City University London

School of Mathematics

Efficient Amortization of Actuarial Gains/Losses and Optimal Funding in Pension Plans

by

M. Iqbal Owadally and Steven Haberman

Actuarial Research Paper No. 133

December 2000

ISBN 1 901615 54 5

Efficient Amortization of Actuarial Gains/Losses and Optimal Funding in Pension Plans

M. Iqbal Owadally* and Steven Haberman

Abstract

Efficient methods of amortizing actuarial gains and losses in final-salary defined benefit pension plans are considered. In the context of a simple model where asset gains and losses emerge as a consequence of random (independent and identically distributed) rates of investment return, it has been shown that direct amortization of such gains and losses over a fixed term leads to more variable funding levels and contribution rates compared with an indirect and proportional form of amortization which 'spreads' the gains and losses. Proportional spreading may be rationalized as the contribution control that optimizes mean square deviations in the contributions and market values of plan assets when the funding process is Markovian and the fund is invested in two assets (a random risky and a riskfree asset). Simulations indicate that, when rates of return follow simple autoregressive AR(1) and moving average MA(1) processes, spreading rather than amortizing gains and losses remains more efficient at achieving secure funding levels and stable contribution rates. Similar results are obtained when a more comprehensive actuarial stochastic investment model (which includes economic wage inflation) is simulated. Efficient ranges of spreading and amortization periods are also determined.

*Correspondence: Department of Actuarial Science and Statistics, City University, Northampton Square, London EC1V 0HB, England. Phone: +44 (0)20 7477 8478. Fax: +44 (0)20 7477 8838. E-mail: iqbal@city.ac.uk. The authors would like to thank Dr. Keith Sharp, University of Waterloo, Ontario, Canada as well as the sponsors of the Actuarial Research Club at the City University, London, England. A research grant from the Actuarial Education and Research Fund of the Society of Actuaries is gratefully acknowledged.

1 Introduction

1.1 A Simplified Model

Consider a *strictly* defined benefit pension plan in which no discretionary or *ad hoc* benefit improvement is allowed except for benefit indexation. Assume that provision is made only for a retirement benefit at normal retirement age based on final salary and that actuarial valuations are carried out with the following features:

1. Actuarial valuations take place at regular intervals of one time period.

2. The actuarial valuation basis is invariant in time.

3. Pension fund assets are valued at market without smoothing (market value $f(t)$).

4. An 'individual' actuarial cost method (or pension funding method) is used, generating an actuarial liability $AL(t)$ and a normal cost $NC(t)$.

A simple model for such a pension plan may be projected forward based on the following:

1. The pension plan population is stationary (deterministic) from the start.

2. Mortality and other decrements are assumed to be contingent as per a life table $\{l'_x\}$.

3. A salary scale exactly reflects promotional, merit-based or longevity-based increases in salaries (and may be incorporated in the life-table, $l_x = s_x l'_x$).

Economic wage inflation (the general increase in wages as measured by a national wages index) may be distinguished from the salary scale. The actuarial valuation basis includes $\{l_x\}$ as well as a valuation discount rate and an assumption as to wage inflation (in order to value final-salary benefits). Actual experience is in accordance with the actuarial valuation assumptions except for inflation and returns on plan assets. The model described above bears similarities to the models described by Trowbridge (1952), Bowers *et al.* (1979), Dufresne (1988, 1989) and Owadally & Haberman (1999).

1.2 Funding Methods and Objectives

Suppose that cash flows occur at the start of the year. The unfunded liability at the start of year $(t,\ t+1)$ is the excess of the actuarial liability over the value of plan assets: $ul(t) = AL(t) - f(t)$. The funding level or funded ratio in the plan is defined as the value of plan assets as a percentage of the actuarial liability. The outcome of an actuarial valuation at the start of year $(t,\ t+1)$ is to recommend a contribution

$$c(t) = NC(t) + adj(t), \tag{1}$$

where $adj(t)$ is a supplementary contribution (or contribution adjustment) paid to amortize past and present experience deviations from actuarial assumptions. These deviations result in actuarial gains or losses. A loss $l(t)$ is the unanticipated change in the unfunded liability over year $(t-1,\ t)$ or the excess of the unfunded liability at time t over the unfunded liability anticipated at time t based on information and the valuation basis at time $t-1$. (A gain is a negative loss.)

The calculation of the contribution is crucial to the dynamics of the pension fund when economic experience is volatile. For example, Winklevoss (1982) finds that "the correct treatment of actuarial gains and losses is critical in stochastic simulations because the effect of random fluctuations in salaries and plan assets impact on costs through the funding of such deviations." The actuarial gain or loss in each year may be individually amortized over a fixed term m:

$$adj(t) = \sum_{j=0}^{m-1} l(t-j)/\ddot{a}_{\overline{m}|}. \tag{2}$$

Alternatively, gains and losses may be spread by paying a proportion $k = 1/\ddot{a}_{\overline{m}|}$ of the unfunded liability:

$$adj(t) = k\,ul(t) = k[AL(t) - f(t)]. \tag{3}$$

Dufresne (1988) states that the method in equation (3) "may be interpreted as 'spreading' the unfunded liability over a period of m years." Trowbridge & Farr (1976, p. 84) refer to "spreading of actuarial gain or loss". The method is described as "Amortization over a Moving Term" by Bowers *et al.* (1979) and as "spreading surpluses/deficits" by Owadally & Haberman (1999). (A deficit is an unfunded liability and a surplus is a negative deficit.) McGill *et al.* (1996, p. 525) describe it as a "*spread* method of dealing with gains and losses" such that "actuarial gains and losses are automatically, and without separate identification, spread" over a future period. Trowbridge & Farr (1976, p. 85) and Bowers *et al.* (1979) show that the spread adjustment is implicit in the Aggregate and Frozen Initial Liability actuarial cost methods, which are also known as "spread-gain" methods.

The unfunded liability in the plan is the accumulation with interest of portions of previously incurred losses (as well as of any initial unfunded liability) that have not been fully paid off and have been deferred. When the supplementary contribution in equation (3) is made to the plan, a spread payment equalling a fraction k of the deferred portion of each incurred loss is settled in respect of each loss (as if the loss were taxed at rate k). Therefore, the excess of the unfunded liability over the supplementary contribution in a given year must equal the present value of the unfunded liability, less the newly emergent loss, in the following year: $u(ul(t) - k\,ul(t)) = ul(t+1) - l(t+1)$, where $u = 1 + i$ and i is the rate at which cash flows in the plan are discounted. Ignoring any initial unfunded liability, it follows that $ul(t) = \sum_{j=0}^{\infty}(1-k)^j u^j l(t-j)$ and

$$adj(t) = \sum_{j=0}^{\infty} k(1-k)^j u^j l(t-j). \tag{4}$$

See Dufresne (1994) for a more precise development of the above. When $k = m = 1$, gains and losses are not deferred and the unfunded liability consists only of the loss that emerged during the past year, that loss being paid off immediately: $ul(t) = adj(t) = l(t)$.

Initial unfunded liabilities, due to past service at plan inception or arising from amendments to actuarial valuation bases or benefit rules, were disregarded in the above. An initial unfunded liability ul_0 at time 0 may be separately amortized or spread. If it is amortized over n years, then a payment of $ul_0/\ddot{a}_{\overline{n}|}$ is required in addition to the supplementary contribution in equation (2) over a finite schedule ($0 \leq t \leq n-1$). When gains and losses are spread and the initial unfunded liability is separately amortized, the supplementary contribution is (Owadally & Haberman, 1994)

$$adj(t) = \begin{cases} ul_0/\ddot{a}_{\overline{n}|} + k[ul(t) - ul_0\ddot{a}_{\overline{n-t}|}/\ddot{a}_{\overline{n}|}], & 0 \leq t \leq n-1, \\ k\,ul(t), & t \geq n, \end{cases} \tag{5}$$

which may be written in terms of the losses emerging from time 1 onwards ($l(t) = 0$ for $t \leq 0$) as

$$adj(t) = \begin{cases} ul_0/\ddot{a}_{\overline{n}|} + \sum_{j=0}^{t} k(1-k)^j u^j l(t-j), & 0 \leq t \leq n-1, \\ \sum_{j=0}^{t} k(1-k)^j u^j l(t-j), & t \geq n. \end{cases} \tag{6}$$

Initial unfunded liabilities may be disregarded in the pension plan model henceforth since they can be separately amortized and have no permanent effect.

Under amortization (equation (2)), gains and losses are paid off in level amounts over a finite term m. Under spreading (equations (3) and (4)), gains and losses are liquidated in perpetuity by means of exponentially declining payments. A large k (or short spreading period m, with $k = 1/\ddot{a}_{\overline{m}|}$) hastens funding as smaller portions of the losses are deferred. Expressing k in equation (3) as $1/\ddot{a}_{\overline{m}|}$ is a convenient device that enables gains and losses to be removed faster if future cash flows are discounted at a higher rate. A loss is asymptotically liquidated when it is spread since the present value of payments made in respect of a unit loss is $\sum_{j=0}^{\infty} k(1-k)^j = 1$ when $m > 1$ (since $0 < d < k = 1/\ddot{a}_{\overline{m}|} < 1$ where $d = i(1+i)^{-1}$).

If actuarial assumptions are unbiased and are realized on average, the loss, or unanticipated change in the unfunded liability, in any year is expected to be zero. Once the initial unfunded liability is completely amortized, the unfunded liability (an accumulation of unpaid losses) is therefore expected to be zero, whether gains and losses are amortized or spread. The uncertain nature of deviations from assumed experience (particularly economic experience) means that gains and losses are volatile, however, and the consequent *variability* of the funding level and of the required contributions must be examined. A motivation for the long-term funding of pension benefits is to maximize the security of these benefits. A reasonable objective is to minimize a second-moment measure of the variability of the funding level or of the unfunded liability in the pension plan. (An ideal measure of security would be based on second and higher moments, would allow for solvency requirements and would not be two-sided, but the variance is a reasonable and tractable approximation.) Another motivation for funding benefits in advance is to stabilize the future contributions required from the plan sponsor and hence reduce the strain on the sponsor's cash flows. Trowbridge & Farr (1976, p. 62) refer to the "smoothness of contributions" as a desirable objective. This is often expressed relative to the total payroll for plan members. Minimizing a second-moment measure of the variability of the contribution or contribution rate (that is, total contribution per payroll dollar) is also a reasonable objective.

The choice between amortizing and spreading gains and losses is discussed, in terms of these objectives and under the modeling assumptions of section 1.1, in the rest of the paper. One notes at the outset that the asymptotic nature of funding under spreading is not a drawback, as remarked by Trowbridge & Farr (1976, p. 85), because gains and losses occur randomly and continually and are never completely defrayed. Gains and losses in the pension plan model arise only as a consequence of unforeseen economic variation from actuarial assumptions, that is, in the returns on plan assets and in inflation on plan liabilities. Contributions may be determined as a level percentage of payroll. The annuities in equation (2) and in $k = 1/\ddot{a}_{\overline{m|}}$ in equation (3) are then calculated at the valuation discount rate net of assumed inflation on wages.

A review of previous work concerning the choice of efficient amortization and spreading periods is undertaken in section 2 and a justification for the result that spreading should be preferred over amortization is provided. This is based on rates of return on plan assets, net of inflation on pension liabilities, being independent from year to year, and in section 3 this assumption is relaxed. Stochastic projections are used in section 4 to investigate if the result holds when both inflation on pension liabilities and returns on plan assets are explicitly modeled. Section 5 summarizes the results and highlights areas where further work is necessary.

2 Random Walks on Pension Fund Assets

2.1 Spreading Gains and Losses

The liabilities of a final-salary pension plan are subjected to economic inflation, both on prices and on wages, since benefits are linked to salary and pensions in payment may also be indexed (or possibly informally enhanced) in line with price inflation. Modeling inflation as well as the returns on various asset classes is not straightforward. For the sake of mathematical tractability, Dufresne (1988, 1989) and Haberman (1994) assume that pensions in payment are indexed with *wage* inflation. All monetary quantities may then be considered net of wage inflation. Under the modeling assumptions made earlier, the payroll, actuarial liability, normal cost and yearly pension benefit outgo (all deflated by wage inflation) are constant. (Alternatively, inflation on salaries could be disregarded for the sake of simplicity, and nominal quantities considered.)

The rate of investment return on pension plan assets net of wage inflation (henceforth termed a *real* rate of return) may be usefully modeled as being independent and identically distributed from year to year (Dufresne, 1988, 1989; Haberman & Sung, 1994). In a simplified pension plan model, such an assumption allows for mathematical tractability, parsimony and a search for optimal or robust performance. Although it may not lead to accurate forecasts of the finances of the pension plan, this assumption allows us to investigate the behavior of the pension fund system when invested in capital markets that are volatile. Frees (1990) views such an assumption as "a useful modification of the traditional deterministic [rate of interest]. This modification permits volatility of interest rates in the model." The random walk assumption accords with the Efficient Market Hypothesis.

The market value of plan assets $f(t)$ and the contribution $c(t)$ are random variables as a consequence of the random real rate of return $i(t)$ on plan assets: $f(t+1) = (1+i(t+1))(f(t)+c(t)-B)$, where B is the benefit outgo. The moments of $f(t)$ and $c(t)$ are derived by Dufresne (1988) when $i(t)$ is independent and identically distributed over time, when liabilities are wage inflation-related, and when gains and losses are spread (equation (3)). The plan is expected to be fully funded eventually: $\mathrm{E}ul(t) \to 0$ and $\mathrm{E}adj(t) \to 0$ as $t \to \infty$, whatever the initial unfunded liability and provided actuarial assumptions are unbiased.

Dufresne (1988) postulates that there are two actuarial objectives in the long-term funding of pension benefits: to maximize the security of these benefits by minimizing the variance of the funding level, and to maximize contribution stability by minimizing the variance of the contribution rate. Dufresne (1988) shows that the stochastic pension funding process becomes stationary in the limit as $t \to \infty$, provided that gains and losses are not spread over very long periods (that is, provided that $k > 1 - 1/\sqrt{\mathrm{E}(1+i(t))^2}$), and he obtains $\mathrm{Var}c(t)$ and $\mathrm{Var}f(t)$ as $t \to \infty$. Since the payroll and actuarial liability are constant (net of wage inflation) under the assumptions set out earlier, $\lim \mathrm{Var}c(t)$ and $\lim \mathrm{Var}f(t)$ represent the long-term variances of the contribution rate and funding level respectively. The following results (labels carry the prefix 'S' for 'spreading') are also obtained by Dufresne (1988).

RESULT S-1 *The variance of the funding level of the stationary (stochastic) pension funding process increases as gains and losses are spread over a longer period.*

RESULT S-2 *As the period over which gains and losses are spread increases, the variance of the contribution rate in the stationary pension funding process initially decreases, attains a minimum (at m_s^*, say), and then increases.*

RESULT S-3 *Based on the criterion of minimizing the variances of the contribution rate and funding level, it is more efficient to spread gains and losses over a period $m \in [1, m_s^*]$.*

Gains and losses are volatile. Spreading them over shorter periods and recognizing them faster ensures that full funded status is reached faster and is maintained. This should improve the security of benefits, as confirmed by Result S-1. It may be thought that spreading, and therefore deferring, the gains and losses over longer periods leads to a smoother and more stable contribution rate pattern. Dufresne's (1988) Result S-2 partly contradicts this. For spreading periods longer than m_s^*, contribution rate stability and fund security cannot be traded off and there will always be a shorter spreading period for which the variances of both funding levels and contribution rates are reduced. It is therefore more efficient to spread gains and losses over $m \in [1, m_s^*]$.

Dufresne (1988) concludes that, under modern economic conditions, $m \in [1, 10]$ is an efficient range over which to spread gains and losses. This result has had some influence on actuarial practice in the United Kingdom. See Wilkie in Dufresne (1994, p. viii). Thornton & Wilson (1992) recommend short spreading periods based partly on Dufresne's (1988) conclusion. Faster defrayal of gains and losses arising from amendments to benefits or to valuation bases, rather than from experience deviations, have also been promoted

(Kryvicky, 1981; Colbran, 1982).

2.2 Amortizing Gains and Losses

Dufresne (1989) also derives the moments of the funding process when gains and losses are amortized (equation (2)), under the same assumptions as above. Again, full funding is expected (when the initial unfunded liability is completely amortized and with unbiased actuarial assumptions) and the funding process has a finite variance and is therefore stable as long as gains and losses are not amortized over very long periods. The following results are obtained by Owadally & Haberman (1999) (labels are prefixed with 'A' for 'amortizing').

RESULT A-1 *The variance of the funding level of the stationary (stochastic) pension funding process increases as gains and losses are amortized over a longer period. Funding levels are less variable under amortization over a fixed term than under spreading over the same term.*

Result A-1 indicates that amortizing gains and losses over shorter periods improves the security of benefits. It is illustrated in the top half of Figure 1 (reproduced from Owadally & Haberman, 1999). The maximum period over which gains and losses may be amortized or spread is constrained. If $m_a = m_s = 1$, then experience deviations are neither spread nor amortized over time and are paid off immediately: equations (2) and (3) lead to identical results.

RESULT A-2 *As the term over which gains and losses are amortized increases, the variance of the contribution rate in the stationary pension funding process initially decreases, attains a minimum (at $m_a^* > m_s^*$), and then increases. Contribution rates are less stable under amortization over a fixed term $m < m_a^*$ than under spreading over the same period.*

The minimum in the contribution rate variance is depicted in the bottom half of Figure 1.

RESULT A-3 *Suppose that gains and losses are amortized. Based on the criterion of minimizing the variances of the contribution rate and funding level, the range of amortization periods* $m \in [1, m_a^*]$ *is efficient.*

We infer from Results A-1 and A-2 in combination that the efficient range of amortization periods exists because for amortization periods $m > m_a^*$ there will always be an amortization period in $[1, m_a^*]$ that yields the same variability in the contribution rate together with less variable funding levels.

RESULT A-4 *According to the objective of minimizing the variance of funding levels and contribution rates in the stationary state, it is more efficient to spread gains and losses than to amortize them: for any two spread and amortization periods such that the funding level has the same variance, the variance of the contribution rate is lower under spreading than under amortization.*

Result A-4 suggests that it is better to recommend contribution rates by spreading gains and losses (equation (3)) than by amortizing them over a fixed schedule (equation (2)). This is depicted in Figure 2 (reproduced from Owadally & Haberman, 1999).

Compare Result A-1 (amortization leads to less variable funding levels than spreading over the same period), Result A-2 (spreading leads to less variable contribution rates than amortization over the same period $m < m_a^*$), Result A-3 (there exists an efficient amortization period range analogous to the efficient spreading period range of Result S-3), and Result A-4 (overall, spreading is more efficient than amortization).

2.3 Proportional Spreading and Optimal Contributions

The efficiency of spreading as compared to direct gain/loss amortization may be explained by the fact that, in the former, the supplementary contribution is proportional to the contemporaneous unfunded liability in the plan (equation (3)). Proportional adjustment represents an optimal form of contribution control when a quadratic measure of the variability of contribution and market value of plan assets is to be minimized. This result is obtained by O'Brien (1987) and Haberman & Sung (1994) assuming random rates of investment return on the plan assets. Boulier *et al.* (1995) obtain a similar result when asset allocation between two assets is also a decision variable and when the pension plan is being valued continuously and indefinitely.

A linear contribution adjustment is also obtained when regular valuations and cash flows occur at discrete intervals and a finite time horizon is assumed. Consider a similar pension plan to the one described in section 1.1. Assume that the fund may be invested in two assets: a risk-less asset earning risk-free rate r and a risky asset earning $r + \alpha(t+1)$ in year $(t,\ t+1)$, where $\alpha(t+1)$ is a random risk premium. Let $y(t)$ be the proportion of the fund invested in the risky asset in year $(t,\ t+1)$, and $1 - y(t)$ be the proportion invested in the risk-less asset. The arithmetic rate of return on the fund in year $(t, t+1)$ is $r + y(t)\alpha(t+1)$. It is further assumed that $\{\alpha(t)\}$ is a sequence of independent and identically distributed random variables over time, with mean $\alpha > 0$ and variance σ^2. It is also simpler to disregard inflation at this stage. Since the plan population is assumed to be stationary, the payroll, actuarial liability and benefit payout are constant. The variability of contributions corresponds to the contribution rate variability while the variability of market values of plan assets corresponds to funding level variability.

The pension fund can be considered as a random system,

$$f(t+1) = [1 + r + y(t)\alpha(t+1)][f(t) + c(t) - B], \tag{7}$$

where the market value of plan assets $f(t)$ is a state variable and $c(t)$ and $y(t)$ are contribution and asset allocation control variables respectively. B represents the retirement benefits paid out yearly. By virtue of the independence over time of $\{\alpha(t)\}$, $f(t)$ exhibits the Markov property: $\mathrm{E}[f(t+1)|W_t] = \mathrm{E}[f(t+1)|f(t), y(t), c(t)]$, where W_t represents all information available up to time t.

The objectives of the funding process are to stabilize contributions, defray any unfunded liabilities and pay off actuarial losses and gains as they emerge. The performance of the pension fund may be judged in terms of the deviations in the values of plan assets and contributions from their desired levels (say FT_t and CT_t respectively) relating to the actuarial liability and normal cost. The 'cost' incurred for any such deviation at time $0 \leq t \leq N-1$ may be defined as

$$C(f(t), c(t), t) = \theta_1(f(t) - FT_t)^2 + \theta_2(c(t) - CT_t)^2 \tag{8}$$

Different weights ($\theta_1 > 0$ and $\theta_2 > 0$) are placed on the twin long-term objectives of fund security and contribution stability. The cost in equation (8) reflects a quadratic utility function. Minimizing the cost also minimizes the risks of contribution instability and of fund inadequacy.

The performance of the fund may be given different importance over time. At the end of the given control period N, a closing cost is incurred if an unfunded liability still exists: $\mathbb{C}_N = \theta_0(f(N) - FT_N)^2$. The *discounted* cost of deviation occurring t years ahead is $\beta^t C(f(t), c(t), t)$, where $0 < \beta < 1$. For $0 \leq t \leq N-1$, the discounted cost-to-go or discounted cost incurred from time t to N is

$$\mathbb{C}_t = \sum_{s=t}^{N-1} \beta^{s-t} C(f(s), c(s), s) + \beta^{N-t}\mathbb{C}_N. \tag{9}$$

An objective criterion for the performance of the pension funding system over period

N may therefore be defined to be

$$\mathrm{E}[\mathbb{C}_0|W_0] = \mathrm{E}\left[\beta^N \mathbb{C}_N + \sum_{s=0}^{N-1} \beta^s C(f(s), c(s), s) \middle| W_0\right]. \tag{10}$$

The value function $J(f(t), t)$ is defined as the minimum, over the remaining asset allocation and contribution decisions, of the expected discounted cost-to-go from time t given information at time t: $J(f(t), t) = \min_\pi \mathrm{E}[\mathbb{C}_t|W_t]$, where $\pi = \{c(t), y(t), c(t+1), y(t+1), \ldots, c(N-1), y(N-1)\}$. Objective criterion (10) may be minimized using the Bellman optimality principle (see e.g. Bertsekas, 1976): the minimizing values of $c(t)$ and $y(t)$ (say, $c^*(t)$ and $y^*(t)$ respectively) in the optimality equation,

$$J(f(t), t) = \min_{c(t), y(t)} \left\{ C(f(t), c(t), t) + \beta \mathrm{E}[J(f(t+1), t+1)|f(t), c(t), y(t)] \right\}, \tag{11}$$

with boundary condition $J(f(N), N) = \mathbb{C}_N = \theta_0(f(N) - FT_N)^2$, are the optimal contribution and asset allocation controls.

The pension planning objectives above were set over a finite period N. The plan is assumed to remain solvent and not discontinued during these N years, so that the funding process does not terminate unexpectedly. When the pension plan is regarded as a going concern, an infinite planning horizon may be usefully envisaged as a reasonable approximation to long-term funding.

No closing cost is incurred in the infinite horizon case. Suppose that the funding process is time-homogeneous, in that the fund and contribution targets are constant. The instantaneous cost at time t is as in equation (8) with $FT_t = FT$, $CT_t = CT$ $\forall t$. The discounted cost-to-go and objective criterion are as in equations (9) and (10) respectively, except that $\mathbb{C}_N = 0$ and an infinite summation of discounted costs is taken. One naturally expects that the control rules that are optimal at time t based on the infinite discounted cost-to-go are also the optimal control rules at times $t+1$, $t+2$,... since the same infinite discounted cost-to-go exists at all times. The dynamic programming algorithm in equation (11) can in fact be shown to converge as $N \to \infty$ (see Bertsekas, 1976, p. 251) as it involves a contraction mapping and the instantaneous costs in equation (8) are non-negative and are discounted. The optimal contribution and asset allocation over an infinite horizon (say, $c^*_\infty(t)$ and $y^*_\infty(t)$ respectively) are the minimizing values of $c(t)$ and $y(t)$ in equation (11) when the value function and the control variables are fixed or time-invariant functions of the state variable $f(t)$.

It is shown in Appendix A that the solution to the Bellman equation (11) in the finite-horizon case is

$$J(f(t),t) = P_t f(t)^2 - 2Q_t f(t) + R_t, \tag{12}$$

where

$$P_t = \theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}_{t+1}P_{t+1}, \tag{13}$$

$$\tilde{P}_{t+1} = [\theta_2(\alpha^2+\sigma^2) + \beta\sigma^2(1+r)^2 P_{t+1}]^{-1}, \tag{14}$$

$$Q_t = \theta_1 FT_t + \theta_2\beta\sigma^2(1+r)\tilde{P}_{t+1}[Q_{t+1} - P_{t+1}(1+r)(CT_t - B)], \tag{15}$$

with boundary conditions $P_N = \theta_0$ and $Q_N = \theta_0 FT_N$ (R_t represents some additional terms independent of $f(t)$).

It is also shown in Appendix A that the equivalent solution in the infinite-horizon case is

$$J(f(t)) = Pf(t)^2 - 2Qf(t) + R, \tag{16}$$

where $P > \theta_1$ is the positive root of the quadratic equation

$$P^2[\beta\sigma^2(1+r)^2] + P[\theta_2(\alpha^2+\sigma^2) - (\theta_1+\theta_2)\beta\sigma^2(1+r)^2] - \theta_1\theta_2(\alpha^2+\sigma^2) = 0 \tag{17}$$

and $P = \lim_{t\to\infty} P_t$, and where

$$Q = [\theta_1 FT + (P-\theta_1)(B - CT)]P(1+r)/(Pr+\theta_1) \tag{18}$$

and $Q = \lim_{t\to\infty} Q_t$, and R contains terms independent of $f(t)$.

Define $\Theta_t = \theta_2(\alpha^2+\sigma^2)\tilde{P}_t = \theta_2(\alpha^2+\sigma^2)/[\theta_2(\alpha^2+\sigma^2)+\beta\sigma^2(1+r)^2P_t]$ in the finite-horizon setting and correspondingly $\Theta = \theta_2(\alpha^2+\sigma^2)/[\theta_2(\alpha^2+\sigma^2)+\beta\sigma^2(1+r)^2P]$ in the infinite-horizon setting. The following proposition is proven in Appendix A.

PROPOSITION 1 *For $0 \le t \le N-1$ over a finite horizon, the optimal contribution is*

$$c^*(t) = \Theta_{t+1}CT_t + (1-\Theta_{t+1})[B - f(t) + Q_{t+1}P_{t+1}^{-1}(1+r)^{-1}] \tag{19}$$

and the optimal amount invested in the risky asset is

$$y^*(t)[f(t)+c^*(t)-B] = [Q_{t+1}P_{t+1}^{-1}(1+r)^{-1} - (f(t)+c^*(t)-B)]\alpha(1+r)(\alpha^2+\sigma^2)^{-1}. \tag{20}$$

The corresponding optimal decisions over an infinite horizon are:

$$c^*_\infty(t) = \Theta CT + (1-\Theta)[B - f(t) + QP^{-1}(1+r)^{-1}], \tag{21}$$

$$y^*_\infty(t)[f(t)+c^*_\infty(t)-B] = [QP^{-1}(1+r)^{-1} - (f(t)+c^*_\infty(t)-B)]\alpha(1+r)(\alpha^2+\sigma^2)^{-1}. \tag{22}$$

Note that since $P_N = \theta_0 > 0$, backward recursion in the Riccati difference equation for P_t formed by equations (13) and (14) shows that $P_t > 0$, $\tilde{P}_t > 0$ for $t \in [1, N]$. Clearly, $0 < \Theta_t < 1$ for $t \in [1, N]$. It is reasonable to assume that the fund and contribution targets in any year are such that $FT_t > 0$ and $CT_t < B$ (as otherwise there is no sense to funding in advance for retirement benefits). Then, $Q_t > 0$ for $t \in [1, N]$ from equation (15) and the boundary condition $Q_N = \theta_0 FT_N > 0$. In the infinite-horizon case, $P > \theta_1 > 0$ and clearly $0 < \Theta < 1$. With $FT > 0$ and $B > CT$, it follows that $Q > 0$ (equation (18)).

It is immediately observed that $\partial c^*(t)/\partial f(t) < 0$ from equation (19). From equation (20), it is easy to show that $\partial y^*(t)/\partial[f(t)+c^*(t)-B]$ is directly proportional to $-Q_{t+1}P_{t+1}^{-1}(1+r)^{-1}$ which is negative. Since $\partial[f(t)+c^*(t)-B]/\partial f(t) = \Theta_{t+1} > 0$, it follows that $\partial y^*(t)/\partial f(t) < 0$. Likewise, $\partial c^*_\infty(t)/\partial f(t) < 0$ and $\partial y^*_\infty(t)/\partial f(t) < 0$.

The optimal proportion invested in the risky asset decreases as $f(t)$ increases, whatever the planning horizon. A similar result is obtained by Boulier *et al.* (1995) and Cairns

(1997) for an infinite horizon and for continuous pension plan valuations. The better the investment performance of the risky asset, the better funded the plan is, the more the pension fund should be invested in the risk-less asset. This is a contrarian strategy that entails buying as the market falls and selling as the market rises. It is reasonable in the sense that, first, liabilities need to be hedged so as to minimize the volatility of both surpluses and contributions and, second, any available surpluses should be 'locked in' by being invested in less risky assets (Exley *et al.*, 1997). Conversely, the optimal strategy requires that an underfunded plan takes a riskier investment position than an overfunded plan, all other things equal. For instance, the assets of a poorly funded immature pension plan (with a young membership) could arguably be invested more aggressively in the early years than the assets of a comparable plan with a healthy surplus. The optimal strategy here may be contrasted with portfolio insurance strategies (Black & Jones, 1988) that require riskier investment as the value of plan assets less some minimum value (possibly determined by a solvency requirement) increases. The contrarian strategy is evidently a consequence of the quadratic utility function implied in criterion (8), which is simplistic as it is symmetric and continuous and does not admit solvency and full funding constraints.

The optimal contribution is linear in $f(t)$, whatever the planning horizon: from equation (19), $c^*(t)$ may be written as $c_0(t) - (1 - \Theta_{t+1})f(t)$, where $1 - \Theta_{t+1} > 0$ while, from equation (21), $c^*_\infty(t)$ may be written as $c_0 - (1 - \Theta)f(t)$, where $1 - \Theta > 0$. The optimal contribution at the start of year $(t, t+1)$ is therefore similar to the contribution calculated when gains and losses are spread (equations (1) and (3)) in that they both depend in a decreasing linear way on the *current* market value of assets. This result is based on the assumption of an efficient market in the risky asset, implying serially independent rates of return and a Markovian funding process. The market value of plan assets represents the *state* of the funding process and, conditional on knowing the current state of funding and current funding decisions, the future evolution of the fund is statistically independent of its past. The optimal contribution is therefore a function of the current state only. This contrasts markedly with the amortization of gains and losses where the contribution (equations (1) and (2)) is a function of unanticipated changes in the state of the funding process over the past m years. This analysis helps to justify the efficiency of spreading over amortization as stated in Result A-4 (which was also based on quadratic or second-moment criteria) despite the simplifying assumptions of a quadratic utility function and of zero inflation.

3 Dependent Rates of Return

The results of the previous sections are limited by the assumption of independent real rates of return from year to year. Rates of return on pension fund assets are likely to be statistically dependent. Markets may not be efficient over the long term and rates of return on several asset classes have been found to be correlated over time, as is demonstrated by the statistical analysis of Panjer & Bellhouse (1980), Fama & French (1988), Wilkie (1995) among others. Whether or not markets are efficient, not all the securities held by the fund will typically be traded every year and some dependence in the returns from individual securities will occur (Vanderhoof, 1973). This is particularly the case where debt securities are held to match certain liability cash flows. Even when the asset portfolio is actively managed, many securities will be held for over a year. McGill *et al.* (1996, p. 663) report that "in a volatile business environment, a third or a half of a common stock (equity) portfolio may turn over within a one-year period." At this stage, consideration of dynamic asset allocation among several asset classes is suppressed and the rate of return on the pension fund is modeled.

3.1 Spreading Gains and Losses

It is of interest to consider whether Results S-1–A-4 hold when dependent rates of return are assumed. The two most common time series of rates of return applied in mathematical actuarial models are the autoregressive (Pollard, 1971; Panjer & Bellhouse, 1980; Bellhouse & Panjer, 1981; Dhaene, 1989) and the moving average processes (Frees, 1990).

Haberman (1994) assumes that logarithmic rates of return are stationary Gaussian autoregressive processes of order 1 and 2 [AR(1), AR(2)] and that gains and losses are spread, while Haberman & Wong (1997) make the assumption of logarithmic rates of return that are stationary Gaussian moving average processes of order 1 and 2 [MA(1), MA(2)]. They derive the moments of the pension funding process. Their numerical work for the simple AR and MA rates of return demonstrates that:

1. Result S-1 appears to hold.
2. Result S-2 appears to hold when the rate of return process exhibits moderate autocorrelation (under practical conditions).
3. An efficient range of spread period therefore exists (Result S-3) for moderately autocorrelated rates of return.

The efficient range appears to become more restricted as the variance of the rate of return process increases, as well as for more positive correlation from year to year in the rate of return process. As the variance and/or correlation from year to year of the rate of return increase beyond some threshold, the efficient range vanishes and minimum contribution rate and funding level variability are yielded when gains and losses are not spread into the future but paid off immediately ($m = 1$).

3.2 Amortizing Gains and Losses

It is more difficult to obtain closed-form solutions for the moments of the pension funding process when gains and losses are being amortized over a fixed term and logarithmic rates of return follow simple MA and AR processes. Stochastic simulations may be performed to verify whether Results A-1–A-4 hold in these cases.

In the following, 2000 scenarios are simulated with a time horizon of 300 years each. (The randomization routine generates the same set of 300×2000 random numbers so that sampling error does not occur when results are compared.) A simple final-salary pension plan as in section 1.1 is assumed. Pensions in payment are assumed to be indexed with economic wage inflation. All quantities may therefore be considered net of wage inflation, and since the pension plan population is stationary, the liability structure of the pension plan is stable in time. The payroll in real dollars (net of wage inflation) is constant. The valuation discount rate (net of wage inflation) is assumed to be 5%. The actuarial liability (AL), normal cost (NC) and yearly benefit outgo (B) (all deflated by wage inflation) are also constant and hold in equilibrium ($B = NC + AL \times 4.76\%$). The standard deviation of the funding level is calculated ($\sqrt{\mathrm{Var}} F/AL$) while the standard deviation of the contribution rate (contribution per payroll dollar) is proportional to $\sqrt{\mathrm{Var}} C/NC$.

3.3 AR(1) Rates of Return

The logarithmic rate of return process (net of wage inflation) is first projected as a stationary Gaussian autoregressive process of order 1:

$$\delta(t+1) - \delta = \varphi(\delta(t) - \delta) + e(t+1), \tag{23}$$

where $|\varphi| < 1$ and $\{e(t)\}$ is a sequence of zero-mean independent and identically normally distributed variables. The process is stationary from the start. The arithmetic rate of return (net of wage inflation) in year $(t-1,\ t)$ is $\exp(\delta(t))-1$ and, in the simulations, its mean is 5% (i.e. equal to the valuation discount rate net of salary inflation) and its standard deviation is 20%.

The scaled standard deviations of the funding level and contribution rate are shown in Table D.1 (for an independent and identically distributed process and $\varphi=0$), Tables D.2, D.3, D.4 (for processes that are positively autocorrelated at lag one and $0<\varphi<1$), and Tables D.5 and D.6 (for processes that are negatively autocorrelated at lag one and $-1<\varphi<0$).

Variance of the Funding Level (Results S-1 and A-1). The numerical data in these tables show that the variance of the funding level increases as spreading and amortization periods increase. Both Results S-1 and A-1 appear to be borne out.

Variance of the Contribution Rate (Results S-2 and A-2). For lightly autocorrelated rates of return ($\varphi=+0.3$, $+0.5$, -0.1), both Results S-2 and A-2 appear to hold. For instance, when $\varphi=+0.5$ in Table D.3, $m_s^*=3$, $m_a^*=5<m_s^*$. For $m<5$, contributions are more variable when gains and losses are amortized, but for $m>5$, contributions are more variable when gains and losses are spread. These features are also displayed in Figures 3 and 4 for $\varphi=+0.3$ and -0.1 respectively (cf. second graph in Figure 1). General features, rather than a precise determination of m_a^* and m_s^*, are sought.

When the autocorrelation in the rate of return process is more extreme, Results S-2 and A-2 do not hold. For example, when $\varphi=+0.8$ (Table D.4), contribution rate variability increases monotonically as gains and losses are spread over longer periods, as indicated by the results of Haberman (1994). Contribution rates also become more variable as gains and losses are amortized over longer terms, although they are more stable than if gains and losses had been spread over the same term. For rates of return that are highly positively autocorrelated at lag one, it is then efficient to pay off any gain or loss immediately. Conversely, it appears that when rates of return are very negatively autocorrelated at lag one ($\varphi = -0.3$, Table D.6), varying either the spreading or amortization period always involves a tradeoff between funding level and contribution rate variability and there is no efficient range for m (as also demonstrated in the case of spreading by Haberman, 1994).

Spread and Amortization Periods (Results S-3 and A-3). Results S-3 and A-3 therefore appear to be robust for moderately autocorrelated rates of return. For example, when $\varphi = +0.5$ (Table D.3), it is better to spread gains and losses over $m_s^* = 3$ years or less, or amortize them over $m_a^* = 5$ years or less. If longer periods are used, the variability of contribution rates may be the same as when the recommended shorter periods are used, but funding levels will be more variable. It is also generally apparent that m_s^* and m_a^* decrease as the autocorrelation at lag one in the rate of return increases (as φ increases).

Efficiency (Result A-4). The variance of the contribution rate is plotted against the variance of the funding level in Figures 5 and 6 for the values of φ considered above. For any two (possibly different) spread and amortization periods for which funding levels are equally variable, contribution rates will be less variable when proportional spreading is employed rather than fixed-term amortization. For all the values of φ considered, it is better to recommend contributions by spreading gains and losses than by amortizing them over a fixed term. Result A-4 appears to hold true.

3.4 MA(1) Rates of Return

The logarithmic rate of return process (net of wage inflation) is next projected as a stationary Gaussian moving average process of order 1:

$$\delta(t) - \delta = e(t) - \phi e(t-1), \tag{24}$$

where $\{e(t)\}$ is a sequence of zero-mean independent and identically normally distributed variables and $|\phi| < 1$ so that the process is invertible and may be expressed as an autoregressive process. The arithmetic rate of return (net of wage inflation) in year $(t-1,\, t)$ is $\exp(\delta(t)) - 1$ and, in the simulations, its mean is 5% (i.e. equal to the valuation discount rate net of salary inflation) and its standard deviation is 20%.

The scaled standard deviations of the funding level and contribution rate are shown in Table D.1 (for an independent and identically distributed process and $\phi = 0$), Tables D.7, D.8 (for processes that are negatively autocorrelated at lag one and $0 < \phi < 1$), and Tables D.9 and D.10 (for processes that are positively autocorrelated at lag one and $-1 < \phi < 0$). The numerical results closely parallel those for AR(1) rates of return. (Note that the autocorrelation at lag one of the rate of return process increases as φ increases in the stationary AR(1) process with $|\varphi| < 1$ considered earlier, but it decreases as ϕ increases in the invertible MA(1) process with $|\phi| < 1$.)

Variance of the Funding Level (Results S-1 and A-1). The funding level appears to become more variable as spreading and amortization periods are increased.

Variance of the Contribution Rate (Results S-2 and A-2). The results concerning the variability of the contribution rate also appear to hold for a moderately autocorrelated (at lag 1) MA(1) rate of investment return process. (They hold for ϕ = -0.5, -0.3, 0, +0.1.)

Spread and Amortization Periods (Results S-3 and A-3). Efficient spreading and amortization period ranges (Result A-3) do emerge for the moderately autocorrelated (at lag 1) processes, and m_s^* and m_a^* appear to decrease as the autocorrelation at lag one increases (as ϕ decreases).

Efficiency (Result A-4). The variance of the contribution rate is plotted against the variance of the funding level in Figure 7 for the values of ϕ considered above: it is better to recommend contributions by spreading gains and losses than by amortizing them. Result A-4 appears to hold.

3.5 AR(p) Logarithmic Rates of Return

Rates of return on pension plan assets that are AR(p), $p \in \mathrm{N}$, $p \geq 2$, have not been simulated. In the case when gains and losses are spread, first-order approximations seem to show that Results S-1–S-3 will hold for moderately autocorrelated logarithmic rates of return.

Haberman (1994) assumes AR(1) and AR(2) logarithmic real rates of return and gain/loss spreading and shows that exact closed form solutions cannot be obtained for the stationary first and second moments of the pension funding process. His numerical analysis shows that first-order approximations are accurate when the rate of return process is not strongly autocorrelated. His approach may be generalized for an AR(p), $p \in \mathrm{N}$, rate of return process.

Consider a simplified pension plan similar to the one described in section 1.1 and assume that retirement benefits are indexed with economic wage inflation, so that all monetary quantities are assumed to be deflated by wage inflation, as in section 3.2. The payroll, actuarial liability (AL), normal cost (NC), and yearly benefit outgo (B) (all deflated by wage inflation) are therefore constant. AL and NC are calculated at a valuation discount rate (net of wage inflation) of i (with $d = i(1+i)^{-1}$) and equilibrium means that $B = dAL + NC$.

The logarithmic rate of return on the pension plan assets in year $(t,\, t+1)$ is $\delta(t+1)$ and is projected to be stationary Gaussian AR(p), $p < \infty$:

$$\delta(t+1) - \delta = \varphi_1(\delta(t) - \delta) + \varphi_2(\delta(t-1) - \delta) + \cdots + \varphi_p(\delta(t-p+1) - \delta) + e(t+1), \quad (25)$$

where $\{e(t)\}$ is a sequence of zero-mean independent and identically normally distributed variables. $\delta(t)$ is stationary and we may let $\mathrm{E}\delta(t) = \delta$ and $\mathrm{Var}\delta(t) = \sigma^2 \ \forall t$.

It is well known that

$$\mathrm{Cov}[\delta(t), \delta(t-h)] = \sigma^2 \sum_{i=1}^{p} A_i G_i^h, \quad (26)$$

for $h \geq 0$ where the characteristic equation

$$z^p - \varphi_1 z^{p-1} - \varphi_2 z^{p-2} - \cdots - \varphi_p = 0 \quad (27)$$

has p distinct roots $\{G_i\}$ and $\{A_i\}$ is defined by $\mathrm{Cov}[\delta(t),\ \delta(t-h)]$ for $h \in [0, p-1]$ characterizing the stationary AR(p) process (25). Stationarity implies that

$$|G_i| \ < \ 1 \quad \text{for } i \in [1, p]. \quad (28)$$

The market value of the fund (deflated by wage inflation) follows the recurrence relationship below:

$$f(t+1) = \exp(\delta(t+1))[f(t) + c(t) - B]. \quad (29)$$

Gains and losses are spread and contributions $c(t)$ are calculated from equations (1) and (3). Noting that $B = dAL + NC$ and letting $k = 1/\ddot{a}_{\overline{m}|} = 1 - Q$ and $R = (1 - Q - d)AL$, the substitution $f(t) + c(t) - B = Qf(t) + R$ may be made. Therefore, if f_0 is the initially known market value of assets at time 0,

$$f(t) = f_0 Q^t \exp\left[\sum_{u=1}^{t} \delta(u)\right] + R \sum_{s=0}^{t-1} Q^{t-s-1} \exp\left[\sum_{u=s+1}^{t} \delta(u)\right]. \quad (30)$$

Define

$$c = \exp\left[\delta + \frac{\sigma^2}{2}\sum_i A_i \frac{1+G_i}{1-G_i}\right], \tag{31}$$

$$w = \exp\left[\delta + \frac{3\sigma^2}{2}\sum_i A_i \frac{1+G_i}{1-G_i}\right], \tag{32}$$

$$z_i = \sigma^2 A_i G_i (1-G_i)^{-2}. \tag{33}$$

The following proposition is proven in Appendix B.

PROPOSITION 2 *Provided that* $|Qc| < 1$,

$$\lim_{t\to\infty} \mathrm{E}f(t) \approx e^{-\sum_i z_i} Rc/(1-Qc), \tag{34}$$

$$\lim_{t\to\infty} \mathrm{E}c(t) = NC + (1-Q)(AL - \lim \mathrm{E}f(t)). \tag{35}$$

Provided also that $Q^2cw < 1$,

$$\lim_{t\to\infty} \mathrm{E}f(t)^2 \approx e^{-3\sum_i z_i} \frac{2R^2Qc^2w}{(1-Q^2cw)(1-Qc)} + e^{-4\sum_i z_i} \frac{R^2cw}{1-Q^2cw}, \tag{36}$$

$$\lim_{t\to\infty} \mathrm{Var}c(t) = (1-Q)^2 \lim \mathrm{Var}f(t). \tag{37}$$

The approximations in equations (34) and (36) are reasonably accurate when the terms $|A_iG_i|$ are small, i.e. when the rate of return process $\{\delta(t)\}$ exhibits weak autocorrelation—see equation (26). (Better approximations are described in the Appendix.) c, w and z_i (in equations (31), (32) and (33) respectively) correspond to the definitions of Haberman (1994) for $p = 1, 2$. The forms of the first and second moments generalize the forms obtained by Haberman (1994) for $p = 1, 2$.

It may be observed that the second moments of the pension funding process depends on the gain/loss spreading period through Q alone and not through c, w and $\{z_i\}$. The structure of the limiting second moments (equation (36)) of the pension funding process for $p = 1, 2$ is retained for $p \in \mathrm{N}$, $p < \infty$ and for $\{\delta(t)\}$ with small autocovariance over time.

It may therefore be surmised that the conclusion (obtained from the previous sections and from Haberman's (1994) numerical analysis) concerning the existence of an efficient range of *spreading* periods, when rates of return are AR(1) and are weakly autocorrelated, holds generally for $p \in \mathrm{N}$, $p < \infty$. (In addition, it would be expected that m_s^* reduces as the variance of the rate of return increases and as the rate of return over time is more positively correlated.)

This simple analysis lends further support to Results S-1, S-2 and S-3. There is a strong indication that they hold when logarithmic rates of return on pension plan assets are weakly autocorrelated AR(p) (for finite integral p) processes. Further work is required to establish if the corresponding results hold when gains and losses are amortized.

4 Asset-Liability Modeling

In this section, a number of assumptions are relaxed.

1. Pensions in payment are not indexed.
2. Stochastic inflation on prices and wages is assumed and pensions are a fraction of final salary.
3. Two asset classes (equity and long-term Government debt) are considered. A proportional rebalancing strategy between the two asset types is assumed, with income from an asset being reinvested in that asset.

The asset-liability projection basis that is used is described by Wilkie (1995), with appropriate parameters for three countries. The statistical methodology, historical data and economic theory employed by Wilkie (1995) are exhaustively discussed in the actuarial literature (see Wilkie (1995) for relevant references). This projection basis is used because it gives a fairly realistic *indication* of the relationship between various economic variables that are relevant to pension funding.

The projections described in this section incorporate inflation explicitly and pensions are not assumed to be indexed with inflation. This is more realistic than considering *real* rates of return, as was done in the previous section. Maynard (1992, p. 245) shows that nominal rates of return have been more volatile than rates of return net of price inflation and that this adds to the volatility in the funding of benefits that are not indexed with inflation. The distinction between inflation on wages and on prices, which is important for final-salary pensions, is also made in our projections. Stochastic projections are carried out 2000 times over 300 years and the standard deviations of the funding level and contribution rate are calculated.

Wilkie's (1995) economic time series are essentially linear and autoregressive. Similar results to those of section 3 may therefore be anticipated. The feature of an efficient range of spreading periods is indeed reproduced by Haberman & Smith (1997) through simulations of Wilkie's (1995) time series based on U.K. economic data. They observe numerically that contribution rate variability decreases and then increases as gains and losses are spread over longer periods.

In all the projections below, assets are valued at market and the Projected Unit Credit method with a time-invariant valuation basis is used to value the liabilities. Contributions are calculated so as to be a level percentage of payroll. The relevant parameters of the Wilkie (1995) series are listed in Appendix C.1.

4.1 U.K. Projections

The projection basis described by Wilkie (1995) is originally designed and parameterized for U.K. economic data. Some important features of our simulation study are listed hereunder:

Economic projection assumptions. The pension fund is invested in two asset classes: U.K. equities & irredeemable Government bonds (or "gilt-edged securities"). Payroll increases in line with economic wage inflation every year.

Economic valuation assumptions. Wage inflation is assumed at 6.5% and a real (net of wage inflation) discount rate of 4.5% is assumed.

Demographic projection assumptions. Mortality follows English Life Table No. 14 for males. There is no early retirement and no salary scale.

Demographic valuation assumptions. Demographic valuation and projection assumptions are identical. Demographic experience does not deviate from the actuarial valuation basis and gains and losses emerge only as a result of unforeseen economic experience.

The standard deviations of the funding level and contribution rate at the time horizon of the simulation study are shown in Tables D.11 and D.12 for a 60:40 and a 80:20 equity:bond portfolio respectively. These portfolios are typical in practice and no suggestion as to their being optimal or otherwise is being made.

Variance of the Funding Level (Results S-1 and A-1). The numerical data in these tables satisfy both Results S-1 and A-1. Funding levels become more variable as m increases. Amortizing gains and losses over any given period m leads to lower variability in the funding level than if they were spread over the same m.

Variance of the Contribution Rate (Results S-2 and A-2). Results S-2 and A-2 are also observed to hold. As spreading and amortization periods increase, contribution rates become more stable, and then gradually less stable. This is shown in Figure 8 for the 80:20 equity:bond portfolio.

Spread and Amortization Periods (Results S-3 and A-3). As gains and losses are spread over a longer period, more stable recommended contribution rates and more variable funding levels are obtained. But beyond a certain spreading period, both contribution rates and funding levels become more variable. The same is true of amortization periods. Efficient ranges of spread and amortization periods, in which contribution stability and funding security may be traded off each other, therefore exist. This agrees with Results S-3 and A-3.

Efficiency (Result A-4). The variance of the contribution rate is plotted against the variance of the funding level in Figure 9 for both 60:40 and 80:20 equity:bond portfolios. For both portfolios, it is better to recommend contributions by spreading gains and losses than by amortizing them over a fixed term. Result A-4 appears to hold.

4.2 U.S. Projections

The projection basis described by Wilkie (1995) is also parameterized for U.S. economic data, but does not include a wage inflation model. Some of the important assumptions made in the projections are listed hereunder:

Economic projection assumptions. The pension fund is invested in two asset classes: U.S. equities & long Treasury bonds. Payroll increases at a constant 4.5% every year.

Economic valuation assumptions. Wage inflation is assumed at 4.5% and a real (net of wage inflation) discount rate of 4.5% is assumed.

Demographic projection assumptions. Mortality follows the 1983 Group Annuitant Mortality table for males. There is no early retirement and no salary scale.

Demographic valuation assumptions. As demographic projection assumptions. Demographic experience does not deviate from the actuarial valuation basis and gains and losses emerge only as a result of random asset returns.

The standard deviations of the funding level and contribution rate at the time horizon of the simulation study are shown in Table D.13 for a 60:40 equity:bond portfolio. McGill *et al.* (1996, p. 665) state that "a 60-40 split between equities and fixed [income securities] is common" in North America. Results S-1–A-3 are evidently borne out. The minima at $m_s^* \approx 7$ and $m_a^* \approx 13$ are shown in Figure 10. This suggests that gains and losses should be spread over periods no longer than 7 years and should be amortized over no more than 13 years. The gain/loss amortization period range of up to 5 years prescribed by the Employee Retirement Income Security Act, 1974, for single-employer pension plans is well within our suggested range, but the range of gain/loss amortization periods for multi-employer plans under ERISA is between 1 and 15 years (source: McGill *et al.*, 1996, p. 597).

Figure 11 shows that it is more efficient to recommend contribution rates by spreading rather than by amortizing gains and losses. Result A-4 appears to hold again. For the most stable funding levels and contribution rates, gains and losses should be paid off by being spread rather than directly amortized. Statutory and regulatory requirements must of course be given consideration in practice.

4.3 Canada Projections

The projection basis described by Wilkie (1995) is also parameterized for Canadian economic data, but does not comprise a wage inflation model. The model of Sharp (1993) for Canadian wage inflation is used (see Appendix C.2). Some important features of the simulation study are listed below:

Economic projection assumptions. The pension fund is invested in two asset classes: Canadian equities & Canada long bonds. Payroll increases in line with economic wage inflation every year.

Economic valuation assumptions. Wage inflation is assumed at 4.5% and a real (net of wage inflation) discount rate of 4.5% is assumed.

Demographic projection assumptions. Mortality follows the 1983 Group Annuitant Mortality table for males. There is no early retirement and no salary scale.

Demographic valuation assumptions. As demographic projection assumptions. Demographic experience does not deviate from the actuarial valuation basis and gains and losses emerge only as a result of unforeseen economic experience.

The variances of the funding level and contribution rate are listed in Tables D.14 and D.15 for a 60:40 and a 40:60 equity:bond portfolio respectively and displayed in Figure 12. Results S-1–A-4 appear to hold.

5 Conclusion

The funding of final-salary defined benefit pension plans was considered and methods of amortizing actuarial gains and losses arising from unforeseen economic experience were investigated within simple models. Dufresne (1988) and Owadally & Haberman (1999) show that there exist efficient periods over which to spread or amortize gains and losses Spreading the gains and losses by paying a proportion of the current unfunded liability appears to yield less volatile funding levels and more stable contribution rates than amortizing past and present gains/losses. This might be explained by the fact that the optimal contribution, based on a quadratic utility function and irrespective of the optimization period, for a pension fund invested in one risk-free and one random risky asset resembles the contribution calculated when gains and losses are spread. Both are a function of the current level of funding rather than the past and present gains or losses.

These results depend on the assumption that rates of return on the fund are independent from year to year. Stochastic projections simulating Gaussian AR(1) and MA(1) logarithmic rates of return indicated that the results of Dufresne (1988) and Owadally & Haberman (1999) are robust when rates of return are moderately autocorrelated. An approximate analysis for more general and weakly autocorrelated AR(p) logarithmic rates of return allowed us to extrapolate and seemed to show that Dufresne's (1988) conclusion holds. The uncertain nature of final-salary pension liabilities was ignored in the foregoing, either by assuming that pension liabilities (active and retired) increase in line with economic wage inflation or by disregarding inflation altogether. Simulations of pension plan assets and liabilities based on published actuarial stochastic time series models of equities, long-term Government debt and wage inflation in three separate jurisdictions and based on typical asset portfolios appeared to support our hypotheses. Gains and losses ought to be amortized over no more than 13 years, but more stable funding levels and contribution rates emerge if gains and losses are spread over suggested periods of no more than 7 years.

Possible areas for further work include: investigating logarithmic rates of return that are more general MA(q) processes, shown by Frees (1990) to be fairly mathematically tractable; considering a more realistic utility function incorporating solvency and full funding constraints; modeling other asset classes, particularly shorter-term and inflation-indexed bonds, and using asset allocation strategies other than proportional rebalancing (e.g. constant proportion portfolio insurance); valuing plan assets at an average of market values rather than pure market value; introducing a dynamic valuation basis possibly based on corporate bond yields; and investigating efficient amortization mechanisms for expensing purposes.

References

BELLHOUSE, D.R. AND PANJER, H.H. 1981. "Stochastic Modelling of Interest Rates with Applications to Life Contingencies—Part II," *Journal of Risk and Insurance* 48: 628–637.

BERTSEKAS, D.P. 1976. *Dynamic Programming and Stochastic Control.* New York: Academic Press.

BLACK, F. AND JONES, R. 1988. "Simplifying portfolio insurance for corporate pension plans," *Journal of Portfolio Management* 14(4): 33–37.

BOULIER, J.-F., TRUSSANT, E. AND FLORENS, D. 1995. "A Dynamic Model for Pension Funds Management," *Proceedings of the 5th AFIR International Colloquium*, Leuven, Belgium, 1: 361–384.

BOWERS, N.L., HICKMAN, J.C. AND NESBITT, C.J. 1979. "The Dynamics of Pension Funding: Contribution Theory," *Transactions of the Society of Actuaries* 31: 93–122.

BOYLE, P.P. 1976. "Rates of Return as Random Variables," *Journal of Risk and Insurance* 43: 693–711.

CAIRNS, A.J.G. 1997. "A Comparison of Optimal and Dynamic Control Strategies for Continuous-time Pension Fund Models," *Proceedings of the 7th AFIR International Colloquium*, Cairns, Australia, 1: 309–326.

COLBRAN, R.B. 1982. "Valuation of Final-salary Pension Schemes," *Journal of the Institute of Actuaries* 109: 359–385.

DHAENE, J. 1989. "Stochastic Interest Rates and Autoregressive Integrated Moving Average Processes," *ASTIN Bulletin* 19: 131–138.

DUFRESNE, D. 1988. "Moments of Pension Contributions and Fund Levels when Rates of Return are Random," *Journal of the Institute of Actuaries* 115: 535–544.

DUFRESNE, D. 1989. "Stability of Pension Systems when Rates of Return are Random," *Insurance: Mathematics and Economics* 8: 71–76.

DUFRESNE, D. 1994. *Mathématiques des Caisses de Retraite.* Montréal, Canada: Éditions Supremum.

EXLEY, C.J., MEHTA, S.J.B. AND SMITH, A.D. 1997. "The Financial Theory of Defined Benefit Pension Schemes," *British Actuarial Journal* 3: 835–966.

FAMA, E.F. AND FRENCH, K.R. 1988. "Dividend Yields and Expected Stock Returns," *Journal of Financial Economics* 22: 3–25.

FREES, E.W. 1990. "Stochastic Life Contingencies with Solvency Considerations," *Transactions of the Society of Actuaries* 42: 91–148.

HABERMAN, S. 1994. "Autoregressive Rates of Return and the Variability of Pension Contributions and Fund Levels for a Defined Benefit Pension Scheme," *Insurance: Mathematics and Economics* 14: 219–240.

HABERMAN, S. AND SMITH, D. 1997. "Stochastic Investment Modelling and Pension Funding: a Simulation Based Analysis," Actuarial Research Paper No. 102. Department of Actuarial Science and Statistics, The City University, London, England.

HABERMAN, S. AND SUNG, J.-H. 1994. "Dynamic Approaches to Pension Funding," *Insurance: Mathematics and Economics* 15: 151–162.

HABERMAN, S. AND WONG, L.Y.P. 1997. "Moving Average Rates of Return and the Variability of Pension Contributions and Fund Levels for a Defined Benefit Scheme," *Insurance: Mathematics and Economics* 20: 115–135.

KRYVICKY, R.C. 1981. "The Funding of Negotiated Pension Plans," *Transactions of the Society of Actuaries* 33: 405–472.

MAYNARD, J.C. 1992. "Financing Defined-Benefit Pension Plans with Indexed Benefits," *Transactions of the Society of Actuaries* 44: 193–246.

MCGILL, D.M., BROWN, K.N., HALEY, J.J. AND SCHIEBER, S.J. 1996. *Fundamentals of Private Pensions,* 7th ed. Philadelphia, Pennsylvania: University of Pennsylvania Press.

O'BRIEN, T. 1987. "A Two-Parameter Family of Pension Contribution Functions and Stochastic Optimization," *Insurance: Mathematics and Economics* 6: 129–134.

OWADALLY, M.I. AND HABERMAN, S. 1999. "Pension Fund Dynamics and Gains/Losses Due to Random Rates of Investment Return," *North American Actuarial Journal* 3 (3): 105–117.

PANJER, H.H. AND BELLHOUSE, D.R. 1980. "Stochastic Modelling of Interest Rates with Applications to Life Contingencies," *Journal of Risk and Insurance* 47: 91–110.

POLLARD, J.H. 1971. "On Fluctuating Interest Rates," *Bulletin de l'Association Royale des Actuaires Belges* 66: 68–97.

SHARP, K.P. 1993. "Modeling Canadian Price and Wage Inflation," *Proceedings of the Canadian Institute of Actuaries* 24: 565–587.

THORNTON, P.N. AND WILSON, A.F. 1992. "A Realistic Approach to Pension Funding," *Journal of the Institute of Actuaries* 119: 229–312.

TROWBRIDGE, C.L. 1952. "Fundamentals of Pension Funding," *Transactions of the Society of Actuaries* 4: 17–43.

TROWBRIDGE, C.L. AND FARR, C.E. 1976. *The Theory and Practice of Pension Funding.* Homewood, Illinois: Richard D. Irwin.

VANDERHOOF, I.T. 1973. "Choice and Justification of an Interest Rate," *Transactions of the Society of Actuaries* 25: 417–458.

WILKIE, A.D. 1995. "More on a Stochastic Asset Model for Actuarial Use," *British Actuarial Journal* 1: 777–964.

WINKLEVOSS, H.E. 1982. "PLASM: Pension Liability and Asset Simulation Model," *Journal of Finance* 37: 585–594.

Appendix A

Solution of Bellman Equation and Proof of Proposition 1

Finite-horizon case. Consider first a finite horizon. Let

$$\Phi_t = f(t) + c(t) - B, \tag{38}$$

$$\Psi_t = 1 + r + y(t)\alpha, \tag{39}$$

and note that

$$y(t)^2 = \alpha^{-2}\Psi_t^2 - 2\alpha^{-2}(1+r)\Psi_t + \alpha^{-2}(1+r)^2, \tag{40}$$

$$(c(t) - CT_t)^2 = \Phi_t^2 - 2(f(t) - B + CT_t)\Phi_t + (f(t) - B + CT_t)^2. \tag{41}$$

Since $\alpha(t)$ is independent and identically distributed over time, it follows from equation (7) that

$$\mathrm{E}[f(t+1)|f(t)] = \Psi_t\Phi_t, \tag{42}$$

$$\mathrm{Var}[f(t+1)|f(t)] = \sigma^2\Phi_t^2 y(t)^2, \tag{43}$$

and using equation (40)

$$\mathrm{E}[f(t+1)^2|f(t)] = (1+\sigma^2\alpha^{-2})\Phi_t^2\Psi_t^2 - 2\sigma^2\alpha^{-2}(1+r)\Phi_t^2\Psi_t + \sigma^2\alpha^{-2}(1+r)^2\Phi_t^2. \tag{44}$$

The Bellman optimality equation (equation (11)) is

$$J(f(t), t) = \min_{c(t), y(t)} J, \tag{45}$$

where

$$J = \theta_1(f(t) - FT_t)^2 + \theta_2(c(t) - CT_t)^2 + \beta\mathrm{E}[J(f(t+1), t+1)|f(t)], \tag{46}$$

with boundary condition, at time $t = N$,

$$J(f(N), N) = \theta_0(f(N) - FT_N)^2. \tag{47}$$

A trial solution for equation (45) is

$$J(f(t),t) = P_t f(t)^2 - 2Q_t f(t) + R_t. \tag{48}$$

The boundary condition in equation (47) certainly satisfies the trial solution, with $P_N = \theta_0$ and $Q_N = \theta_0 F T_N$.

Proceeding by induction, suppose that the right hand side of equation (48) is a solution of the Bellman equation (45) at $t+1$. Then,

$$\begin{aligned}
&\mathrm{E}[J(f(t+1),t+1)|f(t)] \\
&\qquad = P_{t+1}\mathrm{E}[f(t+1)^2|f(t)] - 2Q_{t+1}\mathrm{E}[f(t+1)|f(t)] + R_{t+1} \\
&\qquad = (1+\sigma^2\alpha^{-2})P_{t+1}\Phi_t^2\Psi_t^2 - 2[Q_{t+1}\Phi_t + \sigma^2\alpha^{-2}(1+r)P_{t+1}\Phi_t^2]\Psi_t \\
&\qquad\quad + \sigma^2\alpha^{-2}(1+r)^2 P_{t+1}\Phi_t^2 + R_{t+1}.
\end{aligned} \tag{49}$$

where use is made of equations (42) and (44).

J may be written as a quadratic expression in Ψ_t, Φ_t and $f(t)$, by substituting equations (41) and (49) into equation (46):

$$\begin{aligned}
J = {} & [\beta(1+\sigma^2\alpha^{-2})P_{t+1}\Phi_t^2]\Psi_t^2 - 2\beta[Q_{t+1}\Phi_t + \sigma^2\alpha^{-2}(1+r)P_{t+1}\Phi_t^2]\Psi_t \\
& + [\theta_2 + \beta\sigma^2\alpha^{-2}(1+r)^2 P_{t+1}]\Phi_t^2 - 2\theta_2(f(t) - B + CT_t)\Phi_t \\
& + \beta R_{t+1} + \theta_2(f(t) - B + CT_t)^2 + \theta_1(f(t) - FT_t)^2.
\end{aligned} \tag{50}$$

Upon completing the squares in Ψ_t and Φ_t,

$$J = \Psi_t^A(\Psi_t - \Psi_t^B)^2 + \Phi_t^A(\Phi_t - \Phi_t^B)^2 + P_t f(t)^2 - 2Q_t f(t) + R_t, \tag{51}$$

where P_t and Q_t are as in section 2.3, R_t represents additional terms independent of $f(t)$, and

$$\Psi_t^A = \beta(1+\sigma^2\alpha^{-2})P_{t+1}\Phi_t^2, \tag{52}$$

$$\Psi_t^B = \frac{\sigma^2\alpha^{-2}(1+r)P_{t+1}\Phi_t + Q_{t+1}}{(1+\sigma^2\alpha^{-2})P_{t+1}\Phi_t}, \tag{53}$$

$$\Phi_t^A = \tilde{P}_{t+1}^{-1}(\alpha^2+\sigma^2)^{-1}, \tag{54}$$

$$\Phi_t^B = \theta_2(\alpha^2+\sigma^2)\tilde{P}_{t+1}(f(t) - B + CT_t) + \beta\sigma^2(1+r)\tilde{P}_{t+1}Q_{t+1}], \tag{55}$$

$$\tilde{P}_{t+1} = [\theta_2(\alpha^2+\sigma^2) + \beta\sigma^2(1+r)^2P_{t+1}]^{-1}. \tag{56}$$

J has a unique minimum in Ψ_t and Φ_t provided $\Psi_t^A > 0$ and $\Phi_t^A > 0$. It is sufficient that $P_t > 0$ for $t \in [1, N]$ for both these conditions to be satisfied (since $P_t > 0 \Rightarrow \tilde{P}_t > 0$). (It is assumed that the plan is partially funded and invests in the two assets at all times and $\Phi_t = f(t) + c(t) - B > 0$.) The minimum occurs when $\Psi_t = \Psi_t^B$ and $\Phi_t = \Phi_t^B$ simultaneously. There is a direct linear relationship between $c(t)$ and Φ_t (equation (38)) and between $y(t)$ and Ψ_t ($\alpha > 0$ in equation (39)). Therefore, $\min_{c(t),y(t)} J = P_t f(t)^2 - 2Q_t f(t) + R_t$ which is in the form postulated in the trial solution (48). Since the solution holds for $t = N$, it holds for $t \in [1, N]$. Since $P_N = \theta_0 > 0$, then $P_t > 0$ for $t \in [1, N]$ and the sufficient condition for the existence of a single minimum is satisfied.

Let $\Theta_t = \theta_2(\alpha^2+\sigma^2)\tilde{P}_t$. Then, $1 - \Theta_{t+1} = \beta\sigma^2(1+r)^2\tilde{P}_{t+1}P_{t+1}$, from equation (56).

$$\Phi_t^* = \Phi_t^B = \Theta_{t+1}(f(t) + CT_t - B) + (1 - \Theta_{t+1})Q_{t+1}P_{t+1}^{-1}(1+r)^{-1}. \tag{57}$$

Using equation (38), $c^*(t) = \Phi_t^* + B - f(t)$ is readily obtained. Replacing $\Phi_t = \Phi_t^*$ in the right hand side of equation $\Psi_t = \Psi_t^B$ gives

$$\Psi_t^* = \frac{Q_{t+1} + \sigma^2\alpha^{-2}(1+r)P_{t+1}\Phi_t^*}{(1+\sigma^2\alpha^{-2})P_{t+1}\Phi_t^*}. \tag{58}$$

Now, $[1 + r + \alpha y^*(t)][f(t) + c^*(t) - B] = \Psi_t^*\Phi_t^* = [Q_{t+1} + \sigma^2\alpha^{-2}(1+r)P_{t+1}(f(t) + c^*(t) - B)](1+\sigma^2\alpha^{-2})^{-1}P_{t+1}^{-1}$, from which equation (20) follows.

It is also possible to express $y^*(t)$ independently of $c^*(t)$ by using equations (39) and (58), giving

$$\alpha y^*(t) = \Psi_t^* - (1+r) = \frac{Q_{t+1} - (1+r)P_{t+1}\Phi_t^*}{(1+\sigma^2\alpha^{-2})P_{t+1}\Phi_t^*}, \tag{59}$$

which, upon substitution of Φ_t^* from equation (57), readily yields

$$y^*(t) = \frac{\alpha\Theta_{t+1}(1+r)[Q_{t+1} - P_{t+1}(1+r)(f(t)+CT_t-B)]}{(\alpha^2+\sigma^2)[(1-\Theta_{t+1})Q_{t+1} + \Theta_{t+1}P_{t+1}(1+r)(f(t)+CT_t-B)]}. \tag{60}$$

Infinite-horizon case. The proof for the infinite-horizon case follows closely the preceding proof for the finite-horizon case and is only sketched.

CT_t and FT_t are constant and may be replaced by CT and FT. The value function does not depend directly on time and $J(f(t),t)$ and $J(f(t+1),t+1)$ may be written as $J(f(t))$ and $J(f(t+1))$. The optimality equation has no boundary condition and backwards induction is not necessary. Since the optimality equation does converge, its solution in the infinite-horizon case is the steady-state or equilibrium solution of the finite-horizon case. A suitable trial solution is $J(f(t)) = Pf(t)^2 - 2Qf(t) + R$, where $P_t \to P$, $Q_t \to Q$ and $R_t \to R$ as $t \to \infty$. All P_t, Q_t and R_t in the preceding proof may be replaced by constant P, Q and R. $\mathrm{E}[J(f(t+1))|f(t),c(t),y(t)]$ and J are as in equations (49) and (50), with the appropriate constant terms described above. J may be simplified to a quadratic in Ψ_t and Φ_t. It turns out that J may be minimized, leaving a quadratic in $f(t)$, thereby confirming the trial solution.

$$\begin{aligned} J &= \Psi^A(\Psi_t - \Psi^B)^2 + \Phi^A(\Phi_t - \Phi^B)^2 + f(t)^2[\theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}P] \\ &\quad - 2f(t)[\theta_1 FT + \theta_2\beta\sigma^2(1+r)P(Q-(1+r)P(CT-B))] + R, \end{aligned} \tag{61}$$

where

$$\tilde{P} = [\theta_2(\alpha^2 + \sigma^2) + \beta\sigma^2(1+r)^2 P]^{-1}, \tag{62}$$

$$\Psi^A = \beta(1 + \sigma^2\alpha^{-2})P\Phi_t^2, \tag{63}$$

$$\Psi^B = \frac{\sigma^2\alpha^{-2}(1+r)P\Phi_t + Q}{(1+\sigma^2\alpha^{-2})P\Phi_t}, \tag{64}$$

$$\Phi^A = \tilde{P}^{-1}(\alpha^2 + \sigma^2)^{-1}, \tag{65}$$

$$\Phi^B = \theta_2(\alpha^2 + \sigma^2)\tilde{P}(f(t) - B + CT) + \beta\sigma^2(1+r)\tilde{P}Q], \tag{66}$$

and R includes additional terms independent of $f(t)$ and not required here.

By comparing the coefficients of $f(t)^2$ and $f(t)$ in equation (61) with those in the trial solution, it is clear that P satisfies $P = \theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}P$, which may be rewritten as equation (17), while Q satisfies $Q = \theta_1 FT + \theta_2\beta\sigma^2(1+r)\tilde{P}[Q - (1+r)P(CT - B)]$. This may be simplified into the form given in equation (18) by noting that $\theta_2\beta\sigma^2(1+r)\tilde{P} = (P - \theta_1)/(P + Pr)$ given that $P = \theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}P$.

J in equation (61) has a unique minimum in Ψ_t and Φ_t provided that $\Psi^A > 0$ and $\Phi^A > 0$. It is sufficient that $P > 0$ for both these conditions to hold. (The plan is assumed to be partially funded at all times and $\Phi_t > 0$.) Since the coefficient of P^2 and the constant term in the quadratic equation (17) are positive and negative respectively, P must have one negative real root and one positive real root, the latter being the admissible solution. Then, $\tilde{P} > 0$ in equation (62), and $P > \theta_1$ since $P = \theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}P$. J is minimized when $\Psi_t = \Psi^A$ and $\Phi_t = \Phi^A$ simultaneously and, exploiting again the direct linear relationship between Φ_t and $c(t)$ and between Ψ_t and $y(t)$, the minimizing values of $c(t)$ and $y(t)$, denoted respectively by $c^*_\infty(t)$ in equation (21) and by $y^*_\infty(t)$ in equation (22), may be found as in the finite-horizon case. Again, $y^*_\infty(t)$ may also be written as

$$y^*_\infty(t) = \frac{\alpha\Theta(1+r)[Q - P(1+r)(f(t) + CT - B)]}{(\alpha^2 + \sigma^2)[(1-\Theta)Q + \Theta P(1+r)(f(t) + CT - B)]}. \tag{67}$$

Appendix B

Proof of Proposition 2

Preliminaries. It is required first to find the moments of the accumulation of 1 when the logarithmic rate of interest $\delta(t)$ follows an AR(p) process. Such expressions have been obtained by Boyle (1976) for independent and identically normally distributed $\{\delta(t)\}$, by Panjer & Bellhouse (1980) for stationary AR(1) and AR(2) processes, and by Bellhouse & Panjer (1981) for conditional AR(1) and AR(2) processes.

Consider the variance of a sum of $\delta(t)$ over a term $t-s$.

$$\begin{aligned}\operatorname{Var}\left[\sum_{u=s+1}^{t}\delta(u)\right] &= 2\sum_{u=s+1}^{t}\sum_{w=s+1}^{u}\operatorname{Cov}[\delta(u),\delta(w)] - \sum_{u=s+1}^{t}\operatorname{Var}\delta(u)\\ &= 2\sigma^2\sum_i A_i\sum_{u=s+1}^{t}\sum_{w=s+1}^{u}G_i^{u-w} - \sigma^2(t-s), \qquad (68)\end{aligned}$$

after substituting equation (26). This may be expanded to

$$\begin{aligned}&\sum_i A_i\left[\frac{2\sigma^2}{1-G_i}(t-s)+\frac{2\sigma^2G_i}{(1-G_i)^2}G_i^{t-s}-\frac{2\sigma^2G_i}{(1-G_i)^2}\right]-\sigma^2(t-s)\\ &\qquad=\sum_i\frac{1+G_i}{1-G_i}A_i\sigma^2(t-s)+\sum_i\frac{2\sigma^2G_iA_i}{(1-G_i)^2}G_i^{t-s}-\sum_i\frac{2\sigma^2G_iA_i}{(1-G_i)^2}, \qquad (69)\end{aligned}$$

where use is made of the fact that $\sum_i A_i = 1$.

The covariance of sums of $\delta(t)$ over different terms may also be found.

$$\operatorname{Cov}\left[\sum_{u=s+1}^{t}\delta(u),\ \sum_{w=\tau+1}^{t}\delta(w)\right]=\operatorname{Var}\left[\sum_{u=s+1}^{t}\delta(u)\right]+\operatorname{Cov}\left[\sum_{u=s+1}^{t}\delta(u),\ \sum_{w=\tau+1}^{s}\delta(w)\right], \qquad (70)$$

$(s>\tau)$ where the second term on the right hand side is

$$\begin{aligned}\sum_{u=s+1}^{t}\sum_{w=\tau+1}^{s}\operatorname{Cov}[\delta(u),\delta(w)] &= \sigma^2\sum_i A_i\sum_{u=s+1}^{t}\sum_{w=\tau+1}^{s}G_i^{u-w}\\ &=\sum_i\frac{\sigma^2A_iG_i}{(1-G_i)^2}[1-G_i^{t-s}-G_i^{s-\tau}+G_i^{t-\tau}], \qquad (71)\end{aligned}$$

where equation (26) has again been used.

Therefore,

$$\begin{aligned}
&\mathrm{Var}\left[\sum_{u=s+1}^{t}\delta(u)+\sum_{w=\tau+1}^{t}\delta(w)\right]\\
&=\mathrm{Var}\left[\sum_{u=s+1}^{t}\delta(u)\right]+\mathrm{Var}\left[\sum_{w=\tau+1}^{t}\delta(w)\right]+2\mathrm{Cov}\left[\sum_{u=s+1}^{t}\delta(u),\ \sum_{w=\tau+1}^{t}\delta(w)\right]\\
&=3\mathrm{Var}\left[\sum_{u=s+1}^{t}\delta(u)\right]+\mathrm{Var}\left[\sum_{w=\tau+1}^{t}\delta(w)\right]+2\mathrm{Cov}\left[\sum_{u=s+1}^{t}\delta(u),\ \sum_{w=\tau+1}^{s}\delta(w)\right] \qquad (72)
\end{aligned}$$

and using equations (69) and (71), this may be simplified to

$$\sigma^2[3(t-s)+(t-\tau)]\sum_{i=1}^{p}\frac{1+G_i}{1-G_i}A_i-\sum_{i=1}^{p}\frac{2\sigma^2 A_i G_i}{(1-G_i)^2}[3-2G_i^{t-s}+G_i^{s-\tau}-2G_i^{t-\tau}]. \qquad (73)$$

$\{\delta(t)\}$ is a Gaussian process. Since

$$\mathrm{E}\left[\sum_{u=s+1}^{t}\delta(u)+\sum_{w=\tau+1}^{t}\delta(w)\right]=(t-s)\delta+(t-\tau)\delta, \qquad (74)$$

and given the variance in equation (73), we can write

$$\begin{aligned}
&\mathrm{E}\exp\left[\sum_{u=s+1}^{t}\delta(u)+\sum_{w=\tau+1}^{t}\delta(w)\right]\\
&\qquad=c^{t-\tau}w^{t-s}e^{-3\sum_i z_i}\exp\left\{\sum_i\left[2z_iG_i^{t-s}+2z_iG_i^{t-\tau}-z_iG_i^{s-\tau}\right]\right\}, \qquad (75)
\end{aligned}$$

where c, w and z_i are as in equations (31), (32) and (33) respectively.

Likewise,

$$\mathrm{E}\exp\left[\sum_{u=s+1}^{t}\delta(u)\right]=c^{t-s}e^{-\sum_i z_i}\exp\left[\sum_i z_iG_i^{t-s}\right], \qquad (76)$$

$$\mathrm{E}\exp\left[2\sum_{u=s+1}^{t}\delta(u)\right]=(cw)^{t-s}e^{-4\sum_i z_i}\exp\left[4\sum_i z_iG_i^{t-s}\right]. \qquad (77)$$

First Moments. Applying equation (76) to equation (30),

$$\mathrm{E}f(t)=f_0(Qc)^te^{-\sum_i z_i}\exp\left[\sum_i z_iG_i^t\right]+RQ^{-1}e^{-\sum_i z_i}\sum_{s=0}^{t-1}(Qc)^{t-s}\exp\left[\sum_i z_iG_i^{t-s}\right]. \qquad (78)$$

Given that $\delta(t)$ is stationary and therefore inequality (28) holds, it follows that $\exp(\sum_i z_i G_i^s) \leq \exp(\sum_i |z_i G_i|)$ for $s \in \mathrm{N}$.

$$\mathrm{E}f(t) \leq f_0(Qc)^t e^{-\sum_i z_i} \exp\left[\sum_i |z_i G_i|\right] + RQ^{-1}e^{-\sum_i z_i} \exp\left[\sum_i |z_i G_i|\right] \sum_{s=0}^{t-1}(Qc)^{t-s} \quad (79)$$

and the right hand side of the above is convergent as $t \to \infty$, provided that $|Qc| < 1$. $\mathrm{E}f(t)$ is bounded above and since all terms in equation (78) are non-negative, $\mathrm{E}f(t)$ is convergent as $t \to \infty$.

The first order approximation to $\lim \mathrm{E}f(t)$ is obtained by replacing $\exp(\sum_i z_i G_i^{t-s})$ by $1 + \sum_i z_i G_i^{t-s}$ in equation (78). Provided $|Qc| < 1$ (and given $|G_i| < 1$), it follows that

$$\lim_{t\to\infty} \mathrm{E}f(t) \approx e^{-\sum_i z_i} Rc \left[(1 - Qc)^{-1} + \sum_{i=1}^{p} z_i G_i (1 - QcG_i)^{-1}\right], \quad (80)$$

which may be further approximated as in equation (34). This is a reasonable approximation when the terms $z_i G_i$ are small in magnitude, i.e. when $|A_i G_i|$ are small, which implies that $\{\delta(t)\}$ exhibits weak autocorrelation (equation (26)). The first moment of $c(t)$ is straightforward given equations (1) and (3).

Second Moments. When both sides of equation (30) are squared and expanded and expectation is taken,

$$\begin{aligned}\mathrm{E}f(t)^2 = {} & f_0^2 Q^{2t} \mathrm{E}\exp\left[2\sum_{u=1}^{t}\delta(u)\right] + 2f_0 RQ^{t-1}\sum_{s=0}^{t-1} Q^{t-s}\mathrm{E}\exp\left[\sum_{u=1}^{t}\delta(u) + \sum_{w=s+1}^{t}\delta(w)\right] \\ & + 2R^2Q^{-2}\sum_{s=1}^{t-1}\sum_{\tau=0}^{s-1} Q^{t-s}Q^{t-\tau}\mathrm{E}\exp\left[\sum_{u=s+1}^{t}\delta(u) + \sum_{w=\tau+1}^{t}\delta(w)\right] \\ & + R^2Q^{-2}\sum_{s=0}^{t-1} Q^{2(t-s)}\mathrm{E}\exp\left[2\sum_{u=s+1}^{t}\delta(u)\right] \quad (81)\end{aligned}$$

The third term on the right hand side of equation (81) may be simplified, upon application of equation (75), to

$$2R^2Q^{-2}e^{-3\sum_i z_i}\sum_{s=1}^{t-1}\sum_{\tau=0}^{s-1}(Q^2cw)^{t-s}(Qc)^{s-\tau}\exp\left\{\sum_i\left[2z_iG_i^{t-s} + 2z_iG_i^{t-\tau} - z_iG_i^{s-\tau}\right]\right\}. \quad (82)$$

Again, $\delta(t)$ is stationary and equation (28) holds. Consequently,

$$\exp\left(\sum_i 2z_i G_i^s\right) \le \exp\left(\sum_i 2|z_i G_i|\right), \tag{83}$$

$$\exp\left(-\sum_i z_i G_i^s\right) \le \exp\left(\sum_i |z_i G_i|\right) \tag{84}$$

for $s \in \mathrm{N}$. Therefore, the third term on the right hand side of equation (81) is bounded above by

$$2R^2Q^{-2}e^{-3\sum_i z_i}\sum_{s=1}^{t-1}\sum_{\tau=0}^{s-1}(Q^2cw)^{t-s}(Qc)^{s-\tau}\exp\left[\sum_i 5|z_iG_i|\right], \tag{85}$$

which, as $t \to \infty$, converges provided $|Qc| < 1$ and $Q^2cw < 1$. Since all terms in (82) are non-negative, the third term on the right hand side of equation (81) converges as $t \to \infty$.

To first order, the third term on the right hand side of equation (81) is

$$\begin{aligned}
&2R^2Q^{-2}\sum_{s=1}^{t-1}\sum_{\tau=0}^{s-1}(Qw)^{t-s}(Qc)^{t-\tau}e^{-3\sum_i z_i}\left\{1+\sum_i\left[2z_iG_i^{t-s}+2z_iG_i^{t-\tau}-z_iG_i^{s-\tau}\right]\right\}\\
&=2R^2Q^{-2}e^{-3\sum_i z_i}\left\{\sum_{s=1}^{t-1}\sum_{\tau=0}^{s-1}(Q^2cw)^{t-s}(Qc)^{s-\tau}+\sum_i 2z_i\sum_{s=1}^{t-1}\sum_{\tau=0}^{s-1}(Q^2cwG_i)^{t-s}(Qc)^{s-\tau}\right.\\
&\left.+\sum_i 2z_i\sum_{s=1}^{t-1}\sum_{\tau=0}^{s-1}(Q^2cwG_i)^{t-s}(QcG_i)^{s-\tau}-\sum_i z_i\sum_{s=1}^{t-1}\sum_{\tau=0}^{s-1}(Q^2cw)^{t-s}(QcG_i)^{s-\tau}\right\},
\end{aligned} \tag{86}$$

which converges as $t \to \infty$ to

$$\begin{aligned}
&e^{-3\sum_i z_i}2R^2Qc^2w\left\{(1-Q^2cw)^{-1}(1-Qc)^{-1}+\sum_i z_iG_i\Big[2(1-Q^2cwG_i)^{-1}(1-Qc)^{-1}\right.\\
&\left.+2G_i(1-Q^2cwG_i)^{-1}(1-QcG_i)^{-1}-(1-Q^2cw)^{-1}(1-QcG_i)^{-1}\Big]\right\},
\end{aligned} \tag{87}$$

provided $|Qc| < 1$ and $Q^2cw < 1$. This may be further approximated by

$$e^{-3\sum_i z_i}2R^2Qc^2w(1-Q^2cw)^{-1}(1-Qc)^{-1}. \tag{88}$$

A similar procedure (and substitution of equation (77)) shows that the last term on the right hand side of equation (81) is also bounded above provided $Q^2cw < 1$ and its limit may be approximated by

$$e^{-4\sum_i z_i} R^2 cw \left[(1 - Q^2 cw)^{-1} + \sum_i 4 z_i G_i (1 - Q^2 cw G_i)^{-1} \right] \tag{89}$$

or further by

$$e^{-4\sum_i z_i} R^2 cw (1 - Q^2 cw)^{-1}. \tag{90}$$

The first two terms on the right hand side of equation (81) vanish provided $|Qc| < 1$, $Q^2cw < 1$ as well as $|Q| < 1$. The last condition is redundant since $Q = 1 - 1/\ddot{a}_{\overline{m|}}$. The approximations above are suitable when the terms $|A_iG_i|$ are small, i.e. when the serial correlation of $\{\delta(t)\}$ is small.

Hence, we have shown that $\mathrm{E}f(t)^2$ is convergent as $t \to \infty$ when $\delta(t)$ follows a stationary AR(p) process, $p \in \mathrm{N}$, $p < \infty$, provided $|Qc| < 1$ and $Q^2cw < 1$. Combining expressions (88) and (90) yields equation (36). The numerical analysis of Haberman (1994) shows that this is an accurate approximation for AR(1) and AR(2) processes with moderate autocorrelation. A better approximation is given by adding expressions (87) and (89).

Finally, an approximation to $\lim \mathrm{Var} f(t)$ may be obtained from equations (34) and (36). $\lim \mathrm{Var} c(t) = k^2 \lim \mathrm{Var} f(t)$ given equations (1) and (3).

Appendix C
Economic Time Series

C.1 Parameters in Wilkie's (1995) Time Series

The time series for price and wage inflation, equity dividend growth and yield, and long bond yields are described extensively by Wilkie (1995). The parameters of these series are given in Tables D.16–D.19 where the standard parameter names (as defined by Wilkie, 1995) associated with these economic series have been used. The following parameters were used for U.K. wage inflation: $WMU = 2.14\%$, $WSD = 2.33\%$, $WW1 = 0.6021$, $WW2 = 0.2671$, $WA = 0$. The time series were initialized at their equilibrium values.

C.2 Sharp's (1993) Time Series of Canadian Wage Inflation

Sharp (1993) models the arithmetic (rather than geometric or logarithmic) rate of wage inflation (say $AW(t)$) and finds evidence that the arithmetic rate of wage inflation on Canadian data is crosscorrelated with the arithmetic rate of price inflation. If the geometric rate of price inflation based on Canadian data is $I(t)$, then

$$AW(t) = 0.408(\exp(I(t)) - 1) + WN(t), \tag{91}$$

$$WN(t) = 3.5\% + 0.703(WN(t-1) - 3.5\%) + 1.7\% WZ(t), \tag{92}$$

where $WZ(t)$ is a unit normal variate, independent from year to year.

Appendix D

Tables

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	26.5%	24.3%	46.31%	58.31%
5	34.5%	29.6%	37.95%	47.98%
$m_s^* \approx 10$	54.6%	42.0%	33.65%	39.56%
$m_a^* \approx 15$	79.4%	54.0%	36.43%	37.78%
20	122.9%	67.2%	46.56%	38.50%
25	232.8%	82.2%	78.74%	40.93%

Table D.1: Independent and identically distributed rates of return, $\varphi = \phi = 0$.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	34.6%	30.5%	61.24%	75.83%
$m_s^* \approx 5$	51.0%	41.2%	54.77%	67.08%
$m_a^* \approx 7$	69.3%	52.0%	56.57%	61.24%
10	109.5%	64.8%	67.08%	63.25%
15	273.9%	96.4%	122.47%	70.71%
20	‡	148.3%	‡	89.44%
25	‡	214.5%	‡	111.80%

Table D.2: AR(1) logarithmic rates of return, $\varphi = +0.3$. ‡ indicates divergence.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
2	31.3%	27.4%	80.62%	90.83%
$m_s^* \approx 3$	43.6%	34.6%	77.46%	88.03%
4	57.4%	44.7%	79.06%	86.60%
$m_a^* \approx 5$	74.2%	52.9%	82.16%	85.15%
6	97.5%	61.6%	90.83%	86.60%
7	122.5%	70.7%	104.88%	88.03%
8	167.3%	81.9%	122.47%	94.87%

Table D.3: AR(1) logarithmic rates of return, $\varphi = +0.5$.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
$m^* = 1$	19.1%	19.1%	95.26%	95.26%
2	40.0%	30.3%	102.47%	100.00%
3	79.4%	44.7%	141.42%	110.68%
4	‡	66.3%	‡	136.93%

Table D.4: AR(1) logarithmic rates of return, $\varphi = +0.8$. ‡ indicates divergence.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	24.5%	23.5%	43.01%	54.77%
5	30.7%	27.6%	33.91%	43.87%
$m_s^* \approx 10$	44.7%	37.4%	27.84%	34.28%
15	54.8%	44.7%	28.28%	31.62%
$m_a^* \approx 20$	80.6%	53.9%	30.82%	31.22%
25	104.9%	63.2%	34.64%	32.02%
30	130.4%	72.1%	40.62%	33.17%

Table D.5: AR(1) logarithmic rates of return, $\varphi = -0.1$.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	21.0%	20.5%	36.74%	47.43%
5	24.9%	23.0%	27.39%	36.06%
7	28.5%	25.9%	23.45%	30.41%
10	33.2%	29.3%	20.62%	26.93%
15	40.0%	34.6%	18.71%	23.98%
20	46.9%	38.7%	18.03%	21.79%
25	52.0%	43.6%	18.01%	21.79%

Table D.6: AR(1) logarithmic rates of return, $\varphi = -0.3$.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	24.3%	23.5%	42.44%	54.77%
5	30.6%	27.7%	33.62%	43.59%
10	45.1%	36.7%	27.82%	33.91%
$m_s^* \approx 15$	60.1%	44.6%	27.55%	31.22%
$m_a^* \approx 20$	77.7%	52.0%	29.69%	30.41%
25	102.2%	61.6%	34.51%	31.62%
30	148.1%	71.4%	45.88%	32.40%

Table D.7: MA(1) logarithmic rates of return, $\phi = +0.1$.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	20.1%	20.2%	35.21%	46.64%
5	23.5%	22.6%	25.82%	34.28%
10	30.5%	27.2%	18.80%	24.49%
15	35.7%	31.2%	16.39%	21.79%
20	39.5%	35.0%	15.10%	19.36%
25	42.0%	38.6%	14.18%	19.10%

Table D.8: MA(1) logarithmic rates of return, $\phi = +0.3$.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	32.5%	29.7%	56.87%	72.46%
5	45.9%	38.7%	50.47%	61.24%
$m_s^* \approx 7$	60.8%	46.9%	50.00%	57.01%
$m_a^* \approx 10$	90.1%	56.6%	55.59%	54.77%
12	119.7%	65.6%	64.32%	54.77%
15	209.8%	80.6%	96.18%	57.01%

Table D.9: MA(1) logarithmic rates of return, $\phi = -0.3$.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	35.2%	31.6%	61.62%	79.06%
$m_s^* \approx 5$	51.5%	41.2%	56.57%	67.08%
7	70.8%	52.0%	58.27%	63.25%
$m_a^* \approx 9$	96.9%	60.0%	64.92%	63.22%
10	114.6%	64.8%	70.65%	63.25%
13	219.7%	82.5%	111.39%	65.19%
15	766.2%	94.9%	351.43%	67.08%

Table D.10: MA(1) logarithmic rates of return, $\phi = -0.5$.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	13.6%	13.6%	24.30%	24.30%
3	16.5%	16.4%	10.28%	13.78%
5	19.4%	18.6%	7.58%	10.48%
10	25.3%	22.8%	5.49%	7.29%
15	30.4%	26.4%	4.84%	6.23%
20	35.2%	29.6%	4.60%	5.66%
$m_s^* \approx 25$	39.9%	32.5%	4.55%	5.36%
30	44.6%	35.1%	4.60%	5.22%
35	49.1%	37.7%	4.70%	5.15%
40	53.6%	40.3%	4.83%	5.17%
$m_a^* \approx 45$	57.9%	42.8%	4.99%	5.12%
50	62.0%	45.1%	5.16%	5.16%
55	66.0%	47.4%	5.34%	5.33%
60	69.8%	50.1%	5.52%	5.50%
65	73.4%	53.5%	5.70%	5.68%
70	76.7%	56.7%	5.87%	5.77%
75	79.7%	59.5%	6.03%	5.87%

Table D.11: U.K. projections with a 60:40 equity:bond portfolio.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	16.7%	16.7%	29.67%	29.67%
3	20.3%	20.1%	12.57%	16.82%
5	23.9%	22.9%	9.31%	12.83%
10	32.4%	28.0%	7.00%	8.83%
$m_s^* \approx 15$	41.1%	33.0%	6.52%	7.74%
20	51.3%	37.7%	6.68%	7.16%
25	64.3%	42.3%	7.28%	7.01%
30	81.2%	47.0%	8.29%	6.97%
$m_a^* \approx 35$	102.2%	51.8%	9.65%	7.11%
40	127.4%	57.0%	11.33%	7.41%
45	156.5%	62.3%	13.29%	7.46%
50	188.9%	67.6%	15.47%	8.09%
60	258.2%	83.8%	20.09%	10.25%
70	323.9%	111.1%	24.40%	11.84%

Table D.12: U.K. projections with an 80:20 equity:bond portfolio.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	15.7%	15.7%	42.46%	42.46%
3	23.2%	22.1%	21.79%	28.87%
5	29.8%	27.1%	17.45%	23.54%
$m_s^* \approx 7$	37.0%	30.8%	16.12%	21.04%
10	52.0%	35.9%	16.84%	17.84%
$m_a^* \approx 13$	75.9%	42.0%	20.04%	17.40%
15	99.9%	46.4%	23.74%	17.41%
17	135.7%	51.4%	29.57%	17.56%
20	235.0%	60.8%	45.97%	19.00%

Table D.13: U.S. projections with a 60:40 equity:bond portfolio.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	14.0%	14.0%	37.55%	37.55%
3	18.0%	17.6%	16.81%	22.50%
5	21.7%	20.5%	12.69%	17.47%
10	30.5%	25.6%	9.87%	12.57%
$m_s^* \approx 13$	36.3%	28.5%	9.62%	11.46%
15	40.7%	30.4%	9.71%	11.02%
17	45.4%	32.3%	9.94%	10.66%
$m_a^* \approx 20$	53.3%	35.0%	10.46%	10.29%
23	62.0%	38.1%	11.16%	10.64%
25	68.3%	40.3%	11.71%	10.69%
30	85.7%	46.9%	13.30%	11.21%

Table D.14: Canada projections with a 60:40 equity:bond portfolio.

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	11.3%	11.3%	30.47%	30.47%
3	15.1%	14.2%	14.12%	18.33%
5	19.5%	16.8%	11.46%	14.62%
$m_s^* \approx 7$	24.5%	19.3%	10.69%	13.06%
10	33.5%	23.3%	10.89%	12.02%
13	45.2%	27.6%	12.01%	11.74%
$m_a^* \approx 15$	54.5%	30.6%	13.05%	11.73%
20	83.6%	40.1%	16.42%	12.46%

Table D.15: Canada projections with a 40:60 equity:bond portfolio.

	QMU	QSD	QA
U.K.	4.73%	4.27%	0.5773
U.S.	3%	3.5%	0.65
Canada	3.4%	3.2%	0.64

Table D.16: Parameters for price inflation given by Wilkie (1995).

	YMU	YSD	YA	YW
U.K.	3.77%	15.52%	0.5492	1.794
U.S.	4.3%	21%	0.7	0.5
Canada	3.75%	19%	0.7	1.17

Table D.17: Parameters for equity dividend yields given by Wilkie (1995).

	DMU	DSD	DY	DB	DD	DW
U.K.	1.57%	6.71%	-0.1761	0.5733	0.1344	0.5793
U.S.	1.55%	9%	-0.35	0.5	0.38	1.0
Canada	0.1%	7%	-0.11	0.58	0.26	0.19

Table D.18: Parameters for equity dividend growth given by Wilkie (1995).

	CMU	CSD	CY	$CA1$	CD	CW
U.K.	3.05%	18.53%	0.3371	0.90	0.045	1.0
U.S.	2.65%	21%	0.07	0.96	0.058	1.0
Canada	3.7%	18.5%	0.1	0.95	0.04	1.0

Table D.19: Parameters for long bond yields given by Wilkie (1995).

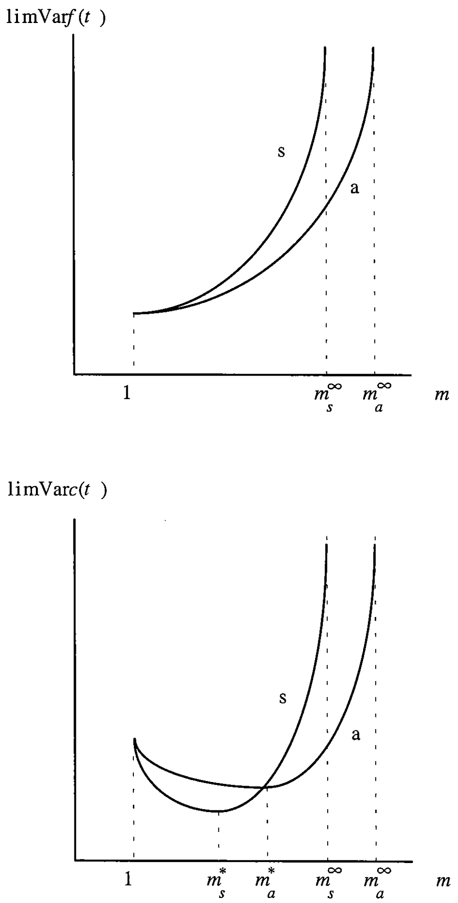

Figure 1: Label 's': spreading. Label 'a': amortization.

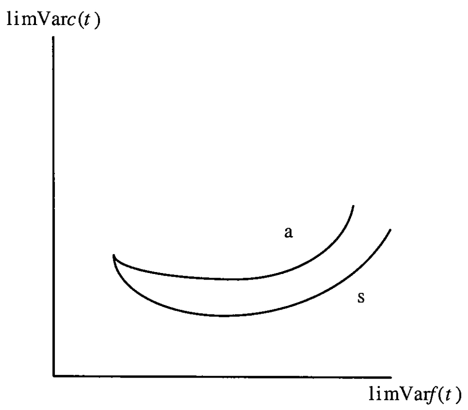

Figure 2: Spreading ('s') is more efficient than amortization ('a').

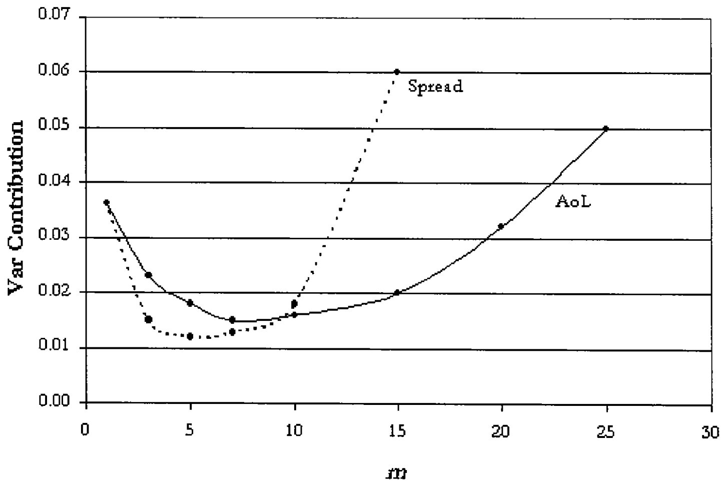

Figure 3: AR(1) logarithmic rates of return, $\varphi = +0.3$.

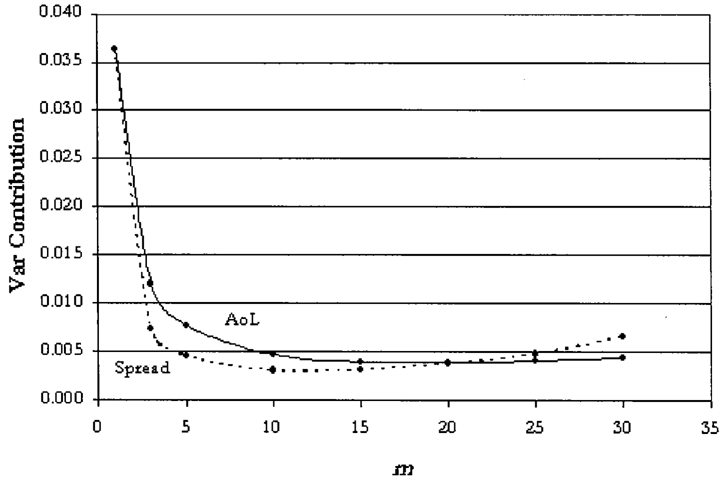

Figure 4: AR(1) logarithmic rates of return, $\varphi = -0.1$.

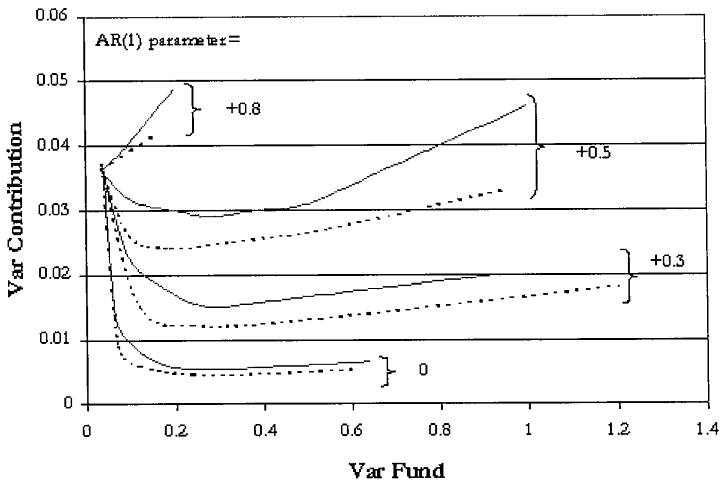

Figure 5: AR(1) logarithmic rates of return ($\varphi \geq 0$): spreading is more efficient than amortization. Broken lines: Spreading. Continuous lines: Amortization.

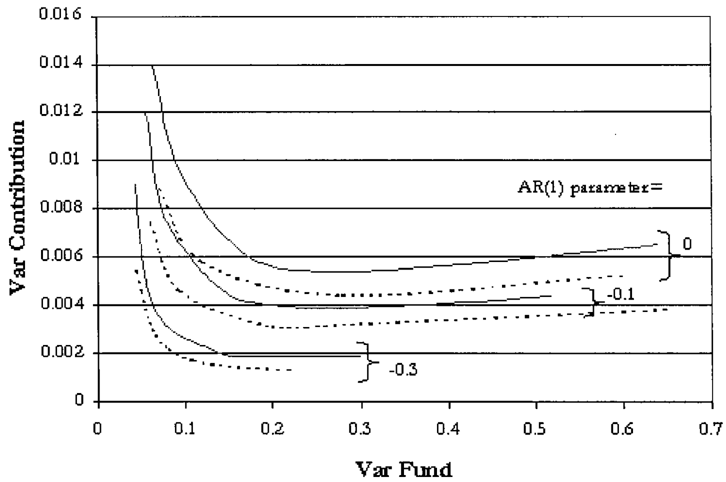

Figure 6: AR(1) logarithmic rates of return ($\varphi \leq 0$): spreading is more efficient than amortization. Broken lines: Spreading. Continuous lines: Amortization.

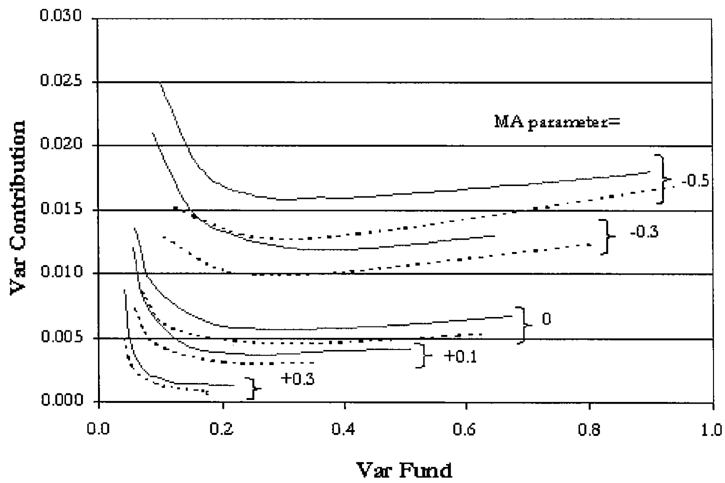

Figure 7: MA(1) logarithmic rates of return: spreading is more efficient than amortization. Broken lines: Spreading. Continuous lines: Amortization.

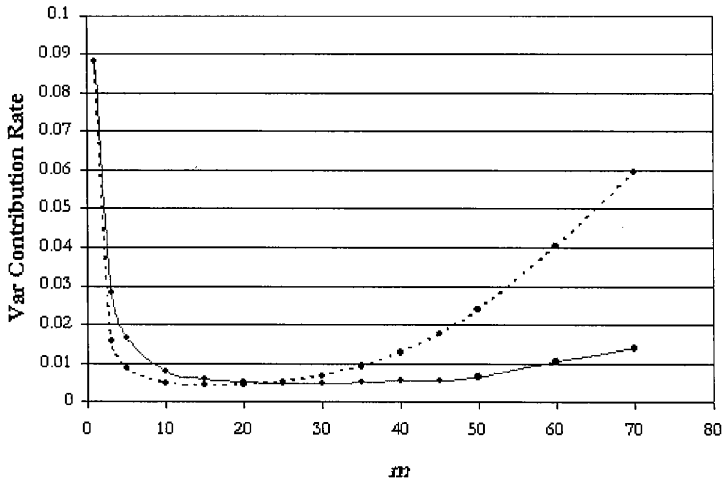

Figure 8: U.K. projections with an 80:20 equity:bond portfolio. Broken line: Spreading. Continuous line: Amortization.

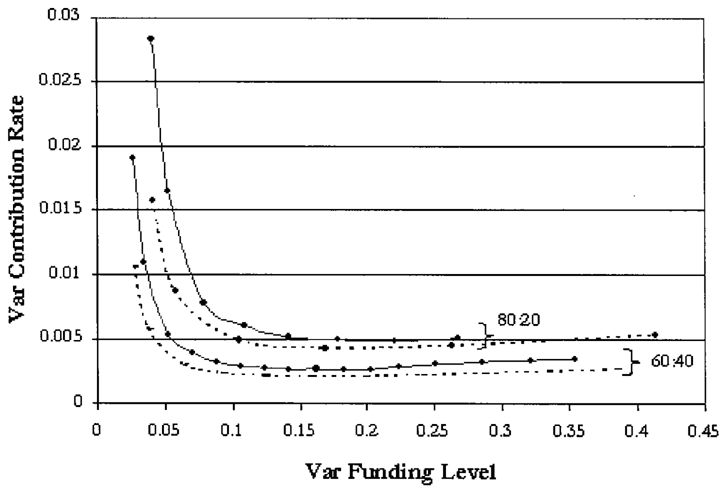

Figure 9: U.K. projections with 60:40 and 80:20 equity:bond portfolios: spreading is more efficient than amortization. Broken lines: Spreading. Continuous lines: Amortization.

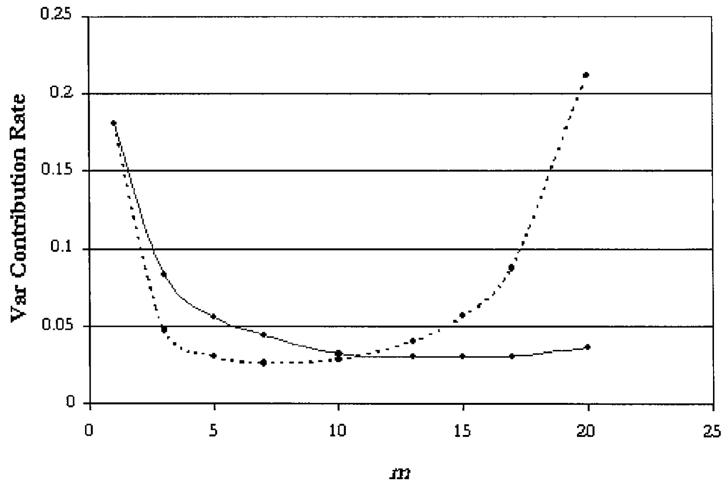

Figure 10: U.S. projections with a 60:40 equity:bond portfolio. Broken line: Spreading. Continuous line: Amortization.

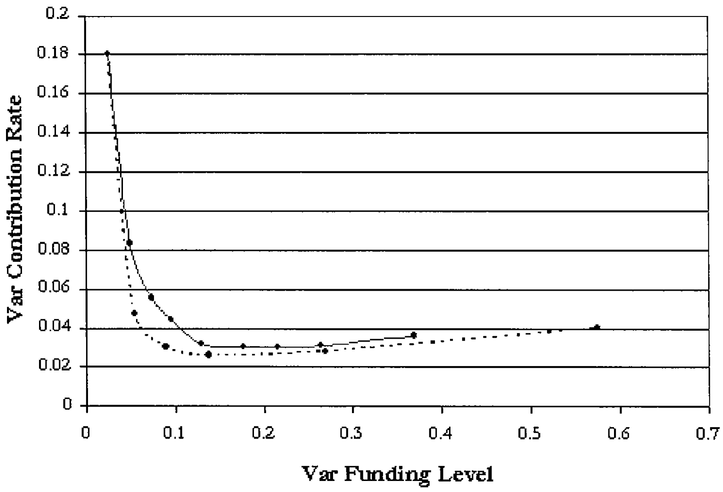

Figure 11: U.S. projections and a 60:40 equity:bond portfolio: spreading is more efficient than amortization. Broken line: Spreading. Continuous line: Amortization.

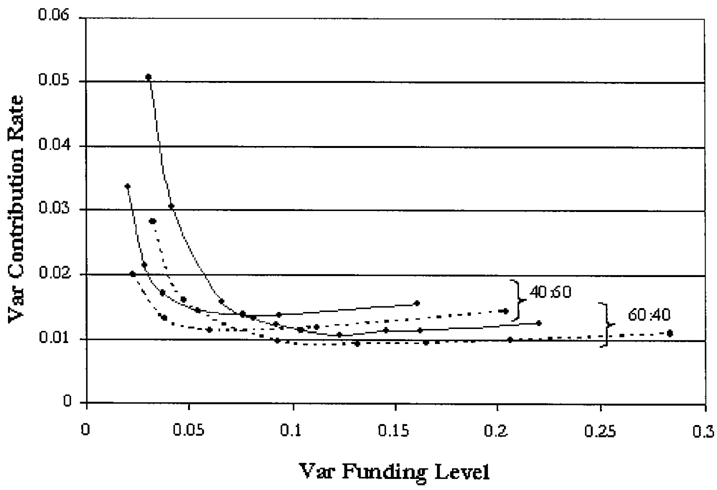

Figure 12: Canada projections with 60:40 and 40:60 equity:bond portfolios: spreading is more efficient than amortization. Broken lines: Spreading. Continuous lines: Amortization.

CITY
University

DEPARTMENT OF ACTUARIAL SCIENCE AND STATISTICS

Actuarial Research Papers since 1995

70. Huber P. A Review of Wilkie's Stochastic Investment Model. January 1995. 22 pages.
ISBN 1 874 770 70 0

71. Renshaw A.E. On the Graduation of 'Amounts'. January 1995. 24 pages.
ISBN 1 874 770 71 9

72. Renshaw A.E. Claims Reserving by Joint Modelling. December 1994. 26 pages.
ISBN 1 874 770 72 7

73. Renshaw A.E. Graduation and Generalised Linear Models: An Overview. February 1995. 40 pages. ISBN 1 874 770 73 5

74. Khorasanee M.Z. Simulation of Investment Returns for a Money Purchase Fund. June 1995. 20 pages. ISBN 1 874 770 74 3

75. Owadally M.I. and Haberman S. Finite-time Pension Fund Dynamics with Random Rates of Return. June 1995. 28 pages. ISBN 1 874 770 75 1

76. Owadally M.I. and Haberman S. Stochastic Investment Modelling and Optimal Funding Strategies. June 1995. 25 pages. ISBN 1 874 770 76 X

77. Khorasanee M.Z Applying the Defined Benefit Principle to a Defined Contribution Scheme. August 1995. 30 pages. ISBN 1 874 770 77 8

78. Sebastiani P. and Settimi R. Experimental Design for Non-Linear Problems. September 1995. 13 pages. ISBN 1 874 770 78 6

79. Verrall R.J. Whittaker Graduation and Parametric State Space Models. November 1995. 23 pages. ISBN 1 874 770 79 4

80. Verrall R.J. Claims Reserving and Generalised Additive Models. November 1995. 17 pages.
ISBN 1 874 770 80 8

81. Nelder J.A. and Verrall R.J. Credibility Theory and Generalized Linear Models. November 1995. 15 pages. ISBN 1 874 770 81 6

82. Renshaw A.E., Haberman S. and Hatzopoulos P. On The Duality of Assumptions Underpinning The Construction of Life Tables. December 1995. 17 Pages. ISBN 1 874 770 82 4

83. Chadburn R.G. Use of a Parametric Risk Measure in Assessing Risk Based Capital and Insolvency Constraints for With Profits Life Insurance. March 1996. 17 Pages.
ISBN 1 874 770 84 0

84. Haberman S. Landmarks in the History of Actuarial Science (up to 1919). March 1996. 62 Pages. ISBN 1 874 770 85 9

85. Renshaw A.E. and Haberman S. Dual Modelling and Select Mortality. March 1996. 30 Pages. ISBN 1 874 770 88 3

86. Booth P.M. Long-Term Care for the Elderly: A Review of Policy Options. April 1996. 45 Pages. ISBN 1 874 770 89 1

87. Huber P.P. A Note on the Jump-Equilibrium Model. April 1996. 17 Pages. ISBN 1 874 770 90 5

88. Haberman S and Wong L.Y.P. Moving Average Rates of Return and the Variability of Pension Contributions and Fund Levels for a Defined Benefit Pension Scheme. May 1996. 51 Pages. ISBN 1 874 770 91 3

89. Cooper D.R. Providing Pensions for Employees with Varied Working Lives. June 1996. 25 Pages. ISBN 1 874 770 93 X

90. Khorasanee M.Z. Annuity Choices for Pensioners. August 1996. 25 Pages. ISBN 1 874 770 94 8

91. Verrall R.J. A Unified Framework for Graduation. November 1996. 25 Pages. ISBN 1 874 770 99 9

92. Haberman S. and Renshaw A.E. A Different Perspective on UK Assured Lives Select Mortality. November 1996. 61 Pages. ISBN 1 874 770 00 X

93. Booth P.M. The Analysis of Actuarial Investment Risk. March 1997. 43 Pages. ISBN 1 901615 03 0

94. Booth P.M., Chadburn R.G. and Ong A.S.K. Utility-Maximisation and the Control of Solvency for Life Insurance Funds. April 1997. 39 Pages. ISBN 1 901615 04 9

95. Chadburn R.G. The Use of Capital, Bonus Policy and Investment Policy in the Control of Solvency for With-Profits Life Insurance Companies in the UK. April 1997. 29 Pages. ISBN 1 901615 05 7

96. Renshaw A.E. and Haberman S. A Simple Graphical Method for the Comparison of Two Mortality Experiences. April 1997. 32 Pages. ISBN 1 901615 06 5

97. Wong C.F.W. and Haberman S. A Short Note on Arma (1, 1) Investment Rates of Return and Pension Funding. April 1997. 14 Pages. ISBN 1 901615 07 3

98. Puzey A S. A General Theory of Mortality Rate Estimators. June 1997. 26 Pages. ISBN 1 901615 08 1

99. Puzey A S. On the Bias of the Conventional Actuarial Estimator of q_x. June 1997. 14 Pages. ISBN 1 901615 09 X

100. Walsh D. and Booth P.M. Actuarial Techniques in Pricing for Risk in Bank Lending. June 1997. 55 Pages. ISBN 1 901615 12 X

101. Haberman S. and Walsh D. Analysis of Trends in PHI Claim Inception Data. July 1997. 51 Pages. ISBN 1 901615 16 2

102. Haberman S. and Smith D. Stochastic Investment Modelling and Pension Funding: A Simulation Based Analysis. November 1997. 91 Pages. ISBN 1 901615 19 7

103. Rickayzen B.D. A Sensitivity Analysis of the Parameters used in a PHI Multiple State Model. December 1997. 18 Pages. ISBN 1 901615 20 0

104. Verrall R.J. and Yakoubov Y.H. A Fuzzy Approach to Grouping by Policyholder Age in General Insurance. January 1998. 18 Pages. ISBN 1 901615 22 7

105. Yakoubov Y.H. and Haberman S. Review of Actuarial Applications of Fuzzy Set Theory. February 1998. 88 Pages. ISBN 1 901615 23 5

106. Haberman S. Stochastic Modelling of Pension Scheme Dynamics. February 1998. 41 Pages. ISBN 1 901615 24 3

107. Cooper D.R. A Re-appraisal of the Revalued Career Average Benefit Design for Occupational Pension Schemes. February 1998. 12 Pages. ISBN 1 901615 25 1

108. Wright I.D. A Stochastic Asset Model using Vector Auto-regression. February 1998. 59 Pages. ISBN 1 901615 26 X

109. Huber P.P. and Verrall R.J. The Need for Theory in Actuarial Economic Models. March 1998. 15 Pages. ISBN 1 901615 27 8

110. Booth P.M. and Yakoubov Y. Investment Policy for Defined Contribution Pension Scheme Members Close to Retirement. May 1998. 32 Pages ISBN 1 901615 28 6

111. Chadburn R.G. A Genetic Approach to the Modelling of Sickness Rates, with Application to Life Insurance Risk Classification. May 1998. 17 Pages. ISBN 1 901615 29 4

112. Wright I.D. A Stochastic Approach to Pension Scheme Funding. June 1998. 24 Pages. ISBN 1 901615 30 8

113. Renshaw A.E. and Haberman S. Modelling the Recent Time Trends in UK Permanent Health Insurance Recovery, Mortality and Claim Inception Transition Intensities. June 1998. 57 Pages. ISBN 1 901615 31 6

114. Megaloudi C. and Haberman S. Contribution and Solvency Risk in a Defined Benefit Pension Scheme. July 1998. 39 Pages ISBN 1 901615 32 4

115. Chadburn R.G. Controlling Solvency and Maximising Policyholders' Returns: A Comparison of Management Strategies for Accumulating With-Profits Long-Term Insurance Business. August 1998. 29 Pages ISBN 1 901615 33 2

116. Fernandes F.N. Total Reward - An Actuarial Perspective. August 1998. 25 Pages. ISBN 1 901615 34 0

117. Booth P.M. and Walsh D. The Application of Financial Theory to the Pricing of Upward Only Rent Reviews. November 1998. 23 Pages. ISBN 1 901615 35 9

118. Renshaw A.E. and Haberman S. Observations on the Proposed New Mortality Tables Based on the 1991-94 Experience for Male Permanent Assurances. February 1999. 40 Pages. ISBN 1 901615 36 7

119. Velmachos D. And Haberman S. Moving Average Models for Interest Rates and Applications to Life Insurance Mathematics. July 1999. 27 Pages. ISBN 1 901615 38 3

120. Chadburn R.G. and Wright I.D. The Sensitivity of Life Office Simulation Outcomes to Differences in Asset Model Structure. July 1999. 58 Pages. ISBN 1 901615 39 1

121. Renshaw A.E. and Haberman S. An Empirical Study of Claim and Sickness Inception Transition Intensities (Aspects of the UK Permanent Health Insurance Experience). November 1999. 35 Pages. ISBN 1 901615 41 3

122. Booth P.M. and Cooper D.R. The Tax Treatment of Pensions. April 2000. 36 pages. ISBN 1 901615 42 1

123. Walsh D.E.P. and Rickayzen B.D. A Model for Projecting the number of People who will require Long-Term Care in the Future. Part I: Data Considerations. July 2000. 37 pages. ISBN 1 901615 43 X

124. Rickayzen B.D. and Walsh D.E.P. A Model for Projecting the number of People who will require Long-Term Care in the Future. Part II: The Multiple State Model. July 2000. 27 pages. ISBN 1 901615 44 8

125. Walsh D.E.P. and Rickayzen B.D. A Model for Projecting the number of People who will require Long-Term Care in the Future. Part III: The Projected Numbers and The Funnel of Doubt. July 2000. 61 pages. ISBN 1 901615 45 6

126. Cooper D.R. Security for the Members of Defined Benefit Pension Schemes. July 2000. 23 pages. ISBN 1 901615 45 4

127. Renshaw A.E. and Haberman S. Modelling for mortality reduction factors. July 2000. 32 pages. ISBN 1 901615 47 2

128. Ballotta L. and Kyprianou A.E A note on the α-quantile option. September 2000. ISBN 1 901615 49 9

129. Spreeuw J. Convex order and multistate life insurance contracts. December 2000. ISBN 1 901615 50 2

130. Spreeuw J. The Probationary Period as a Screening Device. December 2000. ISBN 1 901615 51 0

131. Owadally M.I. and Haberman S. Asset Valuation and the Dynamics of Pension Funding with Random Investment Returns. December 2000. ISBN 1 901615 52 9

132. Owadally M.I. and Haberman S. Asset Valuation and Amortization of Asset Gains and Losses in Defined Benefit Pension Plans. December 2000. ISBN 1 901615 53 7

133. Owadally M.I. and Haberman S. Efficient Amortization of Actuarial Gains/Losses and Optimal Funding in Pension Plans. December 2000. ISBN 1 901615 54 5

<u>Statistical Research Papers</u>

1. Sebastiani P. Some Results on the Derivatives of Matrix Functions. December 1995. 17 Pages. ISBN 1 874 770 83 2

2. Dawid A.P. and Sebastiani P. Coherent Criteria for Optimal Experimental Design. March 1996. 35 Pages. ISBN 1 874 770 86 7

3. Sebastiani P. and Wynn H.P. Maximum Entropy Sampling and Optimal Bayesian Experimental Design. March 1996. 22 Pages. ISBN 1 874 770 87 5

4. Sebastiani P. and Settimi R. A Note on D-optimal Designs for a Logistic Regression Model. May 1996. 12 Pages. ISBN 1 874 770 92 1

5. Sebastiani P. and Settimi R. First-order Optimal Designs for Non Linear Models. August 1996. 28 Pages. ISBN 1 874 770 95 6

6. Newby M. A Business Process Approach to Maintenance: Measurement, Decision and Control. September 1996. 12 Pages. ISBN 1 874 770 96 4

7. Newby M. Moments and Generating Functions for the Absorption Distribution and its Negative Binomial Analogue. September 1996. 16 Pages. ISBN 1 874 770 97 2

8. Cowell R.G. Mixture Reduction via Predictive Scores. November 1996. 17 Pages. ISBN 1 874 770 98 0

9. Sebastiani P. and Ramoni M. Robust Parameter Learning in Bayesian Networks with Missing Data. March 1997. 9 Pages. ISBN 1 901615 00 6

10. Newby M.J. and Coolen F.P.A. Guidelines for Corrective Replacement Based on Low Stochastic Structure Assumptions. March 1997. 9 Pages. ISBN 1 901615 01 4.

11. Newby M.J. Approximations for the Absorption Distribution and its Negative Binomial Analogue. March 1997. 6 Pages. ISBN 1 901615 02 2

12. Ramoni M. and Sebastiani P. The Use of Exogenous Knowledge to Learn Bayesian Networks from Incomplete Databases. June 1997. 11 Pages. ISBN 1 901615 10 3

13. Ramoni M. and Sebastiani P. Learning Bayesian Networks from Incomplete Databases. June 1997. 14 Pages. ISBN 1 901615 11 1

14. Sebastiani P. and Wynn H.P. Risk Based Optimal Designs. June 1997. 10 Pages. ISBN 1 901615 13 8

15. Cowell R. Sampling without Replacement in Junction Trees. June 1997. 10 Pages. ISBN 1 901615 14 6

16. Dagg R.A. and Newby M.J. Optimal Overhaul Intervals with Imperfect Inspection and Repair. July 1997. 11 Pages. ISBN 1 901615 15 4

17. Sebastiani P. and Wynn H.P. Bayesian Experimental Design and Shannon Information. October 1997. 11 Pages. ISBN 1 901615 17 0

18. Wolstenholme L.C. A Characterisation of Phase Type Distributions. November 1997. 11 Pages. ISBN 1 901615 18 9

19. Wolstenholme L.C. A Comparison of Models for Probability of Detection (POD) Curves. December 1997. 23 Pages. ISBN 1 901615 21 9

20. Cowell R.G. Parameter Learning from Incomplete Data Using Maximum Entropy I: Principles. February 1999. 19 Pages. ISBN 1 901615 37 5

21. Cowell R.G. Parameter Learning from Incomplete Data Using Maximum Entropy II: Application to Bayesian Networks. November 1999. 12 Pages ISBN 1 901615 40 5

Department of Actuarial Science and Statistics

Actuarial Research Club

The support of the corporate members

Computer Sciences Corporation
Government Actuary's Department
Guardian Insurance
Hymans Robertson
KPMG
Munich Reinsurance
PricewaterhouseCoopers
Swiss Reinsurance
Watson Wyatt

is gratefully acknowledged.

ISBN 1 901615 54 5