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Mortality Risk and the Valuation of Annuities with Guaranteed Minimum Death Benefit Options: Application to the Italian Population.

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Abstract

In this note, we describe the payoff of Guaranteed Minimum Death Benefit options (GMDB) embedded in annuity contracts and discuss their valuation using data for the Italian male population as a case study. These put options have stochastic maturity dates due to the involuntary exercise at the moment of death. We value the GMDB as a weighted average price of a set of deterministic put options with different maturity dates, where the weights are the probability of death at every date. We take into account the mortality risk and investigate the sensitivity of the price of the option to changes in mortality probability using both deterministic and stochastic approaches.

Keywords: Guaranteed Minimum Death Benefit option, mortality risk, stochastic mortality model, Variable Annuity.

1. Introduction

The Variable Annuity (VA) market has increased considerably in the past decade, when bullish markets and low interest rate have tempted investors to look for higher returns. Variable Annuities, whose benefits are based on the performance of a underlying fund, are very attractive, because they provide participation in the stock market and also partial protection against the downside movements of interest rates or the equity market. The typical VA is a unit-linked deferred annuity contract, which is normally purchased by a single premium payment up-front which is invested in one of several funds. The VA also typically contains some embedded guarantees. One of these is the Guaranteed Minimum Death Benefit, which is an increasing-strike put option with a stochastic maturity date. If the insured dies during the deferment period, the beneficiary obtains a death benefit, that is equal, in the basic form of the product, to the maximum of the invested premium and the account value linked to the fund. An enhanced version of the product returns at least the originally investment accrued at a minimally guaranteed interest rate or the account value, if greater. These guarantees are paid for by the policyholder in the form of a perpetual fee that is deducted regularly from the account value linked to the underlying assets.

The purpose of this study is define a fair price for a GMDB in a market consistent manner and describe how the value of a GMDB evolves over time and in the presence of mortality changes. Our work develops the standard pricing model of mathematical finance and uses the Black and Scholes formula to price this insurance contract. The approach follows the recent actuarial literature on the valuation of VA products: Bauer et al. (2006); Coleman at al. (2005); Milevsky and Posner (2001), Milevsky and Salisbury(2002). Thus, Milevsky and Posner (2001) price various types of guaranteed minimum death benefit treated as a *Titanic Option* and find that in general these products are overpriced in the market. Milevsky and Salisbury (2002) adopt a framework for the valuation of GMDB where the insured has a *Real Option to Lapse*, i.e. the possibility to surrender the policy.

The contribution of this work is the study of the impact of mortality risk on the value of a GMDB under both deterministic and stochastic approaches. At first, we use the methodology of tilting to modify the observed probability of mortality and the projection is realized using assumptions based on historical data. Recently, it has become evident that deterministic mortality projections are inadequate, because unanticipated changes over time in the mortality rates have been observed. For this reason, a stochastic mortality approach is necessary in order to avoid underestimation or overestimation of the expected present value of insurance and annuity contracts. In this note, we propose a simplified version of the stochastic model suggested by Cox and Lin (2005) and developed by Ballotta, Esposito, Haberman (2006). We provide a detailed application to the Italian market, where the first Variable Annuity has been issued in September 2007 with a GMDB option.

The note is organized as follows: in section 2 we describe the product. Section 3 develops the model for the pricing of a GMDB. In Section 4, we study the impact of mortality risk on the value of the contract and show an application to Italian data following a deterministic framework. Mindful of the limits of this approach, we develop, in the section 5, a simulation-based stochastic mortality model and consider the effects on the GMDB. Concluding remarks are offered in section 6.

2. Product description

The GMDB provides for the beneficiary a guaranteed benefit at the time of death that may increase as the fund value grows. It is a put option with a stochastic maturity. There are many kinds of option:

- the basic form of a death benefit is the *Return of Premium Death Benefit*, that ensures the maximum of the current account value at time of death and the single premium paid;

- in the case of a *Roll-up* option, then the minimum benefit is equal to the single premium compounded with a constant interest rate (the *roll-up rate*);

- an enhanced version of the option provides a *rising-floor guarantee*: then the returns is at least the premium paid accrued at a minimally certain interest rate and the payoff is

$\mathrm{Max}[\ \mathrm{Min}\ [\ S_0 e^{rT}, MS_0\ ], S_T\ ]$

where r is the continuously compounded fixed guaranteed rate and M is the cap on the guaranteed return;

- when the contract contains an *Annual Ratchet Death Benefit*, the minimum amount guaranteed is compared every years with the account value, and then this that becomes the new amount guaranteed if it is greater.

- when there is a *look back* guarantee, a guaranteed death benefit is based on a suitably defined highest anniversary account value; some policies offer an annual reset, others require a five year wait and so on. The payoff is Max $[S_{ti}, S_T]$, where t_i is a defined anniversary.

In general, we can classify the GMDB in two groups: a) "interest guarantees" which refer to a contract in which the amount guaranteed is the premium accumulated at a fixed rate of return; and b) "market guarantees" which ensure the highest market return during a certain period. Most Variable Annuities provide a combination of both categories.

In this work we consider a Variable Annuity within a simple *Roll-up* GMDB, and we assume that the policyholder does not have an option to lapse, for the sake of simplicity. The policyholder pays a single premium P, that is invested in a fund; we denote the account value by W_t. As far as the put option is concerned, in contrast to the other derivatives where payments are made on acquisition, the GMDB option is paid by deducting a fixed proportional amount from the account value on a continuous basis. Milevsky and Posner (2001) calculate the fair charge considering that its expected present value has to be equal to the value of a put option with a stochastic maturity date. We note that American options also have a stochastic maturity, but the methodology used to price these derivatives cannot be used for the GMDB, because there is a difference between the two products: in the first case, the investor decides when he exercises the option, in the second one the put will expire at the moment of death. For this reason, the only way to price a GMDB is based on its decomposition into other simpler instruments, as we illustrate in the next section.

3. The model

Let T_x be the future lifetime random variable expressed in continuous time, $F_x(t)$ be its *cdf* and $f_x(t)$ be its *pdf*; therefore, for an individual aged x the probability of death before time t is

$$F_x(t) = P\,(T_x \leq t) = 1 - {}_tp_x = 1 - \exp\left\{-\int_0^t \mu(x+s)ds\right\} \quad (1)$$

where μ denotes the force of mortality.

Let V_t be the account value at time t linked to fund value. Following the standard assumptions in the literature, we model the evolution of the account as:

$$dV_t = (\mu - \delta)V_t dt + \sigma V_t dW_t \quad (2)$$

where W_t is a standard Brownian motion, μ is the drift rate, δ is the charge paid for the GMDB option.

The risk neutral process for V_t is:

$$dV_t = (r - \delta)V_t dt + \sigma V_t dW_t^Q \quad (3)$$

where r is the risk free rate and W_t^Q is a Brownian motion under a new Girsanov transformed measure Q. The solution of the SDE is:

$$V_t = V_0 e^{(r-\delta-\sigma^2/2)t+\sigma W_t^Q} \quad (4).$$

Now we describe the GMDB payoff. At the random date of death τ the beneficiary will receive

$$D_\tau^p = \max(e^{g\tau}V_0, V_\tau) = e^{g\tau}\max(V_0 - e^{-g\tau}V_\tau, 0) + V_\tau \quad (5)$$

where g is the guaranteed rate.

The value of the GMDB option at τ is the sum of the fund value and a put option whose strike price is the initial value V_0, with an underlying asset V_τ discounted by the guaranteed growth rate g. Since the maturity is stochastic and τ and V_τ are independent, the present value of GMDB is given by the expectations under τ and V_τ:

$$D_0^p = E_t\left\{E^Q\left\{e^{-r\tau}D_\tau^p \middle| \tau = t\right\}\right\} \quad (6)$$

If we fixe the date τ, we have at τ an European Option, whose value can be calculated with Black and Scholes formula. Therefore, the previous formula can be interpreted as a decomposition of the actual value of GMDB into the actual value of a continuous sequence of European put options. Substituting the expression for D_τ^p

$$D_0^p = E_t\left\{E^Q\left\{e^{-(r-g)\tau}\max(V_0 - e^{-g\tau}V_\tau, 0) + e^{-r\tau}V_\tau \middle| \tau = t\right\}\right\} \quad (7)$$

We can observe that

$$E^Q\left\{e^{-r\tau}V_\tau\right\} = e^{-\delta\tau}V_0 \quad (8)$$

since we have supposed that V_t is a geometric Brownian motion with drift equal to r-δ and so its expected value is:

$$E_{(0)}^{Q}\{V_\tau\} = e^{(r-\delta)\tau}V_0 \quad (9)$$

Consequently:

$$\begin{aligned} D_0^p &= E_t\left\{E^Q\left\{e^{-(r-g)\tau}\max(V_0 - e^{-g\tau}V_\tau, 0) + e^{-r\tau}V_\tau \middle| \tau = t\right\}\right\} = \\ &= E_t\left\{E^Q\left\{e^{-(r-g)\tau}\max(V_0 - e^{-g\tau}V_\tau, 0) + e^{-\delta\tau} \middle| \tau = t\right\}\right\} \end{aligned} \quad (10)$$

We observe that for a fixed date T

$$\begin{aligned} &E^Q\left\{e^{-(r-g)T}\max(V_0 - e^{-gT}V_T, 0) + e^{-\delta T}\right\} \equiv \\ &\equiv V_0\left(\left[e^{-\tilde{r}T}N(-d_2) - e^{-\delta T}N(-d_1) + e^{-\delta T}\right]\right) \equiv \\ &\equiv V_0\left[BS\left(\tilde{r}, \delta, \sigma, T) + e^{-\delta T}\right)\right] \end{aligned} \quad (11)$$

where $d_1 = \dfrac{(\tilde{r} - \delta + \sigma^2/2)\sqrt{T}}{\sigma}$; $d_2 = d_1 - \sigma\sqrt{T}$; $\tilde{r} = r - g$; N(.) is the cumulative probability function for a random variable normally distributed.

If we consider both the expectations, we obtain:

$$D_0^p = \int_0^{\omega-x} f_x(t)V_0\left[BS\left(\tilde{r}, \delta, \sigma, t) + e^{-\delta t}\right)\right]dt \quad (12)$$

In the discrete case we have:

$$D_0^p = \sum_{t=1}^{\omega-x} {}_t p_x q_{x+t} V_0\left[BS(\tilde{r}, \delta, \sigma.t) + e^{-\delta t}\right] \quad (13)$$

for a policyholder aged x at inception of the contract.

Thus, the value of the GMDB is a weighted average of the values of ω-x European put options, where the weights are the postponed probability of death in t, i.e. the probability of survival until t and death between t and t+1.

4. The impact of mortality on the GMDB value: a deterministic approach

In this section, we illustrate the relationship between the GMDB value and the age of policyholder at the inception of the contract.

We price a simple form of the death benefit; we consider g equal to 0, so that the GMDB option ensures the maximum of the current account value at the beginning of the year of death and the single premium paid is given by:

$$D_0 = \sum_{t=1}^{\omega-x} {}_t p_x q_{x+t} \max(V_t, V_0) e^{-rt} = \sum_{t=1}^{\omega-x} {}_t p_x q_{x+t} \left[\max(V_0 - V_t, 0) + V_t\right] e^{-rt} \qquad (14)$$

and where the parameters in the model take the following specific values: the risk free rate r is 7%, δ=1% , the underlying volatility σ is 10%, the strike V_0 is 100 and the fund value follows a geometric Brownian motion. We make reference to the Black and Scholes framework for option pricing.

We consider two different mortality tables based on the experience of the Italian male population for 2001 and 2004. Figure 1 shows the probability of survival and mortality rate for a policyholder aged 50 occurred in the 2004. The graphs show the characteristic features. We note a kink in the q_{50+t} curve for value of t equal to 55.

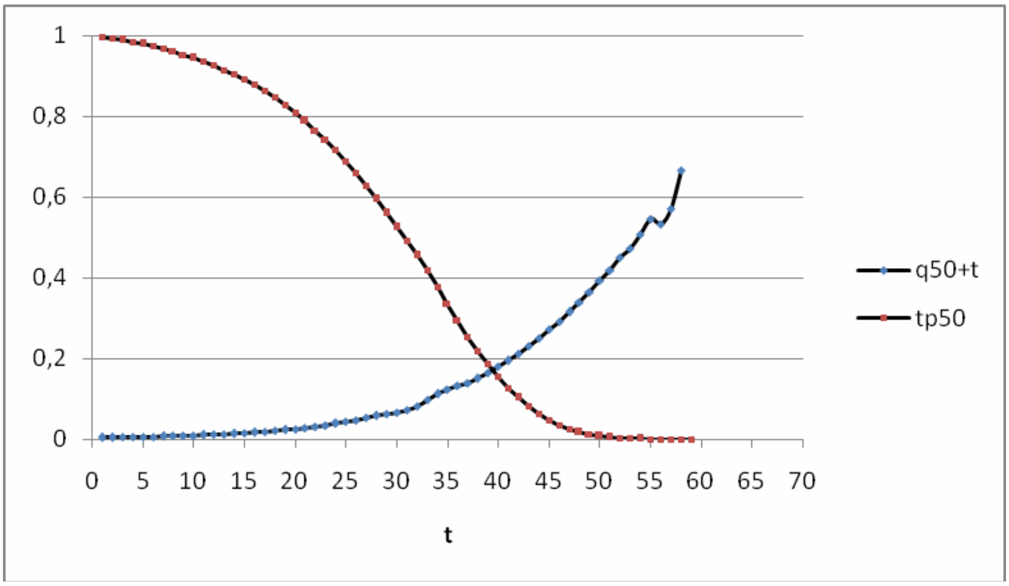

Figure 1: mortality probability functions

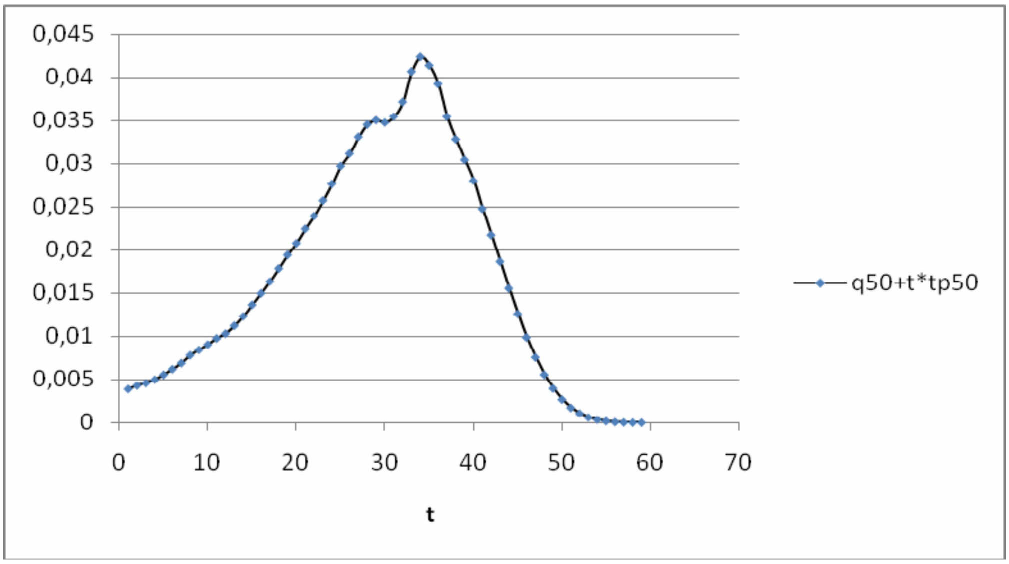

Figure 2: the postponed mortality probability function for a policyholder aged 50

The function ${}_t p_x q_{x+t}$ for discrete values of t represents the probability function of the discrete random variable K_x for t=0,1,2…Thus ${}_t p_x q_{x+t} = \Pr[t < K_x \le t+1]$. Figure 2 shows an unusual feature: it has two modes at t equal to 29 and 34. It depends on the fluctuation in the fitted curve of q_{50+t} , which has a rather strange behaviour between the ages of 79 and 84[1].

Let $F_x(t)$ be the cumulative distribution function (cdf) of the random variable time to death for Italian male policyholders aged x based on the 2004 mortality table, as in equation (1).

We operate a tilting of $F_x(t)$ to create a new function F*(x), characterized by a reduction of mortality:

$$F_x^*(t) = h[F_x(t)] \quad (15)$$

where h is modeled on a historical basis and projects forward the same reduction of mortality that happened between 2001 and 2004. We can think of $F_x^*(t)$ as an

[1] The 2-modal feature can be found also in a recent Belgian males table (see Pitacco et al. (2008)).

adjusted mortality cumulative distribution function, which takes into account projected improvements in life expectancy. Figure 3 provides an example of the tilted cdf from age 0 onwards.

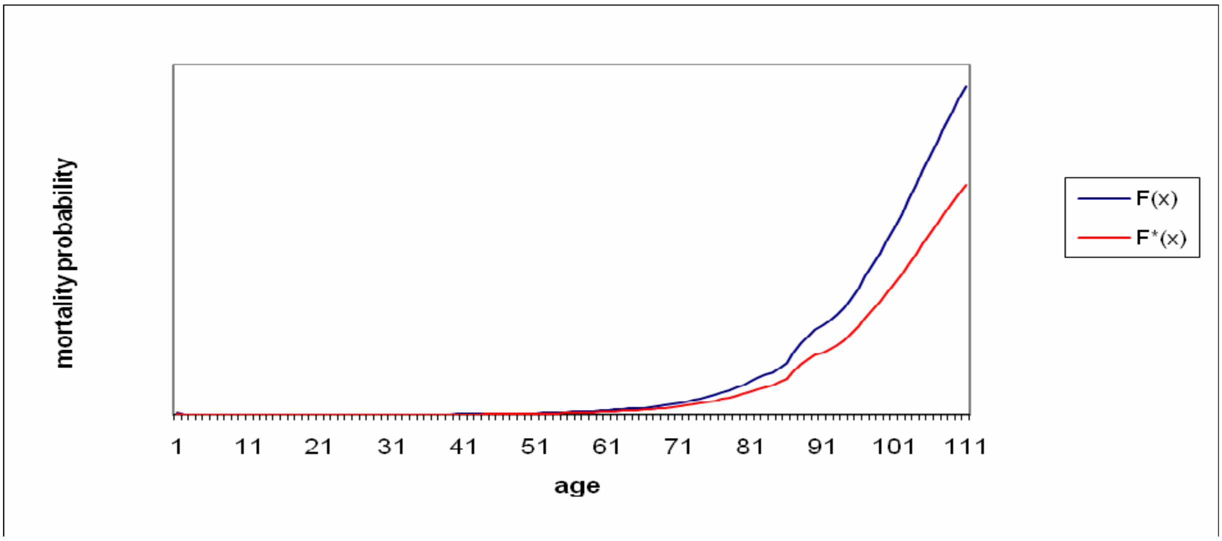

Figure 3: Tilting of mortality Cdf

The assumption is strong: for the sake of simplicity, we are assuming that there will be in the future the same improvement in life expectancy that occurred in the past 3 years.

In our application, we use the above procedure in order to derive a modified probability function at each age between 50 and 95. In Figure 4, we report only the discrete probability function for a policyholder aged 50 at inception.

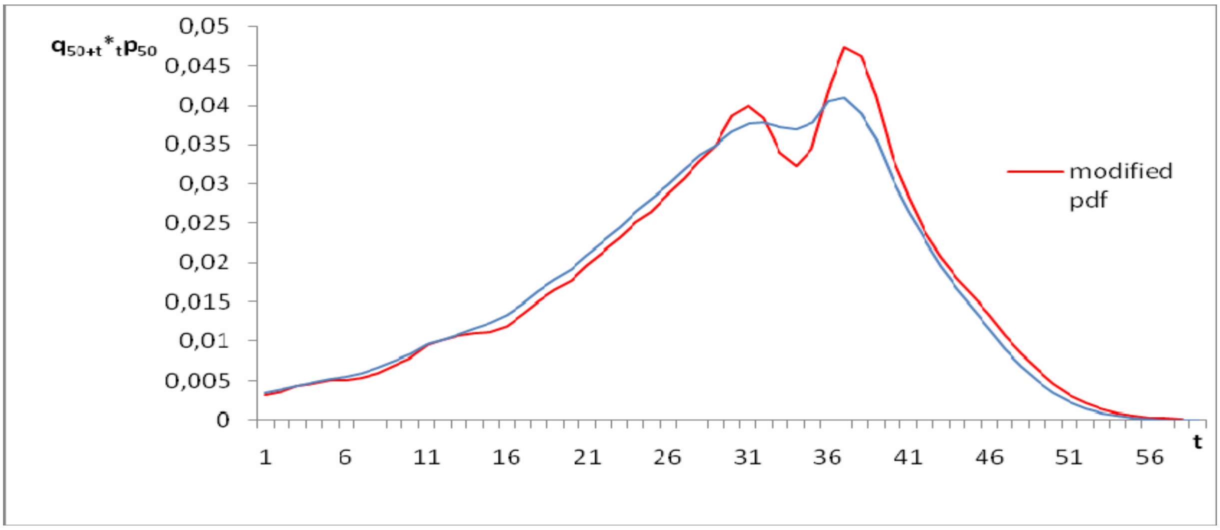

Figure 4 : Tilting of discrete mortality probability function

We calculate the GMDB value for different policyholders with ages from 50 to 95 at inception. At first, we consider only the discrete mortality probability density function in order to study the way in which the GMDB value varies when the age of policyholder at the inception of the contract increases. Then, we analyze the impact of mortality improvements on the GMDB value.

The GMDB option is composed of a sequence of put options with different maturities. For example, we report the calculation of the GMDB value for a policyholders aged 50:

$$D_0(50) = \sum_{t=1}^{\omega-50} {}_t p_{50} q_{50+t} (Max(V_0 - V_t; 0) + V_t) e^{-rt} \tag{16}$$

where BS_t is a put option with maturity t.

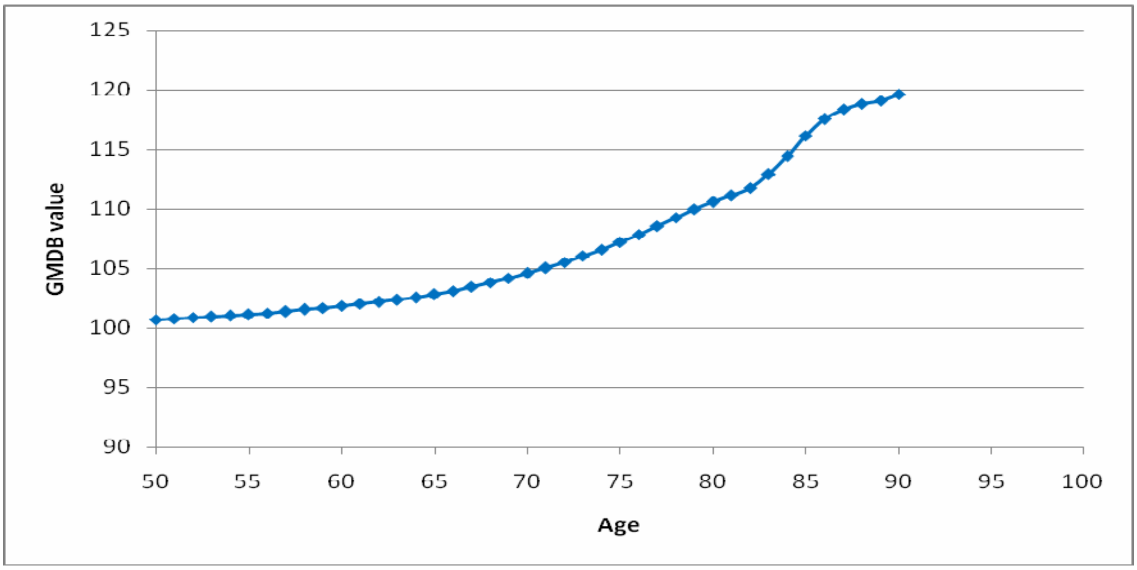

Figure 5: The GMDB value and dependency on age at inception under the real mortality probability function

In order to study the relation between the GMDB value and age at inception, we need to take into account two different effects: on one hand, the weights change because the probability function changes with age; on the other hand, as age at inception increases, the number of put options that compose the GMDB product decreases. Moreover, we have to consider that the value of the put decreases with time. The combination of these effects generates the relation represented in figure 3: as age at inception increases the value of the GMDB increases.

Next, we compare the value of the GMDB under the real and modified probability functions for different policyholders aged between 50 and 95 at the inception of the contract: see Figure 6. In order to explore the consequences of an improvement in life expectancy on the GMDB value, we have to take into account the fact that the probability function changes in response to two different effects: at each time point the survival probability increases and the mortality probability decreases. As we can see from figure 4, the second effect prevails between ages 50 and 80 and between ages 82 and 86. For this reason, the GMDB values under the modified probability function are smaller than that under the real probability function at almost all of the ages considered.

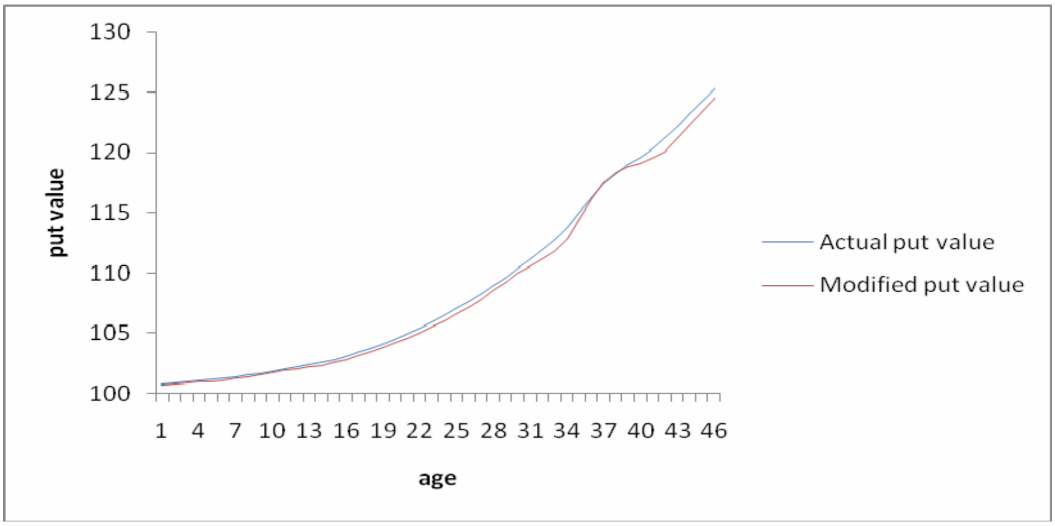

Figure 6 : The comparison of GMDB value under real and modified mortality probability function

At the end of this section, we reflect upon what happens if g is different from zero. In this case, the GMDB option provides the maximum of the current account value at the beginning of the year of death and the single premium capitalized at the rate g:

$$D_0 = \sum_{t=1}^{\omega-x} {}_t p_x q_{x+t} \max(V_t, V_0 e^{gt}) e^{-rt} = \sum_{t=1}^{\omega-x} {}_t p_x q_{x+t} \left[e^{gt} \max(V_0 - e^{-gt} V_t; 0) + V_t \right] e^{-rt} \tag{17}$$

As g increases the spot price of the underlying ($e^{-gt}V_t$) decreases and the value of each put option increases; furthermore, it is capitalized at the rate g, so as g increases the GMDB value increases. Figure 7 shows the relationship between the guaranteed rate g and the GMDB value for a policyholder aged 50.

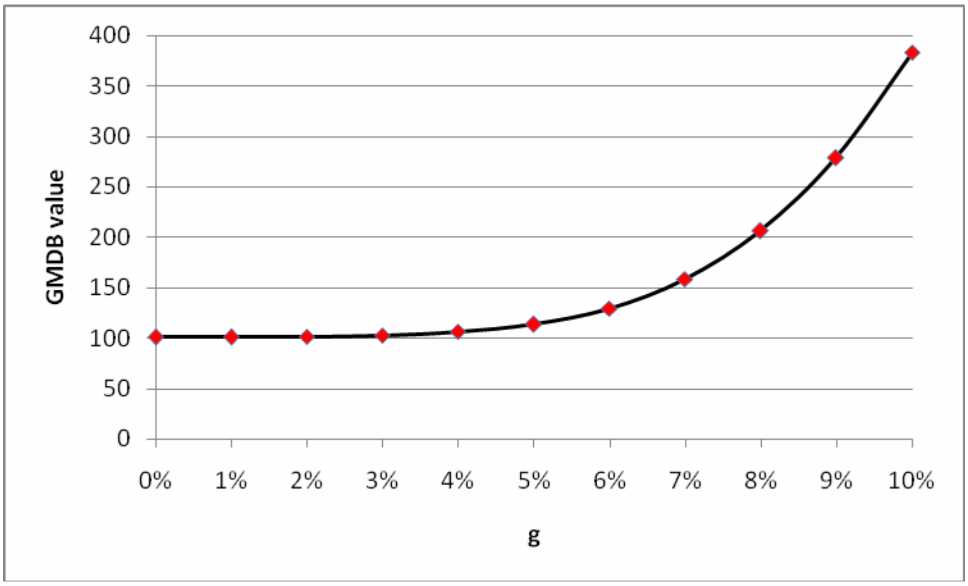

Figure 7: The relation between the GMDB value and g for a policyholder aged 50

5. The impact of mortality risk on the GMDB value: a stochastic approach

In the previous section we have modified the mortality distribution using a tilting method based on historical observations. Recently, it has become evident that deterministic mortality projections are an inadequate approach to dealing with risk, i.e. unanticipated changes over time in the mortality rates and other indices. For this reason, a stochastic mortality approach is necessary in order to avoid underestimation or overestimation of expected present value of life insurance contracts with a significant mortality component. In this section, we propose a simplified version of the stochastic model suggested by Cox and Lin (2005) and developed by Ballotta, Esposito, Haberman (2006).

Our calculation is based on the survival model used before; our purpose is to develop an adjusted survival model (or mortality table), which takes into account possible mortality shocks. In this regard, we estimate the expected value of the number of survivors at age x+t, $E[\, l(x+t)\,]$, in a stochastic framework. It is possible to prove that $l(x+t)$ is approximately distributed as a normal random variable with mean equal to $l(x)\ {}_tp_x$ and variance equal to $l(x)\ {}_tp_x(1-{}_tp_x)$. However, the latest actuarial literature highlights that the empirical data show perturbations in the survival probabilities due to random shocks. Accordingly, we simulate the survival probabilities adjusted for shocks as follows:

$$p'_{x+t} = p_{x+t}^{(1-\varepsilon_t)} \quad (18)$$

where ε_t is the shock in the expected probability at time t. Ballotta, Esposito, Haberman (2006) assume that ε_t follows a beta distribution with parameter a and b and the sign of the shocks depends on the random number k(t) simulated from the uniform distribution U(0,1). In particular, we set:

$$\varepsilon(t) \quad \text{if} \quad k(t) < c$$

$$-\varepsilon(t) \text{ if} \quad k(t) \geq c \quad (19)$$

where c is a parameter which depends on the user's expectation of the future mortality trend.

The importance of assigning a random sign to ε_t is that, in this way, the model captures not only the long period variations in mortality rates, but also the short period fluctuations due to exceptional circumstances.

In our application, we consider two opposite cases for the value of c: c = 1 and c = 0. In the first case, there will be improvements in life expectancy at every date; in other words, all shocks are expected to be positive. Conversely, in the second case further improvements of an already high expectancy of life are impossible and all shocks are expected to be negative. So, we simulate the value of p'_x for a policyholder aged x = 50 at inception of the contract under the two different hypotheses and then we calculate the expected number of survivors l'(x+t+1) as follows:

$$l'(x+t+1) = l(x+t) * p'(x+t) \quad (20)$$

We are then able to calculate the other mortality functions that we need.

In order to analyze the impact of different variations in mortality probabilities, we consider two different expected value for ε_t:

$$E[\varepsilon_t] = 0{,}10$$
$$E[\varepsilon_t] = 0{,}30$$

We carry out two calculation procedures: in the first one, we fix a = 0,5 and b = 4,5, so that shocks have expected value equal to 0,10 and standard deviation equal to 0,12; in the second one, we fix a = 1,5 and b = 3,5, so that shocks have expected value equal to 0,30 and standard deviation equal to 0,19. In both cases, we simulate 1000 paths of evolution of mortality using the Monte Carlo method and consider the alternative hypotheses c = 0 and c = 1. Then, we calculate the price of the GMDB option and compare the results under the different scenarios.

At first, we report the graphics relating to only one path simulated under the hypothesis $E[\varepsilon_t] = 0{,}30$, in order to reflect upon the impact on the GMDB value of an improvement or a worsening in life expectancy; afterwards, we show the more general results of our simulations.

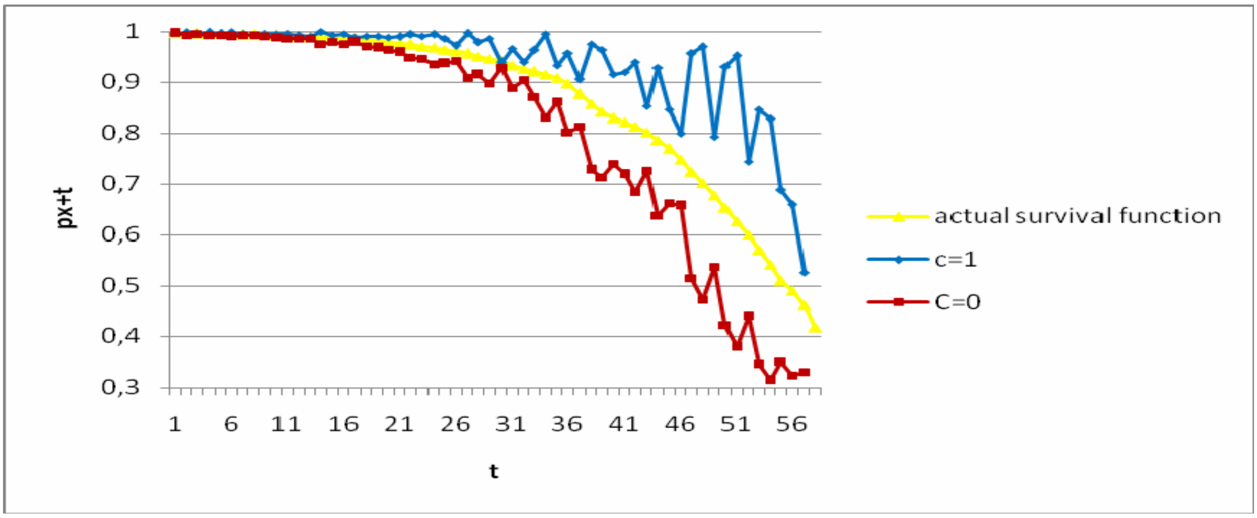

Figure 8: The comparison between the actual survival function and the survival functions simulated under the hypothesis c = 0 and c = 1

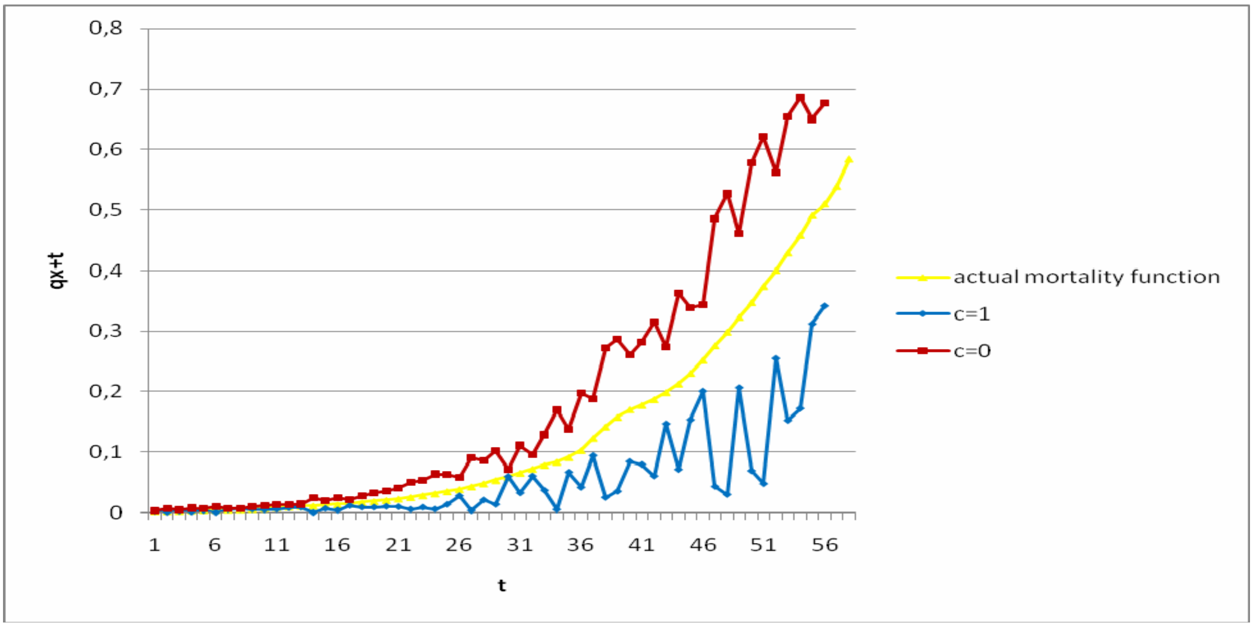

Figure 9: The comparison between the actual mortality function and the mortality functions simulated under the hypothesis c = 0 and c = 1

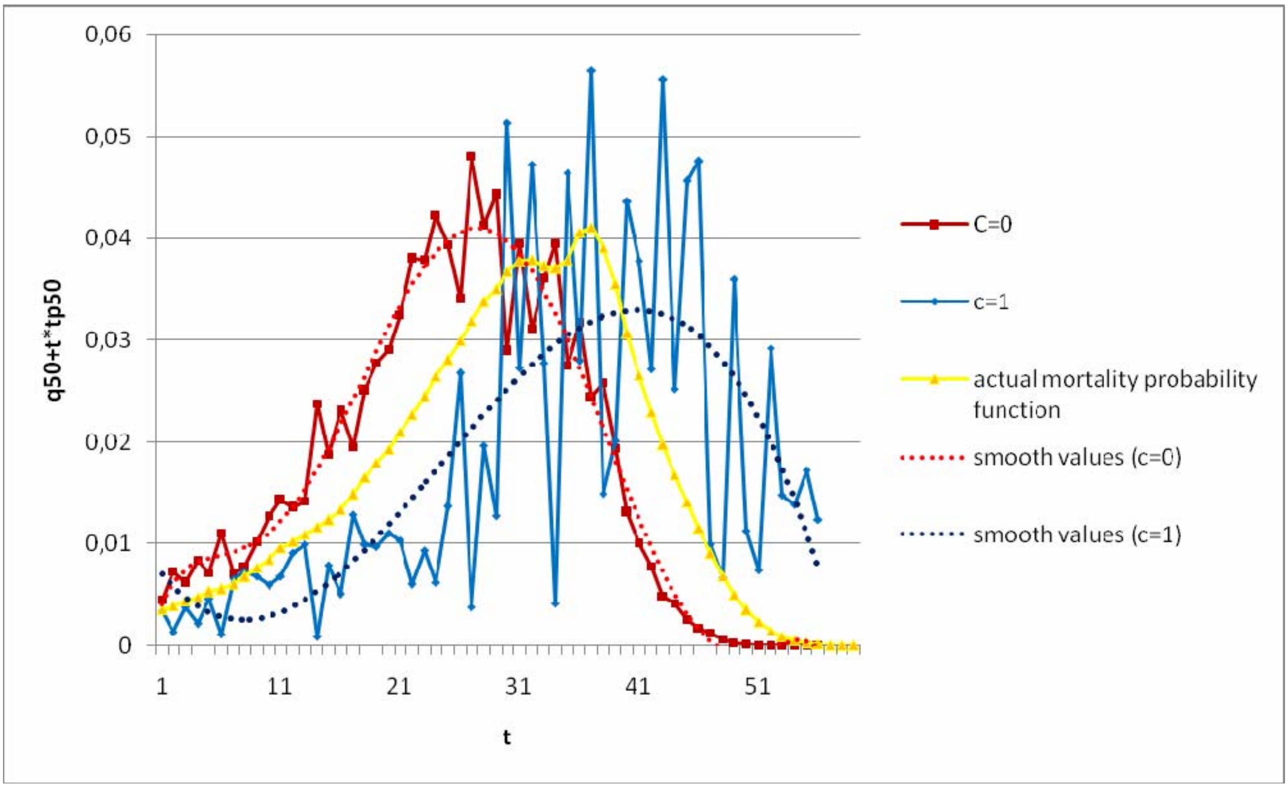

Figure 10: the comparison between the actual mortality probability function for a policyholder aged 50 and the simulated distribution under the hypothesis c = 0 and c = 1

In the figure 8, we compare the actual survival function[2] with those simulated under the hypothesis c = 1 and c = 0. In the first case, we expect that there will be only improvements in life expectancy and, consequently, the simulated function lies above the actual survival function. Instead, in the second case we expect there will be only deteriorations and the simulated function lies below the actual survival function. In the same manner, in Figure 9 we compare the actual mortality function with those simulated under hypothesis c = 1 and c = 0 and we see a complementary picture.

The purpose of this simulation is to quantify the impact of mortality risk on the GMDB value; in this regard, we have to consider the projected postponed probabilities of death[3]. In Figure 10, we compare the actual mortality probability function for a policyholder aged 50 and the simulated distribution under the hypotheses c = 1 and c = 0. We have to keep in mind that the probability function changes because of two different effects: if c = 1the survival probability increases and the mortality probability decreases at every time point; on the contrary, if c = 0 the mortality probability decreases and the survival probability increases. The consequences are that, under the hypothesis c = 0, the probability function is translated so that the left tail bacomes fatter and the right tail less fat than fot the actual probability function[4]. On the contrary, if c = 1, the probability function is translated so that the left tail becomes less fat and the right tail more fat than for the actual probability function. The consequence is that there will be improvementes in life expectancy, the probability of death during a given year will decrease at younger ages and will increase at older ages.

The effects on the GMDB value[5] are described in Figure 11:

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[2] The actual survival function, which we refer, is based on SIM2002 mortality table.

[3] In Figure 8 we have reported only the results relating to a policyholder aged 50 at inception of the contract, but we have simulated the postponed probability of death for every age of inception between 50 and 110.

[4] In Figure 10 we have constructed a smooth function with a polynomial regression to make this translation more clear.

[5] We still refer to the basic form of the GMDB option, that ensures the maximum of the current account value at time of death and the single premium paid.

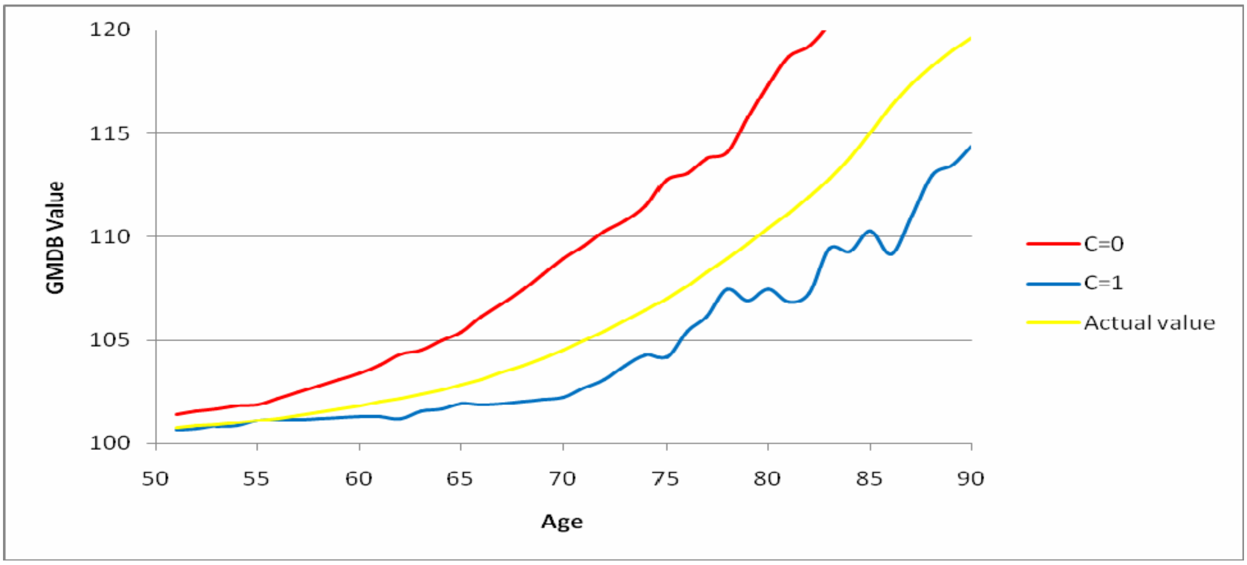

Figure 11: The comparison between actual and simulated GMDB value

Under the hypothesis c = 1, the weights of the valuation formula (i.e. the mortality probability function) are lower at the beginning and higher at the end of the time period than the actual weights; consequently, the earlier put options, that have a large value, are weighted less than under the actual distribution and the final put options, that have a small value, are weighted more. Furthermore, in the valuation formula there is also a term linked to the fund value ($V_t * e^{-rt}$), which decreases as t increases[6]. It is weighted less than under the actual distribution during the first years, when it is higher, and it is weighted more at later time, when it is smaller. For these reasons, if there will be improvements in life expectancy the GMDB value will decrease and the liabilities of the insurer will shrink. On the contrary, under the hypothesis c = 0, the weights of the valuation formula are higher at the beginning and lower at the end than the actual weights; consequently, the earlier put options, that have a large value, are weighted more than under the actual distribution and the final put options, that have a small value, are weighted less. Furthermore, the term linked to the fund value is weighted more than under the actual distribution during the first years, when it is higher, and it is weighted less at later time, when it is smaller. For these reasons, if

[6] In this application, we have considered a risk neutral process for Vt, whit a drift rate (r-δ)=(7%-1%), so the term in the valuation formula $V_t * e^{-rt}$ decreases as t increases.

there were a worsening in life expectancy the GMDB value will increase and the liabilities of the insurer will rise[7].

Up to this time we have shown the results for a particular single simulated path of mortality. Now, we report the more general results from our simulations. We have simulated 1000 values of ε_t for each t from a beta distribution, and then we have calculated the mean of the shocks at every time and, on this base, have calculate the expected postponed probabilities of death. Subsequently, we have considered the extreme shocks that can occur by choosing upper and lower percentiles. In particular, we have cut the beta distribution at the 95th and 5th percentile and have projected the postponed probabilities of death under both scenarios. The results are shown in the Figure 12-15:

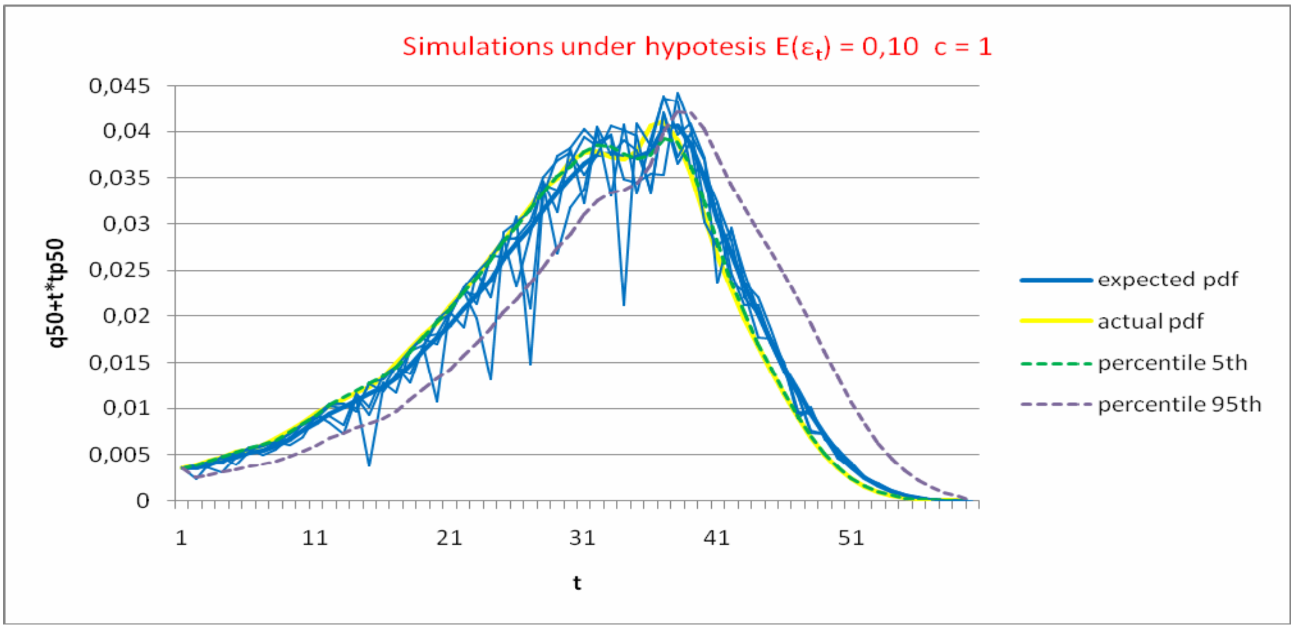

Figure 12a: Simulated mortality probability function under the hypothesis $E(\varepsilon_t)$ = 0,10 and $\sigma(\varepsilon_t)$= 0,12; c = 1

[7] We point out that the effects of an improvement or a worsening in life expectancy can be different as the assumptions change; for example, if the drift of the process of the fund value is higher than r, the value of $V_t{*}e^{-rt}$ increases as t increases. In order to study what happens under the hypothesis c=0 and c=1 it is necessary to observe the interaction between the variations of the value of put option, of $V_t{*}e^{-rt}$ and of the weights in the valuation formula. However, a complete description of this interaction is outside of the scope of this work.

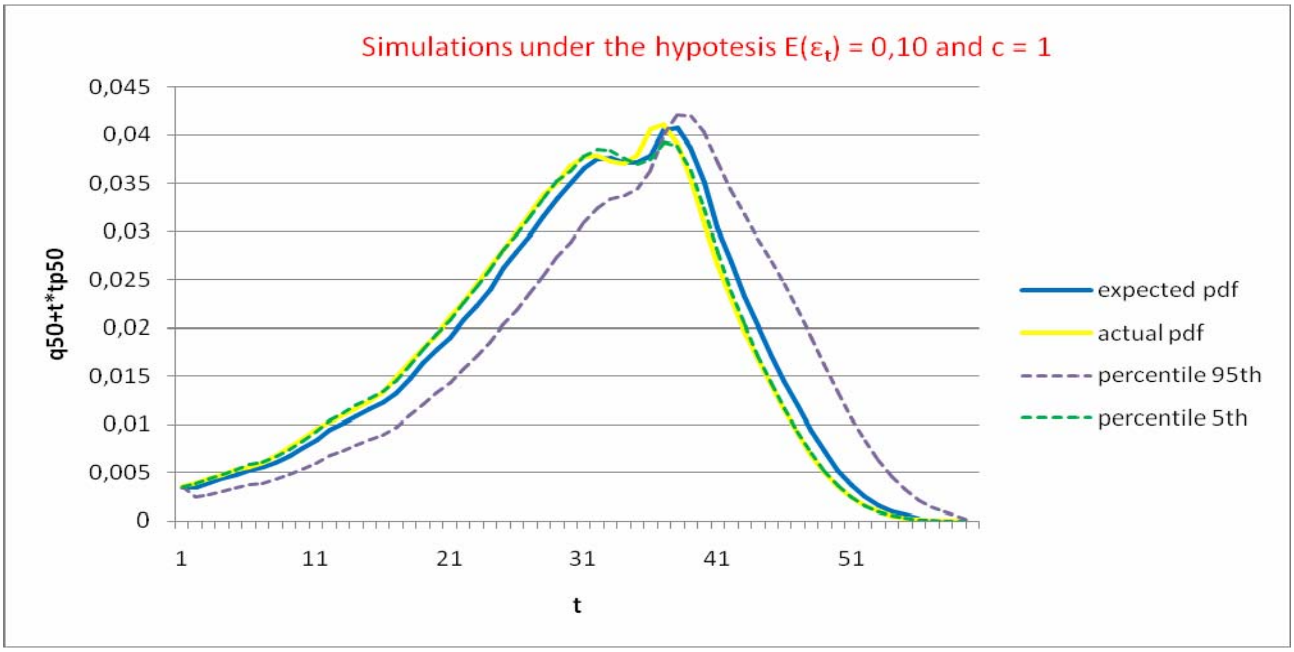

Figure 12 b: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\varepsilon_t) = 0{,}10$ and $\sigma(\varepsilon_t) = 0{,}12$; c = 1

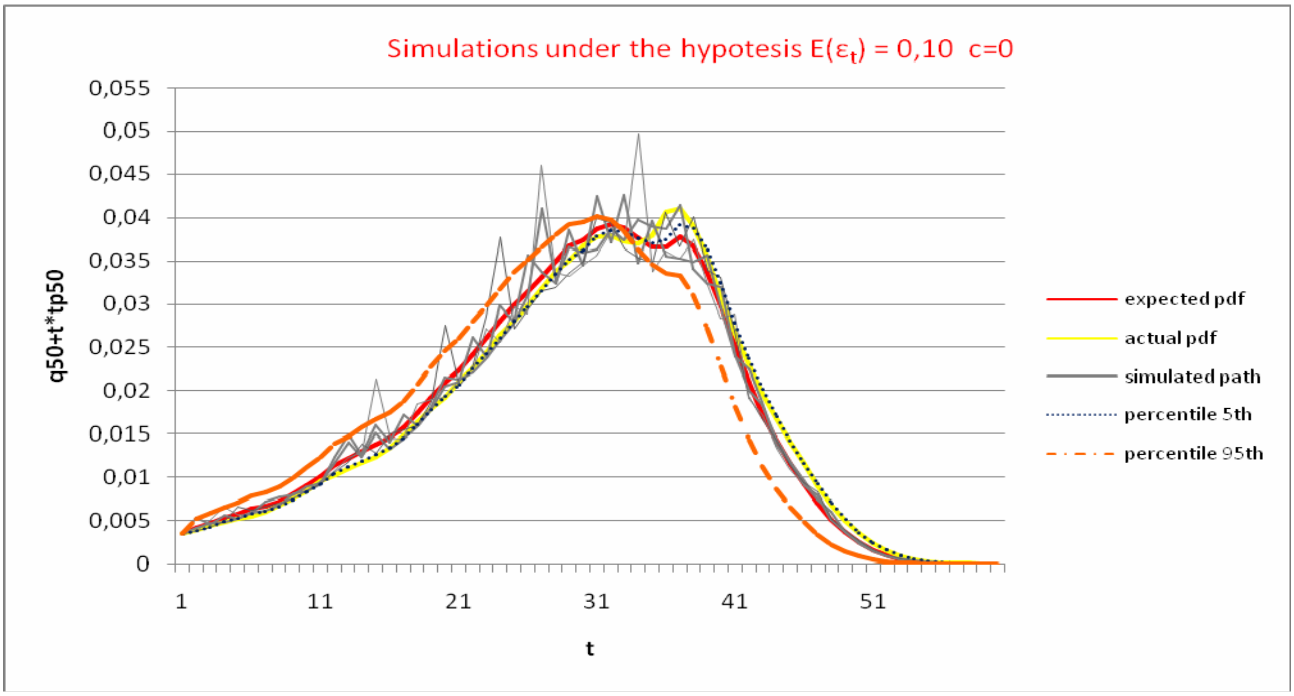

Figure 13a: Simulated mortality probability function under the hypothesis $E(\varepsilon_t) = 0{,}10$ and $\sigma(\varepsilon_t) = 0{,}12$; c = 0

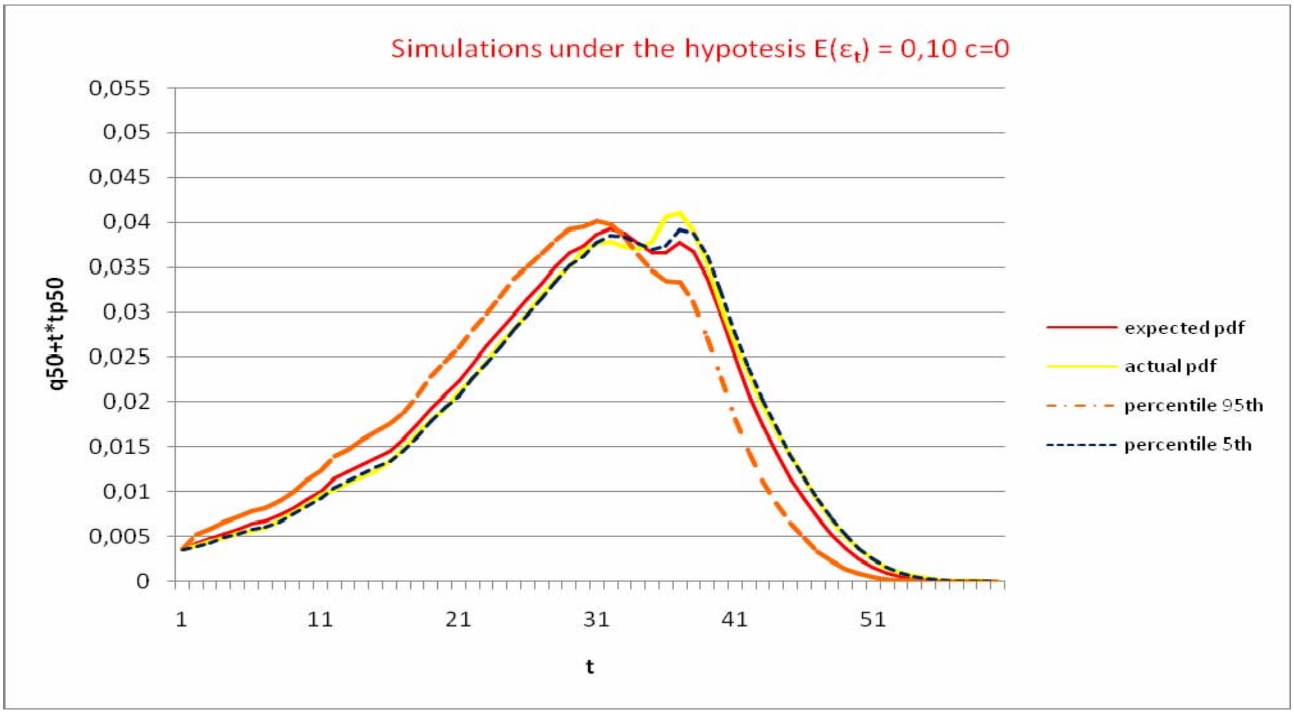

Figure 13 b: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\varepsilon_t) = 0{,}10$ and $\sigma(\varepsilon_t) = 0{,}12$; c = 0

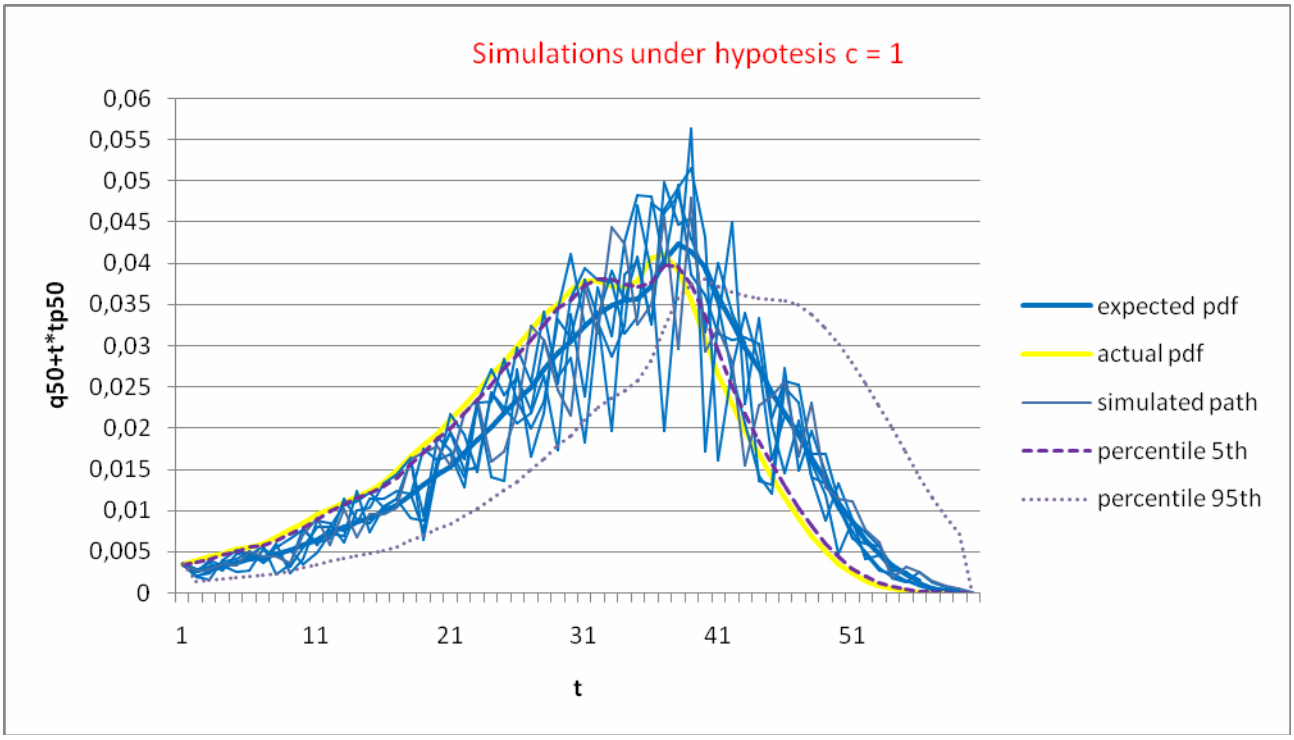

Figure 14a: Simulated mortality probability function under the hypothesis $E(\varepsilon_t) = 0{,}30$ and $\sigma(\varepsilon_t) = 0{,}19$; c = 1

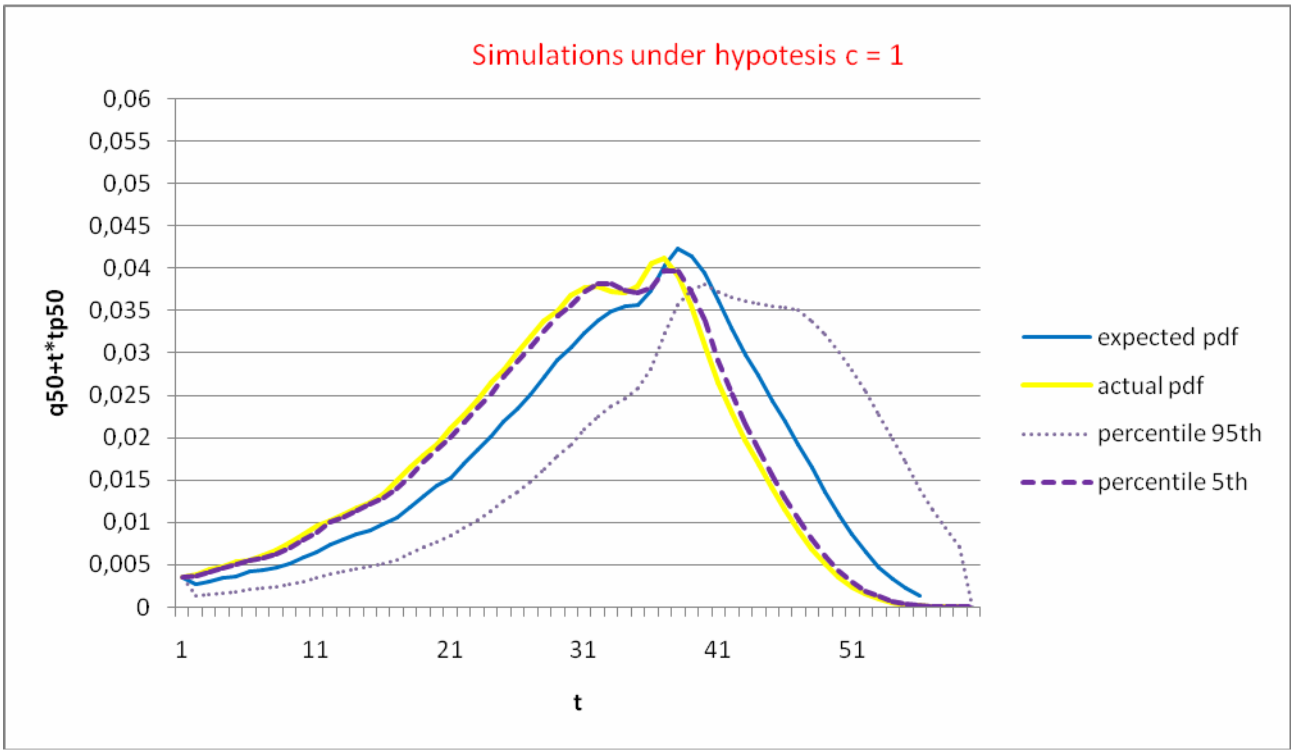

Figure 14b: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\varepsilon_t) = 0{,}30$ and $\sigma(\varepsilon_t) = 0{,}19$; c = 0

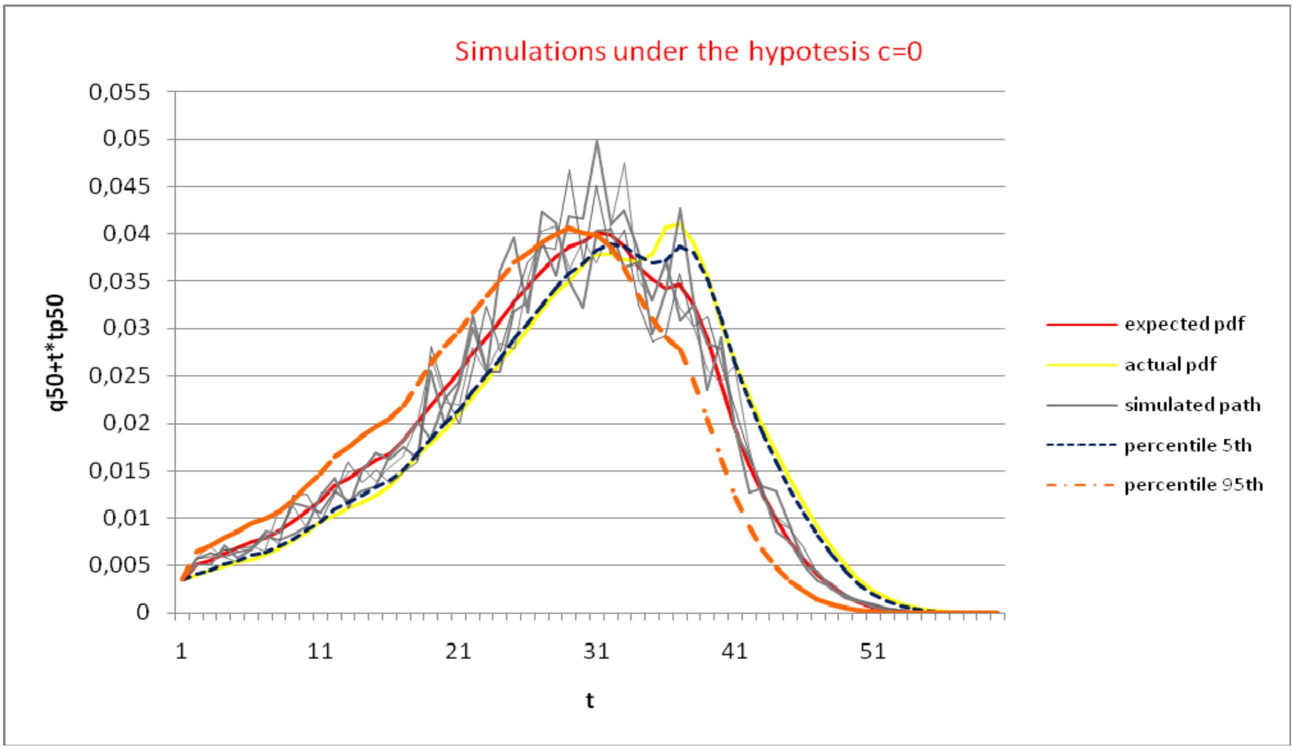

Figure 15a: Simulated mortality probability function under the hypothesis $E(\varepsilon_t) = 0{,}30$ and $\sigma(\varepsilon_t) = 0{,}19$; c = 0

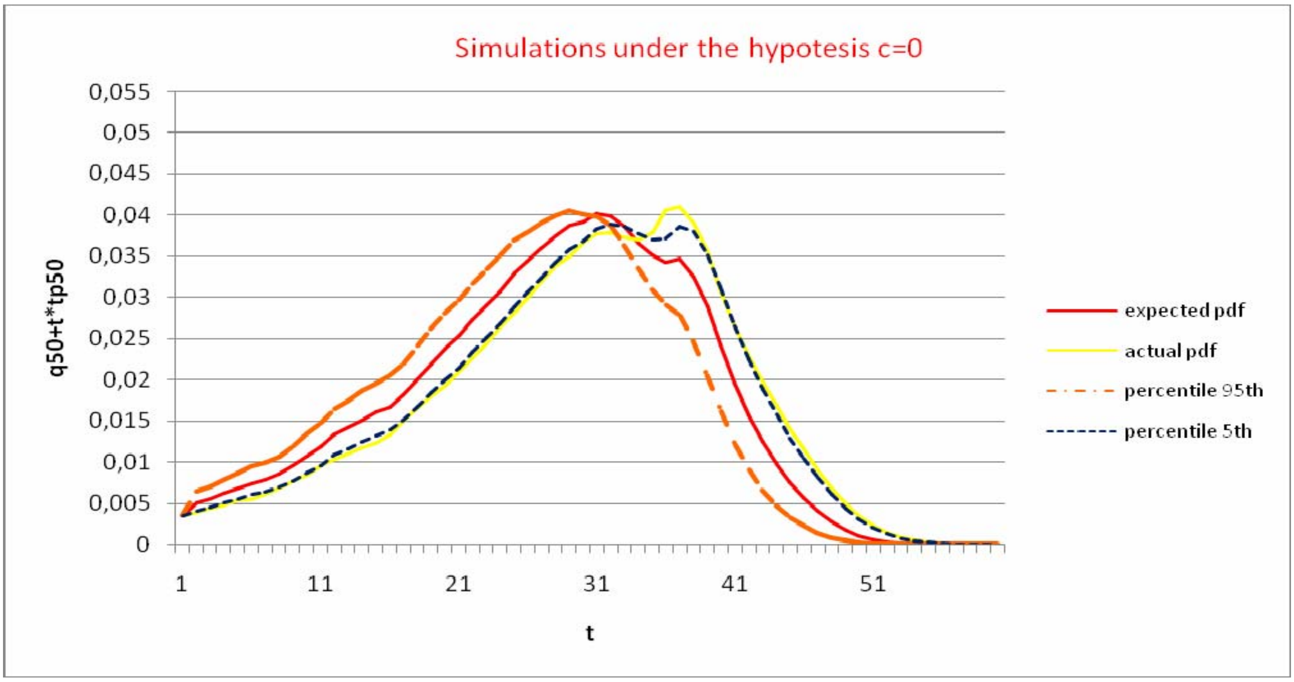

Figure 15b: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\varepsilon_t) = 0{,}30$ and $\sigma(\varepsilon_t) = 0{,}19$; c = 0

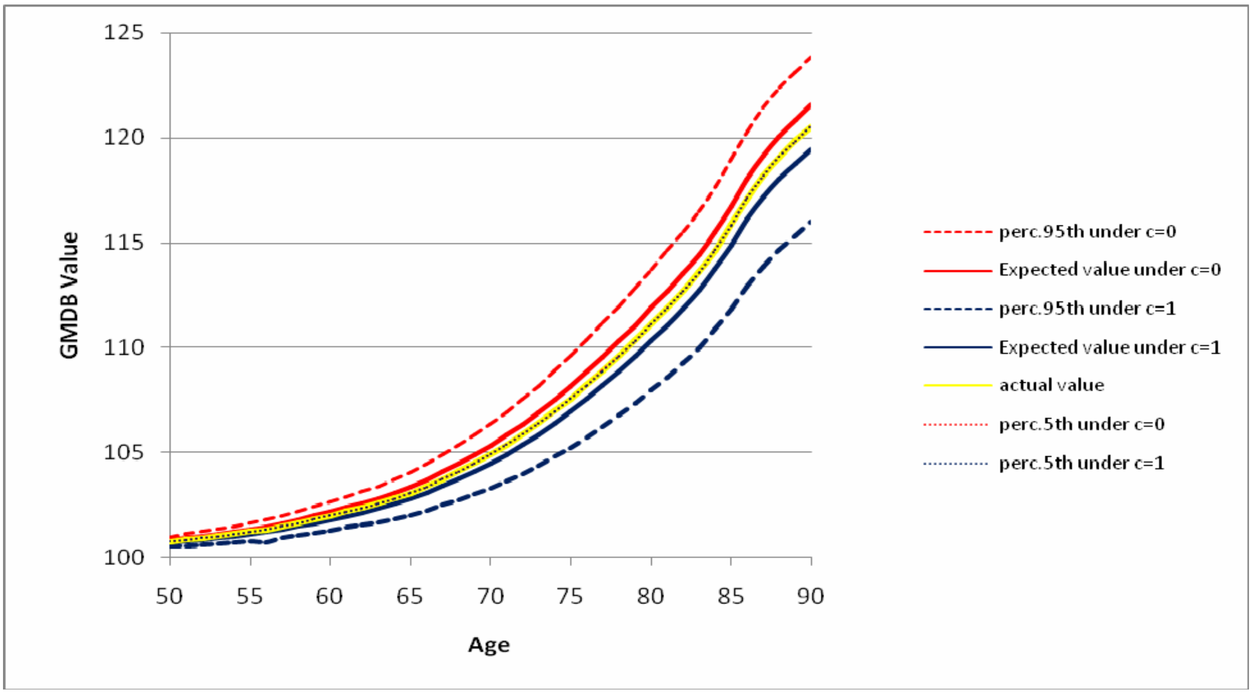

Figure 16: GMDB Value under the hypothesis $E(\varepsilon_t) = 0{,}10$ and $\sigma(\varepsilon_t) = 0{,}12$

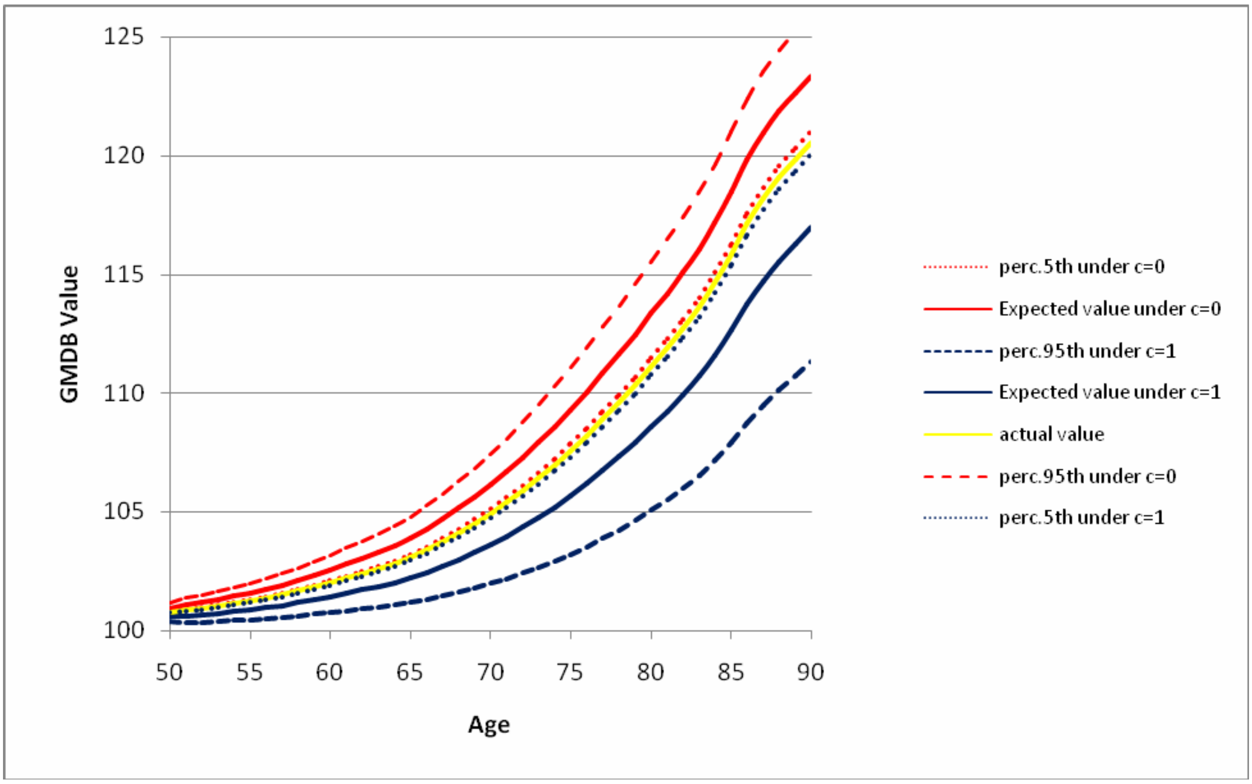

Figure 17: GMDB Value under the hypothesis $E(\varepsilon_t) = 0{,}30$ and $\sigma(\varepsilon_t) = 0{,}19$

We note from Figure 16 and 17 that, with a probability of 0,95, the GMDB value fluctuates between the dashed bands; therefore, we can easily derive a measure of Value at Risk for the product. We point out that greater is the expected value of the shocks larger is the impact on the GMDB value.

5.1 Sensitivity analysis

Up to this time we have considered the expected impact of mortality on the GMDB value; now we carry out a sensitivity analysis, in which we analyze the effect of changes of variance in the distribution of mortality shocks. In particular, we fix a and b such that the shocks have a beta distribution with expected value equal to and standard deviation twice as much those of the previous example; therefore we set a = 0,056 and b = 0,5, so that $E(\varepsilon_t) = 0,10$ and $\sigma(\varepsilon_t) = 0,24$.

As in the prior procedure of calculation, we simulate 1000 values of ε_t for every t from the new beta distribution, and then we calculate the largest shocks that can occur with a probability of 95%. The results are shown Figure 18-19:

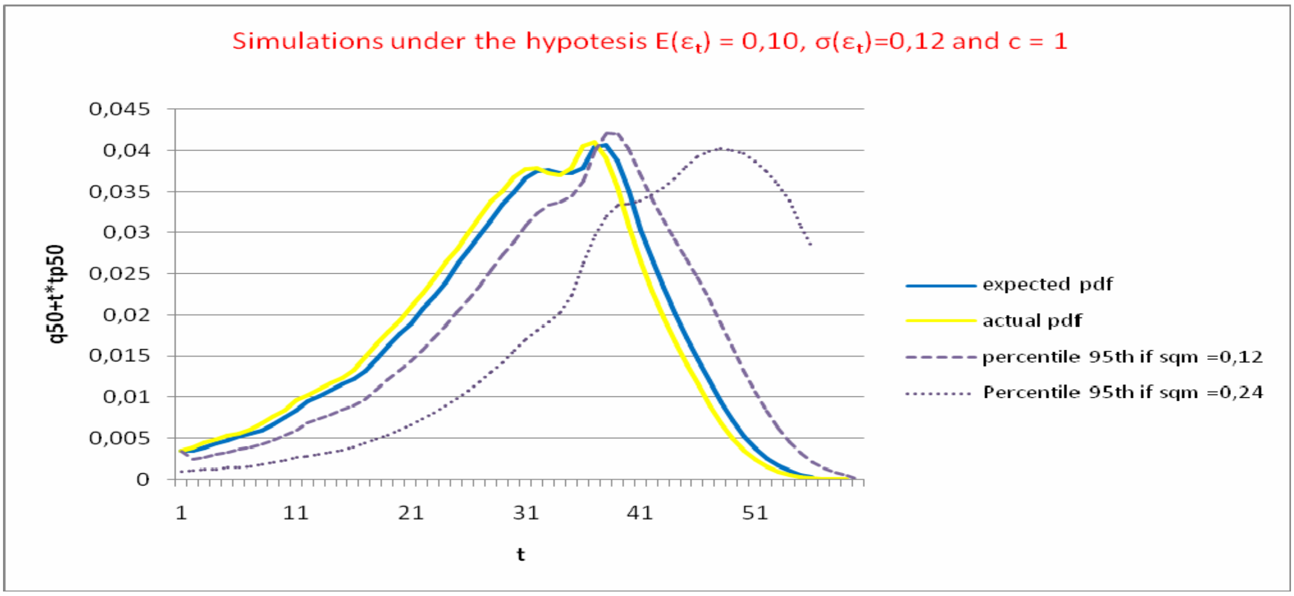

Figure 18: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\varepsilon_t) = 0,30$ and $\sigma(\varepsilon_t) = 0,24$; c = 1

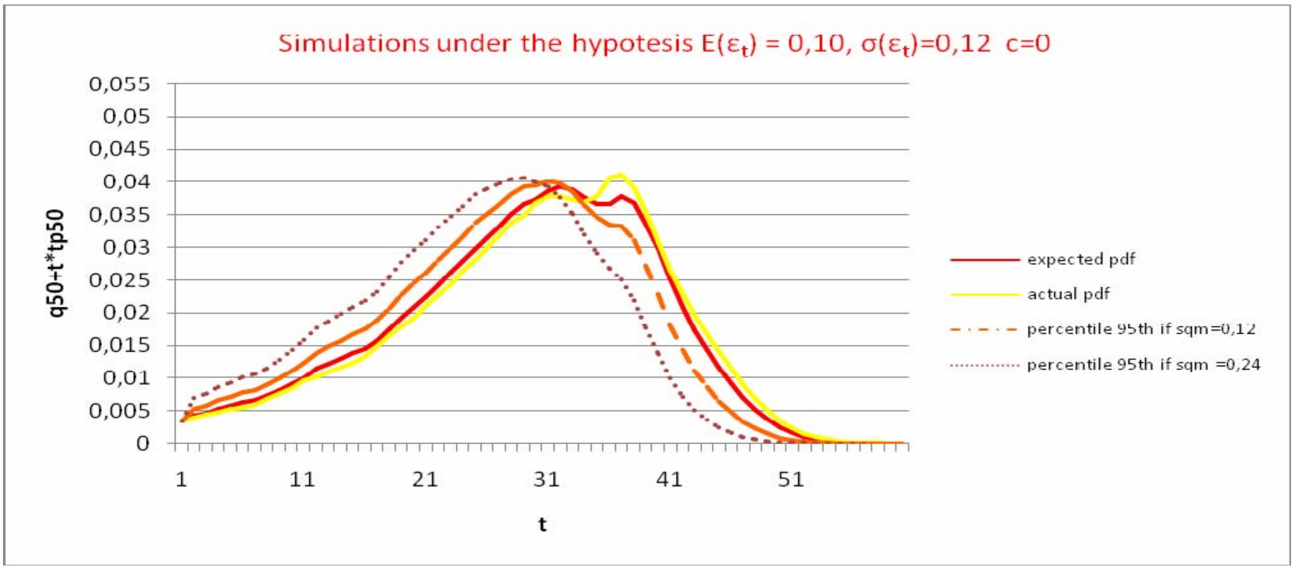

Figure 19: Actual, expected and prudential projected mortality probability function under the hypothesis $E(\varepsilon_t) = 0{,}30$ and $\sigma(\varepsilon_t) = 0{,}24$; c = 0.

If we consider a new beta distribution with the same expected value as before but with double the standard deviation, the simulated pdf under the considered prudential scenario moves to the right under the hypothesis c=1 and to the left under the hypothesis c=0. The consequences for the GMDB value are illustrated in Figure 20.

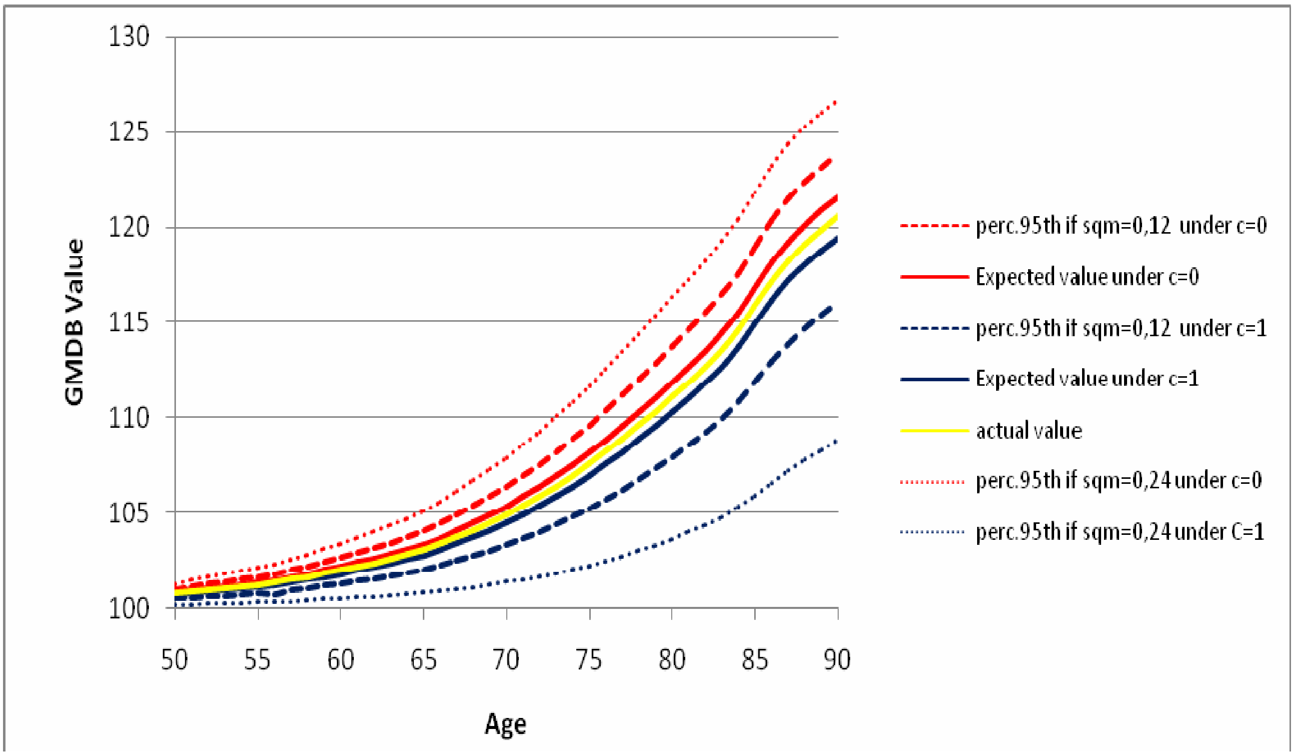

Figure 20: GMDB Value under the hypothesis $E(\varepsilon_t) = 0{,}10$, $\sigma(\varepsilon_t) = 0{,}12$ and $E(\varepsilon_t) = 0{,}10$, $\sigma(\varepsilon_t)= 0{,}24$

We point out that greater is the variance of shocks the larger is the possible oscillation of GMDB value around the expected value and the higher is the risk.

6. Conclusion

In this note we have described Guaranteed Minimum Death Benefit options embedded in Variable Annuities. We have dealt with the problem of valuation of these put options, which have stochastic maturity due to the involuntary exercise at the moment of death. We have introduced a theoretical model for the valuation of GMDB as a weighted average price of a set of deterministic put options with different maturity dates, where the weights are the deferred probabilities of death at each date. The contribution of this work has been to analyze the impact of mortality on the value of the GMDB with an application based on Italian data. We have shown that this product is sensitive to mortality risk, which impacts on the GMDB value through the weights in the valuation formula. We also need to keep in mind that the value of puts decreases with maturity. Since the fluctuation in the GMDB value depends on the interaction of all of the abovementioned factors, it is necessary to implement a simulation to measure and manage mortality risk. The results obtained in this work are not general, but depend on the hypothesis about the parameters of the financial and mortality models. Moreover, our valuation formula, Eq. (14), relates to an expected present value obtained by the methodology of risk-neutral valuation. It would be interesting to study the full distribution of the random present value of the GMDB option and the impact of mortality risk on it.

In the light of the analysis presented here, we identify areas where there is scope for further work. A limitation of the model developed is the assumption of a flat yield curve; we have made this hypothesis for the sake of simplicity and a complete description of the financial market was outside of the scope of this work, being focused on mortality risk. Certainly, in a further work the model can be improved by introducing an additional hypothesis of a stochastic interest rate term structure. Furthermore, the sensitivity analysis shown in section 5 can be extended in order to examine the sensitivity of the GMDB value to each parameter of the financial and mortality model.

One problem left open is the definition of an efficient risk management strategy for the GMDB option. The valuation formula expressed in Eq.(14) shows that this product is affected by financial risk, due to the changes in the fund value and in the level of interest rates over time, and by mortality risk. The hedging of financial risk is troublesome because of the long maturity of these contracts; this feature increases in the presence of the longevity risk. Also, our study highlights that the mispricing due to neglecting mortality improvements or worsening is noticeable over the long-term horizon. For this reason, a stochastic mortality approach is necessary in order to avoid underestimation or overestimation of the expected present value of this insurance contract which has a significant mortality component.

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