it is easily shown that

$$F_j(t) = f(t) + \sum_{k=0}^{j-1} u^k l(t-k), \tag{13}$$

$$F_j(t) = u[F_j(t-1) + c(t-1) - B] - u^j l(t-j). \tag{14}$$

The smoothed actuarial value of plan assets at time $t \geq 1$ is defined generically as

$$F(t) = (1-\lambda)\left\{ f(t) + \sum_{j=1}^{\infty} \lambda^j F_j(t) \right\}, \tag{15}$$

for $0 \leq \lambda < 1$. It is an exponentially weighted average allowing for cash flows and interest.

The actuarial asset value may be reinterpreted by substituting $F_j(t)$ from equation (11) into equation (15),

$$\begin{aligned} F(t) &= \sum_{j=0}^{\infty} \lambda^j (1-\lambda) u^j f(t-j) + \sum_{j=1}^{\infty} \lambda^j (1-\lambda) \sum_{k=1}^{j} u^k [c(t-k) - B] \\ &= \sum_{j=0}^{\infty} \lambda^j (1-\lambda) u^j f(t-j) + \sum_{j=1}^{\infty} (\lambda u)^j [c(t-j) - B], \end{aligned} \tag{16}$$

for $t \geq 1$. The method could be termed an "Exponentially Weighted Infinite Average of Market", by contrast with the arithmetic "Moving Average of Market" mentioned by the Committee on Retirement Systems Research (1998), as it averages over the market values of plan assets whilst allowing for cash flows and the time value of money. (The smoothing weights sum to unity since $\sum_{j=0}^{\infty} \lambda^j (1-\lambda) = 1$.)

The *anticipated* actuarial value of assets at time t based on actuarial valuation assumptions and the actuarial value at time $t-1$ is, from equations (6) and (16),

$$\begin{aligned} F^A(t) &= u[F(t-1) + c(t-1) - B] \\ &= \sum_{j=1}^{\infty} \lambda^{j-1}(1-\lambda)u^j f(t-j) + \sum_{j=1}^{\infty} \lambda^{j-1}u^j[c(t-j) - B]. \end{aligned} \tag{17}$$

By comparison with $F(t)$ in equation (16), it is clear that, for $t \geq 1$,

$$F(t) = \lambda F^A(t) + (1-\lambda)f(t), \tag{18}$$

which exhibits the method of valuation of plan assets as a "Weighted Average" of the market value of plan assets and of the written-up anticipated actuarial value.

The asset valuation method may be rewritten as

$$F(t) = F^A(t) + (1-\lambda)[f(t) - F^A(t)], \tag{19}$$

(for $t \geq 1$) which is one instance of the "Write-up" method with an adjustment. The actuarial value of plan assets is the written-up anticpated value, with recognition of a fraction of the difference between the market and anticipated or written-up values of plan assets.

It is equally clear that, for $t \geq 1$,

$$F(t) = f(t) + \lambda[F^A(t) - f(t)]. \tag{20}$$

This represents the asset valuation method as an "Adjusted Market" method. The actuarial value of plan assets is equal to the market value, smoothed by an adjustment equal to a fraction of the difference between the anticipated actuarial value and the market value.

One can also define the asset valuation method in terms of the asset losses $\{l(t)\}$ in the pension plan. Replacing $F_j(t)$ from equation (14) into equation (15) yields

$$\begin{aligned} F(t) &= (1-\lambda)f(t) \;+\; u\sum_{j=1}^{\infty}\lambda^j(1-\lambda)F_j(t-1) \;+\; u\lambda[c(t-1)-B] \\ &\quad - \sum_{j=1}^{\infty}\lambda^j(1-\lambda)u^j l(t-j) \\ &= uF(t-1) \;+\; (1-\lambda)f(t) \;-\; u(1-\lambda)f(t-1) \;+\; u\lambda[c(t-1)-B] \\ &\quad - \sum_{j=1}^{\infty}\lambda^j(1-\lambda)u^j l(t-j) \end{aligned} \tag{21}$$

and upon substituting from equation (12),

$$F(t) = F^A(t) - \sum_{j=0}^{\infty}\lambda^j(1-\lambda)u^j l(t-j), \tag{22}$$

for $t \geq 1$. This expresses the asset valuation method as a "Write-up" method, with the adjustment explicitly defined in the losses $\{l(t)\}$.

From equations (19) and (22), the smoothed loss as defined in equation (8) is therefore

$$L(t) = (1-\lambda)[F^A(t) - f(t)] \tag{23}$$

$$= \sum_{j=0}^{\infty}\lambda^j(1-\lambda)u^j l(t-j). \tag{24}$$

Equation (24) shows that the present value of losses $\{u^j l(t-j)\}$ is exponentially smoothed. A unit loss is smoothed by recognizing $\{\lambda^0(1-\lambda),\ \lambda^1(1-\lambda),\ \lambda^2(1-\lambda),\ \dots\ \}$ together with interest, in successive years and *in perpetuity*. For a unit loss that emerged j years ago, a total of $1-\lambda^{j+1}$ has been recognized (along with interest) and separately amortized.

The "Adjusted Market" method in equation (20) may be rewritten, using equations (23) and (24), as

$$F(t) = f(t) + \sum_{j=0}^{\infty}\lambda^{j+1}u^j l(t-j), \tag{25}$$

(for $t \geq 1$) which expresses the asset valuation method as a "Deferred Recognition" method. (Equation (25) may also be obtained directly by replacing $F_j(t)$ in equation (13) into the basic definition for the actuarial asset value in equation (15).) Equation (24) shows that a fraction $1-\lambda^{j+1}$ of a unit loss that emerged j years ago has been recognized (and separately amortized). Successive portions $\{\lambda^{j+1}(1-\lambda),\ \lambda^{j+2}(1-\lambda),\ \dots\ \}$ (totalling λ^{j+1}) are yet to be recognized (with interest) in future years. In other words, a fraction λ^{j+1} of a loss j years ago has been deferred and, in the "Deferred Recognition" method, is added (with interest) to the market value of assets.

The perpetual deferral of declining portions of asset gains and losses is not as inadvisable as may first appear. Smoothing using an infinite exponentially weighted average is perhaps more natural than averaging over a finite moving interval. Gains and losses emerge randomly and continually and are never completely liquidated in any case.

It may be observed that the asset valuation methods above continually adjust the actuarial value of assets relative to the market value. One of the methods listed by Winklevoss (1993, p. 174) is identical, except that the adjustment is discontinuous: the actuarial value of assets equals $F^A(t)$ when $F^A(t)$ is within a corridor of values around $f(t)$, and otherwise is a suitably adjusted written-up value similar to $F(t)$ in equation (19). The asset valuation method, in the form described in equation (18), is also akin to a method suggested by Dyson & Exley (1995). Their method involves an exponential smoothing of a certain funding ratio (that is, the asset value as a fraction of the actuarial liability) without accounting for cash flows or interest. The asset valuation method that we have described is also sufficiently general to encompass certain methods described by Jackson & Hamilton (1968), Ferris & Welch (1996) and Aitken (1994, p. 289).

Finally, it is clear that the various asset valuation methods described in equations (16), (18), (19), (20), (22) and (25) are identical, if initial conditions are ignored. The initial conditions may be chosen arbitrarily so that identity follows.

The pension system is initialized as follows. Let the initial market value of plan assets be $f(0) = f_0$ with certainty and the initial unfunded liability (at market) be $ul(0) = ul_0 =$

$AL - f_0$. The pension fund is marked-to-market at time $t = 0$ so that $F(0) = f_0$ and $UL(0) = ul_0$ with probability 1. For $t \leq 0$, assume also that $l(t) = 0$, $c(t) = NC$ and $f(t-1) = vf(t) - NC + B$ (with $f(0) = f_0$).

5 Supplementary Contributions

The method of funding for pension benefits is not completely specified until the treatment of gains and losses is described. Gains and losses are not amortized directly over a fixed term but are instead indirectly spread forward over a moving term, as discussed by Trowbridge (1952), Trowbridge & Farr (1976, p. 85), Bowers *et al.* (1979), Dufresne (1988) and McGill *et al.* (1996, p. 525). They show that this gain and loss adjustment technique is implicit in the Aggregate and Frozen Initial Liability actuarial cost methods which are known as "spread gain" methods (Berin, 1989, p. 63; Aitken, 1994, p. 326). The spread adjustment of gains and losses may also be used with individual actuarial cost methods, such as the Projected Unit Credit method.

The initial unfunded liability ul_0, arising at plan inception or from amendments to valuation methods or plan benefits, may be implicitly spread forward or it may be explicitly amortized over M years (say) by payments

$$P(t) = \begin{cases} ul_0/\ddot{a}_{\overline{M}|}, & 0 \leq t \leq M-1, \\ 0, & t \geq M. \end{cases} \tag{26}$$

The unamortized part of ul_0 is

$$U(t) = \begin{cases} ul_0 \, \ddot{a}_{\overline{M-t}|}/\ddot{a}_{\overline{M}|}, & 0 \leq t \leq M-1, \\ 0, & t \geq M. \end{cases} \tag{27}$$

Note that

$$P(t) = U(t) - vU(t+1), \tag{28}$$

where we define $U(n) = U(n+1) = \cdots = 0$.

The supplementary contribution or adjustment $adj(t)$ is (Owadally & Haberman, 1999):

$$adj(t) = (1-K)[UL(t) - U(t)] + P(t), \qquad 1-K = 1/\ddot{a}_{\overline{m}|}. \tag{29}$$

The adjustment is such that level payments over a spreading period of m years would amortize the plan unfunded liability (based on the smoothed actuarial value of assets) in excess of the unamortized part of ul_0. Note how this fits in with, and historically probably derives from, the calculation of the contribution rate in the discounted-income approach described in section 1. The annuities in equations (26), (27) and (29) are evaluated at rate i. If assets are valued at market only, that is $F(t) = f(t)\ \forall t$, then the model reduces to the one investigated by Dufresne (1988) (with the minor exception that he does not consider the separate amortization of the initial unfunded liability).

The method of indirectly spreading gains and losses is chosen because the proportional supplementary contribution involved in equation (29) appears intuitively to lead to smoother contribution rates by contrast with fixed-term gain/loss amortization schedules. To explore this further, replace $F(t)$ by $UL(t)$ (equation (4)) and $c(t)$ by $adj(t)$ (equation (10)) into equations (6) and (8) to obtain a recurrence relation for the smoothed unfunded liability, for $t \geq 1$,

$$UL(t) - u \cdot UL(t-1) = L(t) - u \cdot adj(t-1), \tag{30}$$

and simplify by further substituting $adj(t)$ from equation (29) and using equation (28):

$$(UL(t) - U(t)) - uK(UL(t-1) - U(t-1)) = L(t). \tag{31}$$

Hence, for $t \geq 0$,

$$UL(t) - U(t) = \sum_{j=0}^{\infty} (uK)^j L(t-j), \tag{32}$$

and from equation (29),

$$adj(t) - P(t) = \sum_{j=0}^{\infty} K^j (1-K) u^j L(t-j). \tag{33}$$

Note the symmetry in equations (24) and (33). Asset gains and losses (at market) are smoothed twice, first by the asset valuation method and then by the gain/loss adjustment method. When the spreading adjustment is used, equation (33) shows that the supplementary contribution is an exponentially weighted infinite average of smoothed losses ($\sum_{j=0}^{\infty} K^j(1-K) = 1$, since $0 \leq K < v < 1$ for $m \geq 1$). An identical smoothing mechanism is employed in the asset valuation and gain/loss adjustment methods that are considered here.

The smoothness of contribution rates when the spreading adjustment is used within the Aggregate and Frozen Initial Liability methods is indeed observed by Trowbridge & Farr (1976, p. 62). Hennington (1968) states that "The smoothness of the annual contribution is determined not only by the method for determining asset value but also by the actuarial funding method. [...] An actuarial cost method involving a spreading of actuarial gains and losses makes it easier to use some of the market value methods". Berin (1989, p. 28) notes that the valuation of assets at market is less "risky" if a spread-gain funding method is used. Owadally & Haberman (1999) also find that spreading is more efficient than the direct amortization of gains and losses in the sense that more stable contribution rates and funding levels may be achieved.

6 Pension Funding with Asset Valuation and Gain/Loss Spreading

The pension funding process with asset valuation and spreading of gains and losses may now be examined under the simplifying assumptions set out in section 3. A simple recurrence relation can be set up as follows. First, note from equations (6) and (18) that

$$F(t) = \lambda u[F(t-1) + c(t-1) - B] + (1-\lambda)f(t), \tag{34}$$

for $t \geq 1$. This can be rewritten in terms of the unfunded liabilities, using equations (1), (3), (4) and (10), as

$$UL(t) = \lambda u[UL(t-1) - adj(t-1)] + (1-\lambda)ul(t). \tag{35}$$

Gains and losses are indirectly spread forward and the contribution adjustment term may be replaced using equations (28) and (29) to give, for $t \geq 1$,

$$UL(t) - U(t) = \lambda Ku[UL(t-1) - U(t-1)] + (1-\lambda)[ul(t) - U(t)]. \tag{36}$$

Since $ul(0) = UL(0) = U(0) = ul_0$, it is clear that for $t \geq 0$,

$$UL(t) - U(t) = (1-\lambda)\sum_{j=0}^{t}(\lambda Ku)^{t-j}[ul(j) - U(j)]. \tag{37}$$

The outcome of an actuarial valuation at $t \geq 0$ is therefore to recommend a contribution payment of

$$\begin{aligned} c(t) &= NC + (1-K)(UL(t) - U(t)) + P(t) && (38) \\ &= NC + (1-K)(1-\lambda)\sum_{j=0}^{t}(\lambda Ku)^{t-j}[ul(j) - U(j)] + U(t) - vU(t+1), && (39) \end{aligned}$$

from equations (10), (28) and (29).

We may now obtain a recurrence relationship for the unfunded liability $ul(t)$ $(t \geq 0)$ by substituting $c(t)$ from equation (39) and $ul(t)$ from equation (3) into equation (2), and by using equation (1):

$$\begin{aligned} &(ul(t+1) - U(t+1)) - (AL - U(t+1)) \\ &\quad = (1 + r(t+1))\Bigg[(ul(t) - U(t)) - (1-K)(1-\lambda)\sum_{j=0}^{t}(\lambda Ku)^{t-j}(ul(j) - U(j)) \\ &\qquad\qquad - v(AL - U(t+1))\Bigg]. \qquad (40) \end{aligned}$$

Suppose now that the initial unfunded liability is not separately amortized and is indirectly spread forward, i.e. $P(t) = U(t) = 0 \ \forall t$. It is easily established from equation (36) that

$$UL(t) = (1-\lambda)\sum_{j=0}^{t}(\lambda K u)^{t-j} ul(j) + \lambda(\lambda K u)^t ul_0 \tag{41}$$

and hence that

$$c(t) = NC + (1-K)(1-\lambda)\sum_{j=0}^{t}(\lambda K u)^{t-j} ul(j) + (1-K)\lambda(\lambda K u)^{t-j} ul_0 \tag{42}$$

and

$$\begin{aligned} ul(t+1) - AL = (1+r(t+1))\Bigg[ul(t) - (1-K)(1-\lambda)\sum_{j=0}^{t}(\lambda K u)^{t-j} ul(j) \\ - vAL - \lambda(1-K)(\lambda K u)^t ul_0\Bigg]. \end{aligned} \tag{43}$$

If the initial unfunded liability is disregarded (assumed to be zero or to have been completely amortized), then symmetry in K and λ holds over time in the pension funding process. The values of K and λ could be interchanged in equations (39), (40) (42) and (43) without affecting the dynamics of the pension funding process (ignoring ul_0). This confirms that the methods of asset valuation and of asset gain/loss treatment under consideration have a similar smoothing function.

7 Moments of the Pension Funding Process

Suppose now that the rate of return $r(t)$ on pension plan assets in year $(t-1,\, t)$ is projected to be random (independent and identically distributed), with mean r and variance σ^2. Such a projection assumption simplifies reality but does introduce volatility and reflect market efficiency. It is convenient to define $d = i/(1+i)$, $d_r = r/(1+r)$, as well as

$$\theta \ = \ (1-K)(1-\lambda)/(1-\lambda K u). \tag{44}$$

The long-term expected values of various variables in the pension fund are shown in Proposition 1, which is proven in Appendix A.

PROPOSITION 1 *If* $i > -100\%$, $r > -100\%$, $\lambda K(1+i)(1+r) < 1$ *and* $\theta > d_r$, *then*

$$\lim_{t\to\infty} \mathrm{E}l(t) = -AL(d_r - d)(d-\theta)u/(d_r - \theta), \tag{45}$$

$$\lim_{t\to\infty} \mathrm{E}ul(t) = AL(d_r - d)/(d_r - \theta), \tag{46}$$

$$\lim_{t\to\infty} \mathrm{E}UL(t) = AL\theta(d_r - d)/(1-K)(d_r - \theta), \tag{47}$$

$$\lim_{t\to\infty} \mathrm{E}f(t) = AL(d-\theta)/(d_r - \theta), \tag{48}$$

$$\lim_{t\to\infty} \mathrm{E}F(t) = AL - AL\theta(d_r - d)/(1-K)(d_r - \theta), \tag{49}$$

$$\lim_{t\to\infty} \mathrm{E}c(t) = NC + AL\theta(d_r - d)/(d_r - \theta). \tag{50}$$

The symmetry between K and λ in the first moments (except in those of the actuarial values $F(t)$ and $UL(t)$ which involve only smoothing through asset valuation and not through gain/loss adjustment) is again evident. When unsmoothed market values of plan assets are used ($\lambda = 0$) then $\theta = 1 - K$, in which case the results reduce exactly to those obtained by Dufresne (1988). When asset gains and losses are not amortized or spread but are paid off immediately ($m = 1$, $K = 0$) then $\theta = 1 - \lambda$, in which case the results mirror those of Dufresne (1988) with λ exactly replacing K.

Simpler results follow if the actuarial assumption i as to the rate of return on plan assets is unbiased and equals the mean rate of return r.

COROLLARY 1 *Suppose that* $r = i$. *Then* $\mathrm{E}l(t) = 0\ \forall t$. *Furthermore, if there is no initial unfunded liability* ($ul_0 = 0$)*, then* $\mathrm{E}ul(t) = \mathrm{E}UL(t) = 0\ \forall t$, $\mathrm{E}f(t) = AL\ \forall t$ *and* $\mathrm{E}c(t) = NC\ \forall t$. *If the initial unfunded liability is separately amortized by additional* *payments, as in equation* (26), *then*

$$\mathrm{E}ul(t) = \mathrm{E}UL(t) = \begin{cases} ul_0\, \ddot{a}_{\overline{M-t}|}/\ddot{a}_{\overline{M}|}, & 0 \le t \le M-1, \\ 0, & t \ge M, \end{cases} \tag{51}$$

$$\mathrm{E}f(t) = \mathrm{E}F(t) = \begin{cases} AL - ul_0\, \ddot{a}_{\overline{M-t}|}/\ddot{a}_{\overline{M}|}, & 0 \le t \le M-1, \\ AL, & t \ge M, \end{cases} \tag{52}$$

$$\mathrm{E}c(t) = \begin{cases} NC + ul_0/\ddot{a}_{\overline{M}|}, & 0 \le t \le M-1, \\ NC, & t \ge M. \end{cases} \tag{53}$$

If the initial unfunded liability is not separately amortized then, provided that $r = i > -100\%$, $0 \le K < v$ *and* $0 \le \lambda < v$,

$$\lim_{t\to\infty} \mathrm{E}ul(t) = \lim_{t\to\infty} \mathrm{E}UL(t) = 0, \tag{54}$$

$$\lim_{t\to\infty} \mathrm{E}f(t) = \lim_{t\to\infty} \mathrm{E}F(t) = AL, \tag{55}$$

$$\lim_{t\to\infty} \mathrm{E}c(t) = NC. \tag{56}$$

See Appendix B for a proof. If the actuarial assumption as to returns on plan assets is a best estimate and is borne out by experience on average, then no gain or loss is expected to emerge. The set of asset valuation methods defined in equations (16), (18), (19), (20), (22) and (25) therefore exactly satisfies the criterion for consistency as stipulated for example in the Standard of Practice for Valuation of Pension Plans of the Canadian Institute of Actuaries (1994, para. 5.01).

After the initial unfunded liability is defrayed, the plan is expected to remain fully funded and no supplementary contribution beyond the normal cost is paid on average. Even if the initial unfunded liability is not being separately identified and amortized over a finite period, the plan is expected to become fully funded eventually. But this will only happen if the unfunded liability is recognized and liquidated fast enough relative to interest paid on the initial unfunded liability: pension liabilities will ultimately be fully funded if asset values are not excessively smoothed ($0 \leq \lambda < v$) and gains and losses are not spread over very long periods ($0 \leq K < v$).

Full funding occurs only asymptotically in spread-gain or aggregate actuarial cost methods but, as pointed out by Trowbridge & Farr (1976, p. 85) and others, this is not necessarily objectionable since random experience deviations occur all the time and a perfectly zero unfunded liability is never achieved except by chance.

The second moments of the pension funding process are considered in Proposition 2. The simplifying assumption is made henceforth that the assumed rate of return is unbiased ($r = i$). The variance of the random rate of return on plan assets is $\sigma^2 = \mathrm{Var}r(t)$. Furthermore, define $q = \mathrm{E}(1 + r(t))^2 = u^2 + \sigma^2$ and $V_\infty = \sigma^2 v^2 AL^2/Q$, where

$$\begin{aligned} Q &= (1 - qK^2)(1 - \lambda^2 u^2)(1 - \lambda K u^2) \\ &\quad - \lambda(1 - K)\sigma^2[2K(1 - \lambda^2 u^2) + \lambda(1 - K)(1 + \lambda K u^2)] \qquad (57) \\ &= (1 - q\lambda^2)(1 - K^2 u^2)(1 - \lambda K u^2) \\ &\quad - K(1 - \lambda)\sigma^2[2\lambda(1 - K^2 u^2) + K(1 - \lambda)(1 + \lambda K u^2)]. \qquad (58) \end{aligned}$$

PROPOSITION 2 *Provided that* $r = i > -100\%$, $0 \leq K < v$, $0 \leq \lambda < v$, $Q > 0$ *and*

$$\begin{aligned} &(1 + \lambda^2 K^2 q u^2)(1 + \lambda^3 K^3 \sigma^2 u^2 - \lambda^4 K^4 q u^6) \\ &\qquad > \; 2\lambda^4 K^4 (\lambda + K) q \sigma^2 u^4 + \lambda K (\lambda + K)^2 q u^2 (1 - \lambda^2 K^2 q u^2), \qquad (59) \end{aligned}$$

then

$$\lim_{t\to\infty} \mathrm{Var}l(t) = V_\infty(1-K^2u^2)(1-\lambda^2u^2)(1-\lambda Ku^2), \tag{60}$$

$$\lim_{t\to\infty} \mathrm{Var}f(t) = V_\infty[(1-\lambda Ku^2)(1-\lambda^2K^2u^2)+2\lambda K(1-\lambda)(1-K)u^2], \tag{61}$$

$$\lim_{t\to\infty} \mathrm{Var}F(t) = V_\infty(1-\lambda)^2(1+\lambda Ku^2), \tag{62}$$

$$\lim_{t\to\infty} \mathrm{Var}c(t) = V_\infty(1-K)^2(1-\lambda)^2(1+\lambda Ku^2), \tag{63}$$

$$\lim_{t\to\infty} \mathrm{Cov}[f(t),F(t)] = V_\infty(1-\lambda)[1+\lambda K(1-K-\lambda)u^2], \tag{64}$$

$$\lim_{t\to\infty} \mathrm{Cov}[f(t),c(t)] = -V_\infty(1-K)(1-\lambda)[1+\lambda K(1-K-\lambda)u^2], \tag{65}$$

$$\lim_{t\to\infty} \mathrm{Cov}[c(t),F(t)] = -V_\infty(1-K)(1-\lambda)^2(1+\lambda Ku^2). \tag{66}$$

See Appendix C for a proof. Again, the moments (except those involving the smoothed actuarial asset value $F(t)$) exhibit symmetry and K and λ can be interchanged. When pure market values of assets are used ($\lambda = 0$) then $Q = 1 - qK^2$ and the second moments are identical to those obtained by Dufresne (1988) ($\lim \mathrm{Var}F(t) = \lim \mathrm{Var}f(t)$). When asset gains and losses are not amortized but are paid off immediately ($K = 0$) then $Q = 1 - q\lambda^2$ and Dufresne's (1988) results are again obtained but with λ exactly replacing K.

8 Effects of Smoothing Asset Values

8.1 Stability

An important property of an asset valuation method is that it should lead to stable and smooth funding for pension obligations. The funding process should at least exhibit finite variance. The stability conditions of Proposition 1 and 2 are *sufficient* for stability. (Necessary and sufficient conditions are discussed in Appendices A and C.) These conditions are realistic in normal economic circumstances and the condition that most constrains the choice of gain/loss spreading period m (or spreading parameter K) and of asset valuation parameter λ is $Q > 0$. (K and m are in a direct one-to-one relationship: see equation (29).) It is shown in the proof of Proposition 2 that it is necessary, but not sufficient, for stability that $K < 1/\sqrt{q}$ and $\lambda < 1/\sqrt{q}$ (see equations (95) and (96) in Appendix C).

Table 1 exhibits the stability constraints in terms of maximum allowable spread periods for various choices of $\{i,\ \sigma,\ \lambda\}$. Table 2 shows maximum allowable smoothing parameters for various choices of $\{i,\ \sigma,\ m\}$. Both tables are based on the stability conditions of Proposition 2. It is easily verified in Tables 1 and 2 that inequalities $K < 1/\sqrt{q}$ and $\lambda < 1/\sqrt{q}$ hold.

It is clear from Table 1 that gains and losses should not be spread over very long periods as this could result in an unstable funding process. This conclusion is also emphasized by Dufresne (1988) who considers only pure market values of assets. Spreading periods should be even shorter if asset values are being smoothed.

Table 2 shows that excessive smoothing of asset values must be avoided, specially if gains and losses are being spread over long periods. Asset valuation and gain/loss adjustment perform a complementary actuarial smoothing function and there is a finite limit to the cumulative amount of smoothing that may be applied.

8.2 Effect on the Smoothed Actuarial Asset Value

A suitable asset valuation method should generate an asset value that is realistic in the sense that it remains fairly close to market values. Furthermore, the asset value should be more stable or less variable than the market value (Berin, 1989, p. 29; Aitken, 1994, p. 289). Proposition 3 states that these properties do indeed hold for the variants of the "Average of Market", "Weighted Average", "Write-up", "Adjusted Market" and "Deferred Recognition" valuation methods described earlier.

PROPOSITION 3 *Provided that the stability conditions of Proposition 2 hold,*

$$\lim_{t\to\infty} \mathrm{E}[f(t) - F(t)]^2 \; < \; \infty, \tag{67}$$

$$\lim_{t\to\infty} \mathrm{Var}F(t) \; \leq \; \lim_{t\to\infty} \mathrm{Var}f(t). \tag{68}$$

Proof. $\mathrm{E}[f(t) - F(t)]^2 = \mathrm{Var}f(t) + \mathrm{Var}F(t) - 2\mathrm{Cov}[f(t), F(t)] + (\mathrm{E}[f(t) - F(t)])^2$ and all the terms on the right hand side are convergent as $t \to \infty$ as shown in Propositions 1 and 2 provided the stability conditions hold. From equations (61) and (62),

$$\lim_{t\to\infty} \mathrm{Var}f(t) - \lim_{t\to\infty} \mathrm{Var}F(t) \;\; = \;\; V_\infty(1 - u^2K^2)\lambda \left[(1-\lambda) + (1 - \lambda^2 u^2 K)\right] \;\; \geq \;\; 0, \tag{69}$$

given the stability conditions in Proposition 2. Equality holds when $\lambda = 0$. □

Inequality (67) shows that the deviation between the smoothed actuarial asset value and the market value of plan assets remains bounded in the mean-square, provided that the amount of smoothing in the asset valuation and gain/loss adjustment methods are constrained as discussed in section 8.1. Excessive averaging of market values (as well as spreading of gains/losses over very long periods) must therefore be avoided. Note in particular that if $\lambda = 1$, a variant of the "Write-up method without adjustment" is being used (Winklevoss, 1993, p. 173; Committee on Retirement Systems Research, 1998, p. 41). The actuarial asset value $F(t)$ in equation (19) does not revert towards the market value $f(t)$ and unless the fund is marked-to-market regularly the actuarial asset value will diverge from the market value of pension plan assets.

Inequality (68) shows that the asset value generated by the set of smoothed asset valuation methods used here is less variable than market value, conditional on stability.

8.3 Effect on the Fund Level

Dufresne (1988) and Owadally & Haberman (1999) consider only pure market values of assets ($\lambda = 0$) but show that spreading or amortizing gains and losses over longer periods lead to more variable fund levels. This is reasonable. As gains and losses are deferred for longer periods, fast enough action is not taken to defray them and the level of funding becomes more volatile. Likewise, one anticipates that heavier smoothing of asset values, which delays the recognition of asset gains and losses, should also adversely affect the security of pension benefits. This is encapsulated in Proposition 4, proven in Appendix D.

PROPOSITION 4 *Provided that the stability conditions of Proposition 2 hold,* $\lim \mathrm{Var} f(t)$ *increases monotonically with both* m *and* λ.

This result is illustrated in Figure 1. The symmetry between asset valuation and gain/loss spreading is clearly exhibited: Figure 1 is symmetrical in the plane $K = \lambda$.

8.4 Effect on the Contribution Rate

Slower recognition and amortization of gains and losses should result in smoother and more stable contribution rates. In the context of pure market values of assets ($\lambda = 0$), Dufresne (1988) shows that spreading gains and losses over longer periods does initially stabilize contributions, but beyond a certain critical period contributions become more variable: $\lim \mathrm{Var} c(t)$ against m has a minimum at m^* corresponding to $K^* = 1/q$. (Owadally & Haberman (1999) also assume that assets are valued at market and prove a similar result when gains and losses are directly amortized rather than indirectly spread.)

An immediate consequence of the congruence between gain/loss spreading and the exponential smoothing asset valuation methods used here is that, if gains and losses are immediately paid off and not spread forward ($m = 1$ or $K = 0$), then $\lim \mathrm{Var} c(t)$ against λ has a minimum at $\lambda^* = 1/q$. Therefore, smoothing beyond a certain amount (weighting the current market value of assets by less than $1 - \lambda^*$) is countereffective as contributions become more variable. (The proof is obtained, as a matter of course, by repeating Dufresne's (1988) proof and replacing all K by λ.)

The combined effect of asset valuation and gain/loss spreading on the stability of contribution rates is investigated in Proposition 5 (proof in Appendix E).

PROPOSITION 5 *Suppose $m > 1$ and $\lambda > 0$. Provided that the stability conditions of Proposition 2 hold,*

1. *as m increases,*

   $\lim \mathrm{Var}c(t)$ *has at least one minimum at some $m < m^*$, provided $0 < \lambda < \lambda^*$;*

   $\lim \mathrm{Var}c(t)$ *increases monotonically, provided either $\lambda \geq \lambda^*$ or $m \geq m^*$;*

2. *as λ increases,*

   $\lim \mathrm{Var}c(t)$ *has at least one minimum at some $\lambda < \lambda^*$, provided $1 < m < m^*$;*

   $\lim \mathrm{Var}c(t)$ *increases monotonically, provided either $m \geq m^*$ or $\lambda \geq \lambda^*$.*

The variation of $\lim \mathrm{Var}c(t)$ with K and λ is illustrated in Figures 2 and 3 and in the second contour plot in Figure 4. The two parts of Proposition 5 are identical except that K and λ are interchanged. The variation of $\lim \mathrm{Var}c(t)$ with K is similar to its variation with λ and Figure 3 is symmetrical about the plane $K = \lambda$. The boomerang-shaped contours of Figure 4 are a further indication of the complementary function of gain/loss adjustment and asset valuation: the same contribution or fund level variability may be achieved by trading off λ and K.

Proposition 5 does not state whether no more than one minimum occurs but numerical work, as illustrated by Figures 2 and 3, does indicate at most one minimum. This suggests that

- $\lim \mathrm{Var}c(t)$ against K exhibits a minimum, except for large enough λ when $\lim \mathrm{Var}c(t)$ increases monotonically;

- $\lim \text{Var}c(t)$ against λ exhibits a minimum, except for large enough K when $\lim \text{Var}c(t)$ increases monotonically.

Thus, in Figure 2, $\lim \text{Var}c(t)$ versus K exhibits a minimum when $\lambda < \lambda^* = 0.82$. The minimum for $\lambda = 0$ is seen to occur at $K = K^* = 0.82$. The minima for $0 < \lambda < \lambda^*$ clearly occur at some $K < 0.82$. But when $\lambda \geq \lambda^* = 0.82$, $\lim \text{Var}c(t)$ versus K has no minimum and increases monotonically. In other words, if asset values are being heavily smoothed ($\lambda \geq \lambda^*$), it is counterproductive to spread gains and losses in an effort to smooth contribution rates further. Likewise, if gains/losses are being spread over long periods ($K \geq K^*$ or $m \geq m^*$), averaging market values of plan assets in an effort to generate smoother contribution rates is counterproductive.

Attention must therefore be paid to the *combined* smoothing effect of gain/loss adjustment and asset valuation.

8.5 Efficient Asset Valuation and Gain/Loss Spreading

It is argued by Dufresne (1988) that maximizing the security of plan members' benefits (by minimizing $\lim \text{Var}f(t)$) and maximizing the stability of contributions required from the plan sponsor (by minimizing $\lim \text{Var}c(t)$) are rational actuarial objectives in pension funding in the long term. Given such objectives, it is possible to go further than in section 8.4 and state that:

PROPOSITION 6 *Under the objectives of minimizing* $\lim \text{Var}f(t)$ *and* $\lim \text{Var}c(t)$,

1. *it is not efficient to smooth asset values by weighting current market value by less than* $1 - \lambda^*$*;*
2. *it is not efficient to adjust gains/losses by spreading them over periods exceeding* m^*.

Proposition 4 states that increasing λ causes $\lim \mathrm{Var} f(t)$ to increase. Proposition 5 states that increasing λ initially causes $\lim \mathrm{Var} c(t)$ to decrease but eventually increasing λ beyond λ^* causes $\lim \mathrm{Var} c(t)$ to increase. Hence, it is inefficient to smooth asset values using $\lambda > \lambda^*$ as there is some other choice of λ for which both $\lim \mathrm{Var} f(t)$ and $\lim \mathrm{Var} c(t)$ may be reduced. By symmetry, the second part of Proposition 6 is also proven.

The second part of Proposition 6 encompasses the conclusions of Dufresne (1988) who investigates the choice of m when pure market values are used ($\lambda = 0$).

Numerical work indicates that $\lim \mathrm{Var} c(t)$ against λ or K has at most one minimum, as discussed in section 8.4. For any given gain/loss spreading period m, it is inefficient to smooth asset values by more than the [$\lim \mathrm{Var} c(t)$]-minimizing value of λ as a lower λ will reduce both $\lim \mathrm{Var} f(t)$ and $\lim \mathrm{Var} c(t)$. If m is long enough and $\lim \mathrm{Var} c(t)$ is strictly increasing with λ, then pure market values should be used. Table 4 lists the [$\lim \mathrm{Var} c(t)$]-minimizing values of λ for various choices of $\{i, \sigma, m\}$. It is efficient to smooth asset values using a value of λ between 0 and the [$\lim \mathrm{Var} c(t)$]-minimizing value in Table 4. The first column of Table 4 ($m = 1$ or $K = 0$) contains the upper bound $\lambda^* = 1/q$. The values in Table 4 are of course lower than the corresponding maximum allowable values of λ for stability in Table 2.

By symmetry, Table 3 shows the longest periods over which gains and losses can be efficiently spread for various choices of $\{i, \sigma, \lambda\}$.

Tables 3 and 4 suggest that pension benefits would be efficiently funded if gains and losses are spread over terms of 1–5 years with a weight of 20–100% placed on the current market value of assets (λ should be at most 80%). Spreading gains and losses over up to 10 years requires the current market value to be weighted by at least 60% for efficiency. (This assumes real rates of return averaging up to 5% with standard deviations of up to 15%).

9 Conclusion

The motivation for the use of special (market-related and discounted-income) methods to value the assets of defined benefit pension plans was discussed in the context of funding valuations. The desirable properties of asset valuation methods were considered and it was argued that the value placed on plan assets must be objectively determined, must be realistic as compared with market values, and must be consistent with liability valuation. Asset valuation must also smooth and stabilize the pension funding process.

The "Average of Market", "Weighted Average", "Deferred Recognition", "Adjusted Market" and "Write-up" methods were defined in terms of exponential smoothing of market values and were shown to be equivalent. A simple pension plan model was described where experience unfolds deterministically except for random investment returns. Asset gains and losses emerge and supplementary contributions are paid to spread them forward in a proportional manner. Several authors have reported that this method leads to greater smoothing. Symmetry between asset gain/loss adjustment and asset valuation was explicitly demonstrated and it was shown that the gains and losses are smoothed twice.

The first two moments of several variables (the intervaluation loss in the plan, the level of contribution required, and the market and smoothed actuarial values of pension plan assets) were obtained. If the assumed rate of return is borne out on average, then no loss is expected to emerge, confirming the consistency of the valuation methods. An important result is that asset valuation and gain/loss adjustment techniques have a complementary function in achieving smoothness in the pension funding process and their combined effect must be considered. Conditions for the funding process to be stable in the mean-square were obtained, restricting the total amount of smoothing through both techniques. The actuarial asset value does not diverge from, and is more stable than, the market value of plan assets if the conditions for stability hold. The total amount of smoothing is further constrained if funding is to be efficient and the long-term variability of both contribution and funding levels is to be minimized. Numerical work appears to indicate that a combination of a gain/loss spreading period of 1–5 years and a 20–100% weighting on the current market value of assets is efficient, as is a combination of 1–10 years and 60–100% respectively.

The analysis undertaken here may be extended in various directions. Pension liabilities are subject to inflation, which could be explicitly modeled, and should be valued using a dynamic valuation basis based on corporate bond yields. The various classes of pension plan assets should also be identified, with more realistic modeling of investment returns and with the possible use of different valuation methods for different classes. Practical regulatory constraints, as mandated by the Internal Revenue Service for example, may also be incorporated and their effects could be explored.

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Appendix A

Proof of Proposition 1

For notational brevity, we define $u = 1 + i$, $v = 1/(1 + i)$, $d = i/(1 + i)$ in terms of i and correspondingly $u_r = 1 + r$, $d_r = r/(1 + r)$ in terms of $r = \mathrm{E}r(t)$.

The rate of return on plan assets $\{r(t)\}$ is a sequence of independent and identically distributed random variables and it follows that $r(t + 1)$ is independent of $r(t)$ and $ul(t)$, $ul(t - 1)$ etc. in equation (40). Hence,

$$\begin{aligned}\mathrm{E}(ul(t+1) - U(t+1)) - (AL - U(t+1))(1 - u_r v)\\ = u_r \mathrm{E}(ul(t) - U(t)) - u_r(1 - K)(1 - \lambda)\sum_{j=0}^{t}(\lambda K u)^{t-j}\mathrm{E}(ul(j) - U(j)),\end{aligned} \quad (70)$$

for $t \geq 0$. We forward-shift equation (70) in time (so that it holds for $t \geq -1$) and upon deducting equation (70) multiplied by $\lambda K u$, we obtain

$$\begin{aligned}\mathrm{E}[ul(t+2) - U(t+2)]\\ - [\lambda K u + u_r - u_r(1 - K)(1 - \lambda)]\mathrm{E}[ul(t+1) - U(t+1)] + \lambda K u u_r \mathrm{E}[ul(t) - U(t)]\\ = AL(1 - \lambda K u)(1 - u_r v) - [U(t+2) - \lambda K u U(t+1)](1 - u_r v),\end{aligned} \quad (71)$$

which holds for $t \geq 0$ and requires $\mathrm{E}(ul(0) - U(0)) = 0$ and an additional initial condition $\mathrm{E}(ul(1) - U(1))$ that may be found from equation (70). These initial conditions have only a transient effect, and we will examine the situation in the limit only and so do not need them.

Were the initial unfunded liability not to be separately paid off but to be spread forward, it is easy to establish by the same method as above, but starting from equation (43), that equation (71) holds (except that $U(t) = 0\ \forall t$ of course).

The characteristic equation of the second order difference equation (71) is

$$P(z) = z^2 - [\lambda K u + u_r - u_r(1 - K)(1 - \lambda)]z + \lambda K u u_r = 0. \quad (72)$$

$U(t) = 0$ as $t \to \infty$, from equation (27). $\mathrm{E}ul(t)$ converges as $t \to \infty$ provided the roots of the characteristic equation (72) are less than unity in magnitude. This will occur if and only if (Haberman, 1992):

$$|\lambda K u u_r| < 1, \tag{73}$$

$$P(z=1) > 0 \quad \Leftrightarrow \quad 1 - u_r - \lambda K u + \lambda K u u_r + u_r(1-K)(1-\lambda) > 0, \tag{74}$$

$$P(z=-1) > 0 \quad \Leftrightarrow \quad 1 + u_r + \lambda K u + \lambda K u u_r - u_r(1-K)(1-\lambda) > 0. \tag{75}$$

A simplifying assumption, which is not restrictive in practice, is now made: both the assumed rate i and the long-term mean rate of return on plan assets r are greater than -100%. Then, $0 \leq Ku < 1$ (Haberman, 1992), since $K = 1 - 1/\ddot{a}_{\overline{m|}}$ where the annuity is calculated at rate i. By definition, $0 \leq \lambda < 1$ and so $0 \leq \lambda K u < 1$.

Condition (73) can be written as $\lambda K u u_r < 1$. As for condition (74), it may be written as $[1 - \lambda K u][1 - u_r(1-\theta)] > 0$, which, since $0 \leq \lambda K u < 1$ and $u_r > 0$, simplifies to $\theta > d_r$. Condition (75) follows directly from the additional assumptions that $u_r > 0$ and $u > 0$ (since $u_r - u_r(1-K)(1-\lambda) > 0$ and $1 + \lambda K u + \lambda K u u_r > 0$).

Given these stability conditions, $\lim \mathrm{E}ul(t)$ and $\lim \mathrm{E}f(t)$ follow immediately. Taking expectations across equation (36) and then taking limits, we find that $\lim \mathrm{E}UL(t) = \lim \mathrm{E}ul(t)(1-\lambda)/(1-\lambda K u)$ provided that $|\lambda K u| < 1$ which holds as shown above. Also, $\lim \mathrm{E}F(t) = AL - \lim \mathrm{E}UL(t)$. From equation (38), it is clear that $\lim \mathrm{E}c(t) = NC + (1 - K) \lim \mathrm{E}UL(t)$. Finally, from equation (9), $\lim \mathrm{E}l(t) = (i - r)[\lim \mathrm{E}f(t) + \lim \mathrm{E}c(t) - B]$ and equation (1) may also be used to simplify the result. $\square$

Appendix B

Proof of Corollary 1

The rate of return $r(t)$ is independent from year to year and hence is independent from $f(t-1)$ and $c(t-1)$ and since $\mathrm{E}r(t) = i$, it follows from equation (9) that $\mathrm{E}l(t) = 0\ \forall t$.

When $r = i$, equation (70) simplifies to

$$\begin{aligned}&\mathrm{E}(ul(t+1) - U(t+1)) \\ &\qquad = u\mathrm{E}(ul(t) - U(t)) - u(1-K)(1-\lambda)\sum_{j=0}^{t}(\lambda K u)^{t-j}\mathrm{E}(ul(j) - U(j)), \quad (76)\end{aligned}$$

for $t \geq 0$. If the initial unfunded liability is separately amortized, then $\mathrm{E}(ul(0) - U(0)) = 0$ is an initial condition. Hence, $\mathrm{E}ul(t) = U(t)$ for $t \geq 0$ with $U(t)$ defined in equation (27). The other moments follow as in Appendix A.

But if the initial unfunded liability is not separately amortized over a finite period but spread forward, we need to establish the conditions for stability. The results of Corollary 1 follow immediately from Proposition 1 when $r = i$ or $d_r = d$ (the subscript r may now be dropped). The stability conditions (73), (74) and (75) simplify respectively to $|\lambda K u^2| < 1$, $(1 - \lambda u)(1 - Ku) > 0$ and $(1 + \lambda u)(1 + Ku) > 0$. Again, under the realistic economic assumption that $i = r > -100\%$, then $0 \leq K < v$ and, since $0 \leq \lambda < 1$, $0 \leq \lambda K u < 1$. Clearly, the condition that $(1 + \lambda u)(1 + Ku) > 0$ does hold. If it is further required that $\lambda < v$, then the conditions that $(1 - \lambda u)(1 - Ku) > 0$ and $|\lambda K u^2| < 1$ will also hold. □

Appendix C

Proof of Proposition 2

For brevity, define $p = \lambda K u$, $q = \mathrm{E}(1+r(t))^2 = u^2 + \sigma^2$. Let the unfunded liability in excess of the unamortized part of the initial unfunded liability be $ul^U(t) = ul(t) - U(t)$ and define $b_t = \mathrm{E}ul^U(t)^2$ and $c_t = \mathrm{E}[ul^U(t)\sum_{j=0}^t p^{t-j} ul^U(j)]$.

Suppose that the initial unfunded liability is amortized separately. Since $r = i$, we have from Corollary 1 that $\mathrm{E}ul^U(t) = 0$ for $t \geq 0$ (equation (51)). Note also the following two identities:

$$\mathrm{E}\left[\sum_{j=0}^{t} p^{t-j} ul^U(j)\right]^2 = 2\sum_{j=0}^{t} p^{2(t-j)} c_j - \sum_{j=0}^{t} p^{2(t-j)} b_j, \tag{77}$$

$$\mathrm{E}\left[ul^U(t+1)p\sum_{j=0}^{t} p^{t-j} ul^U(j)\right] + \mathrm{E}ul^U(t+1)^2 = c_{t+1}. \tag{78}$$

In order to find the second moments, we proceed to square both sides of equation (40) and then take expectations. Note again that $\{r(t)\}$ is a sequence of independent and identically distributed random variables so that $r(t+1)$ is independent of $r(s)$ and $ul(s)$ for $s \leq t$. Upon using equation (77) and collecting like terms, we have, for $t \geq 0$,

$$\begin{aligned} b_{t+1} - qb_t + q(1-K)^2(1-\lambda)^2 \sum_{j=0}^{t} p^{2(t-j)} b_j + 2q(1-K)(1-\lambda)c_t \\ - 2q(1-K)^2(1-\lambda)^2 \sum_{j=0}^{t} p^{2(t-j)} c_j = \sigma^2 v^2 (AL - U(t+1))^2. \end{aligned} \tag{79}$$

A second equation may be found by multiplying equation (40) by $p\sum_{j=0}^t p^{t-j} ul^U(j)$ and adding $ul^U(t+1)^2$ on both sides, and then taking expectations. Upon simplifying by using equations (77) and (78) and collecting like terms, we find that, for $t \geq 0$,

$$c_{t+1} - upc_t + 2up(1-\lambda)(1-K)\sum_{j=0}^{t} p^{2(t-j)} c_j = b_{t+1} + up(1-K)(1-\lambda)\sum_{j=0}^{t} p^{2(t-j)} b_j. \tag{80}$$

Equations (79) and (80) constitute a *linear* dynamic system and, in the limit as $t \to \infty$, the system is not affected by initial conditions, which may therefore be ignored. From equation (27), $U(t) = 0$ for $t \geq M$. We could consider difference equations (79) and (80) for $t \geq M$. Equivalently and more simply, it is convenient to put $U(t) = 0$ or $ul_0 = 0$. Similarly, we can assume that $b_s = c_s = 0$ for $s < 0$. Even if the initial unfunded liability is not separately amortized, $\mathrm{E}ul(t) \to 0$ as long as the stability conditions of Corollary 1 hold, and the limits of b_t and c_t do not depend on the transient values of $\mathrm{E}ul(t)$ (which satisfes a third difference equation obtained by taking expectations across equation (43)) and it can be assumed that $ul_0 = 0$ for simplicity.

It is also convenient to use the forward shift operator E (as distinct from the statistical expectation operator E) where $E^m x_t = x_{t+m}$. See Brand (1966, p. 375). One could alternatively use z-transforms or generating functions. Note that, if $x_s = 0$ for $s < 0$,

$$\sum_{j=0}^{t} a^{t-j} x_j = \sum_{j=-\infty}^{t} a^{t-j} x_j = \sum_{j=-\infty}^{t} a^{t-j} (E^{-1})^{t-j} x_t = E(E-a)^{-1} x_t. \tag{81}$$

In terms of the forward-shift operator, equations (79) and (80) may be written as $\psi(E)b_t + \gamma(E)c_t = \sigma^2 v^2 AL^2(E - p^2)$ and $\varphi(E)c_t = \mu(E)b_t$ respectively for $t \geq 0$, where

$$\psi(E) = (E-q)(E-p^2) + Eq(1-K)^2(1-\lambda)^2, \tag{82}$$

$$\gamma(E) = 2q(1-K)(1-\lambda)(E-p^2) - 2q(1-K)^2(1-\lambda)^2 E, \tag{83}$$

$$\varphi(E) = (E-up)(E-p^2) + 2up(1-K)(1-\lambda)E, \tag{84}$$

$$\mu(E) = E(E-p^2) + up(1-K)(1-\lambda)E. \tag{85}$$

We contrive to obtain equations in terms of b_t and c_t alone. Ignoring initial conditions,

b_t and c_t satisfy $Q(E)b_t = \sigma^2 v^2 AL^2 \varphi(E)$ and $Q(E)c_t = \sigma^2 v^2 AL^2 \mu(E)$ respectively where

$$\begin{aligned}
Q(E) &= [\psi(E)\varphi(E) + \mu(E)\gamma(E)]/(E - p^2) \\
&= (E-q)\varphi(E) + E\gamma(E) + Eq(1-K)^2(1-\lambda)^2(E+up) \\
&= (E-q)(E-up)(E-p^2) + 2q(1-K)(1-\lambda)E[E - p^2 - (1-K)(1-\lambda)E] \\
&\quad + 2u(1-K)(1-\lambda)pE(E-q) + Eq(1-K)^2(1-\lambda)^2(E+up). \qquad (86)
\end{aligned}$$

Assume for now that the limits of b_t and c_t as $t \to \infty$ exist. Then $\lim_{t\to\infty} b_t = \sigma^2 v^2 AL^2 \varphi(1)/Q(1)$ and $\lim_{t\to\infty} c_t = \sigma^2 v^2 AL^2 \mu(1)/Q(1)$. It may be observed that p, $\varphi(E)$, $\mu(E)$, $\psi(E)$, $\gamma(E)$ and $Q(E)$ are symmetrical in K and λ.

$$\varphi(1) = (1 - \lambda K u^2)(1 - \lambda^2 K^2 u^2) + 2\lambda K(1-\lambda)(1-K)u^2, \qquad (87)$$

$$\mu(1) = 1 + \lambda K(1 - K - \lambda)u^2, \qquad (88)$$

$$\gamma(1) = 2q(1-K)(1-\lambda)[1 - \lambda^2 K^2 u^2 - (1-K)(1-\lambda)]. \qquad (89)$$

Define $Q = Q(1)$ and simplify Q by collecting terms in λ.

$$\begin{aligned}
Q = Q(1) &= (1-q)\varphi(1) + \gamma(1) + q(1-K)^2(1-\lambda)^2(1+up), \qquad (90) \\
&= (1 - qK^2) - \lambda K[u^2(1-qK^2) + 2(1-K)\sigma^2] \\
&\quad - \lambda^2[u^2(1-qK^2) + \sigma^2(1-K)^2] + \lambda^3 K u^2[u^2(1-qK^2) + \sigma^2(1-K^2)]. \qquad (91)
\end{aligned}$$

Q may also be simplified into the form given in equation (57). Because K and λ can be interchanged in $Q = Q(1)$, Q may also be written as in equation (58).

Therefore, $\lim b_t = V_\infty \varphi(1)$ and $\lim c_t = V_\infty \mu(1)$ using $V_\infty = \sigma^2 v^2 AL^2/Q$ and assuming that the limits exist. Assuming also that $|p| < 1$, it follows from equation (77) that

$$\lim_{t\to\infty} \mathrm{E}\left[\sum_{j=0}^{t} p^{t-j} u l^U(j)\right]^2 = (2 \lim c_t - \lim b_t)/(1 - p^2) = V_\infty(1 + \lambda K u^2). \qquad (92)$$

The various variances and covariances in Proposition 2 are readily obtained from the above. Recall that $\mathrm{E} u l^U(t) = 0$ for $t \geq 0$ under Corollary 1. Thus, given equation (3),

$\lim \mathrm{Var} f(t) = \lim b_t$. Also, given equations (4) and (37), it follows that

$$\begin{aligned}\mathrm{Cov}[f(t), F(t)] &= \mathrm{Cov}[ul^U(t), UL^U(t)] \\ &= (1-\lambda)\mathrm{Cov}\left[ul^U(t), \sum_{j=0}^{t} p^{t-j} ul^U(j)\right] = (1-\lambda)\mathrm{E}\left[ul^U(t)\sum_{j=0}^{t} p^{t-j} ul^U(j)\right] \quad (93)\end{aligned}$$

and $\lim \mathrm{Cov}[f(t), F(t)] = (1-\lambda)\lim c_t$. From equations (4) and (38), $\lim \mathrm{Cov}[f(t), c(t)] = -(1-K)\lim \mathrm{Cov}[f(t), F(t)]$. It is straightforward from equations (4), (37) and (92) that

$$\mathrm{Var}F(t) = \mathrm{Var}UL(t) = (1-\lambda)^2 \mathrm{Var}\left[\sum_{j=0}^{t} p^{t-j} ul^U(t)\right] = (1-\lambda)^2 \mathrm{E}\left[\sum_{j=0}^{t} p^{t-j} ul^U(j)\right]^2 \quad (94)$$

and $\lim \mathrm{Var}F(t) = V_\infty (1-\lambda)^2(1+\lambda K u^2)$. We use equations (4) and (38) to obtain $\lim \mathrm{Var}c(t) = (1-K)^2 \lim \mathrm{Var}F(t)$ and $\lim \mathrm{Cov}[c(t), F(t)] = -(1-K)\lim \mathrm{Var}F(t)$. Finally, from equation (9), we exploit the independence of $r(t)$ from $f(s)$ and $c(s)$ $(s < t)$ and the fact that $\mathrm{E}l(t) = 0$ (Corollary 1) and use equation (1) to simplify, to find that $\lim \mathrm{Var}l(t) = \sigma^2[\lim \mathrm{Var} f(t) + \lim \mathrm{Var}c(t) + 2\lim \mathrm{Cov}[f(t), c(t)] + v^2 A L^2] = V_\infty[\sigma^2\varphi(1) + \sigma^2(1-K)^2(1-\lambda)^2(1+\lambda K u^2) - 2\sigma^2(1-K)(1-\lambda)\mu(1) + Q]$, which may be further simplified as in equation (60) upon using equations (87)–(90).

It has been assumed thus far that these limits exist. As shown in Appendix A, the definitions of λ and K and the realistic assumption that $i > -100\%$ imply that $0 \leq \lambda K u < 1$ which satisfies the requirement that $|p| < 1$. The limits as $t \to \infty$ of b_t and c_t exist if the magnitude of the roots of $Q(z) = 0$ is less than one. $Q(z)$, in equation (86), can be expanded as a cubic, $Q(z) = z^3 - Az^2 + Bz - C$, where $A = p^2 + q(1-(1-K)(1-\lambda))^2 + up(1-2(1-K)(1-\lambda))$, $B = [up^2 + uq(1-(1-K)(1-\lambda))^2 + pq(1-2(1-K)(1-\lambda))]p$ and $C = up^3 q$. Necessary and sufficient conditions for the roots of $Q(z) = 0$ to be less than unity in magnitude are given by Jury (1964, p. 136) and Marden (1966): condition (a) is $|C| < 1$; condition (b) is $|A + C| < 1 + B$; condition (c) is $|AC - B| < |1 - C^2|$.

We now show that conditions (a), (b) and (c) hold provided that the conditions in Proposition 2 are true. First note that, given that $i > -100\%$, $0 \leq K < v$ and $0 \leq \lambda < v$, then

$$Q > 0 \quad \Rightarrow \quad qK^2 < 1, \tag{95}$$

$$Q > 0 \quad \Rightarrow \quad q\lambda^2 < 1, \tag{96}$$

using equations (57) and (58).

Consider first condition (a). If $i > -100\%$, $0 \le K < v$, $0 \le \lambda < v$ and $Q > 0$ (and using implication (95)), then $|C| = |up^3q| = (\lambda u)^2 qK^2 |\lambda u||uK| < 1$, i.e. condition (a) follows.

Next, consider condition (b). The requirement that $A+C < 1+B$ is of course equivalent to requiring that $Q > 0$. We can show that $A+C > -(1+B)$. Assuming that $i > -100\%$, $0 \le K < v$, $0 \le \lambda < v$, then $C \ge 0$ and

$$\begin{aligned} A &= p^2 + qK^2 + q(1-K)^2\lambda^2 + 2qK(1-K)\lambda + up - 2\lambda(1-\lambda)u^2K(1-K) \\ &= p^2 + qK^2 + q(1-K)^2\lambda^2 + up + 2(1-K)\lambda K(\sigma^2 + \lambda u^2) \qquad \ge 0, \end{aligned}$$

and further

$$\begin{aligned} B &= p[up^2 + uq(1-(1-K)(1-\lambda))^2 + u^2p(1-2(1-K)(1-\lambda)) \\ &\qquad + \sigma^2 p(1-2(1-K)(1-\lambda))] \\ &= upA + \sigma^2p^2(1-2(1-K)(1-\lambda)) \\ &= up[p^2 + qK^2 + q(1-K)^2\lambda^2 + up + 2(1-K)\lambda^2Ku^2] + \sigma^2p^2(1+2\lambda(1-K)) \quad \ge 0. \end{aligned}$$

Hence, $A + C \ge 0 > -1 \ge -(1+B)$.

Finally consider condition (c). If $i > -100\%$, $0 \le K < v$, $0 \le \lambda < v$ and $Q > 0$ (and using implication (95)), then $0 \le C < 1$ and since $A+C < 1+B$, therefore $AC-B < 1-C^2$. The requirement that $AC - B > -(1-C^2)$ is an inequality of the sixth degree in λ and, after collecting terms in λK and $\lambda + K$, may be written as inequality (59).

We have therefore shown that if $i > -100\%$, $0 \le K < v$, $0 \le \lambda < v$, $Q > 0$ and inequality (59) holds, then the limits of the variances and covariances in Proposition 2 do exist. □

Appendix D

Proof of Proposition 4

From Appendix C, $\lim \mathrm{Var} f(t) = \lim b_t = \sigma^2 v^2 A L^2 \varphi / Q$, where $\varphi = \varphi(1)$ in equation (87) and may be written as a polynomial in K as follows: $\varphi = 1 + K\lambda u^2(1-2\lambda) - K^2\lambda u^2(2-\lambda) + K^3\lambda^3 u^4$. In equation (90), Q may be simplified, after replacing $\gamma(1)$ from equation (89), to $Q = (1-q)\varphi + q(1-K)(1-\lambda)\rho$, where ρ may be written as a polynomial in K as follows: $\rho = (1+\lambda) + K(1-\lambda)(1+\lambda u^2) - K^2\lambda(1+\lambda)u^2$.

$\partial[\lim \mathrm{Var} f(t)]/\partial K$ is proportional to $Q\partial\varphi/\partial K - \varphi\partial Q/\partial K$. This may be simplified and is the product of $q(1-\lambda)$ (a positive term under the conditions for stability in Proposition 2) and a second term, $[(1-K)\rho\partial\varphi/\partial K + \varphi\rho - (1-K)\varphi\partial\rho/\partial K]$. The sign of the latter must be examined. This second term may be expanded using φ and ρ as given above. Terms in K may be collected and $2(1-\lambda^2 u^2)$ (also a positive term) may be factored out, and upon collecting terms in λ, we are left with $\lambda(1-K)[1-\lambda u^4 K^3] + K[1-\lambda K u^2][1-\lambda^2 u^2 K]$.

The sign of $\partial[\lim \mathrm{Var} f(t)]/\partial K$ is also the sign of the last expression, which turns out to be non-negative, since each of the bracketed terms in that expression are positive under the stability conditions. $0 \leq \lambda < v$ and $0 \leq K < v$ imply that $\lambda u^4 K^3 = \lambda u (Ku)^3 < 1$, $\lambda K u^2 = (\lambda u)(Ku) < 1$, $\lambda^2 u^2 K = (\lambda u)^2 K < 1$.

Hence, we have shown that $\partial[\lim \mathrm{Var} f(t)]/\partial K > 0$ for $K > 0$ and $\lambda > 0$. By virtue of the direct one-to-one relationship between m and K ($\partial m/\partial K > 0$) and of the symmetry between K and λ we have shown that, as m increases or as λ increases, $\lim \mathrm{Var} f(t)$ increases monotonically. □

Appendix E

Proof of Proposition 5

From equation (63), $\lim \mathrm{Varc}(t) = \sigma^2 v^2 A L^2 (1-K)^2 (1-\lambda)^2 (1+\lambda K u^2)/Q$, where Q may be written in the form given in Appendix D. $\partial[\lim \mathrm{Varc}(t)]/\partial K$ is proportional to $Q\partial[(1-K)^2(1+\lambda K u^2)]/\partial K - [(1-K)^2(1+\lambda K u^2)]\partial Q/\partial K$. This may be expanded into $[(1-q)\varphi + q(1-K)(1-\lambda)\rho](1-K)[\lambda u^2 - 2 - 3\lambda K u^2] - [(1-K)^2(1+\lambda K u^2)][(1-q)\partial\varphi/\partial K - q(1-\lambda)\rho + q(1-\lambda)(1-K)\partial\rho/\partial K]$, from which $1-K$ (a positive term) may be factored out and the resultant expression may be further expanded using φ and ρ, both given in Appendix D as polynomials in K. Terms in K may be collected and after factoring out $2(1-\lambda^2 u^2)$ (also a positive term) an expression, which we denote by π, is left. π may be written as a quadratic in λ or a cubic in K:

$$\pi(\lambda) = -(1-Kq) + \lambda(1-K)q(1-u^2K^2) + \lambda^2 K^2 u^2 [u^2(1-Kq) + (1-K)\sigma^2] \quad (97)$$

$$= -(1-Kq)[1-\lambda^2 K^2 u^4] + \lambda(1-K)q(1-u^2K^2) + \lambda^2 K^2 (1-K) u^2 \sigma^2, \quad (98)$$

$$\pi(K) = -(1-\lambda q) + Kq(1-\lambda) - K^2\lambda(1-\lambda)qu^2 + K^3\lambda u^2(q - \lambda q u^2 - \lambda\sigma^2). \quad (99)$$

The sign of $\partial[\lim \mathrm{Varc}(t)]/\partial K$ depends on the sign of π which must be investigated.

Proof that $m \geq m^* \Rightarrow \lim \mathrm{Varc}(t)$ increases monotonically with m: The last two terms in the right hand side of equation (98) are positive under the stability conditions $0 \leq K < v$ and $0 \leq \lambda < v$. In the first term on the right hand side of equation (98), $1 - \lambda^2 K^2 u^4 = 1 - (\lambda u)^2 (Ku)^2 > 0$. The sign of the first term depends on $1 - Kq$. If $\lambda > 0$ and $m \geq m^*$ (i.e. $1 - Kq \leq 0$), then $\pi(\lambda) > 0$. It follows therefore that $\lim \mathrm{Varc}(t)$ increases with increasing $m \geq m^*$.

Proof that $\lambda \geq \lambda^* \Rightarrow \lim \mathrm{Varc}(t)$ increases monotonically with m: We wish to prove that when $m > 1$ it is sufficient for $\lambda \geq \lambda^*$ for $\pi(\lambda) > 0$. Note that this has already been proven when $m \geq m^*$ in the previous paragraph. We need only consider $m < m^*$ therefore.

From equation (97), $\pi(\lambda)$ is quadratic in λ. When $1 - Kq > 0$ (i.e. when $m < m^*$), the coefficient of λ^2 is positive, so that $\pi(\lambda)$ has a minimum. Further, $\pi(\lambda = 0) < 0$ and $\pi'(\lambda = 0) > 0$. If we show that $\pi(\lambda = \lambda^* > 0) > 0$, then it follows that $\pi(\lambda \geq \lambda^*) > 0$.

$$
\begin{aligned}
&q^2\pi(\lambda = \lambda^* = 1/q) \\
&\quad = u^2K^2[u^2(1 - Kq) + (1 - K)\sigma^2] + q^2(1 - K)(1 - u^2K^2) - q^2(1 - Kq) \\
&\quad = K\left[u^4K - qu^4K^2 + K(1 - K)\sigma^2u^2 + q^2(q - 1) - q^2u^2K(1 - K)\right] \\
&\quad = K(q - 1)\left[-u^2K(u^2 + (1 - K)\sigma^2) + q^2\right] \\
&\quad > K(q - 1)\left[-u^2K(u^2 + (1 - K)\sigma^2) + u^2\right] \quad \text{(since } q^2 > u^2\text{)} \\
&\quad = K(q - 1)u^2\left[1 - K(u^2 + (1 - K)\sigma^2)\right] \\
&\quad \geq K(q - 1)u^2[1 - Kq] \quad \text{(since } u^2 + (1 - K)\sigma^2 \leq q\text{)} \\
&\quad > 0 \quad \text{(when } m < m^*\text{)}.
\end{aligned}
$$

We have shown that $\pi(\lambda \geq \lambda^*) > 0$ or that $\lim \mathrm{Varc}(t)$ increases monotonically with increasing m provided $\lambda \geq \lambda^*$.

Proof that, as m increases, $\lim \mathrm{Varc}(t)$ has at least one minimum, at some $m < m^*$, provided $0 < \lambda < \lambda^*$: Denote the $\lim \mathrm{Varc}(t)$-minimizing value of m (if it exists) as m_1 (with a corresponding K_1). Consider $\pi(K)$ in equation (99). If $\lambda < \lambda^* = 1/q$, then it is clear that $\pi(K = 0) < 0$. In addition, $\pi'(K = 0) > 0$ and $\pi''(K = 0) < 0$. We have shown already that $\pi(K = K^* = 1/q) \geq 0$ from equation (98). $\pi(K)$ in equation (99) is a cubic polynomial in K. Hence, if $\lambda < \lambda^*$, $\pi(K)$ must have at least one root at some $K = K_1$, such that $0 < K_1 < 1/q$ and such that $\pi'(K = K_1) > 0$, i.e. $\pi(K)$ must have at least one minimum at $0 < K_1 < 1/q$. We have shown that, if $0 < \lambda < \lambda^*$, as m increases, $\lim \mathrm{Varc}(t)$ has at least one minimum at some $1 < m_1 < m^*$.

The first part of Proposition 5 is therefore proven. The second part follows by virtue of the symmetry in K and λ. $\square$

σ	i	$\lambda = 0$	$\lambda = 20\%$	$\lambda = 40\%$	$\lambda = 60\%$	$\lambda = 80\%$	$\lambda = 90\%$
0.1	1%	112	111	110	109	104	94
	3%	67	67	66	64	59	47
	5%	51	50	49	47	42	29
	10%	33	33	32	30	23	5
	15%	25	25	25	22	14	
0.15	1%	65	65	64	62	57	48
	3%	45	45	44	42	37	27
	5%	36	36	35	33	27	16
	10%	25	25	24	22	16	1
	15%	21	20	19	17	9	
0.2	1%	42	41	41	39	34	26
	3%	32	32	31	28	24	20
	5%	27	27	24	22	19	9
	10%	20	20	17	15	11	
	15%	17	16	15	13	7	
0.25	1%	29	29	28	26	22	14
	3%	24	24	23	21	16	9
	5%	21	20	20	18	13	5
	10%	16	16	15	13	8	
	15%	14	13	13	11	5	

Table 1: Maximum allowable m for various choices of $\{i,\ \sigma,\ \lambda\}$. Blanks indicate that stability conditions do not hold.

σ	i	$m=1$	3	5	10	15	20	25	30	40	50
0.1	1%	98.5	98.5	98.4	98.4	98.3	98.2	98.0	97.8	97.5	97.0
	3%	96.6	96.6	96.5	96.2	95.9	95.5	94.9	94.2	92.4	88.9
	5%	94.8	94.7	94.6	94.1	93.4	92.5	91.2	89.6	82.9	34.7
	10%	90.5	90.4	90.1	88.8	86.7	83.6	78.0	63.2		
	15%	86.6	86.4	85.9	83.6	79.5	71.4	40.8			
0.15	1%	97.9	97.8	97.7	97.4	97.0	96.6	96.0	95.2	93.1	88.9
	3%	96.1	95.9	95.7	95.0	94.0	92.7	90.9	88.3	73.8	
	5%	94.3	94.0	93.7	92.5	90.7	88.1	84.0	75.5		
	10%	90.1	89.6	89.0	86.2	81.5	71.9	31.7			
	15%	86.2	85.6	84.6	79.8	70.3	31.5				
0.2	1%	97.1	96.9	96.6	95.8	94.5	92.9	90.6	86.8	54.9	
	3%	95.3	94.9	94.5	92.8	90.4	86.6	79.4	58.0		
	5%	93.6	93.0	92.4	89.8	85.7	78.2	55.0			
	10%	89.4	88.6	87.4	82.1	71.5	25.2				
	15%	85.7	84.6	82.7	74.2	49.7					
0.25	1%	96.1	95.6	95.0	92.9	89.6	83.9	70.1			
	3%	94.3	93.6	92.7	89.3	83.3	69.4				
	5%	92.6	91.7	90.5	85.5	75.9	40.5				
	10%	88.6	87.3	85.2	75.9	47.5					
	15%	85.0	83.2	80.2	65.7						

Table 2: Maximum allowable λ (%) for various choices of $\{i, \sigma, m\}$. Blanks indicate that stability conditions do not hold.

σ	i	$\lambda = 0$	$\lambda = 20\%$	$\lambda = 40\%$	$\lambda = 60\%$	$\lambda = 80\%$	$\lambda = 90\%$
0.1	1%	42	41	41	39	36	28
	3%	20	19	19	17	14	3
	5%	13	13	12	11	6	1
	10%	7	7	6	5	1	†
	15%	5	5	4	2	†	
0.15	1%	28	27	27	26	22	14
	3%	16	16	15	14	10	1
	5%	11	11	10	9	4	†
	10%	7	6	6	4	2	†
	15%	5	5	4	2	†	
0.2	1%	19	19	18	17	13	5
	3%	13	13	12	11	6	1
	5%	9	9	9	7	3	†
	10%	6	6	5	4	1	
	15%	5	4	4	2	†	
0.25	1%	14	14	13	12	8	1
	3%	10	10	9	8	4	†
	5%	8	8	7	6	2	†
	10%	6	5	5	3	†	
	15%	5	4	3	2	†	

Table 3: [$\lim \mathrm{Varc}(t)$]-minimizing values of m for various choices of $\{i,\ \sigma,\ \lambda\}$. † indicates that $\lim \mathrm{Varc}(t)$ increases monotonically with m with smallest value at $m = 1$. Blanks indicate instability.

σ	i	$m=1$	3	5	10	15	20	25	30	40	50
0.1	1%	97.1	96.9	96.7	96.0	94.8	91.6	73.1	34.5	3.2	†
	3%	93.4	92.6	91.4	80.6	23.9	†	†	†	†	†
	5%	89.9	87.9	83.8	22.8	†	†	†	†	†	†
	10%	82.0	73.0	35.0	†	†	†	†	†		
	15%	75.0	50.4	4.0	†	†	†	†			
0.15	1%	95.9	95.5	94.7	92.6	83.7	37.9	10.1	†	†	†
	3%	92.3	91.0	88.8	55.4	5.7	†	†	†	†	
	5%	88.9	86.0	79.3	11.1	†	†	†	†		
	10%	81.1	70.2	28.0	†	†	†	†			
	15%	74.3	46.8	1.1	†	†	†				
0.2	1%	94.3	93.4	92.0	81.5	27.1	†	†	†	†	
	3%	90.8	88.6	84.2	25.0	†	†	†	†		
	5%	87.5	83.2	70.7	†	†	†	†			
	10%	80.0	66.0	20.2	†	†	†				
	15%	73.4	42.0	†	†	†					
0.25	1%	92.4	90.3	86.8	39.2	†	†	†			
	3%	89.0	85.1	75.4	3.6	†	†				
	5%	85.8	79.2	55.6	†	†	†				
	10%	78.6	59.9	12.0	†	†					
	15%	72.2	36.2	†	†						

Table 4: [lim Varc(t)]-minimizing values of λ (%) for various choices of $\{i, \sigma, m\}$. † indicates that lim Varc(t) increases monotonically with λ with smallest value at $\lambda = 0$. Blanks indicate instability.

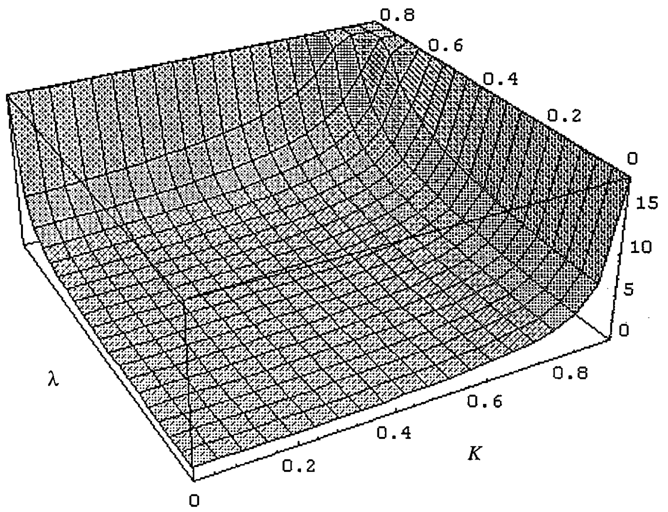

Figure 1: limVar$f(t)$ (scaled) against K and λ. $i = 10\%$, $\sigma = 5\%$.

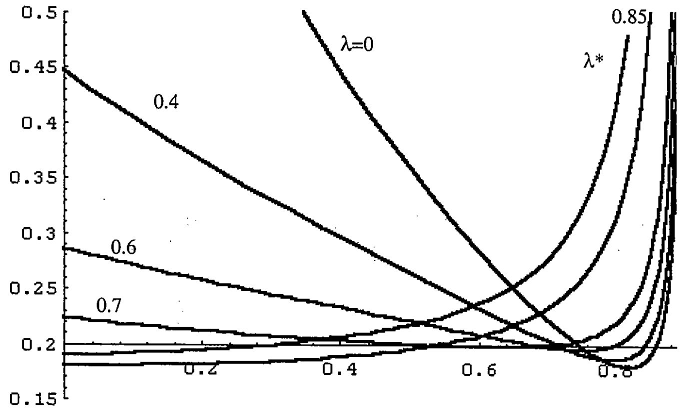

Figure 2: limVarc(t) (scaled) against K for various λ. K and λ can be interposed. $i = 10\%$, $\sigma = 10\%$, $\lambda^* = K^* = 0.82$.

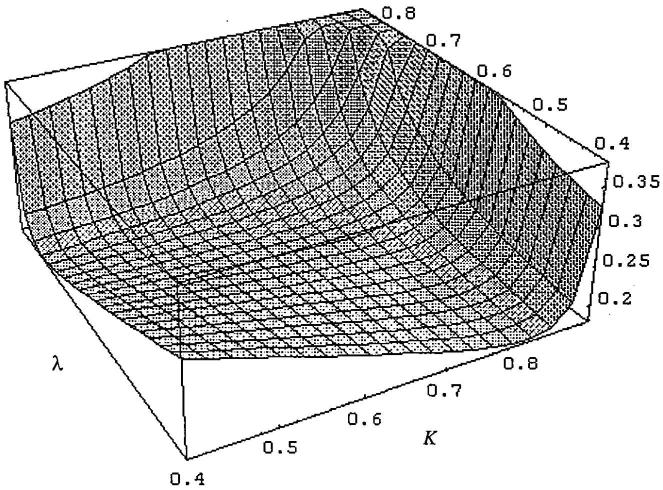

Figure 3: limVarc(t) (scaled) against K and λ. $i = 10\%$, $\sigma = 5\%$.

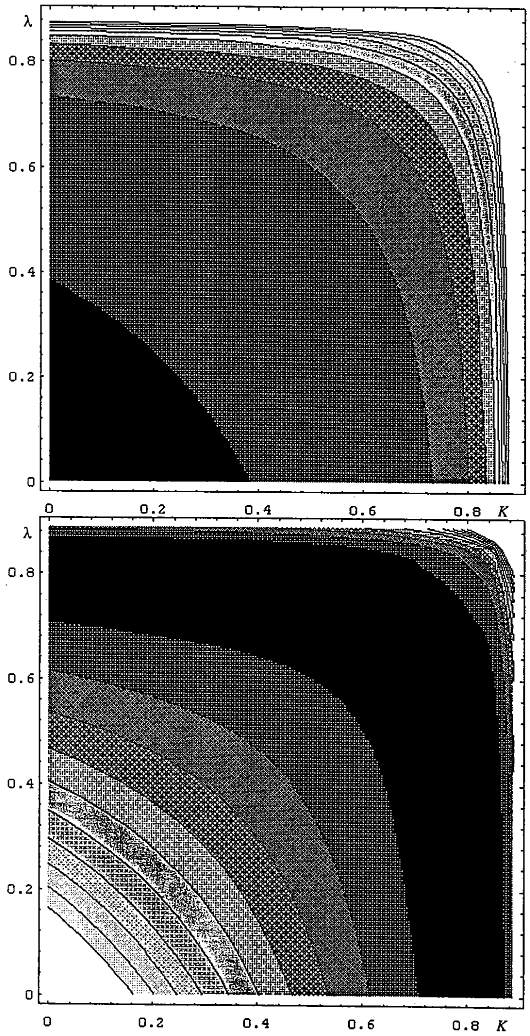

Figure 4: Contour plots of $\lim \mathrm{Var} f(t)$ (above) and $\lim \mathrm{Var} c(t)$ (below) against K and λ. $i = 10\%$, $\sigma = 5\%$. Lighter shading represents higher values.

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