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Modelling the fair value of annuities contracts: the impact of interest rate risk and mortality risk

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Abstract

The purpose of this paper is to analyze the problem of the fair valuation of annuities contracts. The market consistent valuation of these products requires a pricing framework which includes the two main sources of risk affecting the value of the annuity, i.e. interest rate risk and mortality risk. As the IASB has not set any specific guidelines as to which models are the most appropriate for these risks, in this note we consider a range of different models calibrated with historical data. We calculate the fair value of the annuity as a portfolio of zero coupon bonds, each with maturity set equal to the date of the annuity payments; the weights in the portfolio are given by the survival probabilities. Moreover, we focus on the additional information provided by stochastic simulations in order to define a suitable risk margin. The nature of the risk margin is one of the main key issues concerning the IASB and Solvency project.

Keywords: annuity contracts, fair value, market value margin, stochastic mortality

1. Introduction

Following the difficult economic climate that over the past few years has affected the financial stability of the insurance industry, regulators have focussed their attention on the need for risk-sensitive supervision and for transparent financial reporting. These two issues have been taken forward respectively by the IASB European Insurance Project, with the intention of originating a set of international standards for comparable and transparent financial reporting, and by the EU Solvency II Review, which is aimed at reforming the existing solvency rules, thereby improving the resilience of the European insurance industry. These projects are not limited to the European

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Union, as other countries like Switzerland and the USA have already adopted (or are in the process of adopting) similar reporting and solvency requirements.

Both projects support the valuation and management of risks in line with an economic approach, by encouraging firms to adopt a market consistent valuation of assets and insurance liabilities. According to this approach, in order to define the market value of assets and liabilities, observable inputs from deep and liquid markets should be used, whilst the remaining elements should be modelled.

Insurance liabilities are in general not fully traded in the secondary markets, and therefore, according to the regulators' directives, insurance companies need to develop suitable models which incorporate both financial risk and insurance risk, and are market consistent, i.e. are based on the up-to-date information available at the time of valuation. Thus, the financial risk should be quantified using either observable market prices (if available), or sound market valuation models; this information should then be used to generate a distribution for the future cash flows originated from the relevant liabilities. The insurance risk, instead, should be valued by applying a mark to model approach which needs judgement and experience: some assumptions can be set by considering the market view on future trends, but also by taking into account exercise of company judgement. On the basis of the model developed, the unbiased estimate, represented by an expected present value, of the market price of the insurance liabilities can be finally extracted. There is agreement among the regulators on the use of the "risk free" rate of interest as a discount factor.

Due to the fact that the resulting unbiased estimate is clearly affected by model risk and parameter risk, and due to the unavoidable uncertainty of the insurance business, the discussion of both the IASB project and the Solvency II recognises the necessity for an adjustment to the current estimate – this is referred to as the risk margin. Although the concept of a risk margin has theoretical justification, its quantification is difficult. The risk margin for insurance liabilities should be as consistent as possible with observable market prices; however, it is acknowledged that the risk margin includes components related to market variables and non market variables. Practitioners, academics and regulators have proposed various approaches that might be used in estimating risk margins like the percentile and the standard deviation approach.

In the light of these considerations, the purpose of this paper is to analyse some of these aspects for the case of annuity contracts, by extending the work of Ballotta et al. (2006) with specific emphasis on the nature of the risk margin. Hence, in this paper, we develop a pricing framework for annuity contracts incorporating both interest rate risk and mortality risk, which in the late 1990s have caused solvency problems to many UK insurance companies, requiring the setting up of extra reserves.

We note that, although for pricing purposes insurance companies use the current yield curve, in order to quantify the financial risk for projection purposes, the development of a suitable model for the evolution of the interest rate is of key importance. Hence, we consider two alternative pricing frameworks based respectively on the CIR (Cox et al., 1995) model and the HJM model (Heath et al., 1992) for the term structure of interest rates. The parameters of the models are calibrated using the estimates of the UK yield curves published by the Bank of England. Further, mortality risk is taken into consideration by calculating survival probabilities using a modified version of the stochastic mortality model developed by Cox and Lin (2005), which allows for the possible perturbation by mortality shocks of the UK standard mortality tables used by practitioners (for example, the PA90, the PMA80 and PMA92-C20 mortality tables). The risk margin is then calculated using both the percentile approach and the standard deviation method.

In order to assess the reliability of the models for both interest rate risk and mortality risk, we test the proposed framework for the case of a hypothetical contract issued in 1979 to a 65 year old male, and using historical values for both mortality and interest rates. For this contract, we also obtain the evolution over time, until the present day of the "market" value, by using the market yield to maturity of the zero coupon bonds with maturities corresponding to the annuity payment date (Bank of England, 2005) and the mortality tables in force over this period of time.

Numerical results show how critical is the choice of the stochastic model for both sources of risk and, once the structure of the model is chosen, how much its calibration can affect the fair value of insurance liabilities. Consequently, a common reference framework becomes a crucial point in order to promote transparency and comparability.

The paper is organised as follows. Section 2 describes the fair valuation approach for the annuity focusing the attention on the financial models for the dynamic of the term structure. In section 3, we present the stochastic mortality model adopted in this analysis. In sections 4 and 5, we discuss the numerical evidence and we introduce the concept of the risk margin. In section 6, we offer some concluding remarks.

2. A fair valuation approach for the annuity: focus on the models for financial market

In this section we develop a general pricing model to determine the market value of an annuity contract. We start from the definition of an annuity as a series of payments over a set period of time.

If the policyholder is aged x at time t when the contract is started, the expected present value at time t of a life annuity which pays £1 per year can be expressed as

$$a_x(t) = \sum_{k=1}^{w-x-t} {}_k p_x P(t, t+k)$$

where w is the largest survival age, $P(t, t+k)$ is the market price at time t of a zero coupon bond with unit face value and maturity $t+k$, whilst ${}_k p_x$ denotes the probability that a person aged x survives k. Hence, an annuity can be regarded as a portfolio of zero coupon bonds, in which the weights are represented by the future stream of annuity payments, and each bond maturity $t+k$ is set equal to the date of a potential annuity payment. The payments are contingent upon the survival of the policyholder. Further, we assume that the mortality risk is independent of the financial risk
In the remaining of this section, we focus on the modelling of the financial risk, whilst the mortality risk is discussed in section 3.

2.1. The models for the financial market

As mentioned in the previous section, annuities can be considered as a portfolio of zero coupon bonds; therefore, their market price depends on the term structure of interest rates. Hence, in order to obtain a market consistent value of these contracts, insurance companies can use the yield curve available at the time of valuation. However, if the aim of the model is not only to price the annuity contract, but also to set up consistent risk margins and suitable capital requirements, the insurance company needs to implement a stochastic model for the dynamics of the interest rate, which can be used for projection purposes.

Interest modelling has been examined by many authors over the past 30 years, and available models can be classified into two families: the short-term rate models, like the Vasicek model (Vasicek, 1977) or the CIR model (Cox et al., 1985), and the instantaneous forward rate models, i.e. the so-called HJM paradigm (Heath et al., 1992). Short-term rate models require calibration in order to fit the observed term structure of interest rates and volatilities. The HJM framework instead is based on an exogenous specification of the dynamics of the instantaneous, continuously compounded forward rate, in the sense that the initial conditions in the model are given by the current term structure. In this way, the need for a full calibration procedure is removed. In fact, it can be shown that, if we use a deterministic volatility coefficient, the dynamics of the forward rate, and therefore the bond prices, are uniquely determined by this parameter.

Since the choice of a suitable framework is crucial for the type of application discussed in this paper, we consider two alternative models for interest rates and compare their performance against the market term structure. Specifically, we choose the CIR short-term rate model, and the HJM model with exponentially decaying volatility structure.

In the following, consider, as given, a filtered probability space $\left(\Omega, \mathcal{F}, \left(\mathbb{F}_t\right)_{t\geq 0}, \mathbb{P}\right)$; assume a frictionless market, with continuous trading and perfectly divisible securities; further, assume that the full continuum of bond prices for any maturity date is available.

CIR model The general equilibrium approach to term structure modelling developed by Cox et al. (1985) leads to a mean-reverting square root diffusion process for the short rate, of the form

$$dr(t) = \kappa(\theta - r(t))dt + \nu\sqrt{r(t)}dZ(t) \tag{1}$$

where $\theta \in \mathbb{R}^{++}$ is the long-run mean interest rate level, $\kappa \in \mathbb{R}^{++}$ is the speed of mean-reversion and $\nu \in \mathbb{R}^{++}$ is the volatility parameter. Moreover, $(Z(t): t \geq 0)$ is a standard one-dimensional $\mathbb{P}$-Brownian motion. Due to the presence of the square root in the diffusion coefficient, the CIR short rate takes only positive values. Cox et al. (1985) found closed-form solution for the price of a zero coupon bond by a PDE approach; in particular

$$P(t,\tau) = A(t,\tau)e^{-B(t,\tau)r_t} \tag{2}$$

with

$$B(t,\tau) = \frac{2\left(e^{\gamma(\tau-t)} - 1\right)}{(\gamma + \kappa + \eta)\left(e^{\gamma(\tau-t)} - 1\right) + 2\gamma},$$

$$\gamma = \sqrt{(\kappa + \eta)^2} + 2\upsilon^2,$$

$$A(t,\tau) = \left[\frac{2\gamma e^{\frac{(\gamma+\kappa+\eta)(\tau-t)}{2}}}{(\gamma + \kappa + \eta)\left(e^{\gamma(\tau-t)} - 1\right) + 2\gamma}\right].$$

Based on Cox et al. (1985), η represents the "market risk" parameter; following Hull and White (1990), it can be shown that the corresponding market price of interest rate risk is $\lambda(t;r) = \eta\sqrt{r_t/\upsilon}$.

HJM model As mentioned above, the HJM framework models the instantaneous, continuously compounded forward rate, whose dynamics for any fixed maturity T is given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

where α and σ are adapted stochastic processes, and $(W(t): t \geq 0)$ is a standard one-dimensional P -Brownian motion. The short rate can then be calculated as $r(t) := \lim_{T\to t} f(t,T)$. In this paper, we assume a deterministic exponentially decaying structure for $\sigma(t,T)$ the function, which implies that the dynamic of the short rate is Gaussian and given by (see Chiarella and Kwon, 2001)

$$dr(t) = b(a(t) - r(t))dt + \sigma dZ(t) \qquad (3)$$

where

$$a(t) = \frac{\sigma^2}{2b^2}\left(1 - e^{-2bt}\right) + \frac{\sigma}{b}\lambda$$

and λ is the market price of interest rate risk. The corresponding price of a T-zero coupon bond is

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-C(t,T)-\gamma(t,T)(r(t)-f(o,t))}, \qquad (4)$$

$$\gamma(t,T) = \frac{1 - e^{-b(T-t)}}{b} = \int_t^T e^{-b(u-t)}du ,$$

$$C(t,T) = \frac{\sigma^2}{4b}\gamma(t,T)^2\left(1 - e^{-2bt}\right),$$

$$f(0,t) = r_0 e^{-bt} .$$

3. A stochastic approach for mortality risk: an extension of the Cox and Lin model

Given the nature of the insurance contract an adequate stochastic mortality model is necessary in order to avoid underestimation or overestimation of future benefits in the valuation of the expected present values which are required for reserving and pricing.

In this section, we propose a simplified stochastic approach to estimate the survivor function by observing that mortality operates within a complex framework, and is affected by a range of variables such as socioeconomic, medical and environmental variables (for a comprehensive review of studies analysing mortality trends, we refer the reader to Cox and Lin (2005) and the references cited therein).

Our proposed approach is to start with the standard tables produced by the Continuous Mortality Investigation Bureau for use by practitioners in the UK (for example, PA90, PMA80, PA92) and to develop adjusted mortality tables, which take into account possible mortality shocks and an additional source of uncertainty linked to the choice of the table, as proposed by Cox and Lin (2005). On the basis of this idea, we intend to estimate the expected value of number of survivors at age $x+t+1$, $\mathbb{E}\left[l\left(x+t+1\right)\right]$, by analyzing the impact of mortality shocks ε_t and of the source of uncertainty ν_t.

It is shown that the distribution of the number of survivors, $l(x+t)$, is approximately normal with mean equal to $l(x)_t\, p_x$ and variance equal to $l(x)_t\, p_x(1-{}_t p_x)$ where ${}_t p_x$ is the survival probability for reaching the age $(x+t)$ at time *t*, for a person of aged *x* at time 0. However, as underlined by Pollard (1970) and more recently by Jeffery and Olivier (2004), the data for England and Wales show a variation year by year which is far greater than the statistical fluctuations of a binomial variation would generate.

Therefore, we assume that our projections need to incorporate the effect of possible perturbations of future estimates of survival probabilities due to random shocks. Following this line of reasoning, the survival probabilities adjusted for shocks can be defined as:

$$ {}_t p'_x = {}_t p_x^{(1-\varepsilon_t)} $$

where ε_t is the mortality shock expressed as a percentage of the force of mortality. Further, we assume that the mortality shock ε_t at time *t* follows a beta distribution with parameters *a* and *b*. In order to analyze the impact of smaller and greater shocks, we consider the case of different expected values for ε_t such as:

$$ \begin{cases} \mathbb{E}\left[\varepsilon_t\right] = 0.01 \\ \mathbb{E}\left[\varepsilon_t\right] = 0.05 \\ \mathbb{E}\left[\varepsilon_t\right] = 0.10 \end{cases} $$

The assumption that $\mathbb{E}\left[\varepsilon_t\right]=0.01$ reflects the mean of the annual percentage mortality improvement based on the most recent UK mortality tables; we also consider the cases in which $\mathbb{E}\left[\varepsilon_t\right]=0.05$ and $\mathbb{E}\left[\varepsilon_t\right]=0.10$ in order to accommodate also greater shocks.

Further, we assume that the sign of the mortality shock depends on the random number $\kappa_t \sim U\left(0,1\right)$. Specifically, we set

$$\begin{array}{l} \varepsilon\left(t\right) \text{ if } \kappa\left(t\right)<c \\ -\varepsilon\left(t\right) \text{ if } \kappa\left(t\right)\geq c; \end{array} \tag{5}$$

where the value of the parameter *c* depends on the expectation of the future mortality trend, and how it may be affected by future progress in terms of medical, environmental and other factors.

Therefore, by assigning a random sign to ε_t, the proposed model accommodates both secular improvements in mortality rates, as well as temporary deteriorations due to exceptional circumstances. In particular, we consider the following cases for the value of *c*:

- *c* = 0, which reflects the situation in which further improvement of an already high life expectancy is impossible;
- *c* = 0.5, which models the case in which further improvement of an already high life expectancy might be difficult, although not impossible;
- *c* = 0.8, which represents the case in which there is a high probability that the population would continue to experience declining mortality;
- *c* = 1, which represents the case of a population which is certain to continue to experience declining mortality, reflecting a great faith in medicine.

Furthermore, we use an additional source of uncertainty, ν_t, to capture the risk at time *t* from uncertainty in the choice of mortality table; in particular, we assume ν_t to follow a standard normal distribution, although we constrain ν_t to lie on the positive axis. In an extension of this analysis, we are currently considering the effect on the results of different choices for the possible distributions of the random variables ε_t, κ_t and ν_t[1].

It follows that the adjusted expected number of survivors, including mortality shocks and the uncertainty of the mortality table, is:

$$l^{'}\left(x+t+1\right)=l\left(x+t\right)p^{'}\left(x+t\right)+\nu\left(t\right)\sqrt{\left(l^{'}\left(x+t\right)p^{'}\left(x+t\right)\left(1-p^{'}\left(x+t\right)\right)\right)} \tag{5}$$

[1] We note that the purpose of this paper is to focus the attention on the importance of the choice of model assumptions in the context of fair valuation and of the definition of a risk margin, and not necessarily to identify the most correct mortality and financial models.

whilst the new survival probability is estimated as:

$$p''(x+t) = \frac{l'(x+t+1)}{l'(x+t)}$$

3.1 Numerical results and sensitivity analysis

In this section we use the mortality model described in the previous section in order to study the behaviour of the survival probabilities for males over the age of 65, as the contract under consideration in the next sections is a hypothetical 25 year annuity contract issued in 1979 to a male policyholder aged 65 years.

Bearing in mind that our analysis refers to an annuity product, we start our simulations from the tables produced by the Continuous Mortality Investigation Bureau in the UK for insurance companies, considering them as an unbiased position. These tables are the PA90 table, based on data for the period 1967–1970 projected to 1990, PMA80-C10, based on data for the period 1979–1982 projected to 2010 and PMA92-C20, based on data for the period 1991–1994 projected to 2020.

In particular, we analyze the behaviour of the adjusted survival probabilities as a function of the parameters related to the beta distribution, i.e. the mean and variance of the mortality shocks, and the value of c from which the sign of the mortality shock depends.

Specifically, in Figure 1, we consider the survival probabilities for males over the age of 65 until the age of 89, in relation to a hypothetical 25 year annuity contract issued to a male policyholder aged 65 years. In particular, we show the results related to the PA90 mortality table and we provide the difference between this mortality table and the adjusted table reached by varying the parameters of the beta distribution and the values of c (similar results are obtained for the other tables and are available from the authors), based on the assumption that the unbiased position is represented by this mortality table.

By analysing these results, we note that by increasing c the difference between the CMI table and the adjusted table tends to increase and the difference tends to be negative, as the adjusted survival probabilities tend to be higher than CMI ones. As expected, considering the same distribution of the mortality shock, the survival probabilities obtained by assuming c equal to 0.8 and 1 are always higher than those obtained by assuming $c = 0.5$. In the case of c equal to 0.8 and 1 the survival probabilities generally improve increasing the mean of the mortality shock. We do not observe this phenomenon when considering $c = 0.5$ as the probability of a negative mortality shock is higher than in the cases in which c is equal to 0.8 and 1.

The case in which c = 0 requires a different analysis; in this case, in fact, we have a high probability that the difference between the CMI table and the adjusted table has a positive sign, since the only factor which can make this difference become negative is the variable ν_t. According to the model assumptions (see equation (5)), the mortality shock will surely be negative for c = 0.

In Figure 2, we also provide the comparison between the CMI PA90 survival probability values with the adjusted ones referring to ages 65 and 78 years, (similar results apply also for other ages and for other tables; results are available from the authors).

The results shown in Figure 2 confirm those given in Figure 1: by increasing the parameter c, the adjusted survival probabilities become higher than those from the CMI mortality table as the probability that the sign of the mortality shock is negative decreases.

We also observe that, by assuming c = 0, the Continuous Mortality Investigation survival probabilities can also be higher than the adjusted ones, especially by increasing the mean of the beta distribution, since the effect of the mortality shock is stronger than that of the variable ν_t .

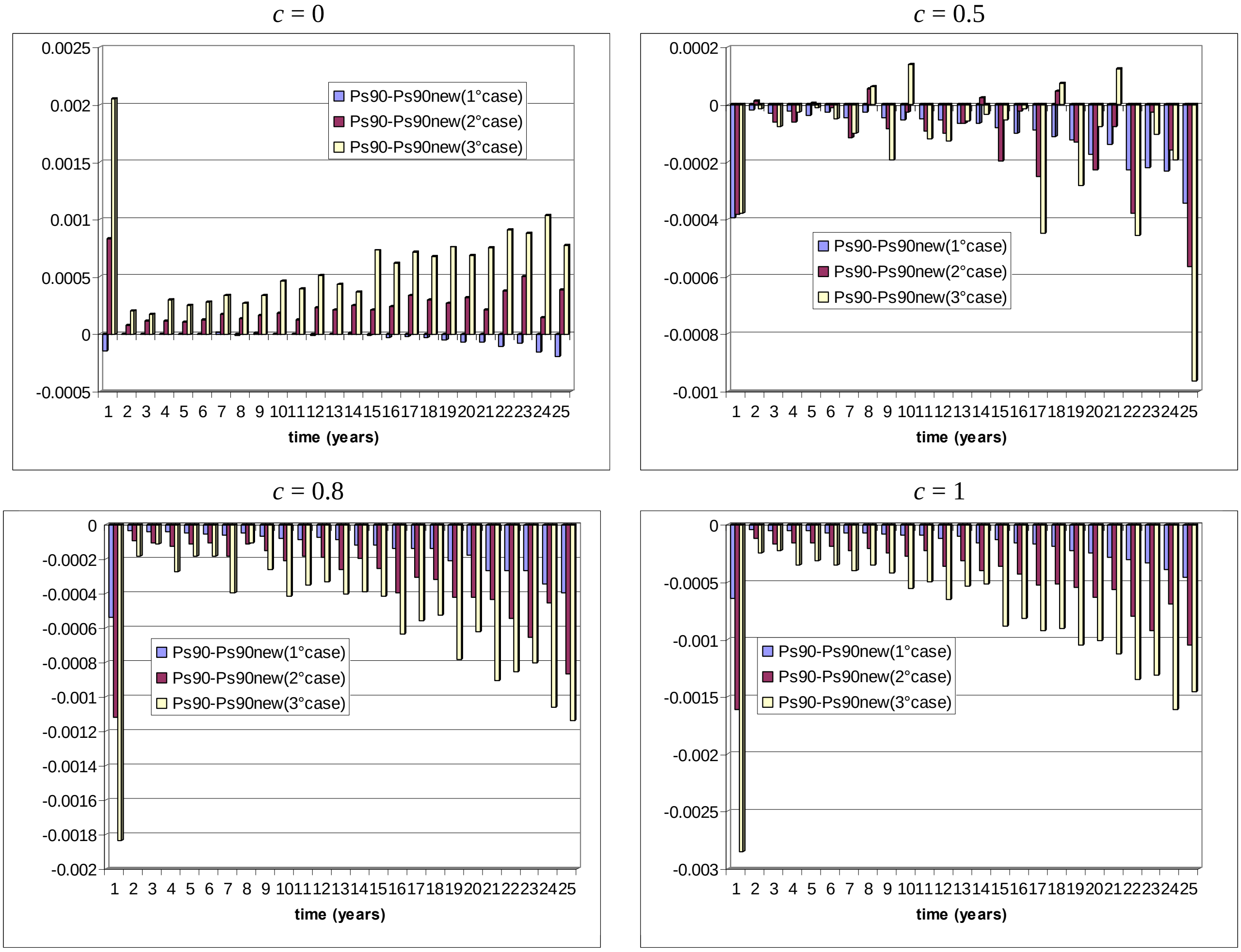
4. Historical analysis

In this section in order to assess the goodness of the models set up in sections 2 and 3, we test the full framework using historical values for mortality and interest rates, in a similar fashion to the study performed by Ballotta and Haberman (2003) for guaranteed annuity options.

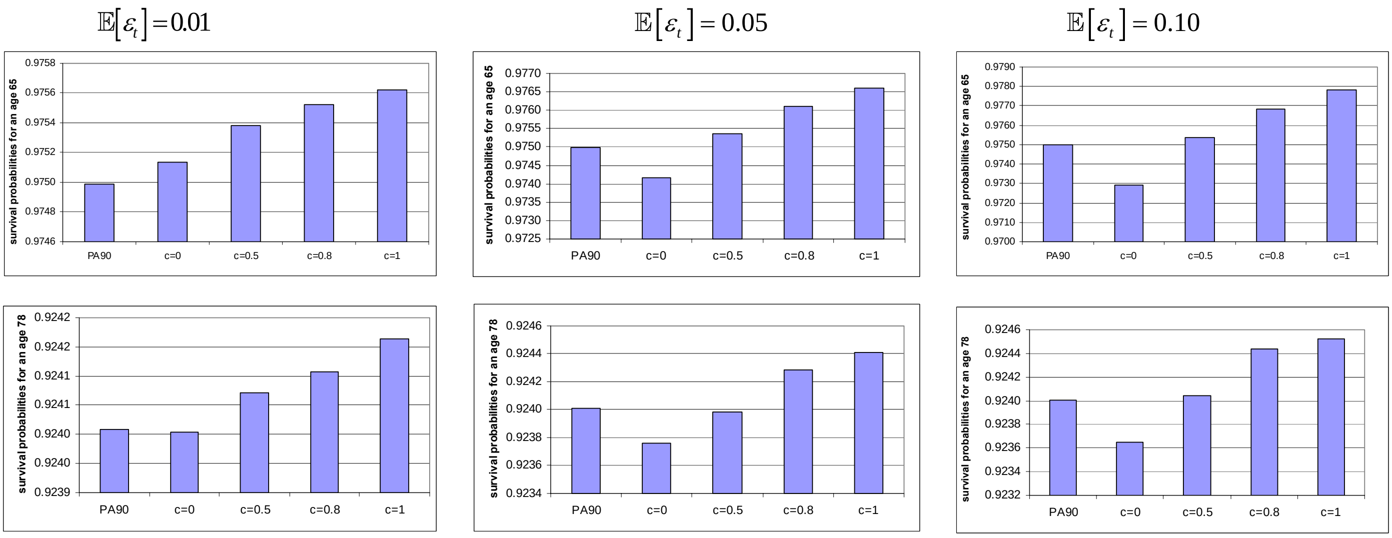
In our analysis we consider a hypothetical 25 year annuity contract issued in 1979 to a male placeholder aged 65 years. In particular, we start by defining for the annuity contract under examination the evolution over time until the present day of the historical market value. We then compare this to the deterministic reserve and the value obtained by using our framework.

The market value of the 25 year annuity contract is obtained by using the market value of the zero coupon bonds with maturities corresponding to the annuity payment dates from 1979 to 2003, and the mortality tables in force over this period of time. Thus, the prices of the zero coupon bonds in each year and for each maturity have been calculated using the prevailing market rates. Further, the survival probabilities are calculated using the PA90 mortality table from 1979 to 1990, the PMA80-C10 over the period 1991-1999, and the PMA92-C20 from year 2000. We also note that, due to the construction of the contract and the type of analysis carried out, the unexpired term of the annuity reduces from 25 year by year. The results presented in Figure 3 show how the market value decreases as we move forward in time.

<u>Figure1:</u> Comparison between PA90 table and adjusted tables obtained by varying the parameters of beta distribution and the values of *c*.
1st case implies a beta distribution with mean equal to 0.01 and variance equal to 0.0081
2nd case implies a beta distribution with mean equal to 0.05 and variance equal to 0.0407
3rd case implies a beta distribution with mean equal to 0.10 and variance equal to 0.0814



<u>Figure 2:</u> Comparison between CMI tables and adjusted tables by varying c and the parameters of beta distribution



The corresponding deterministic mathematical reserves are calculated using standard techniques, i.e. by the expected present value of the future payments, where the discount rate is set equal to the interest rate prevailing in the market at the inception of the contract, which in 1979 was 13% (source: Bank of England). The choice of this rate is justified by the fact that it can be considered as the lower bound for any prudential discount rate. For the mortality rates, we adopt the single-entry mortality tables that were being used in practice over this period of time, as noted above: using the PA90 mortality table from 1979 to 1990, the PMA80-C10 over the period 1991-1999, and the PMA92-C20 from year 2000.

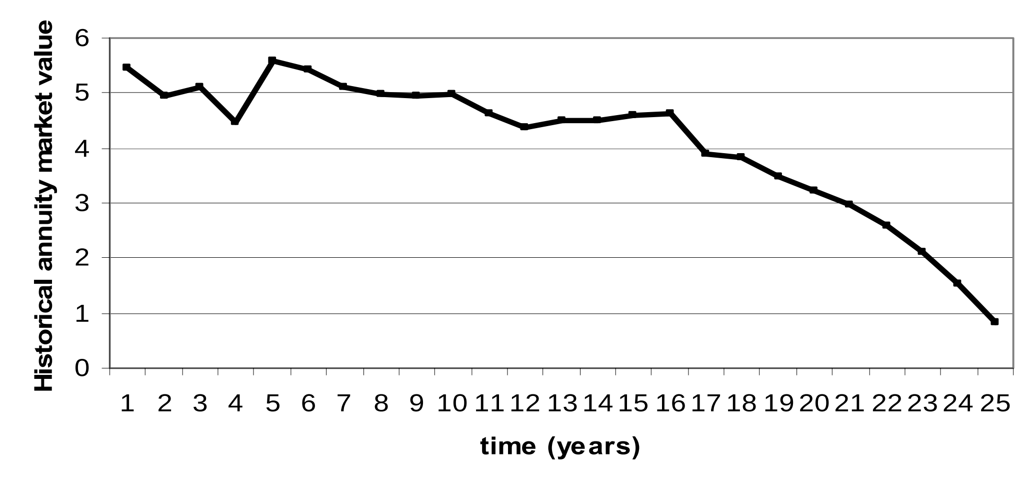
Finally, we compare the “historical annuity market values” with the central estimate of the distribution of the annuity fair value, which is calculated using the financial approaches described in section 2, whose parameters have been calibrated to the estimates of the UK yield curves provided by the Bank of England (the full set of parameters is provided in the Appendix). Survival probabilities are calculated using the corresponding mortality tables as described above.

Hence, Figures 4 and 5 show the comparison among the historical annuity market values, the deterministic reserves and the central estimates of the fair value’s distribution calculated using the CIR model and the HJM model. We note from the results that the annuity fair value calculated on the basis of the CIR model provides the closest estimates to the “historical market value”; however, we recognise that in general the estimated liabilities do not necessarily correspond to the historical annuity prices.

In order to consider in the analysis the mortality risk as well, in Figures 6-9 we show the comparison among the historical annuity market values, the deterministic reserves and the central estimates of the fair value’s distribution calculated using, not only the CIR model and the HJM model, but also the modified version of the stochastic mortality model developed by Lin and Cox (2005) and described in section 3. In particular, we show only the results obtained by assuming $c = 0$ and $c = 1$ as similar conclusions can be obtained for the other assumptions as well (results are available from the authors).

Similarly to the case of the results shown in Figures 4 and 5, we note that in general the estimated liabilities do not reflect the market value. Hence, we conclude that an additional amount should be added to the estimated mean value of the liabilities, in line with that suggested by the IASB Insurance Project and the EU Solvency II Review, which define this amount as risk margin. We observe that the appropriate approach for calculating the MVM is one of the key issues arising from the IASB Insurance Project and the EU Solvency II Review, and is discussed in more detail in the next section.

Figure 3: Historical evolution of the market value of an annuity issued in 1979 to a male policyholder aged 65 years. The market consistent value is calculated in each year using the prices of the corresponding zero coupon bonds for each maturity corresponding to an annuity payment date. (Source for the market yield to maturity: bank of England website).



5. The risk margin

The results of the previous section suggest how the measurement of insurance liabilities needs to incorporate an additional amount arising from the uncertainty naturally associated with the insurance business, and due to the necessary assumptions required in the fair valuation in the absence of a deep liquid market.

As IASB suggests, the risk margin should reflect all of the risks associated with the liability. The risk margin should be explicit in order to improve the quality of the estimate and the transparency of the calculation method, and should consider the risks associated with both market variables (such as interest rates which can be derived from market prices) and non market variables (such as mortality). Therefore, the risk margin should be as consistent as possible with market prices.

The IASB does not prescribe specific techniques in order to estimate the risk margin. However, it is acknowledged, according to the Groupe Consultatif Actuariel Europeen (2006), that the risk margin should be set as an addition to the best estimate, that it should capture uncertainty in parameters, models and trends, that it should be harmonised across Europe and that it should provide a sufficient level of policyholder protection together with capital requirements.

Figure 4: **Comparison of the "historical annuity market values" with the mathematical reserves and the annuity fair values calculated using the CIR model.**

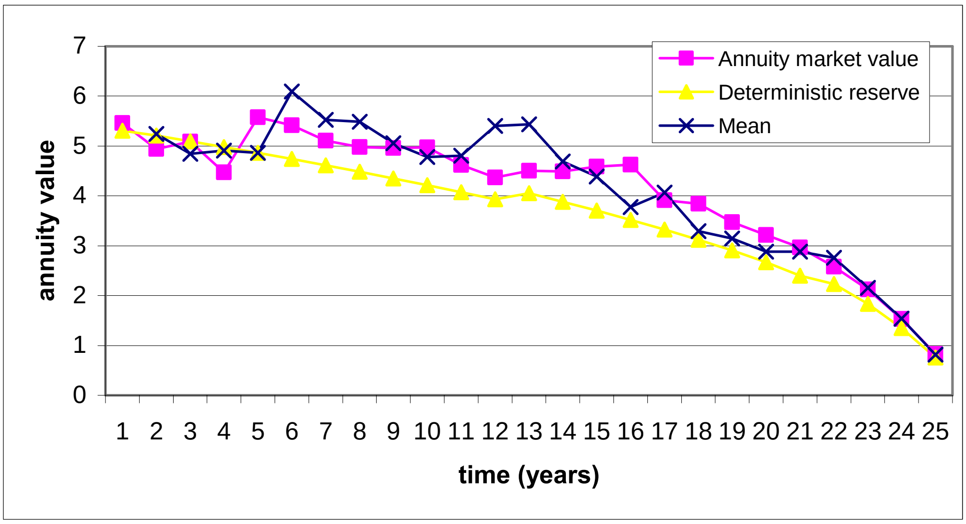
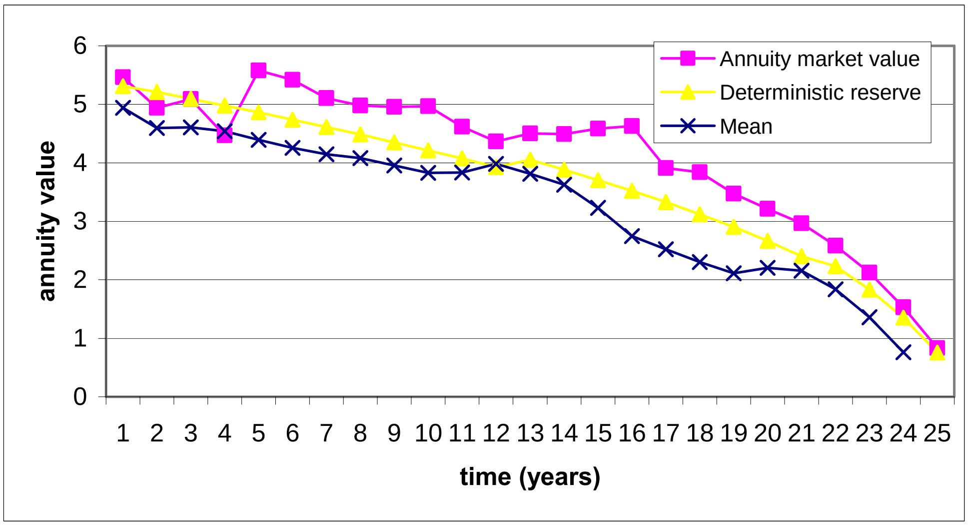
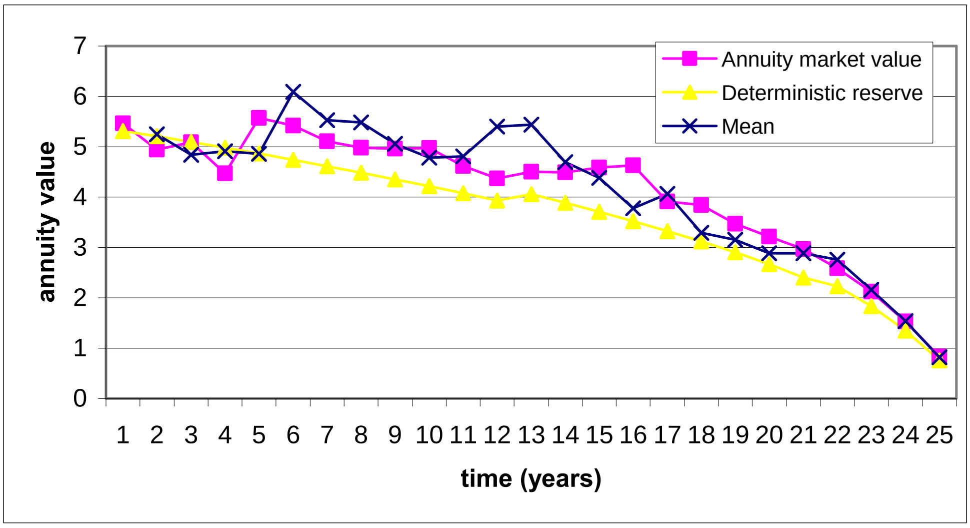


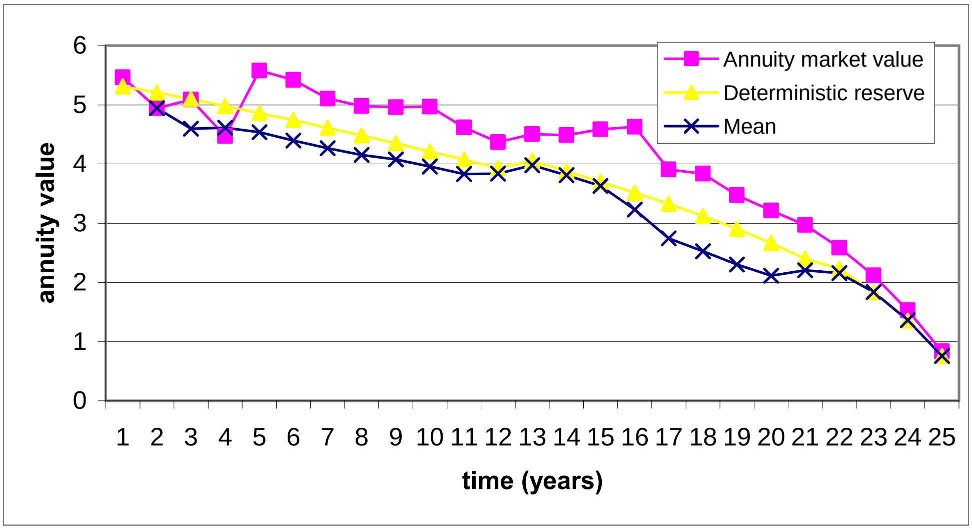
Figure 5: **Comparison of the "historical annuity market values" with the mathematical reserves and the annuity fair values calculated using the HJM model.**



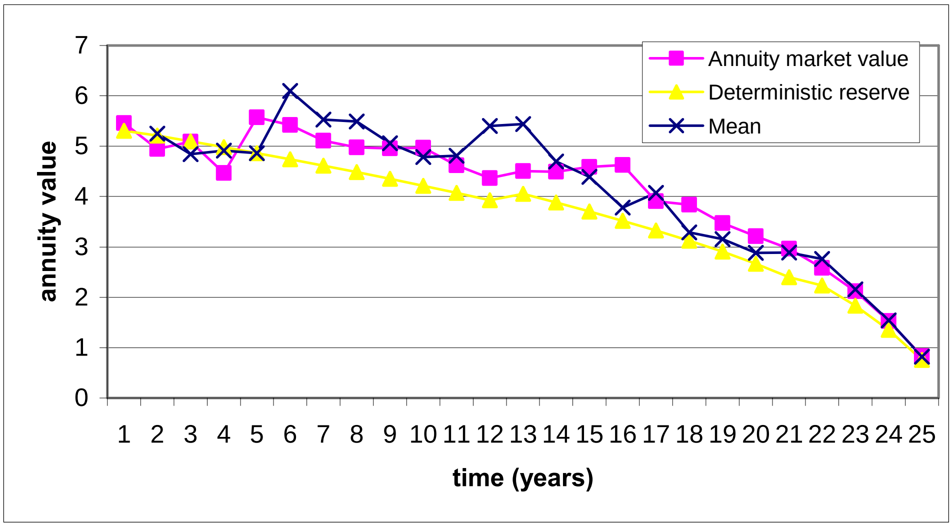
<u>Figure 6:</u> Comparison of the "historical annuity market values" with the mathematical reserves and the annuity fair values calculated using the CIR model and the stochastic mortality model with assumptions *c* = 0 and a mean equal to 0.01 for the beta distribution of the mortality shock parameter.



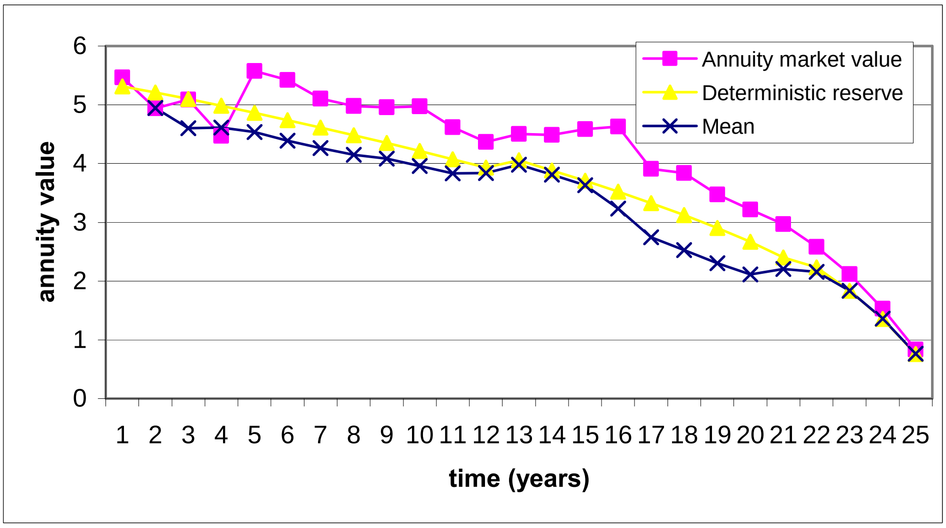
<u>Figure 7:</u> Comparison of the "historical annuity market values" with the mathematical reserves and the annuity fair values calculated using the HJM model and the stochastic mortality model with assumptions *c* = 0 and a mean equal to 0.01 for the beta distribution of the mortality shock parameter.



<u>Figure 8:</u> Comparison of the "historical annuity market values" with the mathematical reserves and the annuity fair values calculated using the CIR model and the stochastic mortality model with assumptions *c* = 1 and a mean equal to 0.01 for the beta distribution of the mortality shock parameter.



<u>Figure 9:</u> Comparison of the "historical annuity market values" with the mathematical reserves and the annuity fair values calculated using the HJM model and the stochastic mortality model with assumptions *c* = 1 and a mean equal to 0.01 for the beta distribution of the mortality shock parameter.



Different approaches are referred to or used by the insurance industry and some insurance regulators. Examples are the cost of capital approach (see the Swiss Solvency Test (FOPI 2004)) which estimates the cost of holding the future required regulatory capital requirement, the percentile approach which requires the fixing of an explicit confidence level, and a moments-based approach which utilises multiples of one or more specific parameters (such as standard deviation, variance and higher moments) of the estimated probability distribution.

In this section we analyse the percentile approach which was taken up by the EU Commission and included in the Solvency 2 Roadmap and provided by CEIOPS as a working hypothesis, and the standard deviation approach which is in line with the Australian Prudential Regulation Authority (APRA).

In both cases, an insurer would need to simulate different scenarios or derive a formula reflecting the probability distribution of cash flows.

5.1 The percentile approach

According to the percentile approach, the margin is calculated as the difference between the liability amount at a prespecified confidence level, and the central estimate of fair value distribution. The problem is to decide which particular percentile should be set as the standard.

The 75% confidence level is the level which is based on the precedent set in Australia. This level has also been taken up by the EU Commission, and included in the Solvency 2 Roadmap, and provided by CEIOPS as a working hypothesis. For completeness, in this study we also consider other percentiles such as the 90^{th} and 95^{th} ones.

Our aim is to analyse the implication of each percentile in order to assess which confidence level makes it possible to capture the historical market values.

In Figure 10, for ease of exposition, we only provide the comparison of the “historical annuity market values” with the 75^{th}, 90^{th}, 95^{th} percentile and the central estimate of the fair value distribution obtained by using the CIR and HJM model and the adjusted survival probabilities. Specifically, we consider the adjusted survival probabilities achieved by assuming c equal to 0 and 1 and a mean equal to 0.01 for the mortality shock (similar results are obtained by considering other mortality assumptions).

From the plots of Figure 10, we note that the closest estimate to the historical market value is given by the 75^{th} percentile in the case of the valuation framework based on the CIR model, and the 90^{th} percentile in the case of the HJM model-based framework. By changing stochastic models and the confidence level, the fair values change significantly.

The results show a strong dependency on the key assumptions for distributions, stochastic models and input parameters. This is recognised by the industry as the main disadvantage of the percentile approach and confirms how clear guidelines on the assumptions backing the fair valuation are necessary in order to guarantee that the results of the percentile approach would be comparable among insurance companies.

5.2 The standard deviation approach

According to the standard deviation approach, the risk margin is a percentage of the standard deviation of the estimated reserve distribution. Thus, the overall estimate of the fair value could be calculated as

$$\mu + k\sigma$$

where μ is the liability's best estimate (represented by the mean value); while k is a percentage of the standard deviation, σ, of the best estimate's distribution.

The problem which arises is the identification of what is the appropriate multiple of the standard deviation capturing effectively the risk. In line with APRA's approach we need to consider at least 50% of the standard deviation, (Collings and White (2001)). For completeness, in this study we also consider other percentages such as 100%, 150% and 200%.

In this section we want to analyse the effect of each percentage in order to assess which multiple makes it possible to better capture the historical market values.

Hence, in Figure 11 we compare the "historical annuity market values" against the mean added to different percentages of the standard deviation of the fair value distribution. In detail, we consider the fair value distribution obtained by using the CIR and HJM model and the adjusted survival probabilities achieved by assuming c equal to 0 and 1 and a mean equal to 0.01 for the mortality shock (similar results are obtained by considering other mortality assumptions).

From the plots of Figure 11 we note that the closest estimate to the historical market value is given by the multiple of standard deviation equal to 0.5 in the case of the valuation framework based on the CIR model, and equal to 1.5 in the case of the HJM model-based framework. Hence, with the standard deviation approach, as for the percentile approach shown in the previous section, we observe that by varying stochastic models, the fair values and the relative risk margin change significantly.

Figure 10: **Comparison of the “historical annuity market values” with the 75th, 90th, 95th percentile and the central estimate of the fair value distribution obtained by using the CIR and HJM model and the adjusted survival probabilities (mean equal to 0.01 for the mortality shock and *c* = 0 and *c* = 1).**

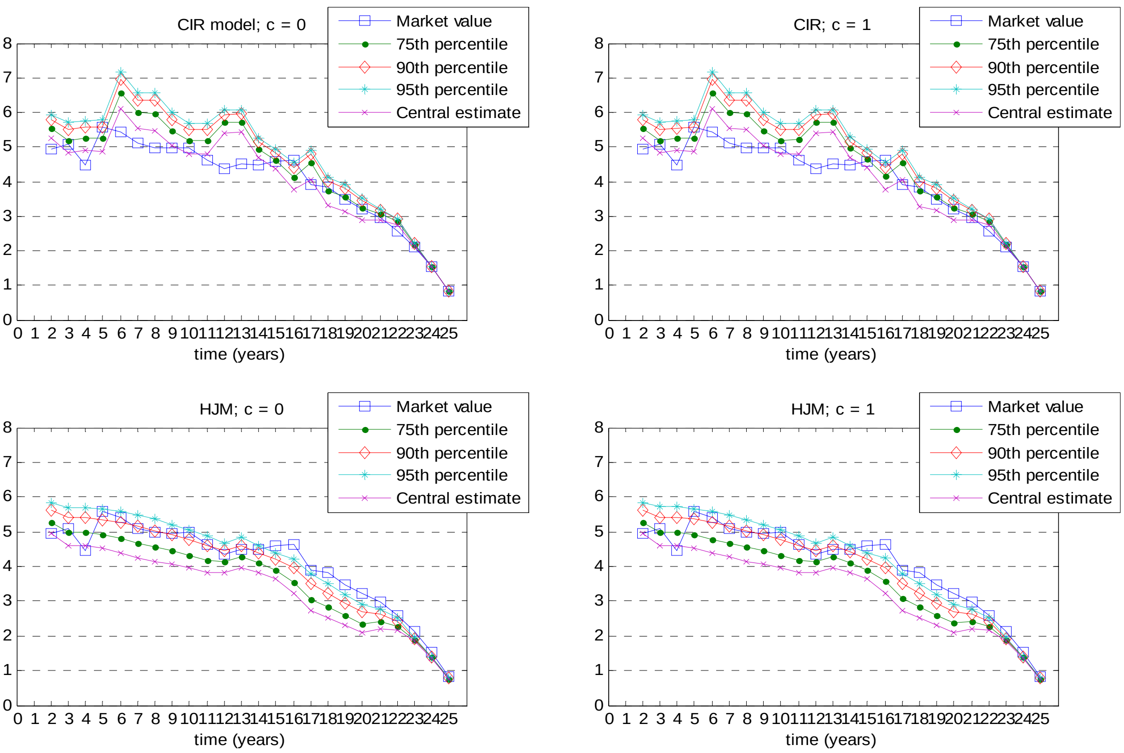
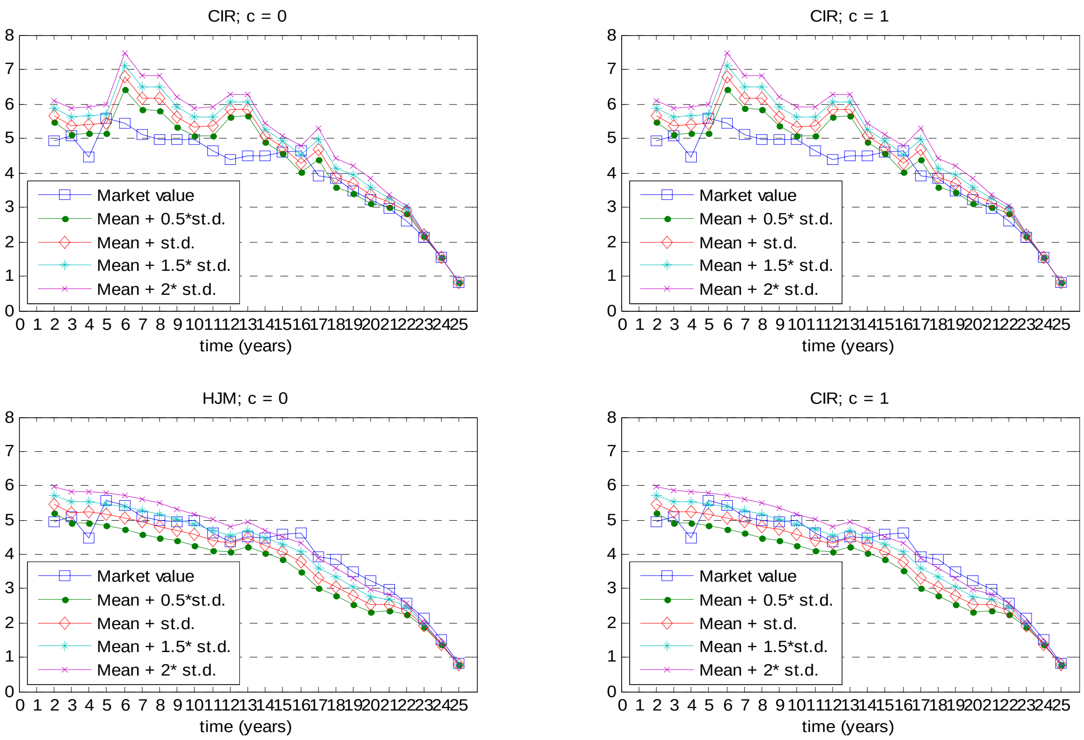


Figure 11: **Comparison of the “historical annuity market values” with the mean added of different percentages of the standard deviation of the fair value distribution obtained by using the CIR and HJM model and the adjusted survival probabilities (mean equal to 0.01 for the mortality shock and *c* = 0 and *c* = 1).**



6. Concluding remarks

In this paper, we develop a market consistent valuation approach for the setting up of the reserve of an annuity contract; the proposed approach incorporates the two main risks affecting the contract: the interest rate risk and the mortality risk.

The behaviour of the annuity contract is analysed by considering two alternative frameworks for interest rates based on the CIR model and the HJM model and a modified version of the stochastic mortality model developed by Cox and Lin. We highlight the importance of the choice of the model assumptions in the context of fair valuation, while not necessarily trying to identify the most correct model for mortality and the financial market.

In the light of the analysis presented here, we identify areas where there is scope for further work such as studying the effect of adopting alternative assumptions for the distributions used in the mortality model.

Numerical results confirm the need, expressed by the IASB and the Solvency II project, for a risk margin in the fair valuation in order to take into account the unavoidable uncertainty of the insurance business and the presence of model risk and parameter risk.

Among the different approaches provided by practitioners, academics and regulators, for the calculation of the risk margin for technical provisions, we consider both the percentile and the standard deviation approach.

An important conclusion of the analysis is the sensitivity of the annuity fair values and the risk margin to the underlying assumptions and choices made as part of calibration.

The results point to the need for clear guidelines and constraints in order to avoid areas of subjectivity, such as models and calibration, which can lead to a wide variation of values from company to company. Consequently, with the objective of guaranteeing more transparency, consistency and comparability, we recognise that a common reference framework for the market consistent valuation of the technical provisions and its components, such as the best estimate and the risk margin, becomes a crucial requirement.

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Appendix

<u>Table 1</u>: Parameter set for the interest rate models introduced in section 2.

	CIR Model				HJM Model		
year	**θ**	**κ**	**ν**	***r*(0)**	**σ**	**b**	***r*(0)**
1979	0.1202	0.10	0.0606	0.14	0.02925	0.10	0.14
1980	0.1226	0.10	0.0588	0.16	0.02925	0.10	0.16
1981	0.1206	0.10	0.0506	0.15	0.02466	0.10	0.15
1982	0.1300	0.10	0.0514	0.11	0.02327	0.10	0.11
1983	0.0768	0.10	0.0549	0.10	0.02424	0.10	0.10
1984	0.0968	0.10	0.0558	0.10	0.02464	0.10	0.10
1985	0.0737	0.10	0.0527	0.13	0.02464	0.10	0.13
1986	0.0977	0.10	0.0513	0.11	0.02320	0.10	0.11
1987	0.1081	0.10	0.0532	0.10	0.02320	0.10	0.10
1988	0.0946	0.10	0.0532	0.10	0.02320	0.10	0.10
1989	0.0414	0.12	0.0418	0.15	0.02118	0.12	0.15
1990	0.0519	0.12	0.0403	0.14	0.02020	0.12	0.14
1991	0.0805	0.12	0.0420	0.12	0.02028	0.12	0.12
1992	0.0877	0.12	0.0456	0.10	0.02085	0.12	0.10
1993	0.1273	0.10	0.0703	0.06	0.02787	0.10	0.06
1994	0.0862	0.08	0.0850	0.06	0.03158	0.08	0.06
1995	0.1370	0.08	0.0896	0.07	0.03386	0.08	0.07
1996	0.1249	0.08	0.0977	0.06	0.03627	0.08	0.06
1997	0.1192	0.08	0.0994	0.07	0.03756	0.08	0.07
1998	0.0579	0.10	0.0770	0.07	0.03143	0.10	0.07
1999	0.0334	0.10	0.0576	0.06	0.02246	0.10	0.06
2000	0.0413	0.12	0.0299	0.06	0.01256	0.12	0.06
2001	0.0409	0.15	0.0142	0.05	0.00628	0.15	0.05
2002	0.0549	0.15	0.0224	0.04	0.00970	0.15	0.04

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