On Double-Boundary Non-Crossing Probability for a Class of Compound Processes with Applications

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Abstract

We develop an efficient method for computing the probability that a non-decreasing, pure jump (compound) stochastic process stays between arbitrary upper and lower boundaries (i.e., deterministic functions, possibly discontinuous) within a finite time period. The compound process is composed of a process modelling the arrivals of certain events (e.g., demands for a product in inventory systems, customers in queuing, or claims/capital gains in insurance/dual risk models), and a sequence of independent and identically distributed random variables modelling the sizes of the events. The events arrival process is assumed to belong to the wide class of point processes with conditional stationary independent increments which includes (non-)homogeneous Poisson, binomial, negative binomial, mixed Poisson and doubly stochastic Poisson (i.e., Cox) processes as special cases. The proposed method is based on expressing the non-exit probability through Chapman-Kolmogorov equations, re-expressing them in terms of a circular convolution of two vectors which is then computed applying fast Fourier transform (FFT). We further demonstrate that our FFT-based method is computationally efficient and can be successfully applied in the context of inventory management (to determine an optimal replenishment policy), ruin theory (to evaluate ruin probabilities and related quantities) and double-barrier option pricing or simply computing non-exit probabilities for Brownian motion with general boundaries.

Keywords: applied probability, doubly stochastic Poisson (i.e., Cox) processes, fast Fourier transform, inventory management under stochastic demand, finite-time non-ruin probability

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1. Introduction

Finding the probability that a stochastic process stays between two boundaries is of importance in many disciplines among which: statistics, for computing p-values and power of goodness-of-fit statistics (see e.g., Doob 1949, Anderson 1960, Durbin 1971, Steck 1971, Siegmund 1986, Bischoff et al. 2003), for constructing confidence intervals for distribution functions (see Wald & Wolfowitz 1939 and Steck 1971); finance and financial engineering, for pricing double-barrier options (see Borovkov & Novikov 2005 and Fusai et al. 2016); actuarial science, in modelling the surplus of an insurance company (see Teunen & Goovaerts 1994, Goffard & Lefèvre 2018 and Dimitrova et al. 2017); economics, in investigating the power of the CUSUM test for structural change (see Krämer et al. 1988), and also in many other fields which can be traced from e.g., Wang & Pötzelberger (2007). Naturally, double-boundary (non-)crossing (abbreviated DB(non-)C) problems have attracted considerable attention in the applied probability literature (see e.g., Buonocore et al. 1990, Lotov 1996, Novikov et al. 1999 and Fu & Wu 2010). Nevertheless, explicit expressions for the probability of non-crossing have not been obtained with very few exceptions (see e.g., Pötzelberger & Wang 2001 and Dimitrova et al. 2017). Approximation methods have been proposed in the case when the stochastic process is a Brownian motion, e.g., by Wang & Pötzelberger (2007) for general boundaries, and by Borovkov & Novikov (2005) and Ycart & Drouilhet (2016) for (piecewise) linear boundaries. Numerical methods to compute the non-exit probability for the case of a homogeneous Poisson process and arbitrary boundaries have been developed by Khmaladze & Shinjikashvili (2001) and Moscovich & Nadler (2017).

Our main contribution in this paper is developing a new method for computing DB(non-)C probability that is applicable for a very large class of models (processes and boundaries). Namely, we consider general boundaries (i.e., arbitrary deterministic functions with possible jump discontinuities) and assume that the underlying process may not necessarily be homogeneous Poisson. It can be any process from the wide class of compound processes in which the arrival process belongs to the large family of *point processes with conditional stationary independent increments* (PPC-SII). This family includes processes that may not necessarily be stationary and have independent increments, such as the Cox process and the mixed Poisson process (see Appendix A) and allow for dependence/clustering of the event arrivals. The method is based on Chapman-Kolmogorov equations, the circular convolution theorem and fast Fourier transform (FFT) and is an extension of the approach of Khmaladze & Shinjikashvili (2001) and Moscovich & Nadler (2017). It can be used to compute extremely efficiently the probability that a compound stochastic process with PPCSII arrivals does not exit the strip between two general boundaries (see Section 4). We highlight that the proposed method has no alternative direct comparator in the current literature in terms of its generality of assumptions and numerical efficiency.

Our second major contribution is in demonstrating that boundary crossing models and the method we develop can be very useful in the context of operations research, in formulating and solving inventory management optimization problems (see Section 4.1), in risk theory in computing ruin probabilities (see Section 4.2) and in finance in pricing barrier options or computing non-exit probabilities for Brownian motion (see Section C of the Supplementary Material).

To the best of our knowledge, we show for the first time that inventory management optimization problems can be elegantly formulated (and solved) by incorporating an appropriate DB(non-)C probability constraint (see Problem 4.1 in Section 4.1). A theoretical contribution is then made by proving the existence of a unique solution of the optimization Problem 4.1 (cf., Proposition 4.1, Section 4.1). A methodological advancement is also made by considering the general PPCSII class which allows (overlapping and non-overlapping) clustering of arrivals. The need of modelling clusters of demand arriving periodically has long been recognized e.g., in the inventory and the supply chain management literature. Clusters of intermittent and lumpy demand typically arise in managing supply of spare parts in the sectors of IT services, aviation, automotive and manufacturing (see e.g., Croston 1972, Willemain et al. 1994b, Ghobbar & Friend 2003, Willemain et al. 2004, Gutierrez et al. 2008, Teunter & Duncan 2009, Teunter et al. 2011 and Berling & Marklund 2013).

As yet another contribution in this paper, we highlight the use of the proposed FFT-based method to compute a certain single-boundary crossing probability, known in the risk and ruin theory as ruin probability. As has been demonstrated (see Embrechts et al. 2004, Kaishev et al. 2008, Dimitrova et al. 2015), computing ruin probability is important in modelling liquidity risk, estimating operational risk and assessing risk capital in insurance and banking, and also in other real-life risk analysis applications among which, flood risk, systems reliability risk and emerging disease spread risk (see Dimitrova et al. 2015). Ruin occurs when the compound process modelling aggregate claims exceeds for the first time the upper boundary (representing the aggregate insurance premium) within a finite time interval. Interpreting the latter as DB(non-)C probability (lower boundary equal to zero) allows us to employ the proposed method to efficiently compute ruin probabilities for any claims arrival model from the PPCSII class and arbitrarily distributed claim sizes. To the best of our knowledge, no such alternative general method, or one specifically for Cox process arrivals (see Example 4.2 in Section 4.2), has been considered in the literature.

Last but not least, we also demonstrate that our proposed FFT-based method is useful in computing the DB(non-)C probability for general boundaries (arbitrary functions with possible jump discontinuities) with respect to a Brownian motion (see Table C.1 in Section C of the Supplementary Material) which could be further utilized in the variety of fields listed above and also summarized by Wang & Pötzelberger (2007). For general boundaries, Borovkov & Novikov (2005) and Ycart & Drouilhet (2016) propose a piecewise linear approximation to the latter, combined with computing multiple conditional probabilities that Brownian motion stays between the corresponding upper and lower linear segments, known as wedge probabilities. Khmaladze & Shinjikashvili (2001) use the convergence of the Poisson process to Brownian motion, as also utilized in our approach, but apply direct convolution instead of FFT. However, none of these approaches has been tested numerically and demonstrated to achieve the generality and numerical efficiency demonstrated by our FFT-based method, as illustrated on an example of multi-step kick-out double-barrier option pricing (see Section C of the Supplementary Material).

The paper is organized as follows. In Section 2, we formulate the DB(non-)C problem. In Section 3, we develop the fast and accurate method for computing the DB(non-)C probability in (2), expressing it through Chapman-Kolmogorov equations, re-expressing the latter in terms of a circular convolution of two appropriate vectors which is then computed applying FFT. In Section 4.1, we show how the method can be used to solve an inventory management optimization problem. In Section 4.2, we apply the method to compute the finite-time probability of non-ruin of an insurance company in the context of risk theory. In Section C of the Supplementary Material, we demonstrate the use of the proposed FFT-based method in computing DB(non-)C probabilities for Brownian motion and pricing double-barrier options. Section 5 concludes the paper.

2. Problem Formulation

In order to introduce the DB(non-)C problem we are concerned with in this paper, let us first introduce some notation. For a point process, ξ, denote by $\xi[u,t]$ the number of points in the interval $[u,t]$, $0 \leq u < t$. We will assume that the process $\xi[0,t] = \xi(t)$ models the consecutive arrival times T_1, T_2, $\ldots$ of certain events, e.g., insurance claims, or capital gains, or quantities of demand, or some other events of interest in a particular application. We will also assume that the sizes of the consecutive events are modelled by the sequence of non-negative, independent, identically distributed (i.i.d.) random variables X_k, $k = 1, 2, ...$, with cumulative distribution function (cdf) $F_X(x)$. We will consider the (compound) process

$$\tau(t) = \sum_{k=1}^{\xi(t)} X_k, \tag{1}$$

modelling the accumulated (up to time t) sizes, X_k, of the events. We denote by $0 < Y_1 < Y_2 < \ldots$ the partial sums $Y_i = \sum_{k=1}^{i} X_k$ with the assumption that at each instant T_i the process $\tau(t)$ jumps at the level Y_i, $i = 1, 2, \ldots$. The jumps, $0 < Y_1 < Y_2 < \ldots$, could be interpreted as reflecting the current level of the cumulative process of interest, τ, e.g., the stochastic demand in an inventory (supply chain) system, or the aggregate claims to an insurance company, or the accumulation of reserves of a bank. We will assume that the random variables X_1, X_2, $\ldots$ and the process $\xi(t)$ are mutually independent and that $\xi(t)$ belongs to the rather general class of PPCSII defined in details in Appendix A (cf., Serfozo 1972). Examples of such processes include: subclass A - point processes with independent increments (PPII) as (non-)homogeneous Poisson and negative binomial process; subclass B - doubly stochastic Poisson processes, known also as Cox processes with mean value process $\nu(t)$, $t \geq 0$, among which (i) mixed Poisson and (ii) order statistics point processes (OSPP) (e.g., Pólya-Lundberg process, linear birth process with immigration, see Appendix A); subclass C - conditional compound Poisson process with respect to $\nu(t)$ given the appropriate σ-algebra. These subclasses of PPCSII cover a broad range of models for the events arrival process $\xi(t)$, used in many fields such as, finance, insurance, operations research, queuing, economics, physics, astronomy and many other.

Thus, we consider the pure-jump stochastic process $\tau(t) = Y_{\xi(t)}$, defined in (1), with right-continuous, non-decreasing trajectories and are interested in the unconditional probability,

$$P\left(g(t) \leq \tau(t) \leq h(t), 0 \leq t \leq z\right), \tag{2}$$

that within a fixed time interval $[0,z]$, the process $\tau(t)$ stays between an upper and a lower deterministic, time-dependent boundaries $h(t)$ and $g(t)$. Formal specification of the boundaries, $h(t)$ and $g(t)$, is provided in Section 3 and their interpretation depends on the application. For example, $h(t)$ and $g(t)$ could model the cumulative insurance premium income and expense outgo, respectively in the insurance and dual risk models considered in the context of ruin theory.

Recall that we are interested in computing the DB(non-)C probability in (2). Let us note that in the (general) case when $\xi(t)$ is assumed a Cox process (see the definition of the subclass B and also C in Appendix A), the process $\tau(t)$ depends on the mean value process $\nu(t)$ of the process $\xi(t)$. Therefore, in order to evaluate the unconditional DB(non-)C probability in (2) with respect to $\tau(t)$, one needs to average over the appropriate σ-algebra $\mathcal{A}$ with respect to which $\nu(t)$ is measurable, i.e.,

$$\begin{aligned} &P(g(t) \leq \tau(t) \leq h(t), 0 \leq t \leq z) \\ &= \int_{\Omega} P(g(t) \leq \tau(t) \leq h(t), 0 \leq t \leq z|\mathcal{A})dP. \end{aligned} \tag{3}$$

In the simpler cases where $\nu(t)$ is deterministic (subclass A), there is no need for averaging and (3) simplifies to (2).

Recall that our aim in the present paper is to develop a numerically efficient method for evaluating the DB(non-)C probability given in (3), under reasonably general assumptions on the process $\tau(t)$ (cf., Equation (1)) including the arrival process $\xi(t)$ and on the boundaries, $g(t)$ and $h(t)$ (see Section 3). Without loss of generality, we will assume that the random variables X_k in the definition (1) of $\tau(t)$ are integer-valued and the random variable V in the definition of $\xi(t)$ as a mixed Poisson process (cf., subclass B.(i) in Appendix A) has a discrete distribution. In case the latter are continuous random variables, it is straightforward to apply discretization using, e.g., the *method of local moment matching* (see Panjer 2006 and the references therein). By appropriately controlling the discretization step and the order up to which moments of the continuous and the approximating discrete distributions are matched, one can achieve a very good accuracy of approximation.

The evaluation of (3) requires efficient evaluation of the conditional probability $P(g(t) \leq \tau(t) \leq h(t), 0 \leq t \leq z|\mathcal{A})$ in which $\tau(t)$ has (conditionally) independent increments. Therefore, without loss of generality, and to avoid complicating the notation, we will formally describe our method in Section 3.1 for the unconditional case, i.e., assuming that the process $\xi(t)$ is from the subclass A of PPII. Then, in Sections 3.2 and 3.3 we describe how it could be utilized for processes from the other subclasses of PPCSII. In Section 4, we will illustrate the evaluation of the DB(non-)C probability in (3) when $\xi(t)$ is a Poisson process (subclass A) and a Cox process (subclass B). Let us note that the method we develop here generalizes the approaches proposed earlier by Khmaladze & Shinjikashvili (2001) and by Moscovich & Nadler (2017) for computing DB(non-)C probabilities in the special case when $\tau(t)$ is a homogeneous Poisson process.

3. A Method to Compute the DB(non-)C Probability

3.1. The case when $\xi(t)$ is a PPII (i.e., is from Subclass A)

As mentioned earlier, without loss of generality, we assume throughout this section that the arrival process $\xi(t)$ is a PPII and that the random variables, X_k, $k = 0, 1, \ldots$ are integer-valued. Also, without loss of generality, we further assume that the two functions $h(t)$ and $g(t)$, $t \in [0, +\infty)$, have the following properties: $h(t)$ and $g(t)$ are non-decreasing functions, such that $h(0) \geq 0$ and $g(0) \leq 0$; $h(t) \geq g(t)$, $\forall t \in [0, +\infty)$; $h(t)$ and $g(t)$ may have jump-discontinuities, and we assume that $h(t)$ is right-continuous and $g(t)$ is left-continuous. If this is not the case, $g(t)$ and $h(t)$ may be replaced by their non-decreasing counterparts without affecting the non-crossing probability (2), see e.g., Lehmann (1998). We will further consider restrictions of $h(t)$ and $g(t)$ on $[0, z]$, $z > 0$, denoted by $h_z(t)$ and $g_z(t)$. For convenience, the subscript z will further be dropped. In view of these definitions, we will assume that the process $\tau(t)$ does not exit if its trajectory touches either one of the two boundaries. Since X_k, $k = 1, 2, ...$, are assumed integer-valued, the process $\tau(t)$ can be viewed as a point process, and we will alternatively use the notation $\tau[u, t]$ to denote the number of points in the interval $[u, t]$, $0 \leq u < t$.

Denote by $l \equiv \lfloor g(z) \rfloor$ and by $n \equiv \lfloor h(z) \rfloor$, where $\lfloor x \rfloor := \max\{v : v \leq x; v \in \mathbb{Z}\}$ and $\mathbb{Z}$ denotes the set of all integers. Denote also by $\lceil x \rceil := \min\{v : v \geq x; v \in \mathbb{Z}\}$. For every integer, $i = 0, 1, \ldots, l$, let $t_i^g = \sup\{t : g(t) \leq i\}$ and for every integer, $i = 0, 1, \ldots, n$, let $t_i^h = \inf\{t : h(t) \geq i\}$. Let $T(g) = \{t_i^g\}$, $i = 0, \ldots, l$ and $T(h) = \{t_i^h\}$, $i = \lfloor h(0) \rfloor, \ldots, n$ and let $0 \equiv t_0 < t_1 < \ldots < t_N \equiv z$ be the ordered set of all distinct points from $T(g) \cup T(h) \cup \{z\}$. We will also use the notation $h(t_i) = h_i$, $g(t_i) = g_i$, $i = 0, ..., N$, $g(t_{N+1}) \equiv g_N$.

It is easy to verify that a non-decreasing step function $f : [0, z] \to \{0, 1, 2, \ldots\}$ satisfies $g(t) \leq f(t) \leq h(t)$, for all $t \in [0, z]$ if and only if it satisfies these conditions at all discrete times $t \in T(g) \cup T(h) \cup \{z\}$, (see e.g., Khmaladze & Shinjikashvili 2001 and Moscovich & Nadler 2017). Therefore, by analogy with Lemma 3.2 of Khmaladze & Shinjikashvili (2001), we have the following proposition.

Proposition 3.1. For the process $\tau[0,t]$ and the boundaries $g(t)$ and $h(t)$, we have

$$P\left(g(t) \leq \tau[0,t] \leq h(t), 0 \leq t \leq z\right) = P\left(\bigcap_{i=1}^{N} \{g(t_{i+1}) \leq \tau[0,t_i] \leq h(t_{i-1})\}\right). \tag{4}$$

Thus, to find the DB(non-)C probability, it suffices to evaluate the probability on the right-hand side of (4). For the purpose, denote

$$Q(s,m) = P\left(g(t) \leq \tau[0,t] \leq h(t),\ \ t \in [0,s] \text{ and } \tau[0,s] = m\right), \tag{5}$$

where $s \in [0,z]$ and $m = 0,1,2,\ldots$. In other words, $Q(s,m)$ is the probability that $\tau[0,s] = m$ and the trajectory of the process $\tau[0,t]$ does not cross the boundaries $g(t)$ and $h(t)$ for $0 \leq t \leq s$.

From (4) and (5), it can be seen that

$$\begin{aligned} P\left(g(t) \leq \tau[0,t] \leq h(t), 0 \leq t \leq z\right) &= P\left(\bigcap_{i=1}^{N} \{g(t_{i+1}) \leq \tau[0,t_i] \leq h(t_{i-1})\}\right) \\ &= \sum_{m=\lceil g(t_{N+1})\rceil}^{\lfloor h(t_{N-1})\rfloor} Q\left(t_N, m\right), \end{aligned} \tag{6}$$

and that, for any $i \in \{0,\ldots,N-1\}$ and any $m \in \{0,1,2,\ldots\}$, $Q(t_{i+1},m)$ is given by the Chapman-Kolmogorov equations

$$Q(t_{i+1},m) = \begin{cases} \sum_{k=\lceil g(t_{i+1})\rceil}^{m} Q(t_i,k) P\left(\tau[t_i,t_{i+1}] = m-k\right), & \text{if } \lceil g(t_{i+2})\rceil \leq m \leq \lfloor h(t_i)\rfloor \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where $Q(t_0,m) = \mathbb{1}_{\{m=0\}}$.

The probability in (6) can then be computed by evaluating the sum $\sum_{m=\lceil g(t_{N+1})\rceil}^{\lfloor h(t_{N-1})\rfloor} Q\left(t_N,m\right)$, using the Chapman-Kolmogorov equations (7), iterating over the values $t_1,\ldots t_N$. Similar approach was suggested by Khmaladze & Shinjikashvili (2001) to compute the DB(non-)C probability in (4) with strict inequalities, for the special case when $t_N \equiv z \equiv 1$, and $\tau[0,t]$ is a homogeneous Poisson process with intensity $n = \lfloor h(z)\rfloor = \lfloor h(1)\rfloor$, (i.e., when $\xi(t)$ in (1) is a homogeneous Poisson process with intensity n and $X_k \equiv 1$, $k = 1,2,\ldots$ with probability one). These authors show that the computational cost in this case is at most $\mathcal{O}(n^3)$. The numerical efficiency of computing DB(non-)C probabilities for a homogeneous Poisson process was recently improved to at most $\mathcal{O}(n^2 \log n)$ by Moscovich & Nadler (2017). They noted that the vector $(Q(t_{i+1}, 0), Q(t_{i+1}, 1), \ldots, Q(t_{i+1}, n))$ in Equation (7), when $\tau[0, t]$ is a homogeneous Poisson process, has the form of a truncated linear convolution of two vectors and one can apply the circular convolution theorem and FFT to compute $Q(t_N, m)$, $m = \lceil g(t_{N+1}) \rceil, \ldots, \lfloor h(t_{N-1}) \rfloor$, which gives $P\left(g(t) \leq \tau[0, t] \leq h(t), 0 \leq t \leq 1\right)$. Here, we apply a similar approach to compute the more general DB(non-)C probability in (4) as follows.

Let us introduce the vectors $\boldsymbol{\tau}(t_0) = \left(P(\tau[t_0, t_1] = 0), \ldots, P(\tau[t_0, t_1] = \lfloor h_0 \rfloor), \underbrace{0, \ldots, 0}_{2n - \lfloor h_0 \rfloor + 1} \right)$ and $\mathbf{Q}(t_0) = (1, \underbrace{0, \ldots, 0}_{2n+1})$. From the Chapman-Kolmogorov equations (7), it can directly be verified that the vector

$$\mathbf{Q}(t_1) := \left(\underbrace{0, \ldots, 0}_{\lceil g_2 \rceil}, Q(t_1, \lceil g_2 \rceil), \ldots, Q(t_1, \lfloor h_0 \rfloor), \underbrace{0, \ldots, 0}_{2n - \lfloor h_0 \rfloor + 1} \right) \equiv (\boldsymbol{\tau}(t_0) \star \mathbf{Q}(t_0)) \circ \mathbf{L}(t_1),$$

where "$\star$" denotes circular convolution, "$\circ$" denotes point-wise multiplication and

$$\mathbf{L}(t_1) = (\underbrace{0, \ldots, 0}_{\lceil g_2 \rceil}, \underbrace{1, \ldots, 1}_{\lfloor h_0 \rfloor + 1 - \lceil g_2 \rceil}, \underbrace{0, \ldots, 0}_{2n - \lfloor h_0 \rfloor + 1}).$$

By analogy, for

$$\boldsymbol{\tau}(t_i) = \left(P(\tau[t_i, t_{i+1}] = 0), \ldots, P(\tau[t_i, t_{i+1}] = \lfloor h_i \rfloor), \underbrace{0, \ldots, 0}_{2n - \lfloor h_i \rfloor + 1} \right)$$

and

$$\boldsymbol{Q}(t_i) = \left(\underbrace{0, \ldots, 0}_{\lceil g_{i+1} \rceil}, Q(t_i, \lceil g_{i+1} \rceil), \ldots, Q(t_i, \lfloor h_{i-1} \rfloor), \underbrace{0, \ldots, 0}_{2n - \lfloor h_{i-1} \rfloor + 1} \right),$$

we have the representation

$$\mathbf{Q}(t_{i+1}) := \left(\underbrace{0,\ldots,0}_{\lceil g_{i+2}\rceil}, Q(t_{i+1},\lceil g_{i+2}\rceil),\ldots,Q(t_{i+1},\lfloor h_i\rfloor),\ \underbrace{0,\ldots,0}_{2n-\lfloor h_i\rfloor+1} \right), \qquad \equiv (\boldsymbol{\tau}(t_i) \star \mathbf{Q}(t_i)) \circ \mathbf{L}(t_{i+1}), \tag{8}$$

where $\mathbf{L}(t_{i+1}) = (\underbrace{0,\ldots,0}_{\lceil g_{i+2}\rceil}, \underbrace{1,\ldots,1}_{\lfloor h_i\rfloor+1-\lceil g_{i+2}\rceil}, \underbrace{0,\ldots,0}_{2n-\lfloor h_i\rfloor+1})$, for $i = 0,1,\ldots,N-1$.

Let us now introduce the Fourier transforms $\mathcal{F}\{\boldsymbol{\tau}(t_i)\}$ and $\mathcal{F}\{\mathbf{Q}(t_i)\}$ with elements, $\sum_{k=0}^{2n+1}\tau(t_i)_k e^{2\pi\imath k\frac{l}{2n+2}}$, and $\sum_{k=0}^{2n+1} Q(t_i)_k e^{2\pi\imath k\frac{l}{2n+2}}$, $l = 0,1,\ldots,2n+1$, respectively. Applying the convolution theorem, for the Fourier transform of the convolution in (8), we have

$$\mathbf{C} = \mathcal{F}\{(\boldsymbol{\tau}(t_i) \star \mathbf{Q}(t_i))\} = \mathcal{F}\{\boldsymbol{\tau}(t_i)\} \circ \mathcal{F}\{\mathbf{Q}(t_i)\} \tag{9}$$

and one can reexpress (8) as

$$\mathbf{Q}(t_{i+1}) = \mathcal{F}^{-1}\{\mathbf{C}\} \circ \mathbf{L}(t_{i+1}), \tag{10}$$

where $\mathcal{F}^{-1}\{\mathbf{C}\}$ is the inverse Fourier transform.

After N iterations over $i = 0,1,\ldots,N-1$ of computing $Q(t_{i+1},m)$, following (10) in conjunction with (9), we obtain $Q(t_N,m) \equiv Q(z,m)$ for $\lceil g_{N+1}\rceil \le m \le \lfloor h_{N-1}\rfloor$ and by summing these values over m, in view of (6), we obtain the required non-exit probability, $P\left(g(t) \le \tau[0,t] \le h(t), 0 \le t \le z\right)$. A key step at each iteration is the application of the FFT algorithm to compute the Fourier transforms $\mathcal{F}\{\boldsymbol{\tau}(t_i)\}$ and $\mathcal{F}\{\mathbf{Q}(t_i)\}$ in (9) which leads to a running time of at most $\mathcal{O}(n\log n)$ for each convolution in (9), leading to a significant speedup of the evaluation of (10) and therefore of the total DB(non-)C probability.

3.2. The case when $\xi(t)$ is a Cox process (i.e., is from Subclass B)

If $\xi(t)$ is a Cox process, in general, it may have uncountably many trajectories and in order to compute the DB(non-)C probability in (3), one needs an efficient method for computing the integral on the right-hand side of (3). In order to achieve that, we combine quasi-Monte Carlo (QMC) simulation with the FFT-based method described in Section 3.1. More precisely, using QMC we first simulate a large number, M, of trajectories of the intensity process $\nu(t)$ and then conditionally on a trajectory $\nu(\cdot,\omega)$, apply the FFT-based method to compute the conditional probability $P(g(t) \le$

$\tau[0,t] \le h(t), 0 \le t \le z|\mathcal{A})$, noting that given $\nu(\cdot,\omega)$, $\tau[0,t]$ is a PPII. The unconditional non-exit probability in (3) is then estimated by averaging over all M conditional probabilities. We illustrate in Section 4.2 that the combined QMC-FFT-based method is remarkably efficient (in terms of speed and accuracy), producing six correct digits after the decimal point in reasonable time.

Let us note that in the case when the intensity process $\nu(t)$ has countably many trajectories, there is no need for simulation and the DB(non-)C probability in (3) is estimated by summing up all conditional probabilities which are very efficiently computed using the FFT-based method of Section 3.1. To elaborate on this, assume that the process $\nu(t)$ has countably many trajectories, $\mu_1(\cdot)$, $\mu_2(\cdot)$, ... We will introduce the events $A_j := \{\omega : \nu(t,\omega) \equiv \mu_j(t), t \ge 0\}, j = 1, 2, \ldots$, which form a complete set of disjoint events and $\cup_{j=1}^{\infty} A_j = \Omega$. Denote by $\mathcal{A}$ the minimal σ-algebra which includes the events $A_j, j = 1, 2, \ldots$. Therefore, for the process $\tau[0,t]$, we have

$$\tau[0,t] = \sum_{k=1}^{\xi(t)} X_k = \begin{cases} \sum_{k=1}^{N(\mu_1(t))} X_k, \text{for } \omega \in A_1, \\ \sum_{k=1}^{N(\mu_2(t))} X_k, \text{for } \omega \in A_2, \\ \ldots \end{cases},$$

and (3) can be rewritten as

$$\begin{aligned} & P(g(t) \le \tau[0,t] \le h(t), 0 \le t \le z) \\ & = \sum_{j=1}^{\infty} P(A_j) P\Big(g(t) \le \tau[0,t] \le h(t), 0 \le t \le z | A_j\Big) \\ & = \sum_{j=1}^{\infty} P(A_j) P\left(g(t) \le \sum_{k=1}^{N(\mu_j(t))} X_k \le h(t), 0 \le t \le z\right). \end{aligned} \tag{11}$$

Since the terms under the sum in (11) can be ordered in descending order of the probabilities $P(A_j)$, summation in (11) can be appropriately truncated to achieve any required accuracy. As mentioned above, even in the case when $\nu(t)$ has uncountably many trajectories, the unconditional DB(non-)C probability in (3) is evaluated with a very good accuracy using the QMC-FFT-based method, achieving five-six correct digits after the decimal place (see Section 4.2).

3.3. The case when $\xi(t)$ is a mixed Poisson (possibly OS) point process

Finally, we consider the case when the arrival process $\xi(t)$ is from subclass B.(i), i.e., is a mixed Poisson process (possibly an OSPP, i.e., from subclass B.(ii)). In the latter case, $\nu(t) = V\lambda(t)$, where $V > 0$ is a (discrete) random variable, and (3) can be rewritten as

$$\begin{aligned} &P(g(t) \le \tau[0,t] \le h(t), 0 \le t \le z) \\ &= \sum_{j \in \mathcal{S}} P(A_j) P\Big(g(t) \le \tau[0,t] \le h(t), 0 \le t \le z | A_j\Big) \\ &= \sum_{j \in \mathcal{S}} P(V = j) P\left(g(t) \le \sum_{k=1}^{N(j\lambda(t))} X_k \le h(t), 0 \le t \le z\right), \end{aligned} \tag{12}$$

where $A_j := \{\omega : V(\omega) = j\}, j \in \mathcal{S}$, and $\mathcal{S}$ is the set of all values the random variable V can take.

We conclude this section by highlighting once again that the FFT-based method described above leads to an extremely fast numerical implementation. For example, as shown in Section 4.2, when $\xi(t)$ is a Cox process, $n = \lfloor h(z) \rfloor = 17$, and the number of simulated trajectories M of the intensity process $\nu(t)$ is 10000, the computation time is about one minute. All the numerical experiments have been performed with C++ running under MacOS High Sierra on a laptop equipped with a 2.3GHz quad-core Intel Core i5 processor and 8GB of RAM.

4. Applications

In this section, we demonstrate how the novel, numerically efficient method, proposed in this paper, can be used to give innovative solutions to the DB(non-)C problems, naturally arising in inventory management. Furthermore, as an application in ruin theory, we demonstrate how the QMC-FFT-based method, described in detail in Section 3.2, is applied to efficiently evaluate a single-boundary non-crossing probability, known as the ruin probability in the context of insurance risk and ruin theory, assuming the claims arrival process, $\xi(t)$ is a Cox process. As noted earlier, to the best of our knowledge, no alternative numerically efficient method has been proposed in the literature that is applicable in this general setting. In addition in Section C of the Supplementary Material, an example on pricing double-barrier options with general boundaries is given.

4.1. Inventory Management Optimization

We will consider an inventory model with stochastic demand. Various demand processes have been studied, starting from homogeneous Poisson and renewal processes modelling demand arrivals, to more general compound Poisson, compound renewal demand processes and compound processes with more general arrivals of demand. For brief review accounts on the corresponding vast literature, we refer to Scarf (1963), Zheng (1992), Presman & Sethi (2006), de Kok et al. (2018) (and references therein) and to papers by Song & Zipkin (1996), Axsäter (2003), Arslan et al. (2007), Bijvank & Johansen (2012), Stenius et al. (2016) and Johansson et al. (2019), where various inventory models with (compound) Poisson demands have been considered. An empirical investigation of the goodness-of-fit of such compound Poisson models in the context of spare parts management has been conducted by Lengu et al. (2014).

We will consider here a simple single-item (single-product) single warehouse periodic review inventory model in which batches of different sizes are shipped (i.e., replenished) from a supplier to the warehouse, over a fixed time horizon, $[0, z]$, with certain (fixed) lead times. The cumulative demand over the period $[0, z]$ is assumed to be modelled by the compound process $\tau(t)$ as introduced in (1), where the demand arrival process, $\xi(t)$, is assumed to be from the family of PPCSII, and X_i are i.i.d. variables modelling the (random) sizes of demand at each point of $\xi(t)$. As noted earlier, $\tau(t)$ belongs to a rather general class of processes since the arrival process $\xi(t)$ includes the Poisson, binomial and negative binomial processes, as well as the mixed Poisson and the Cox processes as special cases. This brings about significant flexibility in modelling the dynamics of real demand processes in various supply chain and inventory management applications, e.g., manufacturing and distribution of goods along the chain from producer to warehouses and further to (a large number of) retailers and market companies. Such situations are typical in production and sales of goods in the construction industry (see Stenius et al. 2016), and more generally in manufacturing, logistics, wholesale, retail and services industries. In the related inventory management applications, it is often more realistic to assume that the intensity of the (stochastic) demand is itself an appropriate stochastic process, a case that is covered by our model if a Cox process is chosen for modelling the demand arrivals (cf., Willemain et al. 1994a).

The upper and lower boundary functions, $h(t)$ and $g(t)$, appearing in (2) are interpreted as follows. The function $g(t)$ is viewed as modelling the minimum demand below which the firm will fail to reach its sales targets and ensure flow of revenue sufficient to cover its operating costs and sustain its business. As for the function $h(t)$, it models the aggregate units of the item replenished throughout the period $[0, z]$. The aggregate replenishment function $h(t)$ is assumed a pure jump (i.e., piecewise constant) function with jumps (batch sizes) at certain (shipment) times. This corresponds to batches of fixed sizes $W_i \geq 0$, $i = 1, 2, ...$ being replenished at appropriate shipment times $0 \leq t_1 < t_2 < t_3 < \cdots < z$. We will denote the class of such piecewise constant functions on $[0, z]$ with $r > 0$ jumps (i.e., $r > 0$ number of shipments) by $\mathcal{H}_r$, and assume that $h(t) \in \mathcal{H}_r$, i.e.,

$$h(t) = W_1 + W_2 \mathbb{1}_{\{t_2 \leq t\}} + \cdots + W_r \mathbb{1}_{\{t_r \leq t\}}, \ 0 \leq t \leq z$$

where W_1 is an amount replenished initially (at time $t_1 = 0$) and $W_2, W_3, ..., W_r$ are amounts to be shipped subsequently at later times $0 < t_2 < t_3 < ... < t_r < z$, with $W = W_1 + W_2 + ... + W_r$ the total number of units ordered at time $t = 0$, for shipment in the period $[0, z]$.

In what follows, we will assume that the lower boundary minimum demand function $g(t)$ is fixed a priory. In particular, without loss of generality we set

$$g(t) = d\ t - b, \ 0 \leq t \leq z, \tag{13}$$

where d, $b > 0$ are constants. Such a choice is reasonable as it allows for an initial "gratis" period (up to time $t = b/d$) with no minimum demand constraint, followed by a period with a linear lower limit demand. Other more complex minimum demand functions which reflect the operating costs related to the particular business (i.e., manufacturing, logistics, sales and services industries, etc.) can be implemented.

As part of our model, we will consider two types of costs, *ordering and holding costs.* The cost of ordering includes the (fixed) costs of placing an order plus the (variable) logistics (shipment) costs related to transportation and reception. We assume that the ordering costs depend on the total amount W and on the number of shipments, r, and are modelled as

$$C_o(W, r) = \begin{cases} 0, & \text{if } W = 0 \\ k + c_s(r)W, & \text{if } W > 0 \end{cases}, \tag{14}$$

where $k > 0$ is the setup cost, i.e., the fixed administrative cost (of placing an order) not dependent on the amount W, and $c_s(r) > 0$ is the shipment cost per unit which in general would depend on the number of shipments in the interval $[0, z]$. More precisely, without loss of generality, we will assume that $c_s(r)$ is a non-decreasing piecewise constant function, i.e.,

$$c_s(r) = \sum_{i=1}^{r} c_i, \; r = 1, 2, ..., \tag{15}$$

where c_1, ..., c_r are non-negative constants.

Remark 4.1. It is natural to assume that the setup (administrative) cost k is constant if multiple batches scheduled to be shipped at fixed future dates, $0 = t_1 < t_2 < t_3, ...$, are pre-ordered with a single order placed at the initial time $t = 0$.

We will further consider *holding costs*, associated with the storage of the inventory until it is used/sold, including different components such as capital costs (the cost of capital tied up, interest on working capital, etc.), storage space costs, services costs (insurance protection, IT, taxes, etc.) and inventory risk costs. More specifically, we assume that the total holding costs are modelled as

$$C_h(t, r) = c_h W_1 t + c_h W_2 (t - t_2)_+ + ... + c_h W_r (t - t_r)_+, \; 0 \le t \le z, \tag{16}$$

where c_h is the holding cost per one unit of product per unit of time, $(y)_+ = \max(0, y)$. Note that $C_h(t, r)$ depends on $h(t)$ and as we are flexible with its choice and the choice of c_h, more detailed holding cost functions, $C_h(t, r)$, that model the dynamics of each of the holding cost components mentioned above, can also be incorporated.

The total ordering and holding costs are then expressed as

$$\begin{aligned} TC(t, r) &= C_o(W, r) + C_h(t, r) \\ &= k + c_s(r) W + c_h W_1 t + c_h W_2 (t - t_2)_+ + ... + c_h W_r (t - t_r)_+, \; 0 \le t \le z, \end{aligned} \tag{17}$$

where a single setup cost k is considered since all batches of sizes $W_1, W_2, ..., W_r$ with shipping times $0 = t_1 < t_2 < ... < t_r < z$ are pre-ordered at time $t = 0$ (see Remark 4.1). We are interested in solving the following optimal replenishment problem.

Problem 4.1. For given k, $c_s(r)$, c_h, ϵ, $g(t)$ and parameters of the demand process $\tau[0, t]$, find the optimal replenishment function $h(t) \in \mathcal{H}_r$, i.e., $\mathbf{W} = (W_1, W_2, ..., W_r)'$, (with $W_1 + W_2 + ... + W_r =$

W), and $\mathbf{t} = (t_2, t_3, ..., t_r)'$, such that

$$\min_{\mathbf{t},\mathbf{W}} \; TC(z, r)$$

is achieved, subject to

$$P(g(t) \leq \tau[0, t] \leq h(t), 0 \leq t \leq z) \geq 1 - \epsilon, \tag{18}$$

where $0 < \epsilon \ll 1$ is a small pre-assigned value.

In other words, by solving Problem 4.1, at time $t = 0$, we optimally select the function $h(t)$ (e.g., the number of shipments, r, batch sizes and future shipment time(s)), so that the total ordering and holding costs incurred at time z (the end of a time period, $[0, z]$) are minimized, while at the same time the probability that within the period, $[0, z]$, the demand does not exceed the cumulative amount of replenished items, $h(t)$, and also does not fall below the minimum demand limit, $g(t)$, is sufficiently large (i.e., is larger than or equal to $1 - \epsilon$).

Note that the demand process, $\tau[0, t]$, multiplied by the unit price of an item can be viewed as the inflowing revenue. Therefore, by solving Problem 4.1, it is ensured that the order policy, $h(t)$, is such that the total costs of ordering and holding inventory are minimized while at the same time the probability of the company staying in business (i.e., revenue not dropping below required minimum) and not incurring loss in revenue due to demand exceeding supply, is sufficiently large (defined by appropriately selecting ϵ). The latter losses are closely related to *stock out costs*, and so by solving Problem 4.1, we are ensuring that the probability of incurring stock out costs is sufficiently small.

Remark 4.2. In order to determine the optimal number of shipments $r > 0$, we solve Problem 4.1 sequentially for fixed values of $r = 1, 2, ...$ and select r that gives the lowest total costs, $TC(z, r)$. Such a value exists since, for fixed total number of units W, the ordering cost $C_o(W, r)$ is typically an increasing function of r and there exist sequences $W_1, ..., W_r$ and $t_1, ..., t_r$ such that the holding cost, $C_h(z, r)$, is a decreasing function of r. The latter statement is easily verified if one analyzes (14), (15) and (16).

Under our approach, it is assumed that (based on past data) the particular real demand (process) is well studied (i.e., possible seasonality, overlapping and non-overlapping clustering and other patterns in the demand have been identified). Based on this, the parameters of the model demand process, $\tau[0, t]$ have been estimated, ensuring it has good predictive power over the future (short) period $[0, z]$. As known, out of sample forecasting ability tends to be higher if the time horizon is shorter. Our approach can therefore be sequentially applied over longer time horizons by splitting the latter into smaller sub-intervals. Let us note also that estimating and forecasting demand is common practice in inventory management. It has for long attracted attention in the inventory and supply chain management literature as can be traced from Chen & Winters (1966), Kwak et al. (1977), Willemain et al. (1994b), Ghobbar & Friend (2003), Willemain et al. (2004), Gutierrez et al. (2008), Teunter & Duncan (2009), Kremer et al. (2011), Teunter et al. (2011).

The following proposition relates to the solution of Problem 4.1.

Proposition 4.1. For a fixed $r > 1$ and total number of units W, there exists a unique solution to Problem 4.1, denoted by $h_{opt}(t)$

$$h_{opt}(t) = W_1^* + W_2^* \mathbb{1}_{\{t_2^* \leq t\}} + \cdots + W_r^* \mathbb{1}_{\{t_r^* \leq t\}},\ 0 \leq t \leq z,$$

with corresponding minimized total costs at time z

$$TC_{opt}(z, r) = k + c_s(r)W + c_h W z - c_h W_2^* t_2^* - c_h W_3^* t_3^* - ... - c_h W_r^* t_r^*,$$

where $W = W_1^* + ... + W_r^*$, and

$$P(g(t) \leq \tau[0, t] \leq h_{opt}(t),\ 0 \leq t \leq z) \geq 1 - \epsilon. \tag{19}$$

The proof of Proposition 4.1 is based on three auxiliary Lemmas, given in Section A of the Supplementary Material. For further graphical illustration of the solution to Problem 4.1 and its sensitivity analysis with respect to model parameters, we refer the reader to Section B of the Supplementary Material.

4.1.1. Numerical Implementation

Let us note that the optimization in Problem 4.1 involves $2(r-1)$ variables, $(t_2, ..., t_r, W_2, ..., W_r)$, and hence, quickly becomes highly multivariate when the number of replenishments, r, increases. While the evaluation of the total costs is straightforward (cf., 17) and requires negligible computation time, the heavy computation burden falls on the multiple evaluations of the non-exit probability in constraint (18). The latter are very efficiently (in terms of speed and accuracy) performed using the FFT-based method, developed in Section 3. Having such a method is crucial as otherwise the computational cost would be prohibitive. Therefore, in order to illustrate how Problem 4.1 is solved numerically, we have developed an efficient optimization algorithm, which heavily exploits the FFT-based method for computing the non-crossing probability in (18) and the properties established by Lemmas A.1 and A.2 (cf., Section A of the Supplementary Material). Due to volume limitations, details related to the latter algorithm are omitted. We have applied it in the following Examples 4.1, as well as Examples B.1 and B.2 of the Supplementary Material, involving a Poisson-Logarithmic demand. The latter model is equivalent to a Negative Binomial demand process allowing clusters of overlapping demand. It is of practical relevance, as demonstrated by Lengu et al. (2014) and Stenius et al. (2016) in the context of management of spare parts and supply of steel sheets in the construction industry, respectively.

Example 4.1. Consider a Poisson-Logarithmic demand model for $\tau[0,t]$. More precisely, let the arrival process $\xi(t)$ be a homogeneous Poisson process with intensity $\lambda = 15$ and let the quantities of demand, X_i, be logarithmically distributed, i.e.,

$$F_X(x) = \sum_{k=1}^{x} P(X_i = k) = \sum_{k=1}^{x} \frac{-1}{\ln(1-\gamma)} \frac{\gamma^k}{k}, \; x = 1, 2, ..., \; 0 < \gamma < 1,$$

with $\gamma = 0.5$. Let also $\epsilon = 0.1$, which corresponds to 90% (and above) probability that the demand $\tau[0,t]$ will stay between the two boundaries, $g(t)$ and $h(t)$, where for the lower boundary $g(t)$ in (13), we have taken $d = 8/0.75$, $b = 8/3$. We consider a unit time interval, i.e., $z = 1$. For the ordering costs function $C_o(W,r)$, defined in (14), we assume that $k = 10$ and $c_s(r)$, defined in (15), is specified by $c_1 = 0.2$, $c_2 = 0.1$, $c_3 = 0.06$, $c_4 = 0.04$, $c_i = 0.03$, $i = 5, 6, ...$ The constant, c_h, defining the holding costs $C_h(t,r)$ (see 16) is taken to be $c_h = 0.9$.

The optimal solutions to Problem 4.1 for fixed values of $r = 1, 2, ..., 5$ and model parameters as in Example 4.1 are presented in Table 1. As can be seen, the optimal number of shipments is $r^* = 2$, with $TC_{opt}(z, r^*) = 46.501$, and optimal replenishment amounts and times $W_1^* = 22$, $W_2^* = 13$ and $t_1^* = 0$, $t_2^* = 0.47$.

In order to gain geometrical insight into the solution of Problem 4.1, e.g., the shape of the domain $D(t_2, ..., t_r, W_2, ..., W_r)$ defined in Lemma A.3 of the Supplementary Material and behaviour of $TC(z,r)$, in Figure 1 we have illustrated graphically the solution of a four dimensional problem,

Table 1: Optimal Solutions to Problem 4.1 for Fixed Values of $r = 1, 2, ..., 5$, for Model Parameters as in Example 4.1.

r	$TC_{opt} - Ex.4.1$	Optimal Solutions
1	48.5	$W = W_1^* = 35$; $t_1^* = 0$;
2	46.501	$W_1^* = 22$, $W_2^* = 13$; $t_1^* = 0$, $t_2^* = 0.47$;
3	47.188	$W_1^* = 17$, $W_2^* = 10$, $W_3^* = 8$; $t_1^* = 0$, $t_2^* = 0.28$, $t_3^* = 0.61$;
4	47.895	$W_1^* = 16$, $W_2^* = 7$, $W_3^* = 6$, $W_4^* = 6$; $t_1^* = 0$, $t_2^* = 0.23$, $t_3^* = 0.46$, $t_4^* = 0.68$;
5	48.567	$W_1^* = 14$, $W_2^* = 6$, $W_3^* = 5$, $W_4^* = 5$, $W_5^* = 5$; $t_1^* = 0$, $t_2^* = 0.17$, $t_3^* = 0.34$, $t_4^* = 0.52$, $t_5^* = 0.71$.

i.e., $r = 3$ and $D(t_2, t_3, W_2, W_3)$. More precisely, we have plotted the surface $TC_{opt}(z = 1, r = 3)$ over the domain determined by the probability constraint (19) (cf., (A.6) and Lemma A.3 in Section A of the Supplementary Material). Along each row, three different values of the variable t_2 are illustrated and also three different values are given along each column for the variable W_1. Thus, on each of the six plots we vary t_3 and W_2.

Figure 1 clearly illustrates the monotonicity of the total cost function $TC(z, r)$, shown in Lemmas A.1 and A.2 in the Supplementary Material. It also illustrates the shape of the domain $D(t_2, t_3, W_2, W_3)$ specified in Lemma A.3. Moreover, the existence and uniqueness of the solution of Problem 4.1, as determined in Proposition 4.1, is also clearly evident. Furthermore, it can be seen that the minimum of the total cost function $TC_{opt}(z = 1, r = 3)$, is attained at the boundary of the domain determined by the probability (19). Note also that the domain becomes smaller when e.g., the size of the first shipment, W_1, increases or the timing of the second shipment, t_2, increases.

4.2. Computing Ruin Probability

In the context of finite-time ruin probability, the upper boundary function $h(t)$ is interpreted as the function which models the cumulative arrival of premiums to an insurance company up to time t, where $h(0) = u \geq 0$ is viewed as the company's initial capital, and $\lim_{t\to\infty} h(t) = \infty$. The stochastic process $\tau[0, t]$, defined in (1), is used to model the aggregate claims process, where $\xi(t)$ is in general assumed a PPCSII modelling the arrival of claims, and their severities are modelled by the sequence of i.i.d. random variables X_k, $k = 1, 2, ...$ defined in (1). Then, the surplus of the

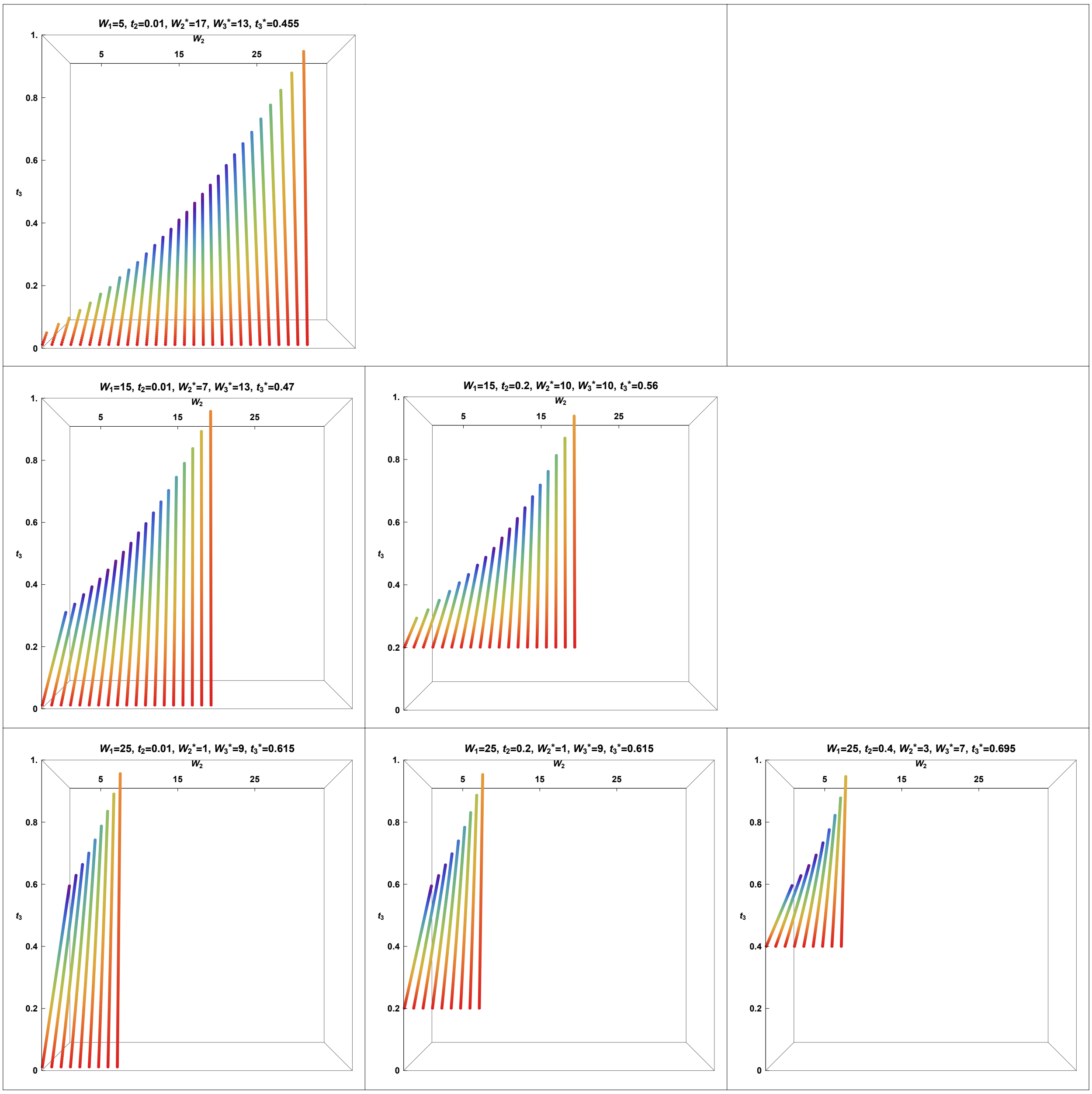

Figure 1: Graphical Illustration of the Total Costs, $TC(z = 1, r = 3)$, for Fixed Values $W_1 = 5, 15, 25$, $t_2 = 0.01, 0.2, 0.4$, and Varying W_2, t_3, Subject to $W = W_1 + W_2 + W_3 = 35$ and $P(g(t) \leq \tau[0, t] \leq h(t), 0 \leq t \leq z) \geq 0.9$. The Red Points on the Plot Represent Higher Values of $TC(z = 1, r = 3)$, and the Dark Blue Points Represent Lower Values of $TC(z = 1, r = 3)$. The Optimal W_2^*, W_3^*, t_3^* Corresponding to a Fixed Value of W_1 and t_2 is also Provided in the Label of Each Plot.

insurance company at time t is given by

$$U(t) = h(t) - \tau[0, t].$$

Ruin is defined as the event when the surplus becomes negative for the first time. Hence, the instant of ruin is $T = \inf\{t > 0 : U(t) < 0\}$, or $T = \infty$ if $U(t) \geq 0$ for all t. Note that T corresponds to the time when the process $\tau[0, t]$ crosses the upper boundary $h(t)$ for the first time. Then, the probability of non-ruin until time z can be written as

$$P(T > z) = P(h(t) - \tau[0, t] \geq 0, \text{ for } 0 \leq t \leq z) = P(\tau[0, t] \leq h(t), \text{ for } 0 \leq t \leq z), \tag{20}$$

which can be seen to coincide with the DB(non-)C probability in (3) with $g(t) \equiv 0$, $\forall t$.

Therefore, one can directly apply the FFT-based method developed in Section 3 to compute the finite-time non-ruin probability in (20). In what follows, we illustrate this for the case when the arrival process $\xi(t)$ is a Cox process. As noted earlier, to the best of our knowledge, no alternative numerically efficient method has been proposed in the literature that is applicable in this general setting of PPCSII arrival process, i.i.d. claim amounts, X_k, $k = 1, 2, \ldots$, with (dis)continuous cdf $F_X(x)$ and general, possibly curvilinear boundary (allowing discontinuities).

Example 4.2. We assume the following surplus process of the insurance company

$$U(t) = h(t) - \tau[0, t] = h(t) - \sum_{k=1}^{\xi(t)} X_k, \tag{21}$$

where X_k, $k = 1, 2, ...$, are i.i.d. random variables, $\xi(t) = N(\nu(t))$ is a Cox process defined as in (24) (cf., Appendix A). Its (cumulative) intensity process is $\nu(t) = \int_0^t \eta(s)ds$ with

$$\eta(t) = \eta + \sum_{n \in \mathbb{N}} J_n b(t - \widehat{T}_n) + a(t), \tag{22}$$

where $\eta > 0$ is assumed to be constant, $\{J_n\}$, $n \in \mathbb{N}$ is an i.i.d. sequence of positive random variables, $b(\cdot)$ is a non-negative function with $b(t) = 0$ for $t < 0$, $\{\widehat{T}_n\}$, $n \in \mathbb{N}$ is the sequence of arrival times of a homogeneous Poisson process of rate ρ, and $a(t)$ is a certain function modelling the response to past events or other perturbations. The process $\xi(t)$ is then referred to as a Cox process with a Poisson (multiplicative) shot noise intensity process (see e.g., Albrecher & Asmussen 2006 for further details of this model).

The intuition of the model in (22) is that claims to an insurance company can be of two types: a "normal" type of claims arriving continuously in time space, modelled by a homogeneous Poisson process with rate η, and the other type of claims caused by certain external events (e.g., natural disasters) arriving discretely in time space at $\{\widehat{T}_n\}, n \in \mathbb{N}$. The latter is modelled by a homogeneous Poisson process with rate ρ.

As shown in (20), the finite-time non-ruin probability, $P(T > z)$, can be viewed as a DB(non-)C probability and therefore, we can use the proposed FFT-based method (cf., Section 3.2) to estimate it as follows. First, we apply QMC to simulate M different trajectories $\mu_j(t)$, $j = 1, \ldots, M$, of the intensity process $\nu(t)$ (cf., (22)). Then, for each simulated trajectory, $\mu_j(t)$, we employ the proposed FFT-based method to compute the probability (cf., (11))

$$P\left(g(t) \leq \sum_{k=1}^{N(\mu_j(t))} X_k \leq h(t), 0 \leq t \leq z\right) = P\Big(N(\mu_j(t)) \leq h(t), 0 \leq t \leq z\Big),$$

with $g(t) \equiv 0, \forall t \geq 0$ and $X_k \equiv 1, \forall k$, which, following the last equation in (11), is estimated as

$$= \sum_{j=1}^{\infty} P(A_j) P(N(\mu_j(t)) \leq h(t), 0 \leq t \leq z) \approx \frac{\sum_{j=1}^{M} P\Big(N(\mu_j(t)) \leq h(t), 0 \leq t \leq z\Big)}{M}.$$

We illustrate this QMC-FFT-based method on Example 4.2. For simplicity, assume that $J_n \equiv 1, \forall n$; $a(t) = 0, \forall t \geq 0$; $b(s) = e^{-\delta s}$ for $s \geq 0$ (recall, $b(s) = 0$ for $s < 0$); and that $\delta = 0.5$, $\rho = \eta = 2.0$, $X_k \equiv 1, \forall k$. We have considered two types of cumulative premium income function $h(t)$, a linear one such that $h(t) = 4t + 1.5$, and a non-linear one such that $h(t) = t^2 + 1.5$. We then have estimated $P(T > 4)$ as follows. For each fixed value of $M = 10^4$, 5×10^4, 10^5, 5×10^5, 10^6, we have run the QMC-FFT-based procedure to compute 30 estimates of $P(T > 4)$, and have then averaged those to obtain the final estimate of $P(T > 4)$. The latter estimates for all values of M and their standard deviations are summarized in Tables 2 and 3 together with CPU times. The standard deviations are also plotted against M in Figure 2. For comparison, we have also implemented plain Monte Carlo (MC) simulation in combination with the FFT-based (i.e., MC-FFT-based) method (CPU times similar to those of QMC-FFT procedure). Lastly, for comparison and verification purposes, we have also implemented a plain MC simulation of the model in Example 4.2, defined by (21) and (22). More precisely, we have first simulated M trajectories from the Poisson (multiplicative) shot noise intensity process $\eta(t)$ and obtained the corresponding trajectories $\mu_j(t)$, $j = 1, \ldots, M$, of $\nu(t)$, following (22). Given each simulated trajectory, $\mu_j(t)$, we have then simulated from a non-homogeneous Poisson process with cumulative intensity, $\mu_j(t)$, in order to estimate the non-ruin probability $P(T > 4)$. Again, for each value of $M = 10^4, 5 \times 10^4, 10^5, 5 \times 10^5, 10^6$, we have repeated the same procedure 30 times and calculated the average values and standard deviations for $P(T > 4)$.

Table 2: Example 4.2 (with $h(t) = 4t + 1.5$) - Average Non-Ruin Probability $P(T > 4)$ and Standard Deviation of 30 Estimates of $P(T > 4)$ Computed Using the QMC-FFT-Based Approach and the MC-FFT-Based Approach with Different Number of Simulated Trajectories, M, of the Cumulative Intensity Process, $\nu(t)$.

Number of simulations, M	10000	50000	100000	500000	1000000
Non-ruin prob. (QMC-FFT)	0.41766773	0.41767023	0.41767670	0.41767393	0.41767405
s.d (QMC-FFT)	0.00012967	0.00003976	0.00002256	5.25×10^{-6}	3.07×10^{-6}
Time(sec)	(1.9)	(9.8)	(19.5)	(100)	(194)
Non-ruin prob. (MC-FFT)	0.41754095	0.41743798	0.41754526	0.41774346	0.41769303
s.d (MC-FFT)	0.00244018	0.00092365	0.00065208	0.00040183	0.00024635
Non-ruin prob. (MC)	0.41624667	0.41731667	0.41704233	0.41771340	0.41780087
s.d (MC)	0.00430595	0.00254454	0.00162873	0.00059441	0.00047338
Time(sec)	(0.08)	(0.39)	(0.76)	(3.88)	(7.69)

Table 3: Example 4.2 (with $h(t) = t^2 + 1.5$) - Average Non-Ruin Probability $P(T > 4)$ and Standard Deviation of 30 Estimates of $P(T > 4)$ Computed Using the QMC-FFT-Based Approach and the MC-FFT-Based Approach with Different Number of Simulated Trajectories, M, of the Cumulative Intensity Process, $\nu(t)$.

Number of simulations, M	10000	50000	100000	500000	1000000
Non-ruin prob. (QMC-FFT)	0.10976089	0.10976233	0.10976455	0.10976367	0.10976381
s.d. (QMC-FFT)	0.00005777	0.00001970	0.00001048	2.23×10^{-6}	1.42×10^{-6}
Time(sec)	(1.9)	(9.4)	(19.3)	(99.2)	(176)
Non-ruin prob. (MC-FFT)	0.10975761	0.10968432	0.10969454	0.10977647	0.10976280
s.d (MC-FFT)	0.00096636	0.00037937	0.00030038	0.00016149	9.66×10^{-5}
Non-ruin prob. (MC)	0.10943667	0.10960200	0.10954233	0.10960720	0.10985887
s.d (MC)	0.00326015	0.00155729	0.00103737	0.00043935	0.00029235
Time(sec)	(0.08)	(0.38)	(0.76)	(3.84)	(7.64)

As can be seen from Tables 2 and 3, as the number of simulations increases, the proposed QMC-FFT-based method provides at least six digits of accuracy after the decimal point. The

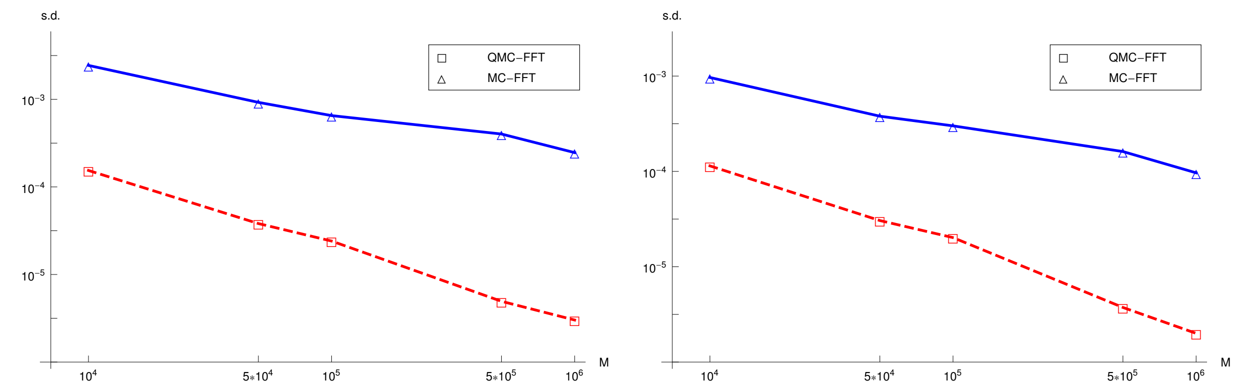

Figure 2: Example 4.2 - Comparison of Standard Deviations of 30 Estimates of $P(T > 4)$ Computed Using the QMC-FFT-Based Approach and MC-FFT-Based Approach, against Different Number of Simulated Trajectories, M, of the Intensity Process, Left-Plot $h(t) = 4t + 1.5$, and Right Plot $h(t) = t^2 + 1.5$.

estimated values from the plain MC simulation of the model in Example 4.2, specified by (21) and (22), confirm the validity of results obtained from the QMC-FFT-based method. However, as can be seen, the QMC-FFT-based method proposed here outperforms the plain MC estimation in terms of speed and accuracy. Moreover, it can be seen from Fig 2 that the convergence is faster for the QMC-FFT-based approach compared to the MC-FFT-based approach (or plain MC approach), which means that for a fixed level of the standard deviation of the estimated non-exit probability $P(T > z)$, the QMC-FFT-based approach is much faster since it requires fewer number of simulations.

5. Conclusion

We have developed a numerically efficient (i.e., fast and accurate) method to compute the probability that a compound process, (defined as in (1)), stays between an upper and a lower deterministic, possibly discontinuous, time-dependent boundaries, within a finite time interval. The compound process is assumed to model the arrivals of certain events of interest, e.g., demands for a product, or capital gains, or insurance claims and the corresponding cumulative size of these events, assuming that their individual sizes (amounts) form a sequence of positive i.i.d. random variables. It is worth highlighting once again that we assume quite generally that the events arrival process belongs to the broad class of *point processes with conditional stationary independent increments*. The latter family includes Poisson, binomial, negative binomial, mixed Poisson and doubly stochastic Poisson (i.e., Cox) processes as special cases. Its richness therefore brings in extended flexibility in applications, as demonstrated in Section 4 and beyond. We therefore highlight this as a methodological contribution of our paper.

The approach we take, thoroughly described in Section 3, leads to an extremely fast algorithm, of order $\mathcal{O}(n^2 \log n)$ for a single non-exit probability (e.g., 0.05 second when $n = 1000$), which has allowed us to compute multiple values of non-exit probability within seconds.

We further show that DB(non-)C problems naturally arise in the context of inventory management, risk and ruin theory, and double-barrier option pricing. We have formulated and solved numerically an interesting inventory management optimization problem (cf., Problem 4.1), which we believe for the first time in the operations research literature involves a DB(non-)C probability as a constraint. By solving the latter problem, we optimally determine the replenishment policy (i.e., the number of shipments and the corresponding optimal times and batch sizes). Since the optimization is highly multivariate, of order twice the number of replenishments, it requires multiple evaluations of the non-exit probability in the constraint (cf., Equation (18)). It is essential to stress once again that this has been possible due to the notable numerical efficiency of the FFT-based algorithm developed in Section 3.

We have also demonstrated how the developed method is applied to efficiently evaluate ruin probabilities in the context of risk theory, assuming the claims arrival process is a Cox process with a Poisson (multiplicative) shot noise intensity process (see Example 4.2, Section 4.2). Combining the FFT-based method with QMC has allowed us to compute ruin probabilities with high accuracy (six correct digits after the decimal point and beyond) and small computation time. This is not the case if one applies the direct MC method which, as known, is slowly convergent (in this particular example, 50 or more times slower than the QMC-FFT-based method for the same level of accuracy), as confirmed by our numerical results (cf., Tables 2 and 3). We once again stress the importance of these results, since ruin probability arises in a number of practically relevant risk quantification tasks, among which assessing liquidity risk and risk capital allocation in insurance and banking, and also valuing reliability risk and flood risk via dam management, and other risk analysis applications (see Dimitrova et al. 2015).

Finally, we have also illustrated numerically how the FFT-based method can be applied in approximating DB(non-)C probabilities for Brownian motion which naturally arise in pricing double-barrier options (see Section C of the Supplementary Material). We highlight once again the applicability of our method for general, possibly curvilinear barriers allowing jump discontinuities (as is the case e.g., for step double barrier options).

One possible way to extend this work is to investigate (based on real data) the practical relevance of alternative demand arrival processes (e.g., Cox process), as part of our flexible model, and to explore how change in the demand process and other characteristics of the model, such as cost functions and lower boundary minimum demand function reflecting operating cost, affect the optimal solution of the inventory management optimization problem.

6. Supplementary Material

The following is given in the Supplementary Material to this paper: (A). Proofs of the results from Section 4.1, Inventory Management Optimization; (B). Graphical illustrations and sensitivity analysis of the solution to Problem 4.1 of Section 4.1; and (C). Application of the proposed FFT-based method in non-exit probabilities for Brownian motion and double-barrier option pricing.

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A. Point processes with conditional stationary independent increments

Following Serfozo (1972) we will define real-valued PPCSII as follows. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\nu(t)$, $t \geq 0$, be a random measure on $(\Omega, \mathcal{F}, P)$ such that $\nu(0) = 0$ *a.s.*, $P(\nu(t) < \infty) = 1$, with trajectories that are non-decreasing and right-continuous. Let $\mathcal{A} = \sigma\{\nu(u) : u \geq 0\}$ be the smallest σ-algebra on Ω with respect to which $\nu(u)$ is measurable. Let $\xi(t)$, $t \geq 0$ be a measurable integer-valued (point) process defined on the same space $(\Omega, \mathcal{F}, P)$ and such that:

(i). for any $0 < s_1 < t_1 < \cdots < s_n < t_n$ and $x_1, \ldots, x_n$,

$$\begin{aligned} &P(\xi(t_1) - \xi(s_1) \leq x_1, \ldots, \xi(t_n) - \xi(s_n) \leq x_n | \mathcal{A}) \\ &= \prod_{k=1}^{n} P(\xi(t_k) - \xi(s_k) \leq x_k | \mathcal{A}), \ a.s., \end{aligned} \tag{23}$$

(ii). for any $0 \leq s \leq t$ and real ζ

$$\mathbb{E}\{\exp[i\zeta(\xi(t) - \xi(s))] | \mathcal{A}\} = \phi(\zeta)^{\nu(t) - \nu(s)}, \ a.s.,$$

where $\phi(\cdot)$ is an infinitely divisible characteristic function and $i = \sqrt{-1}$. We call $\xi(t)$ a PPCSII with respect to $\nu(t)$.

As noted by Serfozo (1972), condition (i) states that, given $\nu(t)$, the process $\xi(t)$ has conditional independent increments. Condition (ii) states that the distribution of $\xi(t) - \xi(s)$ depends on time only through the distribution of $\nu(t) - \nu(s)$ and that $\nu(t)$ and $\phi(\cdot)$ completely determine the behaviour of $\xi(t)$. Note also that the process $\nu(t)$ determining the intensity of arrival of the points of $\xi(t)$ is assumed to be an arbitrary non-decreasing process (see Serfozo 1972 for possible alternatives).

Based on Examples 1-3 of Serfozo (1972), we will highlight the following three important subclasses of the class of PPCSII.

A. If $\nu(t)$ is a deterministic non-decreasing right-continuous function, then (23) holds for the corresponding unconditional probabilities and $\xi(t)$ is a *point process with independent increments* (PPII). This subclass includes several important special cases such as (non-)homogeneous Poisson and negative binomial (NB) processes. For further properties of PPII we refer to e.g., Section 1.5 of Last & Brandt (1995), Chapter 1 of Karr (1991), and Chapter 10 of Daley & Vere-Jones (2007).

B. If $\phi(\xi) = \exp[e^{i\xi} - 1]$, the process $\xi(t)$ is conditionally a non-homogeneous Poisson process with mean value function $\nu(t)$, given the σ-algebra $\mathcal{A}$. Such doubly stochastic Poisson processes, known also as Cox processes are more explicitly defined as

$$\xi(t) = N(\nu(t)), \tag{24}$$

where $N(t)$ is a standard homogeneous Poisson process (with intensity one) and the process $\nu(t)$ is independent of N. In other words, given the trajectory $\nu(\cdot, \omega)$, the consecutive arrival times $T_1 < T_2 < \ldots$ are the consecutive points of a Poisson process $N(\nu(\cdot, \omega))$, with cumulative intensity function, $\nu(t, \omega)$ for $0 \leq t < \infty$.

Cox processes have been extensively studied (see e.g., Grandell 1976, Kallenberg 1997, and Daley & Vere-Jones 2007) and have been used to construct models in finance, insurance, economics and many other fields (see e.g., Lando 1998, Dassios & Jang 2003, Albrecher & Asmussen 2006, Dassios et al. 2015).

B.(i). Let us note that in the special case when $\nu(t) = V\lambda(t)$, where $V > 0$ is a random variable and $\lambda(t) < \infty, t \geq 0$, is a fixed right-continuous, non-decreasing function, the Cox process $N(V\lambda(t))$ defined in (24) is a mixed Poisson process.

B.(ii). Important special cases of mixed Poisson processes are the *point processes with the order statistics property*, extensively studied by many authors among which Crump (1975), Feigin (1979), Puri (1982). Such processes referred to also as *order statistics point processes* (OSPP) have been used to model arrival of claims and other risk events in insurance and finance (see e.g., Lefèvre & Picard 2011, Lefèvre & Picard 2014, Goffard & Lefèvre 2018, Dimitrova et al. 2019). The following definition of a (generalized) OSPP has recently been given by Dimitrova et al. (2019).

Definition A.1. A point process ξ, defined on $(0, \infty)$ with any possibly discontinuous cumulative intensity function $\nu(t)$, is said to have the order statistics (OS) property if, for every $0 < t < \infty$ and $n \geq 0$ such that $P(\xi(t) = n) > 0$, conditional on $\xi(t) = n$, the consecutive arrival times, $0 < T_1 \le \cdots \le T_n \le t$, of ξ coincide in distribution with the order statistics, $Z_{1,n}, \dots, Z_{n,n}$, of n i.i.d. random variables, $Z_1, \dots, Z_n$, with a cdf $F_t(x) = \nu(x)/\nu(t), 0 \le x \le t$, with possible jumps, such that $F_t(0) = 0$ and $F_t(t) = 1$, i.e., $(T_1, \dots, T_n) \stackrel{d}{=} (Z_{1,n}, \dots, Z_{n,n})$.

It is known (cf., Feigin 1979) that, when $\nu(t)$ is continuous, the OSPP ξ has a mixed Poisson representation, i.e., $\xi(t) = N(V\lambda(t))$. Special cases of the OSPP include:

1). Homogeneous Poisson process with parameter $\lambda > 0$, where $\lambda(t) = \lambda t$ and $V \equiv 1$ with probability 1;

2). A Negative Binomial process, i.e.,

$$P(\xi(t) = j) = \binom{\gamma + j - 1}{j} \Big(\frac{t}{b+t}\Big)^j \Big(\frac{b}{b+t}\Big)^\gamma, \ j \ge 0,$$

where $\lambda(t) = t$ and $V \stackrel{d}{=} Gamma(\gamma, b)$;

3). A linear birth process with immigration, of birth rate $b > 0$ and immigration rate $\lambda \ge 0$, where $\lambda(t) = e^{bt} - 1$ and $V \stackrel{d}{=} Gamma(\lambda/b, 1)$ (see e.g., Goffard & Lefèvre 2018 for further details).

C. The third subclass of the PPCSII is obtained when the process $\xi(t)$ is a conditional compound Poisson process with respect to $\nu(t)$ when $\phi(\xi) = \exp[\psi(\xi) - 1]$, where $\psi(\xi)$ is the characteristic function of the size of the jumps of $\xi(t)$, which are assumed independent of the other stochastic components of $\xi(t)$.