A Cash-Flow Approach to Pension Funding

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ABSTRACT

The problem of how to fund a defined-benefit pension plan is detached from the problem of how the cost of such a plan should be recognized. An approach to funding based on the projection of aggregate cash-flows and the explicit modeling of new entrants is presented. It is shown that commonly used funding methods can be derived from the cash-flow approach. A generalized funding method for a plan subject to a stationary distribution of new entrants is derived. It is concluded that plan actuaries might need to modify existing funding methods to incorporate useful information about the expected number and distribution of future entrants.

1. INTRODUCTION

On setting up a defined-benefit pension plan, the plan sponsor is immediately faced with the question of what contributions should be paid into the plan. Before the advent of modern pension accounting standards[1], there was no need to treat this as a separate problem from how to recognize the cost of the plan, as the pension cost charged in the sponsor's financial statements could simply be taken as the contribution paid by the sponsor. Thus, the standard actuarial texts dealing with pension mathematics use the term "actuarial cost method" as a synonym for any method of funding a defined-benefit plan which satisfies certain requirements. Anderson (1992), for example, defines an actuarial cost method as one which:

> ..assigns to each fiscal year a portion of the present value of future benefit payments in such a way as generally to accrue costs over the working lifetimes of employees.

This principle of cost recognition is consistent with accruals accounting and obviously excludes funding strategies such as "pay-as-you go". Anderson points out, however, that there are "many such cost methods in use, each with a different philosophical foundation", and goes on to derive actuarial formulas for several different methods in his textbook. Whilst these "cost methods" are quite justifiable as alternative funding strategies for a defined benefit plan, it was clearly not a satisfactory state of affairs that such discretion should be permitted in the recognition of pension costs in published financial statements, where a key objective is to allow comparisons to be made on a consistent basis. Thus, pension accounting standards have greatly limited the scope for alternative methods of cost recognition, and these restrictions are likely to be tightened further.[2]

Certainly, there are also regulatory constraints which apply to funding. Anderson writes that:

> ..in the US and Canada, and indeed in almost every modern industrialized country, it is not generally legal for a private employer to establish a pension plan which is not properly funded.

"Properly funded" implies a minimum fund which is linked in some way to the accrued liabilities of the plan, and an upper funding limit may also be prescribed by law to prevent the abuse of tax privileges. Such restrictions, however, still allow considerable flexibility, and the

[1] such as SFAS 87 in the US and SSAP 24 in the UK.
[2] See Thornton (1998) for a discussion of proposals for the new international pension accounting standard IAS 19.

This equation states that the rate of increase of the fund assets is equal to the investment income plus the contribution income less the benefit outgo.

For the purpose of projecting into the future, we shall adopt the standard actuarial approach of assuming that some fixed investment return will be earned on the fund assets, by writing $\delta(t) = \delta$. Hence equation (1) can be re-written as:

$$\frac{d}{dt}\left[F(t)\times e^{-\delta t}\right] = \left\lfloor C(t) - B(t)\right\rfloor e^{-\delta t}$$

We now define the current time to be t_0, and project into the future by integrating both sides of the above equation from t_0 to infinity, which gives:

$$\left[F(t)\times e^{-\delta t}\right]_{t_0}^{\infty} = \int_{t_0}^{\infty}\left\lfloor C(t) - B(t)\right\rfloor e^{-\delta t}.dt$$

Consider now the value of the left-hand-side bracket at its upper limit. This value will only be non-zero if the fund is increasing at rate equal to (or faster than) the assumed investment return. Such an eventuality would imply that the plan is being over-funded, so we shall impose the following restriction for all acceptable funding strategies:

$$\underset{t\to\infty}{Lim}\left\{F(t)\times e^{-\delta t}\right\} = 0$$

On applying this condition to the integrated equation we obtain the following expression:

$$F(t_0) + \int_{t_0}^{\infty} C(t)e^{-\delta(t-t_0)}.dt = \int_{t_0}^{\infty} B(t)e^{-\delta(t-t_0)}.dt \qquad (2)$$

The interpretation of above equation is clear: for any acceptable funding strategy we require that the fund assets plus the present value of the future contribution income equals the present value of the future benefit outgo. This simple equation-of-value could have been written down immediately from first principles.

Possible funding solutions

Equation (2) encompasses a wide range of possible funding strategies. At one extreme we have the "pay-as-you-go" solution, given by:

- $F(t) = 0$ for all t
- $C(t) = B(t)$ for all t

At the other extreme we have a "pre-funded" plan for which no further contributions are required, given by:

The structure of the population model is illustrated in the diagram below:

POPULATION MODEL

$n(x,t)$.dx = size of active member cohort at time t

$\longleftrightarrow$

x **x+dx**

Decrements between t and t+dt = $n(x,t).\mu_x$.dx.dt

New entrants between t and t+dt = $g(x,t)$.dx.dt

$n(x$+dt,t+d$t)$.dx = size of active member cohort at time t+dt

$\longleftrightarrow$

x+dt **x+dt+dx**

The above model leads directly to the following equation:

$$[n(x+dt,t+dt)-n(x,t)].dx = [g(x,t)-n(x,t)\mu_x].dx.dt$$

The above equation states that the number of active members in any given age-cohort will change over any element of time by the difference between the new entrants and service table decrements affecting that particular age-cohort. The left-hand-side of the above equation can be re-written as:

$$[n(x+dt,t+dt)-n(x+dt,t)].dx+[n(x+dt,t)-n(x,t)].dx$$
$$=\frac{\partial n}{\partial t}dt.dx+\frac{\partial n}{dx}dt.dx$$

Hence our population model is represented by the following partial differential equation:

$$\frac{\partial n}{\partial t}+\frac{\partial n}{\partial x}=g(x,t)-n(x,t)\mu_x \qquad (3)$$

3.1 SOLUTION FOR A PLAN WITH A FIXED RETIREMENT AGE

Following Bowers *et al* (1997), we shall assume that all the active member are within the age-range $a \le x \le r$, where:

- a = youngest age at which employees are admitted to the plan
- r = fixed retirement age

We now write down the following identities:

$$p(x+dt,t+dt) \equiv p(x,t) + \frac{\partial p}{\partial x}dt + \frac{\partial p}{\partial t}dt$$
$$n(x+dt,t+dt) \equiv n(x,t) + \frac{\partial n}{\partial x}dt + \frac{\partial n}{\partial t}dt$$

Using each of the above identities to substitute for $p(x+\mathrm{d}t,t+\mathrm{d}t)$ and $n(x+\mathrm{d}t,t+\mathrm{d}t)$ in equation (5) leads to the following partial differential equation:

$$\frac{1}{p(x,t)}\left(\frac{\partial p}{\partial x}+\frac{\partial p}{\partial t}-1\right)+\frac{1}{n(x,t)}\left(\frac{\partial n}{\partial x}+\frac{\partial n}{\partial t}\right)+\mu_x=0 \qquad (6)$$

We assume that all active members have zero past service when the plan is established. Furthermore, the average past service at the youngest age, $x = a$, must always be zero. This leads to the following boundary conditions:

- $p(x,0) = 0$ for all x
- $p(a,t) = 0$ for all t

It can be shown that these boundary conditions lead to the following solution of the partial differential equation:

$$p(x,t)=\frac{l_x}{n(x,t)}\int_{x-t}^{x}\frac{n(z,t-x+z)}{l_z}.dz \quad for \quad t \le x-a$$
$$p(x,t)=\frac{l_x}{n(x,t)}\int_{a}^{x}\frac{n(z,t-x+z)}{l_z}.dz \quad for \quad t \ge x-a \qquad (7)$$

If we know the distribution of past service at time t_0, the solution takes the following form in the range $x - t + t_0 > a$.

$$p(x,t)=p(x-t+t_0,t_0)\times\frac{l_x}{l_{x-t+t_0}}\times\frac{n(x-t+t_0,t_0)}{n(x,t)}+\frac{l_x}{n(x,t)}\int_{x-t+t_0}^{x}\frac{n(z,t-x+z)}{l_z}.dz$$

3.3 EXAMPLE OF A STATIONARY POPULATION SOLUTION

We now use our population model to derive the new entrant density function and past service distribution of a plan with a stationary population of active members modeled by a simple parametric function. The active member population can only be stationary if the distribution of new entrants is also stationary, hence we write:

- $n(x,t) = n(x)$
- $g(x,t) = g(x)$

Figure 1: Hypothetical active member density function

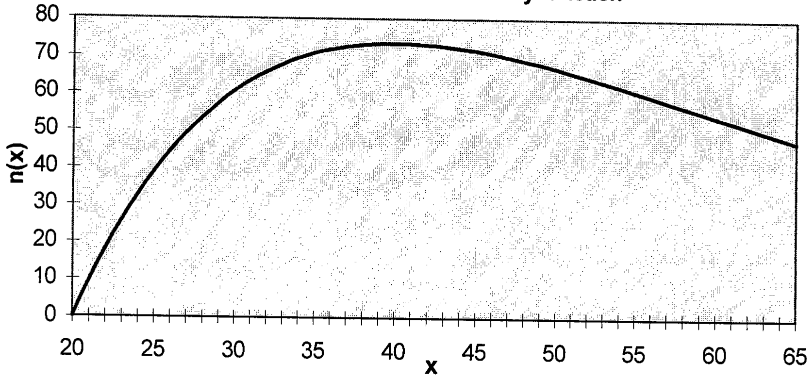


Figure 2: New entrant density function for constant forces of decrement

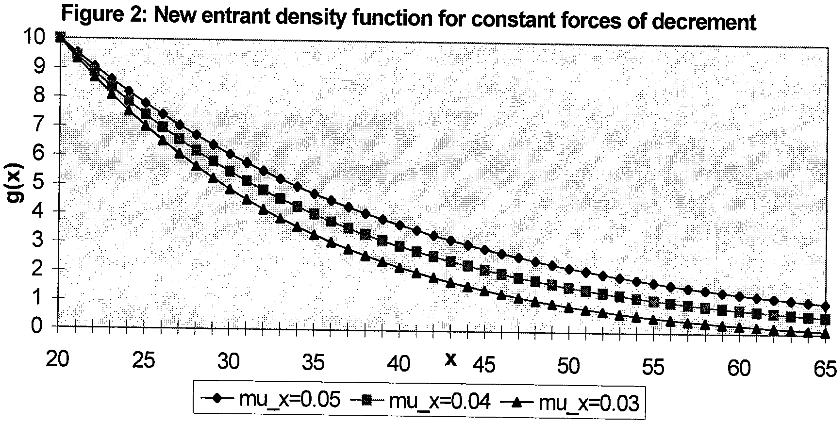
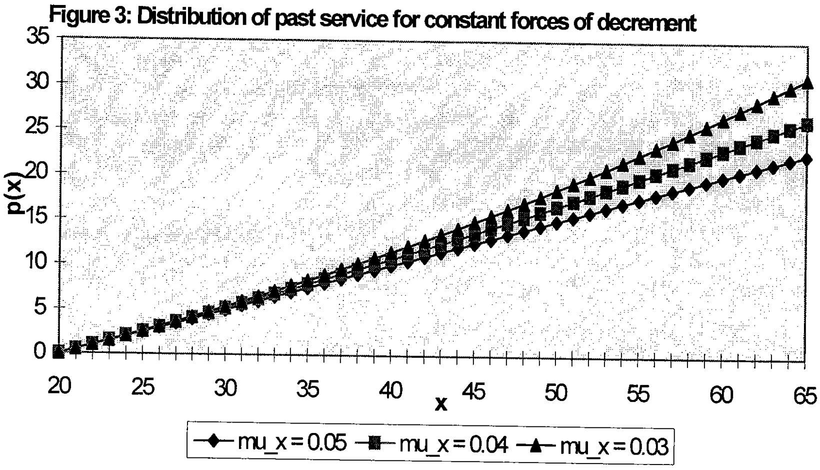


Figure 3: Distribution of past service for constant forces of decrement



$C(t) = C_s$ for all t

where C_s is the "standard contribution" of our chosen funding strategy.

Putting $t_0 = 0$ in equation (2) gives:

$$C_s = \delta \int_0^{\infty} B(t) e^{-\delta t}.dt$$

Hence, the standard contribution is the present value of the future benefit outgo multiplied by the assumed force of interest. As the active member population is stationary, the density function $n(x,t)$ doesn't depend on time, so we can write:

$n(x, t) = n(x)$

Hence from equation (7), the average past service at the retirement age, $x = r$, is given by:

$$p(r,t) = \frac{l_r}{n(r)} \int_{r-t}^{r} \frac{n(z)}{l_z}.dz \quad for \quad t \le r - a$$

$$p(r,t) = \frac{l_r}{n(r)} \int_{a}^{r} \frac{n(z)}{l_z}.dz \quad for \quad t \ge r - a$$

Note that the distribution of past service becomes stationary when $t \ge r - a$, so we can write:

$p(r, t) = p(r)$ for $t \ge r - a$.

<u>Present value of projected benefit outgo</u>

The benefit outgo of the plan at any time is simply the sum of the pensions then being paid to retired employees. Thus, if the pension of a retired member aged $r + z$ at time t is $b(z, t)$, the total benefit outgo at time t is given by:

$$B(t) = k.n(r) \int_0^{\infty} b(z,t).\frac{l_{r+z}}{l_r}.dz$$

The rate of benefit payment, $b(z, t)$, will be zero for employees who attained retirement age before the plan was established, and will otherwise be proportional to the average past service at retirement of any age-cohort of pensioners.

It follows that:

- $b(z, t) = 0$ if $z \ge t$
- $b(z, t) = k \times p(r, t-z)$ if $t-r+a \le z < t$
- $b(z, t) = k \times p(r)$ if $z \le t-r+a$

Substituting the above expression for $b(z, t)$ into the formula for the total benefit outgo gives:

Hence the total contribution at time u is given by:

$$C(u) = c_s \int_{a+u}^{r} n(x,u).dx \quad for \quad u \le r-a$$

$$C(u) = 0 \quad for \quad u \ge r-a$$

where c_s is the standard contribution per active member

To obtain a formula for c_s, we must evaluate the present value of the projected contribution income and the present value of the projected benefit outgo.

Present value of projected contribution income

The present value of the projected contribution income is given by:

$$\int_0^{r-a} C(u).e^{-\delta u}.du = c_s \int_0^{r-a} \int_{a+u}^{r} n(x,u).e^{-\delta u}.dx.du$$

After the plan is closed to new entrants, the active member density function changes over time for any age-cohort purely as a result of the service table decrements, thus:

$$n(x,u) = n(x-u,0) \times \frac{l_x}{l_{x-u}} \quad for \quad x \ge a+u$$

Substituting the above formula into the double-integral gives:

$$c_s \int_0^{r-a} \int_{a+u}^{r} n(x-u,0).\frac{l_x}{l_{x-u}}.e^{-\delta u}.dx.du$$

On applying the substitution $z = x - u$ for the first integration with respect to x, we obtain:

$$c_s \int_0^{r-a} \int_a^{r-u} n(z,0).\frac{l_{z+u}}{l_z}.e^{-\delta u}.dz.du$$

We now reverse the order of integration, which gives:

$$c_s \int_a^r n(z,0) \int_0^{r-z} \frac{l_{z+u}}{l_z}.e^{-\delta u}.du.dz = c_s \int_a^r n(z,0).\bar{a}_{z:\overline{r-z}|}.dz$$

The right-hand-side integral is the sum of the present value of the projected contributions for each active member. This is the usual way of expressing the present value of the total future contribution income under the "individual cost method" approach.

The above formula reflects the fact that the density function for any age-cohort is subject only to service table decrements after the plan is closed to new entrants. The average past service at retirement of this cohort is simply the past service prior to closure of the plan plus the remaining service until retirement, thus:

$$p(r,u-z) = p(r-u+z,0) + u - z$$

We can now deduce that the expression for the benefit outgo of the active members is:

$$B_a(u) = k\int_0^u [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.dz \quad for \quad u \le r-a$$

$$B_a(u) = k\int_{u-r+a}^{u} [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.dz \quad for \quad u \ge r-a$$

Hence the present value of the active member benefit outgo is given by:

$$\int_0^\infty B_a(u).e^{-\delta u}.du = k\int_0^{r-a}\int_0^u [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.e^{-\delta u}.dz.du$$
$$+k\int_{r-a}^{\infty}\int_{u-r+a}^{u} [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.e^{-\delta u}.dz.du$$

It is demonstrated in Appendix 1 that the above expression simplifies to:

$$\int_0^\infty B_a(u).e^{-\delta u}.du = k.\bar{a}_r\int_a^r [p(x,0)+r-x].n(x,0).\frac{l_r}{l_x}.e^{-\delta(r-x)}.dx$$

The integral on the right-hand-side of the above equation is the sum of the present value of the future pensions of the active members, allowing for accrued service at the date of closure and future service up to retirement.

<u>Standard contribution per active member</u>

Returning to the version of equation (2) given at the start of this sub-section, we can use the results derived above to obtain the following expression for the standard contribution per active member:

$$c_s = \frac{k.\bar{a}_r\int_a^r [p(x,0)+r-x].n(x,0).\frac{l_r}{l_x}.e^{-\delta(r-x)}.dx + k.n(r).p(r)\int_0^\infty \frac{l_{r+y}}{l_r}\bar{a}_{r+y}.dy - F(0)}{\int_a^r n(z,0).\bar{a}_{\overline{z:r-z|}}.dz}$$

This corresponds to the normal cost of the aggregate method.

Present value of projected benefit outgo

The first retirements will not occur until $t = r - a$, so the benefit outgo prior to this time will be zero. As all the active members enter at the youngest age, all will have the same past service at retirement, given by $p(r,t) = r - a$. It follows that the projected benefit outgo of the plan will be given by the following expression:

$$B(t) = 0 \quad for \quad t \le r - a$$

$$B(t) = k.(r-a).n_0 \int_0^{t-r+a} \frac{l_{r+z}}{l_r} dz \quad for \quad t \ge r-a$$

Hence the present value of the projected benefit outgo is given by:

$$\int_0^\infty B(t).e^{-\delta t}.dt = k.(r-a).n_0 \int_{r-a}^{\infty} \int_0^{t-r+a} \frac{l_{r+z}}{l_r} e^{-\delta t} dz.dt$$

Reversing the order of integration in the double-integral gives:

$$\int_0^\infty B(t).e^{-\delta t}.dt = k.(r-a).n_0 \int_0^\infty \int_{z+r-a}^{\infty} \frac{l_{r+z}}{l_r} e^{-\delta t} dt.dz$$

We now use the substitution $y = t - z$ for the first integration with respect to t, which gives:

$$\int_0^\infty B(t).e^{-\delta t}.dt = k.(r-a).n_0 \int_0^\infty \int_{r-a}^{\infty} \frac{l_{r+z}}{l_r} e^{-\delta y}.e^{-\delta z} dy.dz$$

$$\Rightarrow \int_0^\infty B(t).e^{-\delta t}.dt = k.(r-a).n_0.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}$$

Standard contribution per active member

As there is no fund at time $t = 0$, we set the present value of the projected contribution income equal to the present value of the projected benefit outgo, which gives:

$$c_s.n_0.\frac{\bar{a}_{a:\overline{r-a}|}}{\delta} = k.(r-a).n_0.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}$$

$$\Rightarrow c_s = \frac{k.(r-a).\bar{a}_r.}{\bar{s}_{a:\overline{r-a}|}}$$

This formula is recognizable as the normal cost of the entry age normal method, where the assumed entry age is at $x = a$. As we have assumed a population structure in which all active members enter at age $x = a$, we have once again obtained a result which is consistent with the appropriate individual cost method.

Benefit outgo for stationary population of active members

We now assume that the active member population is stationary, thus we write:

$n(x,t) = n(x)$

If we have a new plan, the population of deferred pensioners will not initially be stationary, but will become so when $t \geq r-a$. The same is true of the distribution of past service for the deferred pensioners, so we can write:

- $n^{w}(r,t-z) = n(r)$ for $0 \leq z \leq t-r+a$
- $p^{w}(r,t-z) = p(r)$ for $0 \leq z \leq t-r+a$

Thus, our expression for the benefit outgo can be re-written as:

$$B^{w}(t) = k\int_{0}^{t} n^{w}(r,t-z)p^{w}(r,t-z).\frac{l^{w}_{r+z}}{l^{w}_{r}}.dz \quad for \quad t \leq r-a$$

$$B^{w}(t) = k\int_{0}^{t-r+a} n^{w}(r)p^{w}(r).\frac{l^{w}_{r+z}}{l^{w}_{r}}.dz + k\int_{t-r+a}^{t} n^{w}(r,t-z).p^{w}(r,t-z).\frac{l^{w}_{r+z}}{l^{w}_{r}}.dz \quad for \quad t \geq r-a$$

It is demonstrated in Appendix 1 that the present value of the above benefit outgo is:

$$\frac{k.\bar{a}_r}{\delta}\int_{a}^{r} n(x).e^{-\delta(r-x)}\int_{x}^{r}\frac{l_z}{l_x}.\mu^{w}_{z}.\frac{l^{w}_{r}}{l^{w}_{z}}.dz.dx$$

Standard contribution for withdrawal benefit

We now define C^{w}_{s} as the level standard contribution required to fund the withdrawal benefits of the plan. From equation (2), the present value of this standard contribution paid as a perpetuity must equal the present value of the deferred pensioner benefit outgo. Hence, using the result derived above:

$$\frac{C^{w}_{s}}{\delta} = \frac{k.\bar{a}_r}{\delta}\int_{a}^{r} n(x).e^{-\delta(r-x)}\int_{x}^{r}\frac{l_z}{l_x}.\mu^{w}_{z}.\frac{l^{w}_{r}}{l^{w}_{z}}.dz.dx$$

$$\Rightarrow C^{w}_{s} = k.\bar{a}_r\int_{a}^{r} n(x).e^{-\delta(r-x)}\int_{x}^{r}\frac{l_z}{l_x}.\mu^{w}_{z}.\frac{l^{w}_{r}}{l^{w}_{z}}.dz.dx$$

Thus, the standard contribution is the present value of the annual accrual rate of withdrawal benefits summed for all the active members, which is the result we would obtain when calculating the normal cost of the unit credit method for withdrawal benefits. Similar results can be derived for other service table decrements which give rise to ancillary benefits, such disability retirement and death.

$PVB_0 + PVB_1 = (PVC_0 + PVC_1) \times c_s$

From the properties of the unit credit method, we know that the unit credit standard contribution per member, c_0, is the contribution that would fund the future service benefits of a stationary population.

It follows that:

$PVB_0 = PVC_0 \times c_0$

Hence the standard contribution per member for our generalized funding method is given by:

$$c_s = \frac{PVC_0 \times c_0 + PVB_1}{PVC_0 + PVC_1} \qquad (11)$$

Note that c_0 is simply the total unit credit contribution for the plan, as derived at the end of Section 4.1, divided by the initial number of active members.

Present value of benefit outgo and contribution income from surplus new entrants

The present value of the benefit outgo from the surplus new entrants is given by:

$$PVB_1 = \frac{k.\bar{a}_r}{\delta} \int_a^r g_1(x).(r-x).\frac{l_r}{l_x}.e^{-\delta(r-x)}.dx \qquad (12)$$

And the present value of future contributions of one unit per surplus new entrant is given by:

$$PVC_1 = \frac{1}{\delta} \int_a^r g_1(x).\bar{a}_{x:\overline{r-x}|}.dx \qquad (13)$$

Each of the above formulas are derived in Appendix 2. However equations (12) and (13) can also be obtained directly from first principles, as follows.

In both of the above integrals, we can replace $1/\delta$ with the symbol for a perpetuity, ie:

$$\frac{1}{\delta} = \int_0^\infty e^{-\delta t}.dt = \bar{a}_{\overline{\infty}|}$$

Hence equations (12) and (13) can be written as:

$$PVB_1 = \bar{a}_{\overline{\infty}|} \int_a^r \left[k.\bar{a}_r \frac{l_r}{l_x}.(r-x).e^{-\delta(r-x)} \right] g_1(x).dx$$

$$PVC_1 = \bar{a}_{\overline{\infty}|} \int_a^r \bar{a}_{x:\overline{r-x}|}.g_1(x).dx$$

We shall simplify the mathematics required by assuming that the active members are subject to a constant force of decrement given by $\mu = \gamma$, which leads, from equation (9), to the following formula for $g_0(x)$:

$$g_0(x) = Ae^{-\mu(x-a)}$$

As the force of decrement is a constant, we can write:

$$\frac{l_r}{l_x} = e^{-\mu(r-x)}$$

We now assume that the surplus new entrants are some fixed multiple of the new entrants required for a stationary population, ie:

$$g_1(x) = f \times g_0(x) = f \times A \times e^{-\mu(x-a)}$$

where f is a parameter in the range $-1 \leq f \leq \infty$

Thus, the initial population of active members will evolve over a period of $r - a$ years into a stationary population of the same shape as the initial population, but multiplied in number by the factor $1 + f$. When $f = -1$ we will be modeling a plan closed to new entrants, thus the standard contribution will be as for the aggregate method. When $f = 0$ we will be modeling a stationary population, and it is clear from equation (14) that the standard contribution will be as for the unit credit method. As f tends to infinity we will move towards the standard contribution required for a plan with no initial population of members.

Substituting our expressions for $n(x,t_0)$, $\frac{l_r}{l_x}$ and $g_1(x)$ into equation (14) gives:

$$c_s = k \times \frac{f \times (\bar{I}\bar{a})^{\delta}_{\overline{r-a}|} + (\bar{D}\bar{a})^{\delta}_{\overline{r-a}|}}{f \times (\delta + \mu)^{-1} \times (\bar{s}^{\mu}_{\overline{r-a}|} - \bar{a}^{\delta}_{\overline{r-a}|}) + (\bar{I}\bar{s})^{\mu}_{\overline{r-a}|}} \qquad (15)$$

The ratio c_s/c_0 is shown in the table below for different values of f and δ, where the other parameters have the following fixed values:

- $k = 1$
- $a = 20$, $r = 65$
- $\mu = \gamma = 0.05$

Ratio of standard contribution to unit credit contribution

δ \ f	−1	−0.5	0	1	2	5	∞
0.01	1.29	1.03	1.00	0.98	0.98	0.97	0.97
0.02	1.30	1.05	1.00	0.97	0.95	0.94	0.93
0.03	1.31	1.07	1.00	0.95	0.93	0.91	0.88
0.04	1.32	1.08	1.00	0.94	0.91	0.88	0.84
0.05	1.32	1.09	1.00	0.92	0.89	0.85	0.80
0.06	1.31	1.10	1.00	0.91	0.87	0.82	0.76

Figure 4: Active member population which doubles in size

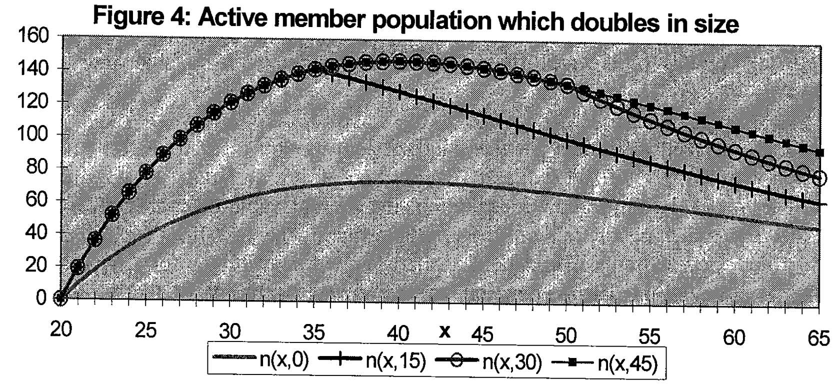


Figure 5: Projected aggregate cash-flows

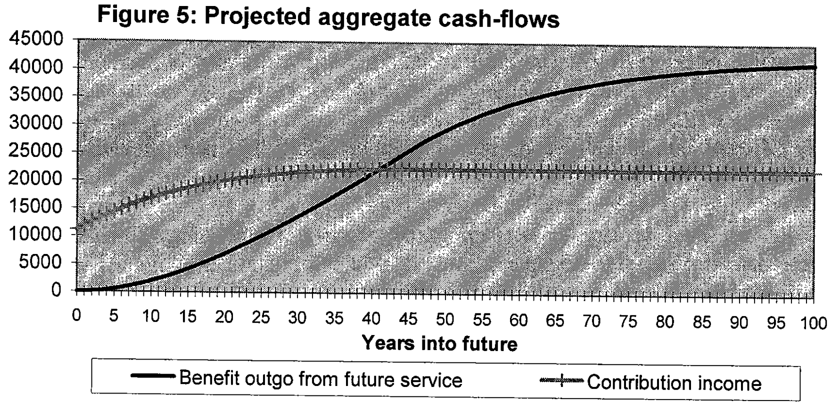
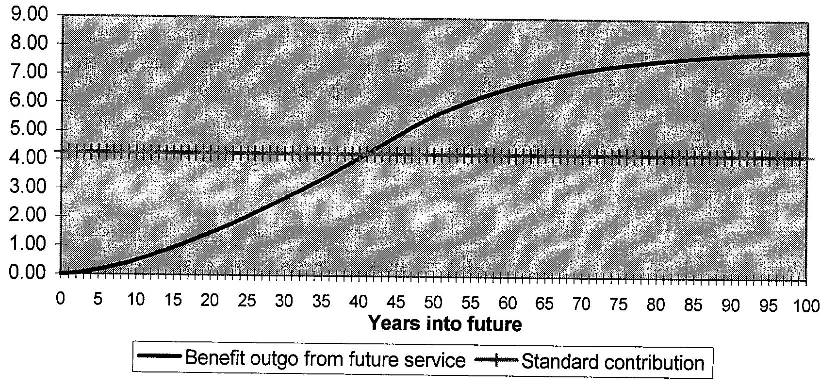


Figure 6: Projected cash-flows per active member



The most obvious further refinements to the generalized funding method presented in this paper would be:

- a non-stationary new entrant density function which could accommodate scenarios in which major short-term changes in the workforce were expected (eg a temporary recruitment drive);
- non-stationary service table decrements which might allow for anticipated changes such as improvements in mortality experience over time;
- the inclusion of alternative strategies for the amortization of surplus or deficit in the projection of the contribution income stream.

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The final stage of the derivation is to substitute the expressions for $p(r,y)$ and $p(r)$ given at the start of Section 4.1, which gives:

$$k.\bar{a}_r \int_0^{r-a} \int_{r-y}^{r} n(z).\frac{l_r}{l_z}.e^{-\delta y}.dz.dy \;+\; k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}\int_a^r n(z).\frac{l_r}{l_z}.dz$$

Reversing the order of integration in the double-integral gives:

$$k.\bar{a}_r \int_a^{r} \int_{r-z}^{r-a} e^{-\delta y}.n(z).\frac{l_r}{l_z}dy.dz \;+\; k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}\int_a^r n(z).\frac{l_r}{l_z}.dz$$

$$= k.\bar{a}_r \int_a^r \left(\frac{e^{-\delta(r-z)} - e^{\delta(r-a)}}{\delta}\right) n(z).\frac{l_r}{l_z}.dz \;+\; k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}\int_a^r n(z).\frac{l_r}{l_z}.dz$$

$$= \frac{k}{\delta}.\bar{a}_r \int_a^r n(z).\frac{l_r}{l_z}.e^{-\delta(r-z)}.dz$$

which is the required formula for the present value of the projected benefit outgo.

Plan closed to new entrants

The expression derived for the present value of the active member benefit outgo was:

$$\int_0^\infty B_a(u).e^{-\delta u}.du = k\int_0^{r-a}\int_0^u [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.e^{-\delta u}.dz.du$$
$$+k\int_{r-a}^{\infty}\int_{u-r+a}^{u} [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.e^{-\delta u}.dz.du$$

Reversing the order of integration in both of the above double-integrals gives the following *three* double integrals, because the second of the above double-integrals must be split into two regions:

$$k\int_0^{r-a}\int_z^{r-a} [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.e^{-\delta u}.du.dz \qquad ①$$

$$+k\int_{r-a}^{\infty}\int_z^{z+r-a} [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.e^{-\delta u}.du.dz \qquad ②$$

$$+k\int_0^{r-a}\int_{r-a}^{z+r-a} [p(r-u+z,0)+u-z].n(r-u+z,0).\frac{l_{r+z}}{l_{r-u+z}}.e^{-\delta u}.du.dz \qquad ③$$

If we inspect the limits of ① and ③, we see that their sum can be expressed one double-integral, which in turn can be combined with ② to give the following double-integral:

Reversing the order of integration in the double-integral gives:

$$k.\bar{a}_r\int_a^r n(z).\mu_z^w.\frac{l_r^w}{l_z^w}\left(\int_{r-z}^{r-a} e^{-\delta y}.p(z,y-r+z).dy\right).dz \;+\; k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}.\int_a^r n(z).p(z).\mu_z^w.\frac{l_r^w}{l_z^w}.dz$$

We now focus our attention on the first-integral of the double-integral, shown in brackets. If we use equation (7) to substitute for $p(z\,,y-r+z)$ we obtain:

$$\int_{r-z}^{r-a} e^{-\delta y}.p(z,y-r+z).dy=\frac{l_z}{n(z)}\int_{r-z}^{r-a} e^{-\delta y}\int_{r-y}^{z}\frac{n(x)}{l_x}.dx.dy$$

Reversing the order of integration in the double-integral above gives:

$$\int_{r-z}^{r-a} e^{-\delta y}.p(z,y-r+z).dy=\frac{l_z}{n(z)}\int_a^z\frac{n(x)}{l_x}.dx\int_{r-x}^{r-a} e^{-\delta y}.dy$$

$$\Rightarrow\int_{r-z}^{r-a} e^{-\delta y}.p(z,y-r+z).dy=\frac{l_z}{n(z)}\int_a^z\frac{n(x)}{l_x}\left(\frac{e^{-\delta(r-x)}-e^{-\delta(r-a)}}{\delta}\right).dx$$

Hence the present value of the deferred pensioner benefit outgo becomes:

$$k.\bar{a}_r\int_a^r l_z.\mu_z^w.\frac{l_r^w}{l_z^w}\int_a^z\frac{n(x)}{l_x}\left(\frac{e^{-\delta(r-x)}-e^{-\delta(r-a)}}{\delta}\right).dx.dz \;+\; k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}.\int_a^r n(z).p(z).\mu_z^w.\frac{l_r^w}{l_z^w}.dz$$

And reversing the order of integration in the double-integral gives:

$$k.\bar{a}_r\int_a^r n(x)\left(\frac{e^{-\delta(r-x)}-e^{-\delta(r-a)}}{\delta}\right)\int_x^r\frac{l_z}{l_x}.\mu_z^w.\frac{l_r^w}{l_z^w}.dz.dx \;+\; k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}.\int_a^r n(z).p(z).\mu_z^w.\frac{l_r^w}{l_z^w}.dz$$

We now turn our attention to the second term in the above expression, again using equation (7) to substitute for the stationary past service function, $p(z)$, of the active members. Thus:

$$k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}.\int_a^r n(z).p(z).\mu_z^w.\frac{l_r^w}{l_z^w}.dz=k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}.\int_a^r l_z.\mu_z^w\frac{l_r^w}{l_z^w}\int_a^z\frac{n(x)}{l_x}.dx.dz$$

Reversing the order of integration in the double-integral gives:

$$k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}.\int_a^r n(z).p(z).\mu_z^w.\frac{l_r^w}{l_z^w}.dz=k.\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}.\int_a^r n(x)\int_x^r\frac{l_z}{l_x}.\mu_z^w.\frac{l_r^w}{l_z^w}.dz.dx$$

Hence the expression for the present value of the deferred pensioner benefit outgo becomes:

$$\int_0^{\infty} B(u).e^{-\delta u}.du = k\int_0^{r-a}\int_0^{u} p(r,u-z).n(r,u-z).\frac{l_{r+z}}{l_r}.e^{-\delta u}.dz.du + k.p(r).n(r)\int_{r-a}^{\infty}\int_0^{u-r+a}\frac{l_{r+z}}{l_r}.e^{-\delta u}.dz.du$$

$$+ \quad k\int_{r-a}^{\infty}\int_{u-r+a}^{u} p(r,t-z).n(r,u-z).\frac{l_{r+z}}{l_r}.e^{-\delta u}.dz.du$$

Reversing the order of integration in each of the above double-integrals gives the following *four* double-integrals (because the third double-integral above must be split into two regions):

$$k\int_0^{r-a}\int_z^{r-a} e^{-\delta u}.p(r,u-z).n(r,u-z).\frac{l_{r+z}}{l_r}.du.dz \qquad ①$$

$$+ \qquad k.p(r).n(r)\int_0^{\infty}\int_{z+r-a}^{\infty} e^{-\delta u}.\frac{l_{r+z}}{l_r}.du.dz \qquad ②$$

$$+ \qquad k\int_0^{r-a}\int_{r-a}^{z+r-a} e^{-\delta u}.p(r,u-z).n(r,u-z).\frac{l_{r+z}}{l_r}.du.dz \qquad ③$$

$$+ \qquad k\int_{r-a}^{\infty}\int_z^{z+r-a} e^{-\delta u}.p(r,u-z).n(r,u-z).\frac{l_{r+z}}{l_r}.du.dz \qquad ④$$

On inspecting the limits of ① and ③, we see that their sum can be expressed as one double-integral, which in turn can be combined with ④ to give another double-integral. Hence, the present value of the projected benefit outgo can be written as:

$$k\int_0^{\infty}\int_z^{z+r-a} e^{-\delta u}.p(r,u-z).n(r,u-z).\frac{l_{r+z}}{l_r}.du.dz \;+\; k.p(r).n(r).\int_0^{\infty}\int_{z+r-a}^{\infty} e^{-\delta u}.\frac{l_{r+z}}{l_r}.du.dz$$

Applying the substitution $y = u - z$ for the integration with respect to u in each of the above double-integrals gives:

$$k\int_0^{\infty} e^{-\delta z}.\frac{l_{r+z}}{l_r}.dz\int_0^{r-a} e^{-\delta y}.p(r,y).n(r,y).dy \;+\; k.p(r).n(r)\int_0^{\infty} e^{-\delta z}\frac{l_{r+z}}{l_r}.dz\int_{r-a}^{\infty} e^{-\delta y}.dy$$

$$= \qquad k.\bar{a}_r\int_0^{r-a} e^{-\delta y}.p(r,y).n(r,y).dy \;+\; k.p(r).n(r).\bar{a}_r.\frac{e^{-\delta(r-a)}}{\delta}$$

We now use equations (4) and (7) to obtain expressions for $p(r,y)\times n(r,y)$ and $p(r)\times n(r)$, each of which will be substituted into the above expression for the present value of the benefit outgo arising from the surplus new entrants.

From equation (4) the active member density function for the surplus new entrants is given by:

$$\Rightarrow \qquad p(r).n(r) = l_r \int_a^r \frac{g_1(x).(r-x)}{l_x} dx$$

On substituting the above expressions for $p(r,y)\times n(r,y)$ and $p(r)\times n(r)$ into the formula for the present value of the benefit outgo we obtain:

$$PVB_1 \quad = k.\bar{a}_r \int_0^{r-a} \int_{r-y}^{r} e^{-\delta y} \frac{l_r.g_1(x).(r-x)}{l_x} dx.dy \; + \; \frac{k.\bar{a}_r.e^{-\delta(r-a)}}{\delta} \int_a^r \frac{l_r.g_1(x).(r-x)}{l_x} dx$$

Reversing the order of integration in the double-integral gives:

$$PVB_1 \quad = k.\bar{a}_r \int_a^r \int_{r-x}^{r-a} e^{-\delta y} \frac{l_r.g_1(x).(r-x)}{l_x} dy.dx \; + \; \frac{k.\bar{a}_r.e^{-\delta(r-a)}}{\delta} \int_a^r \frac{l_r.g_1(x).(r-x)}{l_x} dx$$

$$= k.\bar{a}_r. \int_a^r \frac{l_r.g_1(x).(r-x)}{l_x} \left(\frac{e^{-\delta(r-x)} - e^{-\delta(r-a)}}{\delta} \right) dx \; + \; \frac{k.\bar{a}_r.e^{-\delta(r-a)}}{\delta} \int_a^r \frac{l_r.g_1(x).(r-x)}{l_x} dx$$

$$= \frac{k.\bar{a}_r}{\delta} \int_a^r g_1(x).(r-x).\frac{l_r}{l_x}.e^{-\delta(r-x)}.dx \quad \text{as required.}$$

<u>Present value of contributions of one unit per surplus new entrant</u>

Projecting u years into the future, we define:

- $n(x,u)$ = density function for active members arising from surplus new entrants
- $N(u)$ = total active member population arising from surplus new entrants

$N(u)$ will increase over the first $r - a$ years and then attain its ultimate value. Allowing for the fact that $n(x,u)$ is stationary for all values of x below $a + u$, the expression for $N(u)$ can be written as:

$$N(u) = \int_a^{a+u} n(x).dx + \int_{a+u}^{r} n(x,u).dx \quad for \quad u \le r-a$$

$$N(u) = \int_a^r n(x).dx \quad for \quad u \ge r-a$$

Hence the present value of future contributions of one unit per surplus new entrant is given by:

$$PVC_1 = \int_0^u N(u).e^{-\delta u}.du = \int_0^{r-a} \int_a^{a+u} n(x).e^{-\delta u}.dx.du + \int_0^{r-a} \int_{a+u}^{r} n(x,u).e^{-\delta u}.dx.du + \int_{r-a}^{\infty} \int_a^r n(x).e^{-\delta u}.dx.du$$

Reversing the order of integration in each of the above double-integrals gives:

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